INDIAN INSTITUTE OF TECHNOLOGY GUWAHATI

DEPARTMENT OF MECHANICAL ENGINEERING

Guwahati – 781 039, Assam, India

BY

Nishant Nanasaheb Jagtap

234103329

(j.nishant@iitg.ac.in)

ME 670 Advance Computational Fluid Dynamics (Advance CFD)

Assignment – 2

Guided By

Prof. Atul Soti

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MULTIGRID ALGORITHM

1. Given

1D problem

$$-u''(x) + \sigma u(x) = f(x)$$

- Homogeneous Boundary Conditions
- Finite Difference Method

1.1. DATA

n = 512 (513 points)	
$\omega = 2/3 = 0.66667$	$\nu_1 = \nu_1 = 2$
$\sigma = 1$	$C = \pi^2 k^2 + \sigma$
$f(x) = Csin(k\pi x)$ **Exact Solution** → $u(x) = \frac{C}{\pi^2 k^2 + \sigma} sin(k\pi x)$	
Iterate till Residual 2-norm > 10^{-6}	

K=1	C = 10.869604
K=10	C = 987.96044010893

2. FDM (Finite Difference Method)

0<= x <=1

0<=i<=n (n+1) points

$$h = \frac{1.0}{n}$$

$$x_i = i * h$$

For internal nodes - 0<i<n (n-1) points

2.1. Discretized Equation

$$\frac{-u_{i+1} + 2u_i - u_{i-1}}{h^2} + \sigma u_i = f_i$$

$$LHS = \frac{-u_{i+1} + 2u_i - u_{i-1}}{h^2} + \sigma u_i$$

$$RHS = f_i$$

$$u_i = \frac{h^2 f_i + u_{i+1} + u_{i-1}}{2 + \sigma h^2}$$

$$Residual = RHS - LHS$$

2.2. Code Explanation

m=N = 512 → 513 points

Total number of levels → $level\_\max = \frac{\log(N)}{\log(2)}$ = 9

$1 \leq level \leq 9$

$$n = \frac{N}{2^{level-1}}$$

Arrays→

For the Code, 4 2D arrays are used for each level having size of finest grid-

f→ RHS value array	v → v value array	e → error value array	r → residual value array
f[level_max+1][m+1]	v[level_max+1][m+1]	e[level_max+1][m+1]	r[level_max+1][m+1]
f[10][513]	v[10][513]	e[10][513]	r[10][513]

Relaxations are done by Weighted Jacobi Method or Gauss Seidel Method – Relaxations are done by at start of level while going down the V Cycle and are applied at last while coming back up in the V Cycle

Restriction Formula – All residuals are restricted and stored as RHS (f array) on next grid level which will be used for relaxation

$$v_j^{2h} = \frac{1}{4}\left(v_{2j-1}^h + 2v_{2j}^h + v_{2j+1}^h\right), \qquad 1 \leq j \leq \frac{n}{2} - 1.$$

Prolongation Formula - All errors are prolongated and added to earlier computed solution (v array) on previous grid level which will be used for relaxation

$$\begin{aligned} v_{2j}^h &= v_j^{2h}, \\ v_{2j+1}^h &= \frac{1}{2}\left(v_j^{2h} + v_{j+1}^{2h}\right), \quad 0 \leq j \leq \frac{n}{2} - 1. \end{aligned}$$

The Algorithm is as described below which is further expanded using for loop for V Cycle and Full Multigrid

3. Algorithm

3.1. Two-Grid Correction Scheme

Two-Grid Correction Scheme

$$\mathbf{v}^h \leftarrow MG(\mathbf{v}^h, \mathbf{f}^h).$$

- Relax ν_1 times on $A^h\mathbf{u}^h = \mathbf{f}^h$ on Ω^h with initial guess $\mathbf{v}^h$.
- Compute the fine-grid residual $\mathbf{r}^h = \mathbf{f}^h - A^h\mathbf{v}^h$ and restrict it to the coarse grid by $\mathbf{r}^{2h} = I_h^{2h} r^h$.
- Solve $A^{2h}\mathbf{e}^{2h} = \mathbf{r}^{2h}$ on Ω^{2h}.
- Interpolate the coarse-grid error to the fine grid by $\mathbf{e}^h = I_{2h}^h \mathbf{e}^{2h}$ and correct the fine-grid approximation by $\mathbf{v}^h \leftarrow \mathbf{v}^h + \mathbf{e}^h$.
- Relax ν_2 times on $A^h\mathbf{u}^h = \mathbf{f}^h$ on Ω^h with initial guess $\mathbf{v}^h$.

