Dynamics

Reading: Robot Modeling and Control 7.3

This Lecture

- What are the equations of motion of a robot arm?
- What are the mass matrix, Coriolis matrix, and gravity vector?
- How do we obtain these terms from kinetic and potential energy?
- How can we simulate these dynamics?

Equation of Motion

The **dynamics** of a serial robot arm with n joints is formulated as:

$$\tau = M(\theta)\ddot{\theta} + C\left(\theta, \dot{\theta}\right)\dot{\theta} + g(\theta)$$

- $\boldsymbol{\tau}$ is $n \times 1$ vector of joint torque
- θ is $n \times 1$ vector of joint position
- $\dot{\theta}$ is $n \times 1$ vector of joint velocity
- $\ddot{\theta}$ is $n \times 1$ vector of joint acceleration

Equation of Motion

The **dynamics** of a serial robot arm with n joints is formulated as:

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- $\boldsymbol{M}$ is $n \times n$ **mass matrix**
- $\boldsymbol{C}$ is $n \times n$ **Coriolis matrix**
- $\boldsymbol{g}$ is $n \times 1$ **gravity vector**

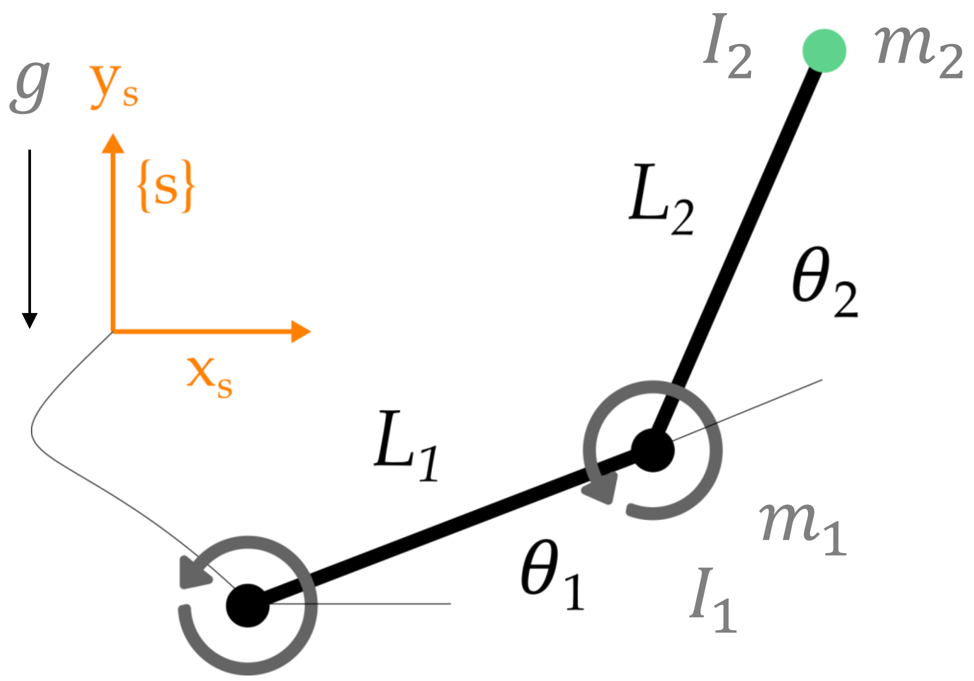
Equation of Motion

The **dynamics** of a serial robot arm with n joints is formulated as:

$$\tau = M(\theta)\ddot{\theta} + C\left(\theta, \dot{\theta}\right)\dot{\theta} + g(\theta)$$

*These dynamics come from applying the **Euler-Lagrange equation** to the kinetic and potential energy of serial robot arms.*

