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ME 670 Advance Computational Fluid Dynamics (Advance CFD)

Assignment – 1

Guided By

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SIMPLE Algorithm on Lid-driven Cavity Problem

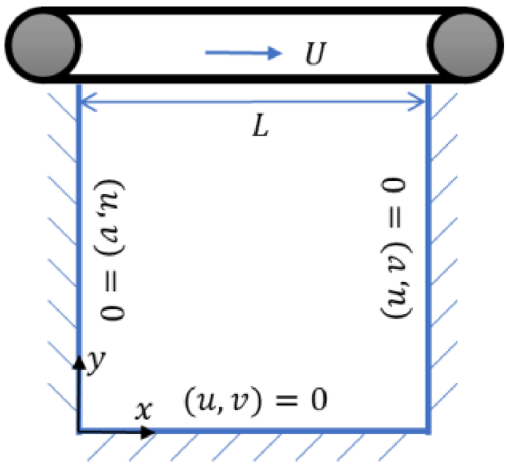


Figure1: A schematic of the lid-drive cavity problem

Given → **LID-DRIVE CAVITY MODEL PROBLEM**

128 X 128 => Staggered Grid => (Number of Pressure CV in x and y direction) → M = N = 128

(129 X 129 => Collocated Grid)

Re = {100.0, 400.0, 1000.0}

(Assume Distance in z direction as 1 and so not considered in the problem)

Governing Equations and Assumptions

Navier' Stokes Equation

(Continuum and Inertial Reference Frame, Newtonian Fluid, Symmetric Stress Tensor, Isotropic, Stokes' Hypothesis)

Continuity Equation	*Momentum Equation*
$\frac{\partial \rho}{\partial t} + \frac{\partial (\rho u_k)}{\partial x_k} = 0$	$\frac{\partial (\rho u_i)}{\partial t} + \frac{\partial (\rho u_i u_j)}{\partial x_j} = \rho F_{vi} + \frac{-\partial \sigma_{ij}}{\partial x_j}$

Where

Total Stress => $\sigma_{ij} = -p\delta_{ij} + 2\mu S_{ij} - \frac{2}{3}\mu\delta_{ij}(\frac{\partial u_k}{\partial x_k})$

Body Force => $\boldsymbol{F_{vi}}$

Assumption

1. No Body Force
2. Steady state
3. Incompressible
4. 2D
5. Constant Viscosity
6. Isothermal

$\boldsymbol{\frac{\partial u_k}{\partial x_k} = 0}$	$\boldsymbol{\frac{\partial(\rho u_i u_j)}{\partial x_j} = \frac{-\partial p}{\partial x_i} + \mu\frac{\partial^2 u_i}{\partial x_i x_j}}$

Momentum Conservation Equations:

x-momentum Conservation Equations	y-momentum Conservation Equations
$\frac{\partial(\rho uu)}{\partial x} + \frac{\partial(\rho uv)}{\partial y} = \frac{-\partial p}{\partial x} + \mu\frac{\partial^2 u}{\partial x^2} + \mu\frac{\partial^2 u}{\partial y^2}$	$\frac{\partial(\rho uv)}{\partial x} + \frac{\partial(\rho vv)}{\partial y} = \frac{-\partial p}{\partial y} + \mu\frac{\partial^2 v}{\partial x^2} + \mu\frac{\partial^2 v}{\partial y^2}$

Continuity Equation	$\frac{\partial u}{\partial x} + \frac{\partial v}{\partial y} = 0$

Non-Dimensional Equation

x-momentum Conservation Equations	y-momentum Conservation Equations
$\frac{\partial(u^*u^*)}{\partial x^*} + \frac{\partial(u^*v^*)}{\partial y^*} = \frac{-\partial p^*}{\partial x^*} + \frac{1}{Re}\frac{\partial^2 u^*}{\partial x^{*2}} + \frac{1}{Re}\frac{\partial^2 u^*}{\partial y^{*2}}$	$\frac{\partial(u^*v^*)}{\partial x^*} + \frac{\partial(v^*v^*)}{\partial y^*} = \frac{-\partial p^*}{\partial y^*} + \frac{1}{Re}\frac{\partial^2 v^*}{\partial x^{*2}} + \frac{1}{Re}\frac{\partial^2 v^*}{\partial y^{*2}}$

where

$$u^* = \frac{u}{U}, v^* = \frac{v}{U}, x^* = \frac{x}{L}, y^* = \frac{y}{L},\ p^* = \frac{p}{\rho U^2}$$

$$\boldsymbol{Re} = \frac{\rho UL}{\mu} = \text{Reynold's Number}$$

Discretized Equations (for FVM)

Let $u^* = u\,, v^* = v\,, x^* = x\,, y^* = y\,, \mathrm{p}^* = \mathrm{p}$ for simplicity & further analysis

Let $dx = dx^* = \Delta x\,;\ dy = dy^* = \Delta y$

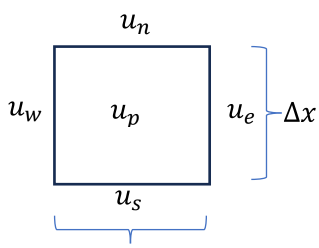
Expressing in Flux Vector →

$$\nabla.(\vec{u}\vec{u}) = \frac{1}{Re}\nabla.(\nabla\vec{u}) + s_\phi \quad ; \quad \nabla.(\vec{J}) = s_\phi \quad ; \quad \vec{J} = (\vec{u}\vec{u}) - \frac{1}{Re}(\nabla\vec{u})$$

$$\iiint \nabla.(\vec{J})\ dV = \iiint s_\phi\ dV \xrightarrow{\text{Gauss Divergence Theorem}} \iint \vec{J}.\vec{ds} = \iiint s_\phi\ dV = \overline{s_\phi}\Delta V$$

$$\vec{J_e}.\vec{S_e} + \vec{J_w}.\vec{S_w} + \vec{J_n}.\vec{S_n} + \vec{J_s}.\vec{S_s} = \overline{s_\phi}\Delta V$$

$$A_e = A_w = \Delta y \qquad A_n = A_s = \Delta x$$



Δy

Integrating Equation on Control Volume ($dV = \Delta x * \Delta x * 1 = dxdy$) for x momentum Equation

$$\iint \frac{\partial(uu)}{\partial x}dxdy + \iint \frac{\partial(uv)}{\partial y}dxdy = \iint \frac{-\partial p}{\partial x}dxdy + \frac{1}{Re}\left[\iint \frac{\partial^2 u}{\partial x^2}dxdy + \iint \frac{\partial^2 u}{\partial y^2}dxdy\right]$$

$$(uA)_e u_e - (uA)_w u_w + (vA)_n u_n + (vA)_s u_s = \left(\frac{1*A_e}{Re*\Delta x}u_e - \frac{1*A_w}{Re*\Delta x}u_w + \frac{1*A_n}{Re*\Delta y}u_n - \frac{1*A_s}{Re*\Delta y}u_s\right) + \overline{s_\phi}\Delta V$$

$$[(uA)u]_w^e - [(vA)u]_s^n = \frac{1}{Re}\left[\left(\frac{\partial u}{\partial x}\right)A\right]_w^e + \frac{1}{Re}\left[\left(\frac{\partial u}{\partial y}\right)A\right]_s^n$$

$$F_e = (uA)_e\ ;\ F_w = (uA)_w\ ;\ F_n = (vA)_n\ ;\ F_s = (vA)_s$$

$$D_e = \left(\left(\frac{\partial}{\partial x}\right)A\right)_e\ ;\ D_w = \left(\left(\frac{\partial}{\partial x}\right)A\right)_w\ ;\ D_n = \left(\left(\frac{\partial}{\partial y}\right)A\right)_n\ ;\ D_s = \left(\left(\frac{\partial}{\partial y}\right)A\right)_s$$

Final Discretized Equation →

$$F_e u_e - F_w u_w + F_n u_n + F_s u_s = (D_e u_e - D_w u_w + D_n u_n - D_s u_s) + \overline{s_\phi}\Delta V$$

Discretization Scheme – Hybrid Differencing Scheme

Hybrid Differencing Scheme =>
Central Difference (When Pe<2) OR First Order Upwind Scheme (When Pe>2) (It ignores Diffusion term)

Coefficients using Hybrid Scheme	
$a_W = max\left[F_w\,,\left(D_w + \frac{F_w}{2}\right),0\right]$	$a_E = max\left[-F_e\,,\left(D_e - \frac{F_e}{2}\right),0\right]$
$a_S = max\left[F_s\,,\left(D_s + \frac{F_s}{2}\right),0\right]$	$a_N = max\left[-F_n\,,\left(D_n - \frac{F_n}{2}\right),0\right]$

Grid Description

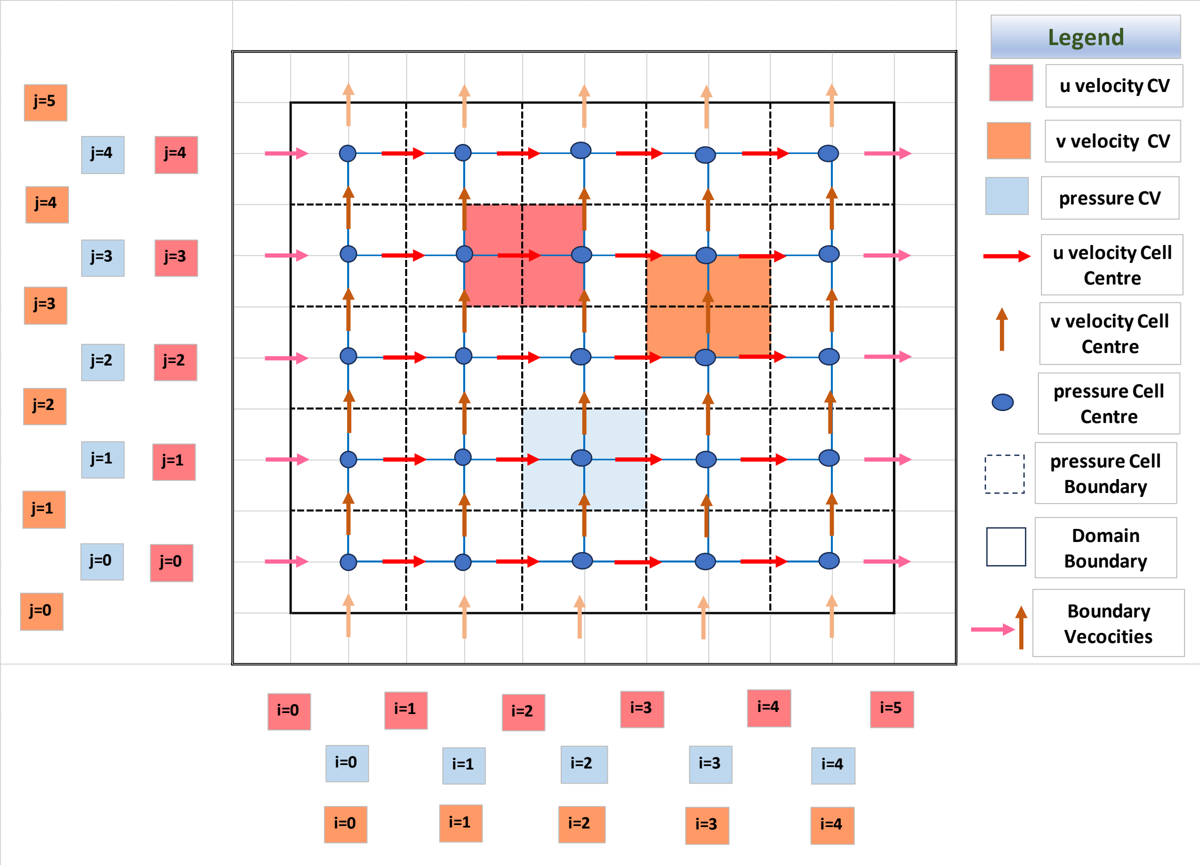
Below is the smaller version of Grid Considered in Code

