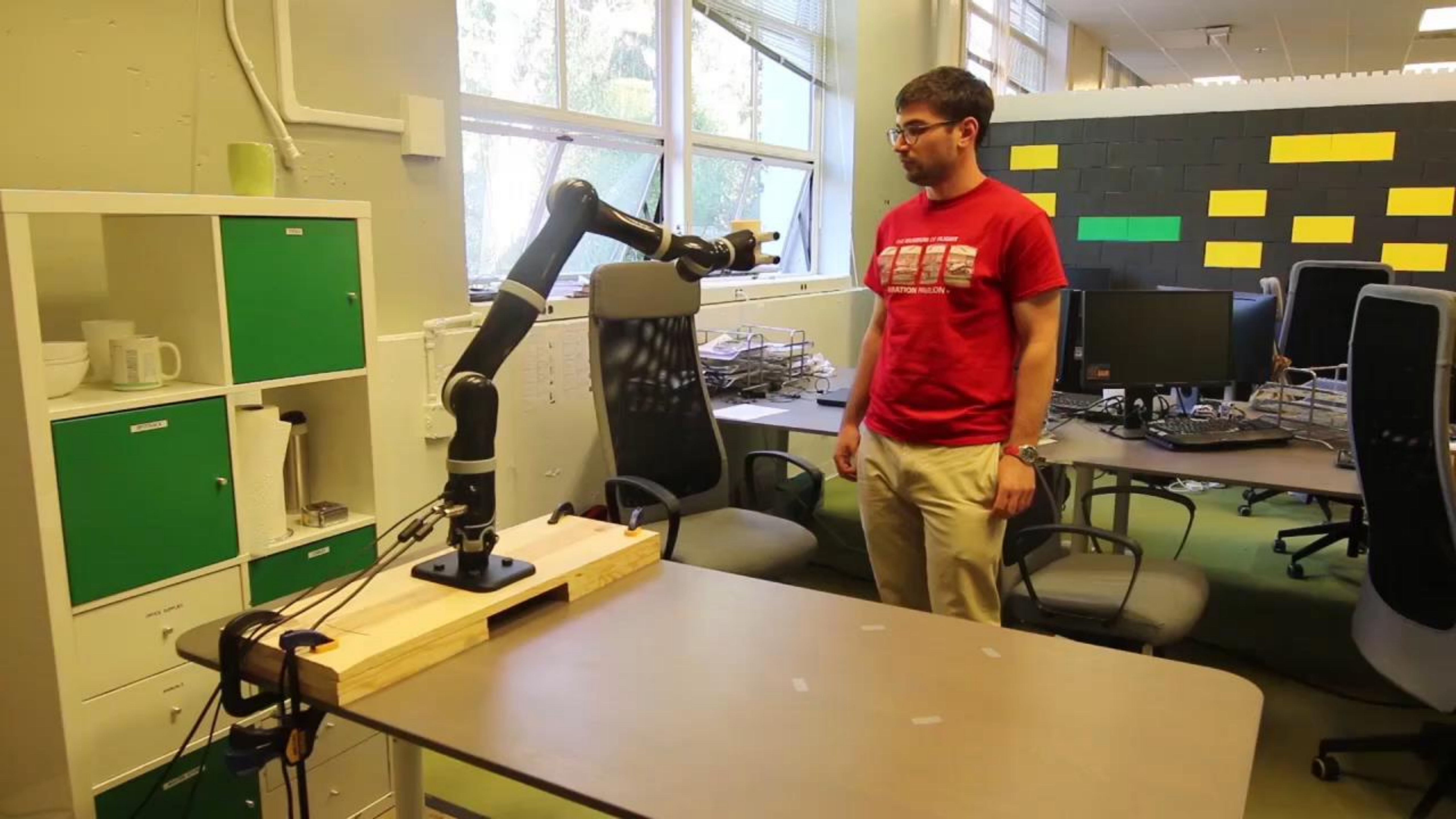
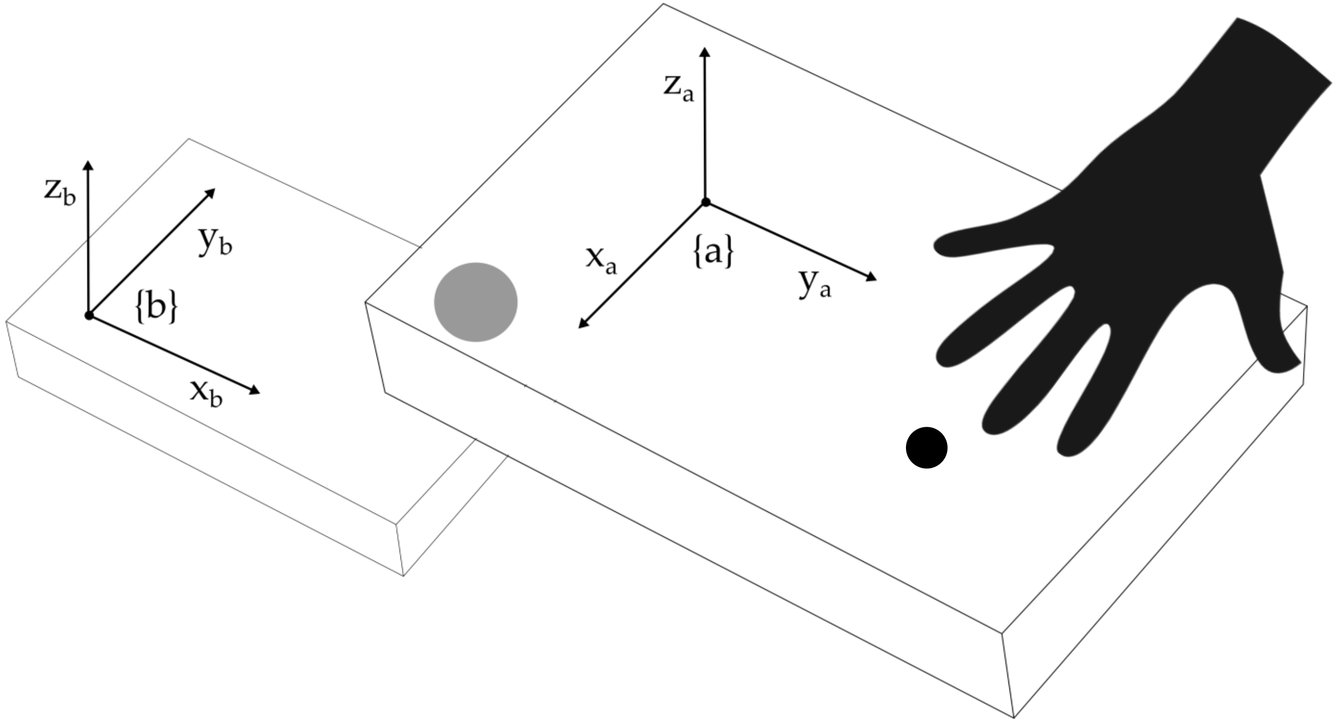
Wrenches

Reading: Modern Robotics 3.4



This Lecture

- What is a wrench?
- How do we convert a wrench between coordinate frames?
- What is the connection between wrenches and twists?

