University of Bristol
Faculty of Engineering
Advanced Control & Dynamics

Coursework

Submitted by

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Declaration

I declare that the coursework submitted is my own and has not (either in whole or part) been submitted towards the award of any other qualification either at UWE or elsewhere. I have fully attributed/referenced all sources of information used during the completion of my assignment, and I an aware that failure to do so constitutes an assessment offence.

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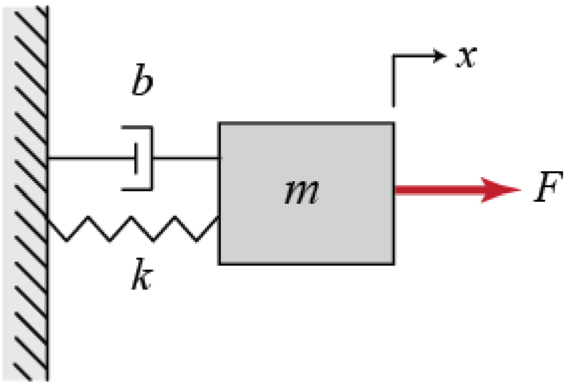
1 Operation of The Plant

A practical engineering plant which would feature similar dynamical behaviour to the theoretical dynamics given in the plant description below represents that of spring damper system.

$$\frac{\mathrm{Y(s)}}{U(s)} = \frac{1}{s^2 + 0.6\mathrm{s} + 4}$$

The mass spring damper system is used for the main purpose of producing oscillations based on some linear input. This system can be used in situations where oscillations are desired such as simulating earthquakes or flight simulation models or to reduce them such as suspension systems of automobiles etc.



By observing the forces acting on the mass the general force equation for a horizontal spring damper system can be computed as given below.

$$\mathrm{F(t)} = \mathrm{m}\frac{d^2x(t)}{dt^2} + \mathrm{c}\frac{\mathrm{d}x(t)}{d\mathrm{t}} + kx(t)$$

Where,
m[kg] – mass
k [N/m] – spring constant (stiffness)
c [Ns/m] – damping coefficient
F [N] – external force acting on the body (input)
x [m] – displacement of the body (output)

Applying the Laplace Transform to the force equation gives the following result:

$$\mathrm{F(s) = \ ms^2X(s) + csX(s)} + k\mathrm{X(s)}$$

$$\mathrm{F(s) = \ X(s)(ms^2 + cs} + k)$$

$$\frac{X(s)}{F(s)} = \frac{1}{\mathrm{m}s^2 + \mathrm{cs} + k}$$

$$H(s) = \frac{1}{\mathrm{m}s^2 + \mathrm{cs} + k}$$

$$\frac{\mathrm{Y(s)}}{U(s)} = \frac{1}{s^2 + 0.6\mathrm{s} + 4}$$

Comparing the transfer function with the desired plant description the values can be observed as

1[kg] – mass
4 [N/m] – spring constant (stiffness)
0.6 [Ns/m] – damping coefficient

To completely understand the system, we need to figure out the state space representation. The general form for state space representation can be represented as shown below:

$$\frac{\mathrm{d}x}{\mathrm{d}t} = \mathrm{A}x + \mathrm{B}u$$

$$y = \mathrm{C}x + \mathrm{D}u$$

$$\frac{\mathrm{d}x}{\mathrm{d}t} = \begin{bmatrix} 1 & 0 \\ -4 & -0.6 \end{bmatrix} x + \begin{bmatrix} 0 \\ 1 \end{bmatrix} u$$

