Problem Set 1

Robotics & Automation
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Instructions. Please write legibly and do not attempt to fit your work into the smallest space possible. It is important to show all work, but basic arithmetic can be omitted. You are encouraged to use Matlab when possible to avoid hand calculations, but print and submit your commented code for non-trivial calculations. You can attach a pdf of your code to the homework, use live scripts or the publish feature in Matlab, or include a snapshot of your code. Do not submit .m files — we will not open or grade these files.

1 Linear Algebra Review

When working with robots — from self-driving cars to assistive arms — we often leverage sets of vectors to describe where our robot is in space, how it moves, and how it reacts to forces. When we use these sets of vectors, one important concept is their *linear independence* or *linear dependence*. A set of vectors $\{v_1, v_2, \dots, v_n\}$ is linearly independent if the coefficients $c_1, c_2, \dots, c_n$ must all be zero in order for $c_1v_1 + c_2v_2 + \dots + c_nv_n$ to equal zero.

1.1 (5 points)

Prove that the following vectors are linearly independent:

$$v_1 = [1, 0, 0]^T, \quad v_2 = [1, 0, 1]^T, \quad v_3 = [0, 1, 1]^T \tag{1}$$

Substitute the vectors into the linear independence definition:

$$c_1 \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix} + c_2 \begin{bmatrix} 1 \\ 0 \\ 1 \end{bmatrix} + c_3 \begin{bmatrix} 0 \\ 1 \\ 1 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix} \tag{2}$$

You now have two options. **Option 1 –** you can solve the system of equations:

$$c_1 + c_2 = 0, \quad c_3 = 0, \quad c_2 + c_3 = 0 \tag{3}$$

and show that c_1, c_2, c_3 are all equal to zero. **Option 2–** rearrange the vectors into a homogeneous matrix of the form $Ac = 0$, so that:

$$\begin{bmatrix} 1 & 1 & 0 \\ 0 & 0 & 1 \\ 0 & 1 & 1 \end{bmatrix} \begin{bmatrix} c_1 \\ c_2 \\ c_3 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix} \tag{4}$$

You can now use the properties of matrix A. For example, if the determinant of A is non-zero, the columns of A are linearly independent. Using Matlab:

$$\det(A) = -1 \neq 0 \tag{5}$$

An equivalent approach is to show that matrix A is full rank. Using Matlab:

$$\text{rank}(A) = 3 \tag{6}$$

1.2 (5 points)

Prove that the following vectors are linearly dependent:

$$v_1 = [1, 1, 0]^T, \quad v_2 = [1, 2, 1]^T, \quad v_3 = [0, 1, 1]^T \tag{7}$$

Use the same approach as in the previous part. Substitute the vectors into the linear independence definition:

$$c_1 \begin{bmatrix} 1 \\ 1 \\ 0 \end{bmatrix} + c_2 \begin{bmatrix} 1 \\ 2 \\ 1 \end{bmatrix} + c_3 \begin{bmatrix} 0 \\ 1 \\ 1 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix} \tag{8}$$

Option 1 – you can solve the system of equations:

$$c_1 + c_2 = 0, \quad c_1 + 2c_2 + c_3 = 0, \quad c_2 + c_3 = 0 \tag{9}$$

where you find that c_1, c_2, c_3 do not need to be zero:

$$c_1 = -c_2, \quad c_3 = -c_2, \quad c_2 \in \mathbb{R} \tag{10}$$

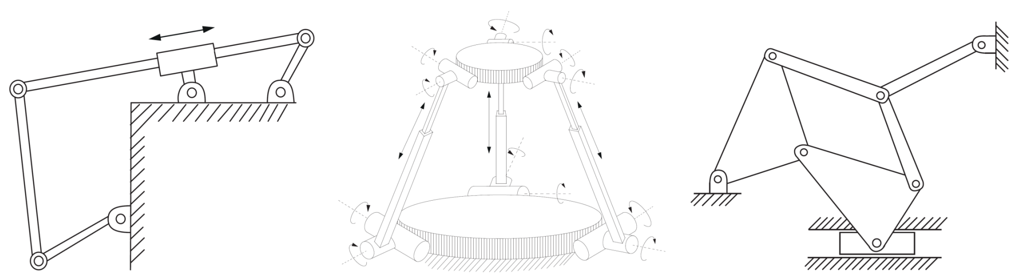
Option 2– rearrange the vectors into a homogeneous matrix of the form $Ac = 0$:

$$\begin{bmatrix} 1 & 1 & 0 \\ 1 & 2 & 1 \\ 0 & 1 & 1 \end{bmatrix} \begin{bmatrix} c_1 \\ c_2 \\ c_3 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix} \tag{11}$$