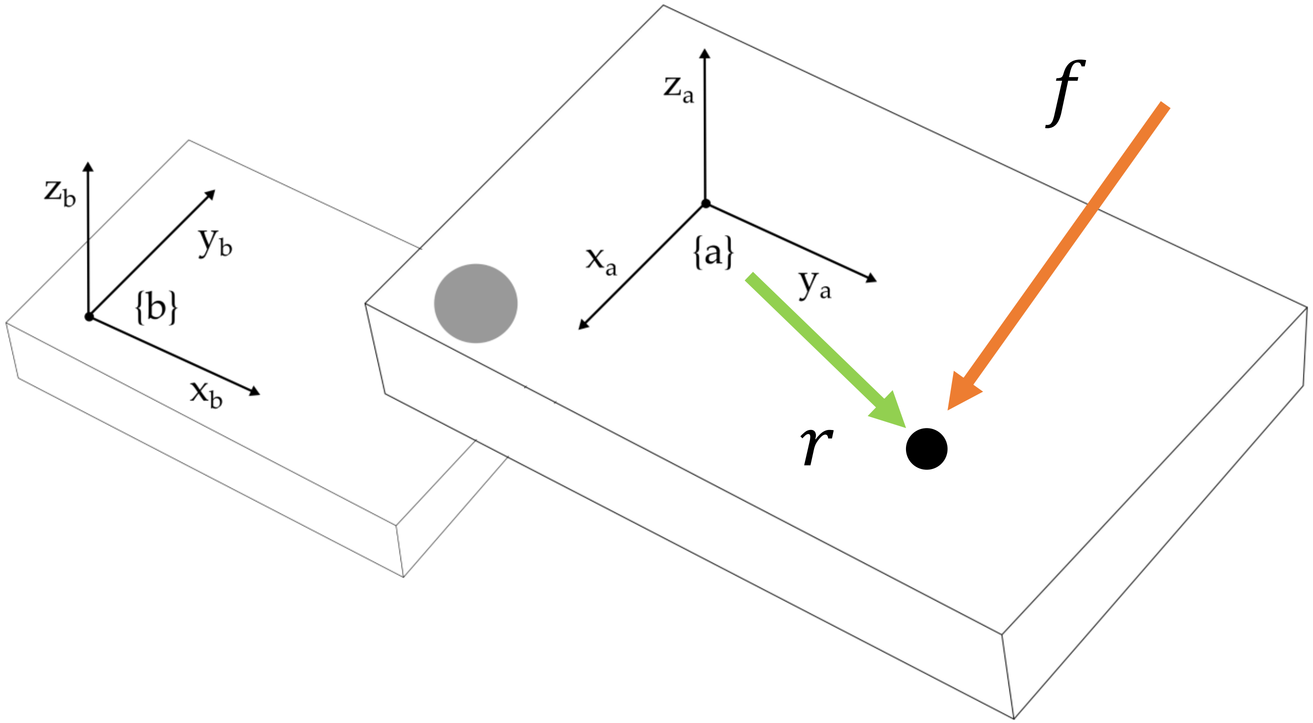


z_b
y_b
{b}
x_b
z_a
x_a
{a}
y_a

Force. f is the force applied at point r.

$$f = \begin{bmatrix} \text{force in } x \\ \text{force in } y \\ \text{force in } z \end{bmatrix}$$

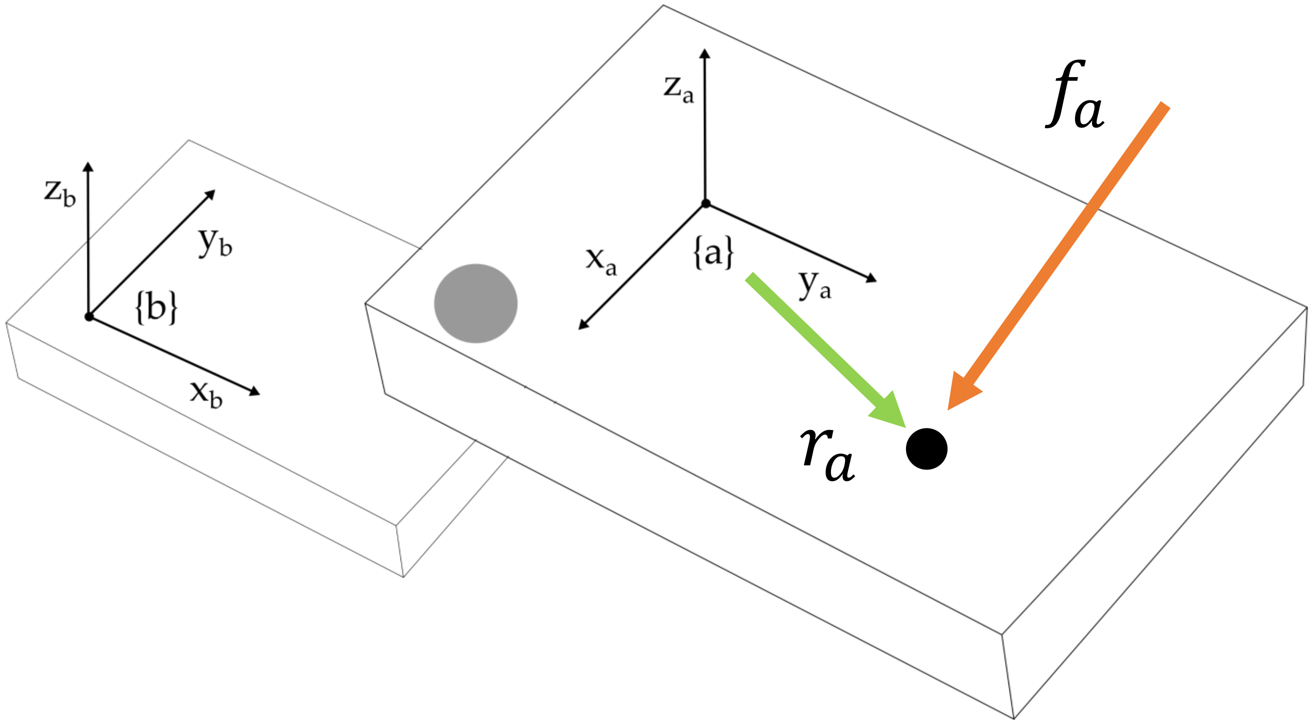
Force is a vector



Force. f is the force applied at point r.

$$f_a = \begin{bmatrix} \text{force in } x_a \\ \text{force in } y_a \\ \text{force in } z_a \end{bmatrix}$$

