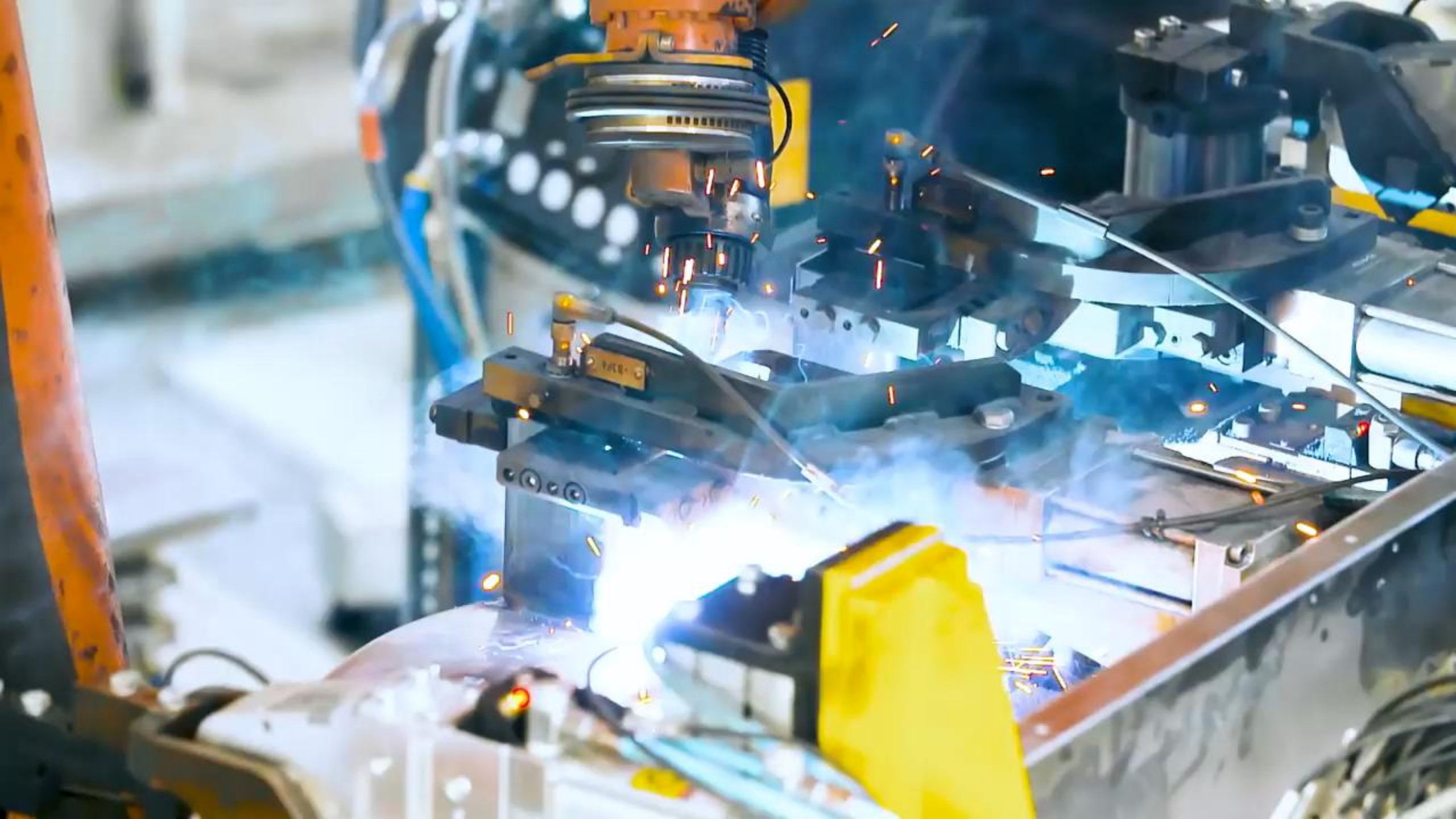


# Dynamics: Example



Reading: Robot Modeling and Control 7.4



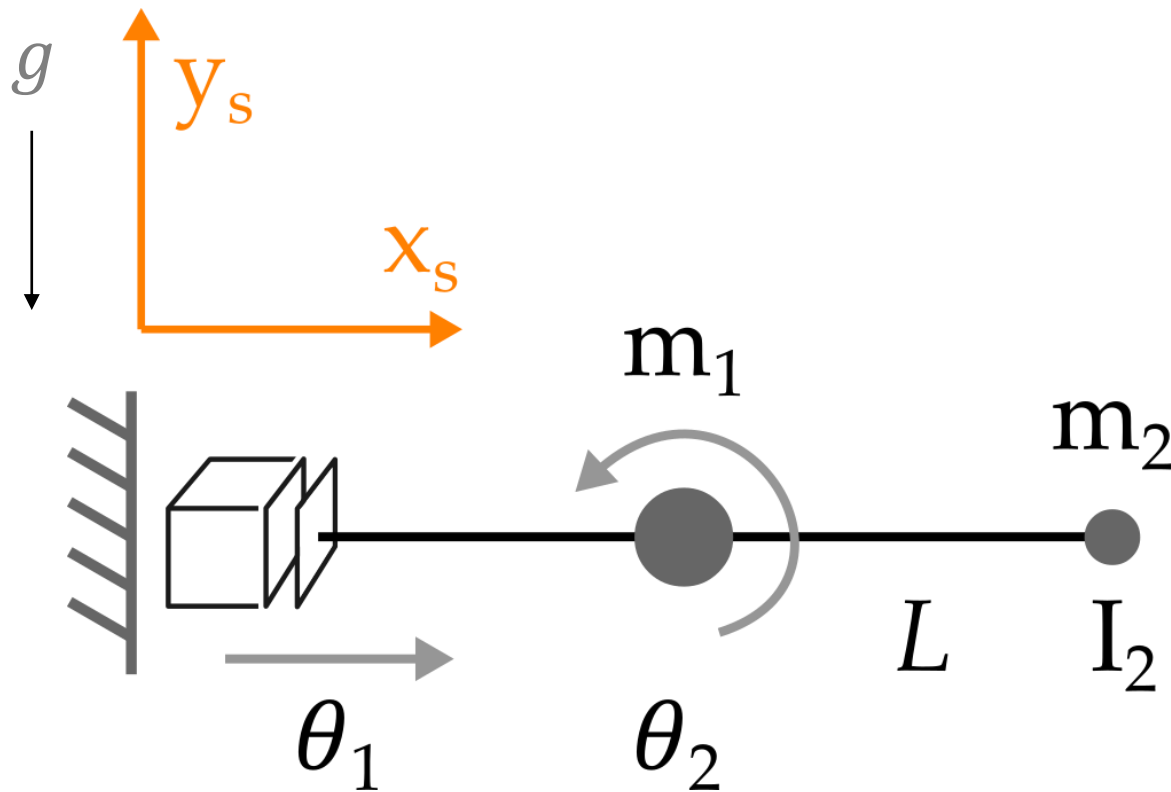
# This Lecture



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- How do we apply the dynamics equation for robot arms?
- Practice dynamics with an example

# Example



## Mass

$m_i$  is the mass of link  $i$

Center of mass at the end of each link

## Inertia

$I_2$  is the inertia of link 2

No inertia needed for link 1

## Gravity

Gravity acts along the  $-y$  axis

# Equation of Motion

The **dynamics** of a serial robot arm with  $n$  joints is formulated as:

$$\tau = M(\theta)\ddot{\theta} + C(\theta, \dot{\theta})\dot{\theta} + g(\theta)$$

- $\tau$  is  $n \times 1$  vector of joint torque
- $\theta$  is  $n \times 1$  vector of joint position
- $\dot{\theta}$  is  $n \times 1$  vector of joint velocity
- $\ddot{\theta}$  is  $n \times 1$  vector of joint acceleration

# Equation of Motion

The **dynamics** of a serial robot arm with  $n$  joints is formulated as:

$$\tau = M(\theta)\ddot{\theta} + C(\theta, \dot{\theta})\dot{\theta} + g(\theta)$$

- $M$  is  $n \times n$  **mass matrix**
- $C$  is  $n \times n$  **Coriolis matrix**
- $g$  is  $n \times 1$  **gravity vector**



A photograph of an industrial manufacturing environment featuring several yellow robotic arms. The arms are mounted on white bases and are positioned over a complex assembly line. The background shows various mechanical components, wiring, and structural elements of the factory. The image has a blue tint and a semi-transparent text overlay.

