Problem Set 4

Robotics & Automation
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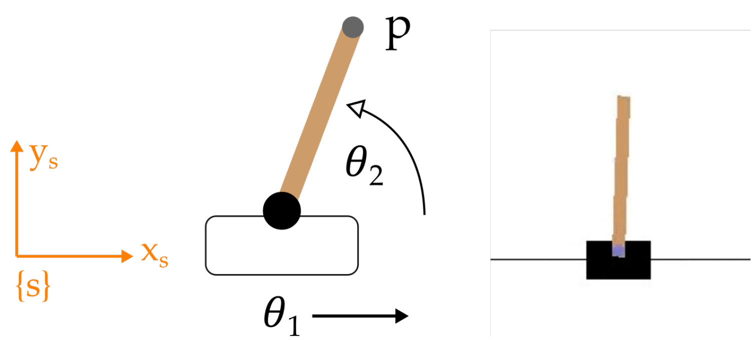
Instructions. Please write legibly and do not attempt to fit your work into the smallest space possible. It is important to show all work, but basic arithmetic can be omitted. You are encouraged to use Matlab when possible to avoid hand calculations, but print and submit your commented code for non-trivial calculations. You can attach a pdf of your code to the homework, use live scripts or the publish feature in Matlab, or include a snapshot of your code. Do not submit .m files — we will not open or grade these files.

1 Properties of Jacobians

1.1 (5 points)



The robot shown above has two joints: a cart, which can slide left and right along a track, and a pole (i.e., an inverted pendulum), which this cart is trying to balance. The pendulum has length L, and the vertical distance from $\{s\}$ to the base of the inverted pendulum is h. Solve for the Jacobian J such that $\dot{p} = J(\theta)\dot{\theta}$. Here p is the position of the end of the pole.

The position of p with respect to frame $\{s\}$ is:

$$p_x = \theta_1 + L\cos(\theta_2) \tag{1}$$

$$p_y = h + L\sin(\theta_2) \tag{2}$$

Taking the derivative of these kinematic equations with respect to time:

$$\dot{p}_x = \dot{\theta}_1 - L\dot{\theta}_2\sin(\theta_2) \tag{3}$$

$$p_y = L\dot{\theta}_2\cos(\theta_2) \tag{4}$$

Now we just need to reorganize the terms into the form $\dot{p} = J(\theta)\dot{\theta}$:

$$\dot{p} = \begin{bmatrix} 1 & -L\sin(\theta_2) \\ 0 & L\cos(\theta_2) \end{bmatrix} \dot{\theta} \tag{5}$$

The matrix above is the Geometric Jacobian J for this cart-pole robot.

1.2 (5 points)

Consider a serial robot arm with three joints. Starting with the spatial twist $[V_s] = \dot{T}T^{-1}$, derive the spatial Jacobian:

$$V_s = J_s(\theta)\dot{\theta}, \quad J_s(\theta) = \begin{bmatrix} S_1 & \mathrm{Ad}_{e^{[S_1]\theta_1}} S_2 & \mathrm{Ad}_{e^{[S_1]\theta_1} e^{[S_2]\theta_2}} S_3 \end{bmatrix} \tag{6}$$

Hint. Let T be a transformation and S be a screw. By definition, $T[S]T^{-1} = [\mathrm{Ad}_T S]$.

The forward kinematics for a robot arm with three joints are:

$$T = e^{[S_1]\theta_1} e^{[S_2]\theta_2} e^{[S_3]\theta_3} M \tag{7}$$

We want to plug this into $[V_s] = \dot{T}T^{-1}$. To do so, we first find $\dot{T}$ and T^{-1}. Remember that taking the inverse reverses the order of the matrices. When taking the time derivative, apply the chain rule:

$$T^{-1} = M^{-1} e^{-[S_3]\theta_3} e^{-[S_2]\theta_2} e^{-[S_1]\theta_1} \tag{8}$$

$$\dot{T} = [S_1]\dot{\theta}_1 e^{[S_1]\theta_1} e^{[S_2]\theta_2} e^{[S_3]\theta_3} M + e^{[S_1]\theta_1} [S_2]\dot{\theta}_2 e^{[S_2]\theta_2} e^{[S_3]\theta_3} M + e^{[S_1]\theta_1} e^{[S_2]\theta_2} [S_3]\dot{\theta}_3 e^{[S_3]\theta_3} M \tag{9}$$

Now substitute both of these equations into $[V_s] = \dot{T}T^{-1}$:

$$\begin{aligned}[V_s] = [S_1]\dot{\theta}_1 e^{[S_1]\theta_1} e^{[S_2]\theta_2} e^{[S_3]\theta_3} M M^{-1} e^{-[S_3]\theta_3} e^{-[S_2]\theta_2} e^{-[S_1]\theta_1} + \\ e^{[S_1]\theta_1} [S_2]\dot{\theta}_2 e^{[S_2]\theta_2} e^{[S_3]\theta_3} M M^{-1} e^{-[S_3]\theta_3} e^{-[S_2]\theta_2} e^{-[S_1]\theta_1} + \\ e^{[S_1]\theta_1} e^{[S_2]\theta_2} [S_3]\dot{\theta}_3 e^{[S_3]\theta_3} M M^{-1} e^{-[S_3]\theta_3} e^{-[S_2]\theta_2} e^{-[S_1]\theta_1}\end{aligned} \tag{10}$$