You can now use the properties of matrix A. Using Matlab:

$$\det(A) = 0, \quad \text{rank}(A) = 2 \tag{12}$$

2 Degrees-of-Freedom



(a) (b) (c)

Imagine that you are developing a new robot. Three very different mechanism designs have been proposed. For each design, you want to determine how many degrees-of-freedom the robot has. In (a) one single link is translating through the indicated slider. In (b) each of the legs are composed of a universal joint at the base, a prismatic joint, and a universal joint joint at the platform.

2.1 (5 points)

How many degrees of freedom does robot (a) have?

Applying Grübler's Formula, we find this robot has **1 DoF**. The planar robot has 6 links (including the ground), 6 revolute joints, and 1 prismatic joint. Each individual joint has one degree of freedom. We plug in with $N = 6$, $K = 7$, and $m = 3$:

$$DoF = m(N - 1 - K) + \sum_{i=1}^{K} f_i \tag{13}$$

$$DoF = 3(6 - 1 - 7) + 7 = 1 \tag{14}$$

2.2 (10 points)

How many degrees of freedom does robot (b) have?

This robot has **3 DoF**. The robot is moving in 3D space, so $m = 6$. There are a total of 8 links: each leg has 2 links, plus one for the top and one for the bottom platform (this bottom platform is ground). There are three joints on each leg: a universal joint at the base, a prismatic joint in the middle, and a universal joint at the top. Universal joints have 2 DoF and prismatic joints have 1 DoF. Plugging everything into Grübler's Formula:

$$DoF = m(N - 1 - K) + \sum_{i=1}^{K} f_i \tag{15}$$

$$DoF = 6(8 - 1 - 9) + 6 \cdot 2 + 3 \cdot 1 = 3 \tag{16}$$

Aside – this is one of the special cases where Grübler's formula is not quite right. If you lock the position of each of the three prismatic joints, according to Grübler's formula this robot should not be able to move; but, in reality, we can still twist the top platform at some specific leg geometries. In general, it's best to treat Grübler's formula as a *lower bound* on the number of degrees-of-freedom.

2.3 (5 points)

How many degrees of freedom does robot (c) have?

Applying Grübler's Formula, we find this robot has **0 DoF**. Recognize that three links intersect at the top right, and so we should interpret the single revolute joint here as *two* revolute joints. With this in mind, the planar robot has 7 links (including the ground), 8 revolute joints, and 1 prismatic joint. Each individual joint has one degree of freedom. We plug in with $N = 7$, $K = 9$, and $m = 3$:

$$DoF = m(N - 1 - K) + \sum_{i=1}^{K} f_i \tag{17}$$

$$DoF = 3(7 - 1 - 9) + 9 = 0 \tag{18}$$

Aside – This robot cannot move!

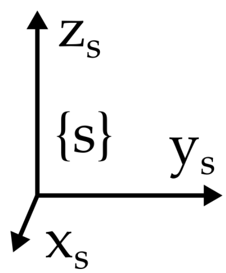
2.4 (5 points)

Once you have chosen your design you will need to use actuators to move the robot. How many actuators do you need to control robot (a)?

You need one actuator for each degree of freedom. Robot (a) therefore needs only **1 actuator**. Additional actuators would make control more challenging.