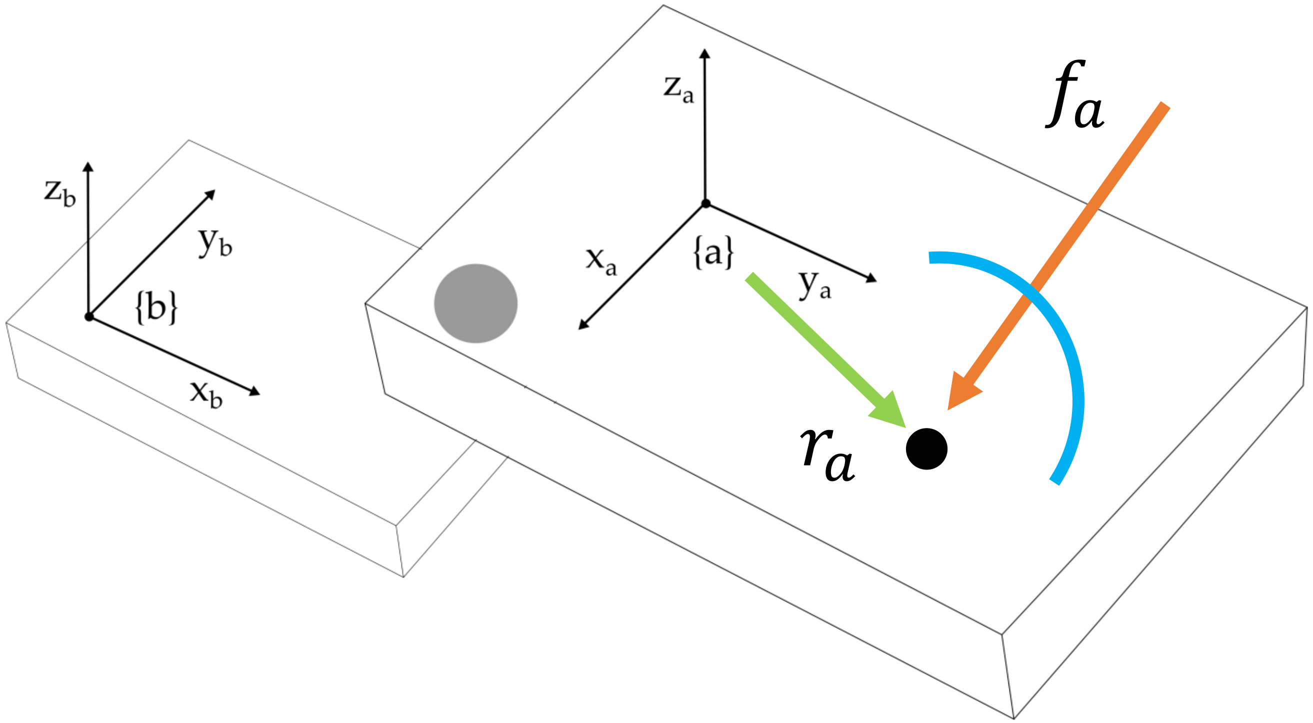
Like any vector, we need a frame of reference



Moment. Force f causes a moment relative to $\{a\}$:

$$m_a = r_a \times f_a$$

Notice the moment arm is the vector from $\{a\}$ to the point of contact.



Wrenches

A wrench is a 6-dimensional **vector**:

$$F = \begin{bmatrix} m \\ f \end{bmatrix}$$

Where $m \in \mathbb{R}^3$ is **moment** and $f \in \mathbb{R}^3$ is **force**.
We use subscripts to denote the frame of reference, so F_a is a wrench in frame $\{a\}$.

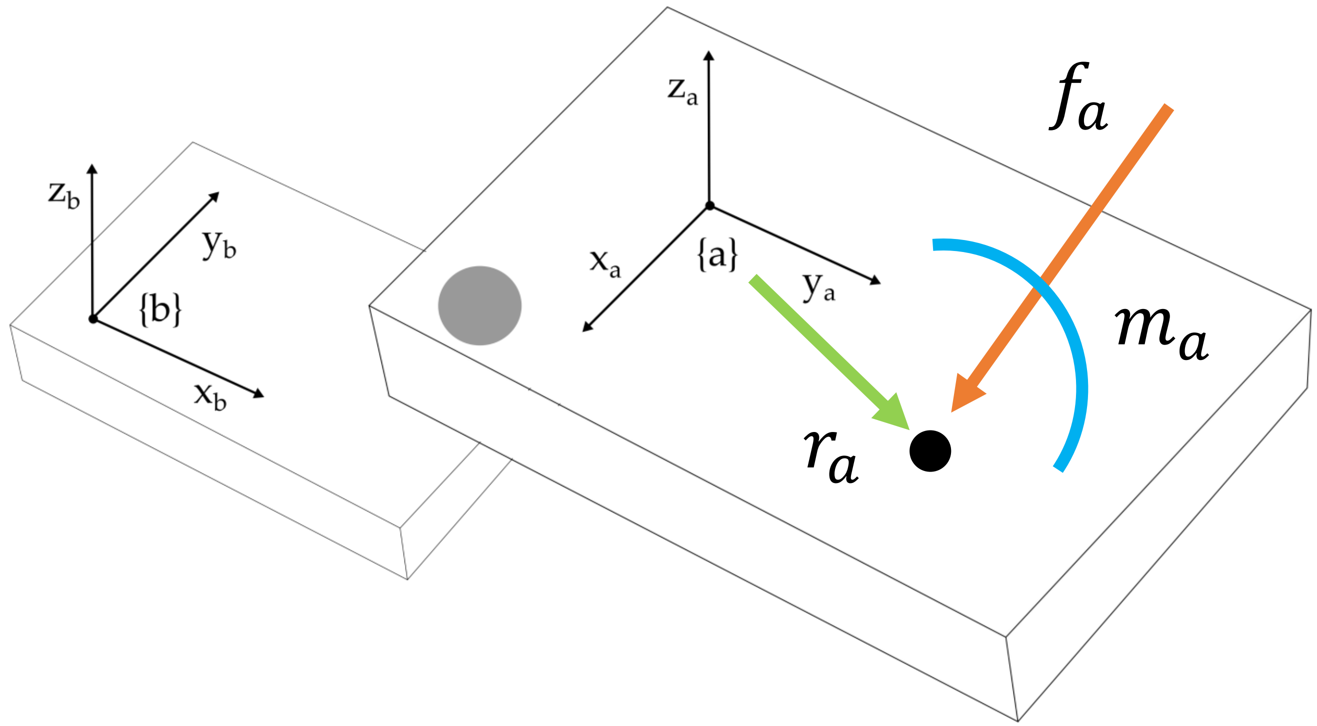
Wrenches

If more than one **wrench** acts on a rigid body, the total wrench is the **sum** of the wrenches.

$$F_a = F_{1a} + F_{2a}$$

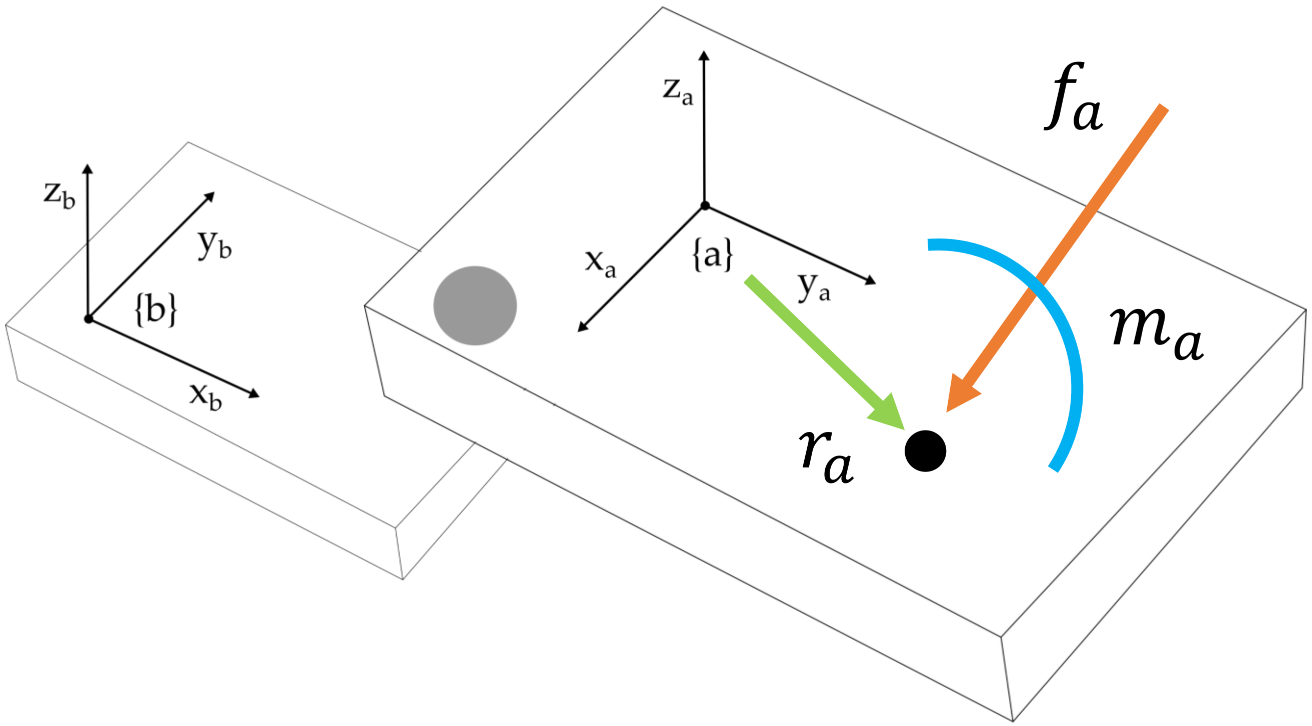
wrench 2 in $\{a\}$

wrench 1 in $\{a\}$



When does the wrench F have force but no moment?

