Kinetic and Potential Energy

Reading: Robot Modeling and Control 7.2

This Lecture

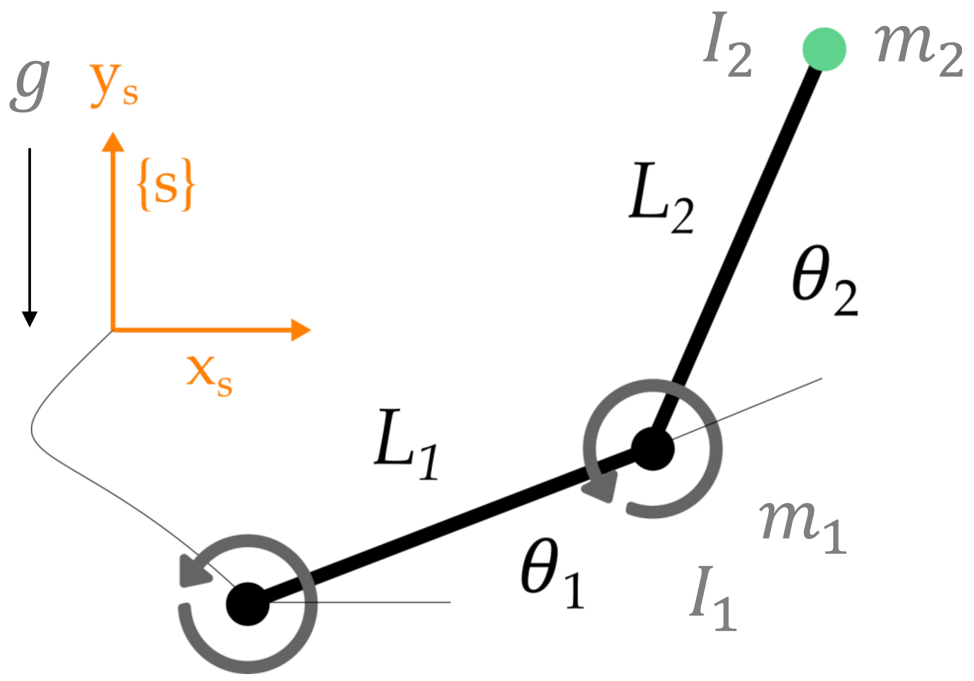
- What is the potential energy of a robot arm?
- What is the kinetic energy of a robot arm?
- How do we calculate kinetic energy using the Jacobian?

Motivation

$$L(\theta, \dot{\theta}) = \underbrace{K(\theta, \dot{\theta})}_{\textit{Kinetic energy}} - \underbrace{P(\theta)}_{\textit{Potential energy}}$$

- **Lagrangian** L is the difference between kinetic and potential energy
- Can convert Lagrangian to dynamics using Euler-Lagrange equation

Motivation



Mass

m_1 is the mass of link 1

m_2 is the mass of link 2

Center of mass at the end of each link

Inertia

I_1 is the inertia of link 1 about the z axis

I_2 is the inertia of link 2 about the z axis

Gravity

Gravity acts along the $-y$ axis

How do we find the **potential energy**?

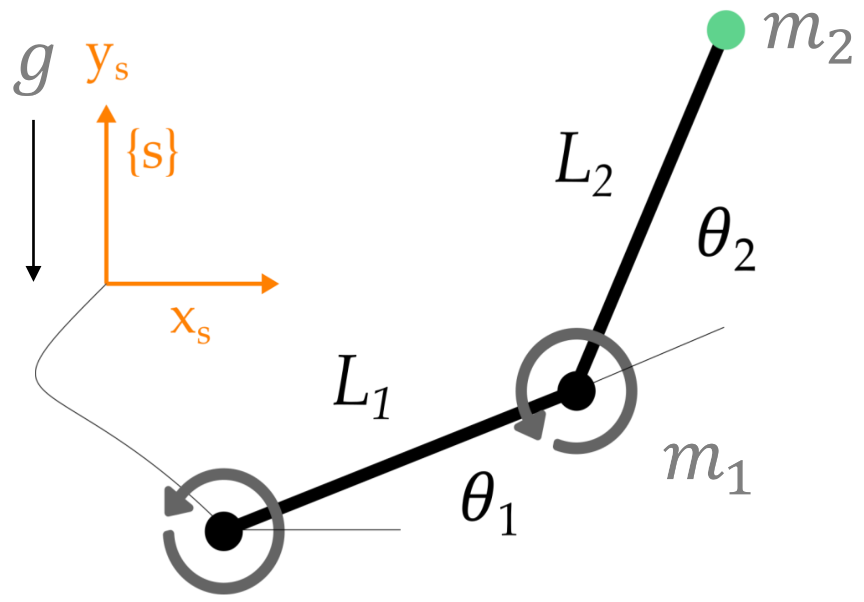
Potential Energy

For standard rigid robot arms, **gravity** is the source of potential energy.

$$P_i(\theta) = g m_i h_i$$

Potential energy of link i

h_i is the height of the center of mass of link i



Potential Energy

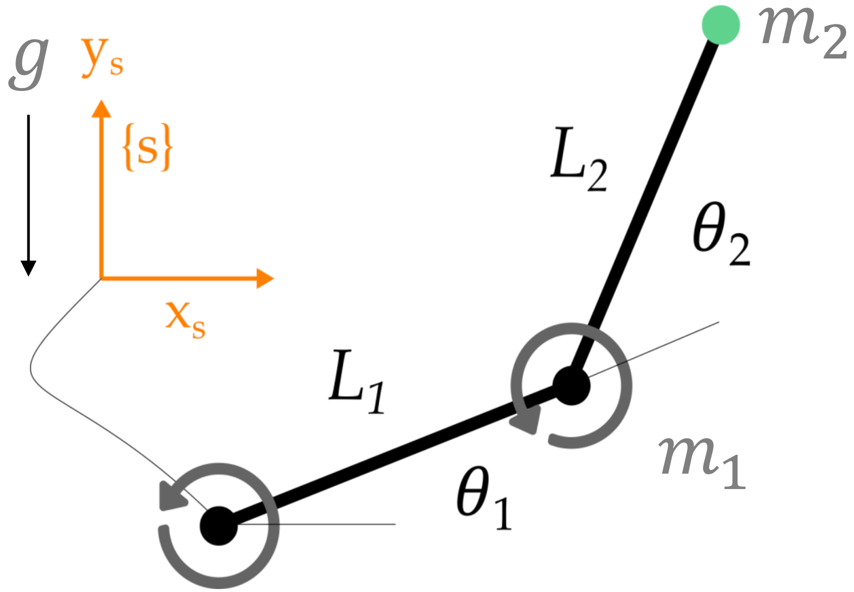
For standard rigid robot arms, **gravity** is the source of potential energy.