3 Properties of Rotation Matrices

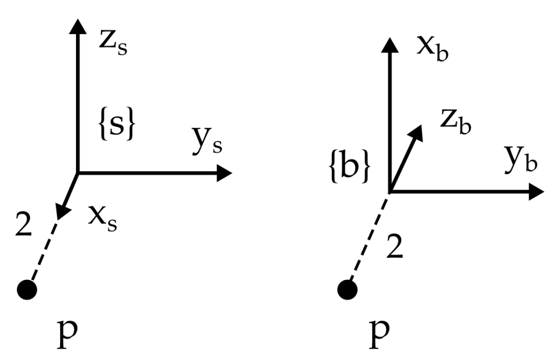
3.1 (5 points)



Using $\{s\}$, draw coordinate frame $\{b\}$ if

$$R_{sb} = \begin{bmatrix} 0 & 0 & -1 \\ 0 & 1 & 0 \\ 1 & 0 & 0 \end{bmatrix} \tag{19}$$

Point p has position $p_b = [0, 0, -2]^T$ in frame $\{b\}$. Draw this point in coordinate frame $\{s\}$. What is its position with respect to $\{s\}$?



See the drawing above. The position of p with respect to $\{s\}$ is $[2, 0, 0]^T$. You can find this using the rotation matrix: $p_s = R_{sb}p_b$

3.2 (5 points)

If R is a rotation matrix, we require that $\det(R) = +1$. Why is this? Your answer should include a drawing of a coordinate frame X where $\det(X) = -1$.

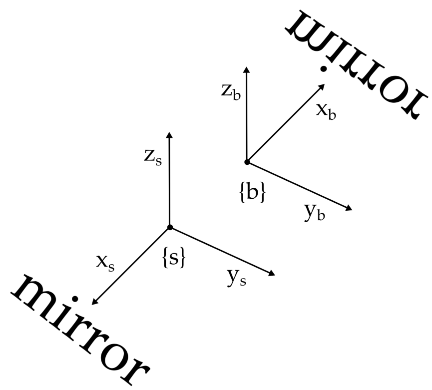


Figure 1: A right-handed frame {s} and left-handed frame {b}. The rigid body "mirror" moves between these frames. This is not a rotation (or even a possible rigid body motion).

For orthonormal matrices X, where $X^T X = I$, the determinant is either $\det(X) = +1$ or $\det(X) = -1$. Matrices with $\det(X) = +1$ correspond to *right-handed* coordinate frames, while matrices where $\det(X) = -1$ are for *left-handed* coordinate frames.

Changing from right-handed to left-handed coordinate frames is not a valid rigid-body rotation. As shown above, this actually *mirrors* or inverts the object, which is not possible for rigid bodies. We therefore restrict rotation matrices to $\det(R) = +1$ so that only right-handed coordinate frames are allowed.

Your answer must mention right- and/or left-handed coordinate frames.

3.3 (5 points)

Given vector x and rotation matrices R_1, R_2, and R_3, prove that $y = R_1 R_2 R_3 x$ has the same magnitude as x. **Hint –** define magnitude (i.e., the length) of vector x as $\|x\| = \sqrt{x^T x}$

Our objective is to prove that x and y have the same magnitude:

$$\|x\| = \sqrt{x^T x} = \sqrt{y^T y} = \|y\| \tag{20}$$

Plugging in for y, and then applying the rotation matrix property $R^T R = I$, we reach:

$$\begin{aligned} y^T y &= (R_1 R_2 R_3 x)^T R_1 R_2 R_3 x \\ &= (x^T R_3^T R_2^T R_1^T) R_1 R_2 R_3 x \\ &= x^T R_3^T R_2^T (R_1^T R_1) R_2 R_3 x \\ &= x^T R_3^T (R_2^T R_2) R_3 x \\ &= x^T (R_3^T R_3) x \\ &= x^T x \end{aligned}$$

Since $y^T y = x^T x$, we know that x and y have the same magnitude. **Aside –** Rotation matrices preserve magnitude! Rotating a vector by any rotation matrix does not change the length of that vector.