When does the wrench F have moment but no force?



When does the wrench F have force but no moment?

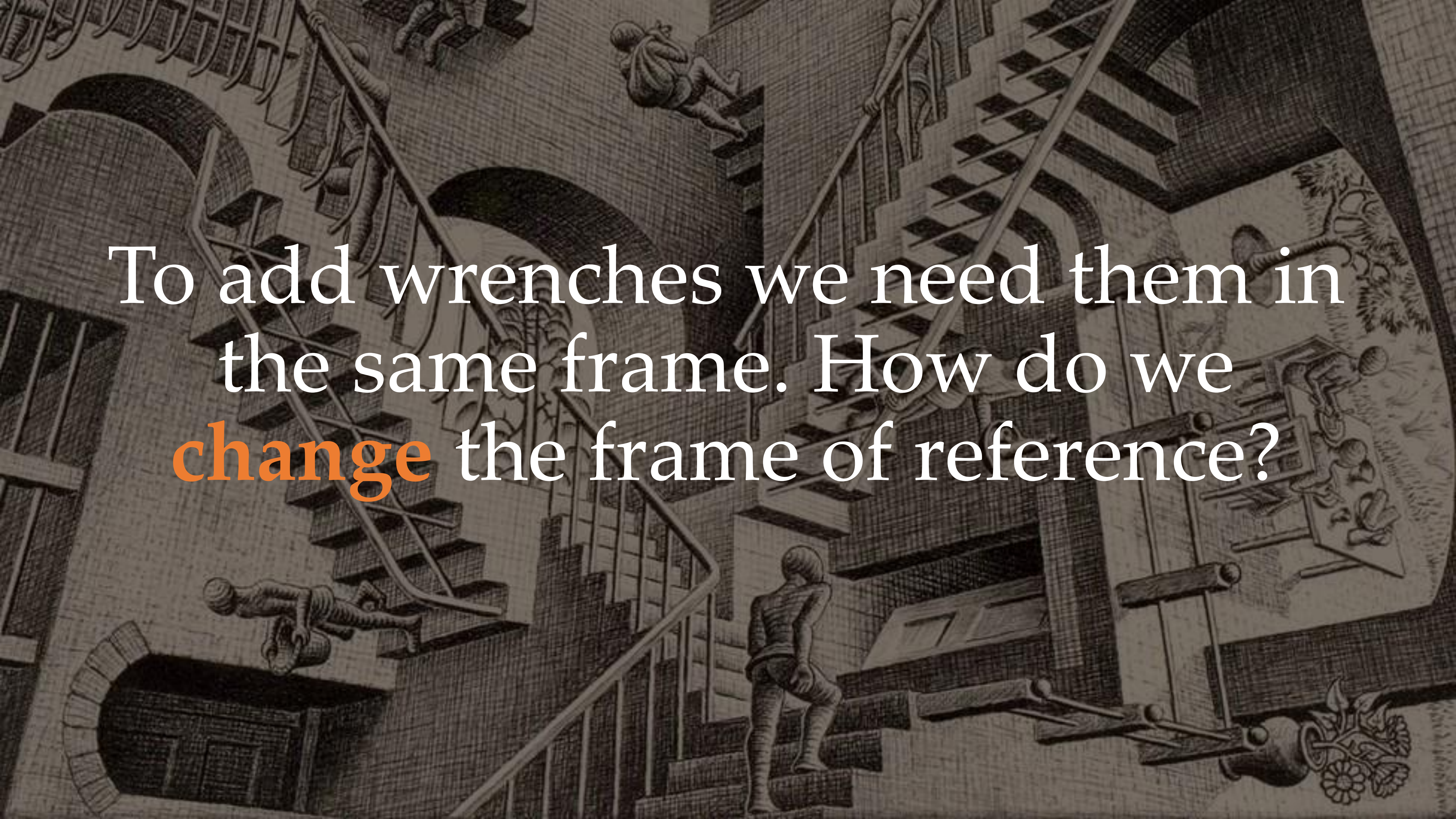
$$F = \begin{bmatrix} m \\ f \end{bmatrix}, \qquad m = 0 \times f$$

when force applied at origin

When does the wrench F have moment but no force?

$$F = \begin{bmatrix} r \times f \\ f \end{bmatrix} + \begin{bmatrix} -r \times -f \\ -f \end{bmatrix}$$

equal but opposite forces
at different locations



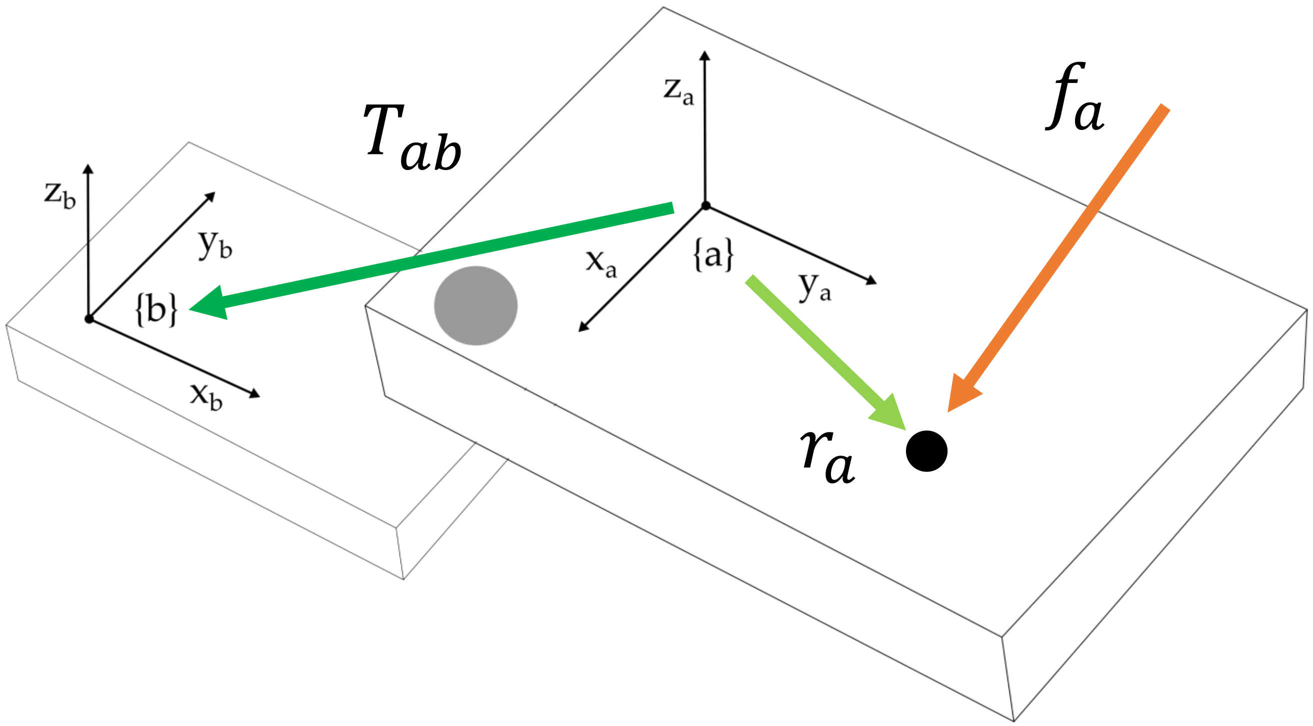
To add wrenches we need them in the same frame. How do we **change** the frame of reference?

