Problem Set 6

Robotics & Automation
Dylan Losey, Virginia Tech

Instructions. Please write legibly and do not attempt to fit your work into the smallest space possible. It is important to show all work, but basic arithmetic can be omitted. You are encouraged to use Matlab when possible to avoid hand calculations, but print and submit your commented code for non-trivial calculations. You can attach a pdf of your code to the homework, use live scripts or the publish feature in Matlab, or include a snapshot of your code. Do not submit .m files — we will not open or grade these files.

For this assignment we are asking you to also submit **videos** of your simulations. Follow the instructions to **label** these videos based on the problem number, and then submit them all within a **single zipped folder**.

1 Wrenches

1.1 (10 points)

Imagine an arbitrary wrench F and coordinate frames $\{s\}$ and $\{b\}$. If $f_s \neq 0$ and $m_s \neq 0$, under what conditions is $\|F_b\| = \|F_s\|$? This question does not refer to the figure below.

Start with the equations that we know:

$$F_b = \begin{bmatrix} f_b \\ m_b \end{bmatrix} \qquad F_s = \begin{bmatrix} f_s \\ m_s \end{bmatrix} \qquad F_b = (\mathrm{Ad}_{T_{sb}})^T F_s \tag{1}$$

Expanding these terms, we reach the following relationships:

$$f_b = R_{sb}^T f_s \qquad m_b = R_{sb}^T [p_{sb}]^T f_s + R_{sb}^T m_s \tag{2}$$

We will separately consider the **force** and the **moment**.

For **force**, we have that $f_b = R_{sb}^T f_s$. But rotation matrices preserve magnitude, so we always have $\|f_b\| = \|f_s\|$. No special conditions are needed here.

For **moment**, we have that $m_b = R^T[p]^T f_s + R^T m_s$. Remembering that rotation matrices preserve magnitude, in order for $\|m_b\| = \|m_s\|$ we must have that $R^T[p]^T f_s = 0$. Recalling that the problem states $f_s \neq 0$, this occurs when either:

- Frame $\{s\}$ and $\{b\}$ share the same origin, so that $p = p_{sb} = 0$
- Vectors p and f_s are parallel, so that the cross product $[p]f_s = 0$

The conditions for $\|F_b\| = \|F_s\|$ are either (a) both coordinate frames are located at the same point or (b) the force is parallel to the vector from $\{s\}$ to $\{b\}$.

Addendum. Choosing p so that $[p]^T f_s = -2m_s$ is also a valid answer. Students are not expected to find this answer, but it is correct if provided.

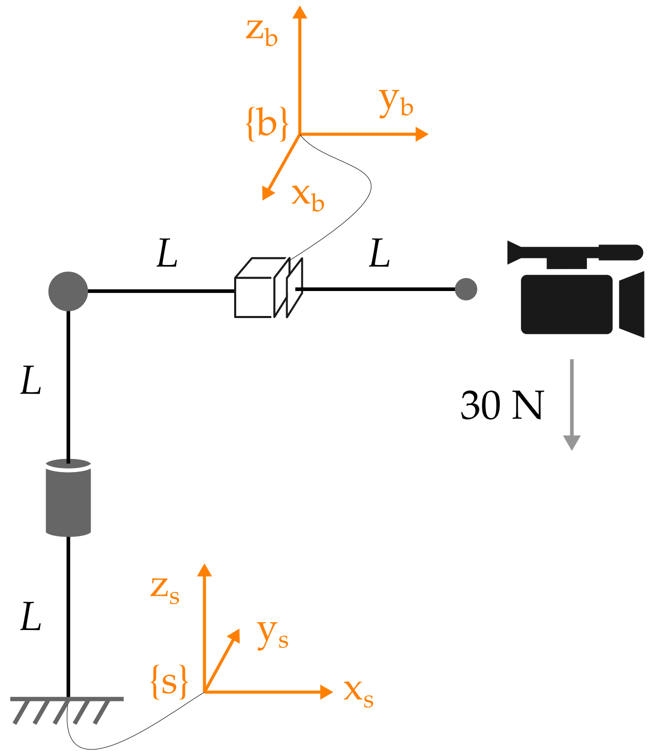


Figure 1: A robot arm holding a camera that weights 30 N.

1.2 (5 points)

Consider the robot in Figure 1. Find the wrench applied by the camera in frame $\{b\}$.