3.4 (5 points)

Given two rotation matrices R_1 and R_2 in 2D space, prove that these rotation matrices are always commutative. When we rotate in a plane, the rotation matrix is:

$$R = \begin{bmatrix} \cos(\theta) & -\sin(\theta) \\ \sin(\theta) & \cos(\theta) \end{bmatrix} \tag{21}$$

Let the rotation angle for R_1 be θ_1, and let the rotation angle for R_2 be θ_2. For these matrices to always be commutative, we must have that:

$$R_1 R_2 = R_2 R_1 \tag{22}$$

Start with the left side of this equation for now. Multiplying $R_1 R_2$, we get:

$$R_1 R_2 = \begin{bmatrix} \cos(\theta_1)\cos(\theta_2) - \sin(\theta_1)\sin(\theta_2) & -\sin(\theta_2)\cos(\theta_1) - \cos(\theta_2)\sin(\theta_1) \\ \sin(\theta_1)\cos(\theta_2) + \cos(\theta_1)\sin(\theta_2) & \cos(\theta_1)\cos(\theta_2) - \sin(\theta_1)\sin(\theta_2) \end{bmatrix} \tag{23}$$

Apply the double angle formulas. Simplifying, we reach:

$$R_1 R_2 = \begin{bmatrix} \cos(\theta_1 + \theta_2) & -\sin(\theta_1 + \theta_2) \\ \sin(\theta_1 + \theta_2) & \cos(\theta_1 + \theta_2) \end{bmatrix} \tag{24}$$

Use the same process for $R_2 R_1$. We get that:

$$R_2 R_1 = \begin{bmatrix} \cos(\theta_1 + \theta_2) & -\sin(\theta_1 + \theta_2) \\ \sin(\theta_1 + \theta_2) & \cos(\theta_1 + \theta_2) \end{bmatrix} \tag{25}$$

Hence, $R_1 R_2 = R_2 R_1$, and planar rotation matrices are always commutative. This is not the case for rotations in 3D space.

4 Implementing Rotation Matrices

4.1 (5 points)

Program the following functions. Here $Rot(x, \theta) =$ `rotx(`θ`)` means we rotate around the x-axis by θ radians.

- `R = rotx(`θ`)`
- `R = roty(`θ`)`
- `R = rotz(`θ`)`

```
function R = rotx(theta)

    R = [1 0 0;
         0 cos(theta) -sin(theta);
         0 sin(theta) cos(theta)];

end
```

See the above functions for my implementation in Matlab.

4.2 (5 points)

Multiply the following rotation matrices:

$$R = Rot(z, \pi/4) Rot(y, -\pi/3) Rot(z, \pi/2) \tag{26}$$

```
function R = roty(theta)

    R = [cos(theta) 0 sin(theta);
        0 1 0;
        -sin(theta) 0 cos(theta)];

end

function R = rotz(theta)

    R = [cos(theta) -sin(theta) 0;
        sin(theta) cos(theta) 0;
        0 0 1];

end
```

Prove that your answer R is also a rotation matrix.

Using the functions implemented above:

$$R = \begin{bmatrix} -0.7071 & -0.3536 & -0.6124 \\ 0.7071 & -0.3536 & -0.6124 \\ 0 & -0.8660 & 0.5 \end{bmatrix} \tag{27}$$

To double check this, you can show that $R^T R = I$ and $\det(R) = +1$. See my code in the following figure.

```
>> R = rotz(pi/4)*roty(-pi/3)*rotz(pi/2)

R =

   -0.7071   -0.3536   -0.6124
    0.7071   -0.3536   -0.6124
    0.0000   -0.8660    0.5000

>> R'*R

ans =

    1.0000    0.0000    0.0000
    0.0000    1.0000    0.0000
    0.0000    0.0000    1.0000

>> det(R)

ans =

     1
```

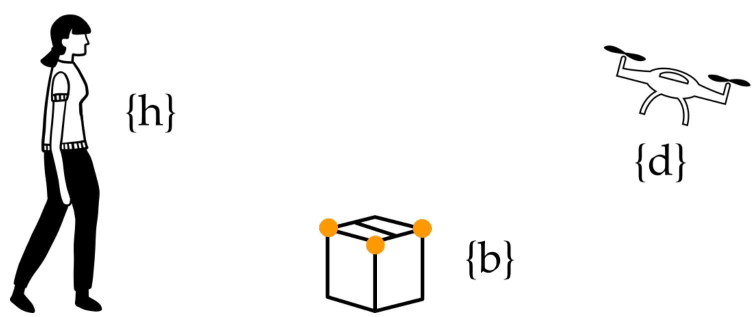
4.3 (5 points)

Using R from the previous part, show that $R^{-1} = R^T$.