Changing Frames

We want to **convert** wrench F_a into F_b

$$F_b = \begin{bmatrix} m_b \\ f_b \end{bmatrix}$$

$$T_{ab} = T = \begin{bmatrix} R & p \\ 0 & 1 \end{bmatrix}$$



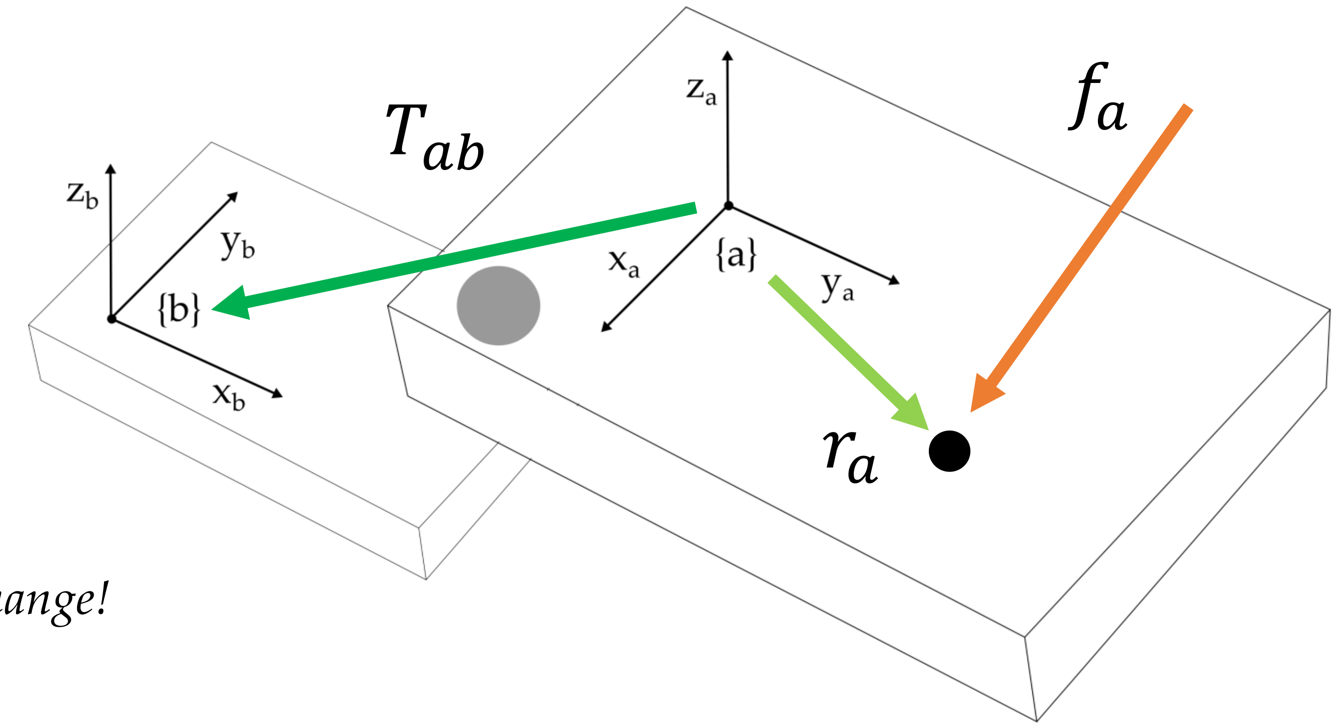
Changing Frames

We want to **convert** wrench F_a into F_b

Force. Remember f_a is vector written in $\{a\}$

$$f_b = R_{ba} f_a = R^T f_a$$

Magnitude of force does not change!