Can you find the  
robot's **dynamics**?

# Mass Matrix

---

We get the  $n \times n$  mass matrix from **kinetic energy**

$$M(\theta) = \sum_{i=1}^n m_i J_{v_i}^T J_{v_i} + J_{\omega_i}^T R_i \mathbf{I}_i R_i^T J_{\omega_i}$$

$$M(\theta) = m_1 J_{v_1}^T J_{v_1} + m_2 J_{v_2}^T J_{v_2} + J_{\omega_2}^T R_2 \mathbf{I}_2 R_2^T J_{\omega_2}$$

First link never rotates, so no rotational kinetic energy



# Mass Matrix

---

First find **Jacobians** for the center of mass of both links

$$M(\theta) = m_1 J_{v_1}^T J_{v_1} + m_2 J_{v_2}^T J_{v_2} + J_{\omega_2}^T R_2 \mathbf{I}_2 R_2^T J_{\omega_2}$$

$$J_{\omega_1} = \begin{bmatrix} 0 & 0 \\ 0 & 0 \\ 0 & 0 \end{bmatrix}, \quad J_{v_1} = \begin{bmatrix} 1 & 0 \\ 0 & 0 \\ 0 & 0 \end{bmatrix}$$

*Second joint has no effect on first center of mass*

# Mass Matrix

---

First find **Jacobians** for the center of mass of both links

$$M(\theta) = m_1 J_{v_1}^T J_{v_1} + m_2 J_{v_2}^T J_{v_2} + J_{\omega_2}^T R_2 \mathbf{I}_2 R_2^T J_{\omega_2}$$

$$J_{\omega_2} = \begin{bmatrix} 0 & 0 \\ 0 & 0 \\ 0 & 1 \end{bmatrix}, \quad J_{v_2} = \begin{bmatrix} 1 & -Ls_2 \\ 0 & Lc_2 \\ 0 & 0 \end{bmatrix}$$

# Mass Matrix

---

Then substitute these Jacobians into our **kinetic energy** equation

$$M(\theta) = m_1 J_{v_1}^T J_{v_1} + m_2 J_{v_2}^T J_{v_2} + J_{\omega_2}^T R_2 I_2 R_2^T J_{\omega_2}$$

$$M(\theta) = \begin{bmatrix} m_1 + m_2 & -Lm_2 s_2 \\ -Lm_2 s_2 & m_2 L^2 + I_2 \end{bmatrix}$$

# Coriolis Matrix

---

We get the  $n \times n$  Coriolis matrix from **mass matrix**

$$C(\theta, \dot{\theta}) = \begin{bmatrix} c_{11} & c_{12} \\ c_{21} & c_{22} \end{bmatrix}$$

$$c_{kj} = \sum_{i=1}^n \frac{1}{2} \left\{ \frac{\partial m_{kj}}{\partial \theta_i} + \frac{\partial m_{ki}}{\partial \theta_j} - \frac{\partial m_{ij}}{\partial \theta_k} \right\} \dot{\theta}_i$$

# Coriolis Matrix

---

$$c_{kj} = \sum_{i=1}^n \frac{1}{2} \left\{ \frac{\partial m_{kj}}{\partial \theta_i} + \frac{\partial m_{ki}}{\partial \theta_j} - \frac{\partial m_{ij}}{\partial \theta_k} \right\} \dot{\theta}_i, \quad M(\theta) = \begin{bmatrix} m_1 + m_2 & -Lm_2 s_2 \\ -Lm_2 s_2 & m_2 L^2 + I_2 \end{bmatrix}$$

$$c_{11} = \frac{1}{2} \left\{ \frac{\partial m_{11}}{\partial \theta_1} + \frac{\partial m_{11}}{\partial \theta_1} - \frac{\partial m_{11}}{\partial \theta_1} \right\} \dot{\theta}_1 + \frac{1}{2} \left\{ \frac{\partial m_{11}}{\partial \theta_2} + \frac{\partial m_{12}}{\partial \theta_1} - \frac{\partial m_{21}}{\partial \theta_1} \right\} \dot{\theta}_2 = 0$$

