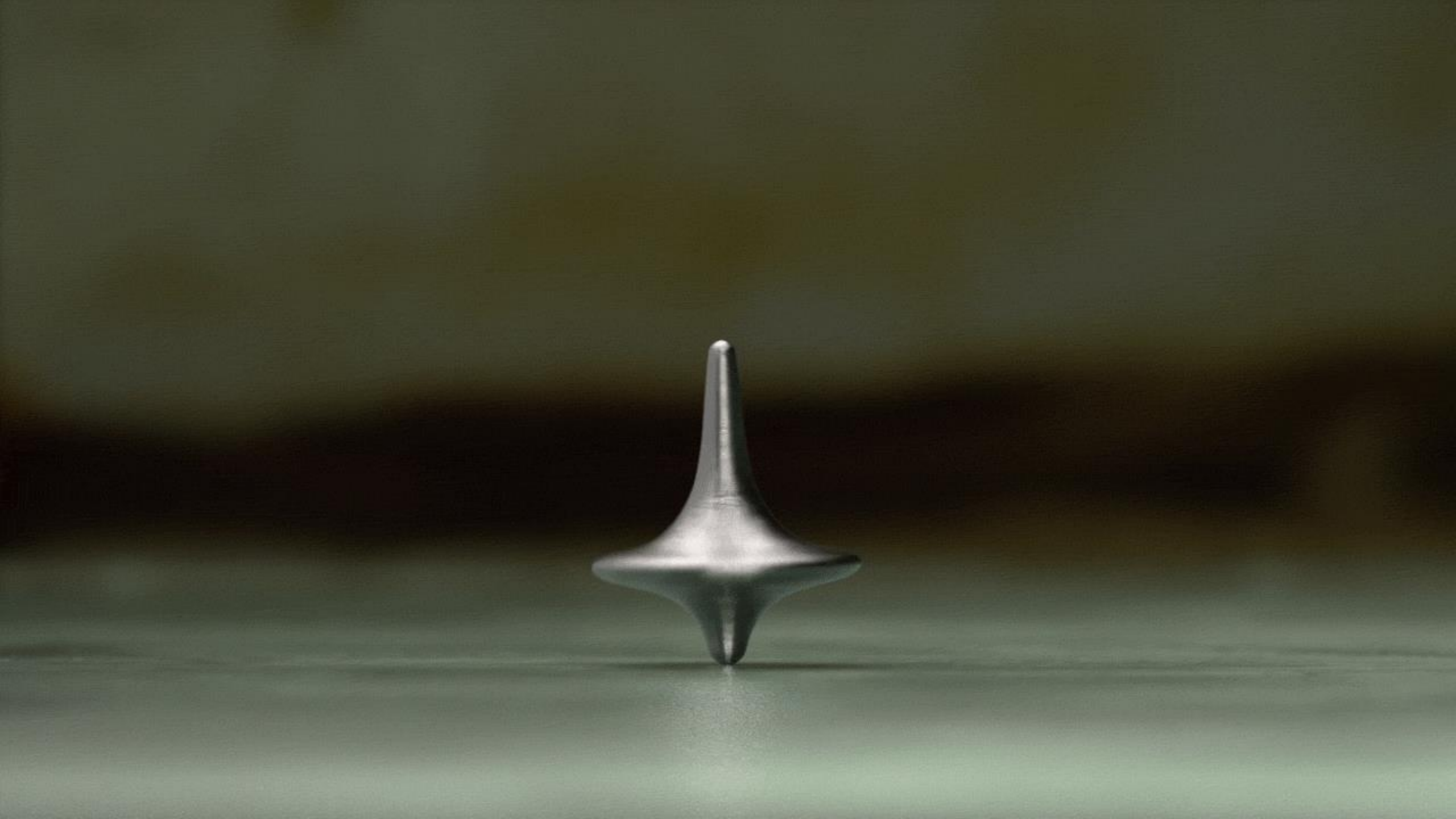
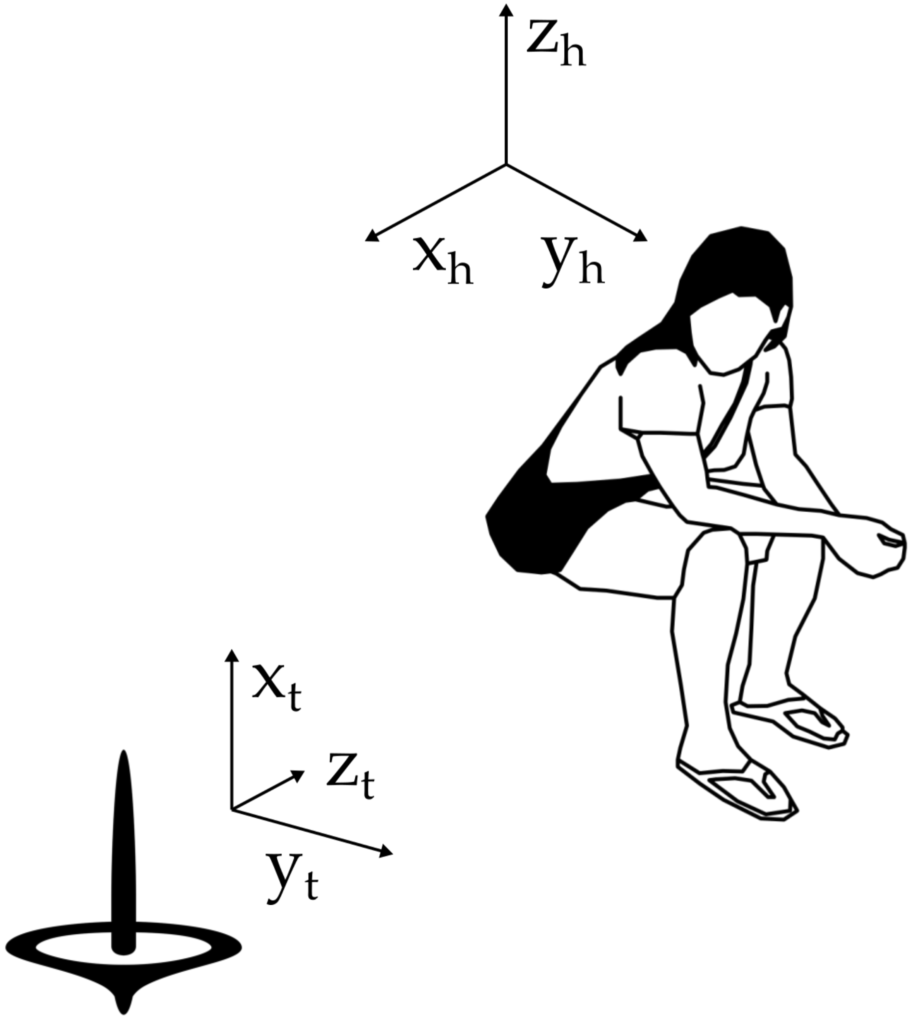
Angular Velocity

