Problem Set 2

Robotics & Automation
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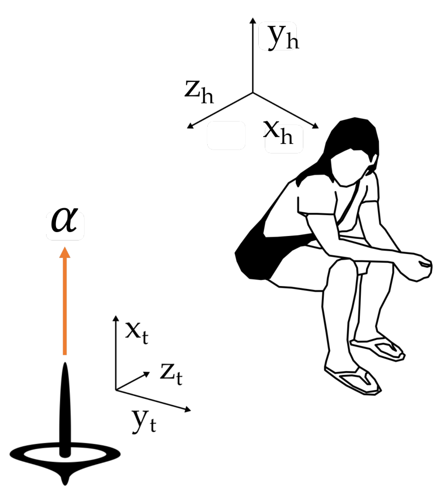
Instructions. Please write legibly and do not attempt to fit your work into the smallest space possible. It is important to show all work, but basic arithmetic can be omitted. You are encouraged to use Matlab when possible to avoid hand calculations, but print and submit your commented code for non-trivial calculations. You can attach a pdf of your code to the homework, use live scripts or the publish feature in Matlab, or include a snapshot of your code. Do not submit .m files — we will not open or grade these files.

1 Angular Velocity

1.1 (5 points)



A top is spinning at α radians per second around the x_t axis. Using the drawing, find ω_t and ω_h. Numerically show that $\omega_h = R_{ht}\omega_t$

The top is spinning around x_t. So the angular velocity expressed in frame $\{t\}$ is:

$$\omega_t = \begin{bmatrix} \alpha \\ 0 \\ 0 \end{bmatrix} \tag{1}$$

The human's y_h axis is aligned with x_t. The angular velocity in frame $\{h\}$ is therefore:

$$\omega_h = \begin{bmatrix} 0 \\ \alpha \\ 0 \end{bmatrix} \tag{2}$$

Another way to get this is by remembering that $\omega_h = R_{ht}\omega_t$ by our subscript cancellation rule (ω is a vector, and can be rotated between frames like any other vector). Looking at the drawing:

$$R_{ht} = \begin{bmatrix} 0 & 1 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & -1 \end{bmatrix} \tag{3}$$

Multiplying this out, we confirm our answer for ω_h:

$$R_{ht}\omega_t = \begin{bmatrix} 0 & 1 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & -1 \end{bmatrix} \begin{bmatrix} \alpha \\ 0 \\ 0 \end{bmatrix} = \begin{bmatrix} 0 \\ \alpha \\ 0 \end{bmatrix} = \omega_h \tag{4}$$

1.2 (5 points)

Explain why ZXX Euler angles do not exist.

ZXX Euler angles correspond to rotating around the z-axis, rotating around the x-axis, and then rotating around the x-axis again:

$$Rot(z, \theta_1)Rot(x, \theta_2)Rot(x, \theta_3) \tag{5}$$

But rotating twice around the same axis is equal to a single rotation:

$$Rot(z, \theta_1)Rot(x, \theta_2)Rot(x, \theta_3) = Rot(z, \theta_1)Rot(x, \theta_2 + \theta_3) \tag{6}$$

In other words, any ZXX rotation matrix is specified by *two* parameters (here θ_1 and $\theta_4 = \theta_2 + \theta_3$). But a minimum of *three* parameters are required to specify an arbitrary rotation. Hence, we don't use ZXX Euler angles because they can only capture a subset of the possible rotations. Put another way, there are many valid rotation matrices that we cannot parameterize with ZXX Euler angles!

1.3 (5 points)

Given the rotation matrix:

$$R = \begin{bmatrix} 0.3642 & -0.7799 & 0.5090 \\ 0.7017 & -0.1296 & -0.7006 \\ 0.6124 & 0.6124 & 0.5 \end{bmatrix} \tag{7}$$

Find the axis-angle representation of R.