---

Only  $m_{21}$  and  $m_{12}$  depend on  $\theta_2$



# Coriolis Matrix

---

$$c_{kj} = \sum_{i=1}^n \frac{1}{2} \left\{ \frac{\partial m_{kj}}{\partial \theta_i} + \frac{\partial m_{ki}}{\partial \theta_j} - \frac{\partial m_{ij}}{\partial \theta_k} \right\} \dot{\theta}_i, \quad M(\theta) = \begin{bmatrix} m_1 + m_2 & -Lm_2 s_2 \\ -Lm_2 s_2 & m_2 L^2 + I_2 \end{bmatrix}$$

$$c_{21} = \frac{1}{2} \left\{ \frac{\partial m_{21}}{\partial \theta_1} + \frac{\partial m_{21}}{\partial \theta_1} - \frac{\partial m_{11}}{\partial \theta_2} \right\} \dot{\theta}_1 + \frac{1}{2} \left\{ \frac{\partial m_{21}}{\partial \theta_2} + \frac{\partial m_{22}}{\partial \theta_1} - \frac{\partial m_{21}}{\partial \theta_2} \right\} \dot{\theta}_2 = 0$$

These terms cancel each other out

# Coriolis Matrix

---

$$c_{kj} = \sum_{i=1}^n \frac{1}{2} \left\{ \frac{\partial m_{kj}}{\partial \theta_i} + \frac{\partial m_{ki}}{\partial \theta_j} - \frac{\partial m_{ij}}{\partial \theta_k} \right\} \dot{\theta}_i, \quad M(\theta) = \begin{bmatrix} m_1 + m_2 & -Lm_2 s_2 \\ -Lm_2 s_2 & m_2 L^2 + I_2 \end{bmatrix}$$

$$c_{12} = \frac{1}{2} \left\{ \frac{\partial m_{12}}{\partial \theta_1} + \frac{\partial m_{11}}{\partial \theta_2} - \frac{\partial m_{12}}{\partial \theta_1} \right\} \dot{\theta}_1 + \frac{1}{2} \left\{ \frac{\partial m_{12}}{\partial \theta_2} + \frac{\partial m_{12}}{\partial \theta_2} - \frac{\partial m_{22}}{\partial \theta_1} \right\} \dot{\theta}_2 = -Lm_2 c_2 \dot{\theta}_2$$

Don't forget to multiply by  $\dot{\theta}_2$

# Coriolis Matrix

---

$$c_{kj} = \sum_{i=1}^n \frac{1}{2} \left\{ \frac{\partial m_{kj}}{\partial \theta_i} + \frac{\partial m_{ki}}{\partial \theta_j} - \frac{\partial m_{ij}}{\partial \theta_k} \right\} \dot{\theta}_i, \quad M(\theta) = \begin{bmatrix} m_1 + m_2 & -Lm_2 s_2 \\ -Lm_2 s_2 & m_2 L^2 + I_2 \end{bmatrix}$$

$$c_{22} = \frac{1}{2} \left\{ \frac{\partial m_{22}}{\partial \theta_1} + \frac{\partial m_{21}}{\partial \theta_2} - \frac{\partial m_{12}}{\partial \theta_2} \right\} \dot{\theta}_1 + \frac{1}{2} \left\{ \frac{\partial m_{22}}{\partial \theta_2} + \frac{\partial m_{22}}{\partial \theta_2} - \frac{\partial m_{22}}{\partial \theta_2} \right\} \dot{\theta}_2 = 0$$

Cancel because  $M$  is symmetric

# Coriolis Matrix

---

We get the  $n \times n$  Coriolis matrix from **mass matrix**

$$c(\theta, \dot{\theta}) = \begin{bmatrix} 0 & -Lm_2 c_2 \dot{\theta}_2 \\ 0 & 0 \end{bmatrix}$$