$$P_i(\theta) = g m_i h_i$$

$$\boxed{P(\theta) = P_1(\theta) + P_2(\theta)}$$

Total potential energy

Sum the potential energy of each link

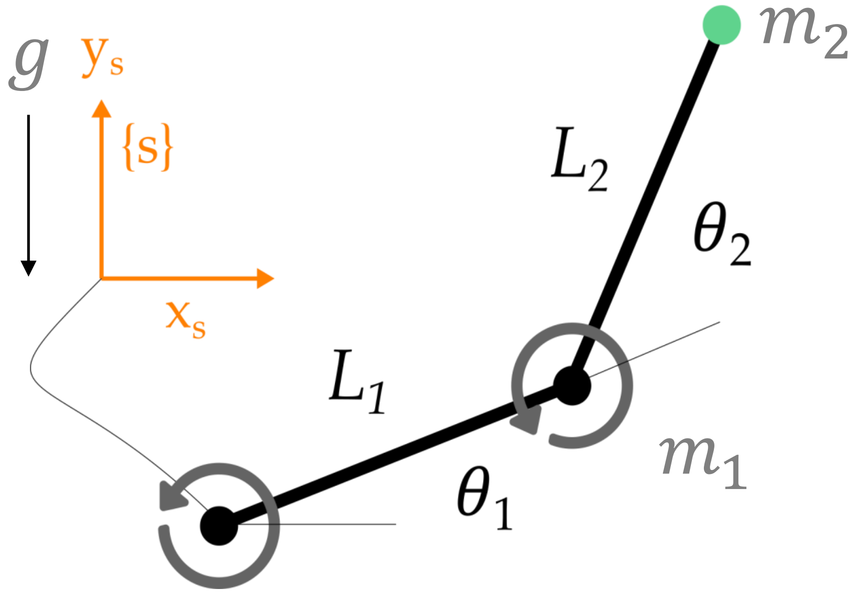


Potential Energy

For standard rigid robot arms, **gravity** is the source of potential energy.

$$P_1(\theta) = g m_1 h_1$$

Reminder: for this robot, center of mass is at the end of the link

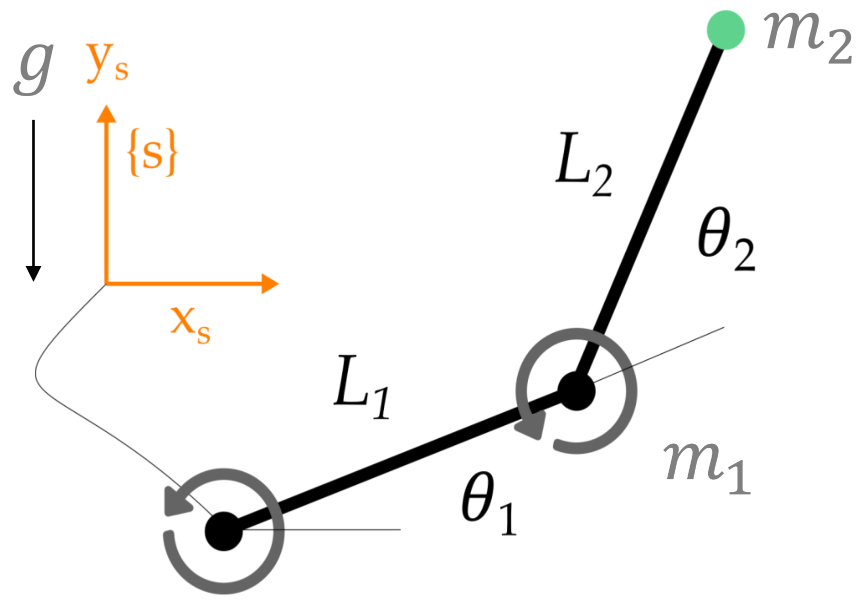


Potential Energy

For standard rigid robot arms, **gravity** is the source of potential energy.

$$P_1(\theta) = g m_1 y_1$$

$$P_1(\theta) = g m_1 L_1 \sin\theta_1$$

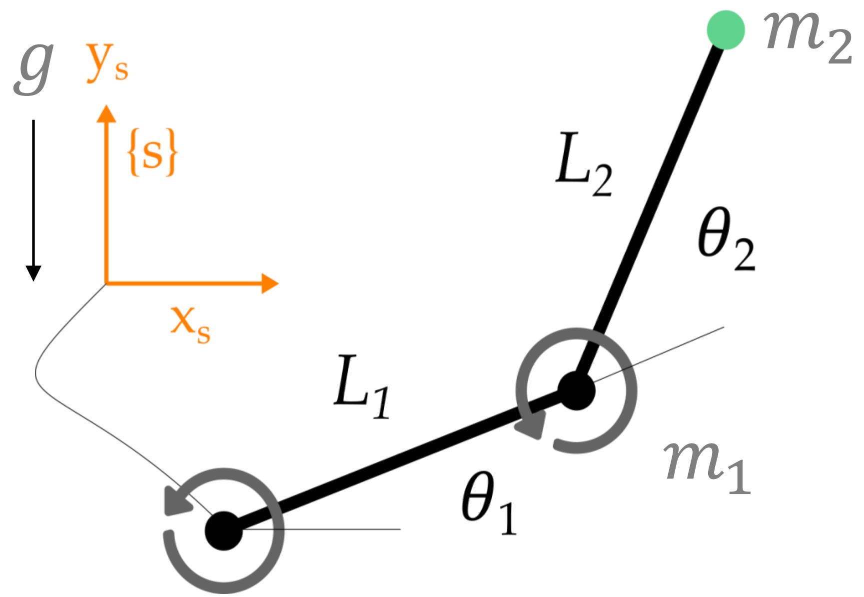


Potential Energy

For standard rigid robot arms, **gravity** is the source of potential energy.