Start with the formula to solve for the angle of rotation:

$$\theta = \cos^{-1}\big(0.5 \cdot (\mathrm{tr}(R) - 1)\big) \tag{8}$$

Remember that the trace is the sum of the diagonal elements of a matrix:

$$\theta = \cos^{-1}\big(0.5 \cdot (0.3642 - 0.1296 + 0.5 - 1)\big) = 1.71 \text{ rad} \tag{9}$$

Next, use θ to get the unit axis of rotation:

$$[\omega] = \frac{1}{2\sin\theta}(R - R^T) \tag{10}$$

Plugging in for θ and R, and using the definition of skew-symmetric matrices:

$$[\omega] = 0.5048 \cdot \begin{bmatrix} 0 & -1.4816 & -0.1034 \\ 1.4816 & 0 & -1.313 \\ 0.1034 & 1.313 & 0 \end{bmatrix} \tag{11}$$

$$\omega = 0.5048 \cdot \begin{bmatrix} 1.313 \\ -0.1034 \\ 1.4816 \end{bmatrix} = \begin{bmatrix} 0.66 \\ -0.05 \\ 0.75 \end{bmatrix} \tag{12}$$

1.4 (5 points)

Write a function that computes the rotation matrix R from an axis ω and an angle θ. Then use that function to find the following rotation matrix:

$$\hat{\omega} = \begin{bmatrix} \sqrt{2}/2 \\ 0 \\ \sqrt{2}/2 \end{bmatrix}, \quad \theta = \pi/4 \tag{13}$$

Prove that your result is a rotation matrix.

```
function R = axisangle(omega, theta)
    R = expm(skew(omega) * theta);

    function bracket_x = skew(x)
        bracket_x = [0 -x(3) x(2);
            x(3) 0 -x(1); -x(2) x(1) 0];
    end
end
```

In lecture we solved the first-order ordinary differential equation to obtain:

$$R = e^{[\omega]\theta} \tag{14}$$

See the listed code to implement this equation. Applying this function, we find:

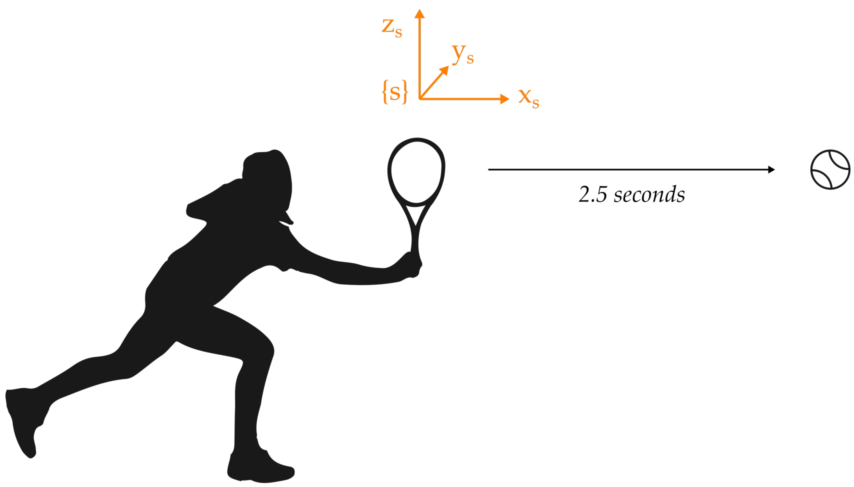
$$R = e^{[\omega]\theta} = \begin{bmatrix} 0.85 & -0.5 & 0.15 \\ 0.5 & 0.71 & -0.5 \\ 0.15 & 0.5 & 0.85 \end{bmatrix} \tag{15}$$

To check that this is a rotation matrix, we must confirm that $R^T R = I$ and $\det(R) = +1$. See the code on the top of next page.

```
>> R = axisangle([1/sqrt(2);0;1/sqrt(2)], pi/4)

R =

    0.8536   -0.5000    0.1464
    0.5000    0.7071   -0.5000
    0.1464    0.5000    0.8536
```

```
>> R'*R

ans =

    1.0000    0.0000   -0.0000
    0.0000    1.0000   -0.0000
   -0.0000   -0.0000    1.0000

>> det(R)

ans =

     1
```

2 Combining Position and Rotation



You just hit a tennis ball. At time $t = 0$, the position and orientation of the tennis ball in your coordinate frame $\{s\}$ are:

$$p(0) = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}, \qquad R(0) = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} \tag{16}$$

The ball is spinning around the $+z_s$ axis at α radians per second, and is flying forward along the $+x_s$ axis at β meters per second.

