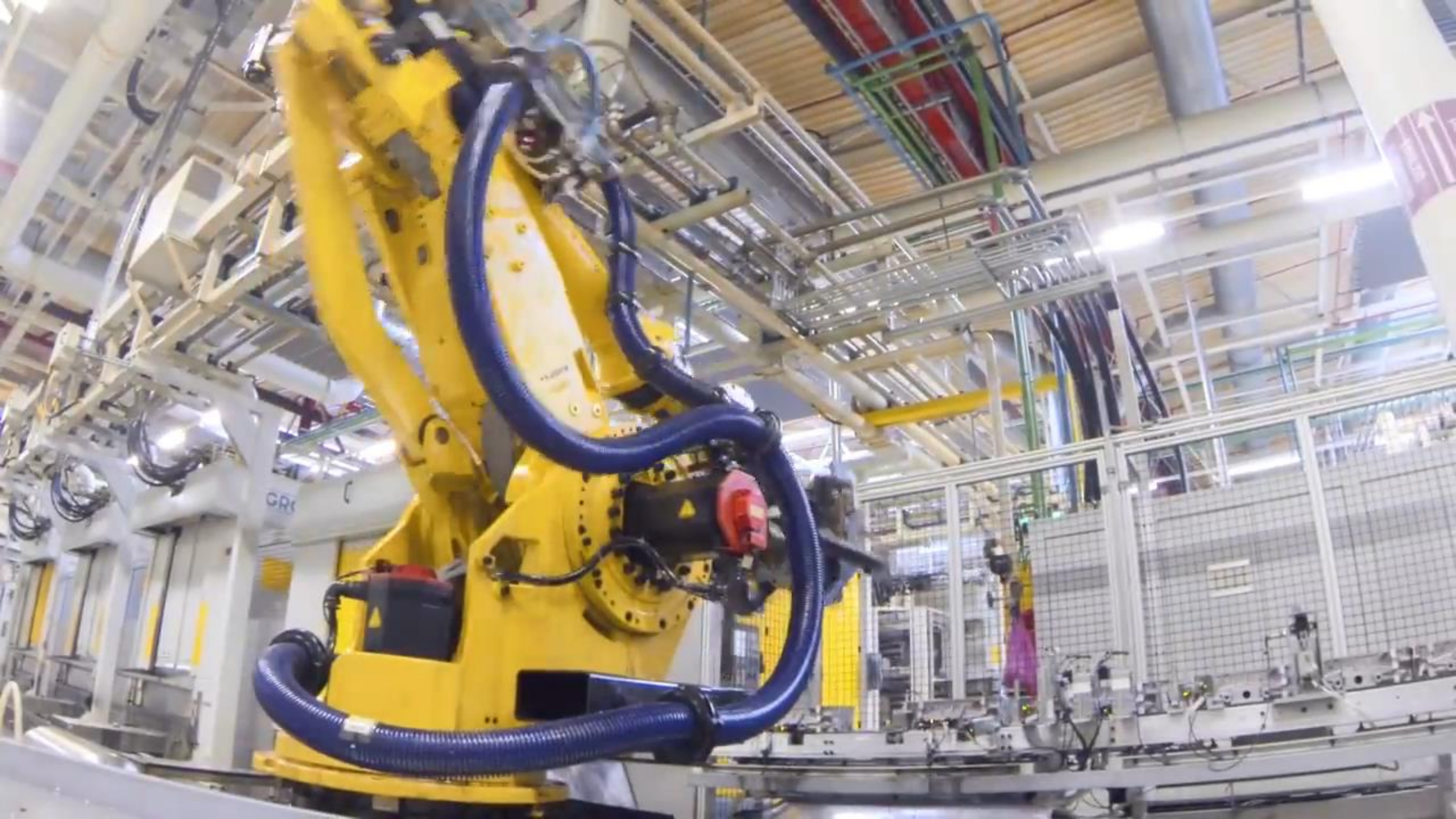
Forward Kinematics

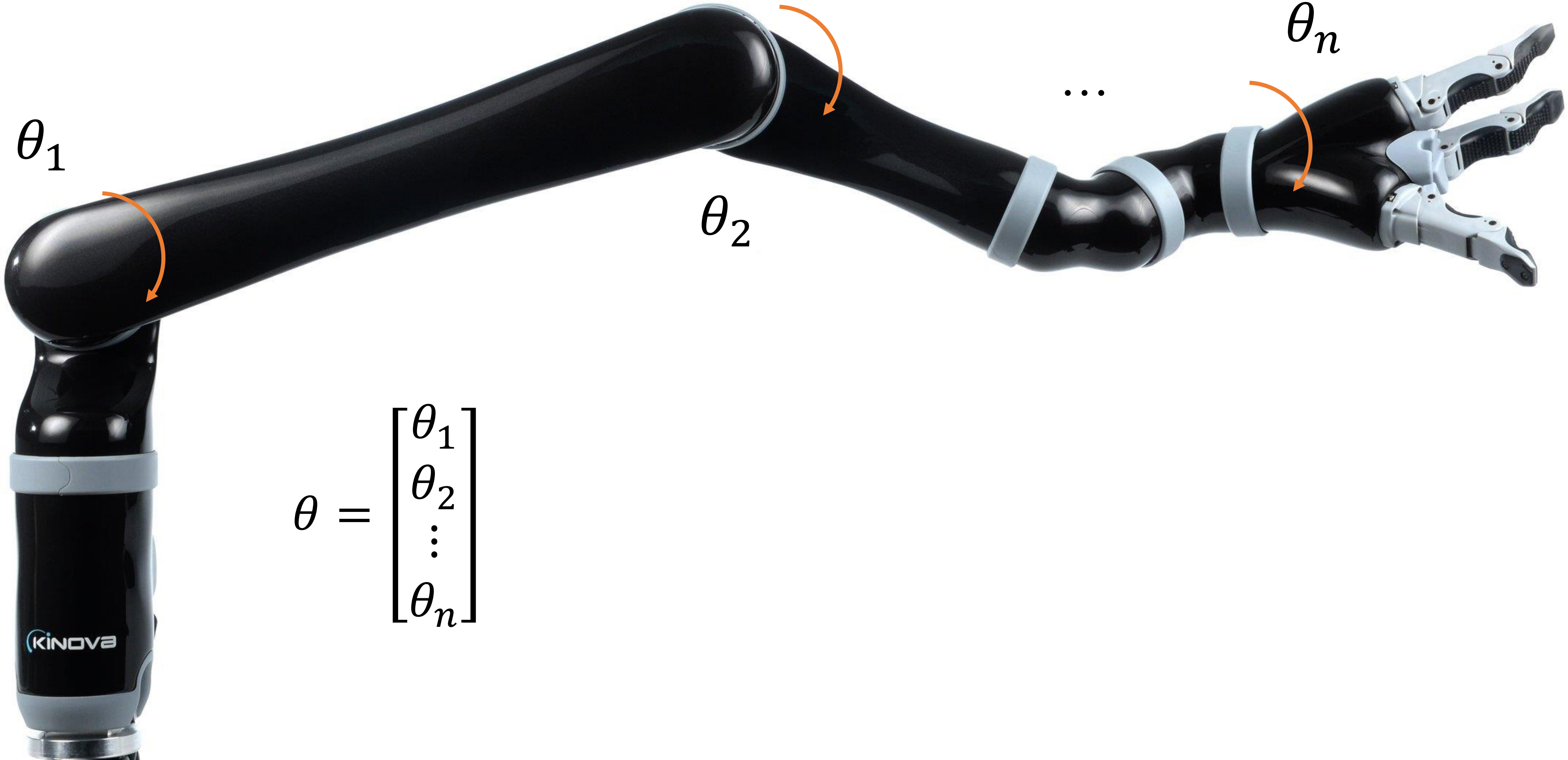
Reading: Modern Robotics 4.1



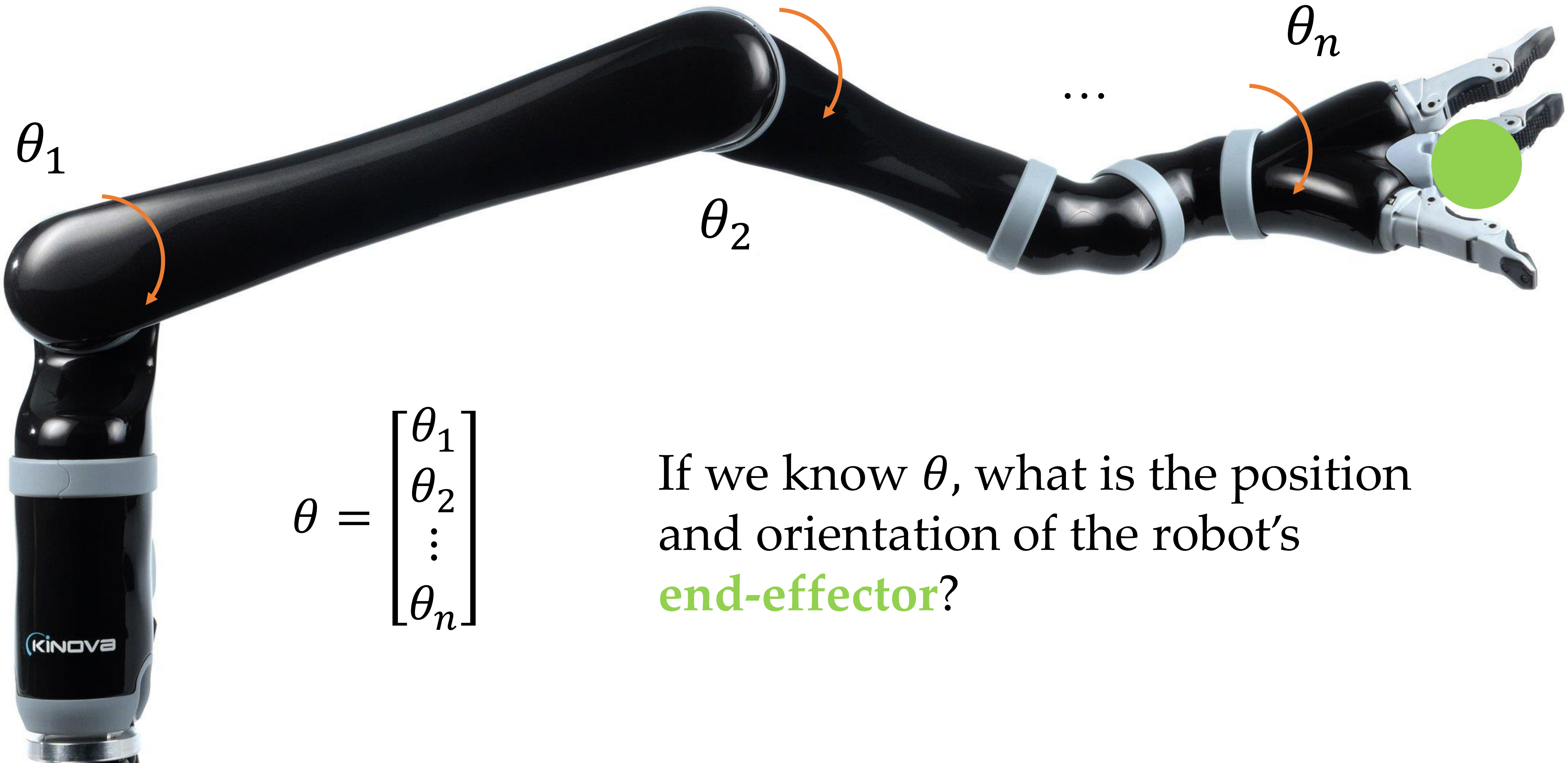
This Lecture

- What is forward kinematics?
- How are screws related to forward kinematics?
- Can we find a general formula for forward kinematics?


