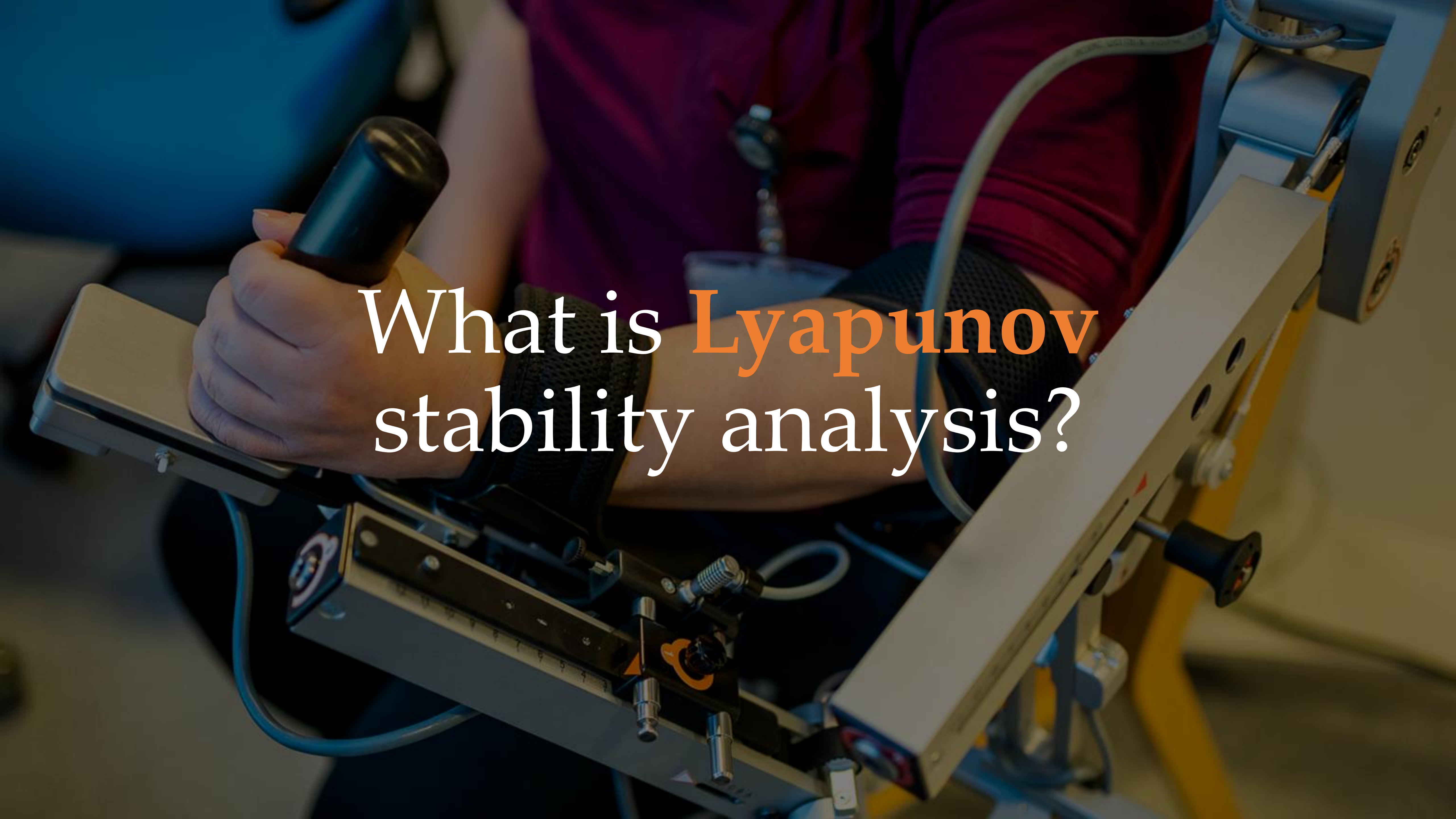
Lyapunov Stability

Reading: Robot Modeling and Control 8.1, Appendix D

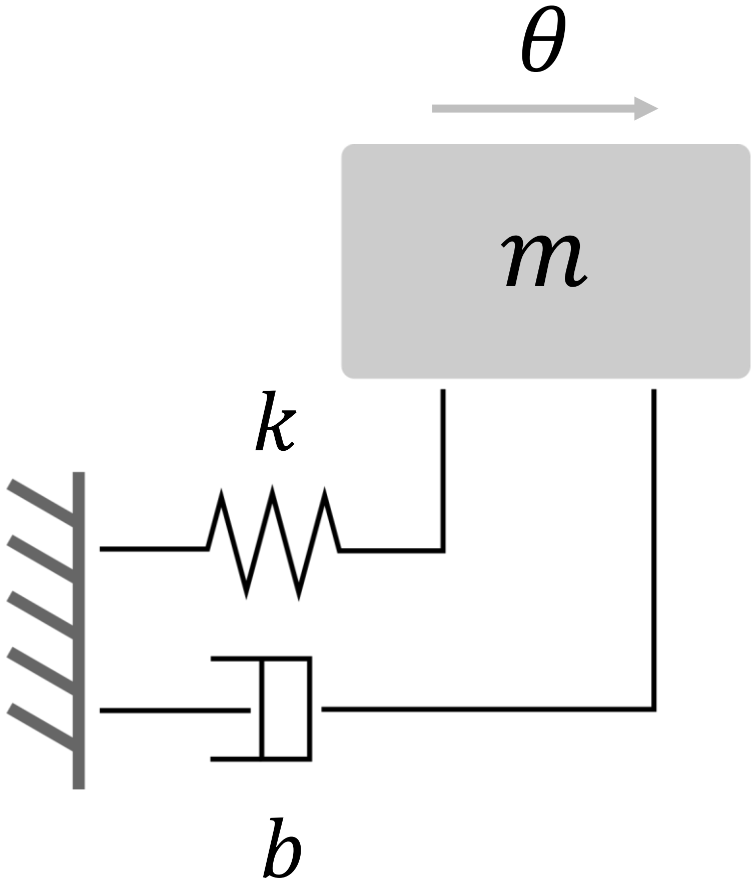
This Lecture

- How do we determine if a dynamical system is stable?
- Introducing Lyapunov stability analysis
- What is multivariable control for robot arms?



What is Lyapunov stability analysis?

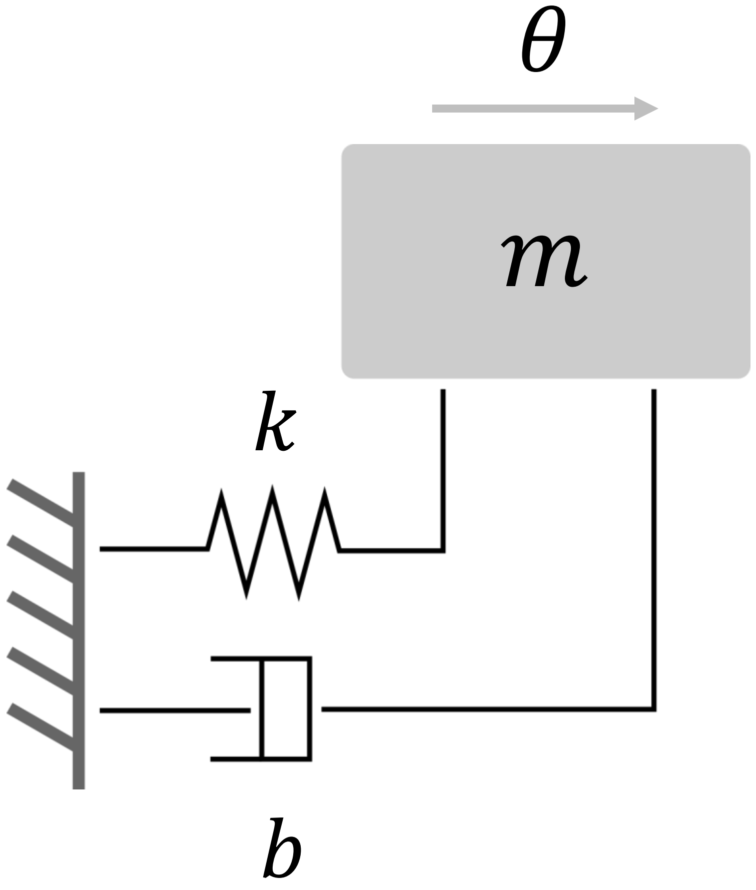
Example



