

Lyapunov Stability



Reading: Robot Modeling and Control 8.1, Appendix D

This Lecture

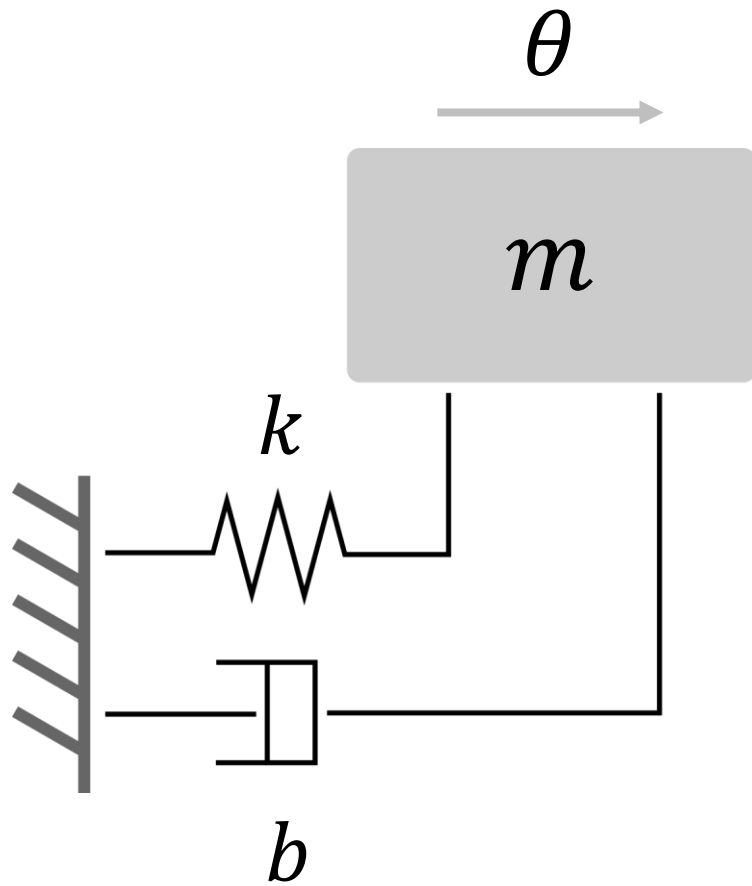


- How do we determine if a dynamical system is stable?
- Introducing Lyapunov stability analysis
- What is multivariable control for robot arms?

A person wearing a red long-sleeved shirt is seated and using a black cylindrical handle attached to a complex mechanical device. The device has various metal components, bolts, and a blue cable. The person's arm is extended, and they are holding the handle firmly. The background is slightly blurred, showing more of the device and the person's torso.

What is **Lyapunov**
stability analysis?

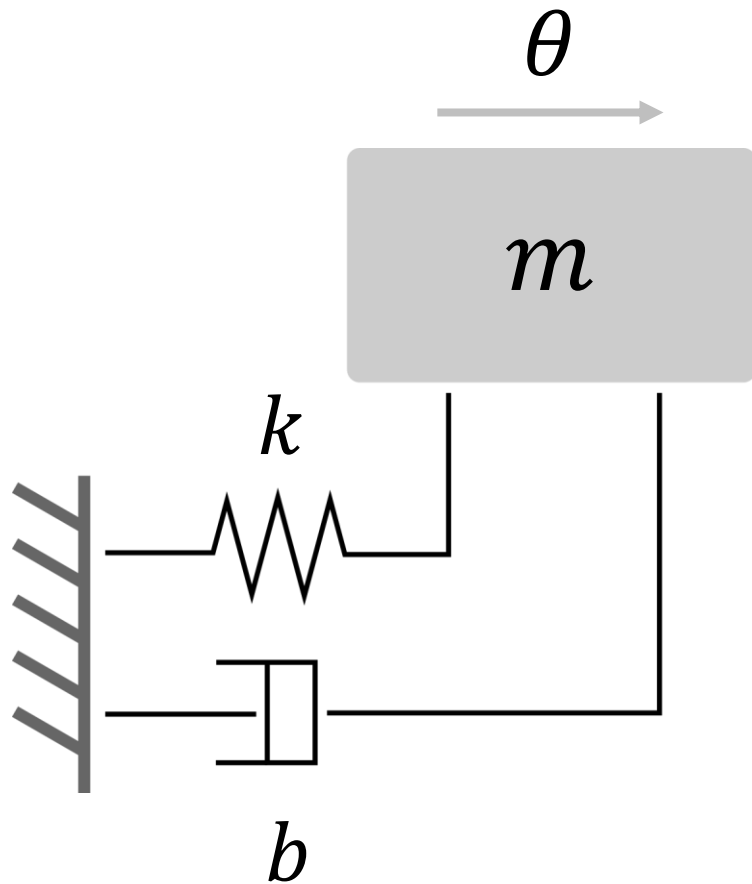
Example



Let's use **Lyapunov stability analysis** to show this mass-spring-damper is stable.