$$y = \begin{bmatrix} 0 & 1 \end{bmatrix} x + [0] u$$

$$A = \begin{bmatrix} 1 & 0 \\ -4 & -0.6 \end{bmatrix}$$

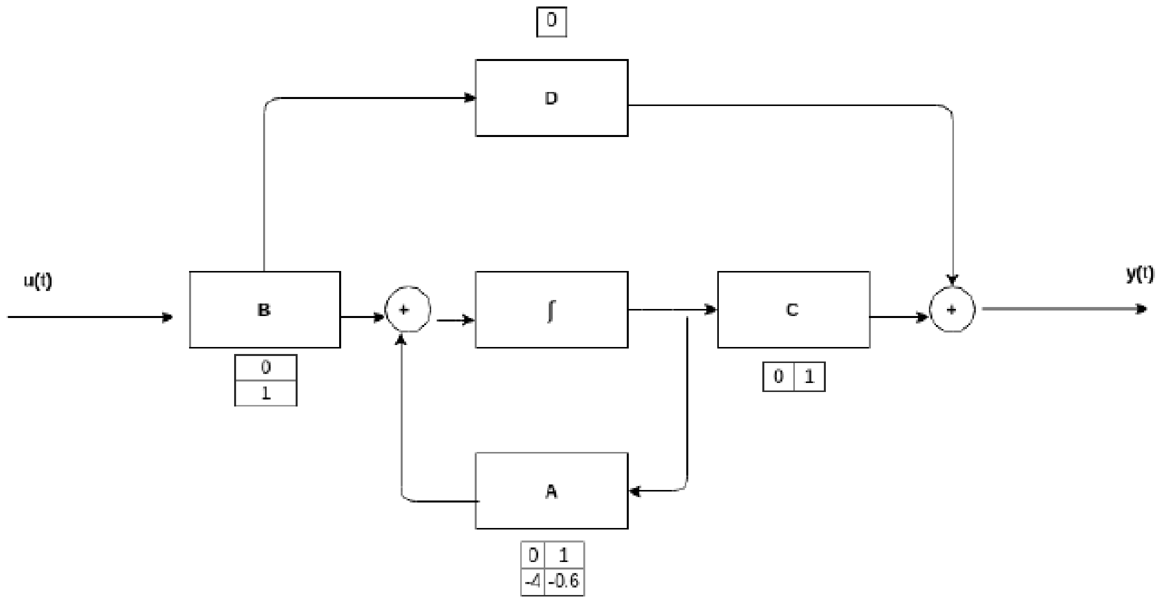
$$\mathrm{B} = \begin{bmatrix} 0 \\ 1 \end{bmatrix}$$