Let's use **Lyapunov stability analysis** to show this mass-spring-damper is stable.

$$m\ddot{\theta} + b\dot{\theta} + k\theta = 0$$

Example

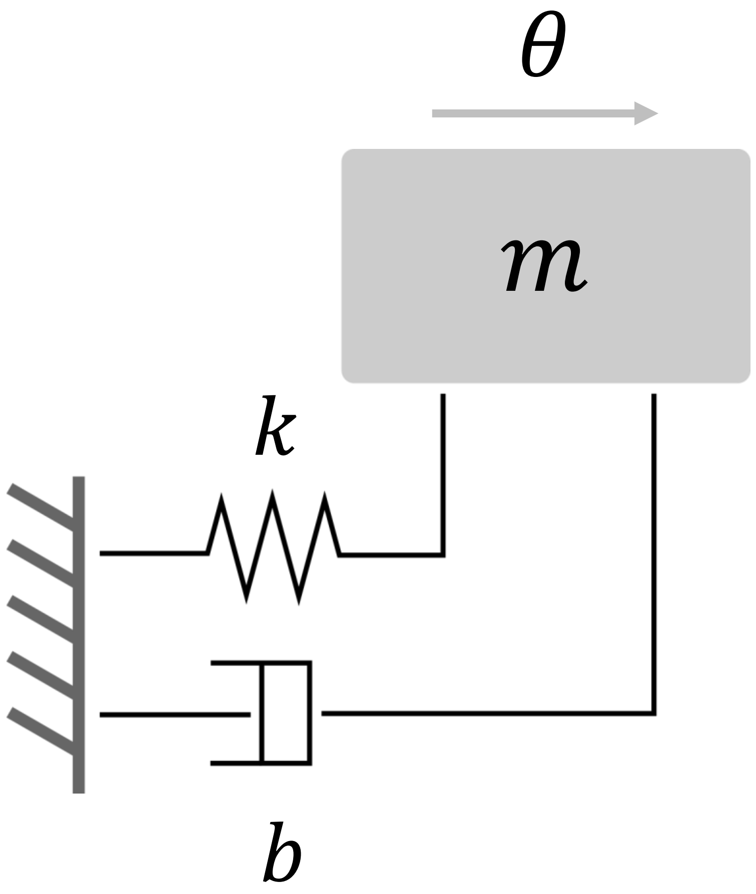


Let's use **Lyapunov stability analysis** to show this mass-spring-damper is stable.

$$v(t) = \frac{1}{2} m \dot{\theta}(t)^2 + \frac{1}{2} k \theta(t)^2$$

Total energy as a function of time.
Note that this is always nonnegative.