$$m\ddot{\theta} + b\dot{\theta} + k\theta = 0$$

Example

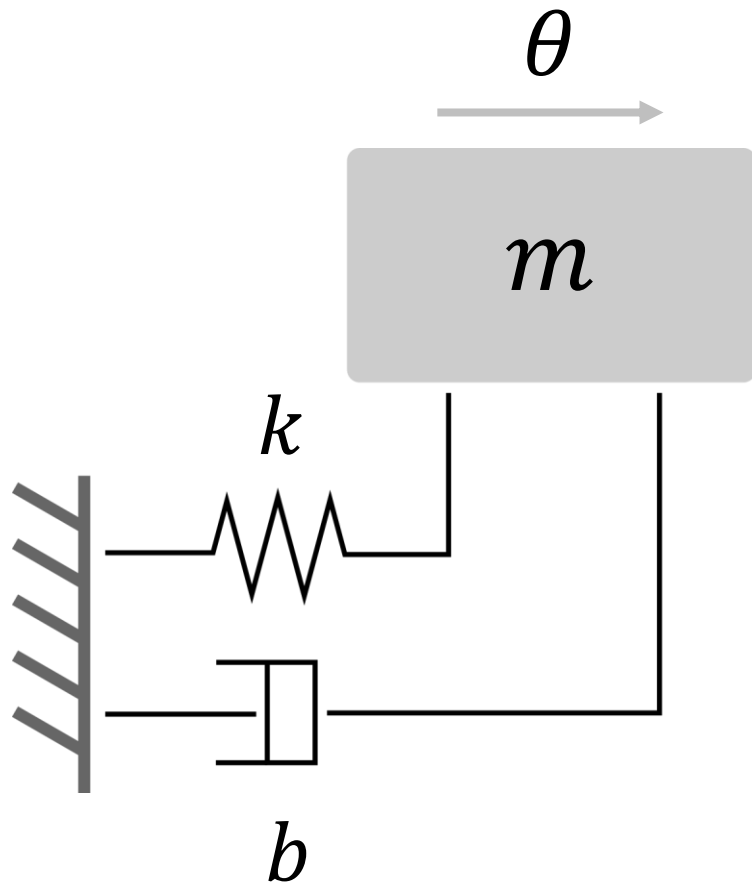


Let's use **Lyapunov stability analysis** to show this mass-spring-damper is stable.

$$v(t) = \frac{1}{2} m \dot{\theta}(t)^2 + \frac{1}{2} k \theta(t)^2$$

Total energy as a function of time.
Note that this is always nonnegative.

Example



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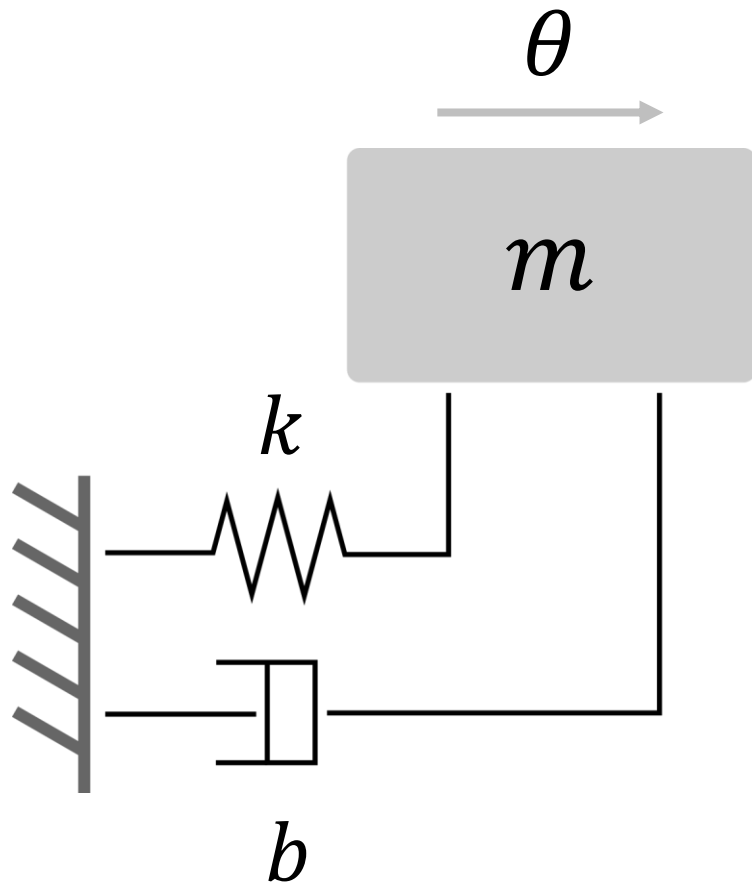
$$\dot{v}(t) = m \ddot{\theta}(t) \dot{\theta}(t) + k \theta(t) \dot{\theta}(t)$$