2.1 (5 points)

What is the position of the tennis ball after 2.5 seconds?

After t seconds the ball will be $\beta \cdot t$ units along the x_s axis (remember that distance is rate $\times$ time). So the position of the ball in meters is:

$$p(2.5) = \begin{bmatrix} 2.5\beta \\ 0 \\ 0 \end{bmatrix} \tag{17}$$

2.2 (5 points)

What is the orientation of the tennis ball after 2.5 seconds?

First get the angular velocity. Expressed in $\{s\}$, we have that:

$$\omega_s = \begin{bmatrix} 0 \\ 0 \\ \alpha \end{bmatrix} \tag{18}$$

After spinning with this velocity for t units of time, the ball's orientation is:

$$R = e^{[\omega_h]t} \tag{19}$$

Plugging in $t = 2.5$ seconds and ω_s radians per second, we get:

$$R(2.5) = \begin{bmatrix} \cos(2.5\alpha) & -\sin(2.5\alpha) & 0 \\ \sin(2.5\alpha) & \cos(2.5\alpha) & 0 \\ 0 & 0 & 1 \end{bmatrix} \tag{20}$$

Hopefully this answer agrees with your intuition: the tennis ball is spinning around the z_s axis, and we obtained a rotation around the z_s axis.

3 Properties of Transformation Matrices

3.1 (5 points)

Given two transformation matrices T_1 and T_2 where:

$$T_1 = \begin{bmatrix} R_1 & p_1 \\ 0 & 1 \end{bmatrix}, \qquad T_2 = \begin{bmatrix} R_2 & p_2 \\ 0 & 1 \end{bmatrix} \tag{21}$$

Find an instance of T_1 and T_2 where the transformation matrices are commutative. In your answer R_1 and R_2 **cannot** be identity, and p_1 and p_2 **cannot** be zero. You also cannot choose $T_1 = T_2$. Alternatively, prove that no such special case exists.

For T_1 and T_2 to be commutative we must have that:

$$T_1T_2 = \begin{bmatrix} R_1R_2 & R_1p_2 + p_1 \\ 0 & 1 \end{bmatrix} = \begin{bmatrix} R_2R_1 & R_2p_1 + p_2 \\ 0 & 1 \end{bmatrix} = T_2T_1 \tag{22}$$

Put another way, we must find rotations and positions such that:

$$R_1R_2 = R_2R_1, \qquad R_1p_2 + p_1 = R_2p_1 + p_2 \tag{23}$$

The trick to answering this question is realizing that (a) two rotations around the same axis are commutative and (b) if p is a vector along the axis of rotation of R, then $Rp = p$. For example, we can choose:

$$R_1 = Rot(z, \theta_1), \quad R_1 = Rot(z, \theta_2), \quad p_1 = \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix}, \quad p_2 = \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix} \tag{24}$$

Then $R_1R_2 = R_2R_1 = Rot(z, \theta_1 + \theta_2)$ and:

$$R_1p_2 = p_2, \quad R_2p_1 = p_1, \quad R_1p_2 + p_1 = R_2p_1 + p_2 = p_1 + p_2 \tag{25}$$

If you are unconvinced, try this in Matlab with $\theta_1 = \pi/4$ and $\theta_2 = -\pi/4$.