KINOVA



$$\theta = \begin{bmatrix} \theta_1 \\ \theta_2 \\ \vdots \\ \theta_n \end{bmatrix}$$



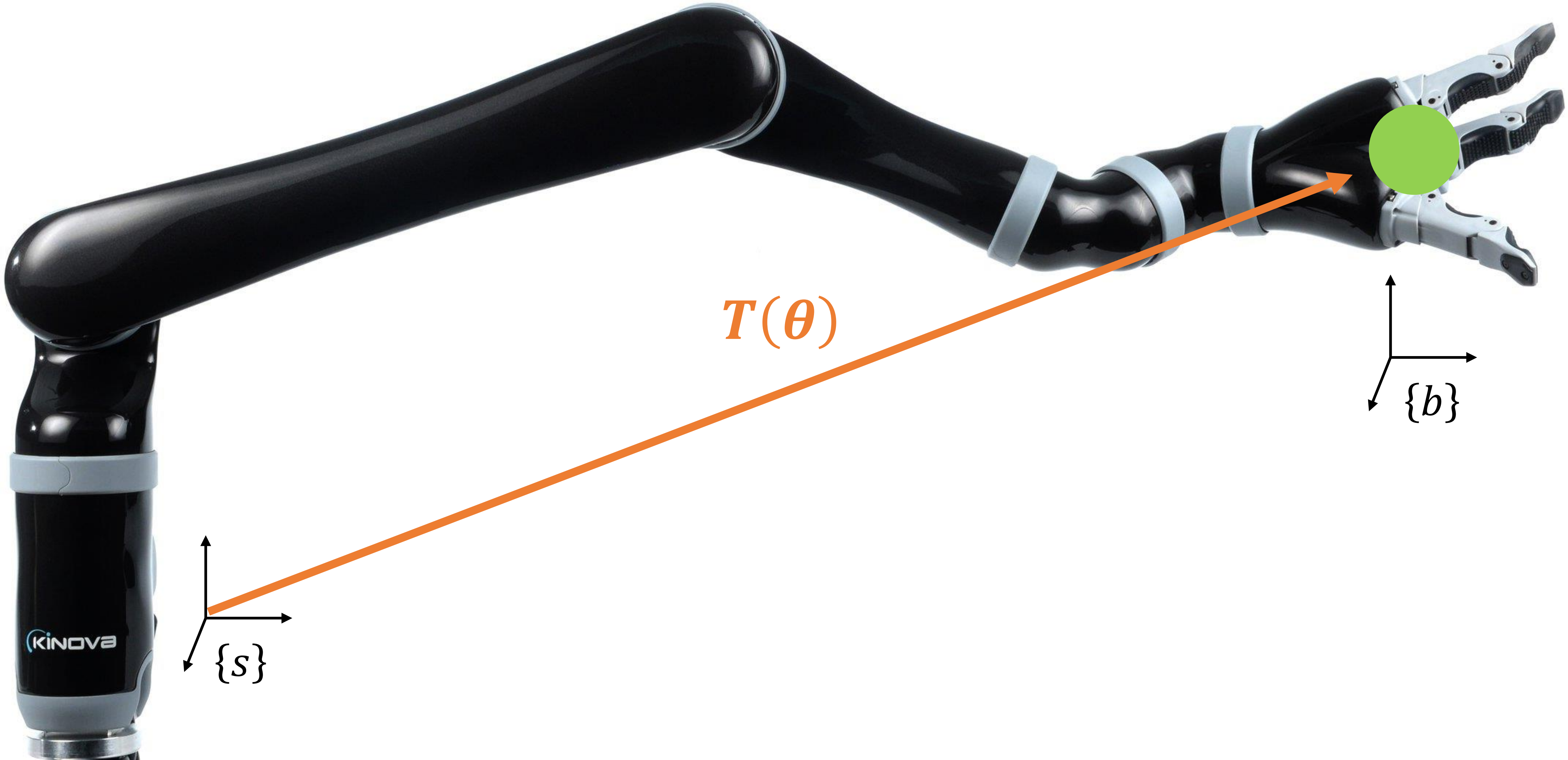
$$\theta = \begin{bmatrix} \theta_1 \\ \theta_2 \\ \vdots \\ \theta_n \end{bmatrix}$$

If we know θ, what is the position and orientation of the robot's **end-effector**?

Given a robot with:

- fixed frame $\{s\}$ at the base
- body frame $\{b\}$ at point of interest

forward kinematics is the mapping $T(\theta)$ from joint values θ to the pose of $\{b\}$ relative to $\{s\}$