Rate of change of the total energy

$\dot{v} > 0$ increasing energy,

$\dot{v} < 0$ decreasing energy,

Example



Let's use **Lyapunov stability analysis** to show this mass-spring-damper is stable.

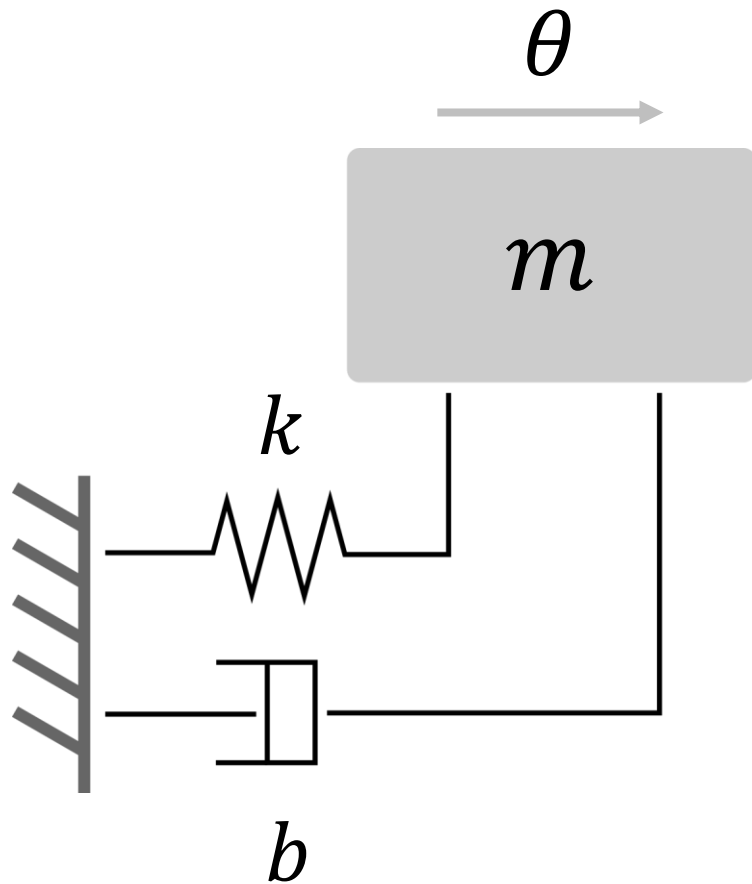
$$m\ddot{\theta} = -b\dot{\theta} - k\theta$$