Example



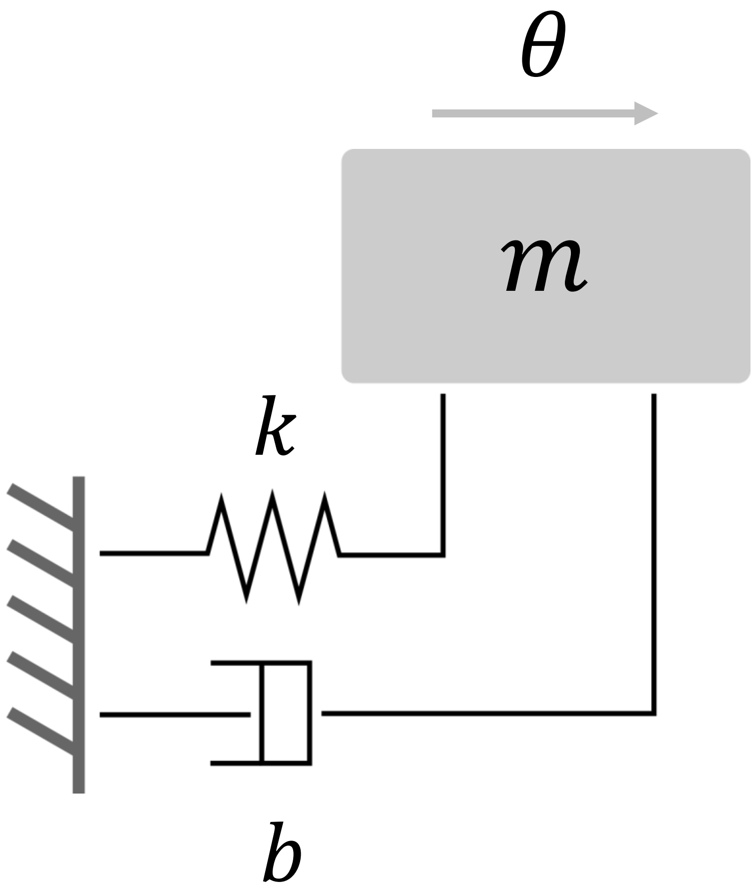
Let's use **Lyapunov stability analysis** to show this mass-spring-damper is stable.

$$v(t) = \frac{1}{2} m\dot{\theta}(t)^2 + \frac{1}{2} k\theta(t)^2$$

$$\dot{v}(t) = m\ddot{\theta}(t)\dot{\theta}(t) + k\theta(t)\dot{\theta}(t)$$

Rate of change of the total energy
$\dot{v} > 0$ increasing energy,
$\dot{v} < 0$ decreasing energy,

Example



Let's use **Lyapunov stability analysis** to show this mass-spring-damper is stable.

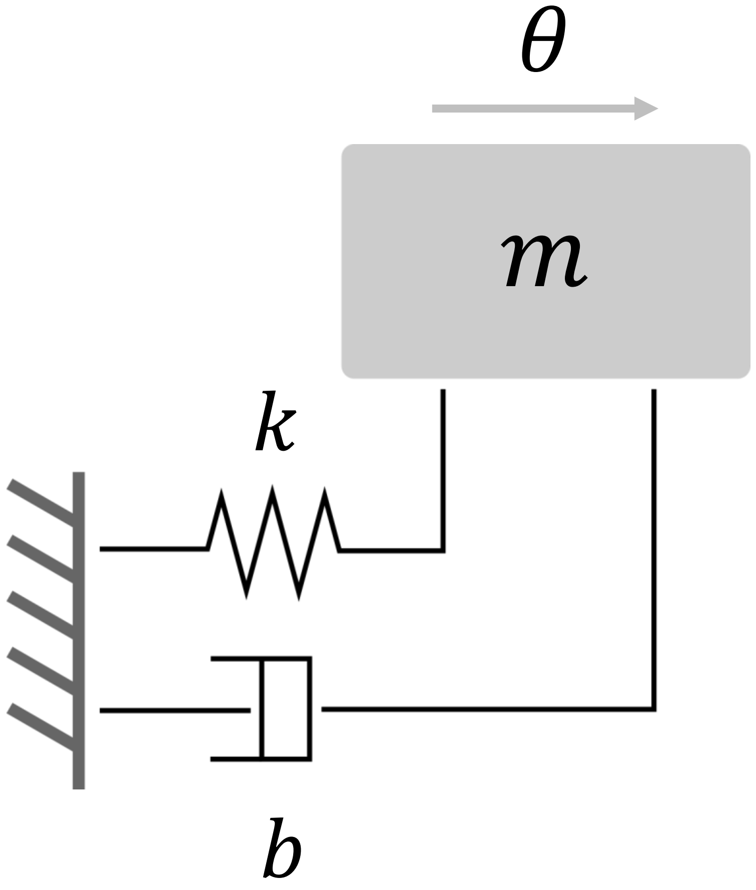
$$m\ddot{\theta} = -b\dot{\theta} - k\theta$$

$$\dot{v} = m\ddot{\theta}\dot{\theta} + k\theta\dot{\theta}$$

$$\dot{v} = \left(-b\dot{\theta} - k\theta\right)\dot{\theta} + k\theta\dot{\theta}$$