# Gravity Vector

---

We get the  $n \times 1$  gravity vector from **potential energy**

$$g(\theta) = \begin{bmatrix} \frac{\partial P(\theta)}{\partial \theta_1} \\ \frac{\partial P(\theta)}{\partial \theta_2} \end{bmatrix}, \quad P(\theta) = gm_1h_1 + gm_2h_2$$

$h_1$  is the height of the  $i$ -th center of mass



# Gravity Vector

---

We get the  $n \times 1$  gravity vector from **potential energy**

$$g(\theta) = \begin{bmatrix} \frac{\partial P(\theta)}{\partial \theta_1} \\ \frac{\partial P(\theta)}{\partial \theta_2} \end{bmatrix}, \quad P(\theta) = gm_2 L s_2$$

# Gravity Vector

---

We get the  $n \times 1$  gravity vector from **potential energy**

$$g(\theta) = \begin{bmatrix} 0 \\ gm_2Lc_2 \end{bmatrix}$$

The image shows a simulated industrial environment with two blue Yaskawa robotic arms. The arms are positioned over a conveyor belt system. The background is a light gray, and the overall scene is presented in a semi-transparent, simulated style. The text 'Let's simulate our robot arm' is overlaid in the center, with 'simulate' in orange and the rest in white.

# Let's simulate our robot arm

## Inspection section

- Assembled smart phone will be inspected
- Shorter cycle time due to synchronized motion between robot and gantry
- Flexible for new 1 production due to hand over of products between two robots

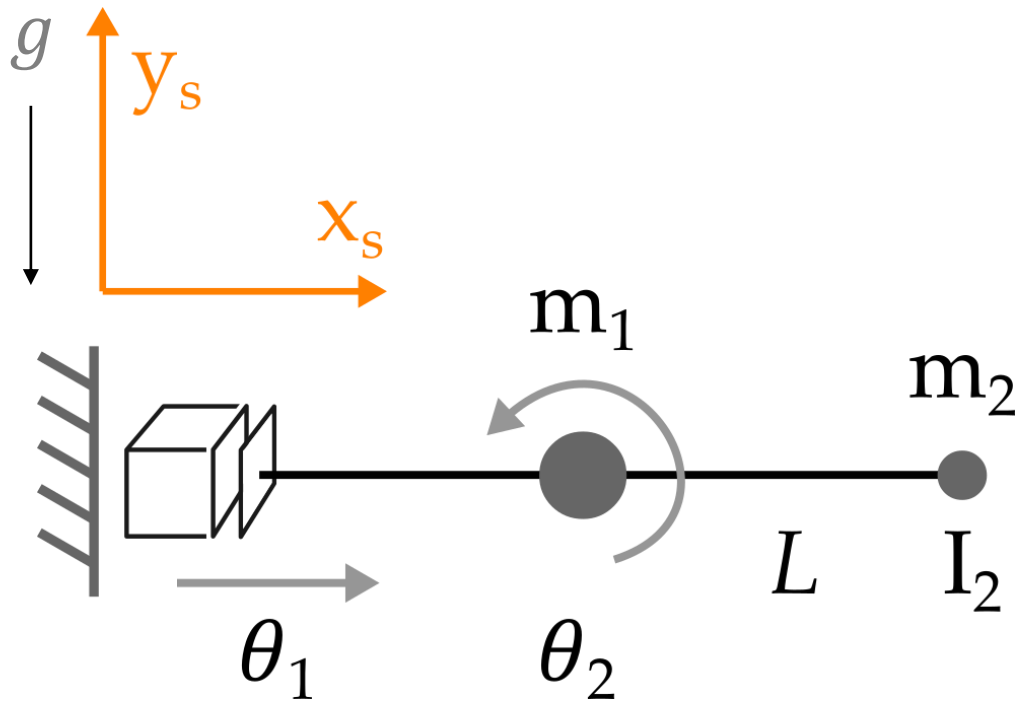
## Kontroll-Bereich

- Montiertes Telefon wird geprüft
- Reduzierte Taktzeit durch synchrone Bewegung zwischen Roboter und Gantry
- Flexible Losgröße-1 Produktion wird durch Wechselschaltung zwischen zwei Robotern ermöglicht