$$\dot{v} = m\ddot{\theta}\dot{\theta} + k\theta\dot{\theta}$$

$$\dot{v} = (-b\dot{\theta} - k\theta)\dot{\theta} + k\theta\dot{\theta}$$

$$\dot{v}(t) = -b\dot{\theta}(t)^2$$

Example

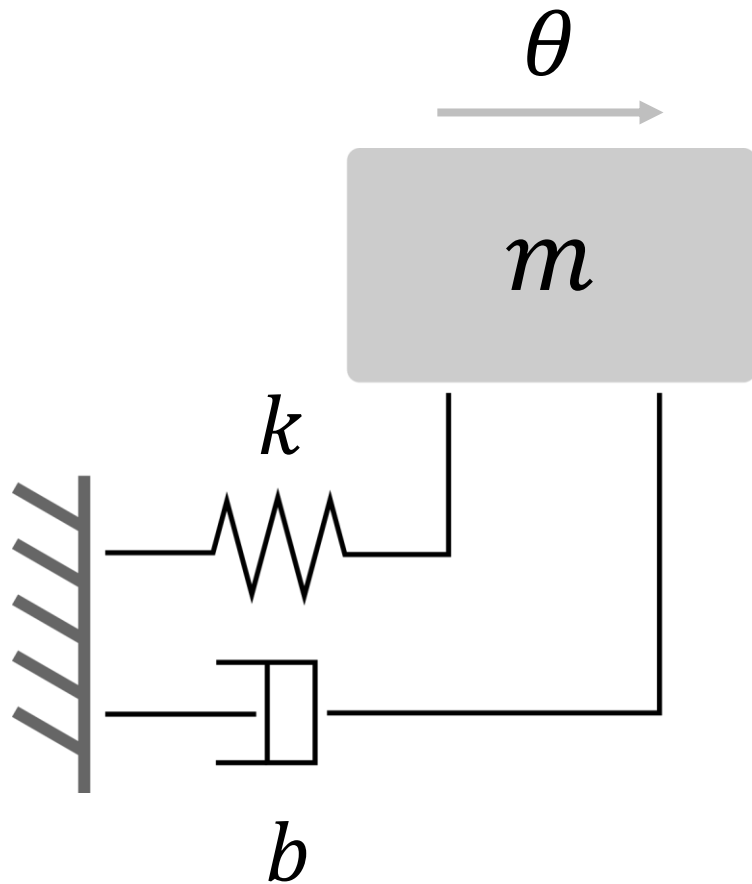


Let's use **Lyapunov stability analysis** to show this mass-spring-damper is stable.

$$\dot{v}(t) = -b\dot{\theta}(t)^2$$

Since $b > 0$, energy always leaving the system
until it comes to rest ($\dot{\theta} = 0$)

Example

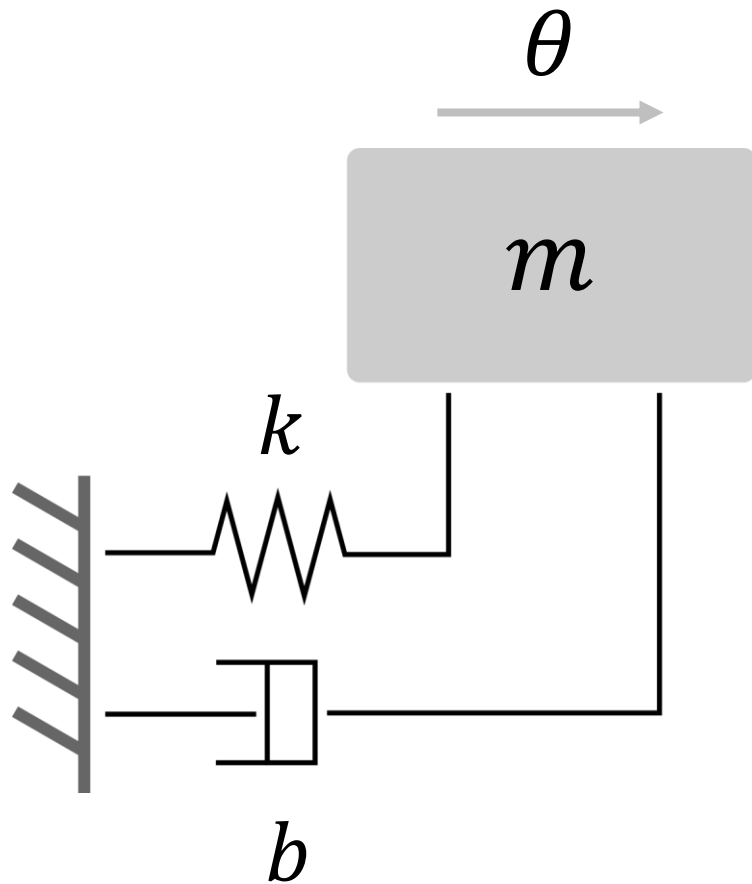


Let's use **Lyapunov stability analysis** to show this mass-spring-damper is stable.

$$\dot{v}(t) = -b\dot{\theta}(t)^2$$
$$m\ddot{\theta} + b\dot{\theta} + k\theta = 0$$

At rest $\dot{\theta} = 0$ and $\ddot{\theta} = 0$, leaving us with $k\theta = 0$. Energy decreases until $\theta = 0$.

