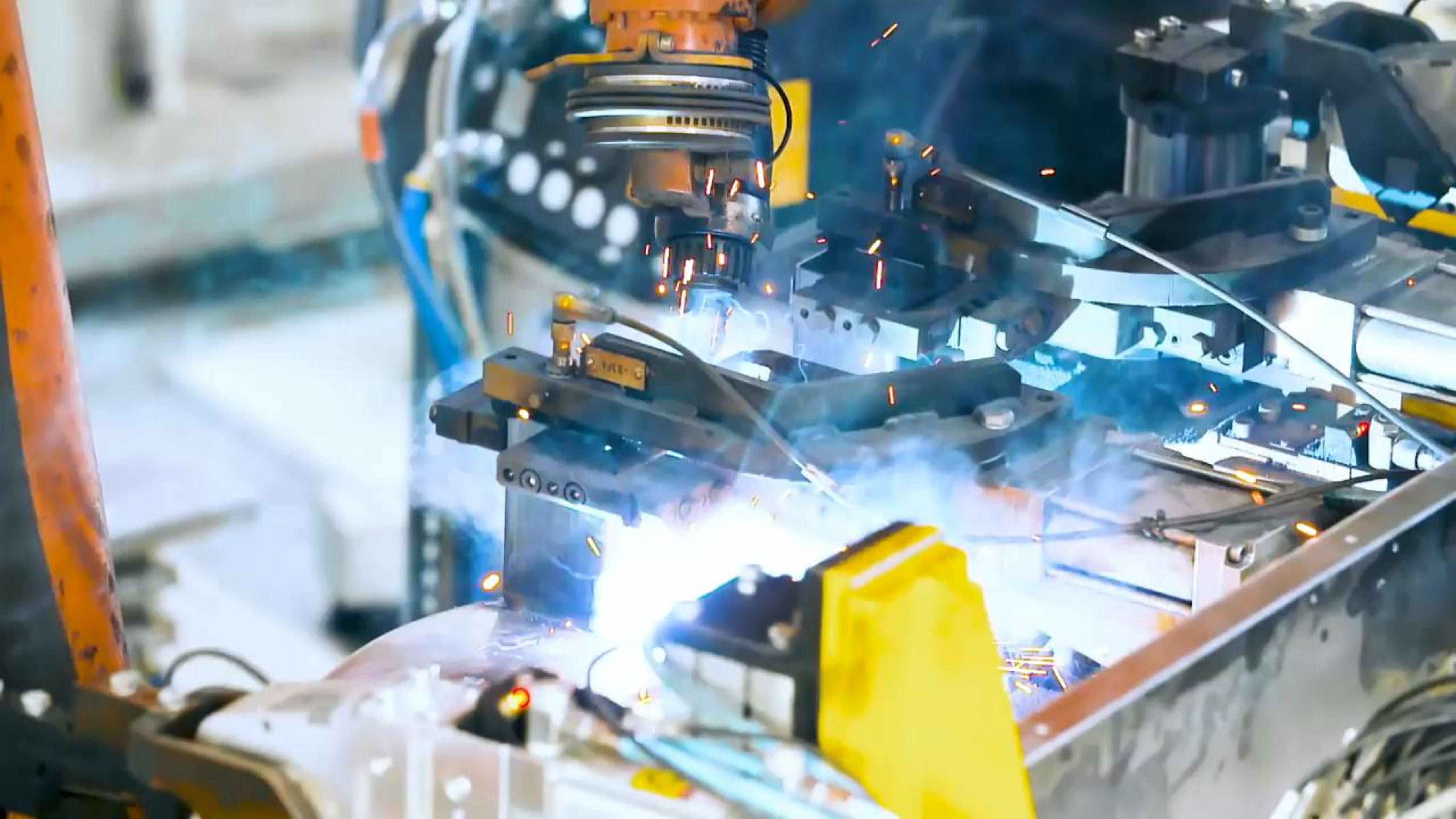
Dynamics: Example

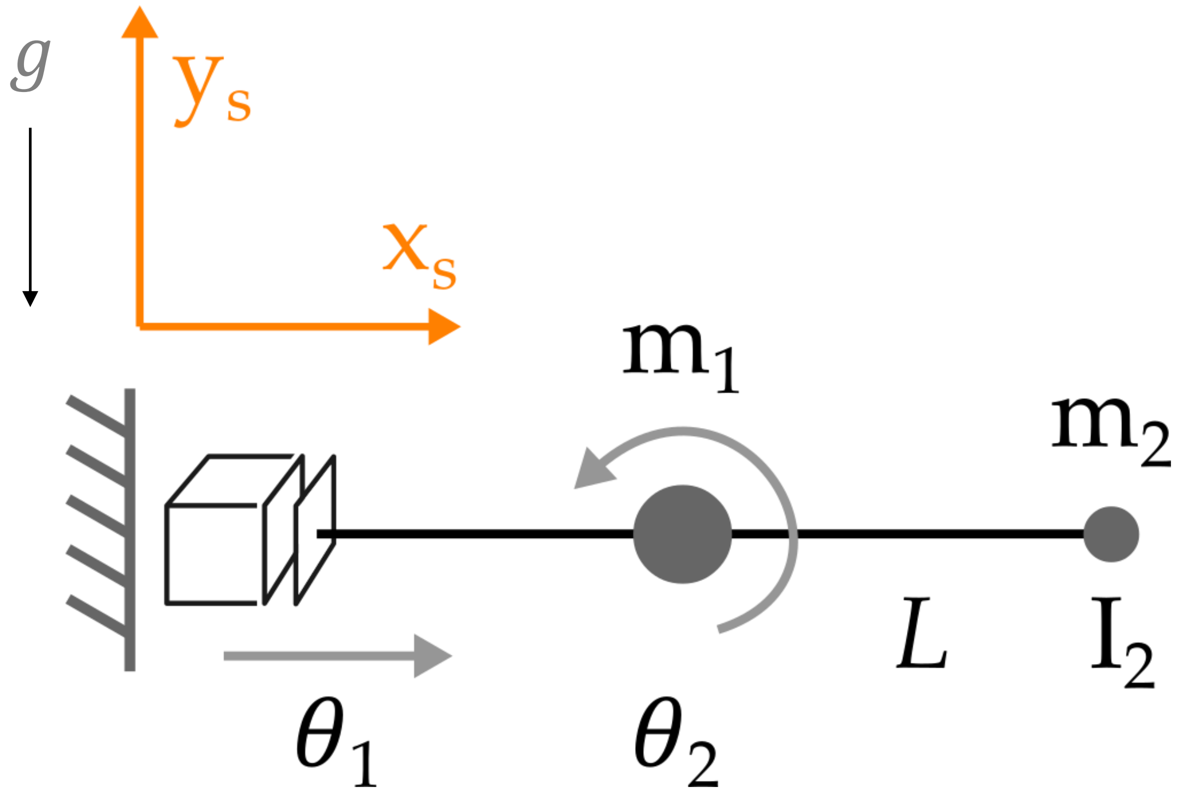
Reading: Robot Modeling and Control 7.4



This Lecture

- How do we apply the dynamics equation for robot arms?
- Practice dynamics with an example

Example



Mass

m_i is the mass of link i

Center of mass at the end of each link

Inertia

I_2 is the inertia of link 2

No inertia needed for link 1

Gravity

Gravity acts along the $-y$ axis

Equation of Motion

The **dynamics** of a serial robot arm with n joints is formulated as:

$$\tau = M(\theta)\ddot{\theta} + C(\theta, \dot{\theta})\dot{\theta} + g(\theta)$$

- $\boldsymbol{\tau}$ is $n \times 1$ vector of joint torque
- θ is $n \times 1$ vector of joint position
- $\dot{\theta}$ is $n \times 1$ vector of joint velocity
- $\ddot{\theta}$ is $n \times 1$ vector of joint acceleration