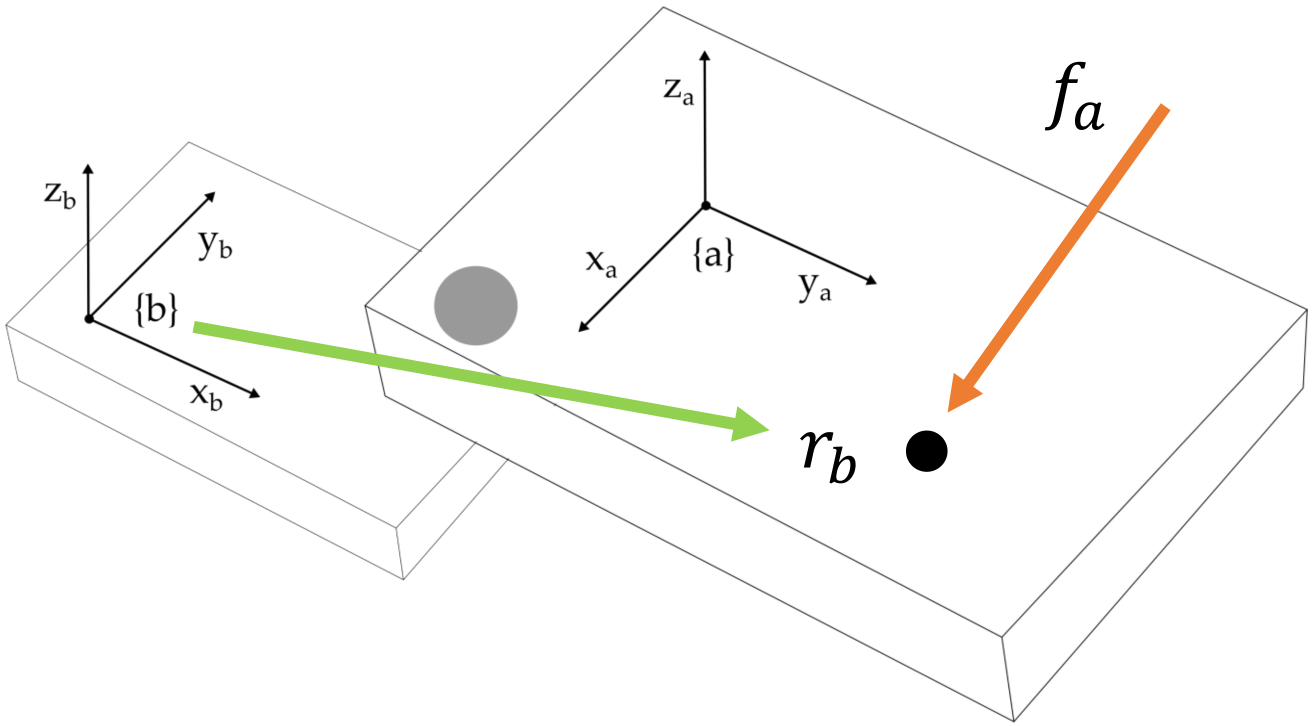


Changing Frames

We want to **convert** wrench F_a into F_b

Moment. Moment arm from $\{b\}$ to point

$$m_b = r_b \times f_b$$

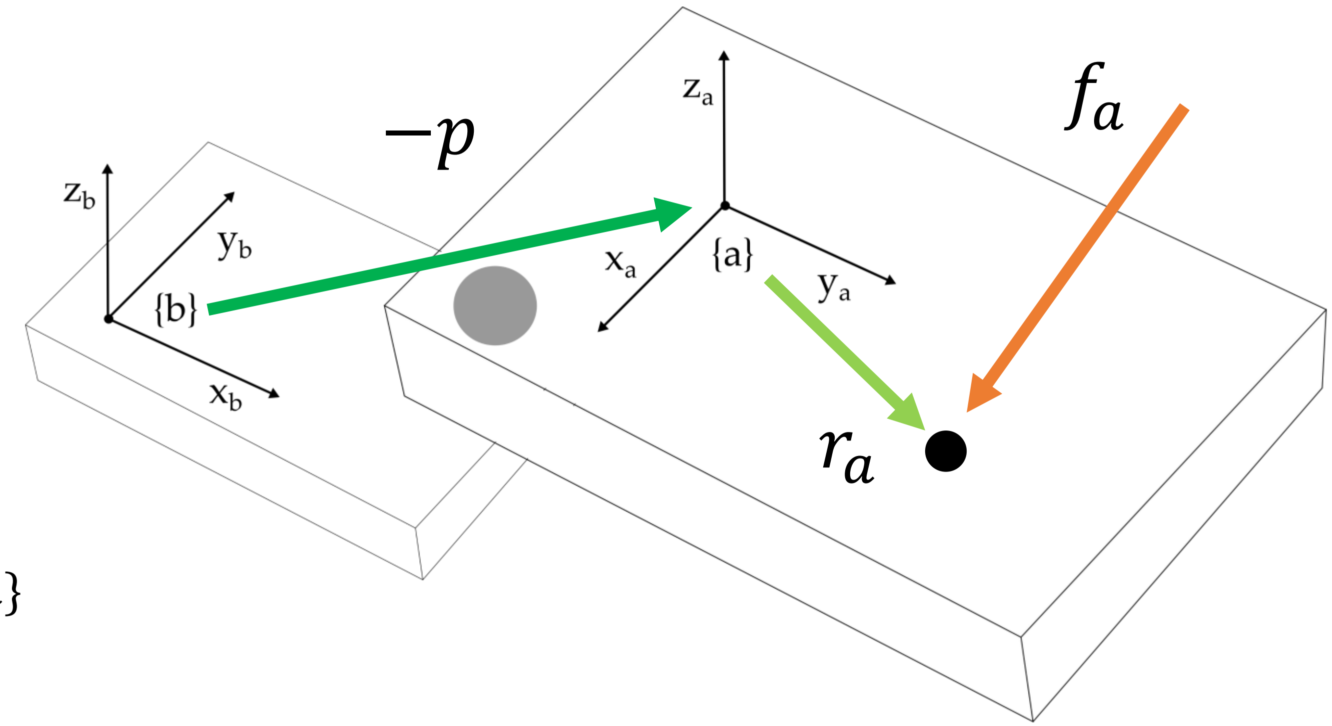


Changing Frames

We want to **convert** wrench F_a into F_b

Moment. Moment arm from $\{b\}$ to point

$$m_b = R^T\big(\underbrace{(-p + r_a)}_{\textit{Moment arm in } \{a\}} \times \underbrace{f_a}_{\textit{Force in } \{a\}}\big)$$