Example



Let's use **Lyapunov stability analysis** to show this mass-spring-damper is stable.

$$\dot{v}(t) = -b\dot{\theta}(t)^2$$

Conclusion. For any initial conditions, the mass-spring-damper will always come to rest at equilibrium $\theta = 0$.

Lyapunov Stability Analysis

Given a dynamical system:

$$\dot{x} = f(x)$$

To show this system is stable, find a generalized energy function $v(x)$ such that:

- $v(x) > 0$ for all values of $x \neq 0$, and $v(0) = 0$
- $v(x)$ has continuous first partial derivatives
- $\dot{v}(x) \leq 0$ (e.g., energy decreases over time)

Lyapunov Stability Analysis

Example. Consider the dynamical system:

$$\dot{x} = -Ax$$

where x is a $n \times 1$ vector and A is a positive definite $n \times n$ matrix. Try:

$$v(x) = \frac{1}{2} x^T x$$

$v(x) > 0$ for all values of $x \neq 0$, and $v(0) = 0$
Partial derivative $\frac{\partial v}{\partial x} = x$ is continuous

Lyapunov Stability Analysis

Example. Consider the dynamical system:

$$\dot{x} = -Ax$$

$$\dot{v}(x) = \frac{d}{dt} \left(\frac{1}{2} x^T x \right) = x^T \dot{x} = \underline{-x^T Ax}$$

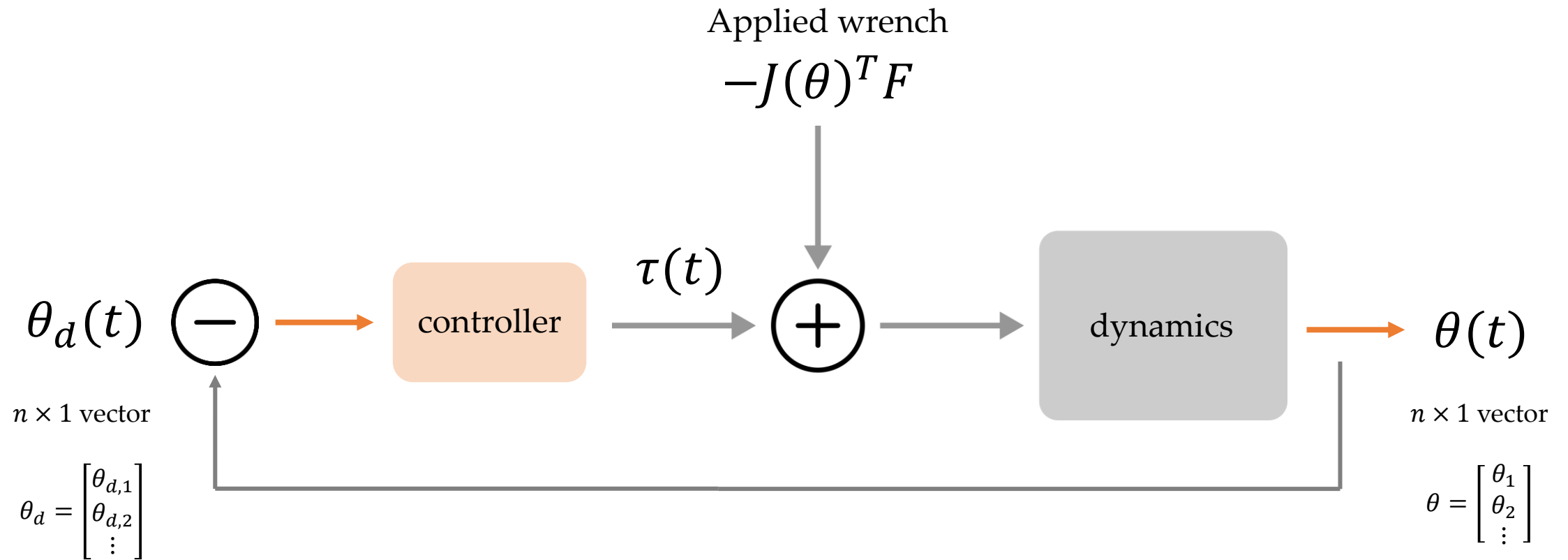
$\dot{v}(x) \leq 0$ because A is positive definite.