$$\dot{v}(t) = -b\dot{\theta}(t)^2$$

Example

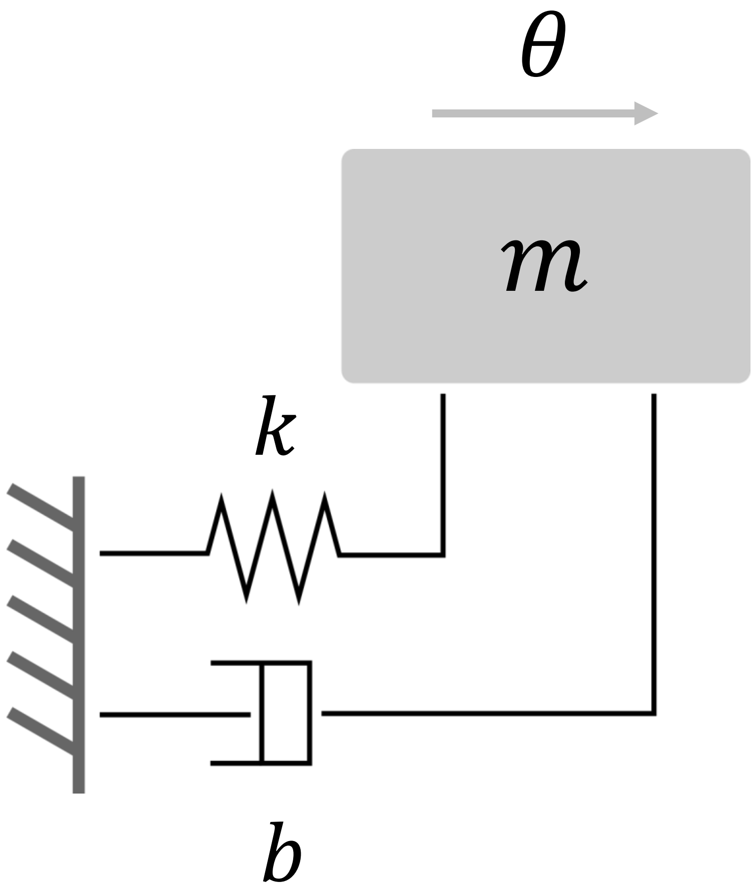


Let's use **Lyapunov stability analysis** to show this mass-spring-damper is stable.

$$\dot{v}(t) = -b\dot{\theta}(t)^2$$

Since $b > 0$, energy always leaving the system until it comes to rest ($\dot{\theta} = 0$)

Example



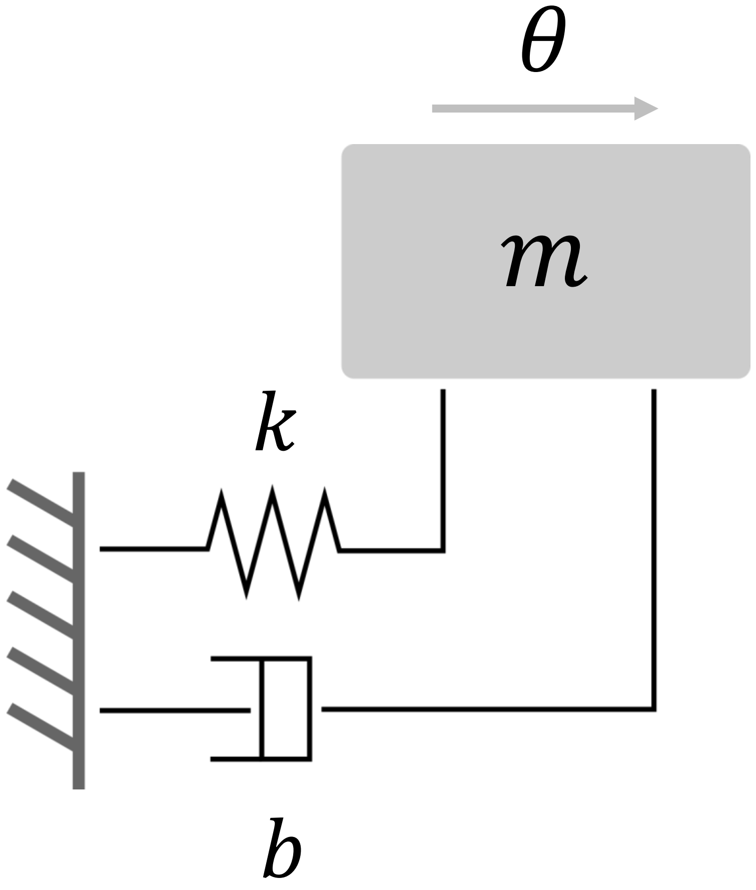
Let's use **Lyapunov stability analysis** to show this mass-spring-damper is stable.

$$\dot{v}(t) = -b\dot{\theta}(t)^2$$

$$m\ddot{\theta} + b\dot{\theta} + k\theta = 0$$

At rest $\dot{\theta} = 0$ and $\ddot{\theta} = 0$, leaving us with $k\theta = 0$. Energy decreases until $\theta = 0$.

Example



Let's use **Lyapunov stability analysis** to show this mass-spring-damper is stable.

$$\dot{v}(t) = -b\dot{\theta}(t)^2$$

Conclusion. For any initial conditions, the mass-spring-damper will always come to rest at equilibrium $\theta = 0$.

