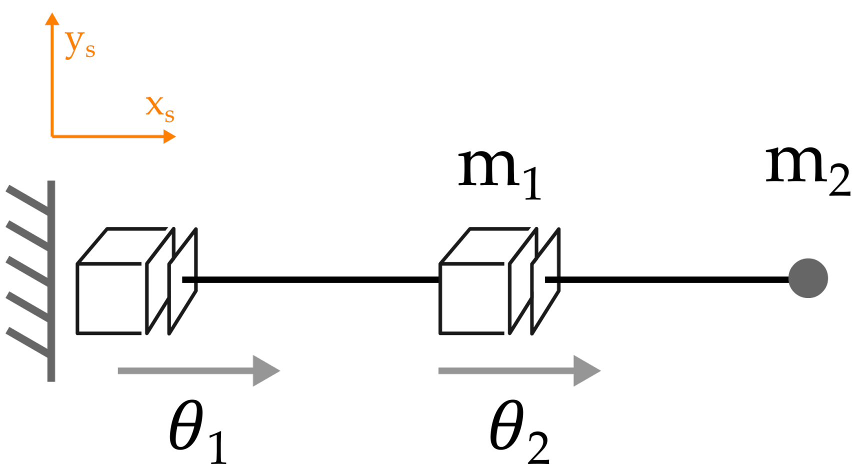
# Practice Set 23

**Robotics & Automation**
Dylan Losey, Virginia Tech

Using your textbook and what we covered in lecture, try solving the following problems. For some problems you may find it convenient to use Matlab (or another programming language of your choice). The solutions are on the next page.

## Problem 1




Get the kinetic and potential energy for the robot shown above. Gravity acts along the $-y$ axis, and each center of mass is at the end of the link.

To get you started, the Jacobian for the first and second center of mass are:

$$J_1 = \begin{bmatrix} 0 & 0 \\ 0 & 0 \\ 0 & 0 \\ 1 & 0 \\ 0 & 0 \\ 0 & 0 \end{bmatrix}, \quad J_2 = \begin{bmatrix} 0 & 0 \\ 0 & 0 \\ 0 & 0 \\ 1 & 1 \\ 0 & 0 \\ 0 & 0 \end{bmatrix} \tag{1}$$

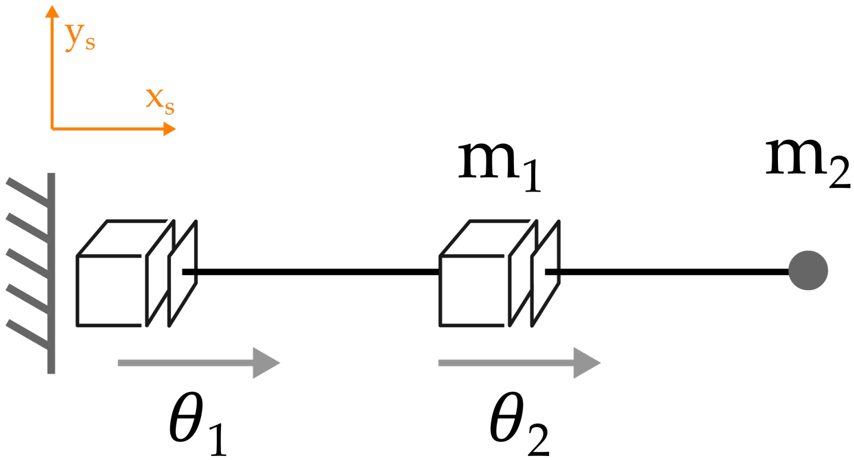
## Problem 2

Apply the Euler-Lagrange equation to find the robot's dynamics. As a reminder, the Lagrangian is $L = K - P$, and the Euler-Lagrange equation is:

$$\tau_i = \frac{d}{dt}\frac{\partial L(\theta,\dot{\theta})}{\partial \dot{\theta}_i} - \frac{\partial L(\theta,\dot{\theta})}{\partial \theta_i} \tag{2}$$

**Hint:** First expand $K(\theta,\dot{\theta})$ so that you have a scalar value for kinetic energy.

## Problem 1




Get the kinetic and potential energy for the robot shown above. Gravity acts along the $-y$ axis, and each center of mass is at the end of the link.