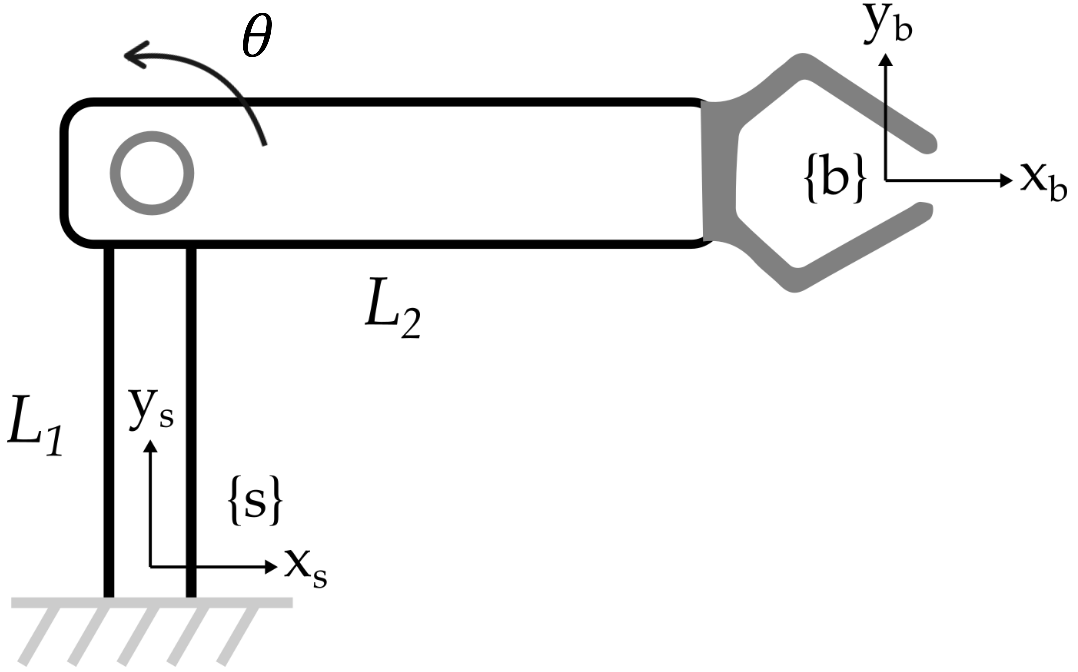


$T(\theta)$
$\{b\}$
$\{s\}$
KINOVA

How do we find the forward kinematics?

Robot with one revolute joint.
We are looking for:

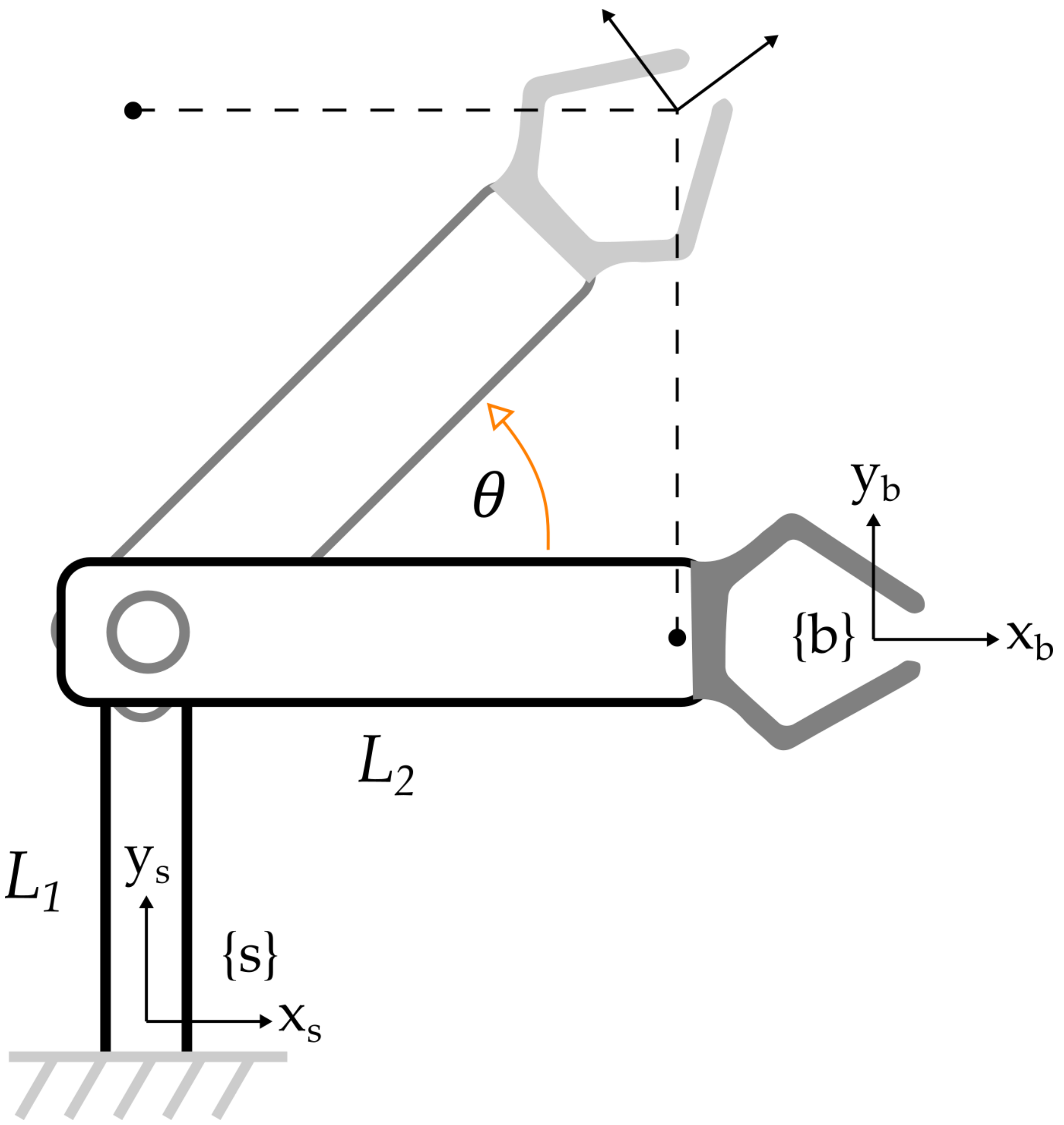
$$T_{sb}(\theta) = \begin{bmatrix} R_{sb}(\theta) & p_{sb}(\theta) \\ 0 & 1 \end{bmatrix}$$