By definition, we know that the body wrench is:

$$F_b = \begin{bmatrix} m_b \\ f_b \end{bmatrix} = \begin{bmatrix} r_b \times f_b \\ f_b \end{bmatrix} \tag{3}$$

Looking at the drawing, we have that f_b is a 30 N force along the negative z_b axis, and $r_b = [0, L, 0]^T$. Plugging in and solving for F_b:

$$F_b = \begin{bmatrix} -30L \\ 0 \\ 0 \\ 0 \\ 0 \\ -30 \end{bmatrix} \tag{4}$$

1.3 (5 points)

Consider the robot in Figure 1. Find the wrench applied by the camera in frame $\{s\}$.

One way to solve this is by converting F_b to F_s. This approach leverages the adjoint operator to change the frame of reference for the wrench.

$$F_s = (\mathrm{Ad}_{T_{bs}})^T F_b \tag{5}$$

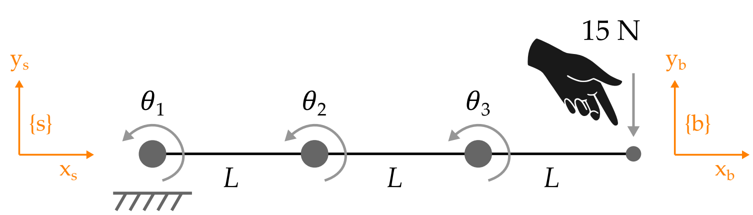
Use the drawing to find T_{sb}:

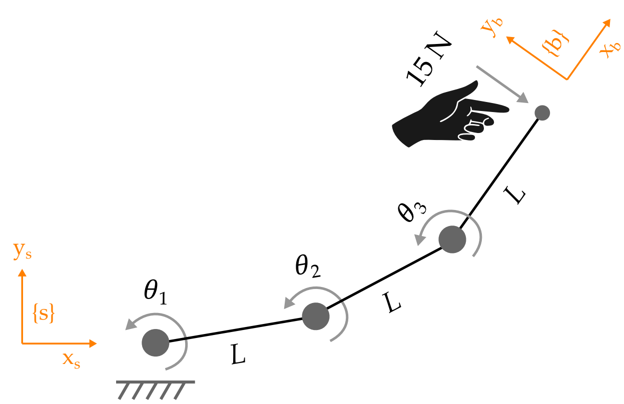
$$T_{sb} = \begin{bmatrix} 0 & 1 & 0 & L \\ -1 & 0 & 0 & 0 \\ 0 & 0 & 1 & 2L \\ 0 & 0 & 0 & 1 \end{bmatrix} \tag{6}$$

Then plug in your answer for F_b and solve:

$$F_s = (\mathrm{Ad}_{T_{sb}^{-1}})^T F_b = \begin{bmatrix} 0 \\ 60L \\ 0 \\ 0 \\ 0 \\ -30 \end{bmatrix} \tag{7}$$

2 Statics





A human is pushing on the end-effector of the 3-DoF robot shown above. (Top Drawing) The human applies a 15 N force along the negative y_b axis. (Bottom Drawing) The human keeps this force aligned with the negative y_b axis as the robot moves.

2.1 (5 points)

What wrench does the robot need to apply at the end-effector to maintain static equilibrium?

First find the wrench that the human applies to the robot. The robot must exert an *equal and opposite* wrench to maintain static equilibrium.

The human's applied wrench in coordinate frame $\{b\}$ is:

$$-F_b = \begin{bmatrix} m_b \\ f_b \end{bmatrix} = \begin{bmatrix} r_b \times f_b \\ f_b \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \\ 0 \\ -15 \\ 0 \end{bmatrix} \tag{8}$$

If we refer to this external wrench as $-F_b$, then the robot needs to apply $+F_b$. To maintain static equilibrium, the robot must apply:

$$F_b = \begin{bmatrix} 0 \\ 0 \\ 0 \\ 0 \\ 15 \\ 0 \end{bmatrix} \tag{9}$$

2.2 (5 points)

Case 1. Let $L = 1$ and let $\theta = [0, \pi/4, \pi/4]^T$.
Find the joint torques needed to balance out the force applied by the human.