$$P_2(\theta) = g m_2 y_2$$

$$P_2(\theta) = g m_2 (L_1 \sin\theta_1 + L_2 \sin(\theta_1 + \theta_2))$$



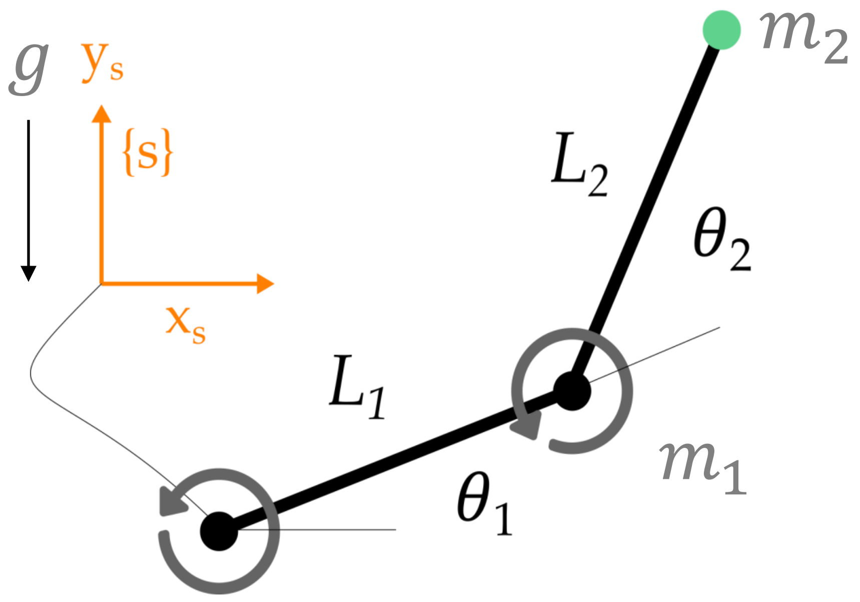
You can get the height by calculating forward kinematics of the center of mass

Potential Energy

For standard rigid robot arms, **gravity** is the source of potential energy.

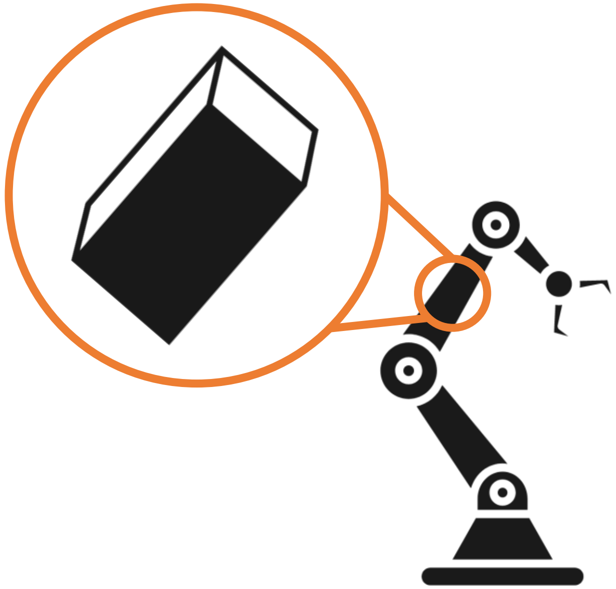
$$P(\theta) = P_1(\theta) + P_2(\theta)$$

$$P(\theta) = g(m_1 + m_2)L_1 \sin\theta_1 + g m_2 L_2 \sin(\theta_1 + \theta_2)$$




How do we find the
kinetic energy?

Kinetic Energy



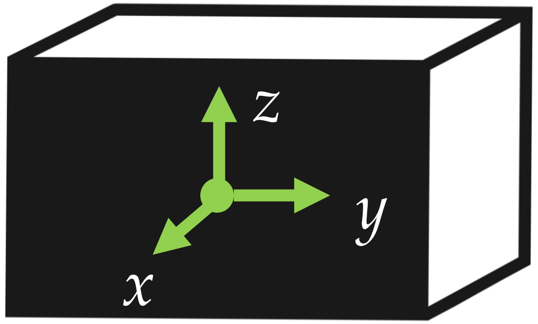
Kinetic energy of rigid body comes from **translation** (at center of mass) and **rotation** (about center of mass)

$$K(\theta, \dot{\theta}) = \underbrace{\frac{1}{2} m v^T v}_{} + \underbrace{\frac{1}{2} \omega^T R \boldsymbol{I} R^T \omega}_{}$$

v is linear velocity of center of mass

ω is angular velocity of center of mass

Kinetic Energy



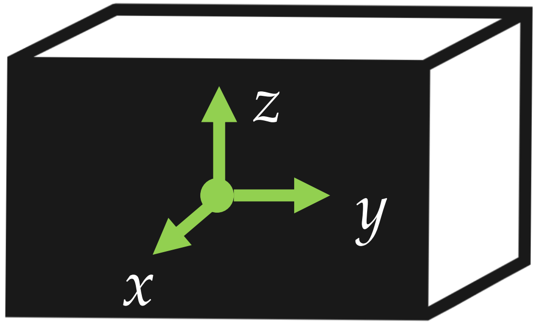
$$\boldsymbol{I} = \begin{bmatrix} I_{xx} & I_{xy} & I_{xz} \\ I_{yx} & I_{yy} & I_{yz} \\ I_{zx} & I_{zy} & I_{zz} \end{bmatrix}$$

Kinetic energy of rigid body comes from **translation** (at center of mass) and **rotation** (about center of mass)

$$K(\theta, \dot{\theta}) = \frac{1}{2} m v^T v + \frac{1}{2} \omega^T \underline{R \boldsymbol{I} R^T} \omega$$