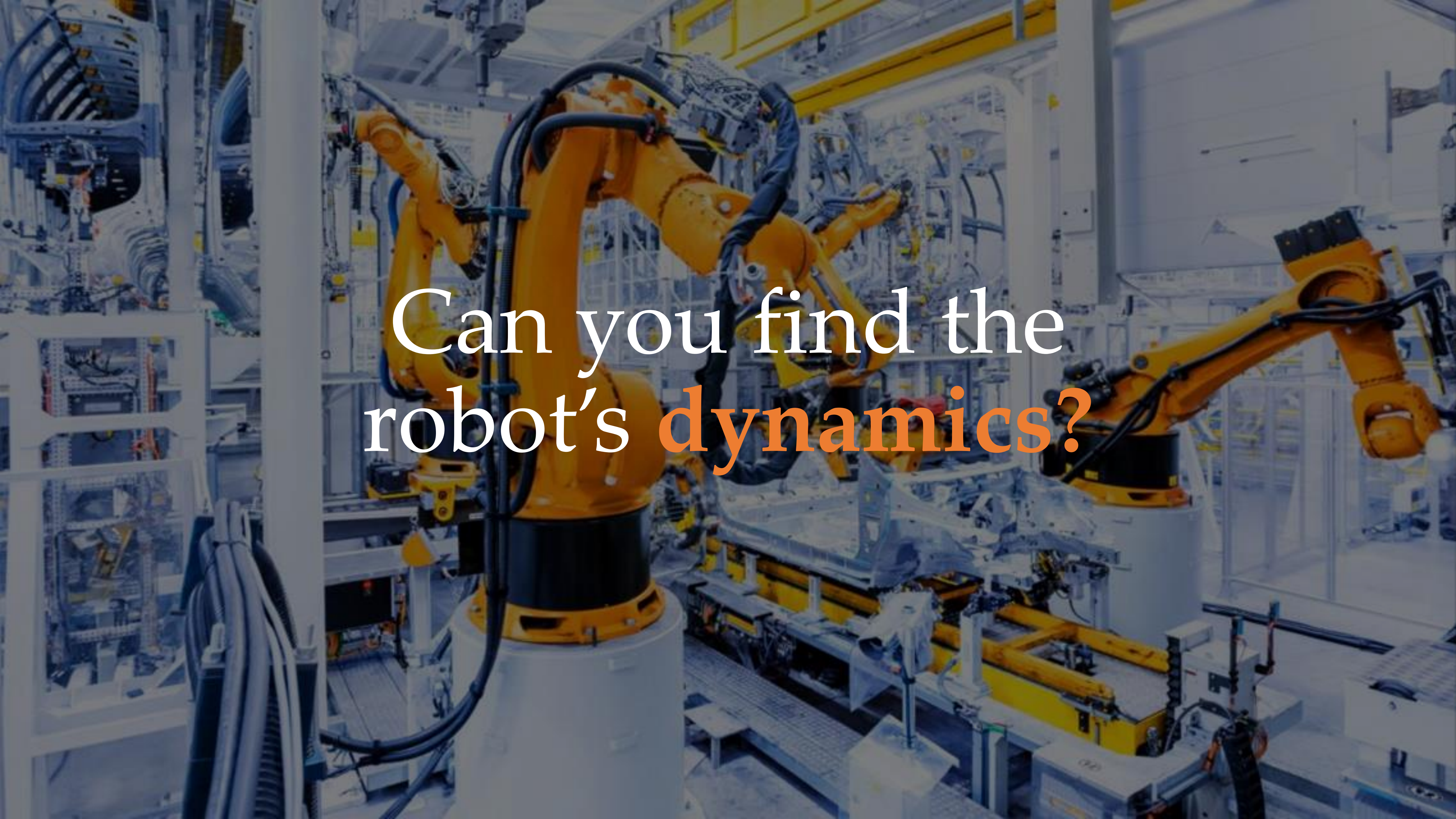
Equation of Motion

The **dynamics** of a serial robot arm with n joints is formulated as:

$$\tau = M(\theta)\ddot{\theta} + C\left(\theta, \dot{\theta}\right)\dot{\theta} + g(\theta)$$

- $\boldsymbol{M}$ is $n \times n$ **mass matrix**
- $\boldsymbol{C}$ is $n \times n$ **Coriolis matrix**
- $\boldsymbol{g}$ is $n \times 1$ **gravity vector**

Can you find the robot's **dynamics**?



Mass Matrix

We get the $n \times n$ mass matrix from **kinetic energy**

$$M(\theta) = \sum_{i=1}^{n} m_i {J_{v_i}}^T J_{v_i} + {J_{\omega_i}}^T R_i \boldsymbol{I}_i {R_i}^T J_{\omega_i}$$

$$M(\theta) = \underline{m_1 {J_{v_1}}^T J_{v_1}} + m_2 {J_{v_2}}^T J_{v_2} + {J_{\omega_2}}^T R_2 \boldsymbol{I}_2 {R_2}^T J_{\omega_2}$$

First link never rotates, so no rotational kinetic energy

Mass Matrix

First find **Jacobians** for the center of mass of both links

$$M(\theta) = m_1 {J_{v_1}}^T J_{v_1} + m_2 {J_{v_2}}^T J_{v_2} + {J_{\omega_2}}^T R_2 \boldsymbol{I}_2 {R_2}^T J_{\omega_2}$$

$$J_{\omega_1} = \begin{bmatrix} 0 & 0 \\ 0 & 0 \\ 0 & 0 \end{bmatrix}, \qquad J_{v_1} = \begin{bmatrix} 1 & 0 \\ 0 & 0 \\ 0 & 0 \end{bmatrix}$$