Several terms in the above equation cancel out. Remember that $\dot{\theta}_i$ is a scalar, and so we can move it around as needed:

$$[V_s] = [S_1]\dot{\theta}_1 + e^{[S_1]\theta_1}[S_2]e^{-[S_1]\theta_1}\dot{\theta}_2 + e^{[S_1]\theta_1} e^{[S_2]\theta_2} [S_3] e^{-[S_2]\theta_2} e^{-[S_1]\theta_1}\dot{\theta}_3 \tag{11}$$

Use the given hint to rewrite this equation in term of adjoints:

$$V_s = S_1\dot{\theta}_1 + \mathrm{Ad}_{e^{[S_1]\theta_1}} S_2\dot{\theta}_2 + \mathrm{Ad}_{e^{[S_1]\theta_1} e^{[S_2]\theta_2}} S_3\dot{\theta}_3 \tag{12}$$

Our final step is to pull out the $\dot{\theta}$ terms to get $V_s = J_s(\theta)\dot{\theta}$:

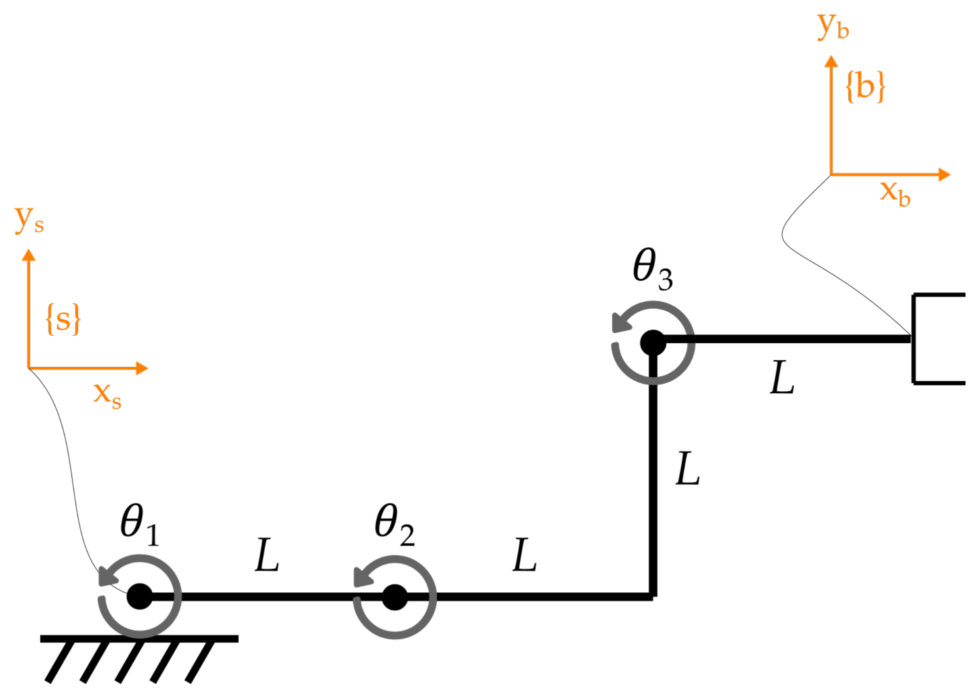
$$V_s = \begin{bmatrix} S_1 & \mathrm{Ad}_{e^{[S_1]\theta_1}} S_2 & \mathrm{Ad}_{e^{[S_1]\theta_1} e^{[S_2]\theta_2}} S_3 \end{bmatrix} \dot{\theta} \tag{13}$$

1.3 (5 points)

Let J be a Jacobian (either space, body, or geometric) such that $V = J(\theta)\dot{\theta}$. Here V is a six-dimensional vector and the serial robot arm has n joints. Under what conditions is J invertible, i.e., when can you find J^{-1}?

To invert a matrix A we must have that (i) A is a square matrix and (ii) A is full rank. This implies the following two conditions that we must have for J to be invertible:

- The robot must have $n = 6$ joints so that $J \in \mathbb{R}^{6\times 6}$ is square.
- The Jacobian must have rank 6 (we cannot be at a singular configuration).



2 Jacobian: Planar Robot

In this problem you will get the Jacobian for the planar robot shown below. Your answers should be in terms of the variables L, θ_1, θ_2, and θ_3.

2.1 (5 points)

Find the forward kinematics for the planar robot. Write out the home position M, the screw axes S_1, S_2, S_3, and the transformation matrix $T(\theta)$.

Start with the transformation matrix M. We find this by writing the orientation and position of frame $\{b\}$ relative to the fixed frame $\{s\}$:

$$M = \begin{bmatrix} 1 & 0 & 0 & 3L \\ 0 & 1 & 0 & L \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix} \tag{14}$$

Next find the screw axes. All three revolute joints are rotating around the z-axis:

$$\omega_1 = \omega_2 = \omega_3 = [0,0,1]^T, \quad q_1 = [0,0,0]^T, \quad q_2 = [L,0,0]^T, \quad q_3 = [2L,L,0]^T \tag{15}$$

Now we can use our twist equations for revolute joints to get each screw axes:

$$S_1 = [0,0,1,0,0,0]^T, \quad S_2 = [0,0,1,0,-L,0]^T, \quad S_3 = [0,0,1,L,-2L,0]^T \tag{16}$$

Applying the product of exponentials formula, we obtain the forward kinematics:

$$T(\theta) = \begin{bmatrix} c_{123} & -s_{123} & 0 & Lc_{123} + Lc_1 + L\sqrt{2}\cos(\theta_1+\theta_2+\pi/4) \\ s_{123} & c_{123} & 0 & Ls_{123} + Ls_1 + L\sqrt{2}\sin(\theta_1+\theta_2+\pi/4) \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix} \tag{17}$$

where $c_{123} = \cos(\theta_1+\theta_2+\theta_3)$ and $s_{123} = \sin(\theta_1+\theta_2+\theta_3)$ as shorthand.

