Pseudoinverse

Reading: Modern Robotics 6.2

This Lecture

- How can we think about inverse kinematics in terms of velocity?
- What is the Jacobian pseudoinverse?
- What is the null space of this pseudoinverse?
- How does this null space relate to redundant robot arms?

How do we **find** the **inverse kinematics**?

Jacobian Pseudoinverse

$$V = J(\theta)\dot{\theta}$$

This maps from joint velocity to end-effector twist…

…but what if we want the *opposite direction*?

Jacobian Pseudoinverse

$$V = J(\theta)\dot{\theta}$$

$$\dot{\theta} = J^{-1}V, \qquad J = J(\theta)$$

Remember J is $6 \times n$ matrix
We can only invert J if **square** matrix,
i.e., if robot has $n = 6$ joints

Jacobian Pseudoinverse

$$V = J(\theta)\dot{\theta}$$

$$\underbrace{\dot{\theta} = J^{+}V,}_{\textit{solve for } J^{+}} \qquad \underbrace{J\dot{\theta} = JJ^{+}V = V}_{\textit{such that } JJ^{+} = I}$$

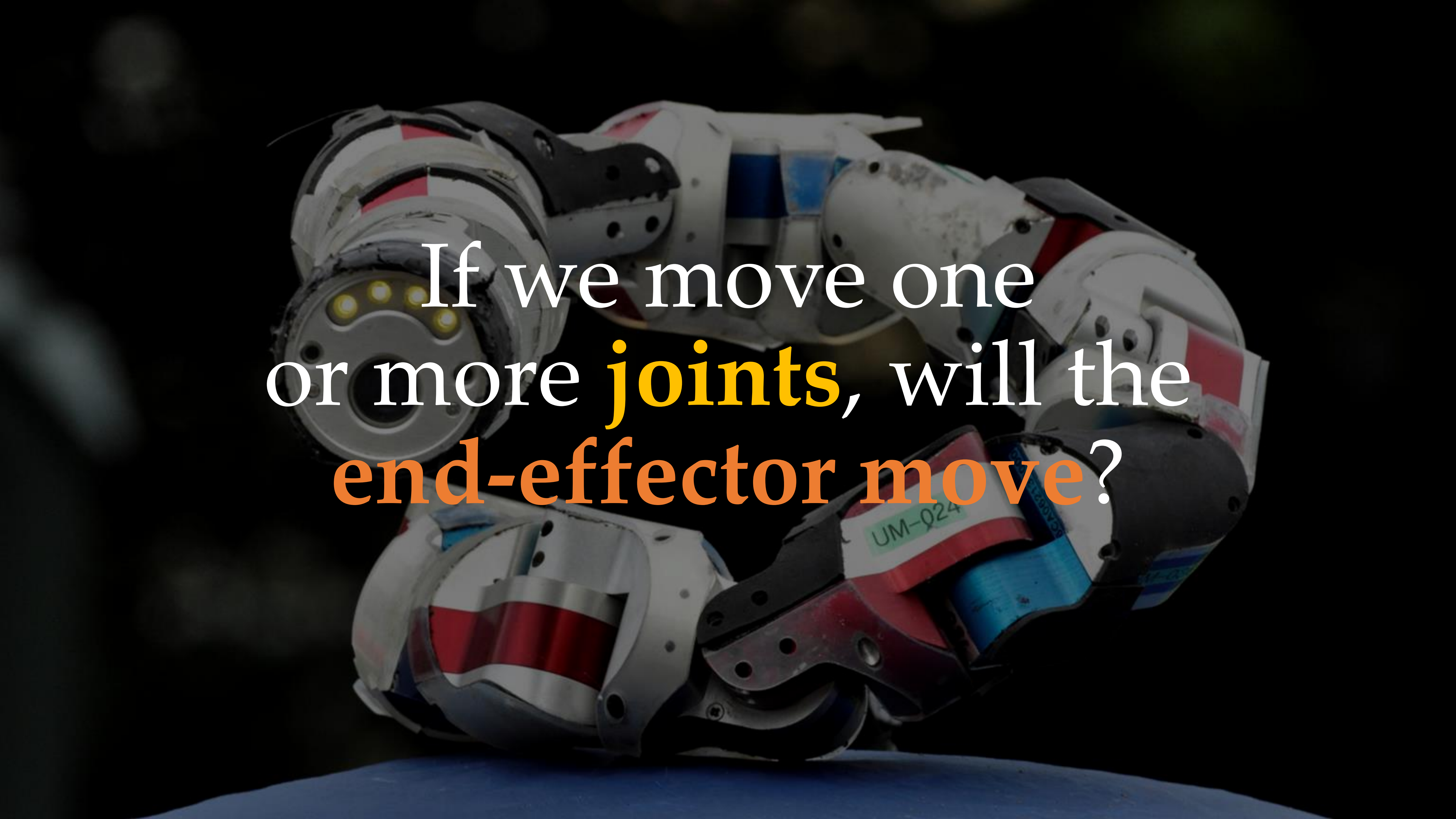
Jacobian Pseudoinverse

$$V = J(\theta)\dot{\theta}$$

$$\dot{\theta} = J^{+}V, \qquad J^{+} = J^{T}(JJ^{T})^{-1}$$

pseudoinverse

This outputs the **joint velocities** we would need to produce a desired **end-effector twist**.