3.2 (5 points)

Write a function that inputs a 4×4 matrix X. This function should output `True` if X is a transformation matrix and `False` if X is not a transformation matrix. Demonstrate that your function works with the following matrices:

$$X_1 = \begin{bmatrix} 1/\sqrt{2} & -1/\sqrt{2} & 0 & -5 \\ 1/\sqrt{2} & 1/\sqrt{2} & 0 & -1.7 \\ 0 & 0 & 1 & -11 \\ 0 & 0 & 0 & 1 \end{bmatrix} \qquad X_2 = \begin{bmatrix} 0.5 & -1/\sqrt{2} & 0.5 & 0 \\ 0.5 & 1/\sqrt{2} & 0.5 & 0 \\ -1/\sqrt{2} & 0 & 1/\sqrt{2} & 0 \\ 0 & 0 & 1 & 1 \end{bmatrix}$$

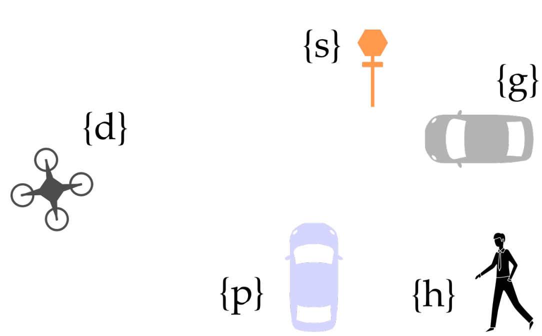
$$X_3 = \begin{bmatrix} -1/\sqrt{2} & -0.5 & 0.5 & -1.1 \\ 1/\sqrt{2} & -0.5 & -0.5 & 4.5 \\ 0 & 1/\sqrt{2} & 1/\sqrt{2} & 1.6 \\ 0 & 0 & 0 & 1 \end{bmatrix}$$

```
function binary = is_transformation(X)

binary = true;
if X(4,1) ~=0 || X(4,2)~=0 || X(4,3)~=0 || X(4,4)~= 1
    binary = false;
end

R = X(1:3, 1:3);
if norm(det(R) - 1) > 1e-3
     binary = false;
end
if norm(R'*R - eye(3)) > 1e-3
     binary = false;
end

end
```

Your code must (a) confirm that the 4-th row of matrix X is $[0, 0, 0, 1]$, and (b) confirm that the 3×3 matrix in the top right corner of X is a rotation matrix.

See the example code shown above. Using this code, we find that X_1 **is** a rotation matrix, X_2 **is not** a rotation matrix, and X_3 **is not** a rotation matrix.

4 Using Transformation Matrices

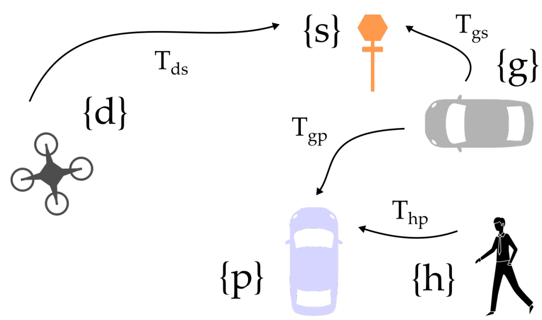


Consider the figure above. Your goal is to find the pose of the human expressed in the drone's coordinate frame. You know:

- the pose of the stop sign with respect to the drone
- the pose of the stop sign with respect to the gray car
- the pose of the purple car with respect to the human
- the pose of the purple car with respect to the gray car

4.1 (5 points)

Sketch a system diagram. Draw arrows to mark the known transformation matrices, and label each arrow.



See the diagram above. Remember that — under our notation — T_{ab} is the pose of frame $\{b\}$ expressed in frame $\{a\}$.