Lyapunov Stability Analysis

Given a dynamical system:

$$\dot{x} = f(x)$$

To show this system is stable, find a generalized energy function $v(x)$ such that:

- $v(x) > 0$ for all values of $x \neq 0$, and $v(0) = 0$
- $v(x)$ has continuous first partial derivatives
- $\dot{v}(x) \leq 0$ (e.g., energy decreases over time)

Lyapunov Stability Analysis

Example. Consider the dynamical system:

$$\dot{x} = -Ax$$

where x is a $n \times 1$ vector and A is a positive definite $n \times n$ matrix. Try:

$$v(x) = \frac{1}{2}x^T x$$

$v(x) > 0$ for all values of $x \neq 0$, and $v(0) = 0$

Partial derivative $\frac{\partial v}{\partial x} = x$ is continuous

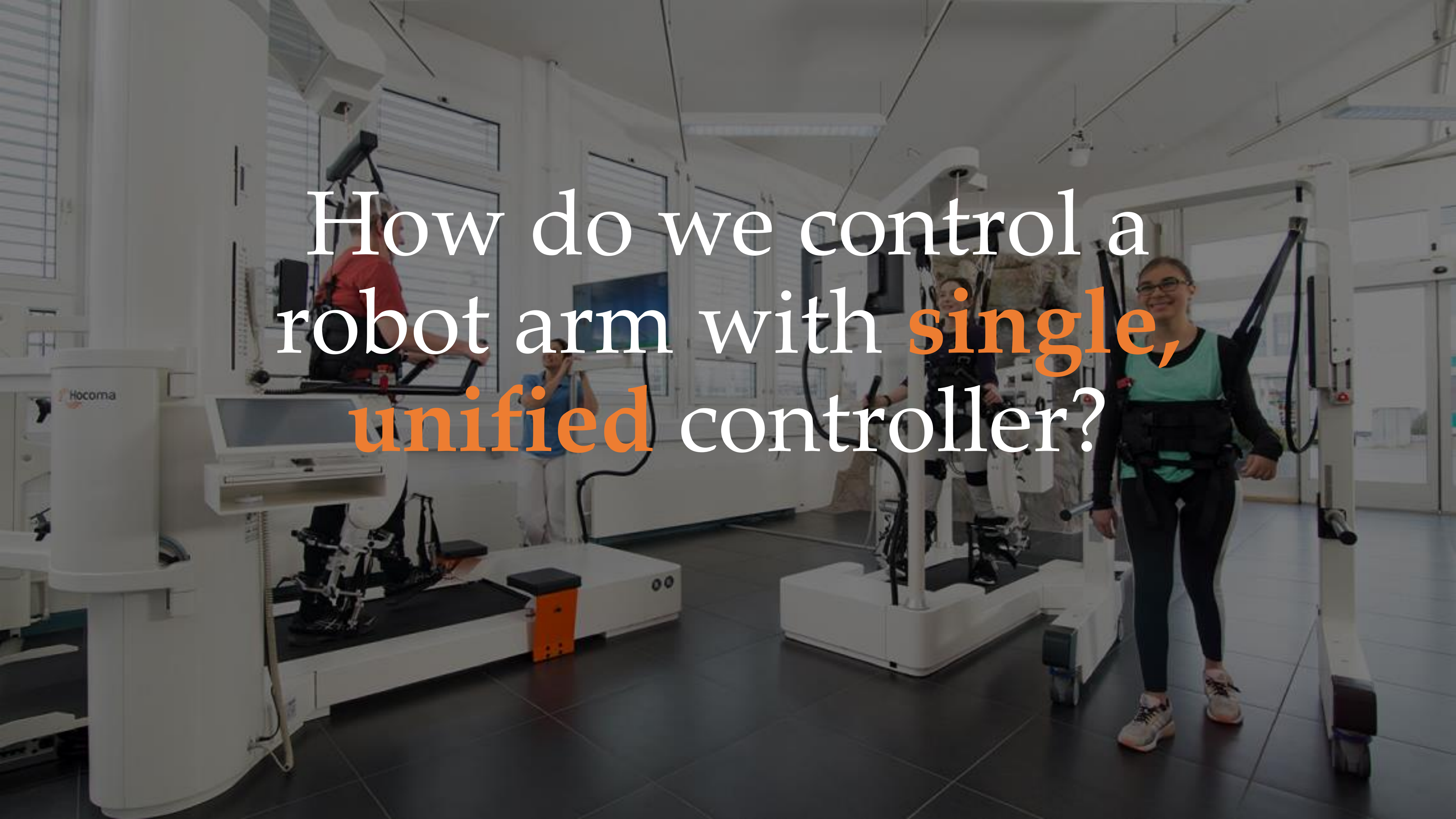
Lyapunov Stability Analysis

Example. Consider the dynamical system:

$$\dot{x} = -Ax$$

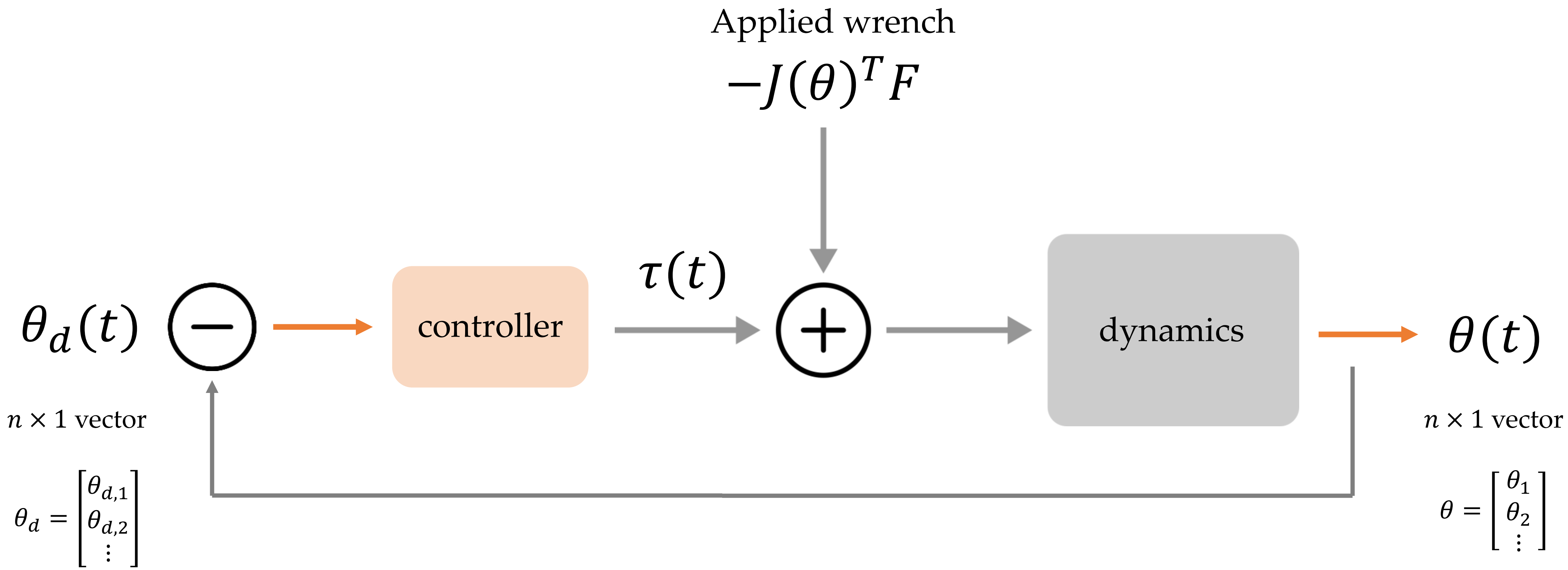
$$\dot{v}(x) = \frac{d}{dt}\left(\frac{1}{2}x^T x\right) = x^T \dot{x} = \underline{-x^T A x}$$

$\dot{v}(x) \leq 0$ because A is positive definite.
Energy decreases until we reach stability at $x = 0$, the system is **stable**.

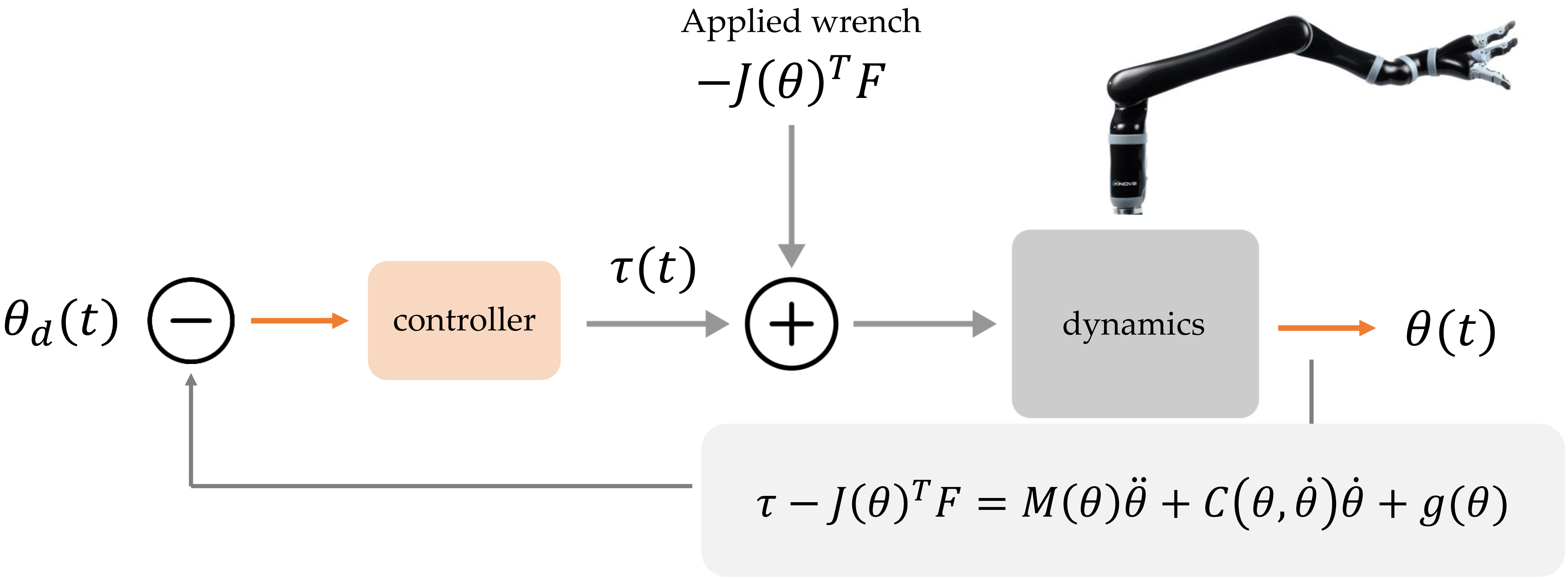


How do we control a robot arm with **single, unified** controller?