For this simple robot, we can solve by hand:

$$R_{sb}(\theta) = \begin{bmatrix} \cos\theta & -\sin\theta & 0 \\ \sin\theta & \cos\theta & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

rotating around z_s axis

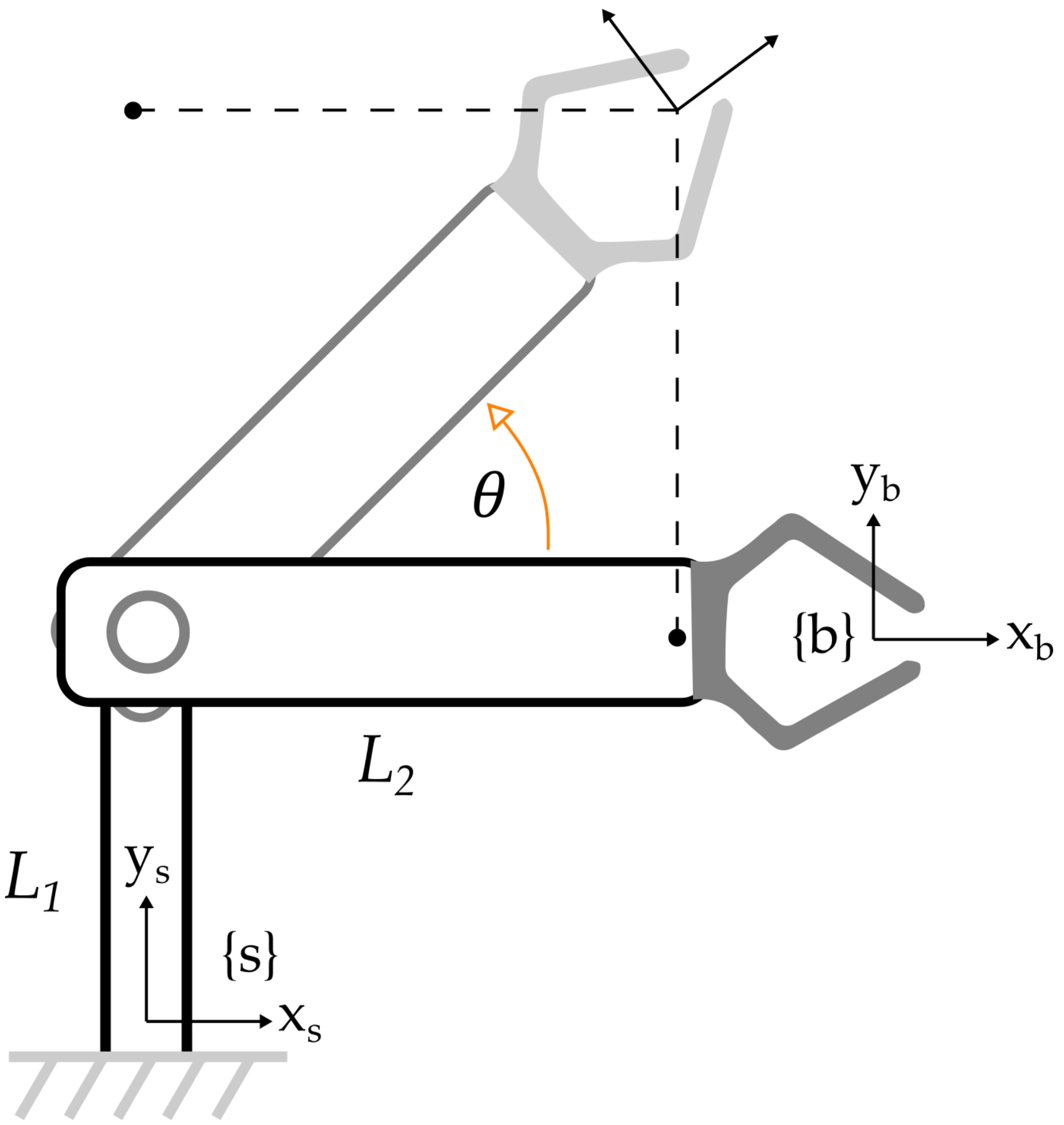


For this simple robot, we can solve by hand:

$$p_{sb}(\theta) = \begin{bmatrix} 0 \\ L_1 \\ 0 \end{bmatrix} + \begin{bmatrix} L_2 \cos\theta \\ L_2 \sin\theta \\ 0 \end{bmatrix}$$