2.2 (5 points)

Find the Space Jacobian $J_s(\theta)$.

```
function Js = JacobianSpace(S, theta)

    T = eye(4);
    for i = 1:length(theta)
        Si = S(:, i);
        Js(:, i) = Adjoint(T) * Si;
        T = T * expm(bracket(Si) * theta(i));
    end

end
```

This is the perfect opportunity to use our `JacobianSpace` function.

```
Js = simplify(JacobianSpace([S1, S2, S3], theta));
```

$$J_s(\theta) = \begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 1 & 1 & 1 \\ 0 & L\sin(\theta_1) & L\sin(\theta_1) + L\sqrt{2}\sin(\theta_1 + \theta_2 + \pi/4) \\ 0 & -L\cos(\theta_1) & -L\cos(\theta_1) - L\sqrt{2}\cos(\theta_1 + \theta_2 + \pi/4) \\ 0 & 0 & 0 \end{bmatrix} \tag{18}$$

2.3 (5 points)

Find the Body Jacobian $J_b(\theta)$.

We convert between Space and Body Jacobians using the adjoint operator:

$$J_b = \mathrm{Ad}_{T(\theta)^{-1}} J_s \tag{19}$$

Remember to leave $T(\theta)$ as a function of θ.

```
Jb = simplify(Adjoint(inv(T)) * Js);
```

$$J_b(\theta) = \begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 1 & 1 & 1 \\ L\sin(\theta_2 + \theta_3) - L\sqrt{2}\cos(\theta_3 + \pi/4) & -L\sqrt{2}\cos(\theta_3 + \pi/4) & 0 \\ L\cos(\theta_2 + \theta_3) + L\sqrt{2}\sin(\theta_3 + \pi/4) + L & L\sqrt{2}\sin(\theta_3 + \pi/4) + L & L \\ 0 & 0 & 0 \end{bmatrix} \tag{20}$$

2.4 (5 points)

Find the Geometric Jacobian $J(\theta)$.

We convert between Body and Geometric Jacobians using the rotation from $T(\theta)$:

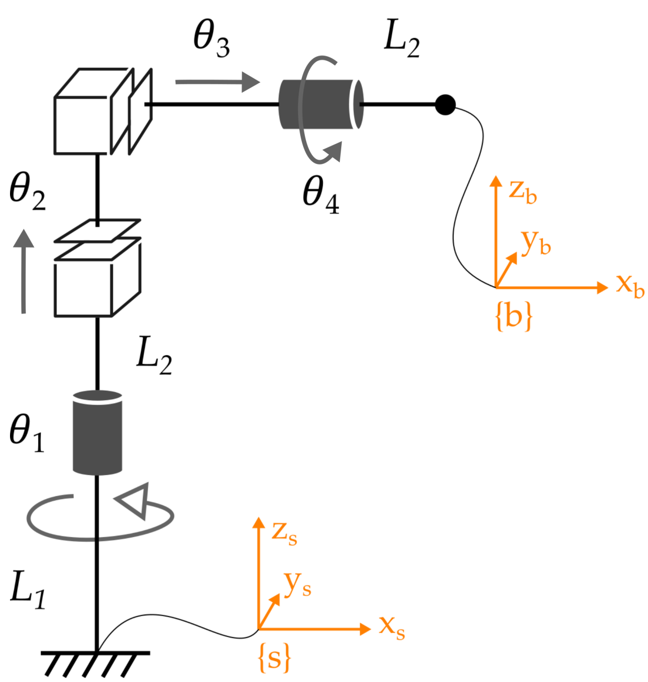
$$J = \begin{bmatrix} R & 0 \\ 0 & R \end{bmatrix} J_b, \quad T(\theta) = \begin{bmatrix} R & p \\ 0 & 1 \end{bmatrix} \tag{21}$$

```
J = simplify([T(1:3,1:3), zeros(3,3); zeros(3,3), T(1:3,1:3)] * Jb);
```

$$J(\theta) = \begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 1 & 1 & 1 \\ -Ls_{123} - Ls_1 - L\sqrt{2}\sin(\theta_1 + \theta_2 + \frac{\pi}{4}) & -Ls_{123} - L\sqrt{2}\sin(\theta_1 + \theta_2 + \frac{\pi}{4}) & -Ls_{123} \\ Lc_{123} + Lc_1 + L\sqrt{2}\cos(\theta_1 + \theta_2 + \frac{\pi}{4}) & Lc_{123} + L\sqrt{2}\cos(\theta_1 + \theta_2 + \frac{\pi}{4}) & Lc_{123} \\ 0 & 0 & 0 \end{bmatrix}$$

To fit this answer cleanly on the page, I am using our $\sin(\cdot)$ and $\cos(\cdot)$ shorthand: $c_{123} = \cos(\theta_1 + \theta_2 + \theta_3)$, $s_{123} = \sin(\theta_1 + \theta_2 + \theta_3)$, and so on.