# Simulation

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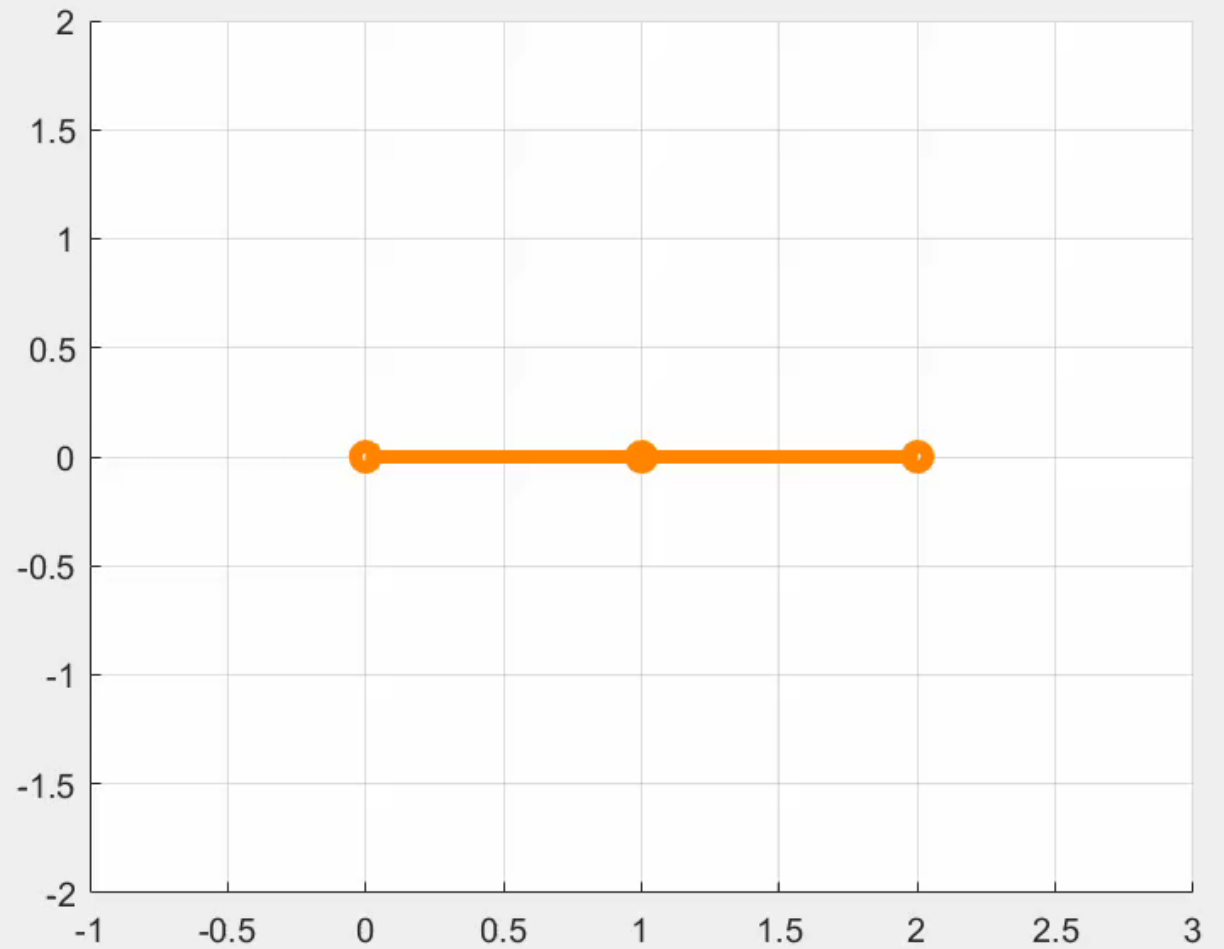


To **simulate**, solve for acceleration then integrate to get velocity and position:

$$\ddot{\theta}^t = M(\theta)^{-1} \left( \tau - C(\theta, \dot{\theta})\dot{\theta} - g(\theta) \right)$$