4.2 (5 points)

Using your diagram, determine the equation for the transformation matrix T_{dh}

Draw a path from $\{d\}$ to $\{h\}$. When that path goes in the opposite direction of a known transformation matrix, take the inverse.

$$T_{dh} = T_{ds} T_{gs}^{-1} T_{gp} T_{hp}^{-1} \tag{26}$$

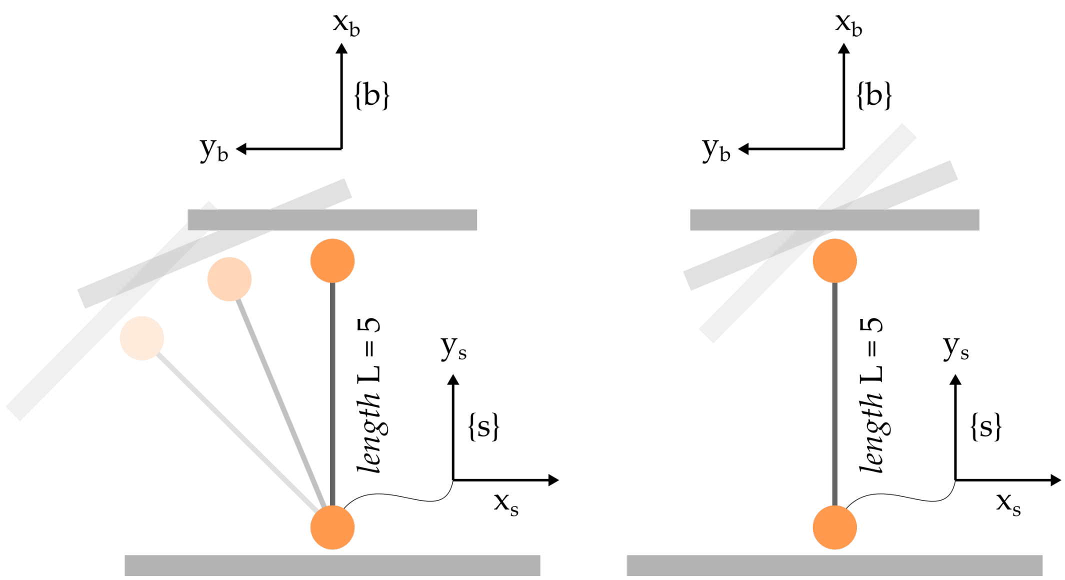
Use the subscript cancellation rule to double check your answer.

5 Fixed and Body Frame Motion

We have a planar robot with two revolute joints. See the figure below: on the left we move the revolute joint at $\{s\}$, and on the right we move the revolute joint at $\{b\}$. For the following parts we will ask your to perform a *sequence* of motions. Solve for the transformation matrix T_{sb} after each motion.

5.1 (5 points)

First rotate the joint at $\{s\}$ by $\pi/4$.



Start by finding T_{sb} before the joint moves. Looking at the sketch:

$$T_{sb} = \begin{bmatrix} 0 & -1 & 0 & 0 \\ 1 & 0 & 0 & 5 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix} \tag{27}$$

Next, recognize that this is a *fixed frame* motion. To find the resulting transformation we pre-multiply by a pure rotation around z_s:

$$T'_{sb} = \begin{bmatrix} \cos(\pi/4) & -\sin(\pi/4) & 0 & 0 \\ \sin(\pi/4) & \cos(\pi/4) & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} 0 & -1 & 0 & 0 \\ 1 & 0 & 0 & 5 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix} = \begin{bmatrix} -0.707 & -0.707 & 0 & -3.54 \\ 0.707 & -0.707 & 0 & 3.54 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}$$

5.2 (5 points)

Then rotate the joint at $\{b\}$ by $-\pi/4$

This is a *body frame* motion. We post-multiply by a pure rotation around z_b:

$$T''_{sb} = T'_{sb} \begin{bmatrix} \cos(-\pi/4) & -\sin(-\pi/4) & 0 & 0 \\ \sin(-\pi/4) & \cos(-\pi/4) & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix} = \begin{bmatrix} 0 & -1 & 0 & -3.54 \\ 1 & 0 & 0 & 3.54 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix} \tag{28}$$

Hopefully this answer is intuitive. After we turned the joint at $\{s\}$ the platform was at an angle. We have now rotated the joint at $\{b\}$ in the opposite direction to level out the platform.