The equation that we want to use is:

$$\tau = J(\theta)^T F \tag{10}$$

where both J and F are expressed in the same frame. I will use the body Jacobian and F_b. You have previously found the body Jacobian for this robot:

$$J_b(\theta) = \begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 1 & 1 & 1 \\ Ls_{23} + Ls_3 & Ls_3 & 0 \\ Lc_{23} + Lc_3 + L & Lc_3 + L & L \\ 0 & 0 & 0 \end{bmatrix} \tag{11}$$

Combining this body Jacobian with F_b from the previous part, we get that:

$$\tau = J_b(\theta)^T F_b = \begin{bmatrix} 25.6 \\ 25.6 \\ 15 \end{bmatrix} \tag{12}$$

2.3 (5 points)

Case 2. Let $L = 1$ and let $\theta = [0, \pi/8, 0]^T$.
Find the joint torques needed to balance out the force applied by the human.

Repeat the process outlined in the previous part, but now use $\theta = [0, \pi/8, 0]^T$ when evaluating the body Jacobian.

$$\tau = J_b(\theta)^T F_b = \begin{bmatrix} 43.9 \\ 30 \\ 15 \end{bmatrix} \tag{13}$$

2.4 (15 points)

Let $\|\tau\|$ be the magnitude (i.e., the length) of the joint torque vector.

- Find a joint position θ that **maximizes** $\|\tau\|$
- Find a joint position θ that **minimizes** $\|\tau\|$

Start by writing τ as a function of θ. Using symbolic variables for θ and $L = 1$:

$$\tau = J_b(\theta)^T F_b = J_s(\theta)^T F_s = \begin{bmatrix} 15(c_{23} + c_3 + 1) \\ 15(c_3 + 1) \\ 15 \end{bmatrix} \tag{14}$$

To answer the question you must solve for values of θ_2 and θ_3 that maximize or minimize the magnitude of τ.

Maximize. Let $\theta = 0$. Then $c_{23} + c_3 + 1 = 3$ and $c_3 + 1 = 2$, and the torque vector is:

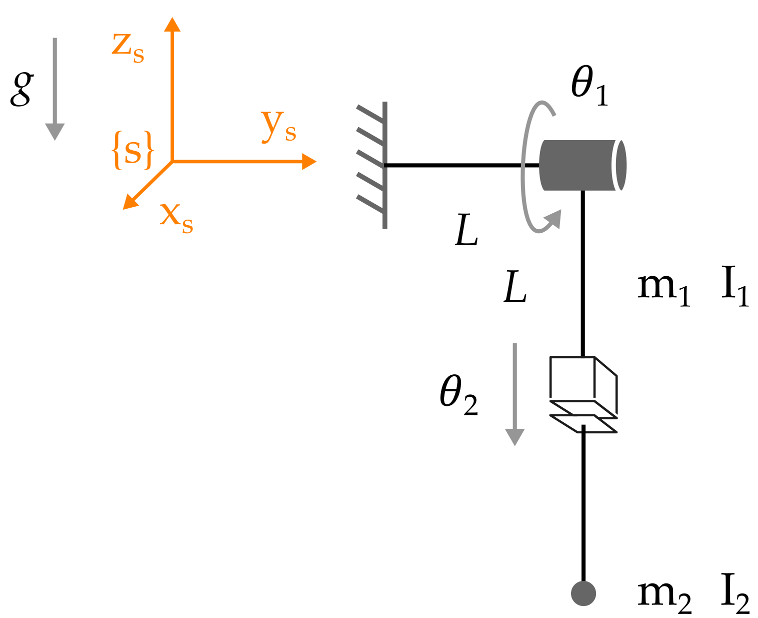
$$\tau = \begin{bmatrix} 45 \\ 30 \\ 15 \end{bmatrix}, \qquad \|\tau\| = 56.1 \tag{15}$$

The intuition here is that the longer the moment arm, the more torque the robot must exert to counteract the human's force.

Minimize. Let $\theta = [0, \pi/2, \pi]^T$. Then $c_{23} = 0$ and $c_3 = -1$. Substituting in, we find that $c_{23} + c_3 + 1 = 0$ and $c_3 + 1 = 0$, so that the torque vector is:

$$\tau = \begin{bmatrix} 0 \\ 0 \\ 15 \end{bmatrix}, \qquad \|\tau\| = 15 \tag{16}$$

3 Dynamics: Revolute-Prismatic



3.1 (20 points)

Find the dynamics for the robot shown above. Your answer should be of the form:

$$\tau = M(\theta)\ddot{\theta} + C(\theta, \dot{\theta})\dot{\theta} + g(\theta) \tag{17}$$