If we move one
or more **joints**, will the
end-effector move?
UM-024

Null Space

$$\dot{\theta} = J^{+}V + (\boldsymbol{I} - \boldsymbol{J}^{+}\boldsymbol{J})\boldsymbol{b}$$

our solution so far

new term where ***b***
is an arbitrary n-length vector

Evaluating this gives us $\dot{\theta}$ with two components:

- $J^{+}V$ is a joint velocity that moves the end-effector at twist V
- $(I - J^{+}J)b$ is an internal joint velocity that does not affect the end-effector twist V

Null Space

$$\dot{\theta} = J^{+}V + (\boldsymbol{I} - \boldsymbol{J}^{+}\boldsymbol{J})\boldsymbol{b}$$

Is this valid? Remember that we must preserve $V = J\dot{\theta}$

$$J\dot{\theta} = JJ^{+}V + (\boldsymbol{J} - \boldsymbol{J}\boldsymbol{J}^{+}\boldsymbol{J})\boldsymbol{b}$$ Multiply both sides by J

$$J\dot{\theta} = V + (\boldsymbol{J} - \boldsymbol{J})\boldsymbol{b}$$ Use property $JJ^{+} = I$

$$J\dot{\theta} = V$$

Null Space

$$\dot{\theta} = J^{+}V + (\boldsymbol{I} - \boldsymbol{J}^{+}\boldsymbol{J})\boldsymbol{b}$$

new term where $\boldsymbol{b}$
is an arbitrary n-length vector

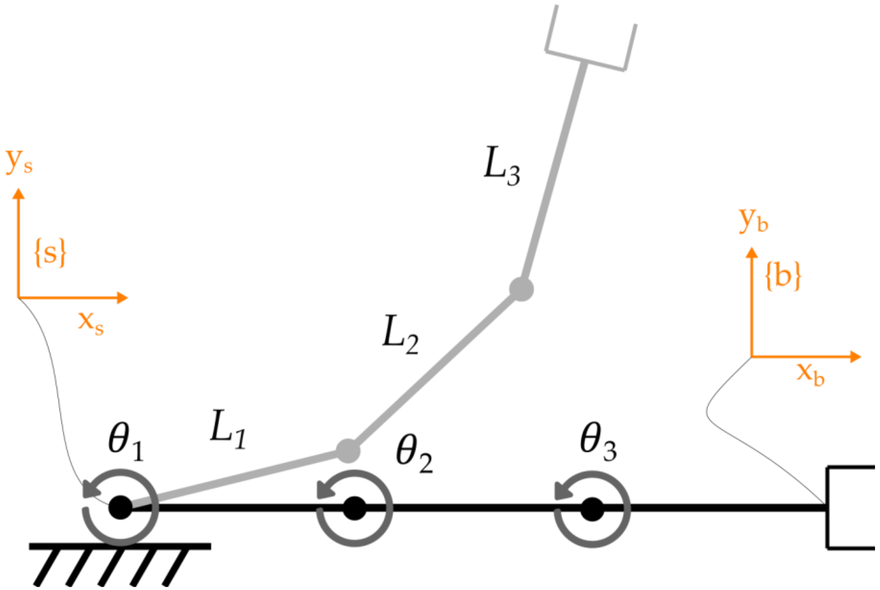
All vectors of the form $(I - J^{+}J)b$ lie in the **null space** of J

No matter what we pick for b, these joint velocities will not move the end-effector

Example 1

3-DoF robot moving in a plane.
What is the null space $(\boldsymbol{I} - \boldsymbol{J}^{+}\boldsymbol{J})\boldsymbol{b}$?

$$\boldsymbol{J}(\boldsymbol{\theta}) = \begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 1 & 1 & 1 \\ -L_1 s_1 - L_2 s_{12} - L_3 s_{123} & -L_2 s_{12} - L_3 s_{123} & -L_3 s_{123} \\ L_1 c_1 + L_2 c_{12} + L_3 c_{123} & L_2 c_{12} + L_3 c_{123} & L_3 c_{123} \\ 0 & 0 & 0 \end{bmatrix}$$