3 Jacobian: $3D$ Robot



In this problem you will get the Jacobian for the robot moving in $3D$ space shown below. Your answers should be in terms of the variables L_1, L_2, θ_1, θ_2, θ_3, and θ_4.

3.1 (5 points)

Find the forward kinematics for the $3D$ robot. Write out the home position M, the screw axes S_1, S_2, S_3, S_4, and the transformation matrix $T(\theta)$.

Start with the transformation matrix M:

$$M = \begin{bmatrix} 1 & 0 & 0 & L_2 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & L_1 + L_2 \\ 0 & 0 & 0 & 1 \end{bmatrix} \tag{22}$$

Next find the four screw axes for the two prismatic joints and two revolute joints:

$$S_1 = [0, 0, 1, 0, 0, 0]^T \tag{23}$$
$$S_2 = [0, 0, 0, 0, 0, 1]^T \tag{24}$$
$$S_3 = [0, 0, 0, 1, 0, 0]^T \tag{25}$$
$$S_4 = [1, 0, 0, 0, L_1 + L_2, 0]^T \tag{26}$$

Applying the product of exponentials formula:

$$T(\theta) = \begin{bmatrix} c_1 & -c_4 s_1 & s_1 s_4 & (L_2 + \theta_3)c_1 \\ s_1 & c_1 c_4 & -c_1 s_4 & (L_2 + \theta_3)s_1 \\ 0 & s_4 & c_4 & L_1 + L_2 + \theta_2 \\ 0 & 0 & 0 & 1 \end{bmatrix} \tag{27}$$

where $c_i = \cos(\theta_i)$ and $s_i = \sin(\theta_i)$ as shorthand.

3.2 (5 points)

Find the Space Jacobian $J_s(\theta)$.

This is the another opportunity to use our `JacobianSpace` function.

```
Js = simplify(JacobianSpace([S1, S2, S3, S4], theta));
```

$$J_s(\theta) = \begin{bmatrix} 0 & 0 & 0 & c_1 \\ 0 & 0 & 0 & s_1 \\ 1 & 0 & 0 & 0 \\ 0 & 0 & c_1 & -(L_1 + L_2 + \theta_2)s_1 \\ 0 & 0 & s_1 & (L_1 + L_2 + \theta_2)c_1 \\ 0 & 1 & 0 & 0 \end{bmatrix} \tag{28}$$

Here s_i is short for $\sin(\theta_i)$, etc.

3.3 (5 points)

Find the Body Jacobian $J_b(\theta)$.

We convert between Space and Body Jacobians using the adjoint operator:

$$J_b = \mathrm{Ad}_{T(\theta)^{-1}} J_s \tag{29}$$

Remember to leave $T(\theta)$ as a function of θ.

```
Jb = simplify(Adjoint(inv(T)) * Js);
```

$$J_b(\theta) = \begin{bmatrix} 0 & 0 & 0 & 1 \\ s_4 & 0 & 0 & 0 \\ c_4 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ (L_2+\theta_3)c_4 & s_4 & 0 & 0 \\ -(L_2+\theta_3)s_4 & c_4 & 0 & 0 \end{bmatrix} \tag{30}$$

Here s_i is short for $\sin(\theta_i)$, etc.

3.4 (5 points)

Find the Geometric Jacobian $J(\theta)$.

We convert between Body and Geometric Jacobians using the rotation from $T(\theta)$:

$$J = \begin{bmatrix} R & 0 \\ 0 & R \end{bmatrix} J_b, \quad T(\theta) = \begin{bmatrix} R & p \\ 0 & 1 \end{bmatrix} \tag{31}$$

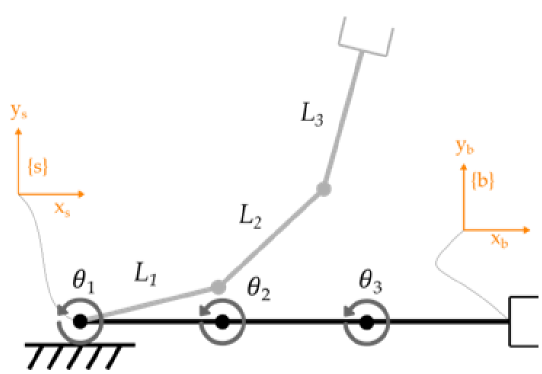
As before, remember to leave R as a function of θ.

```
J = simplify([T(1:3,1:3), zeros(3,3); zeros(3,3), T(1:3,1:3)] * Jb);
```

$$J(\theta) = \begin{bmatrix} 0 & 0 & 0 & c_1 \\ 0 & 0 & 0 & s_1 \\ 1 & 0 & 0 & 0 \\ -(L_2+\theta_3)s_1 & 0 & c_1 & 0 \\ (L_2+\theta_3)c_1 & 0 & s_1 & 0 \\ 0 & 1 & 0 & 0 \end{bmatrix} \tag{32}$$

Here s_i is short for $\sin(\theta_i)$, etc.

4 Interpreting the Jacobian



This problem deals with the planar robot shown above.

4.1 (10 points)

Imagine that you can only actuate one joint right now. In other words, either $\dot{\theta}_1 = 1$ rad/s, or $\dot{\theta}_2 = 1$ rad/s, or $\dot{\theta}_3 = 1$ rad/s. Let $p = p_{sb}$ be the position of the robot's end-effector, and assume that $L_1 = L_2 = L_3$.