- $\boldsymbol{I}$ is 3×3 constant **inertia matrix** in link's body frame
- R is the orientation of the link's body frame

Kinetic Energy



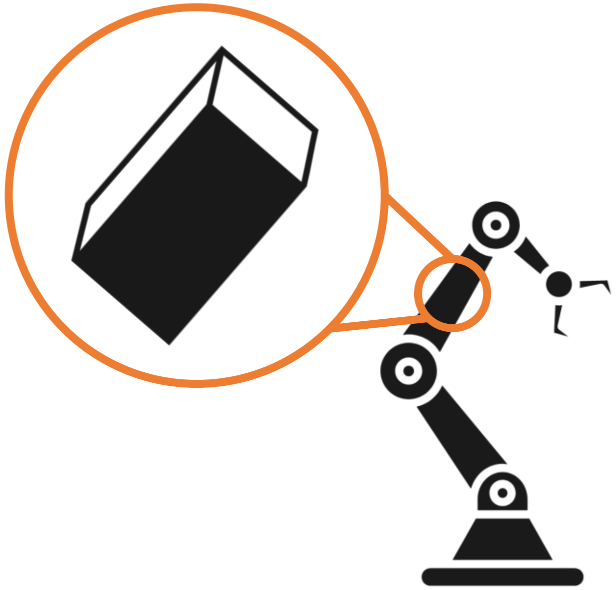
$$\boldsymbol{I} = \begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & I \end{bmatrix}$$

Kinetic energy of rigid body comes from **translation** (at center of mass) and **rotation** (about center of mass)

$$K(\theta, \dot{\theta}) = \frac{1}{2} m v^T v + \frac{1}{2} \omega^T \underline{R \boldsymbol{I} R^T} \omega$$

- $\boldsymbol{I}$ is 3×3 constant **inertia matrix** in link's body frame
- R is the orientation of the link's body frame
- When moving in a **plane**, inertia only about z axis

Kinetic Energy



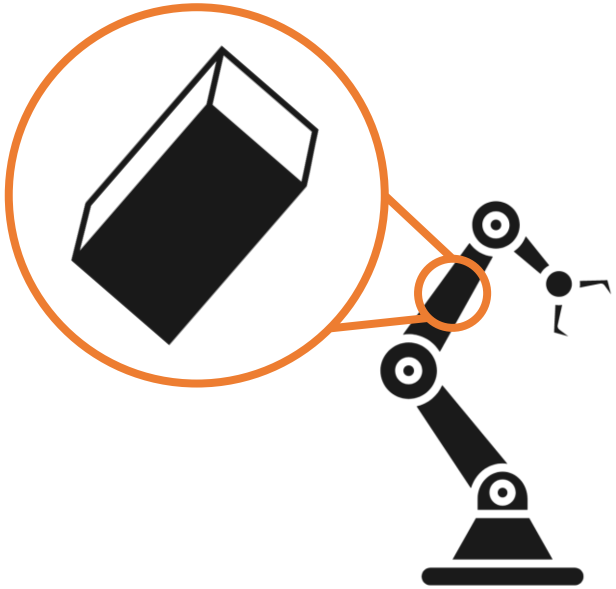
Kinetic energy of rigid body comes from **translation** (at center of mass) and **rotation** (about center of mass)

$$K(\theta, \dot{\theta}) = \frac{1}{2} m v^T v + \frac{1}{2} \omega^T R \boldsymbol{I} R^T \omega$$

$$\underline{V = J\dot{\theta}}, \qquad \begin{bmatrix} \omega \\ v \end{bmatrix} = \begin{bmatrix} J_\omega \\ J_v \end{bmatrix} \dot{\theta}$$

geometric Jacobian

Kinetic Energy



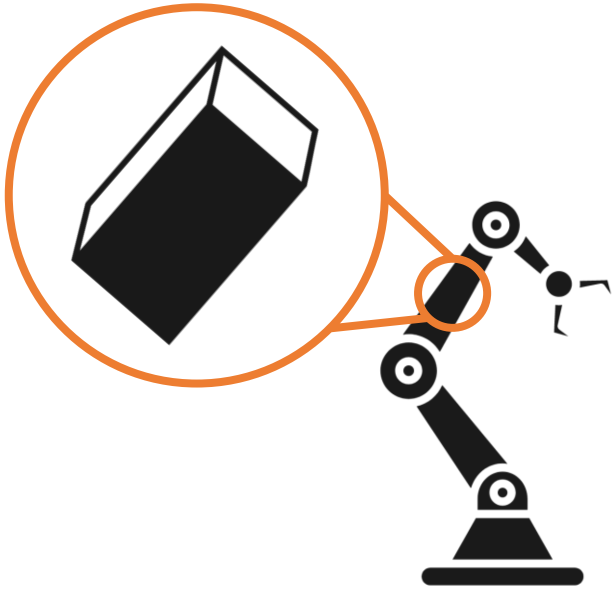
Kinetic energy of rigid body comes from **translation** (at center of mass) and **rotation** (about center of mass)

$$K(\theta, \dot{\theta}) = \frac{1}{2} m v^T v + \frac{1}{2} \omega^T R \boldsymbol{I} R^T \omega$$

$$\omega = J_\omega \dot{\theta}, \qquad v = J_v \dot{\theta}$$

Not new Jacobians, just split geometric Jacobian into two parts

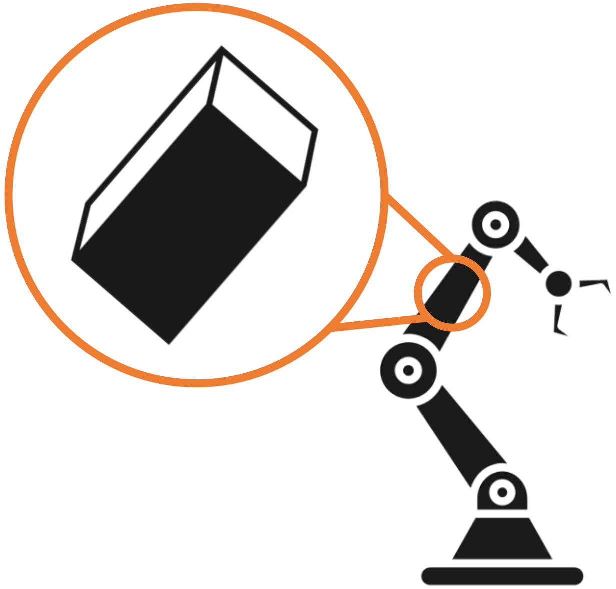
Kinetic Energy



Kinetic energy of rigid body comes from **translation** (at center of mass) and **rotation** (about center of mass)