Here M = N = 5

Pressure Nodes → 5 X 5	
U Velocity → 6 X 5	V Velocity → 5 X 6

The Indexing used is as shown in Figure for staggered grid

Staggered Grid



Actual Grid Dimensions

M=128 N=128

U Velocity →	i = 0 to M (i=0 to 128)	j = 0 to (N-1) (j=0 to 127)
V Velocity →	i = 0 to (M-1) (i= 0 to 127)	j = 0 to N (j=0 to 128)
Pressure →	i = 0 to (M-1) (i=0 to 127)	j = 0 to (N-1) (j=0 to 127)

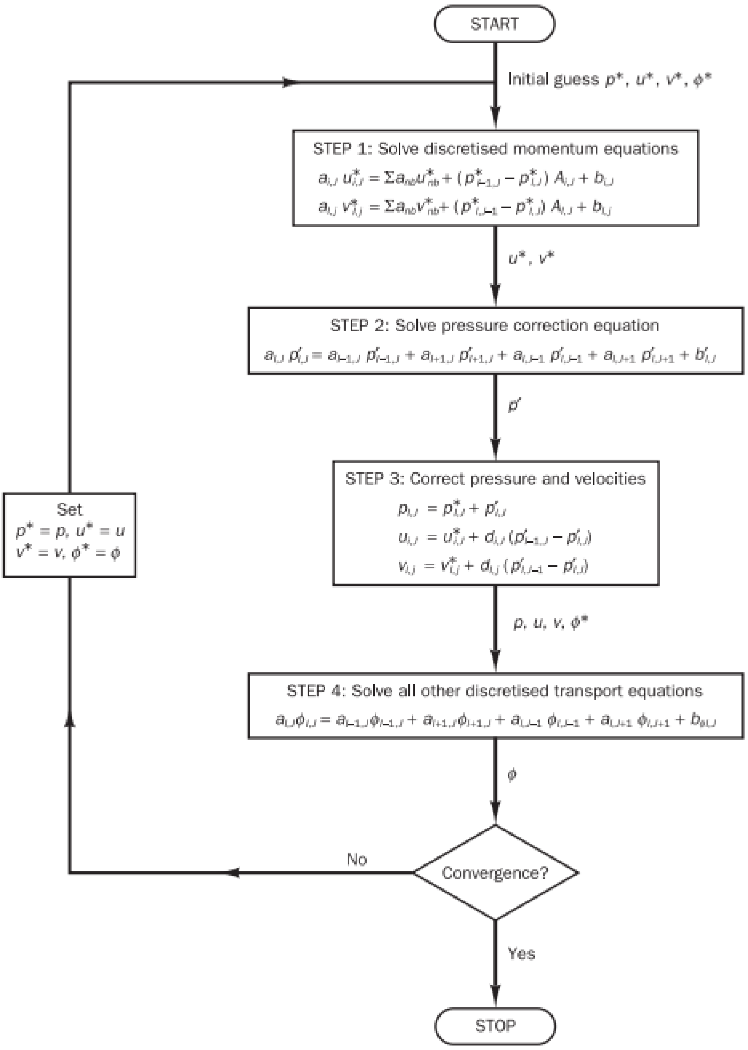
Boundary Conditions

Pressure Boundary Condition → There are no Pressure Boundary Conditions as all Pressure CV centres are inside the Geometry Domain. All pressure nodes in x & y direction are solved (updated) in continued equations.

U Velocity Boundary Condition → The U velocities in x direction at i=0 and i=M are at boundary and have fixed value (Not solved while updating x momentum equation). All u Velocities in y direction are solved (updated) in x momentum equations.

V Velocity Boundary Condition → The V velocities in y direction at j=0 and j=N are at boundary and have fixed value (Not solved while updating x momentum equation). All u Velocities in x direction are solved (updated) in y momentum equations.

SIMPLE Algorithm



Guessed Pressure → p^*

$$u = u^* + u' \;;\; v = v^* + v' \;;\; p = p^* + p'$$

Actual Velocities

$$a_p u_p = \sum_{nb} a_{nb} u_{nb} + \left[\, \Delta y * \left(p^*_{I-1,J} + p^*_{I,J}\right) + S_u \,\right]$$

$$a_p v_p = \sum_{nb} a_{nb} v_{nb} + \left[\, \Delta x * \left(p^*_{I,J-1} + p^*_{I,J}\right) + S_v \,\right]$$

u', v', p' => *Correction in velocities and pressure*

Use p^* to calculate Guessed Velocities → u^*, v^*

$$a_p u^*_p = a_E u^*_E + a_w u^*_w + a_N u^*_N + a_S u^*_S + \left[\, \Delta y * \left(p^*_{I-1,J} + p^*_{I,J}\right) + S_u \,\right]$$

$$a_p u^*_p = \sum_{nb} a_{nb} u^*_{nb} + \left[\, \Delta y * \left(p^*_{I-1,J} + p^*_{I,J}\right) + S_u \,\right]$$

$$\mathbf{a_p v_p^* = a_E v_E^* + a_W v_W^* + a_N v_N^* + a_S v_S^* + [\,\Delta x * (p_{I,J-1}^* + p_{I,J}^*) + S_v\,]}$$

$$\mathbf{a_p v_p^*} = \sum_{nb} a_{nb} v_{nb}^* + [\,\Delta x * (p_{I,J-1}^* + p_{I,J}^*) + S_v\,]$$

With each iteration of SIMPLE Algorithm we try to refine the Guess Velocity and make it closer to True velocity so the correction decreases

Subtracting [2] from [1] we get

$$a_p u_p' = \sum_{nb} a_{nb} u_{nb}' + [\,\Delta y * (p_{I-1,J}' + p_{I,J}')]$$

$$a_p v_p' = \sum_{nb} a_{nb} v_{nb}' + [\,\Delta x * (p_{I,J-1}' + p_{I,J}')\,]$$

As the correction at convergence is zero, we ignore the summation of neighbouring terms to simplify our analysis

$$u_p' = \frac{[\Delta y * (p_{I-1,J}' + p_{I,J}')]}{a_p} = d_{i,J}[\,(p_{I-1,J}' + p_{I,J}')] \qquad d_{i,J} = \frac{A_e \text{ or } A_w \text{ or } \Delta y}{a_p}$$

$$v_p' = \frac{[\Delta x * (p_{I,J-1}' + p_{I,J}')\,]}{a_p} = d_{I,j}[\,(p_{I,J-1}' + p_{I,J}')] \qquad d_{I,j} = \frac{A_n \text{ or } A_s \text{ or } \Delta x}{a_p}$$

$$u_p = u_p^* + u_p' \;;\; v_p = v_p^* + v_p'$$

$$u_p = u_p^* + d_{i,J}[\,(p_{I-1,J}' + p_{I,J}')]$$

$$v_p = v_p^* + d_{I,j}[\,(p_{I,J-1}' + p_{I,J}')]$$

Continuity Equations are solved using u_p , v_p to get p'

Update velocities and pressure using above formulas

Then for next iteration → $u_P^* = u_p$, $v_P^* = v_p$ & $p = p^* + p'$

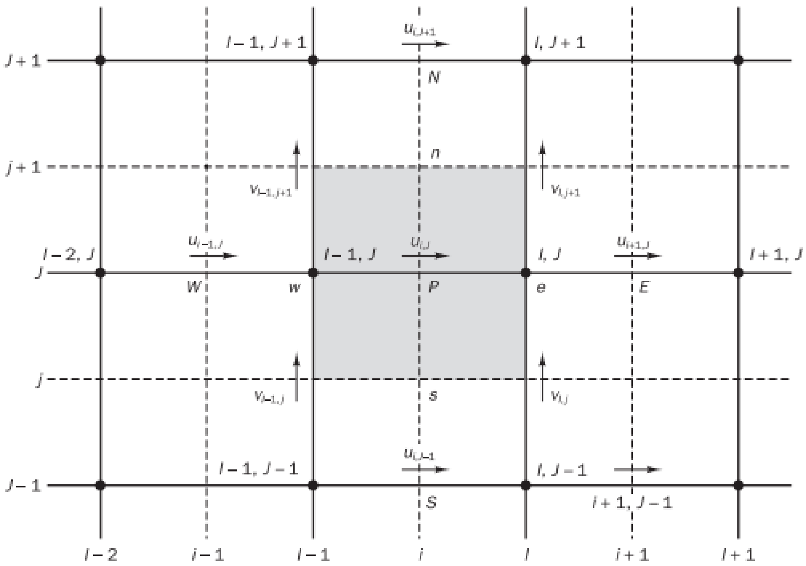
(Algorithm Flowchart is as shown above)

<u>The Discretized Equations solutions are as follows</u>

X Momentum Equation

Equations Solved for All Red Arrows

(For i=1 to M-1) (For j=0 to N-1)



$\mathbf{I = i}$	$\mathbf{J = j}$
i varies in x direction	j varies in y direction

True/Valid for all internal nodes

$$F_e = \frac{1}{2}\left[u^*_{i+1,J} + u^*_{i,J}\right] * \Delta y \qquad F_w = \frac{1}{2}\left[u^*_{i,J} + u^*_{i-1,J}\right] * \Delta y$$

$$F_n = \frac{1}{2}\left[v^*_{I,j+1} + v^*_{I-1,j+1}\right] * \Delta x \qquad F_s = \frac{1}{2}\left[v^*_{I,j} + v^*_{I-1,j}\right] * \Delta x$$

$$D_e = D_w = \frac{\Delta y}{\mathrm{Re} * \Delta x} \qquad D_n = D_s = \frac{\Delta x}{\mathrm{Re} * \Delta y}$$

$$S_p = 0\,; S_u = 0$$

$$a_P = a_W + a_E + a_N + a_S + \left(F_e - F_w + F_n - F_s\right) - S_p$$

(Note: All Coefficients are updated for velocities of earlier/last iteration)

$$Old\ values \rightarrow\ u^* = u^*_{old}\ and\ v^* = v^*_{old}$$

Boundary Conditions

For Lower Boundary	For Upper Boundary
$a_s = 0\,; F_s = 0;$ $S_p = -\frac{2 * \Delta x}{\mathrm{Re} * \Delta y}\,; S_u = 0$	$a_n = 0\,; F_n = 0;$ $S_p = -\frac{2 * \Delta x}{\mathrm{Re} * \Delta y}\,; S_u = \frac{2 * \mathrm{U} * \Delta x}{\mathrm{Re} * \Delta y}$

East and West Boundaries → i=0 and i=M serve as boundary nodes and so no specific boundary conditions needs to be defined

North and South Boundaries → As the Coefficient of neighbouring term at boundary are made zero (since it lies outside the boundary domain), the effect of convection and diffusion is considered through the Sp and Su terms.