$$C = \begin{bmatrix} 0 & 1 \end{bmatrix}$$

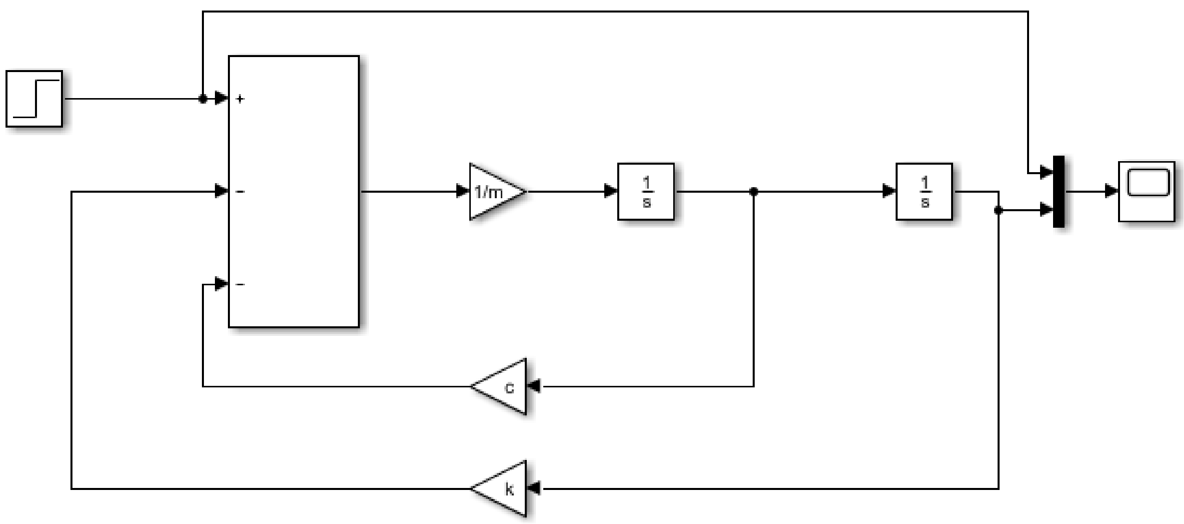
$$D = [0]$$

2 Control System Block Diagrams

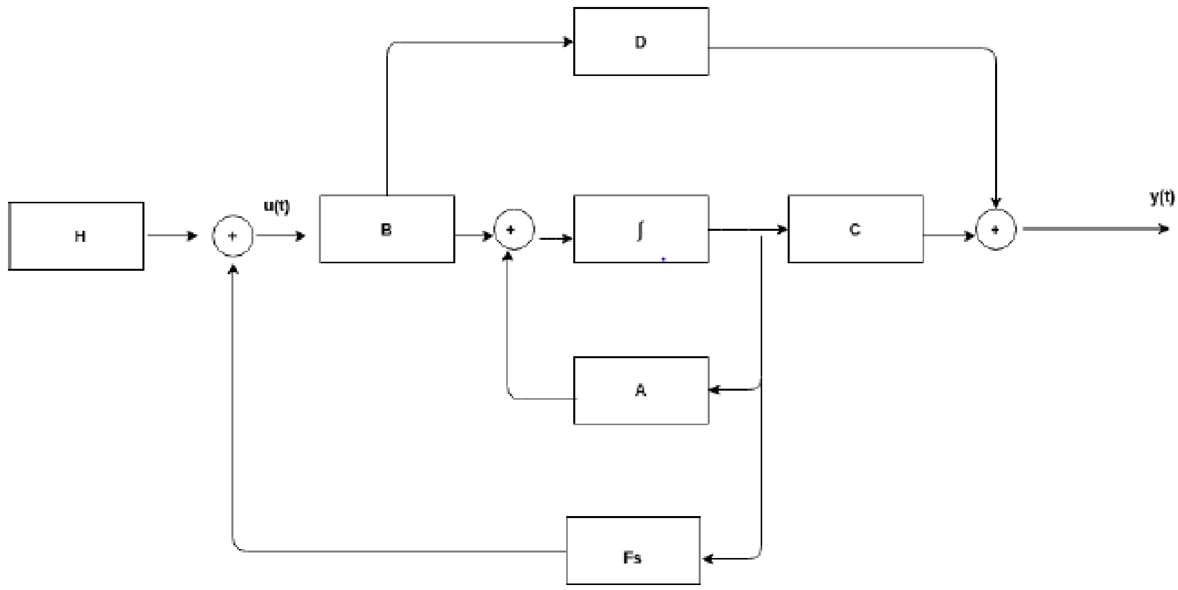
After analysing the state space representation the block diagram can be formed as shown below



Output feedback block diagram



Simulation Output block diagram



State Feedback block diagram

3 Plant Performance

To analyse the plant performance, first we need to compute the closed loop poles of the system. In order to do that we must solve the characteristic equation given below.

$$C \; = s^2 + 0.6\mathrm{s} + 4$$

The given characteristic equation is in the form of a quadratic equation whose roots can be determined using the formula,

$$x \; = \frac{-\mathrm{b} \pm \sqrt{\mathrm{b}^2 - 4\mathrm{ac}}}{2\mathrm{a}}$$

Where the quadratic equation can be written as,

$$ax^2 + \mathrm{bx} + c = 0$$

After solving the characteristic equation, the poles can be determined as,

$$\mathrm{S} = \; -0.3 + 1.97737\mathrm{j}$$

$$\mathrm{S} = \; -0.3 - 1.97737\mathrm{j}$$

Looking at the poles, both are negative which means that the system is stable.

The poles also contain a conjugate pair; therefore, the system is stable but has an oscillation which will decay which makes the system underdamped.