Energy decreases until we reach stability at $x = 0$, the system is **stable**.

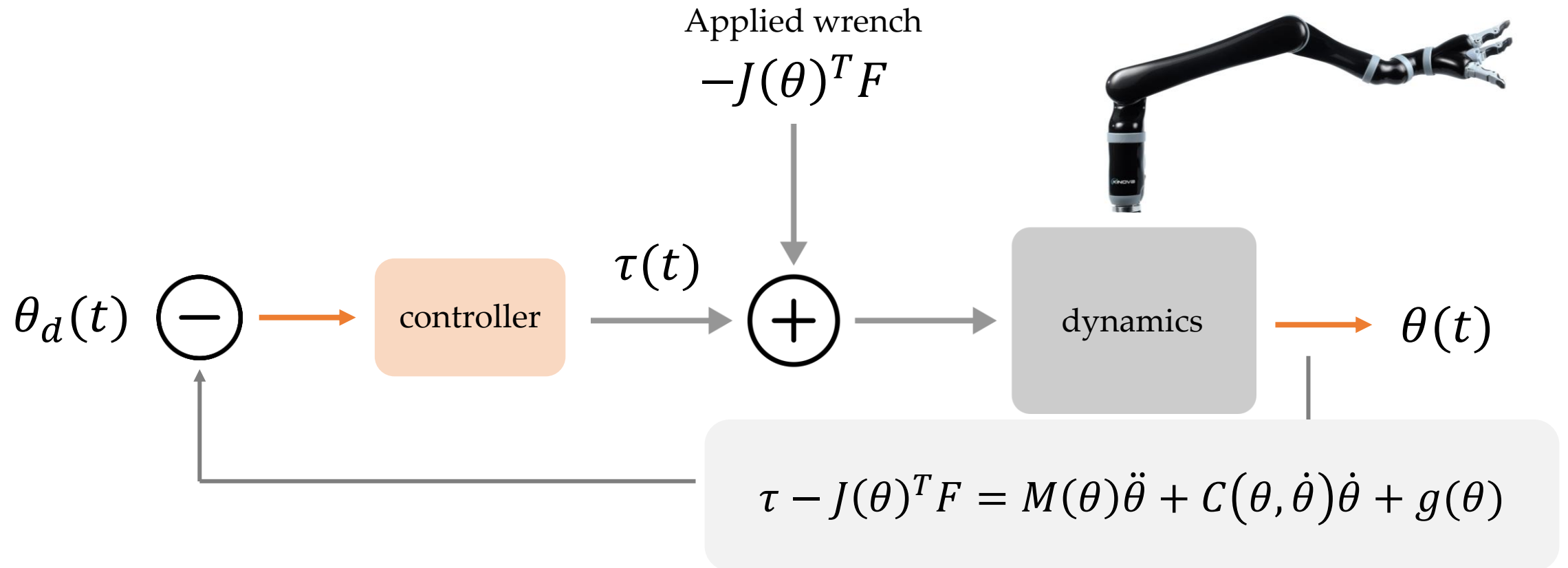
The background image shows a laboratory with several robotic exoskeletons. On the left, a person in a red shirt is seated in a Hocoma exoskeleton, which has a computer monitor and keyboard attached. In the center, another person is standing and operating a device. On the right, a woman in a green vest and black leggings is standing, wearing a full-body exoskeleton. The room has large windows and a tiled floor.

How do we control a
robot arm with **single,
unified** controller?

Multivariable Control



Multivariable Control



Multivariable Control

One industry-standard multivariable robot controller is:

$$\tau = K_P(\theta_d - \theta) - K_D\dot{\theta} + g(\theta)$$

- K_P is a positive definite matrix of proportional gains
- K_D is a positive definite matrix of derivative gains
- $g(\theta)$ is the gravity vector

Multivariable Control

One industry-standard multivariable robot controller is:

$$\tau = K_P(\theta_d - \theta) - K_D\dot{\theta} + g(\theta)$$

Often diagonal gain matrix

$$K_P = \begin{bmatrix} k_{p,1} & & \\ & k_{p,2} & \\ & & \ddots \end{bmatrix}$$

Often diagonal gain matrix

$$K_D = \begin{bmatrix} k_{d,1} & & \\ & k_{d,2} & \\ & & \ddots \end{bmatrix}$$

Gravity compensation.
Remove steady-state
error due to gravity.