U Momentum Equation solved using Gauss Siedel Method

$$u_E^*, u_W^*, u_N^*, u_S^* \text{ are all have updated values of current iterations}$$

$$a_p u_p^* = a_E u_E^* + a_w u_w^* + a_N u_N^* + a_S u_S^* + source = \sum_{nb} a_{nb} u_{nb} + Source$$

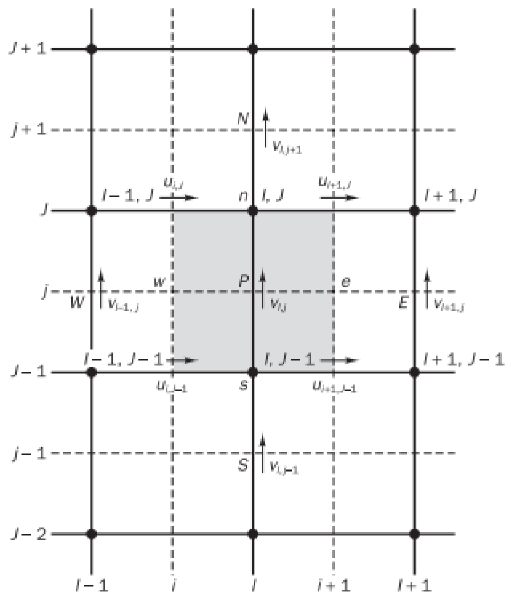
$$source = \Delta y * (p_{I-1,J}^* + p_{I,J}^*) + S_u$$

$$\boldsymbol{a_p u_p^* = a_E u_E^* + a_w u_w^* + a_N u_N^* + a_S u_S^* + [\,\Delta y * (p_{I-1,J}^* + p_{I,J}^*) + S_u\,]}$$

Y Momentum Equation

Equations Solved for All Brown Arrows

(For i=0 to M-1) (For j=1 to N-1)



I = i	J = j
i varies in x direction	j varies in y direction

True/Valid for all internal nodes

$$F_e = \frac{1}{2}[u_{i+1,J}^* + u_{i+1,J-1}^*] * \Delta y \qquad F_w = \frac{1}{2}[u_{i,J}^* + u_{i,J-1}^*] * \Delta y$$

$$F_n = \frac{1}{2}[v_{I,j}^* + v_{I,j+1}^*] * \Delta x \qquad F_s = \frac{1}{2}[v_{I,j-1}^* + v_{I,j}^*] * \Delta x$$

$$D_e = D_w = \frac{\Delta y}{\text{Re}*\Delta x} \qquad D_n = D_s = \frac{\Delta x}{\text{Re}*\Delta y}$$

$$S_p = 0\,; S_u = 0$$

$$a_P = a_W + a_E + a_N + a_S + (F_e - F_w + F_n - F_s) - S_p$$

(Note: All Coefficients are updated for velocities of earlier/last iteration)

$$Old\ values \rightarrow\ u^* = u^*_{old}\ and\ v^* = v^*_{old}$$

Boundary Conditions

For Left Boundary	For Right Boundary
$a_W = 0\,; F_w = 0;$ $S_p = -\frac{2*\Delta\text{y}}{\text{Re}*\Delta\text{x}}\,; S_u = 0$	$a_E = 0\,; F_e = 0;$ $S_p = -\frac{2*\Delta\text{y}}{\text{Re}*\Delta\text{x}}\,; S_u = 0$

North and South Boundaries → j=0 and j=N serve as boundary nodes and so no specific boundary conditions needs to be defined

East and West Boundaries → As the Coefficient of neighbouring term at boundary are made zero (since it lies outside the boundary domain), the effect of convection and diffusion is considered through the Sp and Su terms.

U Momentum Equation solved using Gauss Siedel Method

$$v^*_E, v^*_W, v^*_N, v^*_S\ are\ all\ have\ updated\ values\ of\ current\ iterations$$

$$a_p u^*_p = a_E v^*_E + a_w v^*_w + a_N v^*_N + a_S v^*_S + source = \sum_{nb} a_{nb} v_{nb} + Source$$

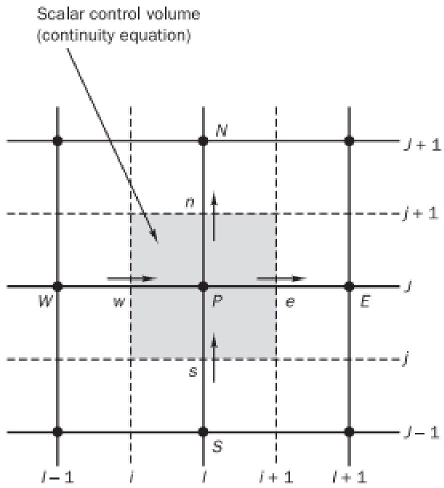
$$source = \Delta y * (p^*_{I,J-1} + p^*_{I,J}) + S_u$$

$$\mathbf{a_p v^*_p = a_E v^*_E + a_w v^*_w + a_N v^*_N + a_S v^*_S + [\,\Delta x * (p^*_{I,J-1} + p^*_{I,J}) + S_u\,]}$$

Pressure Equation

Equations Solved for All Blue Dots

(For i=0 to M-1) (For j=0 to N-1)



I = i	J = j
i varies in x direction	j varies in y direction

Continuity Equation is solved to obtain Pressure Correction Equation.

$$\boldsymbol{u_p = u_p^* + \ d_{i,J}[\,(p'_{I-1,J} + p'_{I,J})]}$$

$$\boldsymbol{v_p = v_p^* + d_{I,j}[\,(p'_{I,J-1} + p'_{I,J})]}$$

$$\iint \frac{\partial u}{\partial x} dxdy + \iint \frac{\partial v}{\partial y} dxdy = 0$$

$$[uA]_w^e + [vA]_s^n = 0$$

$$u_e A_e - u_w A_w + v_n - u_s A_s = 0$$

$$\frac{u_P + u_E}{2} A_e - \frac{u_P + u_W}{2} A_w + \frac{v_P + v_N}{2} A_n - \frac{v_P + v_S}{2} A_s = 0$$

$$u_E^* A_e - u_W^* A_w + v_N^* A_n + v_S^* A_s + d_e A_e (P'_E - P'_p) - d_w A_w (P'_P - P'_W) + d_n A_n (P'_N - P'_p) - d_s A_s (P'_P - P'_S) = 0$$

$$Source = \sum_f F^* = u_E^* A_e - u_W^* A_w + v_N^* A_n + v_S^* A_s$$

$$\mathbf{a_p p'_p = a_E p'_E + a_W p'_W + a_N p'_N + a_S p'_S} - \sum_f \boldsymbol{F^*} = \sum_{nb} \boldsymbol{a_{nb} p_{nb} + Source}$$

$$a_P = a_W + a_E + a_N + a_S$$

$$a_E = d_e A_e \ ;; \ a_W = d_w A_w \ ; \ a_N = d_n A_n \ ; \ a_S = d_s A_s$$

$$d_e = \frac{A_e}{u\ velocity\_coefficients[i+1][j]}$$

$$d_w = \frac{A_w}{u\ velocity\_coefficients[i][j]}$$

$$d_n = \frac{A_n}{v\ velocity\_coefficients[i][j+1]}$$

$$d_s = \frac{A_s}{v\ velocity\_coefficients[i][j]}$$

All the equations as described above are used similarly in the code. Comments are used in the Code wherever required

RESULTS

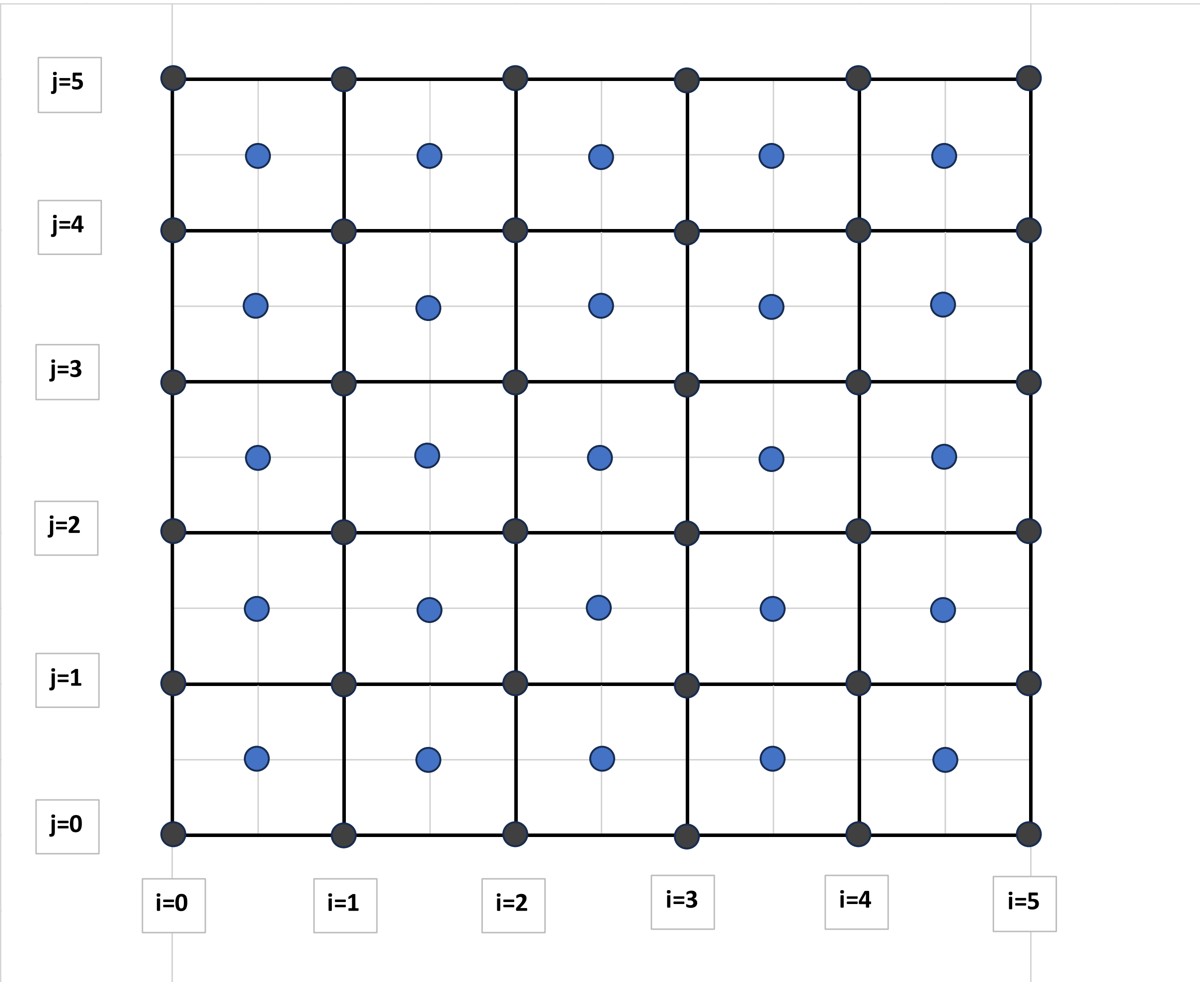
All Results are obtained on collocated Grid

Collocated Grid => 129 x 129 [(M+1) x (N+1)]

After Getting the results, the Collocated Grid is used for all Calculations of Velocities, Vorticity Stream Function, etc as Black nodes a shown below. (Blue nodes indicate the Pressure CV centres)

The Indexing used is as shown in Figure for Collocated grid

Here M = N = 5 (Example)



Error Condition used in Code ➔ **_L2_Norm <= 10^{-5}**

	Re=100	**Re=400**	**Re=1000**
Number of Simple Iterations	3709	3535	14861
Gauss Siedel Iterations for calculating Stream function	4797	4137	3797
Relaxation factor — Alpha_u	0.5	0.5	0.1
Relaxation factor — Alpha_v	0.5	0.5	0.1
Relaxation factor — Alpha_p	0.5	0.5	0.1
Minimum Stream function value	-0.103778	-0.115598	-0.102281
Location of minimum stream function value	(0.617188, 0.742188)	(0.554688, 0.601562)	(0.539062, 0.578125)
Vorticity value at minimum stream function	-3.193425	-2.293424	-1.601248

The location of minimum Stream function can be the centre around which all streamlines are rotating

No relaxation factor used while solving u and v momentum equation and pressure equation.