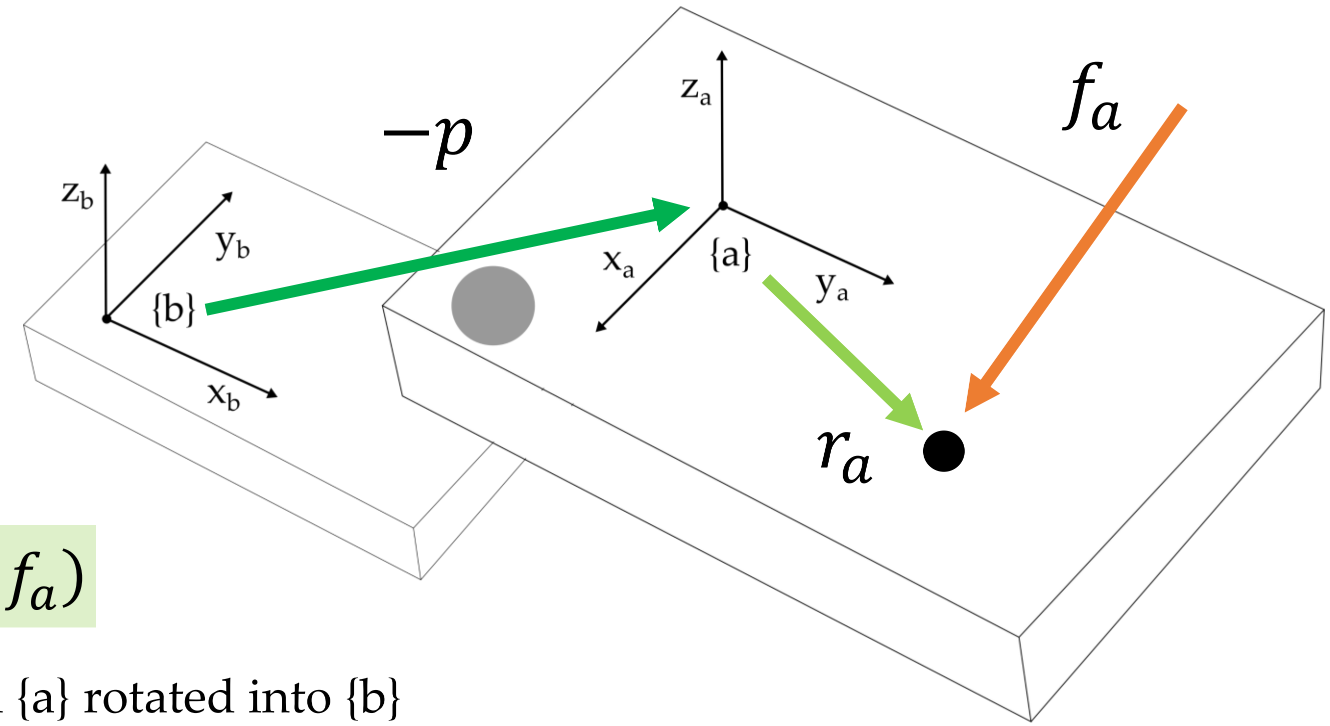
Changing Frames

We want to **convert** wrench F_a into F_b

Moment. Moment arm from $\{b\}$ to point

$$m_b = R^T(-p \times f_a) + R^T(r_a \times f_a)$$

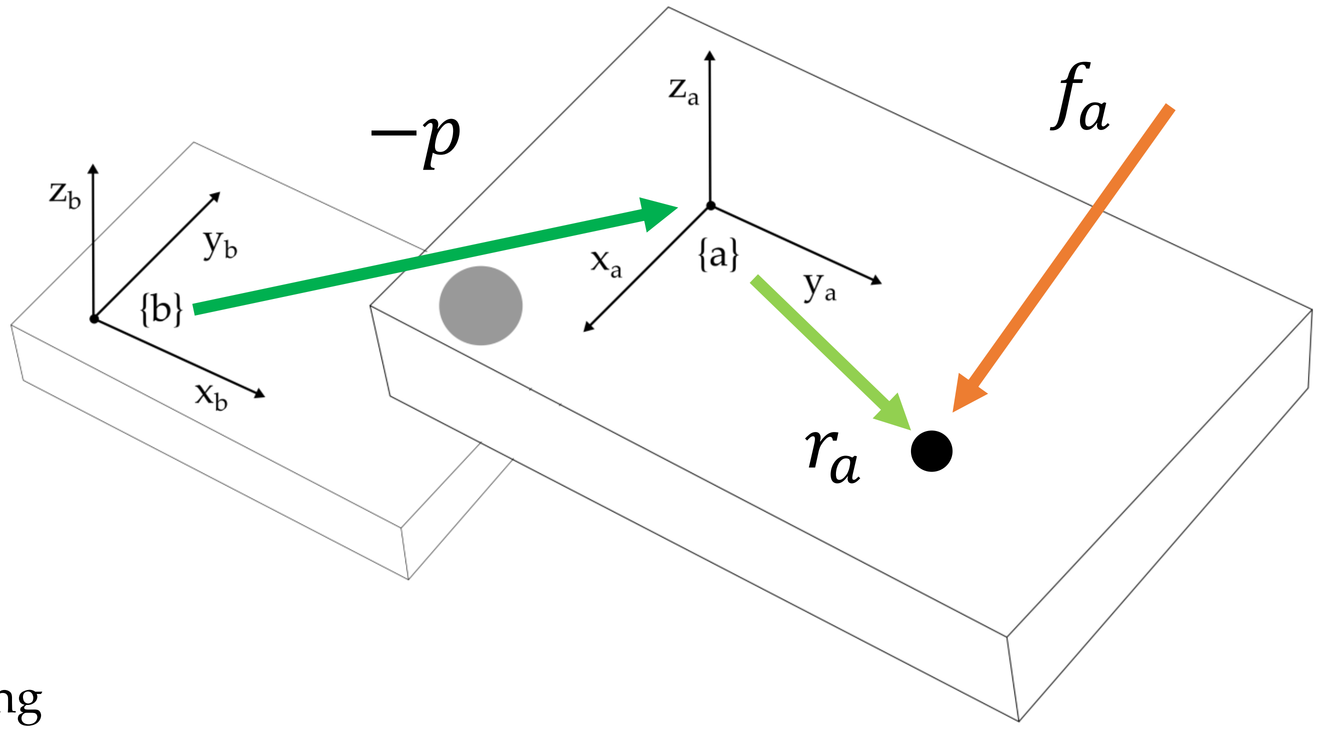
Moment in {a} rotated into {b}



Changing Frames

We want to **convert** wrench F_a into F_b

Moment. Moment arm from $\{b\}$ to point



$$m_b = \boxed{R^T(-p \times f_a)} + R^T m_a$$

Moment caused by changing moment arm

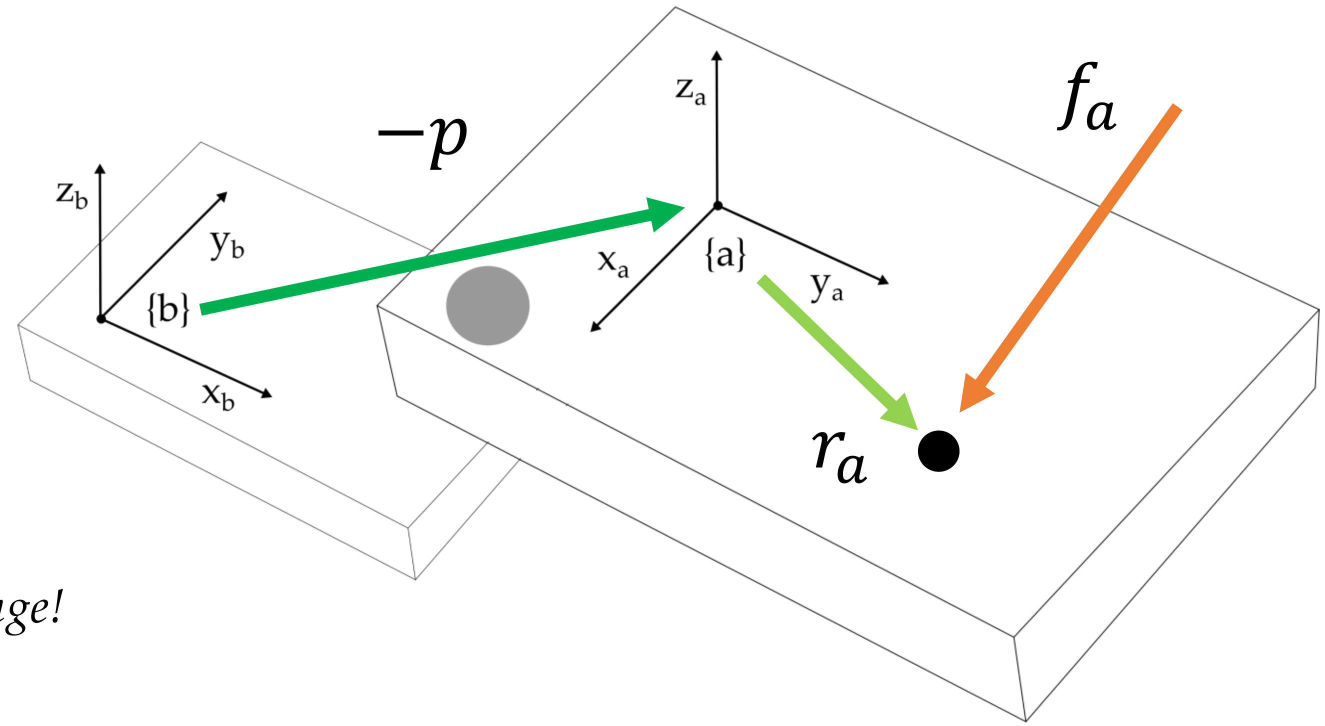
Changing Frames

We want to **convert** wrench F_a into F_b

Moment. Moment arm from $\{b\}$ to point

$$m_b = \underline{R^T [p]^T f_a} + R^T m_a$$

Magnitude of moment can change!