Criterion for state controllability is if the rank of the matrix p is equal to the number of states n =2. The system is said to be state controllable.

$$\text{Where } P = [B \quad AB \ldots A^{n-1}B]$$

As n =2,

$$P = [B \quad AB]$$

Substituting the values for matrices A and B, we get

$$P = \begin{bmatrix} 0 & 1 \\ 1 & -0.6 \end{bmatrix}$$

Rank of P = 2, therefore, the systems is state controllable.

Criterion for output controllability is if the rank of the matrix Q is equal to the number of outputs l =1. The system is said to be output controllable.

$$\text{Where } Q = [CB \quad CAB \ldots CA^{n-1}B]$$

As n = 2,

$$Q = [CB \quad CAB]$$

Substituting the values for C and B we get,

$$Q = [0 \quad 1]$$

Rank of Q is 1 which is equal to the number of outputs. Therefore, the system is output controllable. Hence the system is fully controllable.

A system is said to be completely observable on $t_0 < t < T$ if, for every t_0 and some T, every state vector $x(t_0)$ can be determined from the knowledge of the output vector $y(t)$ on $t_0 < t < T$. In physical terms, a system is completely observable if every transition of the system state eventually affects the output.

In other words, a system is said to be completely observable if the rank of the matrix R is equal to the number of states n =2.

$$\text{Where } R = [C \quad CA \ldots CA^{n-1}]^{T}$$

As n = 2,

$$R = [C \quad CA]^{T}$$

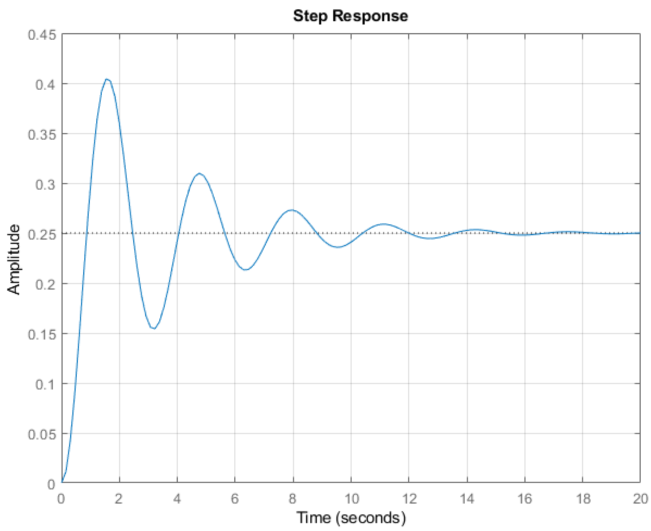
Substituting values for C and A we get,

$$R = \; [[1 \quad 0] \;\; [0 \; 1]]^{T}$$

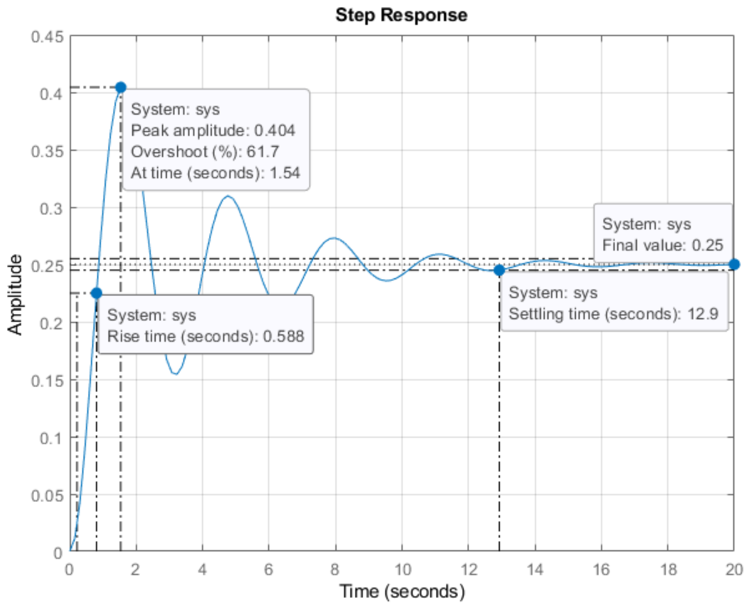
$$R = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$$

Rank of R = 2 = n, therefore system is completely observable. Hence the systems is completely observable and controllable.