- Which joint should you actuate to maximize $|\dot{p}_x|$ if $\theta = [-\pi/8, \pi/4, \pi/8]^T$?
- Which joint should you actuate to maximize $|\dot{p}_x|$ if $\theta = [3\pi/4, -\pi/4, 0]^T$?
- Which joint should you actuate to maximize $|\dot{p}_y|$ if $\theta = [\pi/2, -\pi/8, -\pi/2]^T$?

Start with the geometric Jacobian. We previously found this in lecture:

$$J = \begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 1 & 1 & 1 \\ -L_3 s_{123} - L_2 s_{12} - L_1 s_1 & -L_3 s_{123} - L_2 s_{12} & -L_3 s_{123} \\ L_3 c_{123} + L_2 c_{12} + L_1 c_1 & L_3 c_{123} + L_2 c_{12} & L_3 c_{123} \\ 0 & 0 & 0 \end{bmatrix} \tag{33}$$

We next need to determine which rows of this Jacobian correspond to $\dot{p}_x$ and $\dot{p}_y$

$$\begin{bmatrix} \omega_x \\ \omega_y \\ \omega_z \\ \dot{p}_x \\ \dot{p}_y \\ \dot{p}_z \end{bmatrix} = \begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 1 & 1 & 1 \\ -L_3 s_{123} - L_2 s_{12} - L_1 s_1 & -L_3 s_{123} - L_2 s_{12} & -L_3 s_{123} \\ L_3 c_{123} + L_2 c_{12} + L_1 c_1 & L_3 c_{123} + L_2 c_{12} & L_3 c_{123} \\ 0 & 0 & 0 \end{bmatrix} \begin{bmatrix} \dot{\theta}_1 \\ \dot{\theta}_2 \\ \dot{\theta}_3 \end{bmatrix} \tag{34}$$

Looking at the Jacobian relationship, we want to focus on the **fourth** row for $\dot{p}_x$ and the **fifth** row for $\dot{p}_y$. Expanding, we have the following two relationships:

$$\dot{p}_x = (-L_3 s_{123} - L_2 s_{12} - L_1 s_1)\dot{\theta}_1 + (-L_3 s_{123} - L_2 s_{12})\dot{\theta}_2 + (-L_3 s_{123})\dot{\theta}_3 \tag{35}$$

$$\dot{p}_y = (L_3 c_{123} + L_2 c_{12} + L_1 c_1)\dot{\theta}_1 + (L_3 c_{123} + L_2 c_{12})\dot{\theta}_2 + (L_3 c_{123})\dot{\theta}_3 \tag{36}$$

Since we are given that $L_1 = L_2 = L_3$, each term is scaled by the same constant. We can drop this constant without changing the relative effects:

$$\dot{p}_x \propto (-s_{123} - s_{12} - s_1)\dot{\theta}_1 + (-s_{123} - s_{12})\dot{\theta}_2 + (-s_{123})\dot{\theta}_3 \tag{37}$$

$$\dot{p}_y \propto (c_{123} + c_{12} + c_1)\dot{\theta}_1 + (c_{123} + c_{12})\dot{\theta}_2 + (c_{123})\dot{\theta}_3 \tag{38}$$

We will use these equations to answer the following questions.

Which joint should you actuate to maximize $|\dot{p}_x|$ if $\theta = [-\pi/8, \pi/4, \pi/8]^T$?

Plugging the joint values into the Jacobian, we get:

$$\dot{p}_x \propto -0.707\dot{\theta}_1 - 1.09\dot{\theta}_2 - 0.707\dot{\theta}_3 \tag{39}$$

In this configuration actuating the **second joint** will have the largest impact on the end-effector's velocity along the x_s axis.

Which joint should you actuate to maximize $|\dot{p}_x|$ if $\theta = [3\pi/4, -\pi/4, 0]^T$?

Plugging the joint values into the Jacobian, we get:

$$\dot{p}_x \propto -2.707\dot{\theta}_1 - 2\dot{\theta}_2 - 1\dot{\theta}_3 \tag{40}$$

In this configuration actuating the **first joint** will have the largest impact on the end-effector's velocity along the x_s axis.

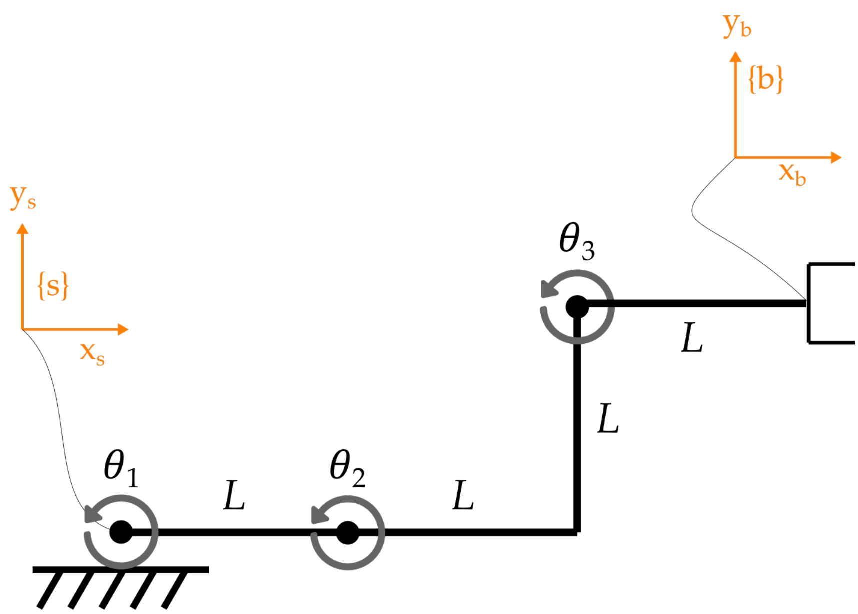
Which joint should you actuate to maximize $|\dot{p}_y|$ if $\theta = [\pi/2, -\pi/8, -\pi/2]^T$?