5.3 (5 points)

Finally rotate the joint at $\{b\}$ by $\pi/2$

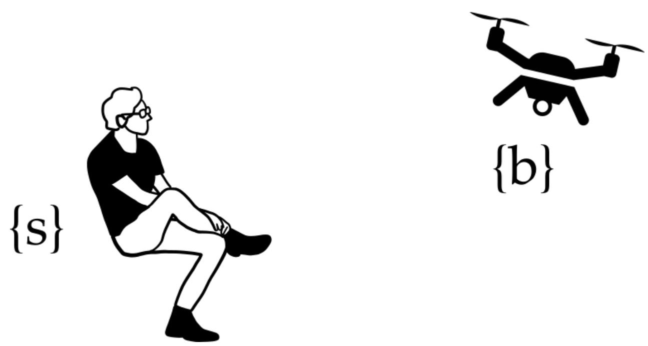
Another *body frame* motion. We post-multiply to get:

$$T_{sb}''' = T_{sb}'' \begin{bmatrix} \cos(\pi/2) & -\sin(\pi/2) & 0 & 0 \\ \sin(\pi/2) & \cos(\pi/2) & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix} = \begin{bmatrix} -1 & 0 & 0 & -3.54 \\ 0 & -1 & 0 & 3.54 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix} \tag{29}$$

If you are stumped by these answers, it may help to draw the planar robot before and after each of the motions.

6 Properties of Twists

6.1 (5 points)



Consider a seated person watching a drone. Let V_s be a spatial twist and let V_b be a body twist. The transformation between the person and drone is currently:

$$T_{sb} = \begin{bmatrix} R & p \\ 0 & 1 \end{bmatrix} \tag{30}$$

- If $\omega_b \neq 0$ and $p \neq 0$, find sufficient condition(s) for $V_s = V_b$

For $V_s = V_b$ we must have $\omega_s = \omega_b$ and $v_s = v_b$. By definition, this means:

$$\omega_s = R\omega_b = \omega_b \tag{31}$$

$$v_s = -[\omega_s]p + \dot{p} = R^T\dot{p} = v_b \tag{32}$$

Both of the these equations hold if **(a)** $R = I$ and **(b)** p and ω_s are parallel vectors, i.e., $\omega_s \times p = 0$.

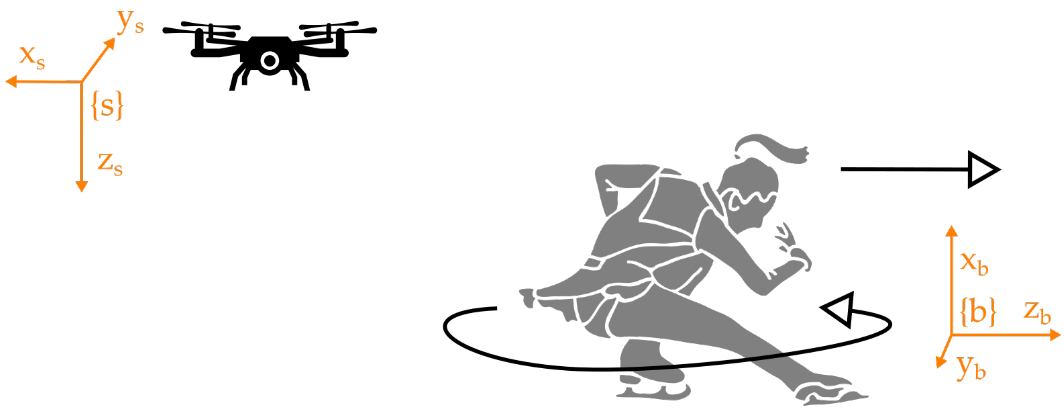
- If $\omega_s \neq 0$ and $p \neq 0$, find the condition(s) that must be met for $\|V_s\| = \|V_b\|$. Here $\|\cdot\|$ denotes the magnitude (or length) of a vector.

Recall that multiplying a vector by a rotation matrix does not change the magnitude of that vector. Hence, $\|\omega_s\| = \|\omega_b\|$ since $\omega_s = R\omega_b$.

