Practice Set 27

Robotics & Automation
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Using your textbook and what we covered in lecture, try solving the following problems. For some problems you may find it convenient to use Matlab (or another programming language of your choice). The solutions are on the next page.

Problem 1

If A is a positive definite matrix and x is a vector, then $x^T A x > 0$ if $x \neq 0$. Similarly, if Z is a negative definite matrix, then $x^T Z x < 0$ when $x \neq 0$. Try this out in Matlab with the example matrices below:

$$A = \begin{bmatrix} 10 & -3 \\ -3 & 2 \end{bmatrix}, \qquad Z = \begin{bmatrix} -5 & 2 \\ 2 & -8 \end{bmatrix} \tag{1}$$

Pick different values of x and check whether $x^T A x > 0$ and $x^T Z x < 0$.

Problem 2

Consider a robot arm with dynamics:

$$\tau = M\ddot{\theta} + B\dot{\theta}, \qquad \tau = -K_P \theta \tag{2}$$

Use Lyapunov's method to show that this system is stable. Note that M does not depend on θ, and thus the Coriolis matrix $C = 0$. Assume the friction matrix B is positive definite. Where does the system come to rest?

Problem 1

If A is a positive definite matrix and x is a vector, then $x^T Ax > 0$ if $x \neq 0$. Similarly, if Z is a negative definite matrix, then $x^T Zx < 0$ when $x \neq 0$. Try this out in Matlab with the example matrices below:

$$A = \begin{bmatrix} 10 & -3 \\ -3 & 2 \end{bmatrix}, \qquad Z = \begin{bmatrix} -5 & 2 \\ 2 & -8 \end{bmatrix} \tag{3}$$

Pick different values of x and check whether $x^T Ax > 0$ and $x^T Zx < 0$.

```
A = [10, -3; -3, 2];
B = [-5, 2; 2 -8];
x = rand(2,1)*2-1;
```

Command Window

```
>> x'*A*x

ans =

    0.4808

>> x'*B*x

ans =

   -0.8716
```

> See code above. You can check if a matrix is positive definite or negative definite using its eigenvalues. If the eigenvalues are all positive, it is positive definite. If the eigenvalues are all negative, it is negative definite.

Problem 2

Consider a robot arm with dynamics:

$$\tau = M\ddot{\theta} + B\dot{\theta}, \qquad \tau = -K_P\theta \tag{4}$$

Use Lyapunov's method to show that this system is stable. Note that M does not depend on θ, and thus the Coriolis matrix $C = 0$. Assume the friction matrix B is positive definite. Where does the system come to rest?

Start by writing a generalized energy function. Finding the right energy function is often an iterative process of guess and check. Here we can try:

$$v(\theta, \dot{\theta}) = \frac{1}{2}\dot{\theta}^T M \dot{\theta} + \frac{1}{2}\theta^T K_P \theta \tag{5}$$

We have $v = 0$ when $\theta = \dot{\theta} = 0$, and otherwise $v > 0$. The partial derivatives with respect to θ and $\dot{\theta}$ are continuous. This satisfies our conditions, so we can proceed to take the time derivative to see how energy changes over time:

$$\dot{v}(\theta, \dot{\theta}) = \dot{\theta}^T M \ddot{\theta} + \dot{\theta}^T K_P \theta \tag{6}$$

Plugging in the dynamics and simplifying:

$$\dot{v}(\theta, \dot{\theta}) = \dot{\theta}^T \big(- K_P \theta - B\dot{\theta} + K_P \theta \big) = -\dot{\theta}^T B \dot{\theta} \tag{7}$$

So v is negative (and energy is decreasing) because B is positive definite. The system comes to equilibrium (energy stops decreasing) when $\dot{\theta}$ rests at 0. This implies $\ddot{\theta} = 0$ and the dynamics at equilibrium become:

$$-K_P \theta = 0 \tag{8}$$

We conclude that equilibrium is $\theta = \dot{\theta} = 0$. This equilibrium aligns with the chosen energy function; $v(\theta, \dot{\theta}) = 0$ when $\theta = \dot{\theta} = 0$.