Multivariable Control



Multivariable Control



Multivariable Control

One industry-standard multivariable robot controller is:

$$\tau = K_P(\theta_d - \theta) - K_D\dot{\theta} + g(\theta)$$

- K_P is a positive definite matrix of proportional gains
- K_D is a positive definite matrix of derivative gains
- $g(\theta)$ is the gravity vector

Multivariable Control

One industry-standard multivariable robot controller is:

$$\tau = K_P(\theta_d - \theta) - K_D\dot{\theta} + g(\theta)$$

Often diagonal gain matrix

$$K_P = \begin{bmatrix} k_{p,1} & & \\ & k_{p,2} & \\ & & \ddots \end{bmatrix}$$

Often diagonal gain matrix

$$K_D = \begin{bmatrix} k_{d,1} & & \\ & k_{d,2} & \\ & & \ddots \end{bmatrix}$$

Gravity compensation.
Remove steady-state
error due to gravity.

Multivariable Control

Plugging this controller into the robot's dynamics:

$$M(\theta)\ddot{\theta} + C(\theta,\dot{\theta})\dot{\theta} + g(\theta) = \tau = K_P(\theta_d - \theta) - K_D\dot{\theta} + g(\theta)$$

$$M(\theta)\ddot{\theta} = -C(\theta,\dot{\theta})\dot{\theta} - K_D\dot{\theta} + K_P(\theta_d - \theta)$$

Is this closed-loop system **stable**?
Will the robot converge to $\theta = \theta_d$?

Multivariable Control

Consider the dynamical system and **generalized energy function**:

$$M(\theta)\ddot{\theta} = -C(\theta, \dot{\theta})\dot{\theta} - K_D\dot{\theta} + K_P(\theta_d - \theta)$$

$$v(\theta, \dot{\theta}) = \frac{1}{2}\dot{\theta}^T M(\theta)\dot{\theta} + \frac{1}{2}(\theta_d - \theta)^T K_P(\theta_d - \theta)$$

We know that M and K_P are positive definite matrices, so $v(\theta, \dot{\theta}) > 0$ for all values of $\theta \neq \theta_d$, $\dot{\theta} \neq 0$, and $v(0,0) = 0$

Multivariable Control

Take the **time derivative** to see whether energy is **decreasing**:

$$\dot{v} = \frac{1}{2}\dot{\theta}^T \dot{M}(\theta)\dot{\theta} + \dot{\theta}^T M(\theta)\ddot{\theta} - \dot{\theta}^T K_P(\theta_d - \theta)$$

$$\dot{v} = \dot{\theta}^T \left(\frac{1}{2}\dot{M}(\theta)\dot{\theta} + \boxed{M(\theta)\ddot{\theta}} - K_P(\theta_d - \theta)\right)$$

Plug in closed-loop dynamics here:
$M(\theta)\ddot{\theta} = -C(\theta, \dot{\theta})\dot{\theta} - K_D\dot{\theta} + K_P(\theta_d - \theta)$

Multivariable Control

Take the **time derivative** to see whether energy is **decreasing**:

$$\dot{v} = \dot{\theta}^T \left(\frac{1}{2} \dot{M}(\theta)\dot{\theta} - C(\theta, \dot{\theta})\dot{\theta} - K_D \dot{\theta} \right)$$

Property. *$\dot{M} - 2C$ is skew symmetric, and therefore* $\frac{1}{2}\dot{\theta}^T(\dot{M} - 2C)\dot{\theta} = 0$