3.2. V-CYCLE MULTIGRID

Algorithm

V-Cycle Scheme

$$\mathbf{v}^h \leftarrow V^h(\mathbf{v}^h, \mathbf{f}^h)$$

- Relax on $A^h\mathbf{u}^h = \mathbf{f}^h$ ν_1 times with initial guess $\mathbf{v}^h$.
- Compute $\mathbf{f}^{2h} = I_h^{2h}\mathbf{r}^h$.
  - Relax on $A^{2h}\mathbf{u}^{2h} = \mathbf{f}^{2h}$ ν_1 times with initial guess $\mathbf{v}^{2h} = \mathbf{0}$.
  - Compute $\mathbf{f}^{4h} = I_{2h}^{4h}\mathbf{r}^{2h}$.
    - Relax on $A^{4h}\mathbf{u}^{4h} = \mathbf{f}^{4h}$ ν_1 times with initial guess $\mathbf{v}^{4h} = \mathbf{0}$.
    - Compute $\mathbf{f}^{8h} = I_{4h}^{8h}\mathbf{r}^{4h}$.

.
.
.

- Solve $A^{Lh}\mathbf{u}^{Lh} = \mathbf{f}^{Lh}$.

.
.
.

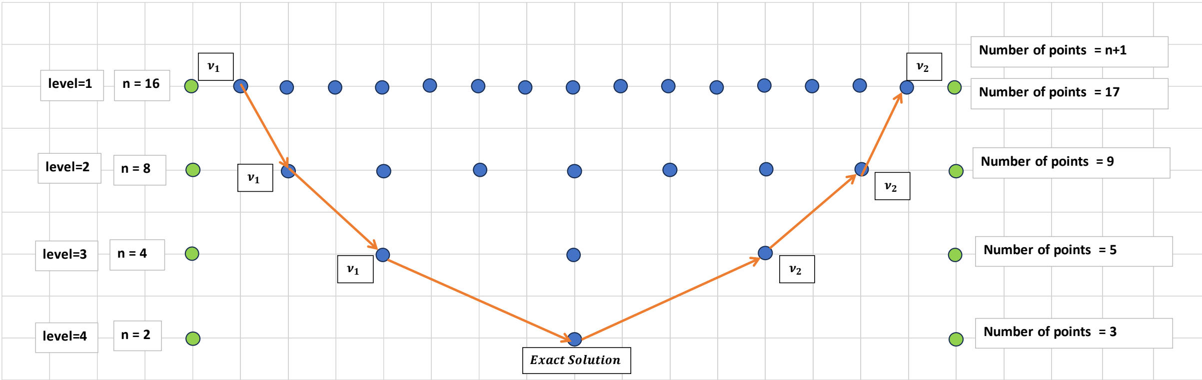
    - Correct $\mathbf{v}^{4h} \leftarrow \mathbf{v}^{4h} + I_{8h}^{4h}\mathbf{v}^{8h}$.
    - Relax on $A^{4h}\mathbf{u}^{4h} = \mathbf{f}^{4h}$ ν_2 times with initial guess $\mathbf{v}^{4h}$.
  - Correct $\mathbf{v}^{2h} \leftarrow \mathbf{v}^{2h} + I_{4h}^{2h}\mathbf{v}^{4h}$.
  - Relax on $A^{2h}\mathbf{u}^{2h} = \mathbf{f}^{2h}$ ν_2 times with initial guess $\mathbf{v}^{2h}$.
- Correct $\mathbf{v}^h \leftarrow \mathbf{v}^h + I_{2h}^h\mathbf{v}^{2h}$.
- Relax on $A^h\mathbf{u}^h = \mathbf{f}^h$ ν_2 times with initial guess $\mathbf{v}^h$.

3.2.1. Description of V-Cycle for n=16

$\nu_1 => \textit{Gauss Siedel iteration at start of each level while Going down the V Cycle}$

$\nu_2 => \textit{Gauss Siedel iteration at each level while Going back up the V Cycle}$

V Cycle Multigrid Algorithm (For n=16)



3.3. FULL MULTIGRID

Algorithm

Full Multigrid V-Cycle

$$\mathbf{v}^h \leftarrow FMG^h(\mathbf{f}^h).$$

Initialize $f^{2h} \leftarrow I_h^{2h} f^h,\ f^{4h} \leftarrow I_{2h}^{4h} f^{2h}, \ldots$.