Reading: Modern Robotics 3.1 – 3.2



This Lecture

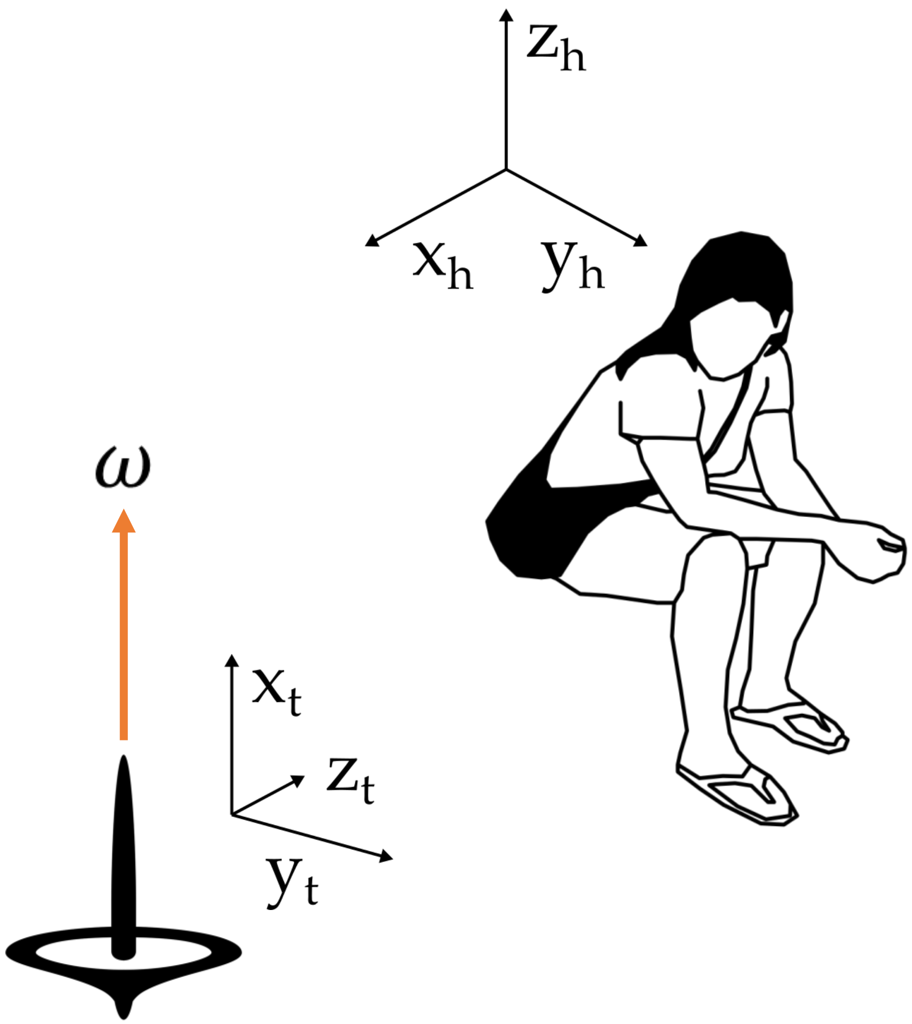
- How do we go from rotation to angular velocity?
- What are other ways to capture rotation?