Plugging the joint values into the Jacobian, we get:

$$\dot{p}_y \propto 1.31\dot{\theta}_1 + 1.31\dot{\theta}_2 + 0.92\dot{\theta}_3 \tag{41}$$

In this configuration the **first joint** and the **second joint** are tied. Both will have the same impact on the end-effector's velocity along the y_s axis.

5 Singularities: Planar Robot



In this problem you will explore singularities and manipulability of the planar robot shown above. You previously found the Jacobian for this robot. The task space of this robot is the (x, y) position of the end-effector and the angle of the end-effector around the z axis.

5.1 (5 points)

Identify the joint position(s) where the robot is at a singular configuration.

Singularities are a physical property of the system, and do not depend on the type of Jacobian (space, body, or geometric). Start with a Jacobian of your choice:

$$J_s(\theta) = \begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 1 & 1 & 1 \\ 0 & L\sin(\theta_1) & L\sin(\theta_1) + L\sqrt{2}\sin(\theta_1+\theta_2+\pi/4) \\ 0 & -L\cos(\theta_1) & -L\cos(\theta_1) - L\sqrt{2}\cos(\theta_1+\theta_2+\pi/4) \\ 0 & 0 & 0 \end{bmatrix} \tag{42}$$

Isolate the rows that correspond to the task space (here planar motion):

$$J_s(\theta) = \begin{bmatrix} 1 & 1 & 1 \\ 0 & L\sin(\theta_1) & L\sin(\theta_1) + L\sqrt{2}\sin(\theta_1+\theta_2+\pi/4) \\ 0 & -L\cos(\theta_1) & -L\cos(\theta_1) - L\sqrt{2}\cos(\theta_1+\theta_2+\pi/4) \end{bmatrix} \tag{43}$$

Singularities occur at joint positions where $\det(J) = 0$. You can also use $\det(JJ^T) = 0$, and will get the same answer. Solving this in Matlab:

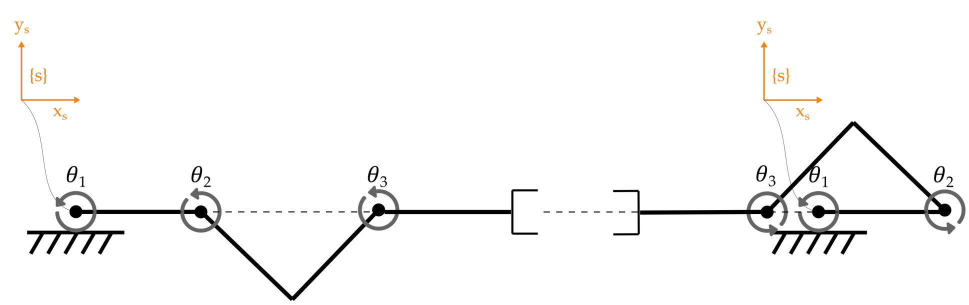
$$\det(J) = 0 \quad \text{if } \theta_2 = 0.5\sin^{-1}(-1) \tag{44}$$

The robot is at a singular configuration when:

$$\theta_2 = -\frac{\pi}{4} \pm 2\pi i \quad \text{or} \quad \theta_2 = \frac{3\pi}{4} \pm 2\pi i \tag{45}$$

5.2 (5 points)

Draw the robot in two different singular configurations.



See the sketch above. In the left drawing $\theta_2 = -\frac{\pi}{4}$ and in the right drawing $\theta_2 = \frac{3\pi}{4}$. In both drawings the robot cannot instantaneously move along the x axis.

5.3 (5 points)

Let $\theta = [\pi/4, -\pi/4, \pi/4]^T$.

- Is this a singular configuration?
- In what direction(s) can the robot move?
- In what direction(s) can the robot not move?

Start with a Jacobian type of your choice. Here I use the space Jacobian:

$$J_s(\theta) = \begin{bmatrix} 1 & 1 & 1 \\ 0 & L\sin(\theta_1) & L\sin(\theta_1) + L\sqrt{2}\sin(\theta_1 + \theta_2 + \pi/4) \\ 0 & -L\cos(\theta_1) & -L\cos(\theta_1) - L\sqrt{2}\cos(\theta_1 + \theta_2 + \pi/4) \end{bmatrix} \tag{46}$$

Plugging in the given $\theta = [\pi/4, -\pi/4, \pi/4]^T$, this becomes:

$$J_s = \begin{bmatrix} 1 & 1 & 1 \\ 0 & \sqrt{2}L/2 & L(\sqrt{2}+2)/2 \\ 0 & -\sqrt{2}L/2 & -L(\sqrt{2}+2)/2 \end{bmatrix} \tag{47}$$

The maximum rank of J_s is three (matching the dimension of the task space). We will use this Jacobian to answer the following questions.

Is this a singular configuration?

`rank(Js) = 2`. Hence, the given value of θ **is a singular configuration** because the Jacobian drops rank at this joint position. You can also answer this problem by taking the determinant of the Jacobian and showing that it is zero at the given θ.

In what direction(s) can the robot move?