Relaxation used only while correcting velocities and pressure

The Value of stream Function would be zero at the boundaries

Equation used for calculating Vorticity

$$\omega = \frac{\partial v}{\partial x} - \frac{\partial u}{\partial y}$$

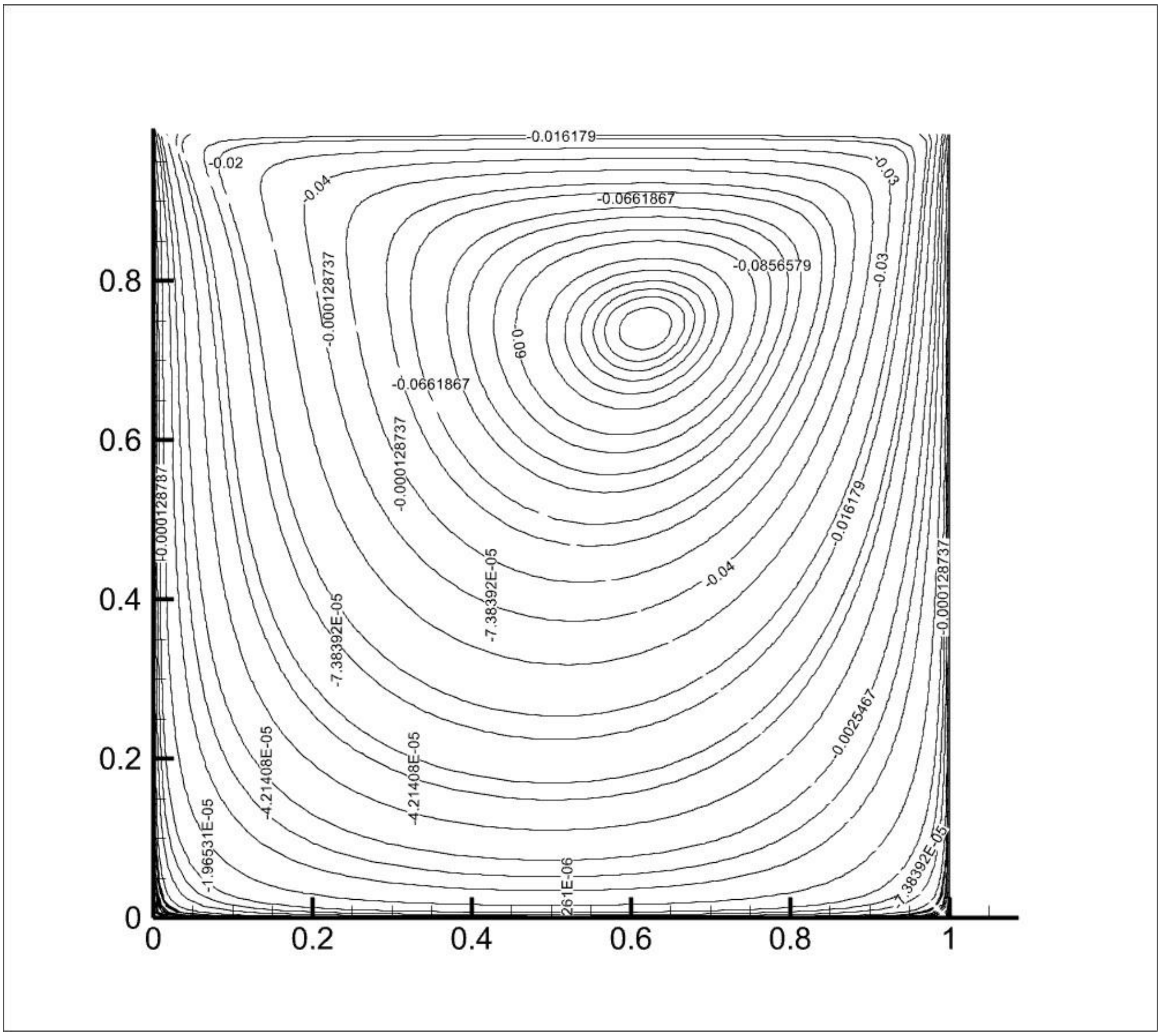
Equation used for calculating Stream Function

$$\Psi_{i,j}^{k+1} = \frac{1}{2 * (1 + \beta^2)} \left([(\Delta x)^2 * \omega_{i,j}^{k+1}] + \beta^2 * \Psi_{i,j-1}^{k+1} + \Psi_{i-1,j}^{k+1} + \Psi_{i+1,j}^{k} + \beta^2 * \Psi_{i,j+1}^{k} \right)$$