Multivariable Control

Plugging this controller into the robot's dynamics:

$$M(\theta)\ddot{\theta} + C(\theta, \dot{\theta})\dot{\theta} + g(\theta) = \tau = K_P(\theta_d - \theta) - K_D\dot{\theta} + g(\theta)$$

$$M(\theta)\ddot{\theta} = -C(\theta, \dot{\theta})\dot{\theta} - K_D\dot{\theta} + K_P(\theta_d - \theta)$$

Is this closed-loop system **stable**?
Will the robot converge to $\theta = \theta_d$?

Multivariable Control

Consider the dynamical system and **generalized energy function**:

$$M(\theta)\ddot{\theta} = -C(\theta, \dot{\theta})\dot{\theta} - K_D\dot{\theta} + K_P(\theta_d - \theta)$$

$$v(\theta, \dot{\theta}) = \frac{1}{2}\dot{\theta}^T M(\theta)\dot{\theta} + \frac{1}{2}(\theta_d - \theta)^T K_P(\theta_d - \theta)$$

We know that M and K_P are positive definite matrices, so $v(\theta, \dot{\theta}) > 0$ for all values of $\theta \neq \theta_d$, $\dot{\theta} \neq 0$, and $v(0,0) = 0$

Multivariable Control

Take the **time derivative** to see whether energy is **decreasing**:

$$\dot{v} = \frac{1}{2} \dot{\theta}^T \dot{M}(\theta) \dot{\theta} + \dot{\theta}^T M(\theta) \ddot{\theta} - \dot{\theta}^T K_P(\theta_d - \theta)$$

$$\dot{v} = \dot{\theta}^T \left(\frac{1}{2} \dot{M}(\theta) \dot{\theta} + M(\theta) \ddot{\theta} - K_P(\theta_d - \theta) \right)$$



Plug in closed-loop dynamics here:

$$M(\theta) \ddot{\theta} = -C(\theta, \dot{\theta}) \dot{\theta} - K_D \dot{\theta} + K_P(\theta_d - \theta)$$

Multivariable Control

Take the **time derivative** to see whether energy is **decreasing**:

$$\dot{v} = \dot{\theta}^T \left(\frac{1}{2} \dot{M}(\theta) \dot{\theta} - C(\theta, \dot{\theta}) \dot{\theta} - K_D \dot{\theta} \right)$$

Property. $\dot{M} - 2C$ is skew symmetric, and therefore $\frac{1}{2} \dot{\theta}^T (\dot{M} - 2C) \dot{\theta} = 0$