vertical offset

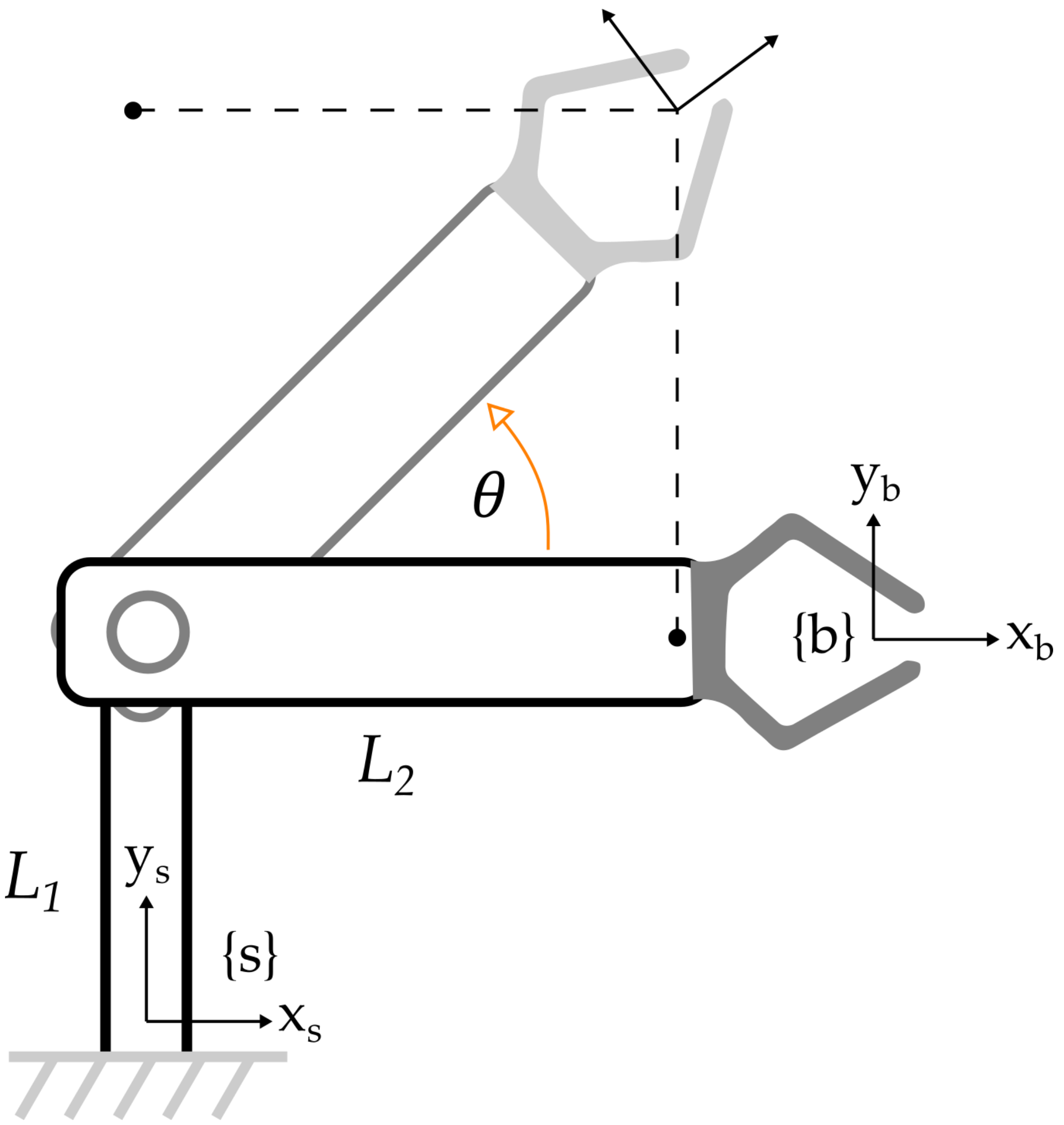
second link rotating in a circle



For this simple robot, we can solve by hand:

$$T(\theta) = \begin{bmatrix} \cos\theta & -\sin\theta & 0 & L_2\cos\theta \\ \sin\theta & \cos\theta & 0 & L_2\sin\theta + L_1 \\ 0 & 0 & 1 & 0 \\ & 0 & & 1 \end{bmatrix}$$

forward kinematics of our robot!

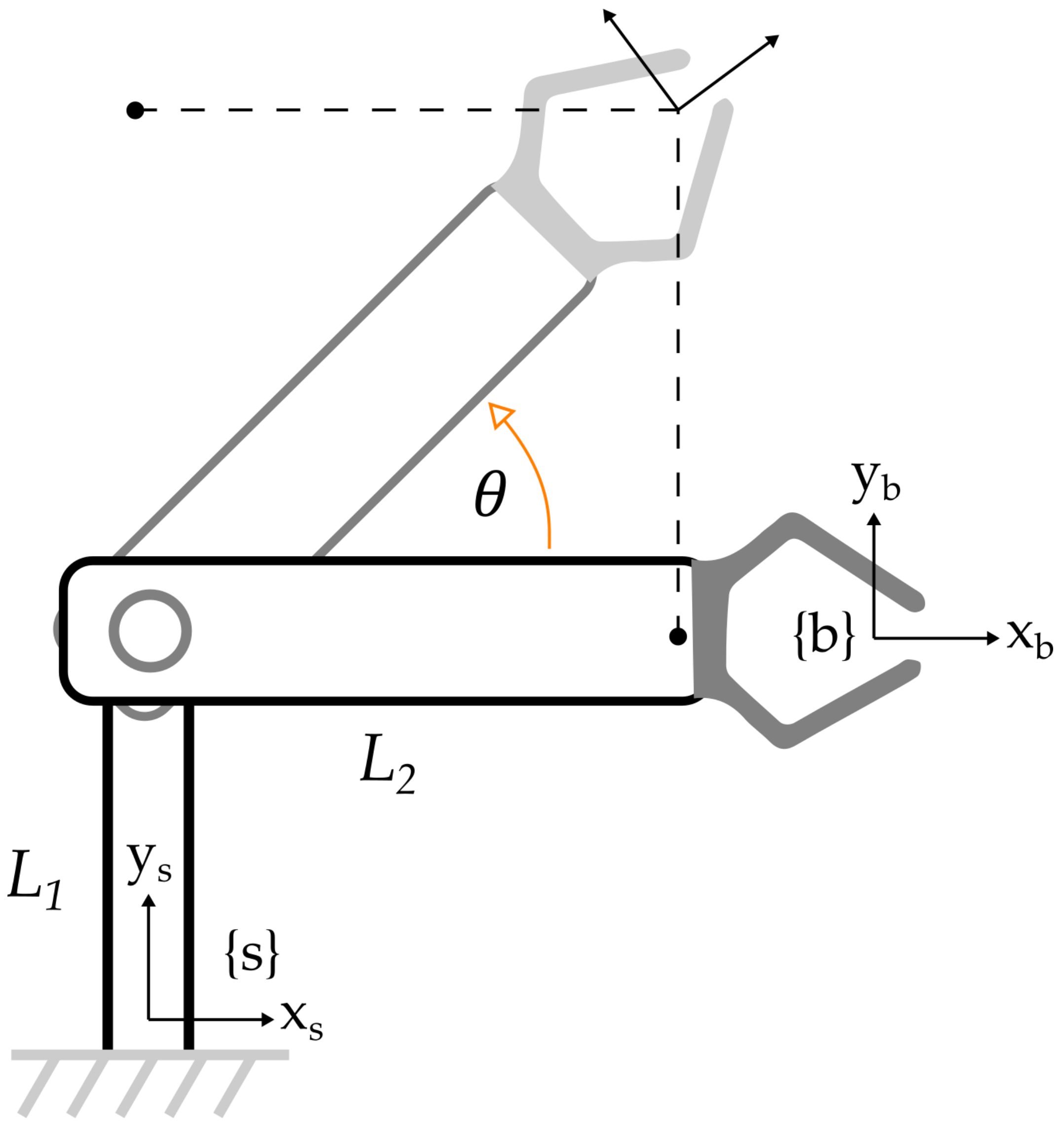


What if we apply screws to find the forward kinematics?