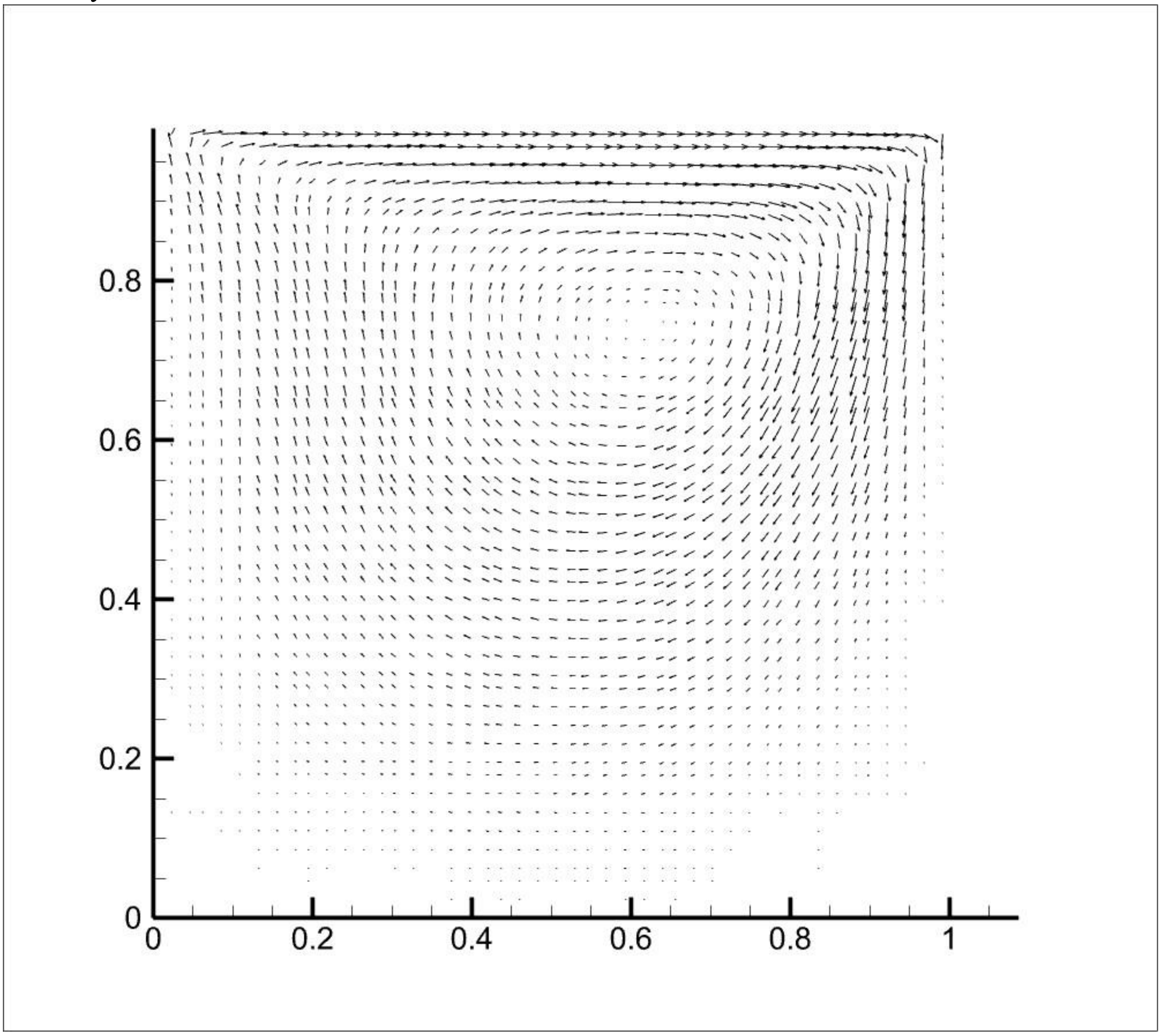
Re = 100

Contours

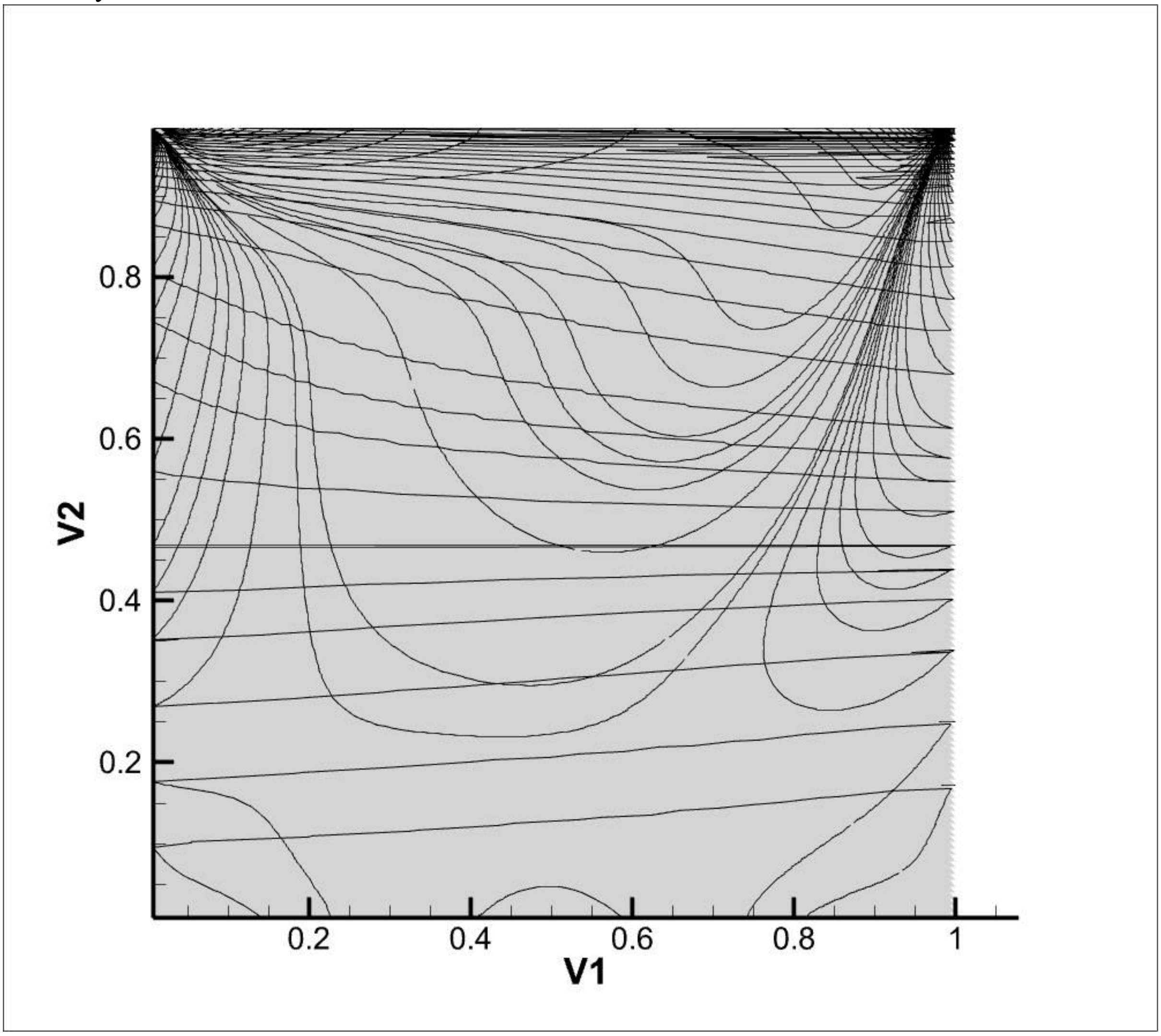
Streamline Contour



Velocity Contour

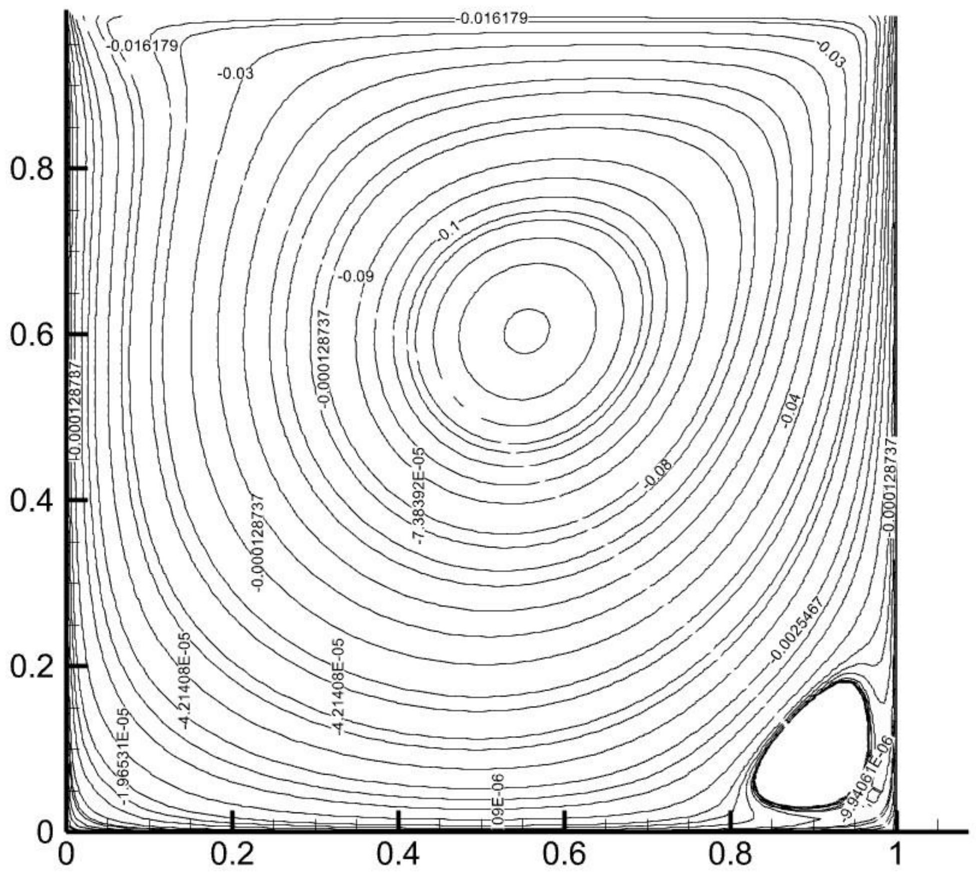


Vorticity Contour

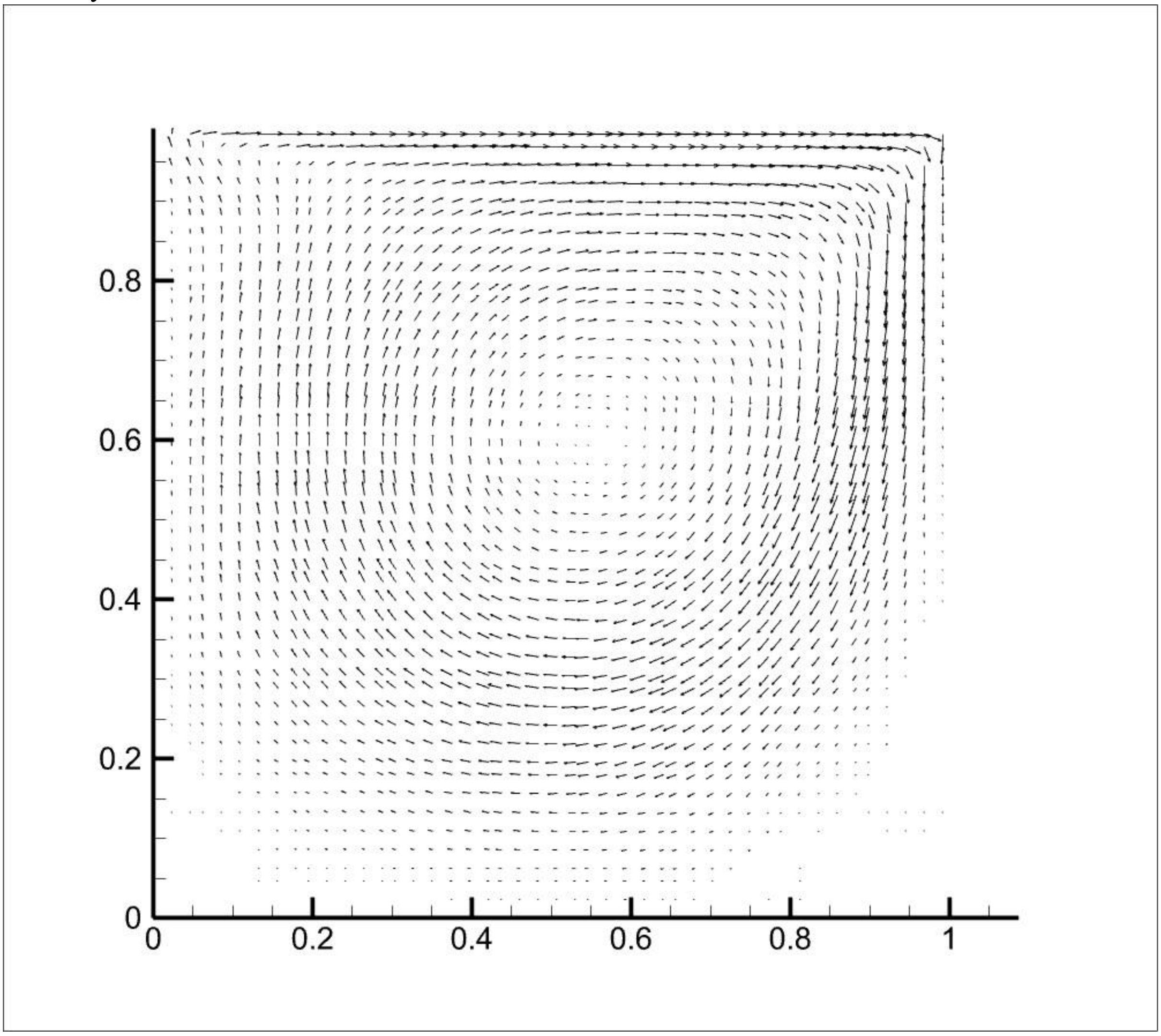


Re = 400

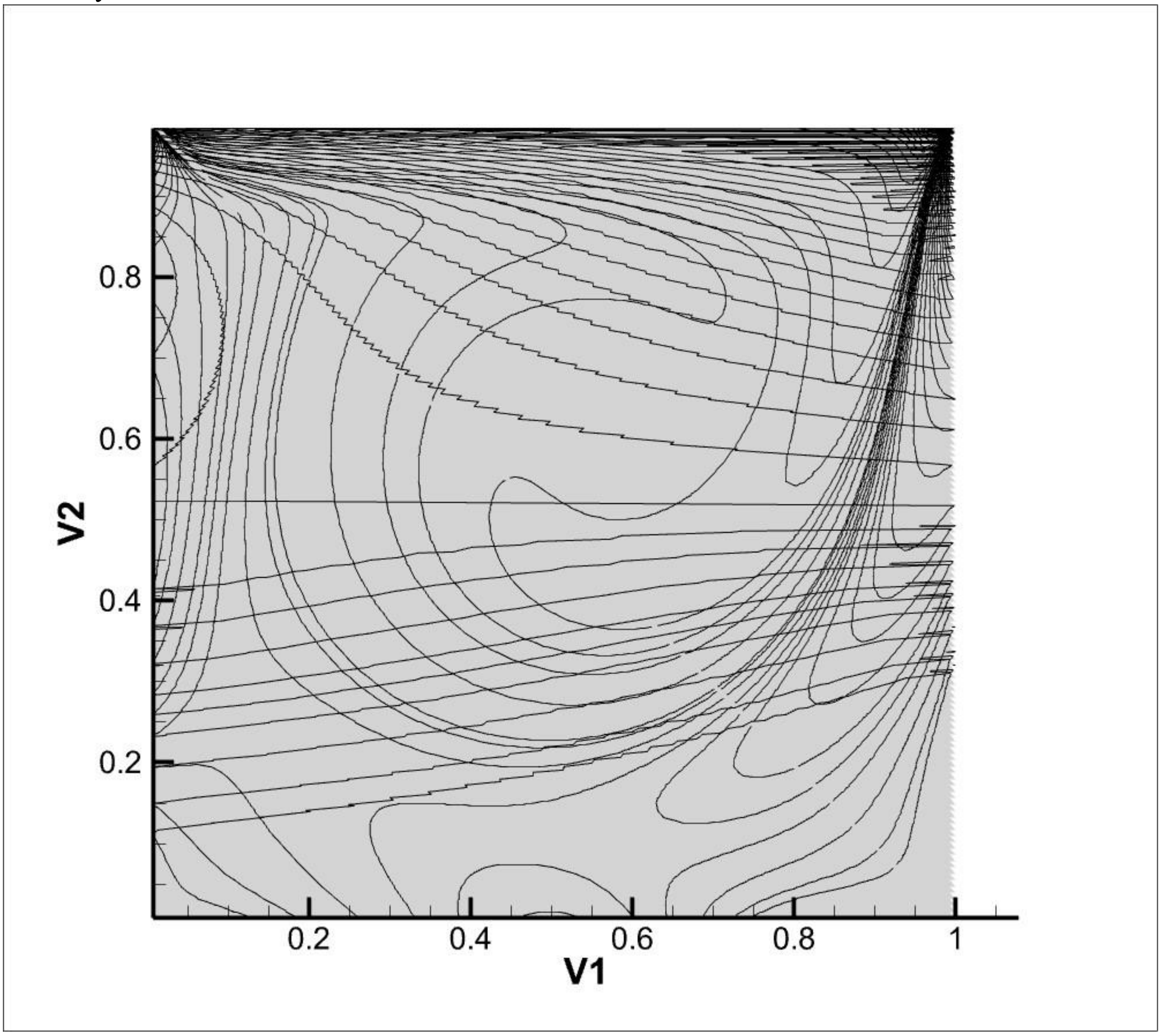
Streamline Contour



Velocity Contour

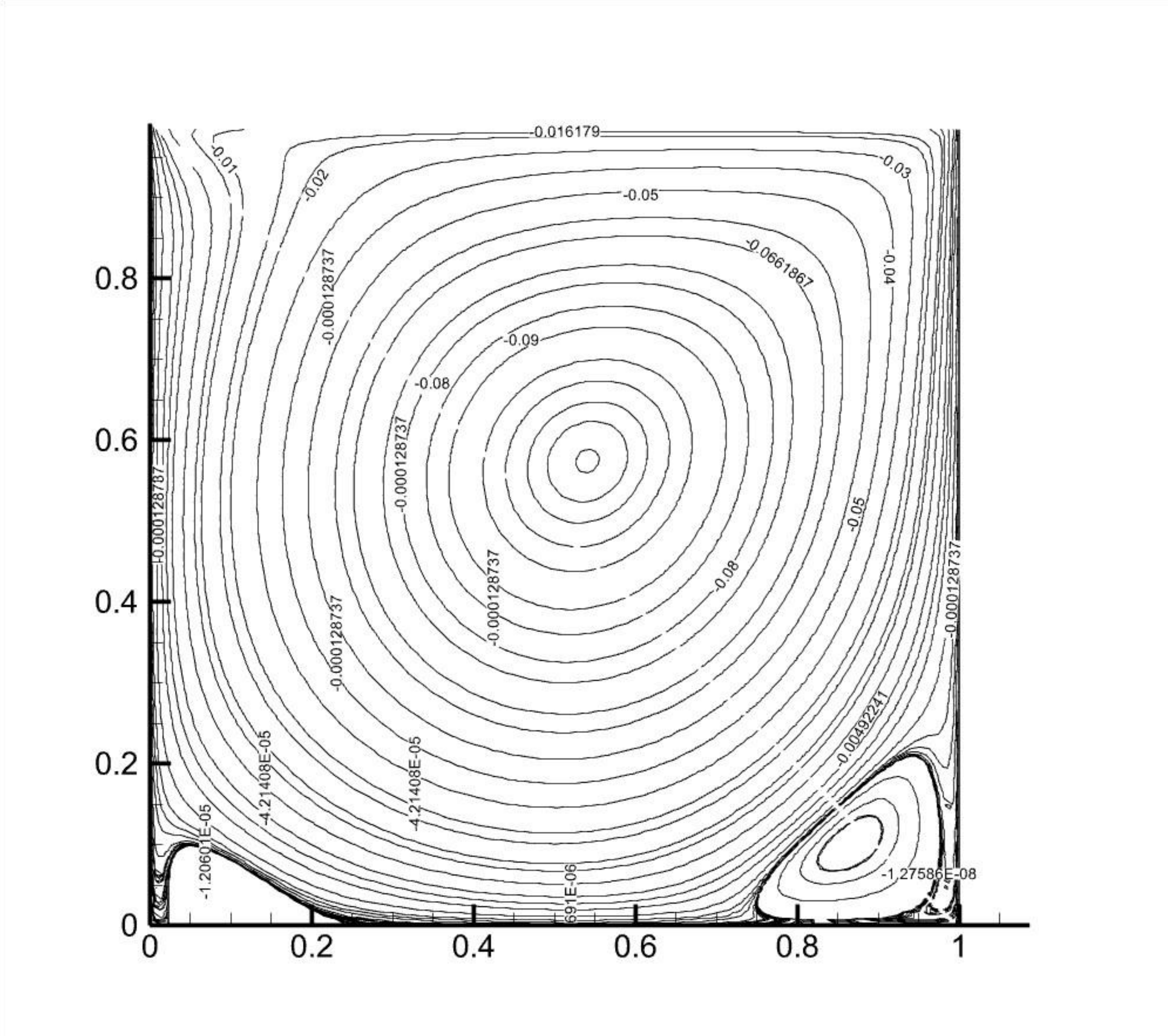


Vorticity Contour

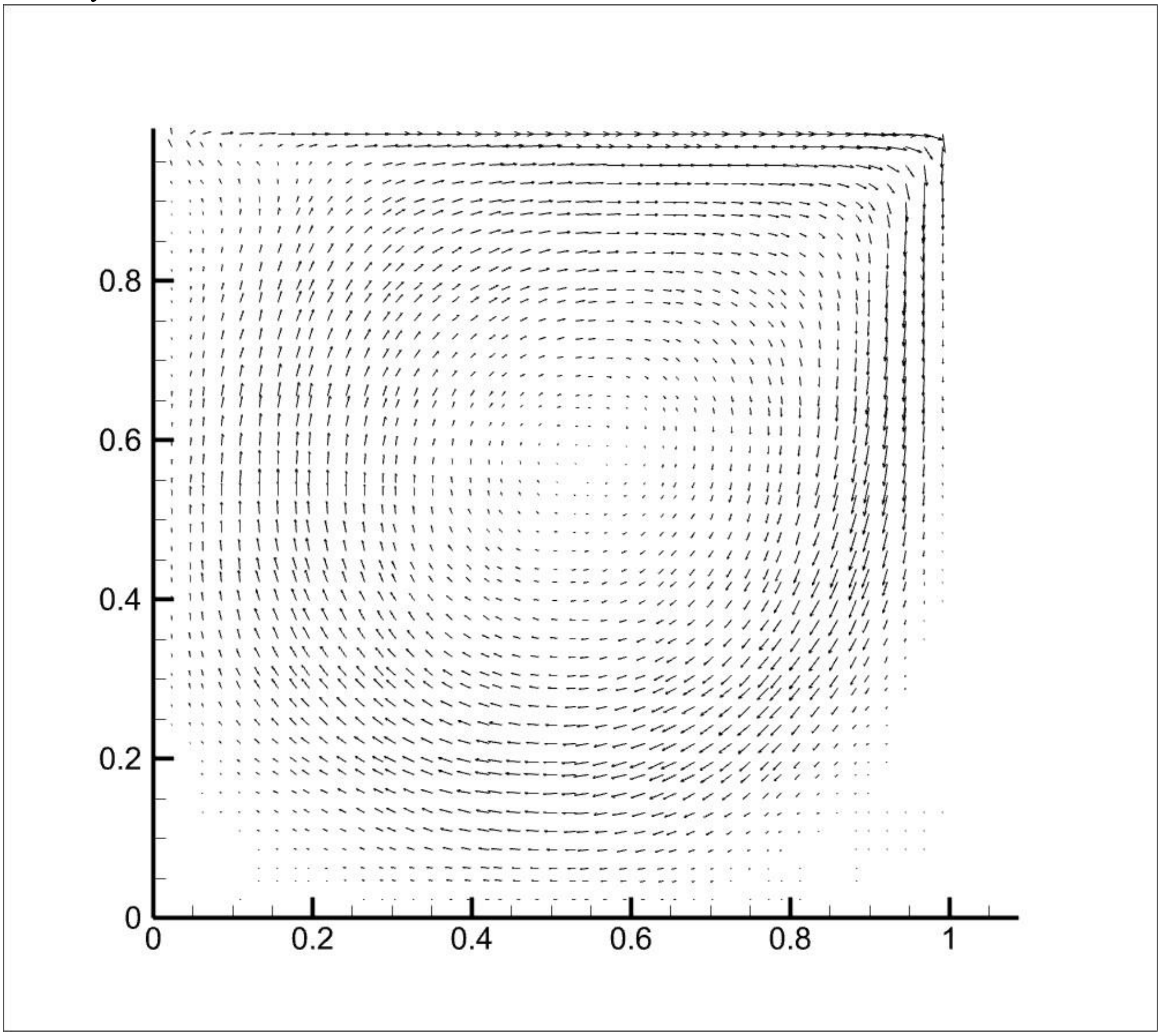