To a given unit step response to the system the output is as shown below.



As expected, the output is a decaying oscillation. Analysing the waveform, the time response of the system can be found.



The rise time of a system is defined as the time taken for the response to rise from 10% to
90% of the steady state response, by observing the waveform the rise time is observed as
0.588 seconds. Other metrics of the oscillation are shown below.

Rise time = 0.588 secs
Peak amplitude = 0.404
Peak time = 1.54
Overshoot (%) = 61.7
Undershoot (%) = 0
Settling time = 12.9 secs
Settling min = 0.15
Settling max = 0.404

4 State Feedback Controller

Pole placement is a method employed in feedback control system theory to place the closed-loop poles of a plant in pre-determined locations in the s-plane. Placing poles is desirable because the location of the poles corresponds directly to the eigenvalues of the system, which control the characteristics of the response of the system. The system must be considered controllable in order to implement this method.

The controller parameters are defined as,

V(s) is the reference input

F_s is the $m*n$ **state feedback gain matrix** to specify the poles of the closed loop system and
H: is the $m*m$ **input feedforward gain matrix** to specify the zeros of the closed loop system.

The transfer function matrix between output *Y(s)* and reference *V(s)* is given by

$$\frac{Y(s)}{V(s)} = G(s) = \left\{[C + DF_s][sI - A - BF_s]^{-1} B + D\right\} H$$

$$= \left\{[C + DF_s]\left[sI - \bar{F}_s\right]^{-1} B + D\right\} H$$

$$= \left\{[C + DF_s]\frac{adj\left[sI - \bar{F}_s\right]}{det\left[sI - \bar{F}_s\right]} B + D\right\} H$$

$$\bar{F}_s = A + BF_s$$

Therefore the controller design can be divided into the assignment of poles and zeros according to some specifications, which also is called pole and zero assignment approach.

$$\Delta(s) = det\left[sI - \bar{F}_s\right]$$

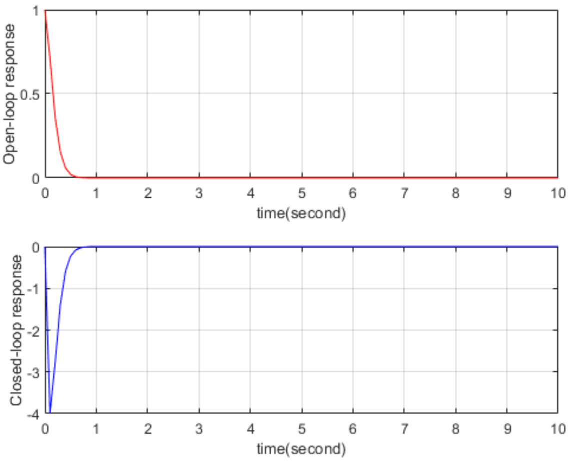
The selection of the input feedforward gain *H* is usually carried out in order to ensure that the actual output *y(t)* is exactly equal to the reference *v(t)* once steady state conditions are reached, i.e. *H* is chosen to counteract the inherent steady state gain of the closed loop system. Therefore, it is straightforward, with reference to eqn (4.4), and assuming a step reference *v(t),* to select

$$H = \left\{[C + DF_s]\left[sI - \bar{F}_s\right]^{-1} B + D\right\}^{-1}_{s=0} = -\left\{[C + DF_s]\left[\bar{F}_s\right]^{-1} B - D\right\}^{-1}$$

Placing poles at -10+3.2j and -10-3.2j

$$F = 1.0e + 04 * [-1.2149 \quad -0.4409]$$

H = 1.2153e+04



5 Observer Design

The objective of this section is to present a method of reconstructing or estimating the state vector from output variables and thereafter to form a procedure of the observer based control.