List the mass matrix, the Coriolis matrix, and the gravity vector. The center of mass for m_1 is located **halfway** between the revolute joint and the prismatic joint. The center of mass for m_2 is at the robot's end-effector. Inertia matrices I_1 and I_2 are:

$$I_1 = \begin{bmatrix} I_{x1} & 0 & 0 \\ 0 & I_{y1} & 0 \\ 0 & 0 & I_{z1} \end{bmatrix}, \quad I_2 = \begin{bmatrix} I_{x2} & 0 & 0 \\ 0 & I_{y2} & 0 \\ 0 & 0 & I_{z2} \end{bmatrix}$$

First we find the geometric Jacobian for the center of mass of each link. The Jacobian for the first center of mass is:

$$J_{\omega_1} = \begin{bmatrix} 0 & 0 \\ 1 & 0 \\ 0 & 0 \end{bmatrix}, \qquad J_{v_1} = \begin{bmatrix} -0.5Lc_1 & 0 \\ 0 & 0 \\ 0.5Ls_1 & 0 \end{bmatrix} \tag{18}$$

The Jacobian for the second center of mass is:

$$J_{\omega_2} = \begin{bmatrix} 0 & 0 \\ 1 & 0 \\ 0 & 0 \end{bmatrix}, \qquad J_{v_2} = \begin{bmatrix} -c_1(L+\theta_2) & -s_1 \\ 0 & 0 \\ s_1(L+\theta_2) & -c_1 \end{bmatrix} \tag{19}$$

Next we use these Jacobians to get the mass matrix. Our formula is:

$$M(\theta) = m_1 J_{v_1}^T J_{v_1} + J_{\omega_1}^T R_1 I_1 R_1^T J_{\omega_1} + m_2 J_{v_2}^T J_{v_2} + J_{\omega_2}^T R_2 I_2 R_2^T J_{\omega_2} \tag{20}$$

Substituting in and simplifying, the **mass matrix** is:

$$M(\theta) = \begin{bmatrix} \frac{m_1 L^2}{4} + m_2(L^2 + 2L\theta_2 + \theta_2^2) + I_{y1} + I_{y2} & 0 \\ 0 & m_2 \end{bmatrix} \tag{21}$$

Next we will get the Coriolis matrix from the mass matrix. Note that only the top left element of M depends on θ. Specifically, m_{11} depends on θ_2:

$$\frac{\partial m_{11}}{\partial \theta_2} = 2m_2\theta_2 + 2m_2 L \tag{22}$$

Use the Christoffel symbols to get the terms of the Coriolis matrix:

$$c_{11} = \frac{1}{2}\frac{\partial m_{11}}{\partial \theta_2}\dot{\theta}_2 = m_2(\theta_2 + L)\dot{\theta}_2 \tag{23}$$

$$c_{21} = -\frac{1}{2}\frac{\partial m_{11}}{\partial \theta_2}\dot{\theta}_1 = -m_2(\theta_2 + L)\dot{\theta}_1 \tag{24}$$

$$c_{12} = \frac{1}{2}\frac{\partial m_{11}}{\partial \theta_2}\dot{\theta}_1 = m_2(\theta_2 + L)\dot{\theta}_1 \tag{25}$$

$$c_{22} = 0 \tag{26}$$

Combining the answers, the **Coriolis matrix** becomes:

$$C(\theta, \dot{\theta}) = \begin{bmatrix} m_2(\theta_2 + L)\dot{\theta}_2 & m_2(\theta_2 + L)\dot{\theta}_1 \\ -m_2(\theta_2 + L)\dot{\theta}_1 & 0 \end{bmatrix} \tag{27}$$

Finally we get the gravity vector. The potential energy of each center of mass is:

$$P_1(\theta) = m_1 g(-0.5Lc_1) \tag{28}$$

$$P_2(\theta) = m_2 g(-Lc_1 - \theta_2 c_1) \tag{29}$$

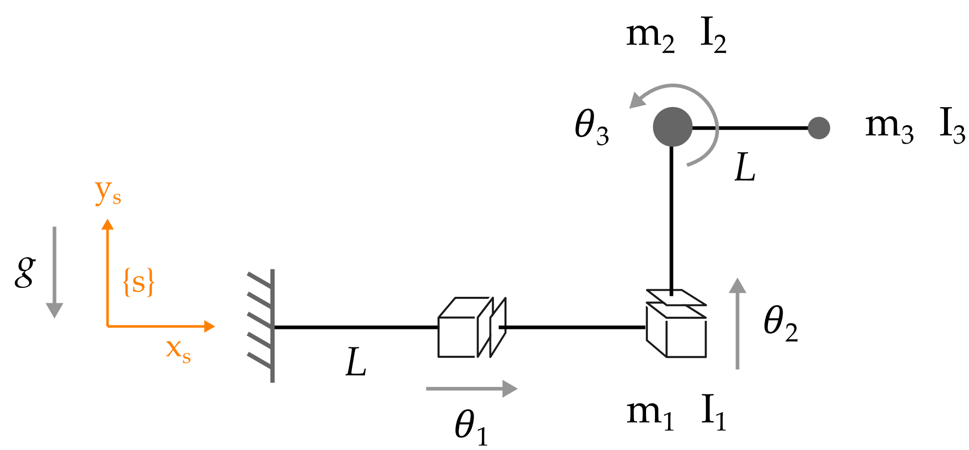
Summing these terms, the potential energy of the robot is:

$$P(\theta) = P_1(\theta) + P_2(\theta) = -gc_1(0.5m_1L + m_2L + m_2\theta_2) \tag{30}$$

Taking the partial derivatives with respect to θ_1 and θ_2, the **gravity vector** is:

$$g(\theta) = \begin{bmatrix} gs_1(0.5m_1L + m_2L + m_2\theta_2) \\ -gm_2c_1 \end{bmatrix} \tag{31}$$

4 Dynamics: Prismatic-Prismatic-Revolute



4.1 (20 points)

Find the dynamics for the robot shown above. Your answer should be of the form:

$$\tau = M(\theta)\ddot{\theta} + C(\theta, \dot{\theta})\dot{\theta} + g(\theta) \tag{32}$$

List the mass matrix, the Coriolis matrix, and the gravity vector. Each center of mass is at the end of the link.

First we find the geometric Jacobian for the center of mass of each link. The Jacobian for the first center of mass is:

$$J_{\omega_1} = \begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}, \qquad J_{v_1} = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix} \tag{33}$$

The Jacobian for the second center of mass is:

$$J_{\omega_2} = \begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}, \qquad J_{v_2} = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 0 \end{bmatrix} \tag{34}$$

The Jacobian for the third center of mass is:

$$J_{\omega_3} = \begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 1 \end{bmatrix}, \qquad J_{v_3} = \begin{bmatrix} 1 & 0 & -Ls_3 \\ 0 & 1 & Lc_3 \\ 0 & 0 & 0 \end{bmatrix} \tag{35}$$

Next we use these Jacobians to get the mass matrix. Our formula is:

$$M(\theta) = \sum_{i=1}^{3} m_i J_{v_i}^T J_{v_i} + J_{\omega_i}^T R_i I_i R_i^T J_{\omega_i} \tag{36}$$

Remember that for a planar robot we only have inertia around the out-of-plane axis, so that $I_i = \text{diag}(0, 0, I_i)$. You may also notice that the first two joints are not rotating, so there is no rotational kinetic energy to consider for joints 1 and 2. Substituting in and simplifying, the **mass matrix** is:

$$M(\theta) = \begin{bmatrix} m_1 + m_2 + m_3 & 0 & -Lm_3s_3 \\ 0 & m_2 + m_3 & Lm_3c_3 \\ -Lm_3s_3 & Lm_3c_3 & m_3L^2 + I_3 \end{bmatrix} \tag{37}$$

Next we will get the Coriolis matrix from the mass matrix. It helps to recognize that no terms depend on θ_1 or θ_2. There are elements of M that depend on θ_3:

$$\frac{\partial m_{13}}{\partial \theta_3} = \frac{\partial m_{31}}{\partial \theta_3} = -Lm_3c_3 \tag{38}$$

$$\frac{\partial m_{23}}{\partial \theta_3} = \frac{\partial m_{32}}{\partial \theta_3} = -Lm_3s_3 \tag{39}$$

Use the Christoffel symbols to get the terms of the Coriolis matrix. Be careful to account for all three joints since here $C(\theta, \dot{\theta})$ is a 3×3 matrix:

$$c_{kj} = \frac{1}{2}\left[\frac{\partial m_{kj}}{\partial \theta_1} + \frac{\partial m_{k1}}{\partial \theta_j} + \frac{\partial m_{1j}}{\partial \theta_k}\right]\dot{\theta}_1 + \frac{1}{2}\left[\frac{\partial m_{kj}}{\partial \theta_2} + \frac{\partial m_{k2}}{\partial \theta_j} + \frac{\partial m_{2j}}{\partial \theta_k}\right]\dot{\theta}_2 + \frac{1}{2}\left[\frac{\partial m_{kj}}{\partial \theta_3} + \frac{\partial m_{k3}}{\partial \theta_j} + \frac{\partial m_{3j}}{\partial \theta_k}\right]\dot{\theta}_3$$