z_h
x_h
y_h
x_t
z_t
y_t

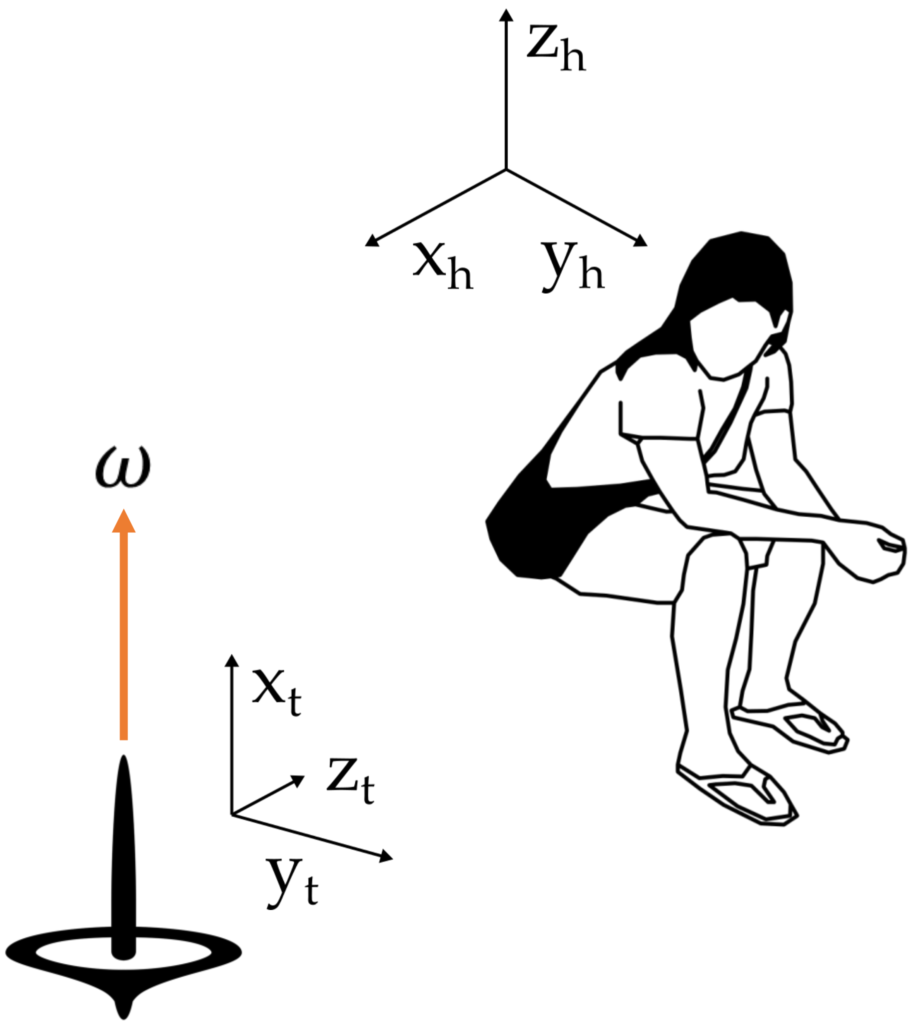
Angular velocity is a **vector**

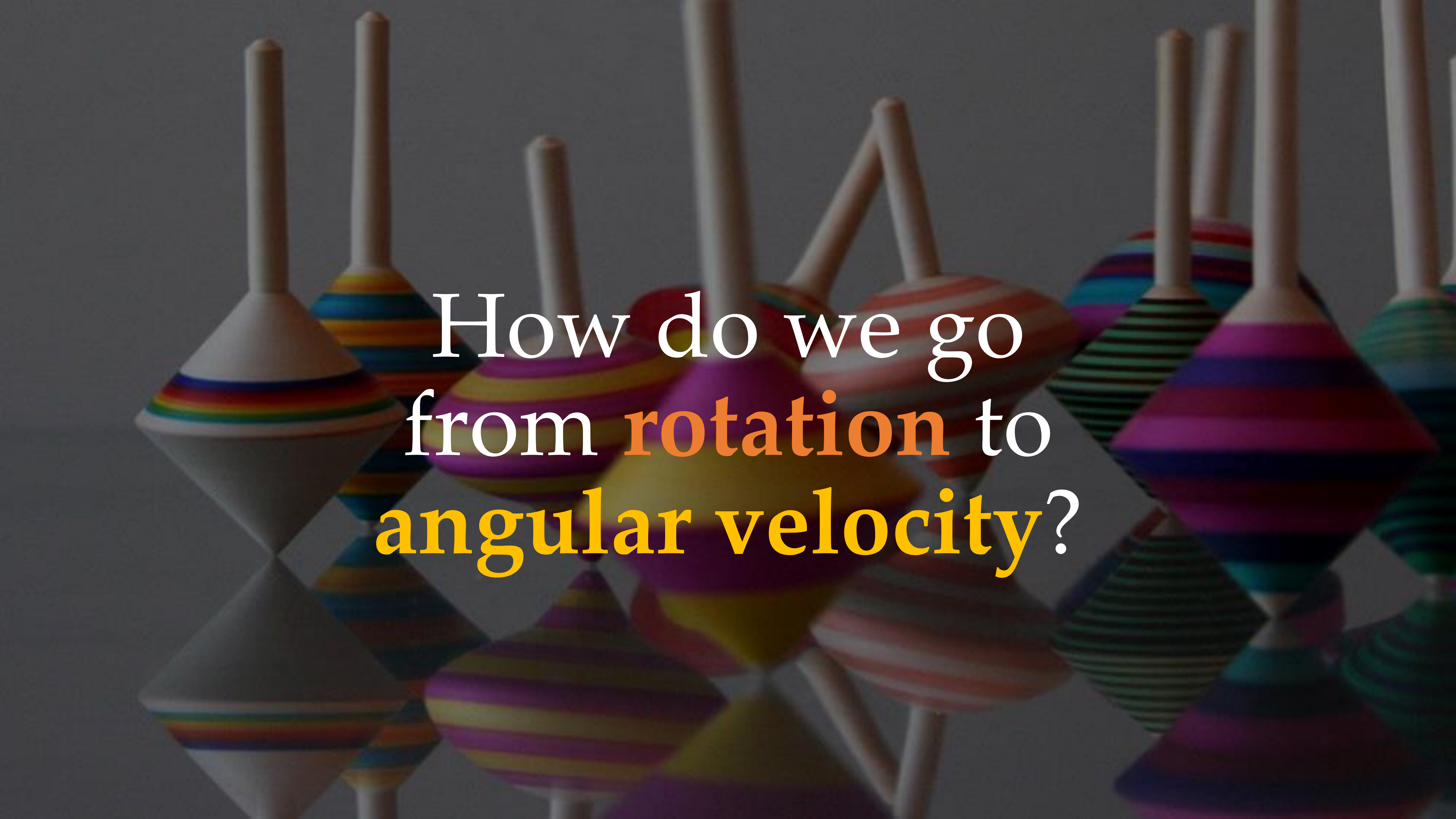
- Direction is the axis the frame is rotating around
- Magnitude is the speed of rotation