To get you started, the Jacobian for the first and second center of mass are:

$$J_1 = \begin{bmatrix} 0 & 0 \\ 0 & 0 \\ 0 & 0 \\ 1 & 0 \\ 0 & 0 \\ 0 & 0 \end{bmatrix}, \quad J_2 = \begin{bmatrix} 0 & 0 \\ 0 & 0 \\ 0 & 0 \\ 1 & 1 \\ 0 & 0 \\ 0 & 0 \end{bmatrix} \tag{3}$$

Start with **potential energy**. The potential energy of the $i$-th joint is $P_i(\theta) = gm_ih_i$, where $h_i$ is the height of that joint's center of mass. For this robot, height corresponds to the $y$-axis. Since the $y$ position of both centers of mass is always zero, this robot has no potential energy:

$$P(\theta) = P_1(\theta) + P_2(\theta) = 0 \tag{4}$$

,

Now focus on **Kinetic Energy**. Start with the equation for the kinetic energy of the $i$-th joint of the robot arm:

$$K_i(\theta, \dot{\theta}) = \frac{1}{2}\dot{\theta}^T \left(m_i J_{v_i}^T J_{v_i} + J_{\omega_i}^T R_i I_i R_i^T J_{\omega_i}\right)\dot{\theta} \tag{5}$$

This robot never rotates, and therefore has no kinetic energy due to rotation. Because $J_{\omega_1} = J_{\omega_2} = 0$, the equation for kinetic energy simplifies to:

$$K(\theta, \dot{\theta}) = K_1(\theta, \dot{\theta}) + K_2(\theta, \dot{\theta}) = \frac{1}{2}\dot{\theta}^T \left(m_1 J_{v_1}^T J_{v_1} + m_2 J_{v_2}^T J_{v_2}\right)\dot{\theta} \tag{6}$$

Plugging in the given Jacobians, we obtain:

$$K(\theta, \dot{\theta}) = \frac{1}{2}\dot{\theta}^T M \dot{\theta}, \quad M = \begin{bmatrix} m_1 + m_2 & m_2 \\ m_2 & m_2 \end{bmatrix} \tag{7}$$

## Problem 2

Apply the Euler-Lagrange equation to find the robot's dynamics. As a reminder, the Lagrangian is $L = K - P$, and the Euler-Lagrange equation is:

$$\tau_i = \frac{d}{dt}\frac{\partial L(\theta,\dot{\theta})}{\partial \dot{\theta}_i} - \frac{\partial L(\theta,\dot{\theta})}{\partial \theta_i} \tag{8}$$

**Hint:** First expand $K(\theta,\dot{\theta})$ so that you have a scalar value for kinetic energy.

Start by using the hint to write out the kinetic energy:

$$K(\theta,\dot{\theta}) = \frac{1}{2}(m_1 + m_2)\dot{\theta}_1^2 + m_2\dot{\theta}_1\dot{\theta}_2 + \frac{1}{2}m_2\dot{\theta}_2^2 \tag{9}$$

We have no potential energy, so the Lagrangian $L = K$. At this point we are ready to apply the Euler-Lagrange equation twice: once for each joint.

$$\tau_1 = \frac{d}{dt}\frac{\partial L(\theta,\dot{\theta})}{\partial \dot{\theta}_1} - \frac{\partial L(\theta,\dot{\theta})}{\partial \theta_1} = \frac{d}{dt}\Big[(m_1 + m_2)\dot{\theta}_1 + m_2\dot{\theta}_2\Big] = (m_1 + m_2)\ddot{\theta}_1 + m_2\ddot{\theta}_2 \tag{10}$$

$$\tau_2 = \frac{d}{dt}\frac{\partial L(\theta,\dot{\theta})}{\partial \dot{\theta}_2} - \frac{\partial L(\theta,\dot{\theta})}{\partial \theta_2} = \frac{d}{dt}\Big[m_2\dot{\theta}_1 + m_2\dot{\theta}_2\Big] = m_2\ddot{\theta}_1 + m_2\ddot{\theta}_2 \tag{11}$$

Our final answer for the dynamics of this robot arm is:

$$\tau_1 = (m_1 + m_2)\ddot{\theta}_1 + m_2\ddot{\theta}_2, \quad \tau_2 = m_2\ddot{\theta}_1 + m_2\ddot{\theta}_2 \tag{12}$$