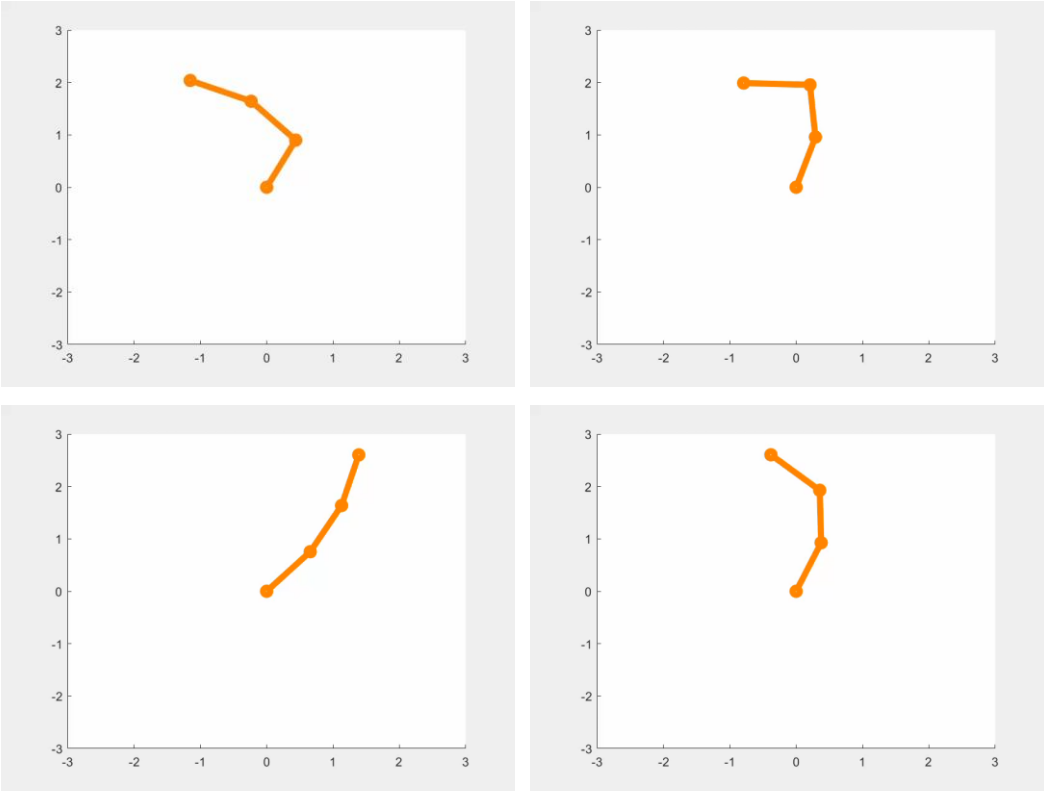
Example 1

3-DoF robot moving in a plane.
What is the null space $(\boldsymbol{I} - \boldsymbol{J}^{+}\boldsymbol{J})\boldsymbol{b}$?