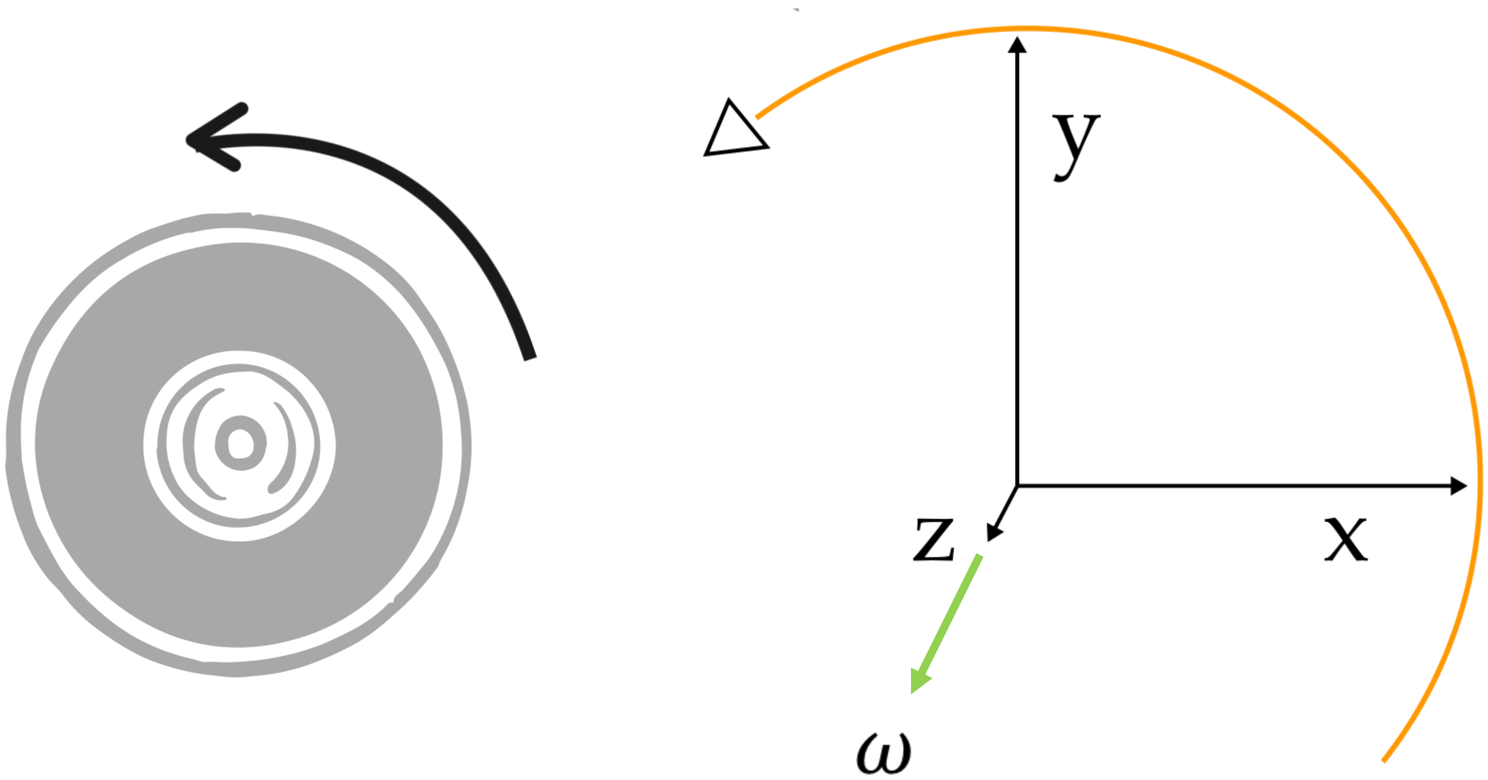


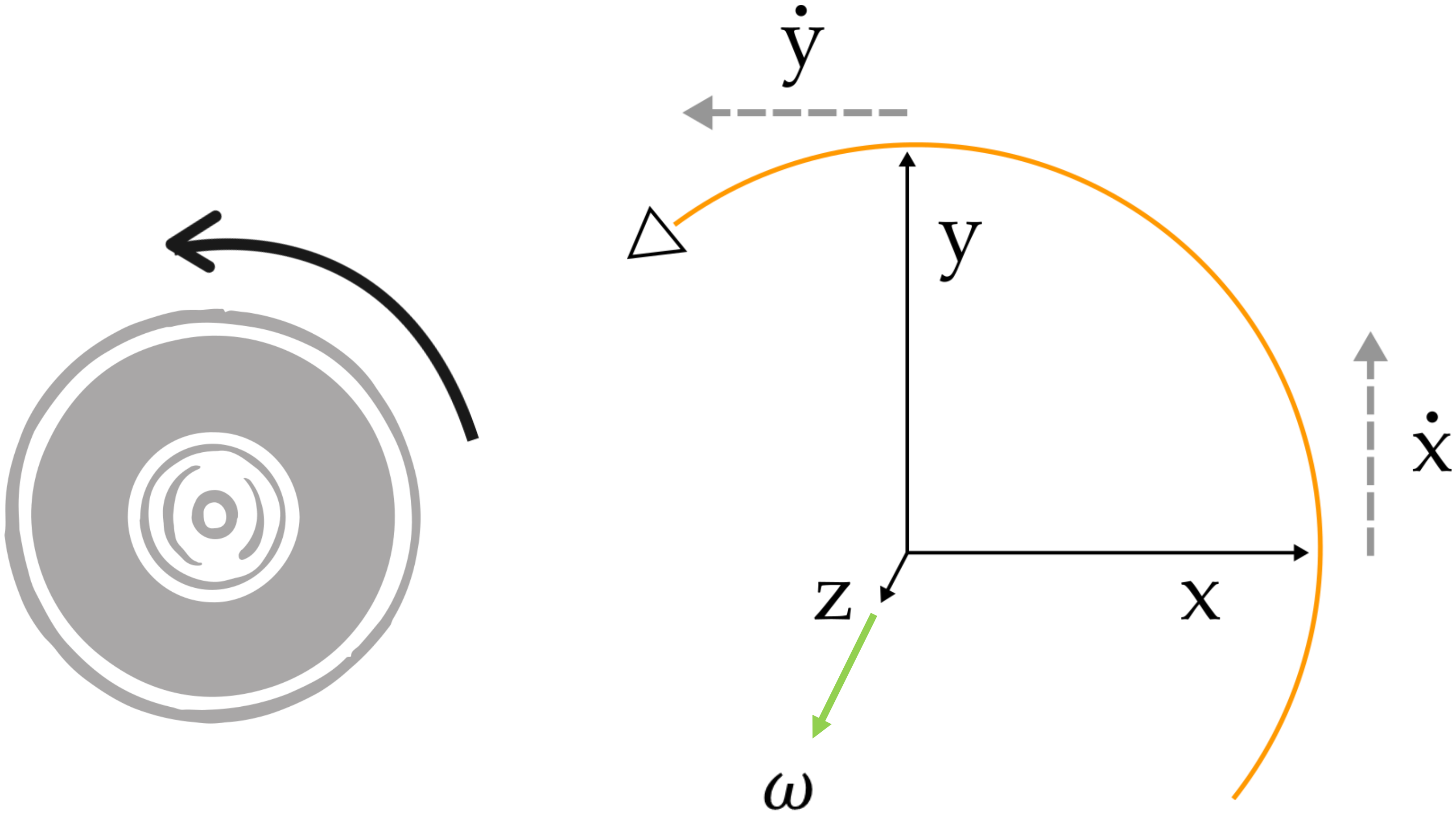
- Angular velocity is a **vector**

$$\omega_h = \begin{bmatrix} 0 \\ 0 \\ 5 \end{bmatrix}, \qquad \omega_t = \begin{bmatrix} 5 \\ 0 \\ 0 \end{bmatrix}$$




How do we go
from rotation to
angular velocity?

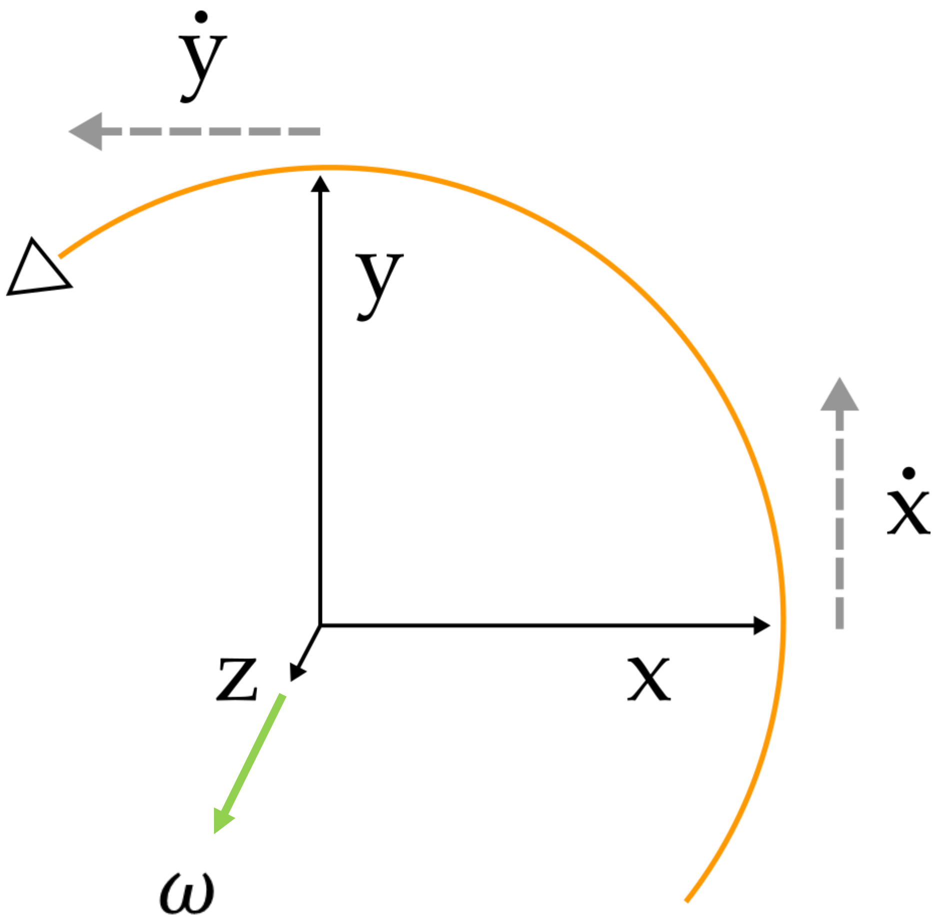

y
z
x
ω



$\dot{y}$
y
$\dot{x}$
z
x
ω

$$\dot{R} = \frac{dR}{dt}$$

$$\dot{R} = [\omega \times x \quad \omega \times y \quad \omega \times z]$$
$$\dot{R} = \omega \times R$$



Let’s introduce an **operator**
to make this a bit easier

Skew-Symmetric Matrix

-

$$x \times y = \begin{bmatrix} 0 & -x_3 & x_2 \\ x_3 & 0 & -x_1 \\ -x_2 & x_1 & 0 \end{bmatrix} \begin{bmatrix} y_1 \\ y_2 \\ y_3 \end{bmatrix} = [x]y$$

