Conditional Game Theory:I

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1 Worked Example

1.1 Battle of the Sexes

Here, we will be concerned with the classical game, 'Battle of the Sexes' [1]. This game is two-player, where assumed that neither player has perfect information of the other. Hence, we are interested in calculating the optimum strategy. Here, we use this example to show the various extensions that are introduced to game theory through conditional game theory. In traditional game theory, the 'Battle of the Sexes' game has the following payoff matrix,

	W	
M	D	B
D	$4,3$	$2,2$
B	$1,1$	$3,4$

Table 1: Payoff matrix for the 'Battle of the Sexes' game

Considering this payoff matrix, we find that there are two stable Nash equilibria, namely (D,D) and (B,B). Therefore, we can claim that this game is left unresolved using traditional methods. In order to address this unsatisfying conclusion, previous work has appealed to cultural assumptions that are applied post hoc. Such assumptions are said to be exogenous; they are added onto the system from outside. In order to avoid the need to add these exogenous effects, we can use conditional game theory (CGT). In what follows, we outline the necessary steps required to recast this game in the language of CGT.

We assume that the game is directed but not bi-directional, ie. the direction of influence goes from one player to the other (but does not apply in reverse). We follow typical conventions and name player 1, W, and player 2, M. Hence, we have the following network of influence as shown in Figure 1.

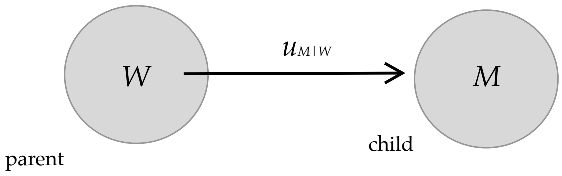

Figure 1: Network of the two players within the 'Battle of the Sexes' game. Note, the influence is uni-directional.

In order to construct the 'Battle of the Sexes' within CGT, we apply several operational definitions. For further information about each definition, we refer the reader to [2].

First, we write out the **parameterised utility mass function**, $u_i(A_i, A_j)$. As the influence network is unidirectional (W directly influences M and not vice-versa), we have $i = W$. We assume that W will put α of their focus into (D, D), which is their favored option, and $1-\alpha$ for (B, B), their second favored option. They will put 0 of their focus into either (B, D) or (B, D). Hence, we have the parameterised utility mass function as,

$$\begin{aligned} u_W(D, D) &= \alpha \qquad & u_W(B, D) &= 0 \\ u_W(D, B) &= 0 \qquad & u_W(B, B) &= (1-\alpha), \end{aligned} \tag{1}$$

where $0 \leq \alpha \leq 1/2$.

Next, we construct the conditional utility mass function, $u_{M|W}(A_i, A_j)$. If the conjecture (ie the action profile) for W is $\{(D, D), (D, B), (B, D)\}$, the M will give all of their attention to (B, B) as this is their preferred option. Alternatively, if the conjecture for W is (B, B), then M will give β of their attention to (D, D) and $1-\beta$ to (B, B).

Using the information above, we are left with the following table outlining the various conditional utilities:

	$u_{M\|W}(a_{MM}, a_{MW}\|a_{WM}, a_{WW})$			
		a_{WM}, a_{WW}		
a_{MM}, a_{MW}	D, D	D, B	B, D	B, B
D, D	1	1	1	β
D, B	0	0	0	0
B, D	0	0	0	0
B, B	0	0	0	$(1-\beta)$

Table 2: Table showing the conditional utility mass function.

We note that each of the columns in Table 2 sum to one. The individual payoff matrices, as conditioned on the conjecture of W can also be constructed. See, for example, 3.

M	W D	B
D	$1, \alpha$	$0, 0$
B	$0, 0$	$0, (1-\alpha)$

M	W D	B
D	β, α	$0, 0$
B	$0, 0$	$(1-\beta), (1-\alpha)$

Table 3: The payoff matrix where M is conditioned on (a) (D, D), represented by first column in Table 2 and (b) (B, B), represented by the fourth column in Table 2.

As discussed above, we are assuming that the 'utility' of W is categorical, ie that their preferences are uninfluenced by the choice of M, and is given by xxx. We use this information and the preceding discussion to develop the **joint utility**,

$$u_{M|W}(a_M, a_W) = u_M(a_M) u_{W|M}(a_W|a_M), \tag{2}$$

which, for our specific case, can be expanded to

$$u_{M|W}\left[(a_{MM}, a_{MW}), (a_{WM}, a_{WW})\right] = u_M(a_{MM}, a_{MW}) u_{W|M}(a_{MM}, a_{MW}|a_{WM}, a_{WW}), \tag{3}$$

This gives the following table outlining the joint utilities,

a_{MM}, a_{MW}	a_{WM}, a_{MM} D, D	D, B	B, D	B, B
D, D	α	0	0	$\beta(1-\alpha)$
D, B	0	0	0	0
B, D	0	0	0	0
B, B	0	0	0	$(1-\alpha)(1-\beta)$

This can be used to construct the ex poste (ie after the conditioning) payoff matrix. We calculate the marginal utility by summing the individual rows,

$$\begin{aligned} v_M(D, D) &= \alpha + \beta(1-\alpha) \qquad & v_M(B, D) &= 0 \\ v_M(D, B) &= 0 \qquad & v_M(B, B) &= (1-\alpha)(1-\beta) \end{aligned} \tag{4}$$

and recall that the the ex ante (ie before the conditioning) utility of W is categorical (so remains unchanged). The ex poste payoff matrix is then given by Table 4.