See the following code.

```
>> R'

ans =

   -0.7071    0.7071    0.0000
   -0.3536   -0.3536   -0.8660
   -0.6124   -0.6124    0.5000

>> inv(R)

ans =

   -0.7071    0.7071    0.0000
   -0.3536   -0.3536   -0.8660
   -0.6124   -0.6124    0.5000
```

5 Using Rotation Matrices



A drone is picking up and delivering packages. In order to locate these packages the drone has an attached imaging system, which can detect the corners of a box. From the drone's perspective the orientation of the box is:

$$R_{db} = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1/2 & -\sqrt{3}/2 \\ 0 & \sqrt{3}/2 & 1/2 \end{bmatrix} \tag{28}$$

A person is also looking at the box. From their perspective the orientation of the box is:

$$R_{hb} = \begin{bmatrix} \sqrt{2}/2 & -\sqrt{2}/2 & 0 \\ \sqrt{2}/2 & \sqrt{2}/2 & 0 \\ 0 & 0 & 1 \end{bmatrix} \tag{29}$$

5.1 (10 points)

What is the orientation of the person relative to the drone?

We are given R_{db} and R_{hb}. We want to find R_{dh}, the orientation of the human relative to the drone. Using the subscript cancellation rule, we have that:

$$R_{dh} = R_{db}R_{bh} \tag{30}$$

Although this is correct, we do not know R_{bh}. Remember that $R_{hb}^T = R_{bh}$, i.e., transposing a rotation matrix switches the frame of reference:

$$R_{dh} = R_{db}R_{hb}^T \tag{31}$$

Plugging in and multiplying:

$$R_{dh} = \begin{bmatrix} 0.7071 & 0.7071 & 0 \\ -0.3536 & 0.3536 & -0.8660 \\ -0.6124 & 0.6124 & 0.5 \end{bmatrix} \tag{32}$$

5.2 (5 points)

What is the orientation of the drone relative to the person?

We want R_{hd}. Taking the transpose switches the frame of reference:

$$R_{hd} = R_{dh}^T \tag{33}$$

$$R_{hd} = \begin{bmatrix} 0.7071 & -0.3536 & -0.6124 \\ 0.7071 & 0.3536 & 0.6124 \\ 0 & -0.8660 & 0.5 \end{bmatrix} \tag{34}$$

5.3 (15 points)

From the human's perspective the position of the box is $p_h = \overrightarrow{hb}_h = [1, 2, 0]^T$. From the drone's perspective the position of the box is $p_d = \overrightarrow{db}_d = [2, 0, -2]^T$.

What is the position of the drone in frame $\{h\}$?

To find the position of the drone in frame $\{h\}$ we must add two components. The vector from the human to the box expressed in frame $\{h\}$, and the vector from the box to the drone in frame $\{h\}$. In other words, we want:

$$\overrightarrow{hd}_h = \overrightarrow{hb}_h + \overrightarrow{bd}_h \tag{35}$$

To get $\overrightarrow{bd}_h$ we need to do two things. (1) Rotate $\overrightarrow{db}_d$ into frame $\{h\}$. (2) The output is a vector from the drone to the box expressed in frame $\{h\}$. We want the vector from the box to the drone. Switch the direction of the vector by taking the negative:

$$\overrightarrow{hd}_h = \overrightarrow{hb}_h - R_{hd}\overrightarrow{db}_d \tag{36}$$

$$p_{hd} = \begin{bmatrix} 1 \\ 2 \\ 0 \end{bmatrix} - \begin{bmatrix} 0.7071 & -0.3536 & -0.6124 \\ 0.7071 & 0.3536 & 0.6124 \\ 0 & -0.8660 & 0.5 \end{bmatrix} \begin{bmatrix} 2 \\ 0 \\ -2 \end{bmatrix} = \begin{bmatrix} -1.6390 \\ 1.8105 \\ 1 \end{bmatrix} \tag{37}$$