Re = 1000

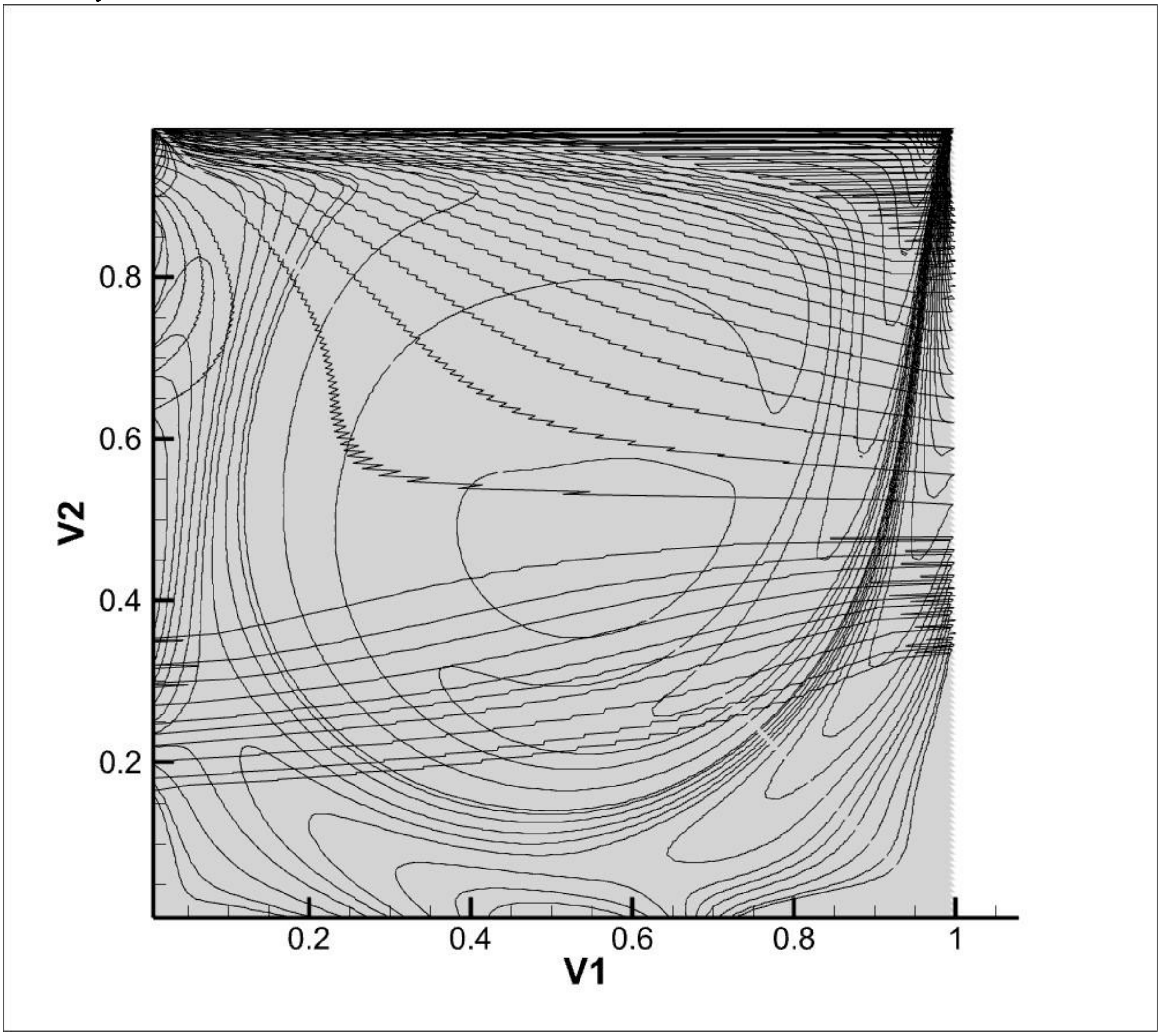
Streamline Contour



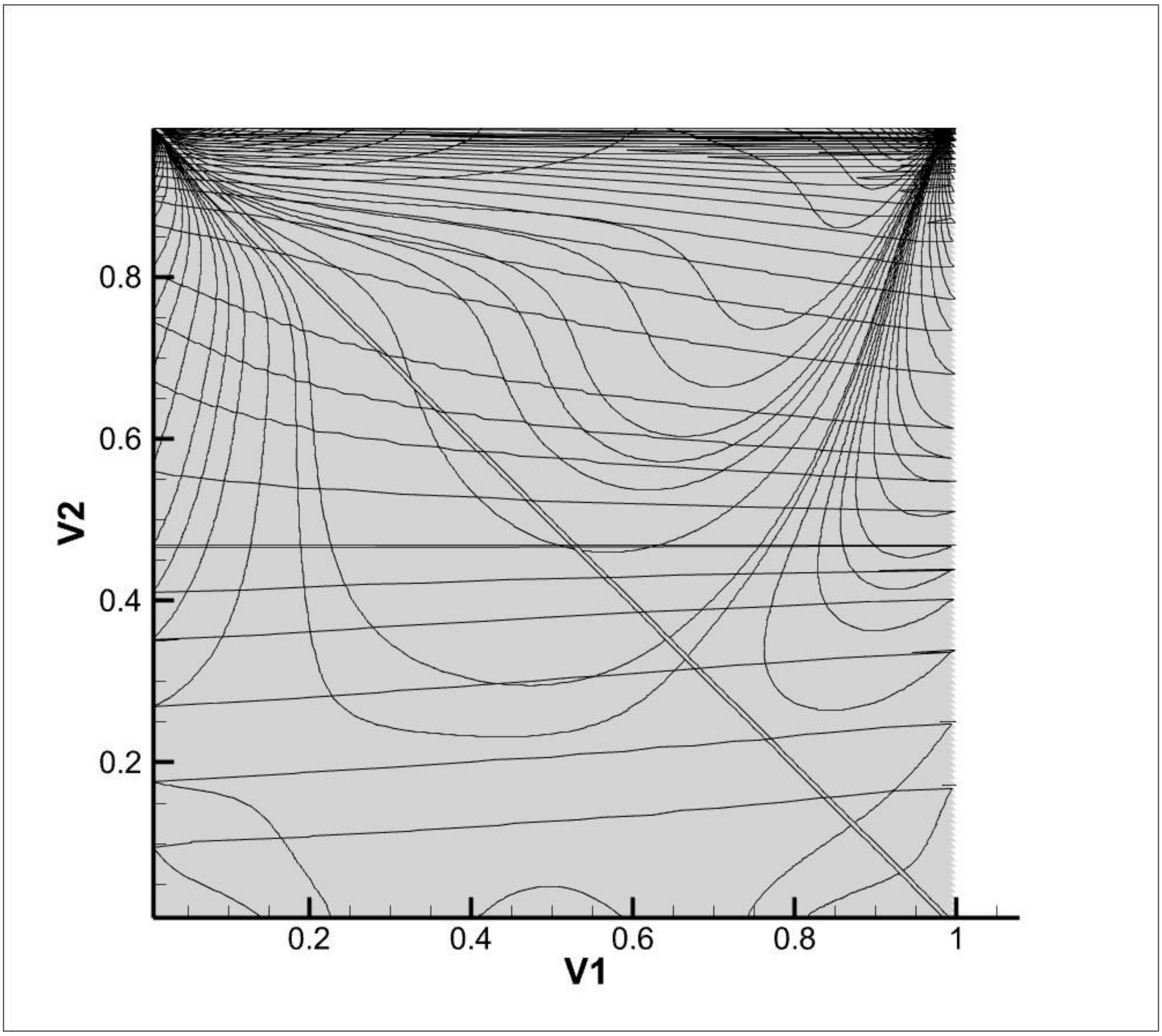
Velocity Contour



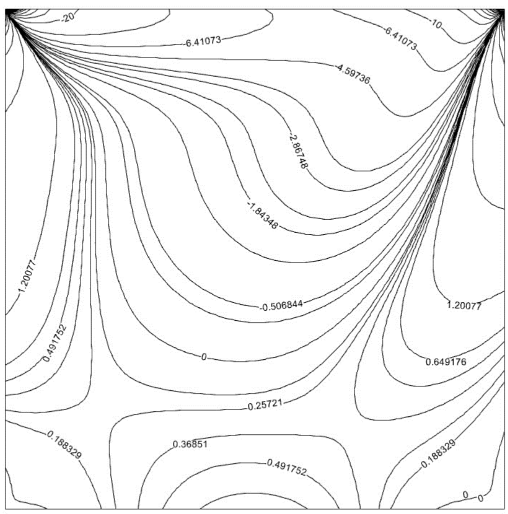
Vorticity Contour

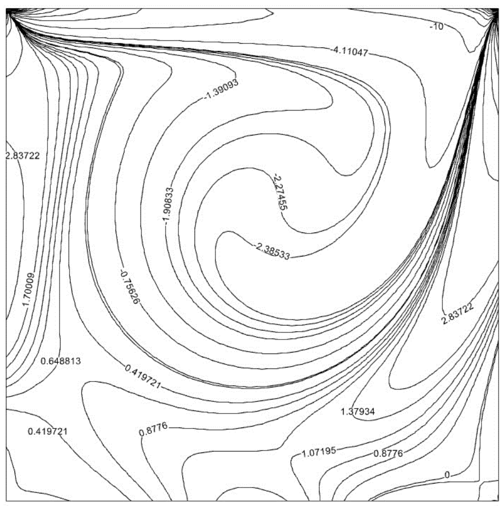


NOTE: Please ignore the distortions and horizontal lines in front as they are caused du to dislocation in top and right age as shown in error image below (The Diagonal double line is the edge which needs to be adjusted to corrected to make distortions invisible)