M	W D	B
D	$\alpha + \beta(1-\alpha), \alpha$	$0, 0$
B	$0, 0$	$(1-\alpha)(1-\beta), (1-\alpha)$

Table 4: The ex poste payoff matrix for $u_{M|W}$.

The above table describes the pay off matrix for both M and W as conditioned on the categorical utility of W. However, to fully describe the game dynamics we need to account for the coordinated response. To do so, we need a theory of how the individual actions fit together, namely the **coordinated utility**. This is given by,

$$
\begin{aligned}
w_M(D,D) &= u_{MW}\left[(D,D),(D,D)\right] + u_{MW}\left[(D,D),(B,D)\right] + u_{MW}\left[(D,B),(D,D)\right] + u_{MW}\left[(D,B),(B,D)\right] \\
w_M(D,B) &= u_{MW}\left[(D,D),(D,B)\right] + u_{MW}\left[(D,D),(B,B)\right] + u_{MW}\left[(D,B),(D,B)\right] + u_{MW}\left[(D,B),(B,B)\right] \\
w_M(B,D) &= u_{MW}\left[(B,D),(D,D)\right] + u_{MW}\left[(B,D),(B,D)\right] + u_{MW}\left[(B,B),(D,D)\right] + u_{MW}\left[(B,B),(B,D)\right] \\
w_M(B,B) &= u_{MW}\left[(B,D),(D,B)\right] + u_{MW}\left[(B,D),(B,B)\right] + u_{MW}\left[(B,B),(D,B)\right] + u_{MW}\left[(B,B),(B,B)\right].
\end{aligned}
$$

Hence, we have

$$
\begin{aligned}
w_{MW}(D,D) &= \alpha & w_{MW}(B,D) &= 0 \\
w_{MW}(D,B) &= \beta(1-\alpha) & w_{MW}(B,B) &= (1-\alpha)(1-\beta)
\end{aligned}
\tag{5}
$$

which implies that the coordinated decision rules are given by,

$$
\begin{aligned}
w_M(D) &= w_{MW}(D,D) + w_{MW}(D,B) = \alpha + \beta(1-\alpha) \\
w_M(B) &= w_{MW}(B,D) + w_{MW}(B,B) = (1-\alpha)(1-\beta).
\end{aligned}
\tag{6}
$$

We note that this can also be arrived at when M and W are entirely disassociated (such that their actions are made independent of the considerations of the other). To show this,we begin by assuming xxx, such that we have

$$
u_W(D) = \alpha \qquad u_W(B) = 1-\alpha. \tag{7}
$$

As M has their conditional utility mass function only considering their own actions (but conditioned on Ws), we have

$$
\begin{aligned}
u_{M|W}(D,D) &= 1 & u_{M|W}(D,B) &= 0 \\
u_{M|W}(B,D) &= \beta & u_{M|W}(B,B) &= (1-\beta)
\end{aligned}
\tag{8}
$$

As above, in this description the actions are both disassociated such that their coordination function is equal to the coordination utility, that is

$$
w_{MW}(a_M, a_W) = u_{M|W}(a_M|a_W)u_W(a_W). \tag{9}
$$

This implies that,

$$\begin{aligned} w_{MW}(D,D) &= \alpha \qquad & w_{MW}(D,B) &= (1-\beta) \\ w_{MW}(B,D) &= 0 \qquad & w_{MW}(B,B) &= (1-\alpha)(1-\beta) \end{aligned} \tag{10}$$

giving,

$$u_M(D) = \alpha + \beta(1-\alpha) \qquad u_M(B) = (1-\alpha)(1-\beta), \tag{11}$$

as in Equation 6.

1.2 Summary

We have now reviewed four different versions of the game.

- *Classical game:* Simplest model, where selfish individuals are assumed to act to maximise their self-interest only.
- *Focal Point:* Extension to the classical game where exogenous considerations are introduced to resolve the game.
- *Fully Sociated game:* The complete sociated game where each action is conditional on the implied preference of the other agents.
- *Disassociated game:* Simplification of above, where we need only consider the preferences of M which are influenced by those of W.

For more information, we refer the reader to [2].

References

[1] Fudenberg, D. and Tirole, J., *Game theory*, MIT Press, (1991)

[2] Stirling, Wynn C, *Theory of conditional games*, Cambridge University Press, (2012)