Second joint has no effect on first center of mass

Mass Matrix

First find **Jacobians** for the center of mass of both links

$$M(\theta) = m_1 {J_{v_1}}^T J_{v_1} + m_2 {J_{v_2}}^T J_{v_2} + {J_{\omega_2}}^T R_2 \boldsymbol{I}_2 {R_2}^T J_{\omega_2}$$

$$J_{\omega_2} = \begin{bmatrix} 0 & 0 \\ 0 & 0 \\ 0 & 1 \end{bmatrix}, \qquad J_{v_2} = \begin{bmatrix} 1 & -Ls_2 \\ 0 & Lc_2 \\ 0 & 0 \end{bmatrix}$$

Mass Matrix

Then substitute these Jacobians into our **kinetic energy** equation

$$M(\theta) = m_1 {J_{v_1}}^T J_{v_1} + m_2 {J_{v_2}}^T J_{v_2} + {J_{\omega_2}}^T R_2 \boldsymbol{I}_2 {R_2}^T J_{\omega_2}$$

$$M(\theta) = \begin{bmatrix} m_1 + m_2 & -Lm_2 s_2 \\ -Lm_2 s_2 & m_2 L^2 + I_2 \end{bmatrix}$$

Coriolis Matrix

We get the $n \times n$ Coriolis matrix from **mass matrix**

$$C(\theta, \dot{\theta}) = \begin{bmatrix} c_{11} & c_{12} \\ c_{21} & c_{22} \end{bmatrix}$$

$$c_{kj} = \sum_{i=1}^{n} \frac{1}{2} \left\{ \frac{\partial m_{kj}}{\partial \theta_i} + \frac{\partial m_{ki}}{\partial \theta_j} - \frac{\partial m_{ij}}{\partial \theta_k} \right\} \dot{\theta}_i$$

Coriolis Matrix

$$c_{kj} = \sum_{i=1}^{n} \frac{1}{2} \left\{ \frac{\partial m_{kj}}{\partial \theta_i} + \frac{\partial m_{ki}}{\partial \theta_j} - \frac{\partial m_{ij}}{\partial \theta_k} \right\} \dot{\theta}_i \,, \qquad M(\theta) = \begin{bmatrix} m_1 + m_2 & -L m_2 \, s_2 \\ -L m_2 \, s_2 & m_2 \, L^2 + I_2 \end{bmatrix}$$

$$c_{11} = \frac{1}{2} \left\{ \frac{\partial m_{11}}{\partial \theta_1} + \frac{\partial m_{11}}{\partial \theta_1} - \frac{\partial m_{11}}{\partial \theta_1} \right\} \dot{\theta}_1 + \frac{1}{2} \left\{ \frac{\partial m_{11}}{\partial \theta_2} + \frac{\partial m_{12}}{\partial \theta_1} - \frac{\partial m_{21}}{\partial \theta_1} \right\} \dot{\theta}_2 = 0$$

Only m_{21} and m_{12} depend on θ_2

Coriolis Matrix

$$c_{kj} = \sum_{i=1}^{n} \frac{1}{2}\left\{\frac{\partial m_{kj}}{\partial \theta_i} + \frac{\partial m_{ki}}{\partial \theta_j} - \frac{\partial m_{ij}}{\partial \theta_k}\right\}\dot{\theta}_i\,, \qquad M(\theta) = \begin{bmatrix} m_1 + m_2 & -Lm_2\, s_2 \\ -Lm_2\, s_2 & m_2\, L^2 + I_2 \end{bmatrix}$$

$$c_{21} = \frac{1}{2}\left\{\frac{\partial m_{21}}{\partial \theta_1} + \frac{\partial m_{21}}{\partial \theta_1} - \frac{\partial m_{11}}{\partial \theta_2}\right\}\dot{\theta}_1 + \frac{1}{2}\left\{\underbrace{\boxed{\frac{\partial m_{21}}{\partial \theta_2}} + \frac{\partial m_{22}}{\partial \theta_1} - \boxed{\frac{\partial m_{21}}{\partial \theta_2}}}_{}\right\}\dot{\theta}_2 = 0$$