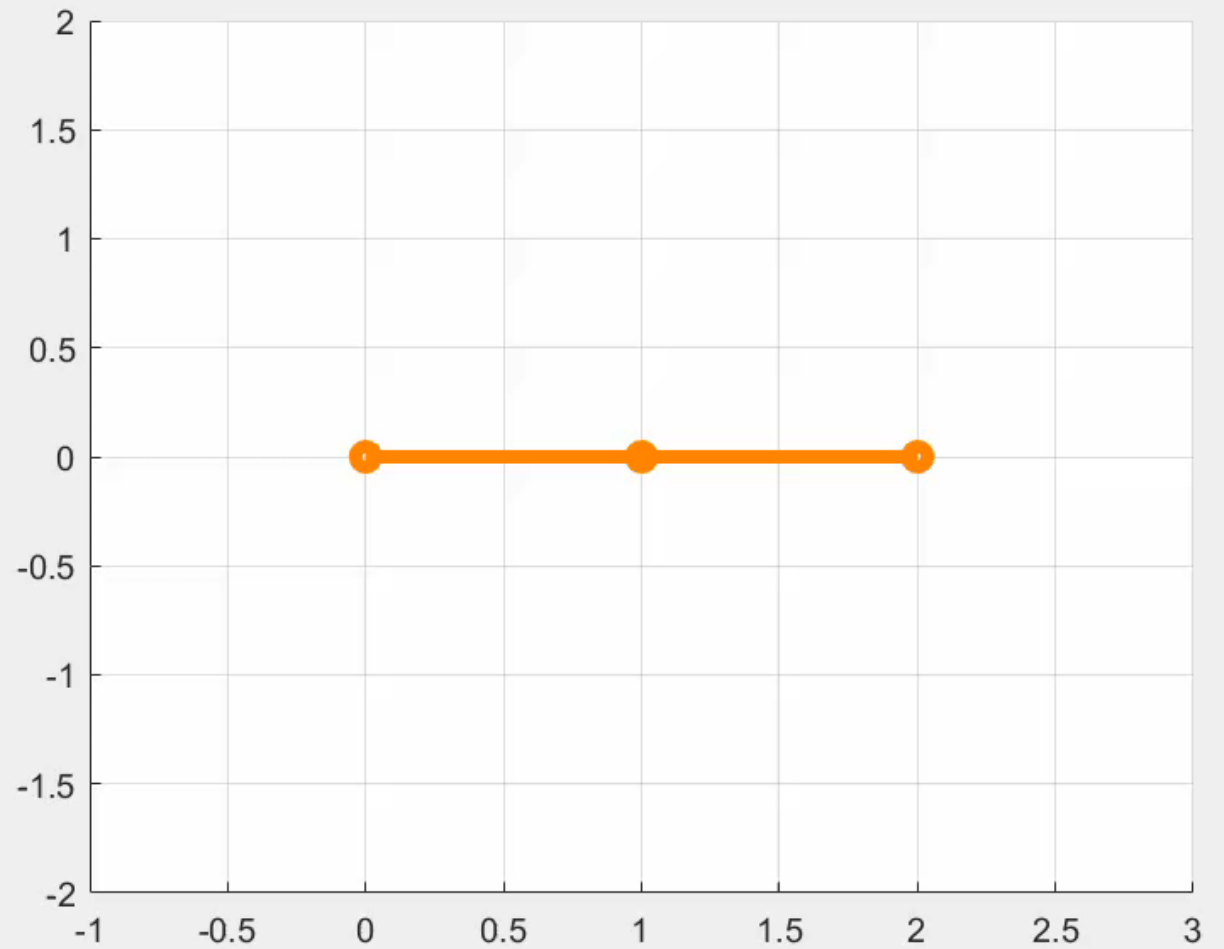
$$\begin{aligned}\dot{\theta}^{t+1} &= \dot{\theta}^t + \Delta T \cdot \ddot{\theta}^t \\ \theta^{t+1} &= \theta^t + \Delta T \cdot \dot{\theta}^t\end{aligned}$$

Simulation





Simulation



# This Lecture



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- How do we apply the dynamics equation for robot arms?
- Practice dynamics with an example

# Next Lecture



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- Now that we have a dynamic model of our robot arm...  
...can we leverage this model to *control* a robot?