$$\dot{v} = -\dot{\theta}^T K_D \dot{\theta}$$

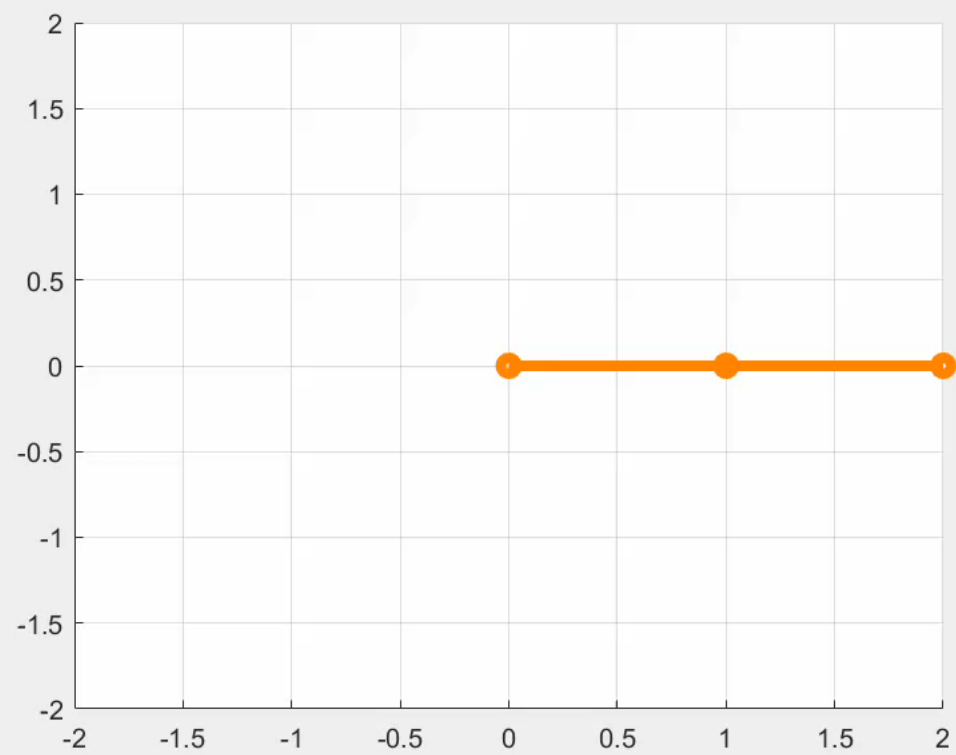
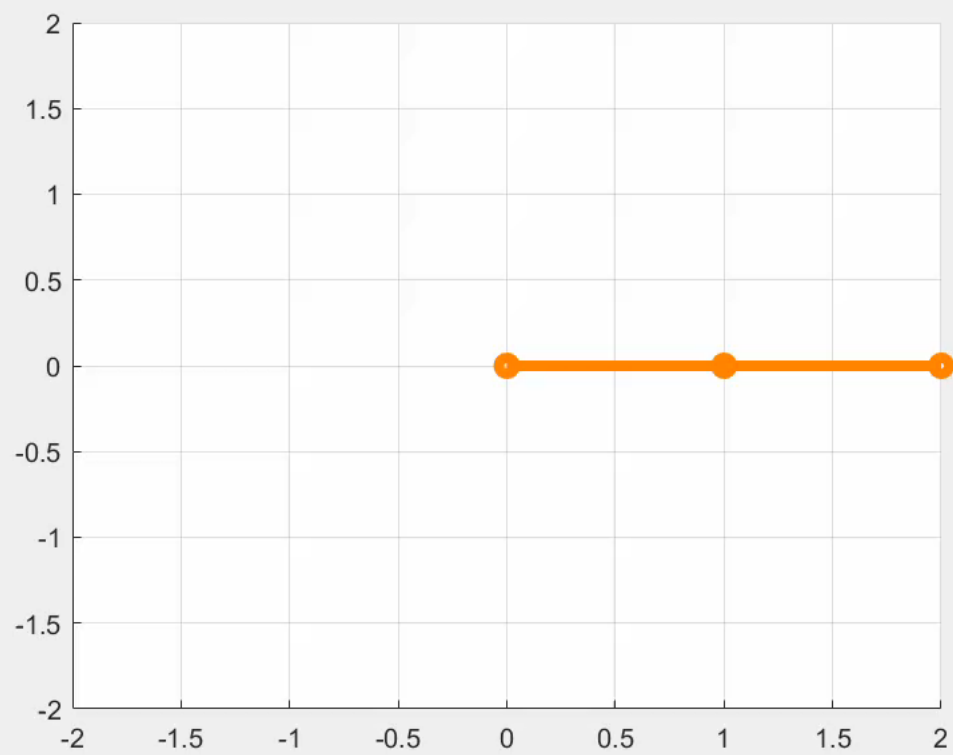
Multivariable Control

Complete stability analysis.

- The time derivative of our energy function is $\dot{v} = -\dot{\theta}^T K_D \dot{\theta}$
- This function is negative definite; energy decreases until $\dot{\theta} = 0$
- If $\dot{\theta} = 0$ is constant, then $\ddot{\theta} = 0$ and the dynamics become:

$$0 = K_P(\theta_d - \theta)$$

- Since K_P is positive definite (nonsingular), we have $\theta_d = \theta$ at equilibrium



Takeaways

Lyapunov's method examines stability of dynamical systems.

To show a system is stable, find a generalized energy function $v(x)$ such that:

- $v(x) > 0$ for all values of $x \neq 0$, and $v(0) = 0$
- $v(x)$ has continuous first partial derivatives
- $\dot{v}(x) \leq 0$ (e.g., energy decreases over time)

One stable multivariable controller for robot arms is:

$$\tau = K_P(\theta_d - \theta) - K_D\dot{\theta} + g(\theta)$$

This Lecture



- How do we determine if a dynamical system is stable?
- Introducing Lyapunov stability analysis
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Next Lecture



- We've focused on controlling the robot to reach a desired position...
... what are some other things we might want to control for?