These terms cancel each other out

Coriolis Matrix

$$c_{kj} = \sum_{i=1}^{n} \frac{1}{2}\left\{\frac{\partial m_{kj}}{\partial \theta_i} + \frac{\partial m_{ki}}{\partial \theta_j} - \frac{\partial m_{ij}}{\partial \theta_k}\right\}\dot{\theta}_i\,, \qquad M(\theta) = \begin{bmatrix} m_1 + m_2 & -Lm_2\, s_2 \\ -Lm_2\, s_2 & m_2\, L^2 + I_2 \end{bmatrix}$$

$$c_{12} = \frac{1}{2}\left\{\frac{\partial m_{12}}{\partial \theta_1} + \frac{\partial m_{11}}{\partial \theta_2} - \frac{\partial m_{12}}{\partial \theta_1}\right\}\dot{\theta}_1 + \frac{1}{2}\left\{\boxed{\frac{\partial m_{12}}{\partial \theta_2} + \frac{\partial m_{12}}{\partial \theta_2}} - \frac{\partial m_{22}}{\partial \theta_1}\right\}\dot{\theta}_2 = -Lm_2\, c_2\dot{\theta}_2$$

Don't forget to multiply by $\dot{\theta}_2$

Coriolis Matrix

$$c_{kj} = \sum_{i=1}^{n} \frac{1}{2} \left\{ \frac{\partial m_{kj}}{\partial \theta_i} + \frac{\partial m_{ki}}{\partial \theta_j} - \frac{\partial m_{ij}}{\partial \theta_k} \right\} \dot{\theta}_i \,, \qquad M(\theta) = \begin{bmatrix} m_1 + m_2 & -Lm_2\, s_2 \\ -Lm_2\, s_2 & m_2\, L^2 + I_2 \end{bmatrix}$$

$$c_{22} = \frac{1}{2} \left\{ \frac{\partial m_{22}}{\partial \theta_1} + \boxed{\frac{\partial m_{21}}{\partial \theta_2} - \frac{\partial m_{12}}{\partial \theta_2}} \right\} \dot{\theta}_1 + \frac{1}{2} \left\{ \frac{\partial m_{22}}{\partial \theta_2} + \frac{\partial m_{22}}{\partial \theta_2} - \frac{\partial m_{22}}{\partial \theta_2} \right\} \dot{\theta}_2 = 0$$

Cancel because M is symmetric

Coriolis Matrix

We get the $n \times n$ Coriolis matrix from **mass matrix**

$$C(\theta, \dot{\theta}) = \begin{bmatrix} 0 & -Lm_2\, c_2 \dot{\theta}_2 \\ 0 & 0 \end{bmatrix}$$

Gravity Vector

We get the $n \times 1$ gravity vector from **potential energy**

$$g(\theta) = \begin{bmatrix} \dfrac{\partial P(\theta)}{\partial \theta_1} \\ \\ \dfrac{\partial P(\theta)}{\partial \theta_2} \end{bmatrix}, \qquad P(\theta) = g m_1 h_1 + g m_2 h_2$$