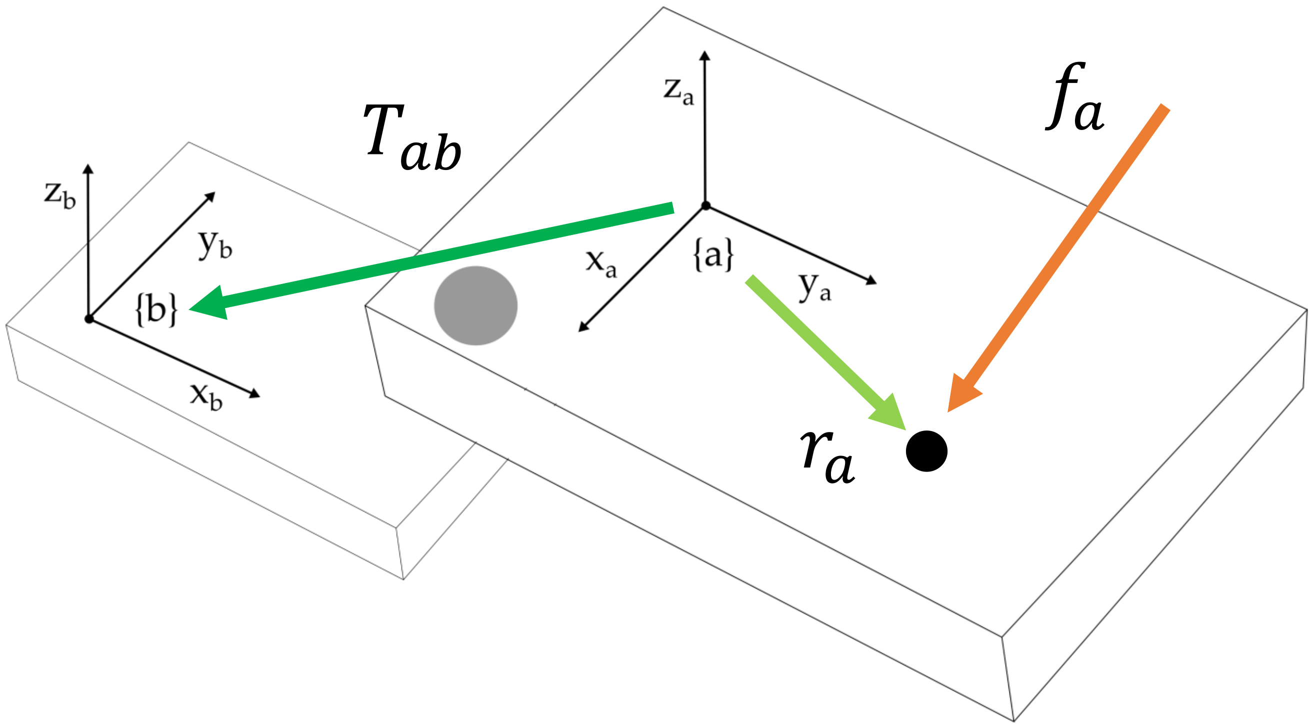


Changing Frames

We want to **convert** wrench F_a into F_b

$$f_b = R^T f_a$$

$$m_b = R^T [p]^T f_a + R^T m_a$$



Power

Power is the product of **effort** (wrench) and **flow** (twist)

$$P = V \cdot F = V^T F$$

Power must be the same regardless of the frame in which it is written. Energy cannot be generated or dissipated by changing our frame of reference

$$\underbrace{{V_b}^T F_b}_{\text{Power in } \{b\}} = \underbrace{{V_a}^T F_a}_{\text{Power in } \{a\}}$$

Power

Power must be the **same** regardless of the frame in which it is written. Energy cannot be generated or dissipated by changing our frame of reference

$$V_b{}^T F_b = V_a{}^T F_a$$

$$V_b{}^T F_b = \left(\mathrm{Ad}_{T_{ab}} V_b\right)^T F_a$$

$$V_b{}^T \left(\underline{F_b = \mathrm{Ad}_{T_{ab}}{}^T F_a}\right)$$

Key step here is $V_a = \mathrm{Ad}_T V_b$

Power

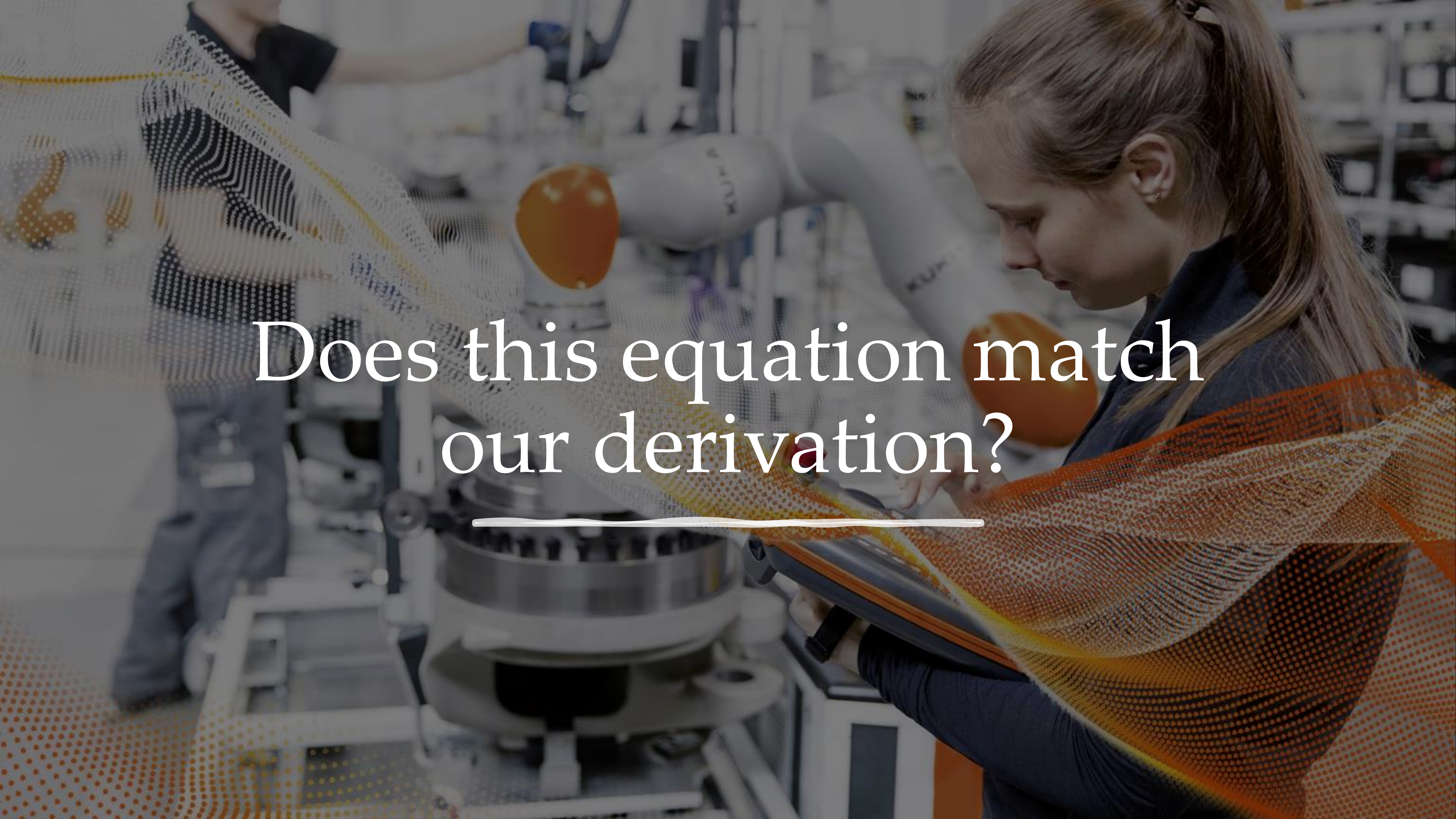
Power must be the **same** regardless of the frame in which it is written. Energy cannot be generated or dissipated by changing our frame of reference

$$V_b{}^T F_b = V_a{}^T F_a$$

$$F_b = \mathrm{Ad}_{T_{ab}}{}^T F_a$$

We have a new equation for changing wrench frames!

Does this equation match our derivation?



Power

$$F_b = {Ad_T}^T F_a$$

$$ {Ad_T}^T F_a = \begin{bmatrix} R^T & R^T[p]^T \\ 0 & R^T \end{bmatrix} \begin{bmatrix} m_a \\ f_a \end{bmatrix} $$

Power

$$F_b = {Ad_T}^T F_a$$

$$F_b = \begin{bmatrix} R^T [p]^T f_a + R^T m_a \\ R^T f_a \end{bmatrix}$$

Takeaways

Wrenches are like twists, but deal with **moment and force** instead of **angular and linear velocity**.

$$V = \begin{bmatrix} \omega \\ v \end{bmatrix}, \qquad V_a = \mathrm{Ad}_{T_{ab}} V_b$$

$$F = \begin{bmatrix} m \\ f \end{bmatrix}, \qquad F_b = \mathrm{Ad}_{T_{ab}}{}^{T} F_a$$

This Lecture

- What is a wrench?
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Next Lecture

- How can use use wrenches to find the statics of our robot arm?