Screws

Last lecture we found that:

$$T(\theta) = e^{[S]\theta}T(0)$$

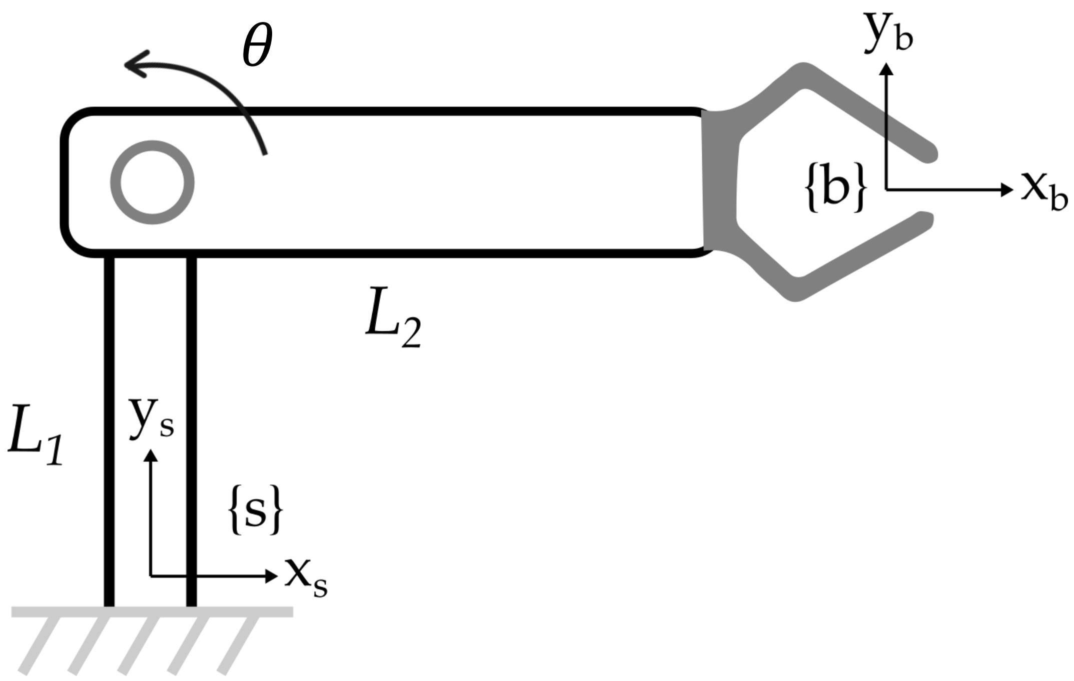


Screws

Home position $T(0)$ is:

$$T(0) = \begin{bmatrix} 1 & 0 & 0 & L_2 \\ 0 & 1 & 0 & L_1 \\ 0 & 0 & 1 & 0 \\ & 0 & & 1 \end{bmatrix}$$

Initial transformation from $\{s\}$ to $\{b\}$

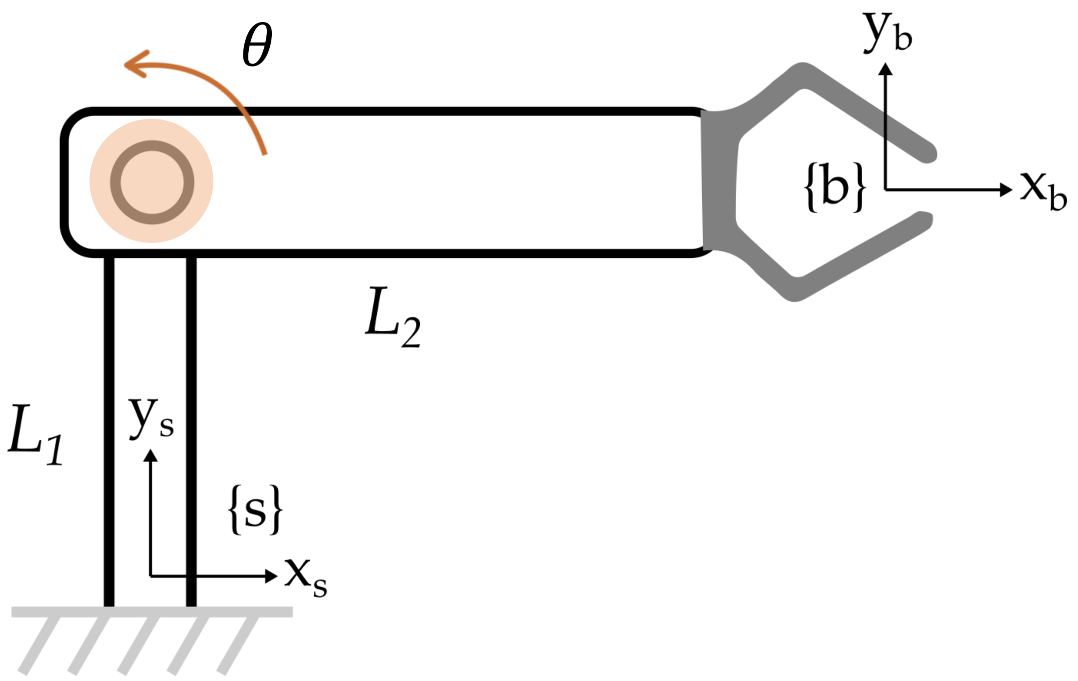


Screws

Screw S for a revolute joint is:

$$S = \begin{bmatrix} \omega_s \\ -\omega_s \times p_{\text{joint}} \end{bmatrix}$$

$$\omega_s = \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix}, \qquad p_{\text{joint}} = \begin{bmatrix} 0 \\ L_1 \\ 0 \end{bmatrix}$$