$$K(\theta,\dot{\theta}) = \frac{1}{2}m\dot{\theta}^T {J_v}^T J_v \dot{\theta} + \frac{1}{2}\dot{\theta}^T {J_\omega}^T R\boldsymbol{I}R^T J_\omega \dot{\theta}$$

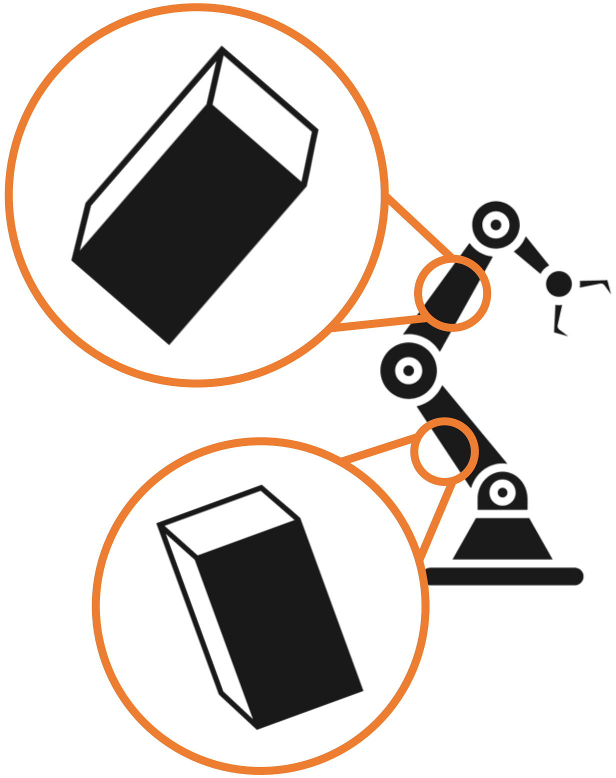
Kinetic Energy



Kinetic energy of rigid body comes from **translation** (at center of mass) and **rotation** (about center of mass)

$$K(\theta,\dot{\theta}) = \frac{1}{2}\dot{\theta}^T\left[m{J_v}^T J_v + {J_\omega}^T R\boldsymbol{I}R^T J_\omega\right]\dot{\theta}$$

Kinetic Energy

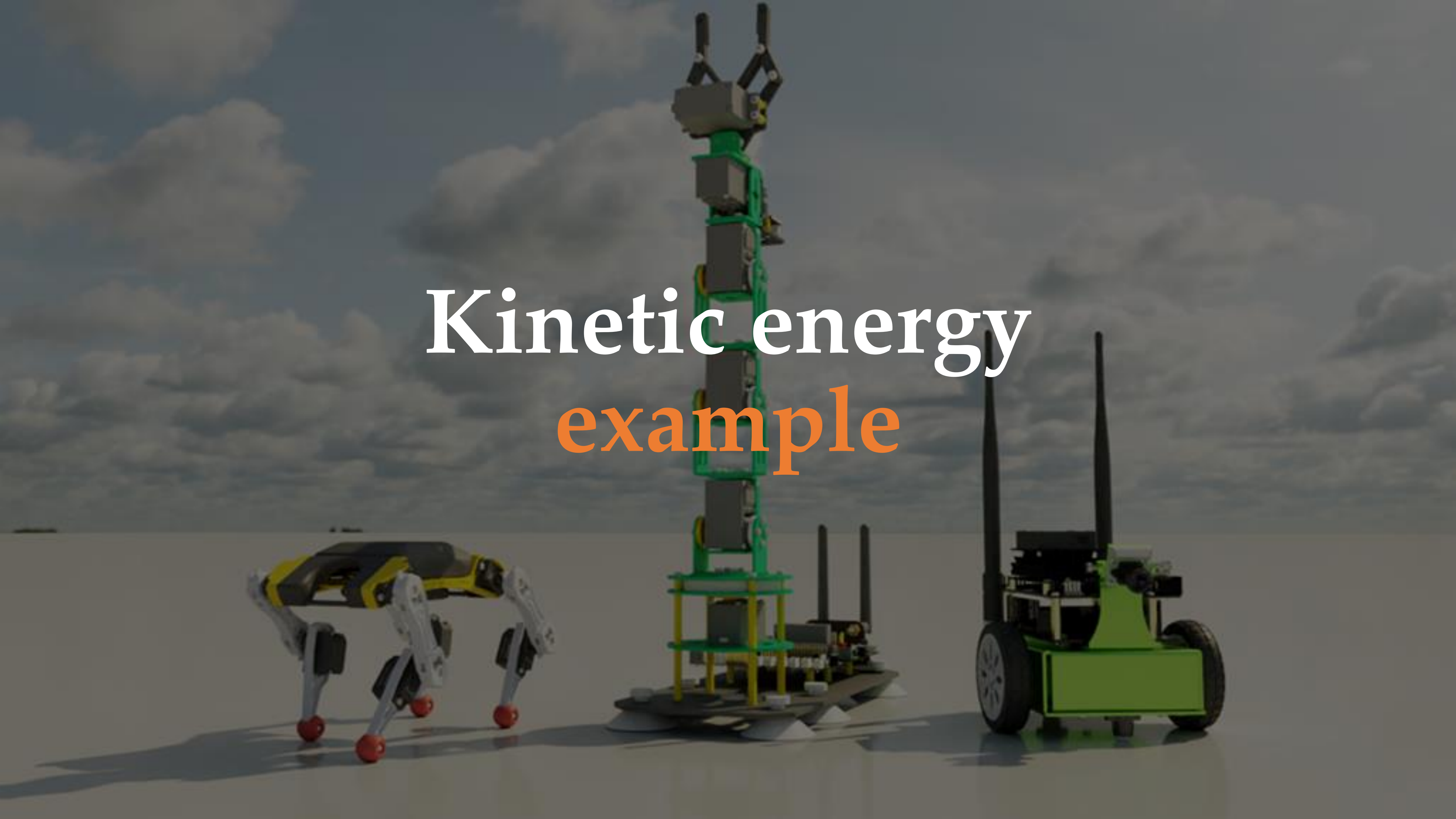


Kinetic energy of rigid body comes from **translation** (at center of mass) and **rotation** (about center of mass)