h_1 is the height of the
i-th center of mass

Gravity Vector

We get the $n \times 1$ gravity vector from **potential energy**

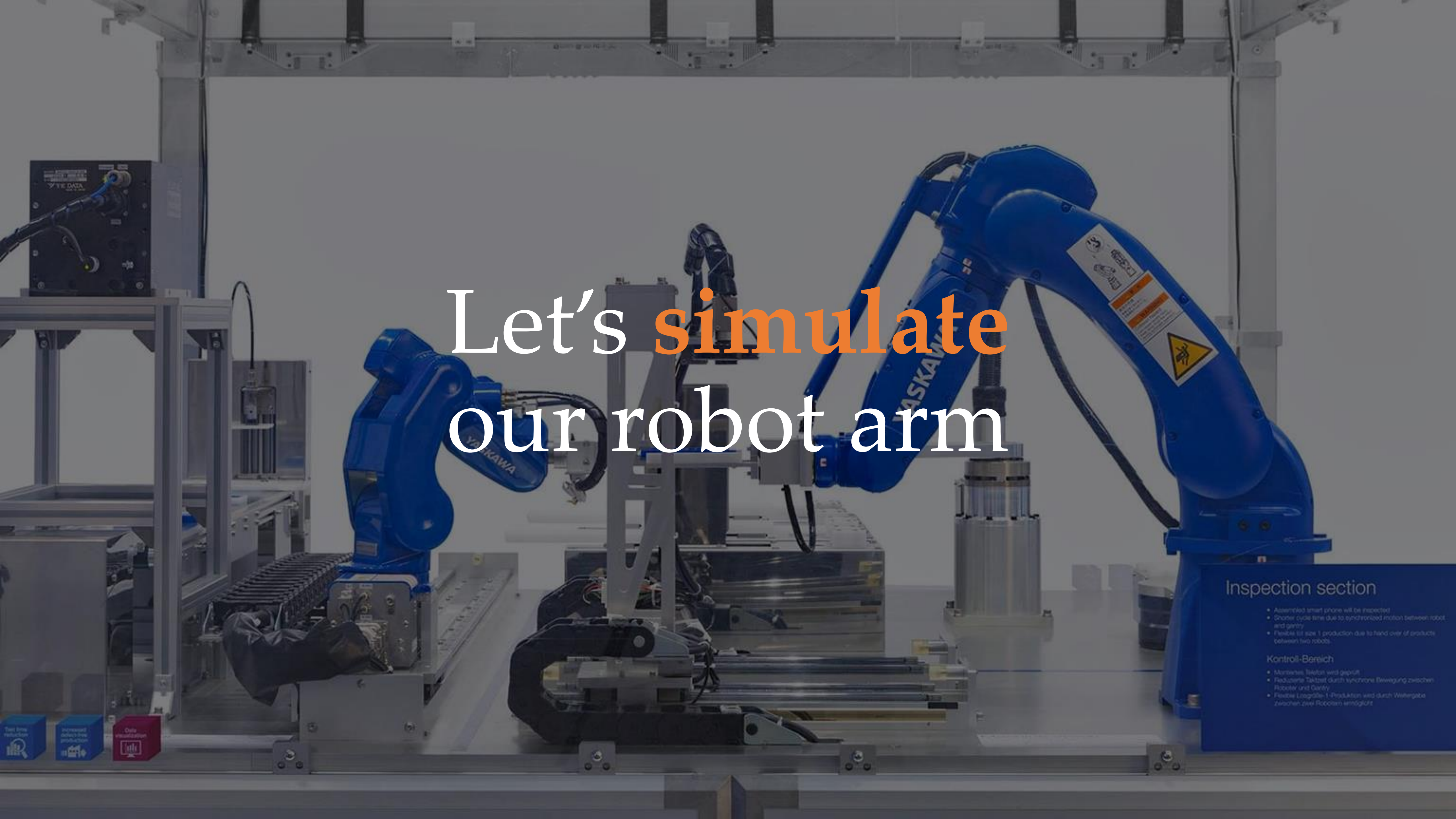
$$g(\theta) = \begin{bmatrix} \dfrac{\partial P(\theta)}{\partial \theta_1} \\ \\ \dfrac{\partial P(\theta)}{\partial \theta_2} \end{bmatrix}, \qquad P(\theta) = g m_2 L s_2$$

Gravity Vector

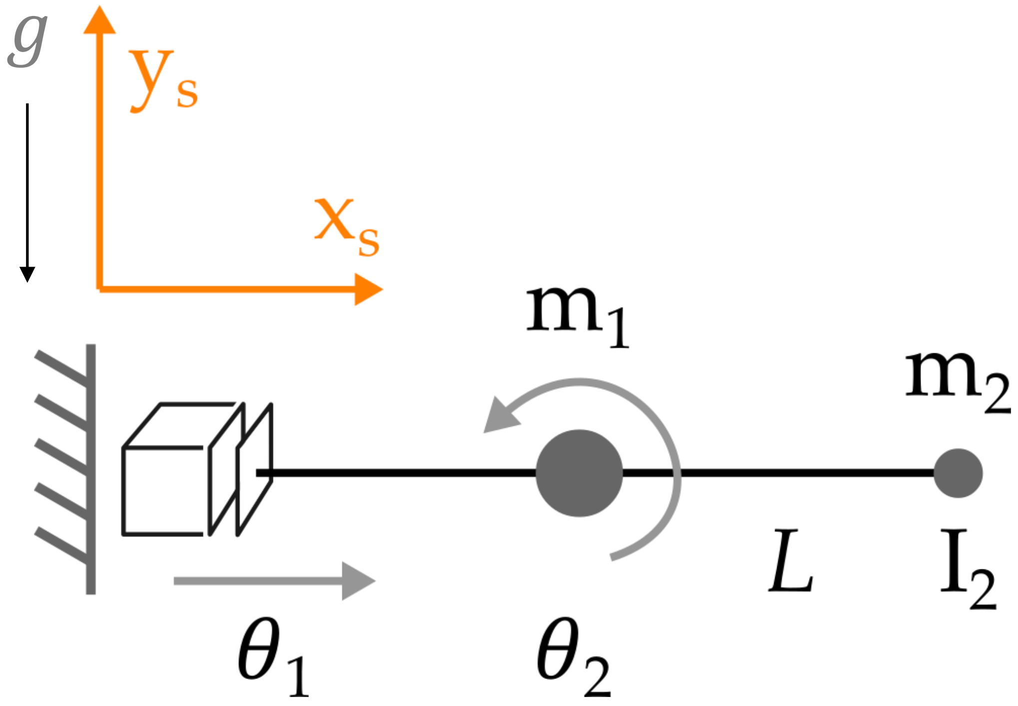
We get the $n \times 1$ gravity vector from **potential energy**