The robot can achieve velocities in range(Js). Here `colspace(Js)` outputs:

$$\begin{bmatrix} 1 & 0 \\ 0 & \sqrt{2}/2 \\ 0 & -\sqrt{2}/2 \end{bmatrix} \tag{48}$$

The robot can **rotate around the z axis** (column 1) and **translate in the direction** $[\sqrt{2}/2, -\sqrt{2}/2]^T$ (column 2). This translation vector corresponds to a line between the $+x$ and $-y$ axes at $-45°$. Your answer should be two column vectors that span rotation around z and translation along a line at $-45°$. The exact values of those vectors do not need to match the ones listed here — for instance, the second vector could equally be $[0, -\sqrt{2}/2, \sqrt{2}/2]^T$.

In what direction(s) can the robot not move?

The robot cannot move in null(Js^T). Here `null(Js')` outputs: $[0, 1, 1]^T$. Hence, the robot **cannot move along a line between the x and y axes at** $45°$. The direction the robot cannot move is orthogonal to the directions the robot can translate.

5.4 (5 points)

Find the joint position(s) that maximize the robot's manipulability.

We want to find joint position(s) that maximize $\sqrt{\det(JJ^T)}$. Starting with the space Jacobian from the previous part:

$$J_s(\theta) = \begin{bmatrix} 1 & 1 & 1 \\ 0 & L\sin(\theta_1) & L\sin(\theta_1) + L\sqrt{2}\sin(\theta_1 + \theta_2 + \pi/4) \\ 0 & -L\cos(\theta_1) & -L\cos(\theta_1) - L\sqrt{2}\cos(\theta_1 + \theta_2 + \pi/4) \end{bmatrix} \tag{49}$$

Evaluating the manipulability metric we reach:

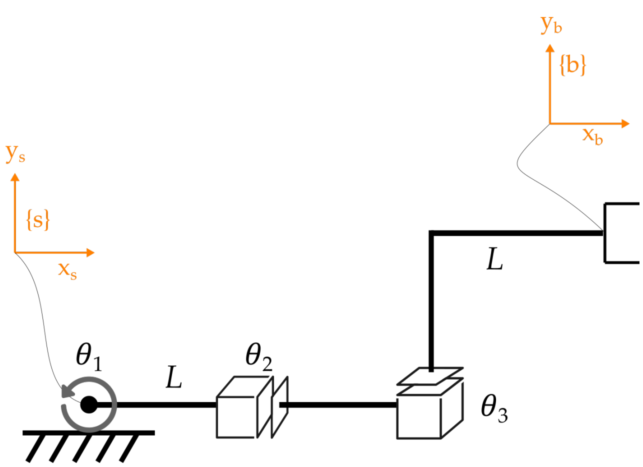
$$\sqrt{\det(JJ^T)} = L^2\sqrt{\sin(2\theta_2) + 1} \tag{50}$$

This term is maximized when we choose θ_2 to maximize the value of $\sin(2\theta_2) + 1$. This occurs when $\theta_2 = \pi/4 + \pi i$. Accordingly, the joint positions that maximize the robot's manipulability are:

$$\theta_2 = \pi/4 + \pi i \tag{51}$$

5.5 (5 points)

Modify the planar robot's design to remove **all singular configuration(s)**. You are allowed to: add joints, change the geometry of the links, and/or change the type of joints. Prove that $\det(JJ^T) \neq 0$ for all θ with your modified robot.



One solution is switching the last two revolute joints into **prismatic joints** as shown above. The first prismatic joint is pointed along the x_s axis, and the second prismatic joint is pointed along the y_s axis. To confirm that the resulting robot does not have any singularities, you must find J and then check $\det(JJ^T)$.

First let's get the screw axes S_1, S_2, S_3 for the new robot arm:

$$S_1 = [0, 0, 1, 0, 0, 0]^T \tag{52}$$

$$S_2 = [0, 0, 0, 1, 0, 0]^T, \quad S_3 = [0, 0, 0, 0, 1, 0]^T \tag{53}$$

Now use the `SpaceJacobian` function to get J_s. Specifically looking at three rows that contributes to planar motion (ω_z, $\dot{p}_x$, and $\dot{p}_y$), we get:

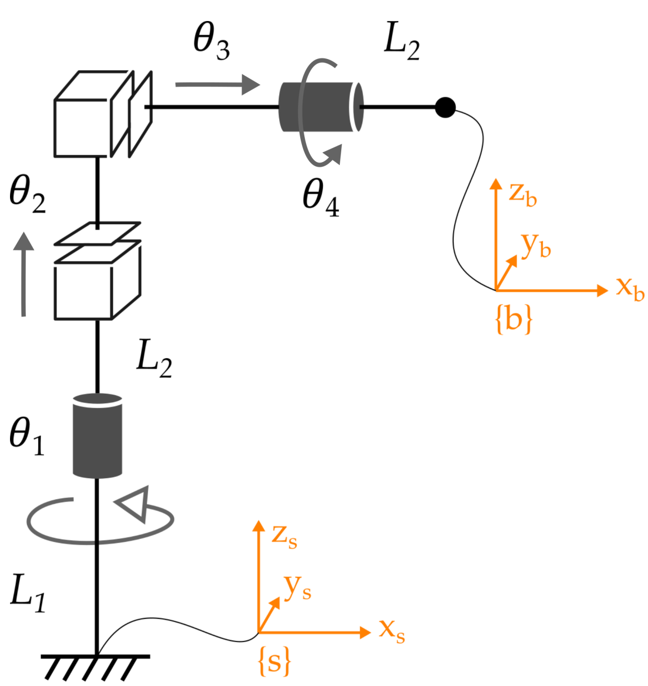
$$J_s = \begin{bmatrix} 1 & 0 & 0 \\ 0 & \cos(\theta_1) & -\sin(\theta_1) \\ 0 & \sin(\theta_1) & \cos(\theta_1) \end{bmatrix} \tag{54}$$