$$K(\theta,\dot{\theta}) = \frac{1}{2}\dot{\theta}^T \left[\sum_{i=1}^{n} m_i {J_{v_i}}^T J_{v_i} + {J_{\omega_i}}^T R_i \boldsymbol{I}_i {R_i}^T J_{\omega_i} \right] \dot{\theta}$$

Must find geometric Jacobian for *each* center of mass (i.e., one Jacobian for each link)

Kinetic energy example

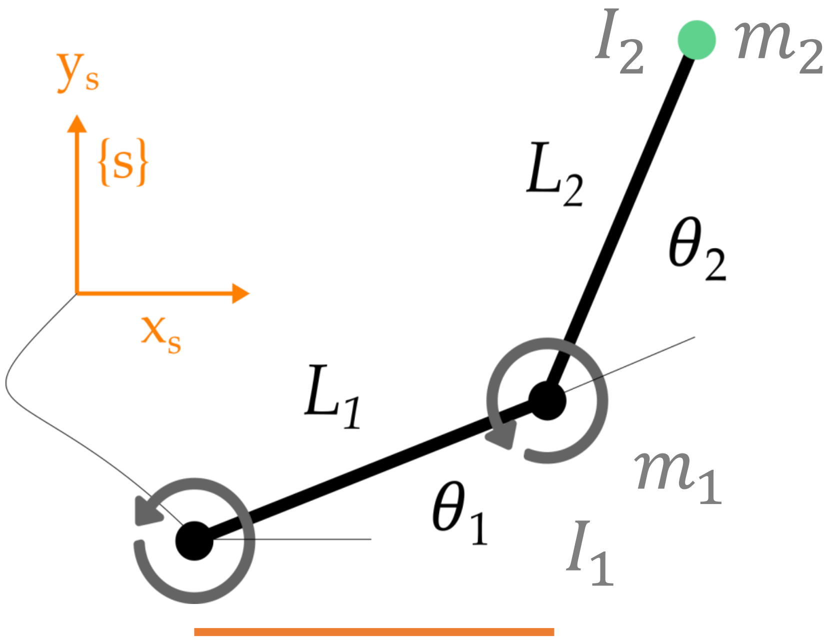


Kinetic Energy

Step 1. Get geometric Jacobian for the center of mass of the 1st link

$$M = \begin{bmatrix} I & \begin{bmatrix} L_1 \\ 0 \\ 0 \end{bmatrix} \\ 0 & 1 \end{bmatrix}, \qquad S_1 = \begin{bmatrix} 0 \\ 0 \\ 1 \\ 0 \\ 0 \\ 0 \end{bmatrix}, \qquad S_2 = \begin{bmatrix} 0 \\ 0 \\ 0 \\ 0 \\ 0 \\ 0 \end{bmatrix}$$

The second joint has no effect on the center of mass for the first link, so $S_2 = 0$

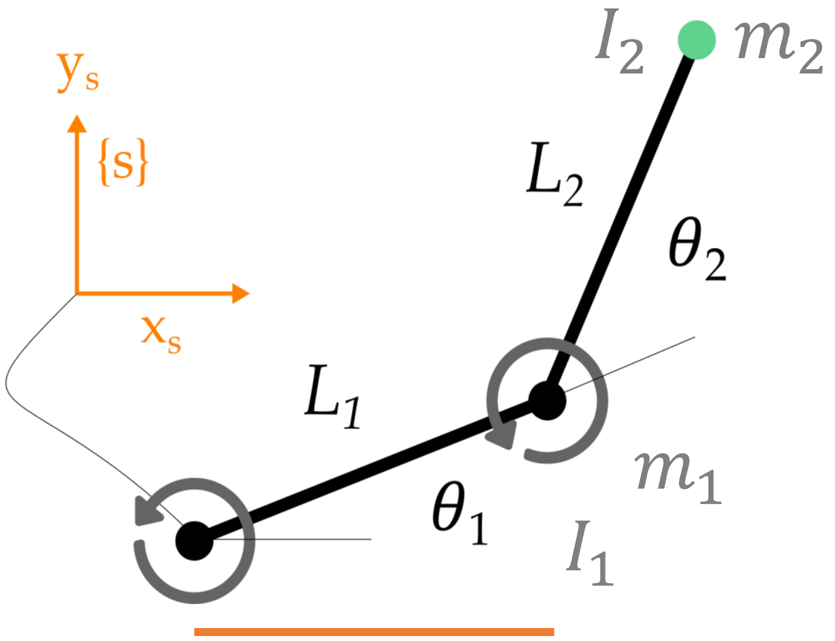


Kinetic Energy

Step 1. Get geometric Jacobian for the center of mass of the 1st link

$$J_{\omega_1} = \begin{bmatrix} 0 & \boxed{0} \\ 0 & \boxed{0} \\ 1 & \boxed{0} \end{bmatrix}, \qquad J_{v_1} = \begin{bmatrix} -L_1 s_1 & \boxed{0} \\ L_1 c_1 & \boxed{0} \\ 0 & \boxed{0} \end{bmatrix}$$

$$R_1 = \begin{bmatrix} c_1 & -s_1 & 0 \\ s_1 & c_1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