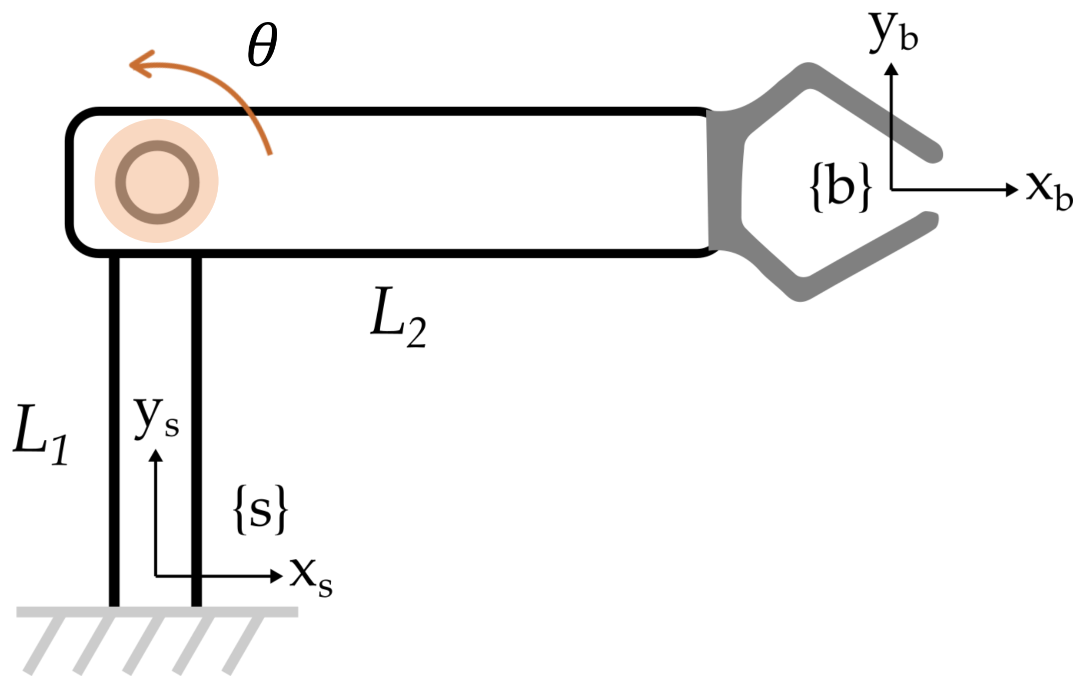


Screws

Screw S for a revolute joint is:

$$S = \begin{bmatrix} 0 \\ 0 \\ 1 \\ L_1 \\ 0 \\ 0 \end{bmatrix}$$

Normalized linear and angular velocity of joint



Screws

Putting it all together:

$$T(\theta) = e^{[S]\theta} T(0)$$

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syms theta L1 L2 real

T_0 = [eye(3), [L2; L1; 0]; 0 0 0 1];
S = [0; 0; 1; L1; 0; 0];
T = expm(bracket(S) * theta) * T_0;

function S_matrix = bracket(S)
    S_matrix = [0 -S(3) S(2) S(4);
                S(3) 0 -S(1) S(5);
                -S(2) S(1) 0 S(6);
                0 0 0 0];
end
```

Screws

Putting it all together:

$$T(\theta) = e^{[S]\theta}T(0)$$

```
T =

[cos(theta), -sin(theta), 0,      L2*cos(theta)]
[sin(theta),  cos(theta), 0, L1 + L2*sin(theta)]
[         0,           0, 1,                  0]
[         0,           0, 0,                  1]
```

forward kinematics of our robot!

We can **extend** this process to robots with ***n* joints**

Product of Exponentials

Forward kinematics of a serial robot arm with n joints is:

$$T(\theta) = e^{[S_1]\theta_1} e^{[S_2]\theta_2} \dots e^{[S_n]\theta_n} M$$

- $M = T(0)$ is just T_{sb} when the robot is in home position
- S_i is the screw for the i-th joint when the robot is in home position

Product of Exponentials

Prismatic Joints

$$S = \begin{bmatrix} 0 \\ v_s \end{bmatrix}$$

v_s is unit vector in the direction of positive translation

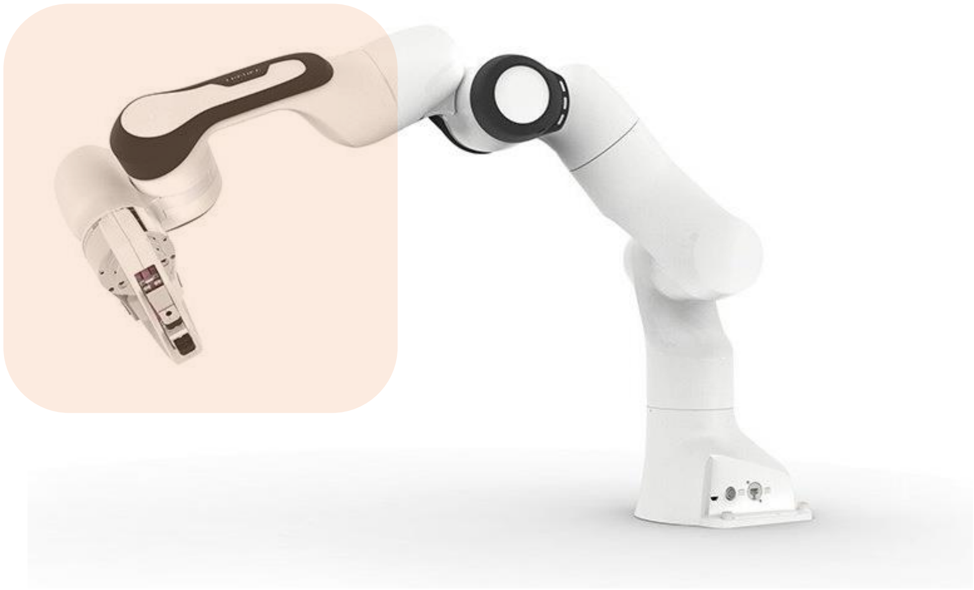
Product of Exponentials

Revolute Joints

$$S = \begin{bmatrix} \omega_s \\ -\omega_s \times q \end{bmatrix}$$

ω_s is unit vector in the direction of the axis of positive rotation

q is vector from $\{s\}$ to the joint axis



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Next Lecture

- Practice forward kinematics with several examples