To find cases where $\|V_s\| = \|V_b\|$ we therefore check when $\|v_s\| = \|v_b\|$. This occurs when p and ω_s are parallel vectors. Hence, the **only** condition that must be met is $\omega_s \times p = 0$.

6.2 (5 points)

You have a video drone that is trying to track the motion of a figure skater. The skater is rotating around the x_b axis at α radians per second, and is moving forward at β meters per second. The drone is c_1 meters above the figure skater and c_2 meters to the left of the skater.



- What is the body twist?

From the drawing we can see that the skater is rotating around the $+x_b$ axis and is translating along the $+z_b$ axis. The body twist is:

$$V_b = \begin{bmatrix} \alpha \\ 0 \\ 0 \\ 0 \\ 0 \\ \beta \end{bmatrix} \tag{33}$$

- What is the spatial twist?

From the drawing the skater is rotating around the $-z_s$ axis and is translating along the $-x_s$ axis. Here $\omega_s = [0, 0, -\alpha]^T$ and $\dot{p} = [-\beta, 0, 0]^T$. We also need to account for the linear velocity at $\{s\}$ due to the angular velocity of the ball. Here $p = [-c_2, 0, c_1]^T$. The spatial twist is:

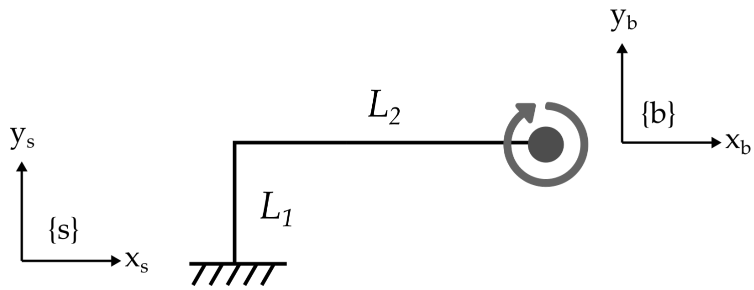
$$V_s = \begin{bmatrix} \omega_s \\ -\omega_s \times p + \dot{p} \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ -\alpha \\ -\beta \\ -c_2\alpha \\ 0 \end{bmatrix} \tag{34}$$

7 Properties of Screws

7.1 (5 points)

Consider the revolute joint shown above. This joint is rotating at α radians per second.

- What is the spatial twist V_s?



Use the right-hand rule to recognize that the revolute joint is rotating around the $-z_s$ axis. Since this a revolute joint, here $\dot{p} = 0$.

$$V_s = \begin{bmatrix} \omega_s \\ -\omega_s \times p \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ -\alpha \\ -L_1\alpha \\ L_2\alpha \\ 0 \end{bmatrix} \tag{35}$$

- What is the screw S?

Here we normalize by using a unit vector for $\hat{\omega}_s$.

$$S = \begin{bmatrix} \hat{\omega}_s \\ -\hat{\omega}_s \times p \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ -1 \\ -L_1 \\ L_2 \\ 0 \end{bmatrix} \tag{36}$$

- Show that your screw S is the "normalized" twist V_s

S is proportional to V_s. Indeed, here we have that $V_s = \alpha S$. The screw S is "normalized" since we have converted ω_s to a unit vector; all other parts of the math are the same.

7.2 (5 points)

Given the screw S for a prismatic joint:

$$S = \begin{bmatrix} 0 \\ v_s \end{bmatrix} \tag{37}$$

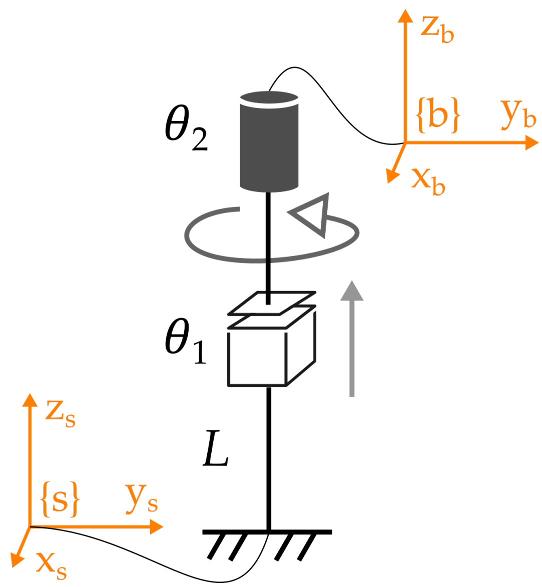
Find the transformation matrix $T = e^{[S]\theta}$. Your answer should be an expression for T that works for any choice of v_s.

For a prismatic joint that is translating in direction v_s the transformation matrix is:

$$T = \begin{bmatrix} I & v_s\theta \\ 0 & 1 \end{bmatrix} \tag{38}$$

Remember that v_s is a 3×1 vector and θ is a scalar. Hopefully this answer is intuitive: a prismatic joint is only translating, so there should be no rotation associated with the motion. The amount we translate is $v_s\theta$.

8 Using Screws



Answer the following questions based on the robot shown below.

8.1 (5 points)

Let S_1 be the screw for the prismatic joint (with joint position θ_1) and let S_2 be the screw for the revolute joint (with joint position θ_2). Write S_1 and S_2.

The first joint is a prismatic joint. Using our formula for S and the drawing:

$$S_1 = \begin{bmatrix} 0 \\ v_s \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \\ 0 \\ 0 \\ 1 \end{bmatrix} \tag{39}$$

where v_s must be a unit vector.

The second joint is a revolute joint. From our formulas for S and the drawing:

$$S_2 = \begin{bmatrix} \omega_s \\ -\omega_s \times p \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 1 \\ 0 \\ 0 \\ 0 \end{bmatrix} \tag{40}$$

where again ω_s must be a unit vector and here ω_s and p are parallel. Remember that these formulas are coming from "normalized" twists.

8.2 (5 points)

Convert the screws to joint motion to find T_{sb} as a function of θ.

```
syms theta1 theta2 L real

T_sb0 = [eye(3), [0;0;L]; 0 0 0 1];
S1 = [0 0 0 0 0 1];
S2 = [0 0 1 0 0 0];

T1 = screw2matrix(S1, theta1);
T2 = screw2matrix(S2, theta2);
T_sb = T1 * T2 * T_sb0;
```

Start by finding T_{sb} before the joint moves. Looking at the sketch:

$$T_{sb}(0) = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & L \\ 0 & 0 & 0 & 1 \end{bmatrix} \tag{41}$$

Next, recognize that we are going to (a) translate along z_s by θ_1 and (b) rotate around z_s by θ_2. Both of these are *fixed frame* motions since they are expressed in frame $\{s\}$. To find the resulting transformation we pre-multiply:

$$T_{sb}(\theta) = T_{trans} \cdot T_{rot} \cdot T_{sb}(0) = e^{[S_1]\theta_1} e^{[S_2]\theta_2} T_{sb}(0) \tag{42}$$

Note that T_{trans} is on the far left since — when we translate joint 1 — we are moving everything after it (including joint 2). Using our code shown above, we get:

$$T_{sb}(\theta) = \begin{bmatrix} \cos(\theta_2) & -\sin(\theta_2) & 0 & 0 \\ \sin(\theta_2) & \cos(\theta_2) & 0 & 0 \\ 0 & 0 & 1 & \theta_1 + L \\ 0 & 0 & 0 & 1 \end{bmatrix} \tag{43}$$

8.3 (5 points)

If $L = 1$, $\theta_1 = 5$, and $\theta_2 = \pi/2$, what is T_{sb}?

Here we plug the given values into our answer from the previous part:

$$T_{sb} = \begin{bmatrix} 0 & -1 & 0 & 0 \\ 1 & 0 & 0 & 0 \\ 0 & 0 & 1 & 6 \\ 0 & 0 & 0 & 1 \end{bmatrix} \tag{44}$$