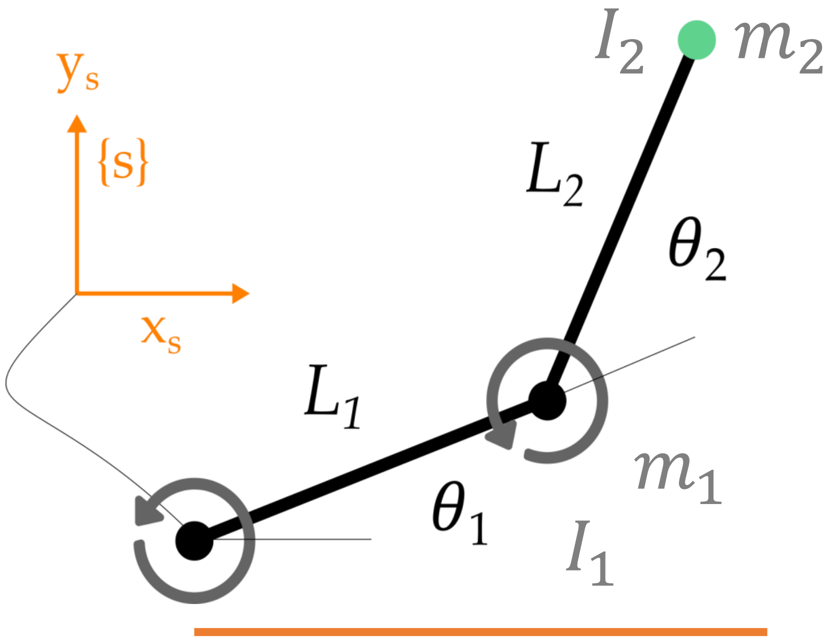


Kinetic Energy

Step 2. Get geometric Jacobian for the center of mass of the 2nd link

$$M = \begin{bmatrix} I & \begin{bmatrix} L_1 + L_2 \\ 0 \\ 0 \end{bmatrix} \\ 0 & 1 \end{bmatrix}, \quad S_1 = \begin{bmatrix} 0 \\ 0 \\ 1 \\ 0 \\ 0 \\ 0 \end{bmatrix}, \quad S_2 = \begin{bmatrix} 0 \\ 0 \\ 1 \\ 0 \\ -L_1 \\ 0 \end{bmatrix}$$

For this robot arm, the second center of mass is at the robot's end-effector

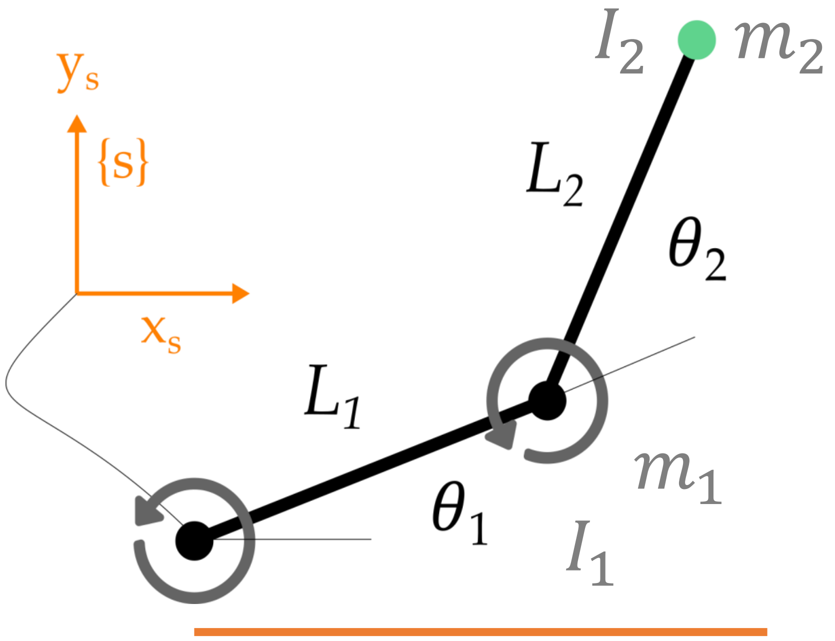


Kinetic Energy

Step 2. Get geometric Jacobian for the center of mass of the 2nd link

$$J_{\omega_2} = \begin{bmatrix} 0 & 0 \\ 0 & 0 \\ 1 & 1 \end{bmatrix}, \qquad J_{v_2} = \begin{bmatrix} -L_2 s_{12} - L_1 s_1 & -L_2 s_{12} \\ L_2 c_{12} + L_1 c_1 & L_2 c_{12} \\ 0 & 0 \end{bmatrix}$$

$$R_2 = \begin{bmatrix} c_{12} & -s_{12} & 0 \\ s_{12} & c_{12} & 0 \\ 0 & 0 & 1 \end{bmatrix}$$



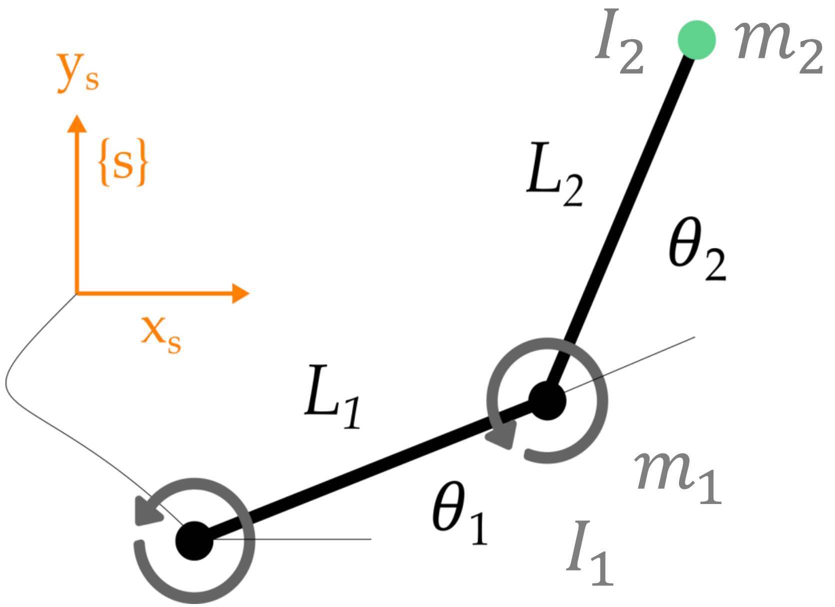
Kinetic Energy

Step 3. Solve for kinetic energy of each link

$$K_1(\theta,\dot{\theta}) = \frac{1}{2}\dot{\theta}^T\left[m_1 {J_{v_1}}^T J_{v_1} + {J_{\omega_1}}^T R_1 \boldsymbol{I}_1 {R_1}^T J_{\omega_1}\right]\dot{\theta}$$