$[x]$ is called a **skew-symmetric** matrix

Skew-Symmetric Matrix

- Given a 3-dimensional vector $x = [x_1 \quad x_2 \quad x_3]^T$

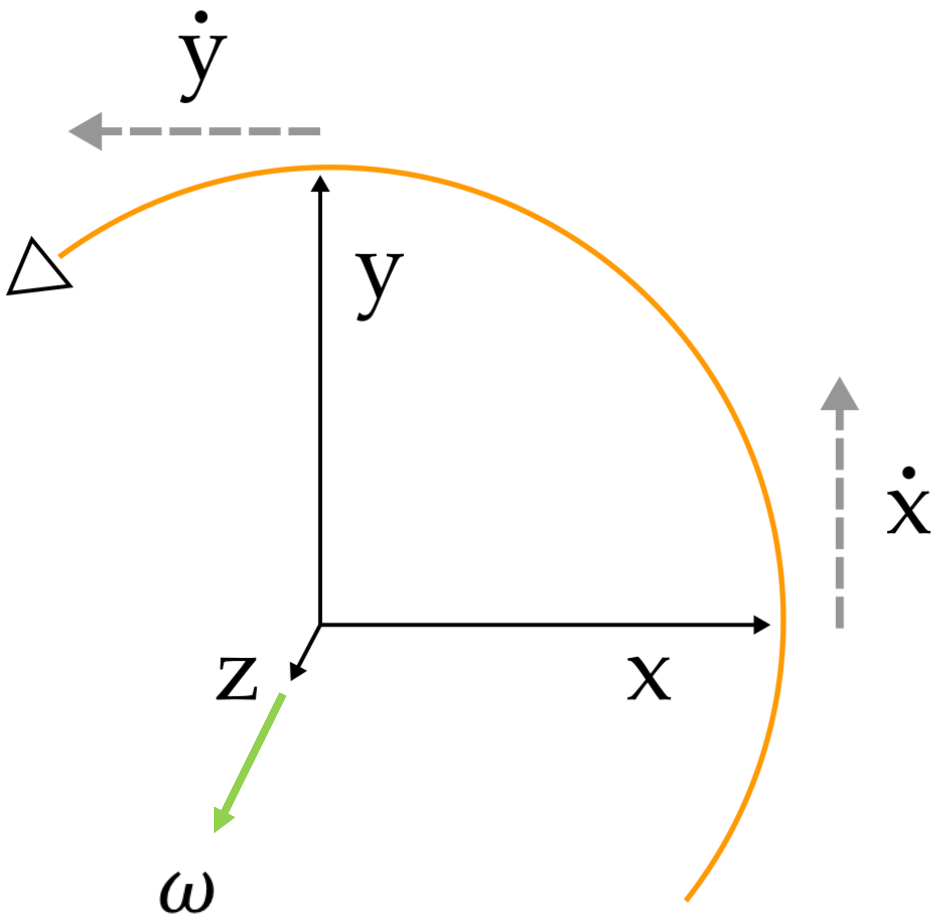
$$[x] = \begin{bmatrix} 0 & -x_3 & x_2 \\ x_3 & 0 & -x_1 \\ -x_2 & x_1 & 0 \end{bmatrix}$$

$[x]$ is called a **skew-symmetric** matrix

$$\dot{R} = \frac{dR}{dt}$$

$$\dot{R} = [\omega \times x \quad \omega \times y \quad \omega \times z]$$

$$\dot{\boldsymbol{R}} = [\boldsymbol{\omega}]\boldsymbol{R}$$



Takeaways

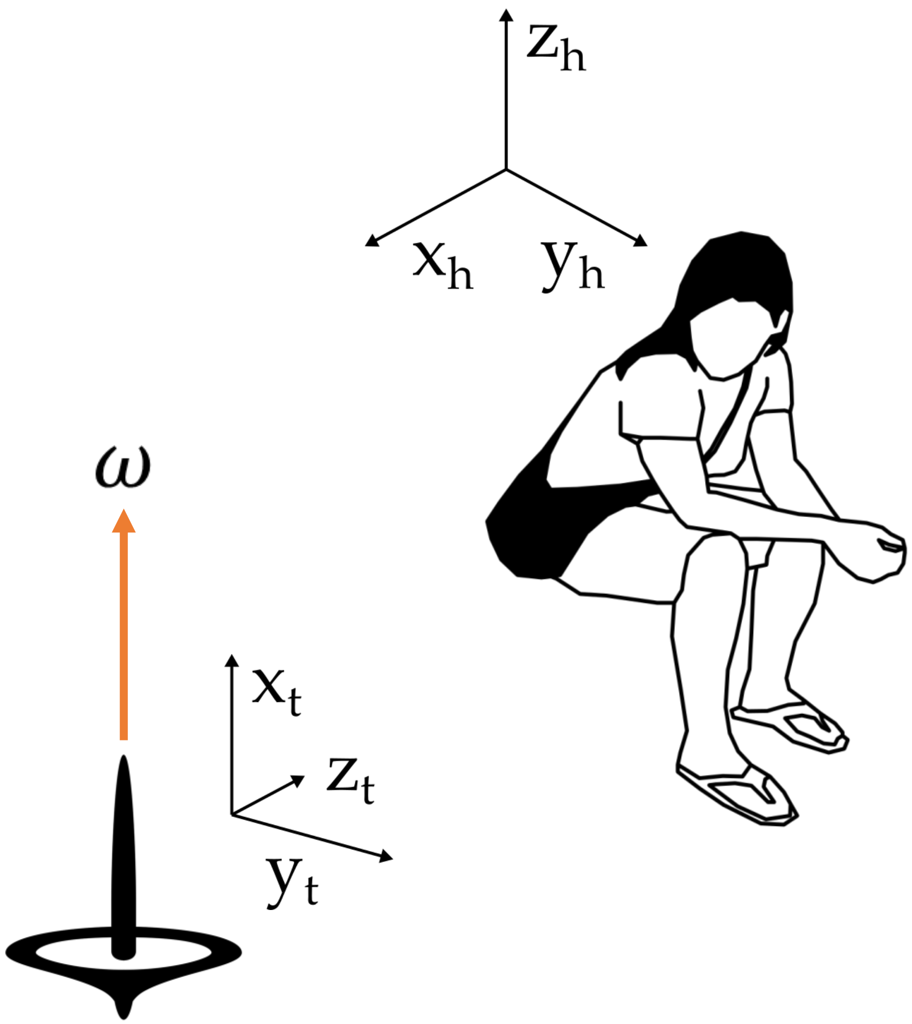
- Use R as shorthand for R_{ht}

We get the following **results**:

$$[\omega_h] = \dot{R}R^T$$

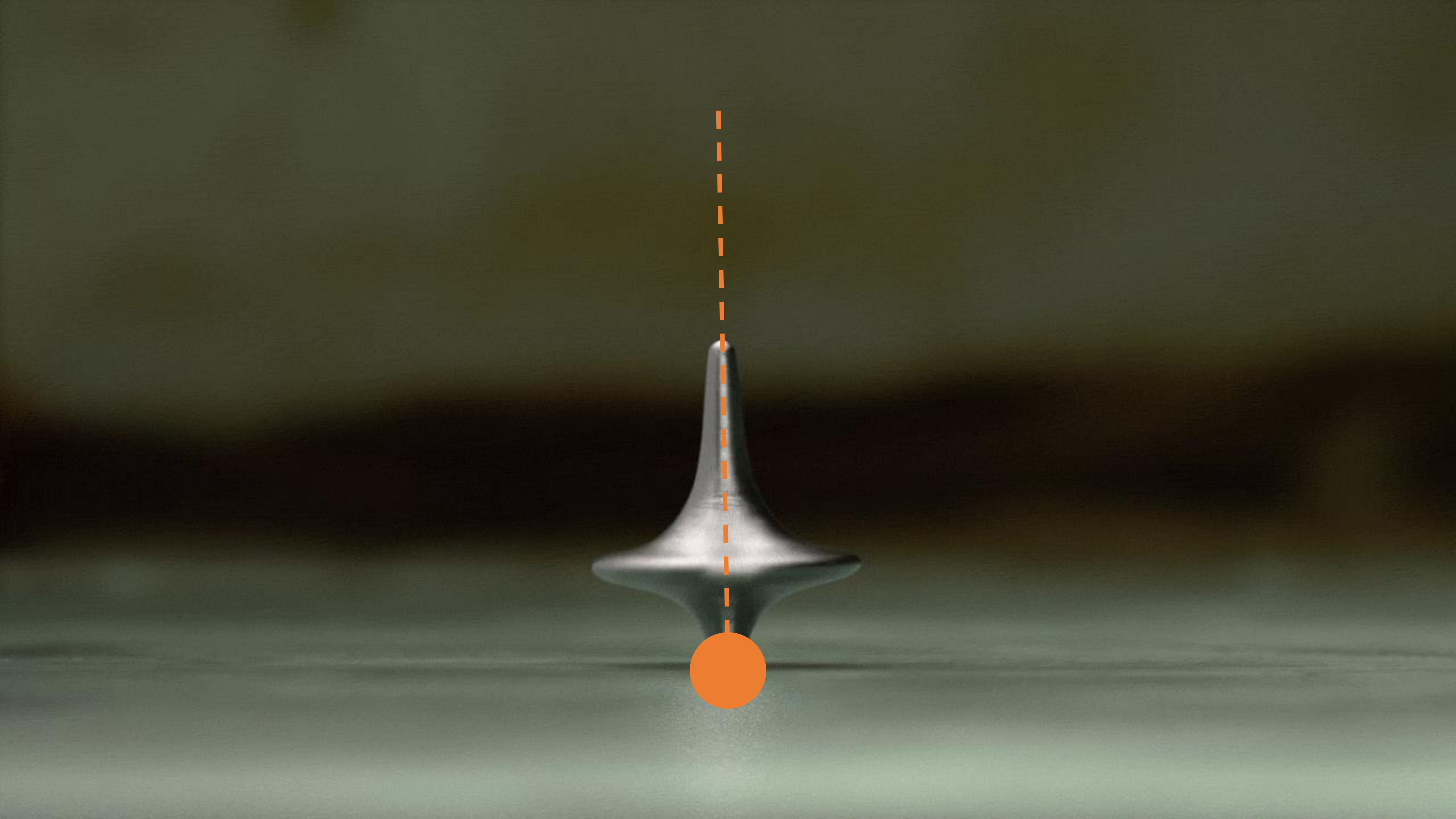
$$[\omega_t] = R^T\dot{R}$$

$$\omega_h = R\omega_t$$



Euler’s Rotation Theorem

Any rigid body motion that leaves one point fixed can be represented by a **single rotation** about an **axis** through the fixed point.



Capturing Rotation

- We just need a 3-dimensional vector to capture rotation…

…but we've been using rotation matrices with ***9 elements?***

$$R = \begin{bmatrix} r_{11} & r_{12} & r_{13} \\ r_{21} & r_{22} & r_{23} \\ r_{31} & r_{32} & r_{33} \end{bmatrix}$$

Remember that we impose $R^T R = I$

- Each column must be **orthogonal** to the others (3 constraints)
- Each column must be a **unit vector** (3 more constraints)

Other Ways to Capture Rotation

- Euler angles
- Axis-angle
- Quaternions

Euler Angles

- We construct R from the product of **three successive rotations**.

XYZ: rotate about x by θ_1, then y by θ_2, then z by θ_3

$$R(\theta_1,\theta_2,\theta_3) = \underbrace{\begin{bmatrix} 1 & 0 & 0 \\ 0 & \cos\theta_1 & -\sin\theta_1 \\ 0 & \sin\theta_1 & \cos\theta_1 \end{bmatrix}}_{Rot(x,\theta_1)} \underbrace{\begin{bmatrix} \cos\theta_2 & 0 & \sin\theta_2 \\ 0 & 1 & 0 \\ -\sin\theta_2 & 0 & \cos\theta_2 \end{bmatrix}}_{Rot(y,\theta_2)} \underbrace{\begin{bmatrix} \cos\theta_3 & -\sin\theta_3 & 0 \\ \sin\theta_3 & \cos\theta_3 & 0 \\ 0 & 0 & 1 \end{bmatrix}}_{Rot(z,\theta_3)}$$

Axis-Angle

- We construct R from an **axis** and **angle**
  - $\hat{\omega}$ is a unit vector (axis we are rotating around)
  - θ is a scalar (angle we want to rotate)
  - Any angular velocity is an axis and angle: $\omega = \hat{\omega}\theta$

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$$\dot{p} = \omega \times p = [\omega]p$$

$$\boxed{p(\theta) = e^{[\omega]\theta} p(0)}$$

$R = e^{[\omega]\theta}$ is a rotation matrix. *See expm in matlab.*

What about the **inverse problem**? Given R, can we find axis $\hat{\omega}$ and angle θ?