The observing problem can be described by considering

$$\hat{x}(t) = f[\hat{x}(\tau), u(\tau), y(\tau),\ t_0 \le \tau \le t]$$

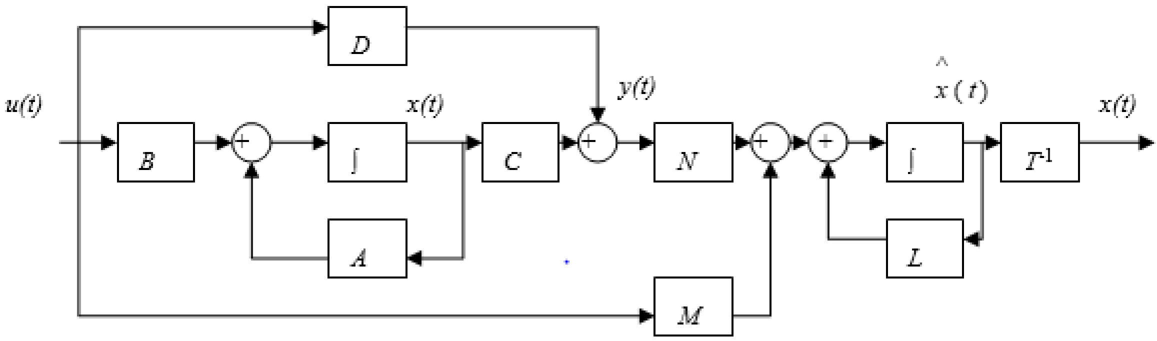
the aim is to evaluate this function *f[.]* such that $\hat{x}(t) \approx Tx(t)$ presents the reconstructed state of the system *S(A, B, C, D)*, where t_0 is the initial time of observation and *T* is a constant transformation matrix. It should be noted that the reconstructed state $\hat{x}(t)$ is a function of the past observations of $y(\tau)$, $t_0 < \tau < t$, and does not depend on future observations of $y(\tau)$, $\tau > t$.

The "simulator" that reconstructs $\hat{x}(t)$ is called a state observer. The observer is a dynamic sub-system in which, with increasing time, the output approaches the state that is to be reconstructed. Once the observer has been evaluated, the controller design procedures presented previously which assume knowledge of complete state vector, may be used by replacing the actual state with the reconstructed state.

$$\dot{\hat{x}}(t) = L\hat{x}(t) + Mu(t) + Ny(t)$$

$$\hat{x}(t_0) = Tx(t_0) \;\; implies \;\; \hat{x}(t) = Tx(t), \;\; \forall \;\; t > t_0$$

The general block diagram of the state observer can be found below



$$\text{Where,} \quad \begin{aligned} NC &= TA - LT \\ M &= TB - ND \end{aligned}$$

From this we can compute N, L and M as

$$L = A - NC$$
$$M = B - ND$$

Results found,

$$N = \begin{bmatrix} 19.4 \\ 94.6 \end{bmatrix}$$

$$M = \begin{bmatrix} 0 \\ 1 \end{bmatrix}$$

$$L = \begin{bmatrix} -19.4 & 1 \\ -98.6 & -0.6 \end{bmatrix}$$

6 References

[1] Brian Douglas . The Fundamentals of Control Theory. 2019 3

[2] Mark A Haidekker. Digital Control System, 2013 .3

[3] State Feedback. http://www.cds.caltech.edu/~murray/books/AM05/pdf/am06-statefbk_16Sep06.pdf

[4] 16.31 Feedback Control Systems. https://ocw.mit.edu/courses/aeronautics-and-astronautics/16-30-feedback-control-systems-fall-2010/lecture-notes/MIT16_30F10_lec11.pdf

[5] Transfer function.

https://www.wolframalpha.com/input/?i=transfer+function+1%2F%28s%5E2-%280.6+*s%29%2B4%29