Plug in the terms we just found.
Here inertia only possible around z axis

$$K_1(\theta,\dot{\theta}) = \frac{1}{2}\dot{\theta}^T\begin{bmatrix} m_1 L_1^2 + I_1 & 0 \\ 0 & 0 \end{bmatrix}\dot{\theta}$$



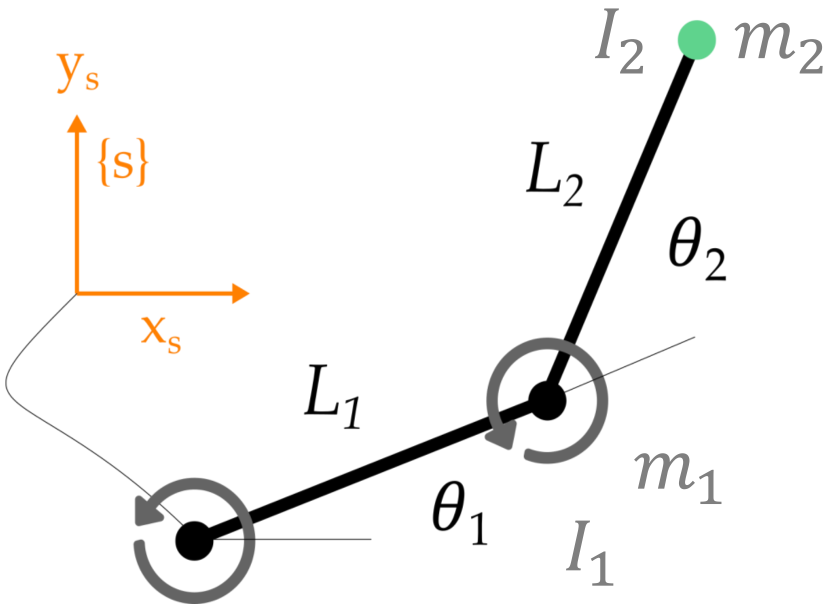
Kinetic Energy

Step 3. Solve for kinetic energy of each link

$$K_2(\theta,\dot{\theta}) = \frac{1}{2}\dot{\theta}^T\left[m_2 {J_{v_2}}^T J_{v_2} + {J_{\omega_2}}^T R_2 \boldsymbol{I}_2 {R_2}^T J_{\omega_2}\right]\dot{\theta}$$

Plug in the terms we just found.
Here inertia only possible around z axis

$$K_2(\theta,\dot{\theta}) = \frac{1}{2}\dot{\theta}^T\begin{bmatrix} m_2(L_1^2 + L_2^2 + 2L_1L_2c_2) + I_2 & m_2(L_2^2 + L_1L_2c_2) + I_2 \\ m_2(L_2^2 + L_1L_2c_2) + I_2 & m_2L_2^2 + I_2 \end{bmatrix}\dot{\theta}$$



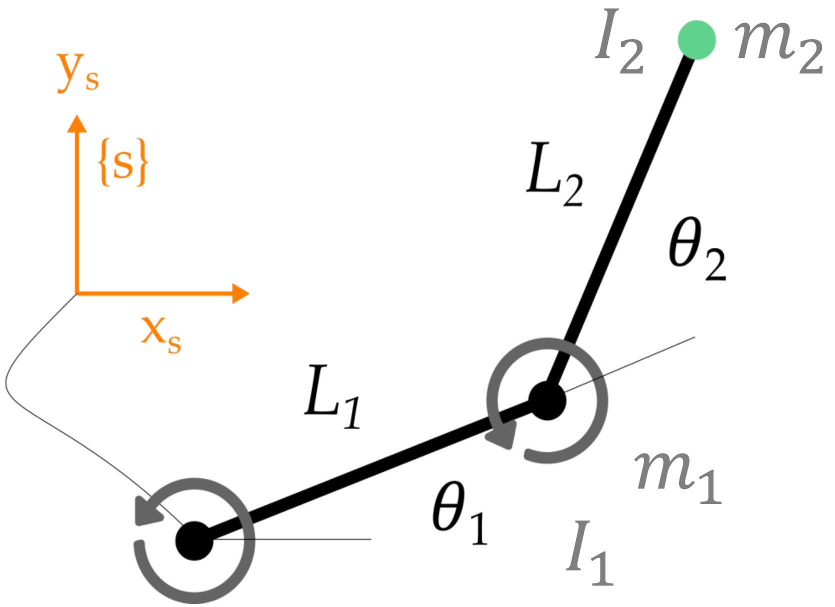
Kinetic Energy

Step 4. Sum to get total kinetic energy

$$K(\theta,\dot\theta) = K_1(\theta,\dot\theta) + K_2(\theta,\dot\theta)$$

$$K(\theta,\dot\theta) = \frac{1}{2}\dot\theta^T M \dot\theta$$

$$M = \begin{bmatrix} m_1L_1^2 + m_2(L_1^2 + L_2^2 + 2L_1L_2c_2) + I_1 + I_2 & m_2(L_2^2 + L_1L_2c_2) + I_2 \\ m_2(L_2^2 + L_1L_2c_2) + I_2 & m_2L_2^2 + I_2 \end{bmatrix}$$



This Lecture

- What is the potential energy of a robot arm?
- What is the kinetic energy of a robot arm?
- How do we calculate kinetic energy using the Jacobian?

Next Lecture

- Now that we have kinetic and potential energy, what are the dynamics?