Combining the answers, the **Coriolis matrix** becomes:

$$C(\theta, \dot{\theta}) = \begin{bmatrix} 0 & 0 & -Lm_3c_3\dot{\theta}_3 \\ 0 & 0 & -Lm_3s_3\dot{\theta}_3 \\ 0 & 0 & 0 \end{bmatrix} \tag{40}$$

Finally we get the gravity vector. The potential energy of each center of mass is:

$$P_1(\theta) = 0 \tag{41}$$

$$P_2(\theta) = m_2 g \theta_2 \tag{42}$$

$$P_3(\theta) = m_3 g(\theta_2 + Ls_3) \tag{43}$$

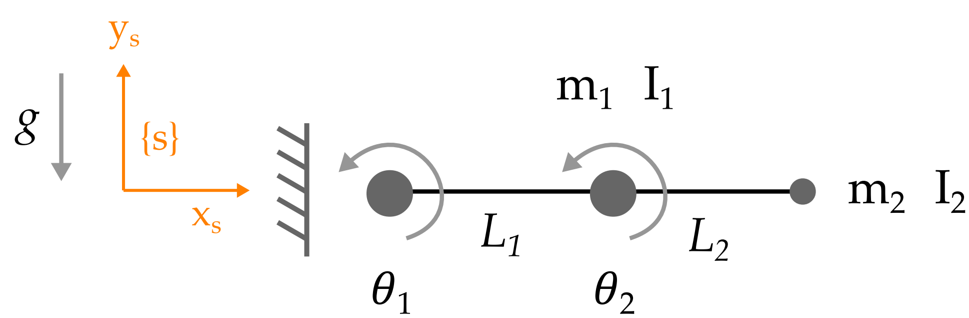
Summing these terms, the potential energy of the robot is:

$$P(\theta) = P_1(\theta) + P_2(\theta) + P_3(\theta) = m_2 g \theta_2 + m_3 g(\theta_2 + Ls_3) \tag{44}$$

Taking the partial derivatives with respect to θ_1, θ_2, and θ_3, the **gravity vector** is:

$$g(\theta) = \begin{bmatrix} 0 \\ g(m_2 + m_3) \\ g m_3 L c_3 \end{bmatrix} \tag{45}$$

5 Simulation



Here you will simulate the dynamics of the planar robot shown above. **We found the dynamics of this robot in lecture**. Start by **downloading** the Matlab file `make_simulation.m` that was provided with this assignment.

5.1 (10 points)

Combine `make_simulation.m` with the dynamics we derived. You will need to get the (x, y) position of each link to plot the robot. Use the given simulation parameters and frame rates; all videos should be 10 seconds in length. Turn in the following MP4 videos:

- Make a simulation where $\tau = [0, 0]^T$ and the robot has no friction. Title this video **Problem5_1.mp4**
- Make a simulation where $\tau = [0, 0]^T$ and the robot has viscous friction $B = I$. Title this video **Problem5_2.mp4**
- Make a simulation where $\tau = [20, 5]^T$ and the robot has viscous friction $B = I$. Title this video **Problem5_3.mp4**

Although not required, I encourage you to play with the parameters (such as mass, inertia, friction, and τ), and see how these parameters affect the simulation.

See the figure below for the part of the simulation code you were responsible for implementing. The videos for each case are uploaded in a separate folder in solutions.

```
% your code here
M = [I1 + I2 + L1^2*m1 + L1^2*m2 + L2^2*m2 + 2*L1*L2*m2*cos(theta(2)),...
    m2*L2^2 + L1*m2*cos(theta(2))*L2 + I2;...
        m2*L2^2 + L1*m2*cos(theta(2))*L2 + I2, m2*L2^2 + I2];
C = [-L1*L2*thetadot(2)*m2*sin(theta(2)),...
    - L1*L2*m2*sin(theta(2))*(thetadot(1) + thetadot(2));...
        L1*L2*thetadot(1)*m2*sin(theta(2)), 0];
G = [g*m2*(L2*cos(theta(1) + theta(2)) + L1*cos(theta(1))) + L1*g*m1*cos(theta(1));
        L2*g*m2*cos(theta(1) + theta(2))];
thetadotdot = M \ (tau - C*thetadot - B*thetadot - G);
```