$$\boldsymbol{I} - \boldsymbol{J}^{+}\boldsymbol{J} = \boldsymbol{0}$$

If not at singularity

For this robot *any joint velocity* causes end-effector motion

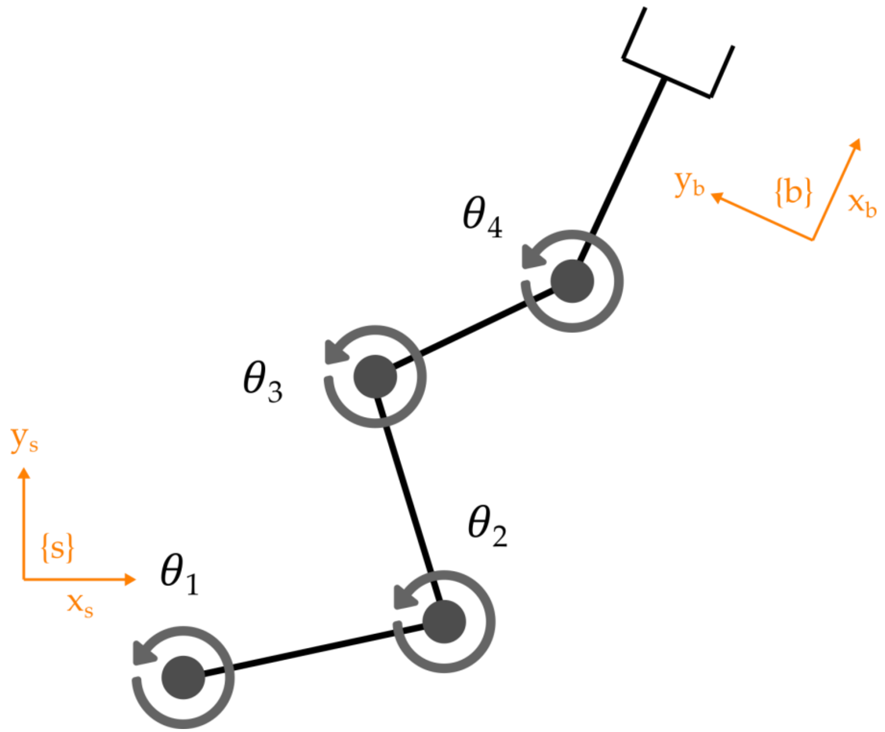


Example 2

4-DoF robot moving in a plane.

This robot is **redundant** because it has more DoF than necessary for its workspace.

What is the null space $(\boldsymbol{I} - \boldsymbol{J}^{+}\boldsymbol{J})\boldsymbol{b}$?



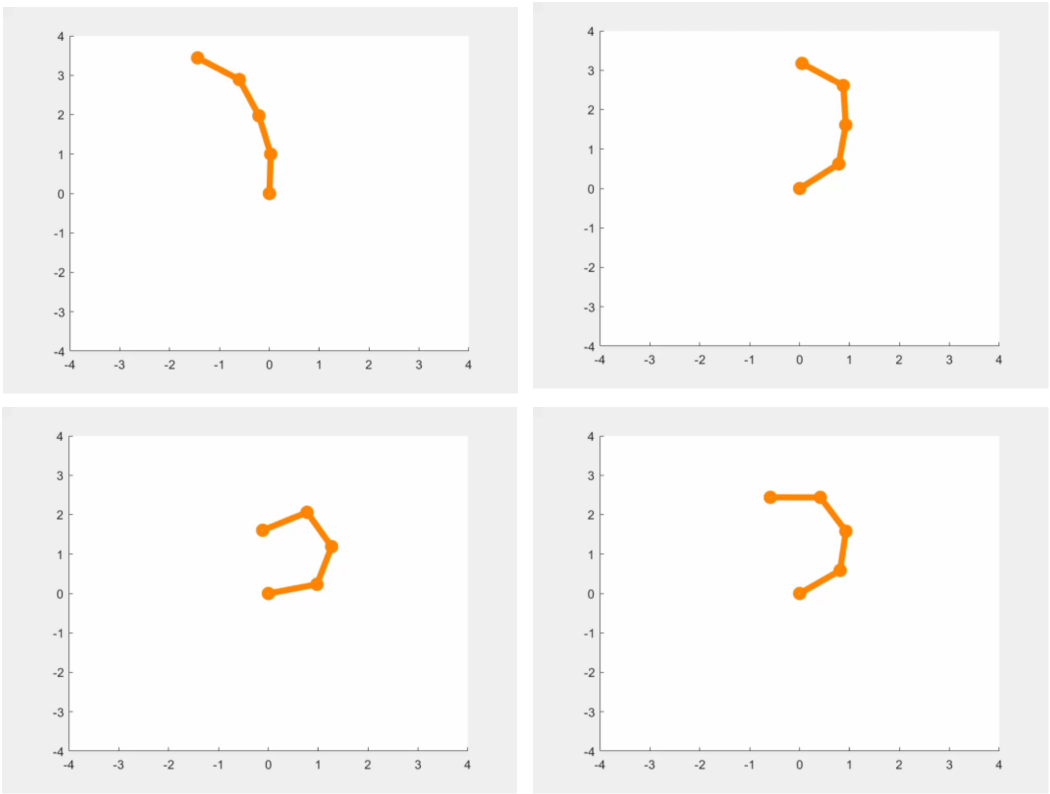
Example 2

4-DoF robot moving in a plane.

This robot is **redundant** because it has more DoF than necessary.

$$\dot{\theta} = (\boldsymbol{I} - \boldsymbol{J}^{+}\boldsymbol{J})\boldsymbol{b}$$

Even though we move the joints, we *do not cause* the end-effector to move!



Takeaways

- We don't have a clear way to go from T_{sb} to θ
- But we can use the Jacobian to go from V to $\dot{\theta}$

$$\dot{\theta} = J^+V + (I - J^+J)b$$

$J^+ = J^T(JJ^T)^{-1}$ is the pseudoinverse.
J^+V converts V to joint velocity $\dot{\theta}$

$(I - J^+J)b$ is the null space. Useful for redundant robots.

This Lecture

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Next Lecture

- Can we use the pseudoinverse to find a robot's inverse kinematics?