$$\boxed{\dot{v} = -\dot{\theta}^T K_D \dot{\theta}}$$

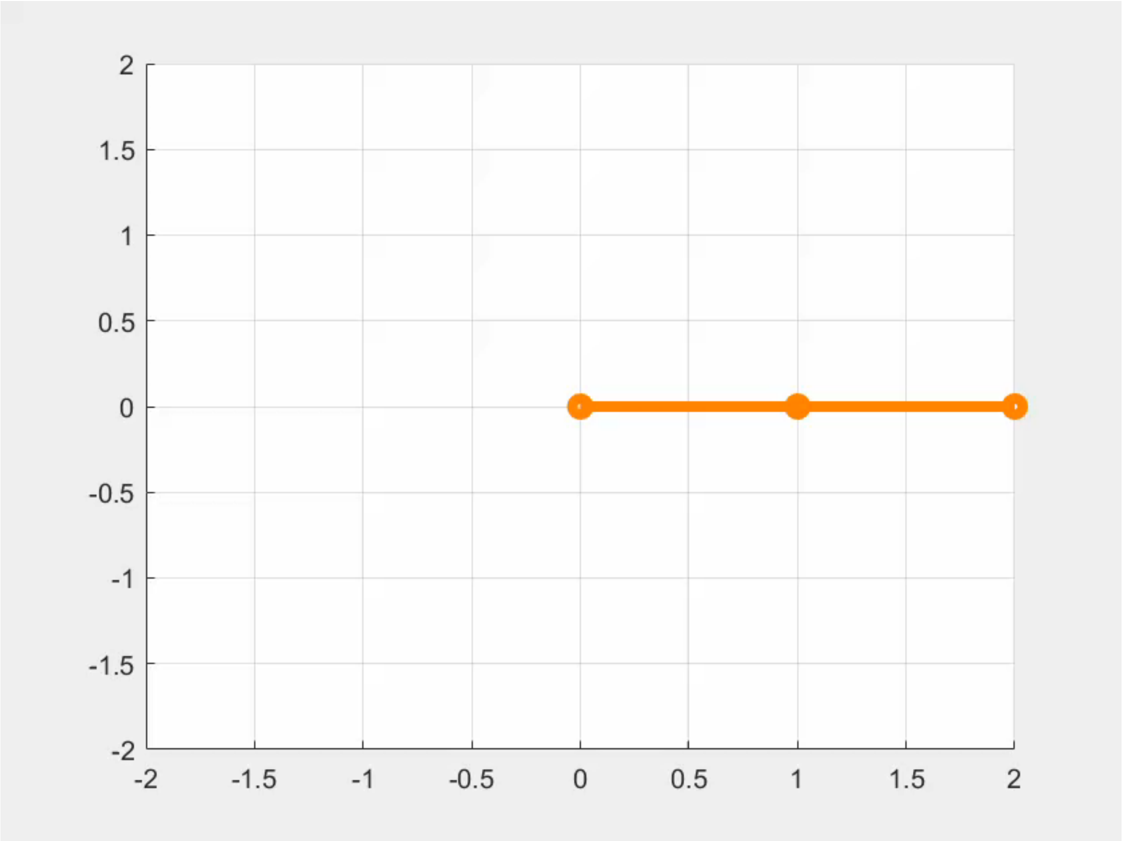
Multivariable Control

Complete stability analysis.

- The time derivative of our energy function is $\dot{v} = -\dot{\theta}^T K_D \dot{\theta}$
- This function is negative definite; energy decreases until $\dot{\theta} = 0$
- If $\dot{\theta} = 0$ is constant, then $\ddot{\theta} = 0$ and the dynamics become:

$$0 = K_P(\theta_d - \theta)$$

- Since K_P is positive definite (nonsingular), we have $\boldsymbol{\theta_d} = \boldsymbol{\theta}$ at equilibrium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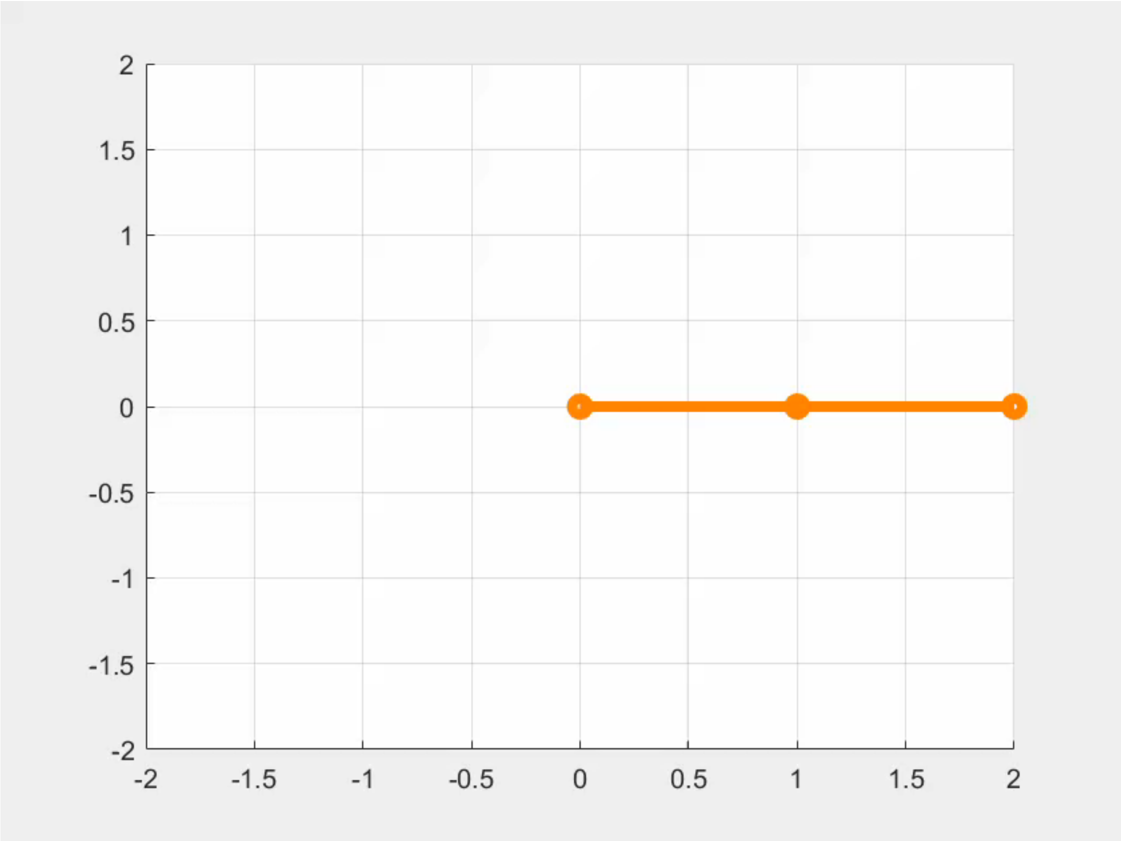
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Takeaways

Lyapunov's method examines stability of dynamical systems.

To show a system is stable, find a generalized energy function $v(x)$ such that:

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- $v(x)$ has continuous first partial derivatives
- $\dot{v}(x) \leq 0$ (e.g., energy decreases over time)

One stable multivariable controller for robot arms is:

$$\tau = K_P(\theta_d - \theta) - K_D\dot{\theta} + g(\theta)$$

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Next Lecture

- We’ve focused on controlling the robot to reach a desired position…

  … what are some other things we might want to control for?