- Solve or relax on coarsest grid.

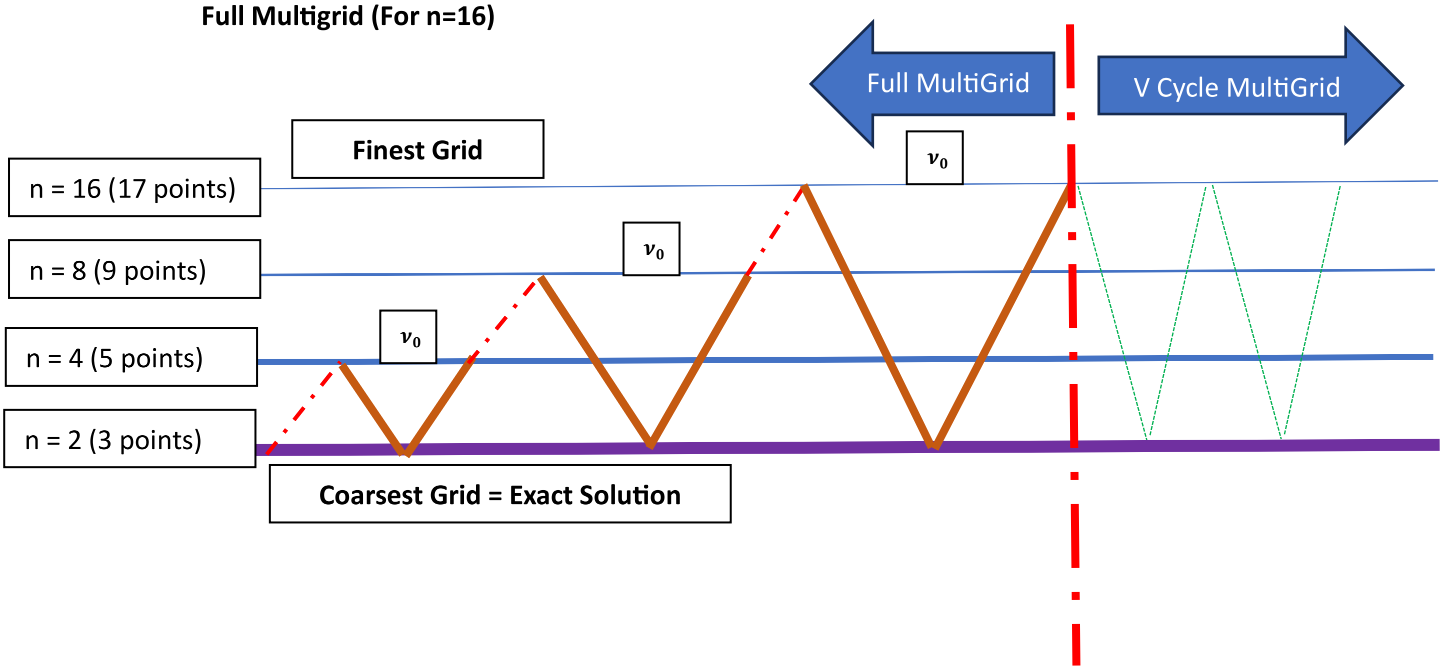
$\vdots$

- $\mathbf{v}^{4h} \leftarrow I_{8h}^{4h} \mathbf{v}^{8h}$.
- $\mathbf{v}^{4h} \leftarrow V^{4h}(\mathbf{v}^{4h}, \mathbf{f}^{4h})\ \nu_0$ times.
- $\mathbf{v}^{2h} \leftarrow I_{4h}^{2h} \mathbf{v}^{4h}$.
- $\mathbf{v}^{2h} \leftarrow V^{2h}(\mathbf{v}^{2h}, \mathbf{f}^{2h})\ \nu_0$ times.
- $\mathbf{v}^{h} \leftarrow I_{2h}^{h} \mathbf{v}^{2h}$.
- $\mathbf{v}^{h} \leftarrow V^{h}(\mathbf{v}^{h}, \mathbf{f}^{h}),\ \nu_0$ times.