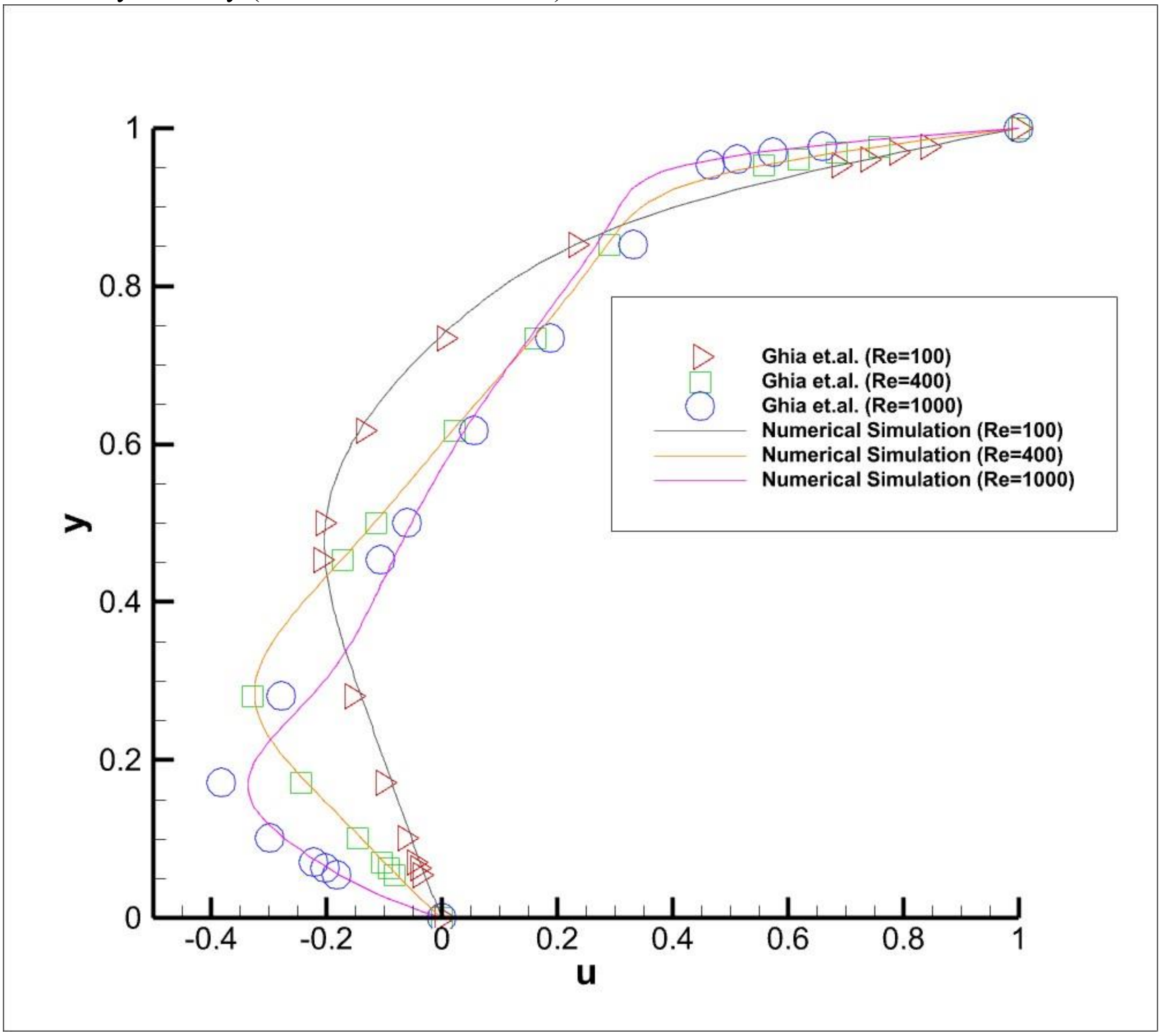
The plots from last year CFD Vorticity contours for Re=100, 400 are as shown below





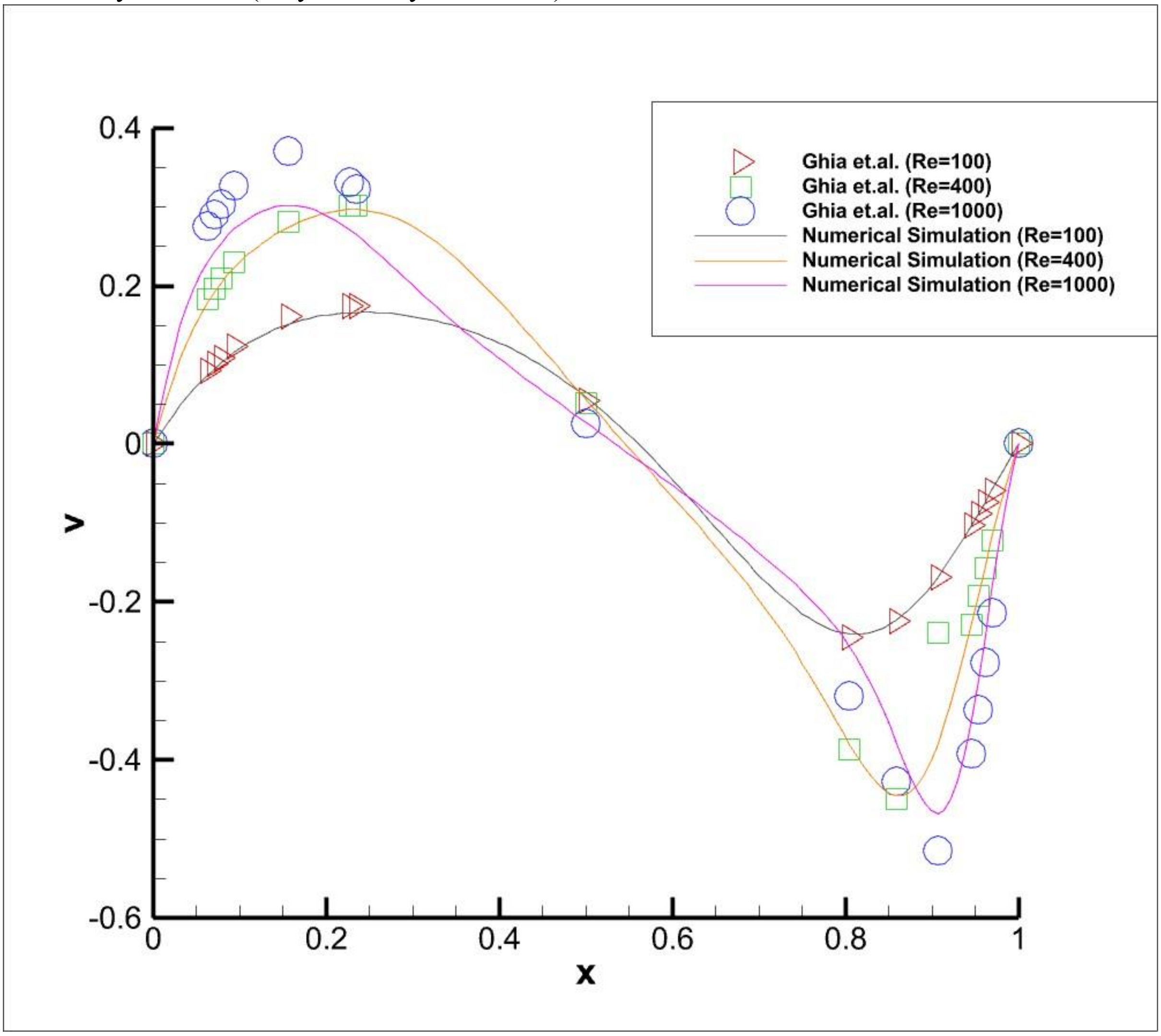
Comparison for various Reynold Number

U velocity versus y (At x=0.5 or x centreline)



x-velocity profile at $x = L/2$

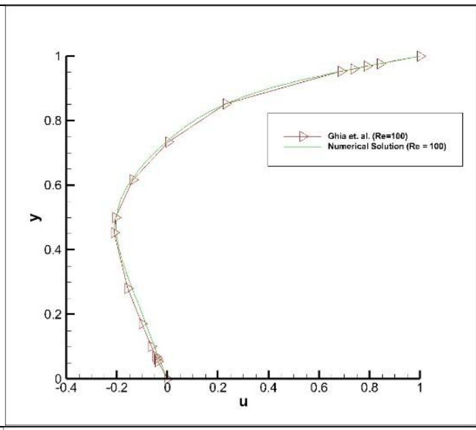
V velocity versus x (At y=0.5 or y centreline)