Our last step is checking whether this robot has any singular configurations. To have no singularities, we need to ensure that $\det(JJ^T) \neq 0$, regardless of θ_1, θ_2, and θ_3:

$$JJ^T = \begin{bmatrix} 1 & 0 & 0 \\ 0 & \cos(\theta_1)^2 + \sin(\theta_1)^2 & 0 \\ 0 & 0 & \cos(\theta_1)^2 + \sin(\theta_1)^2 \end{bmatrix} = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} \tag{55}$$

Hence, the determinant $\det(JJ^T) = 1$ for any value of θ. Our modification therefore removed all singular configuration from the original robot.

6 Singularities: $3D$ Robot



In this problem you will explore singularities and manipulability of the planar robot shown above. You previously found the Jacobian for this robot.

6.1 (5 points)

Identify the joint position(s) where the robot is at a singular configuration. Alternatively, prove that none exist. **Hint:** A singularity occurs when the rank of this robot's Jacobian is less than its maximum rank.

Start with the robot's Jacobian matrix. I picked the geometric Jacobian:

$$J(\theta) = \begin{bmatrix} 0 & 0 & 0 & c_1 \\ 0 & 0 & 0 & s_1 \\ 1 & 0 & 0 & 0 \\ -(L_2 + \theta_3)s_1 & 0 & c_1 & 0 \\ (L_2 + \theta_3)c_1 & 0 & s_1 & 0 \\ 0 & 1 & 0 & 0 \end{bmatrix} \tag{56}$$

The maximum rank of this Jacobian is 4 (you can verify this in Matlab). Singularities occur when rank(J) < 4.

Let's prove that J never drops rank (i.e., the robot does not have any singularities). To do this, we need to show that the columns of J are linearly independent for all values of θ. By definition, a set of vectors $\{v_1, v_2, \ldots, v_n\}$ is linearly independent if the coefficients $\alpha_1, \alpha_2, \ldots, \alpha_n$ must all be zero in order for $\alpha_1 v_1 + \alpha_2 v_2 + \ldots + \alpha_n v_n$ to equal zero. Applying this definition:

$$\alpha_1 \begin{bmatrix} 0 \\ 0 \\ 1 \\ -(L_2 + \theta_3)s_1 \\ (L_2 + \theta_3)c_1 \\ 0 \end{bmatrix} + \alpha_2 \begin{bmatrix} 0 \\ 0 \\ 0 \\ 0 \\ 0 \\ 1 \end{bmatrix} + \alpha_3 \begin{bmatrix} 0 \\ 0 \\ 0 \\ c_1 \\ s_1 \\ 0 \end{bmatrix} + \alpha_4 \begin{bmatrix} c_1 \\ s_1 \\ 0 \\ 0 \\ 0 \\ 0 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \\ 0 \\ 0 \\ 0 \end{bmatrix} \tag{57}$$

This gives the following system of equations:

$$\alpha_4 c_1 = 0, \quad \alpha_4 s_1 = 0, \quad \alpha_1 = 0 \tag{58}$$

$$-\alpha_1(L_2 + \theta_3)s_1 + \alpha_3 c_1 = 0, \quad \alpha_1(L_2 + \theta_3)c_1 + \alpha_3 s_1 = 0, \quad \alpha_2 = 0 \tag{59}$$

Solving this system of equations: $\alpha_1 = \alpha_2 = \alpha_3 = \alpha_4 = 0$. Hence, we conclude that the columns of J are linearly independent for all values of θ, and thus the Jacobian never drops rank and **the robot does not have any singularities**.

6.2 (5 points)

Find the joint position(s) that maximize the robot's manipulability. **Hint:** There are multiple metrics for manipulability. For this problem treat manipulability as the robot's overall movement speed. In other words, at what joint positions can we maximize $V^T V$?

The equation for manipulability we covered in lecture applies to robots that have at least as many degrees of freedom as their task space. But that is not the case for this robot: it is moving in a $3D$ space, but only has four degrees of freedom.

The hint tells us we want $V^T V$ to be as large as possible. Substitute in the Jacobian $V = J\dot{\theta}$ so that: $V^T V = \dot{\theta}^T J^T J \dot{\theta}$. Then explore $J^T J$:

$$J^T J = \begin{bmatrix} L_2^2 + 2L_2\theta_3 + \theta_3^2 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix} \tag{60}$$

Expanding, we have that $V^T V$ (i.e., the squared magnitude of V) is equal to:

$$V^T V = (L_2^2 + 2L_2\theta_3 + \theta_3^2)\dot{\theta}_1^2 + \dot{\theta}_2^2 + \dot{\theta}_3^2 + \dot{\theta}_4^2 \tag{61}$$

We can therefore maximize the end-effector twist by **maximizing the length of** θ_3. The joint positions that maximize the robot's manipulability are: $\theta_3 \longrightarrow \pm\infty$

Intuitively, increasing the length of the third joint increases the "moment arm," so that turning joint one causes the end-effector to move faster.