3.3.1. Description of Full Multigrid for n=16

$$\nu_0 \Rightarrow \textit{Number of V Cycle at each level}$$

Full Multigrid (For n=16)



4. Results

4.1. Question 1 → V Cycle Multigrid

4.1.1. V-Cycle Multigrid for various relaxation methods

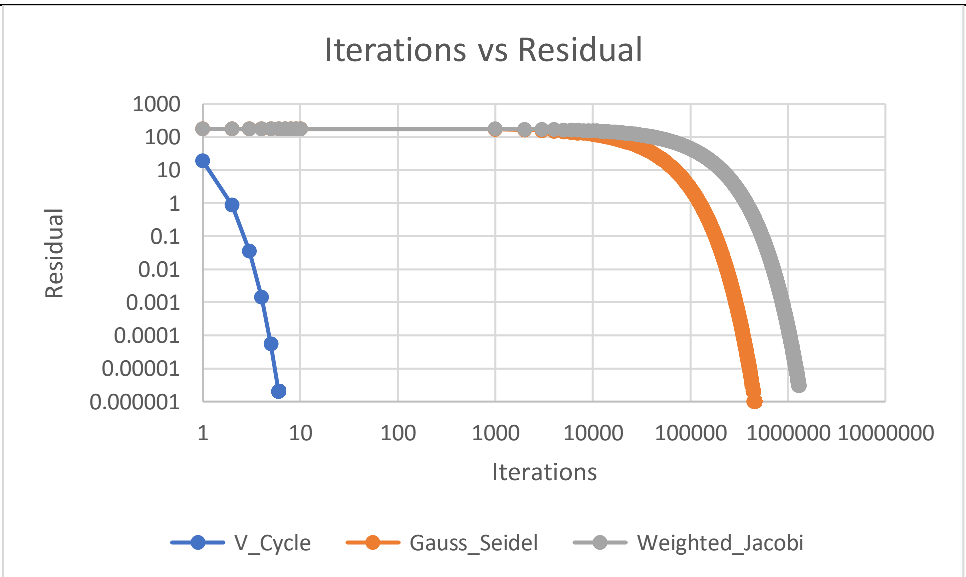
	Number of Iteration	
Algorithm implemented for v_1, v_2 *ietartions at each level*	K=1	K=10
Gauss Seidel	7	8
Weighted Jacobi	9	10

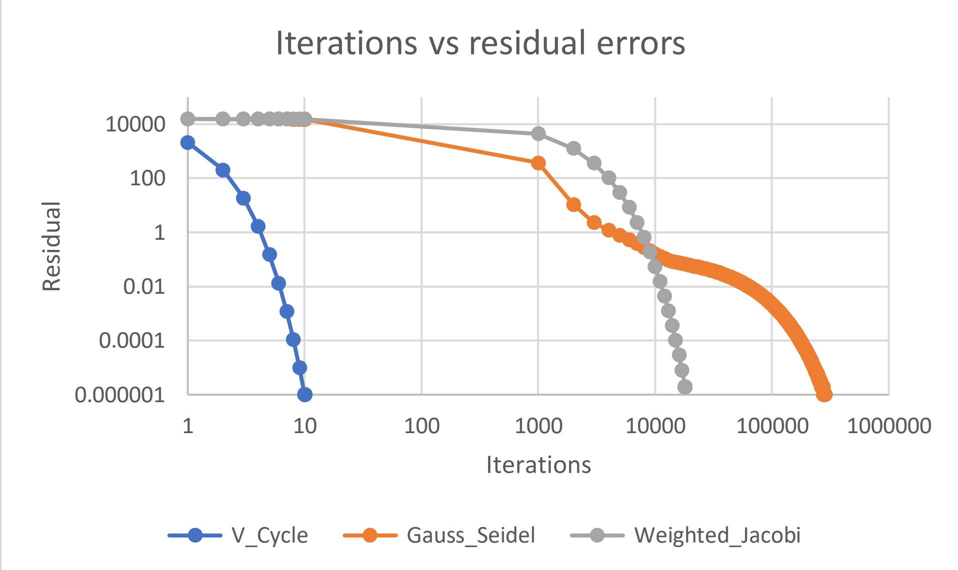
4.1.2. V-Cycle Multigrid vs Normal Iterative Methods