Axis-Angle

- Given R, find the axis and angle:

$$\theta = \cos^{-1}\left(\frac{1}{2}(\mathrm{trace}(R) - 1)\right)$$

$$[\omega] = \frac{1}{2\sin\theta}(R - R^T)$$

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- Example 1

$$Rot\left(x, \frac{\pi}{2}\right) = \begin{bmatrix} 1 & 0 & 0 \\ 0 & \cos\theta & -\sin\theta \\ 0 & \sin\theta & \cos\theta \end{bmatrix} = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 0 & -1 \\ 0 & 1 & 0 \end{bmatrix}$$

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$$[\omega] = \frac{1}{2}\begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & -2 \\ 0 & 2 & 0 \end{bmatrix}, \qquad \omega = \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix}$$

Definition of **skew-symmetric** matrix

Axis-Angle

- Given R, find the axis and angle:

$$\theta = \cos^{-1}\left(\frac{1}{2}(\text{trace}(R) - 1)\right)$$

$$[\omega] = \frac{1}{2\sin\theta}(R - R^T)$$

- Example 2

$$Rot(x,\pi) = \begin{bmatrix} 1 & 0 & 0 \\ 0 & \cos\theta & -\sin\theta \\ 0 & \sin\theta & \cos\theta \end{bmatrix} = \begin{bmatrix} 1 & 0 & 0 \\ 0 & -1 & 0 \\ 0 & 0 & -1 \end{bmatrix}$$

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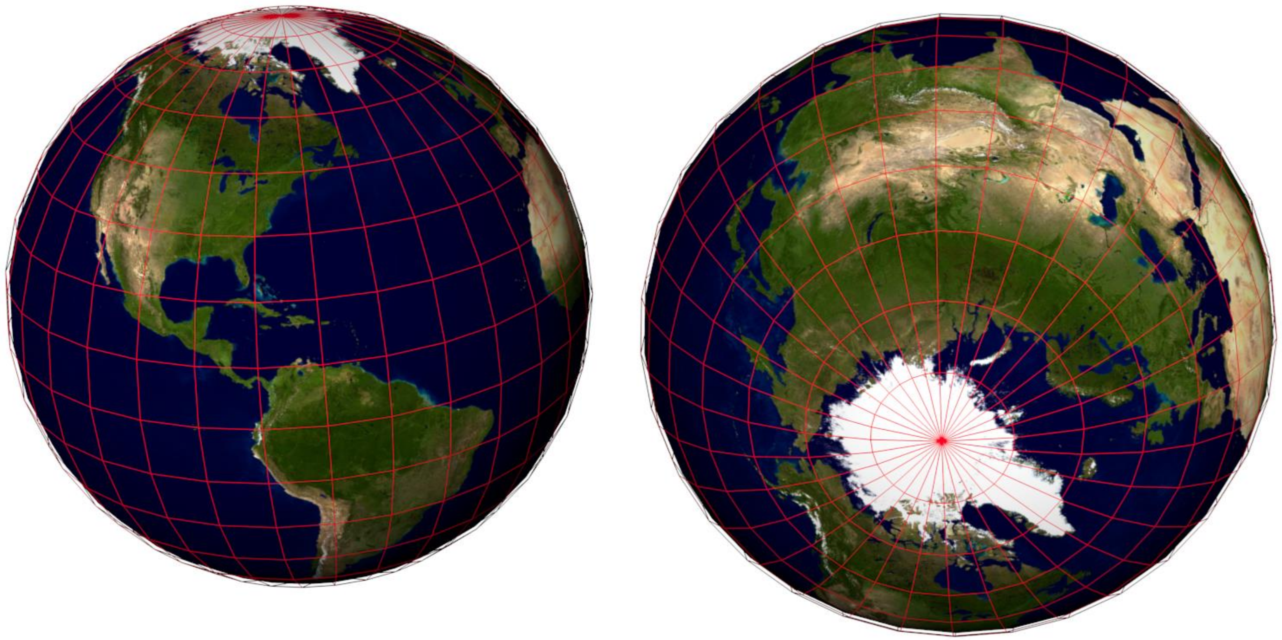
$$\theta = \cos^{-1}\left(\frac{1}{2}(-1 - 1)\right) = \pi$$

$$[\omega] = \frac{1}{2 * 0}(R - R^T)$$

ω is undefined?

Quaternions

- Given Euler Angles or Axis-Angle, we can **always** get R
- Given R, sometimes we cannot find a **unique** axis or Euler Angles



Quaternions

Quaternions avoid this by capturing rotation with **4 parameters**.

$$\boldsymbol{q} = (\boldsymbol{\eta}, \boldsymbol{\varepsilon})$$

- η is a scalar
- $\varepsilon = [\varepsilon_1 \quad \varepsilon_2 \quad \varepsilon_3]^T$ is a vector

$$\eta = \cos\frac{\theta}{2} \qquad \varepsilon = \widehat{\omega}\sin\frac{\theta}{2}$$

Axis-angle to quaternion

This Lecture

- How do we go from rotation to angular velocity?
- What are other ways to capture rotation?

Next Lecture

- How do we combine position and rotation to describe rigid body motion?