[6] Feedback control system design. https://ocw.mit.edu/courses/mechanical-engineering/2-017j-design-of-electromechanical-robotic-systems-fall-2009/lecture-notes/MIT2_017JF09_control.pdf

[7] "state output and feedback control".

https://www.cds.caltech.edu/~murray/courses/cds101/fa04/caltech/am04_ch5-24oct04.pdf

7 Appendix

Matlab code for Spring damper system

```
clc;
clear all;


%% Constant values according to the transfer function
k = 4;
b = 0.6;
m= 1;

% Compute matrices for state space representation

A = [0 1; -k/m -b/m];
B = [0; 1/m];
C = [1 0];
D = [0];


sys = ss(A,B,C,D);

% Observe output for unit step input
step(sys);

%% Analysis the system

%Represent Transfer function to compute poles
num = [1];
den = [1 0.6 4];
[r,p,c] = residue(num,den)

poles = pole(sys);


%% State feedback controller

%input desired parameters for controller

desired_damping_ratio = input('Please Input Desired Damping
Ratio dr = ')
desired_natural_freq = input('Please Input Desired Natural
Frequency nf = ')
Simulation_time_final = input('Please Input Simulation Time
');


for i = 1:2
   M(1,:) = B';
```

```
   M(2,:) = [-det([A(:,2)';B']),det([A(:,1)';B'])];
end
N = [-2*desired_damping_ratio*desired_natural_freq-A(1,1)-
A(2,2),desired_natural_freq^2-det(A)]';
f = M\N;
f = f';

F = f;
h = -inv((C+D*f)*inv(A+B*f)*B-D);

H = h;

disp('The controller matrices are ')
F, H


%% State Observer

% Unit step responses of the open-loop and closed-loop systems
Ac = A+B*f; Bc = B*h; Cc = C+D*f; Dc = D*h; % Closed-loop
matrices

[yc,xc,tc] = step(Ac, Bc, Cc, Dc, 1, Simulation_time_final);
[y,x,t] = step(A, B, C, D, 1, Simulation_time_final);

subplot(2,1,1), plot(t, y), grid, xlabel('time(second)'),
ylabel('Open-loop response')
subplot(2,1,2), plot(tc, yc),
grid,xlabel('time(second)'),ylabel('Closed-loop response')


[m,n] = size(A); % get the dimension of the matrix A
s = zeros(1,n);

disp('The following is the information for the desired oberver
poles')
for i = 1:n
   in_text = ['Please Input Desired Pole s' num2str(i) ' = '];
   s(i) = input(in_text); % for full order observer, n poles
are needed
end

% Obtain the characteristic equation of the observe
ch_ob = poly(s);

% Calculate N for observer x = Lx + Mu + Ny
for i = 1:2
   A_N(1,:) = C;
   A_N(2,:) = [A(2,1)*C(2)-A(2,2)*C(1), A(1,2)*C(1)-
A(1,1)*C(2)];
end
```

```
B_N = [ch_ob(2)+A(1,1)+A(2,2),ch_ob(3)-det(A)]';
N = A_N\B_N;

% Calculate M for observer x = Lx + Mu + Ny
M = B-N*D;
L = A-N*C;

% Result display
disp('The observer matrices are ')
L, M, N
```

Output

```
r =

   0.0000 - 0.2529i
   0.0000 + 0.2529i


p =

  -0.3000 + 1.9774i
  -0.3000 - 1.9774i
```

c =

    []

dr =

    20

nf =

  110.2400

t_final =

   100

The controller matrices are

F =

   1.0e+04 *

   -1.2149   -0.4409

H =

   1.2153e+04

The following is the information for the desired oberver poles
Please Input Desired Pole s1 = -10+3.2j
Please Input Desired Pole s2 = -10-3.2j
The observer matrices are

L =

```
  -19.4000    1.0000
  -98.6000   -0.6000


M =

     0
     1


N =

   19.4000
   94.6000
```