$$g(\theta) = \begin{bmatrix} 0 \\ g m_2 L c_2 \end{bmatrix}$$

Let's **simulate** our robot arm



Simulation

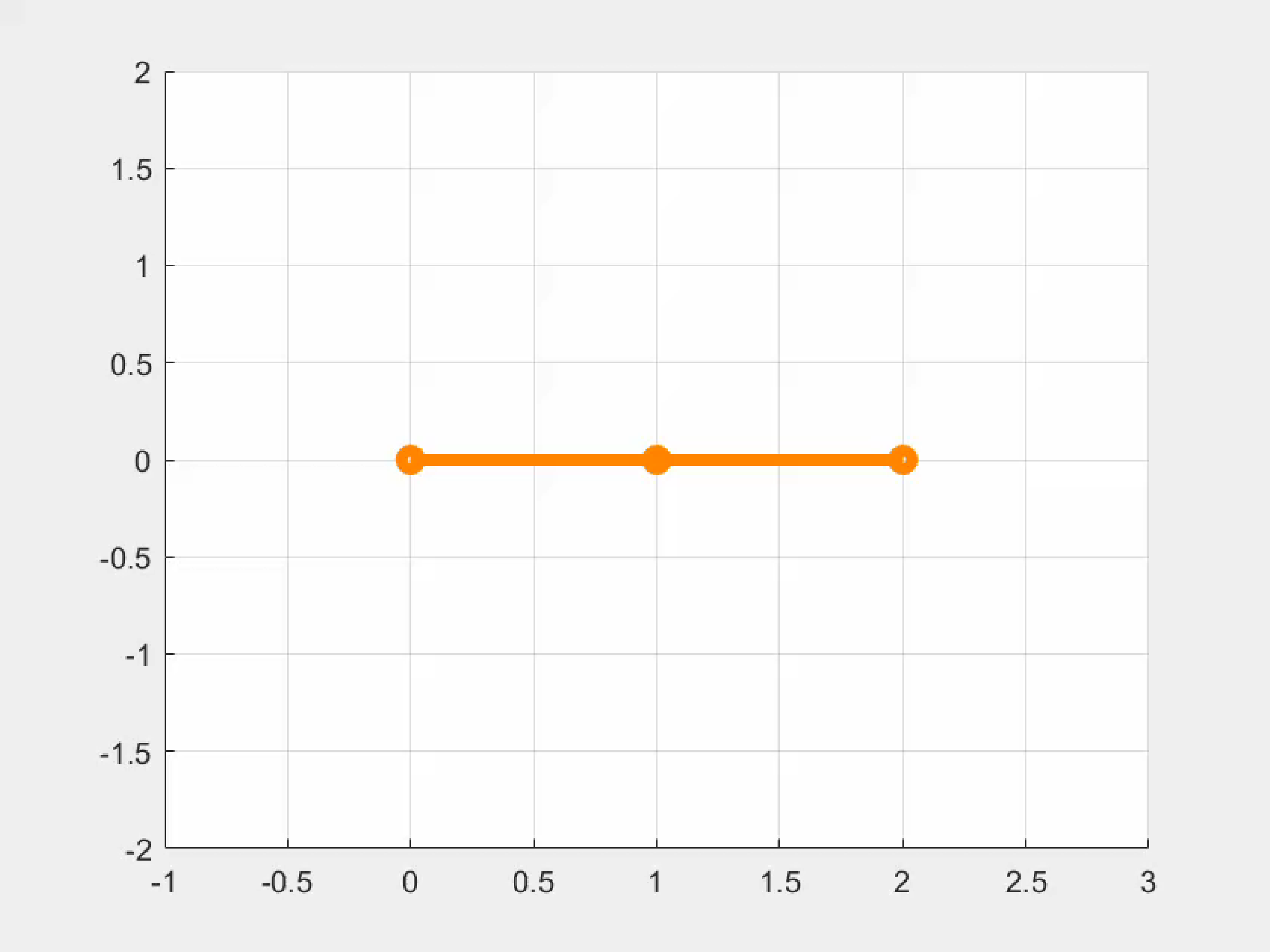


To **simulate**, solve for acceleration then integrate to get velocity and position:

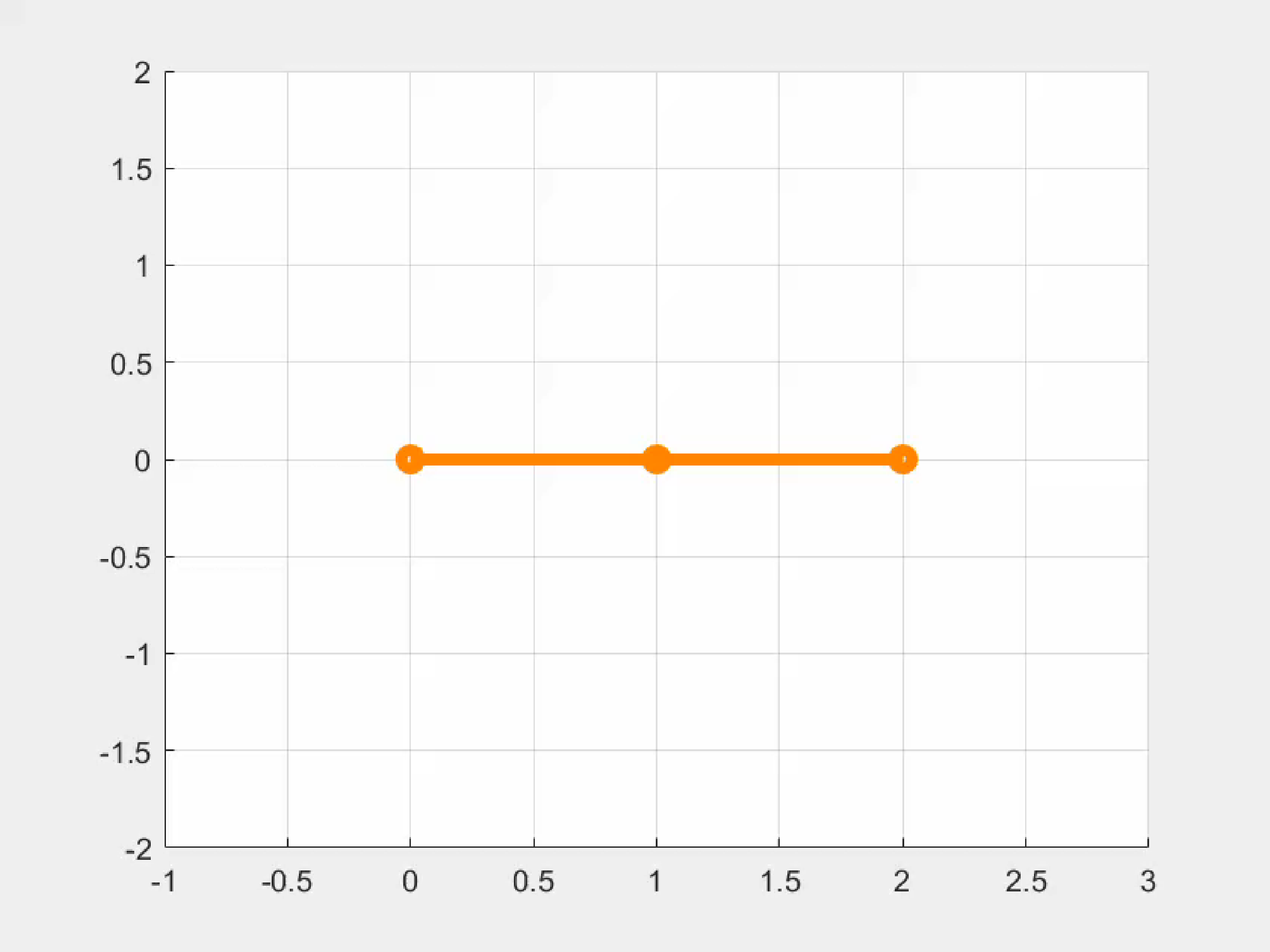
$$\ddot{\theta}^t = M(\theta)^{-1}\left(\tau - C(\theta, \dot{\theta})\dot{\theta} - g(\theta)\right)$$

$$\dot{\theta}^{t+1} = \dot{\theta}^t + \Delta T \cdot \ddot{\theta}^t$$
$$\theta^{t+1} = \theta^t + \Delta T \cdot \dot{\theta}^t$$

Simulation



Simulation



This Lecture

- How do we apply the dynamics equation for robot arms?
- Practice dynamics with an example

Next Lecture

- Now that we have a dynamic model of our robot arm…

  …can we leverage this model to *control* a robot?