y-velocity profile at y = $L/2$

Comparison Table

	x-velocity profile at $x = L/2$	y-velocity profile at y = $L/2$

<table>
<tr>
<td>Re = 100</td>
<td>
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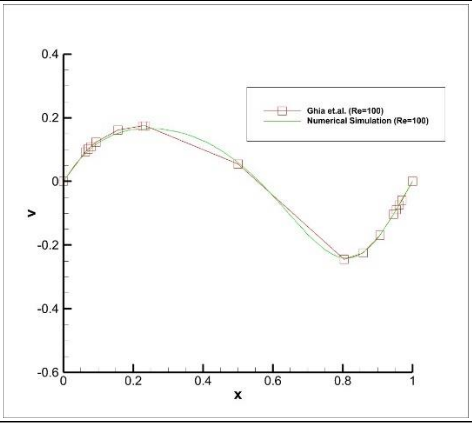
<td>
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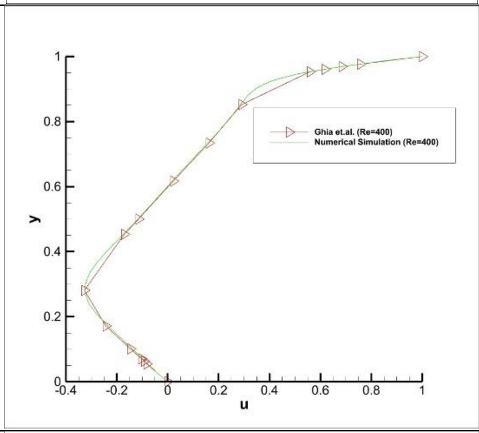
<td>Re = 400</td>
<td>
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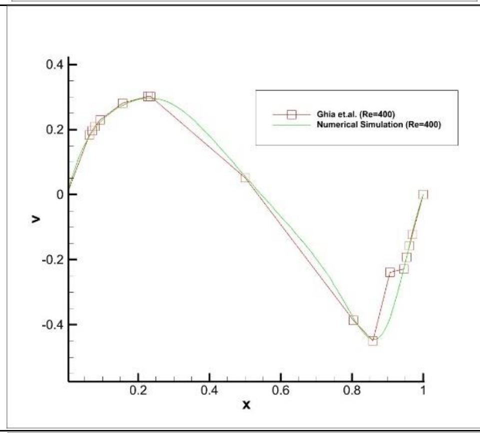
<td>
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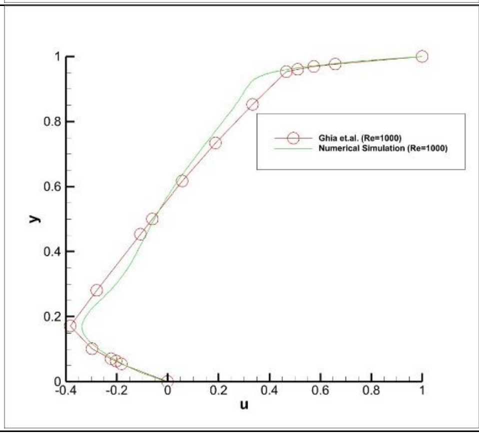
</tr>
<tr>
<td>Re = 1000</td>
<td>
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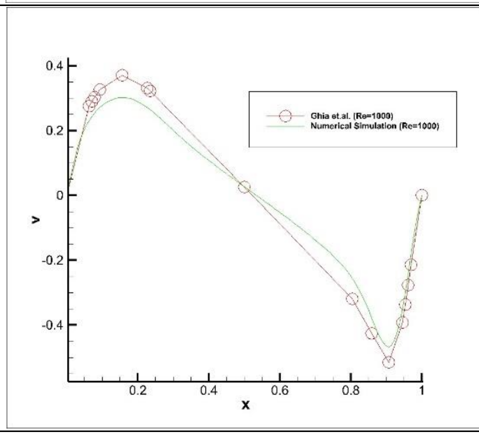
<td>
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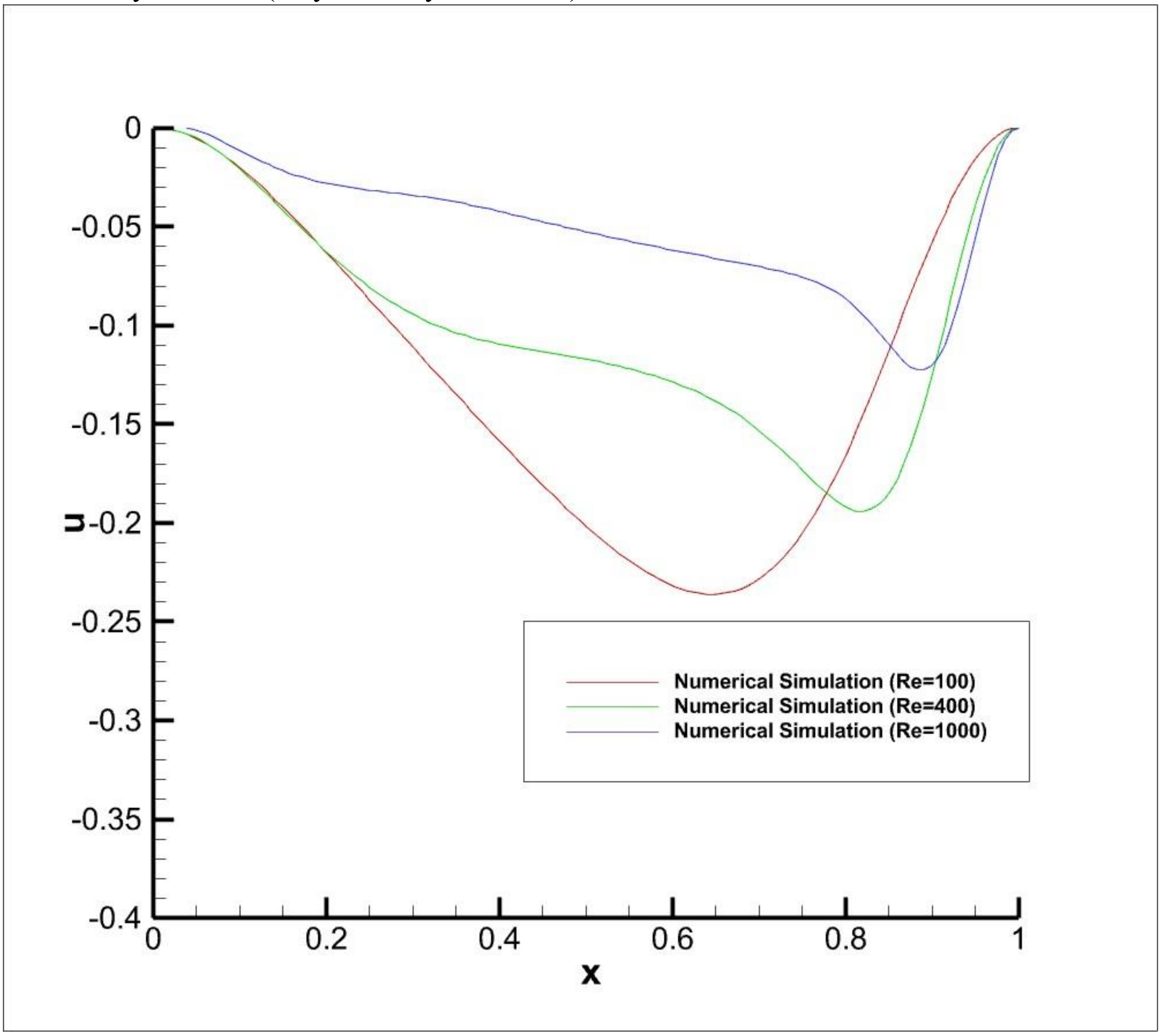
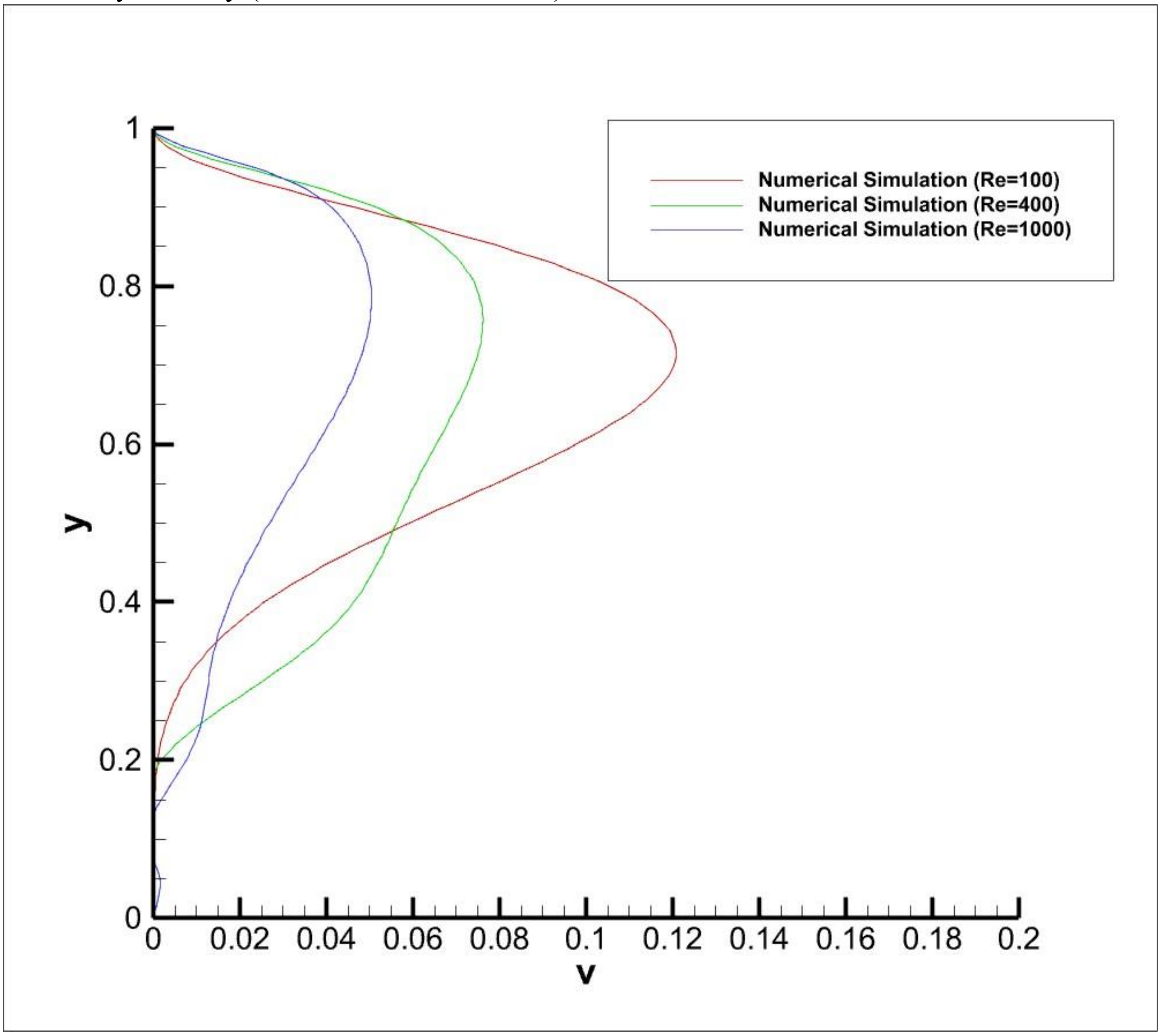
U velocity versus x (At y=0.5 or y centreline)



x-velocity profile at y = $L/2$

V velocity versus y (At x=0.5 or x centreline)



y-velocity profile at $x = L/2$