	Number of Iterations		
	V_Cycle Multigrid (Gauss Seidel)	Weighted Jacobi Method	Gauss Seidel
	Iterated till Residual > 10^{-6}		
K=1	7	1372794	457598
K=10	8	18688	285561

Conclusion: <u>It is clearly seen that Multigrid is much more efficient than normal iterative methods from above table</u>

4.1.3. PLOTS

K = 1	

K = 10	

4.2. Question 2 → Full Multigrid (FMG)

Results for FMG Starting from Lowest Level

4.2.1. K = 1

		Total Number of all V cycle Iteration in 1 FMG Cycle	**Residual after 1 FMG Cycle**	**Total Number of V Cycles after1 FMG**
Number of V Cycle at each level of 1 FMG	Algorithm implemented for $v_1 , v_2 = 2$ $itertions\ at\ each\ level$		**K=1**	
$v_0 = 1$	Gauss Seidel	8	1.9516312678	4
	Weighted Jacobi	8	1.9493234039	4
$v_0 = 2$	Gauss Seidel	16	0.0240847622	3
	Weighted Jacobi	16	0.0240764275	3
$v_0 = 3$	Gauss Seidel	24	0.0002980187	2
	Weighted Jacobi	24	0.0002986768	2

4.2.2. K = 10

		Total Number of all V cycle Iteration in 1 FMG Cycle	**Residual after 1 FMG Cycle**	**Total Number of V Cycles after1 FMG**
Number of V Cycle at each level of 1 FMG	Algorithm implemented for $v_1 , v_2 = 2$ $itertions\ at\ each\ level$		**K=10**	

$\nu_0 = 1$	Gauss Seidel	8	195.8608745361	5
	Weighted Jacobi	8	192.5883211242	7
$\nu_0 = 2$	Gauss Seidel	16	2.5706697081	4
	Weighted Jacobi	16	2.4916103290	6
$\nu_0 = 3$	Gauss Seidel	24	0.0395367160	3
	Weighted Jacobi	24	0.0473146771	5

FMG Code can start from any level as well

We see that Gauss Seidel method is more efficient than Weighed Jacobi for Multigrid

All the residuals after each iteration can be seen in the terminal after running the code

Code Execution:

V Cycle Code Output

```
nishant@nishant-VirtualBox:~/Documents/SEM 2/Advance CFD/Assignment_2$ gcc V_Cycle_Multigrid.c -lm
nishant@nishant-VirtualBox:~/Documents/SEM 2/Advance CFD/Assignment_2$ ./a.out
n = 512 , w = 0.666667 , neu = 2 , sigma = 1.000000 , k = 10 , C = 987.960440
Total number of levels: 9
neu_1 = 2  ,  neu_2 = 2
Which Method You need to use for relaxation->    1 = Weighted Jacobi   OR  2 = Gauss Seidel Jacobi: 2
Iteration No. = 1        Residual = 1843.9372347633
Iteration No. = 2        Residual = 81.9729569318
Iteration No. = 3        Residual = 2.8826502027
Iteration No. = 4        Residual = 0.0968290965
Iteration No. = 5        Residual = 0.0031666130
Iteration No. = 6        Residual = 0.0001033926
Iteration No. = 7        Residual = 0.0000034362
Iteration No. = 8        Residual = 0.0000001160
The number of V Cycle Iterations: 8
nishant@nishant-VirtualBox:~/Documents/SEM 2/Advance CFD/Assignment_2$
```

Full Multigrid Code Output

```
n = 512 , w = 0.666667 , neu = 2 , sigma = 1.000000 , k = 10 , C = 987.960440
Total number of levels: 9
Please enter number of V Cycle iteration (1 or 2 or 3) at each leve in 1 Full Multigrid: 3
neu_1 = 2  ,  neu_2 = 2
neu_0 = 3
Which Method You need to use for relaxation->    1 = Weighted Jacobi   OR  2 = Gauss Seidel Jacobi: 2
Error = 0.0050148973
Total V Cycles in 1 FMG = 24     Residual = 0.0016272236
----------After 1 FMG Cycle, Now applying V Cyles-----------
Iteration No. = 1        Residual = 0.0000463656
Iteration No. = 2        Residual = 0.0000013363
Iteration No. = 3        Residual = 0.0000000385
The total number of V Cycle Iterations after 1 FMG: 3
nishant@nishant-VirtualBox:~/Documents/SEM 2/Advance CFD/Assignment_2$
```