Example



Mass

m_1 is the mass of link 1

m_2 is the mass of link 2

Center of mass at the end of each link

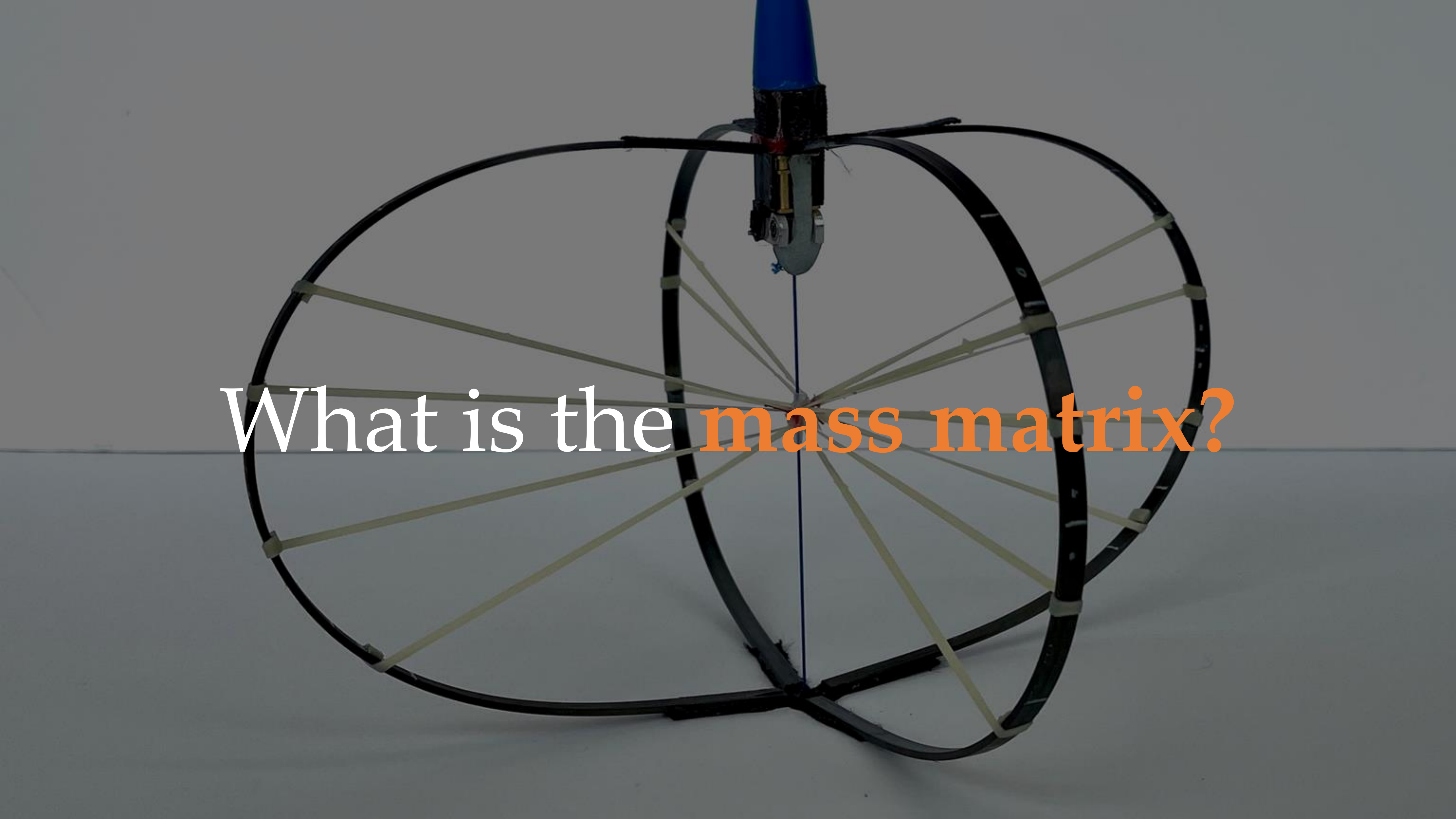
Inertia

I_1 is the inertia of link 1 about the z axis

I_2 is the inertia of link 2 about the z axis

Gravity

Gravity acts along the $-y$ axis


What is the mass matrix?

Mass Matrix

We get the $n \times n$ mass matrix from **kinetic energy**

$$K(\theta, \dot{\theta}) = \frac{1}{2}\dot{\theta}^T M \dot{\theta}$$

$$M(\theta) = \sum_{i=1}^{n} m_i {J_{v_i}}^T J_{v_i} + {J_{\omega_i}}^T R_i \boldsymbol{I}_i {R_i}^T J_{\omega_i}$$

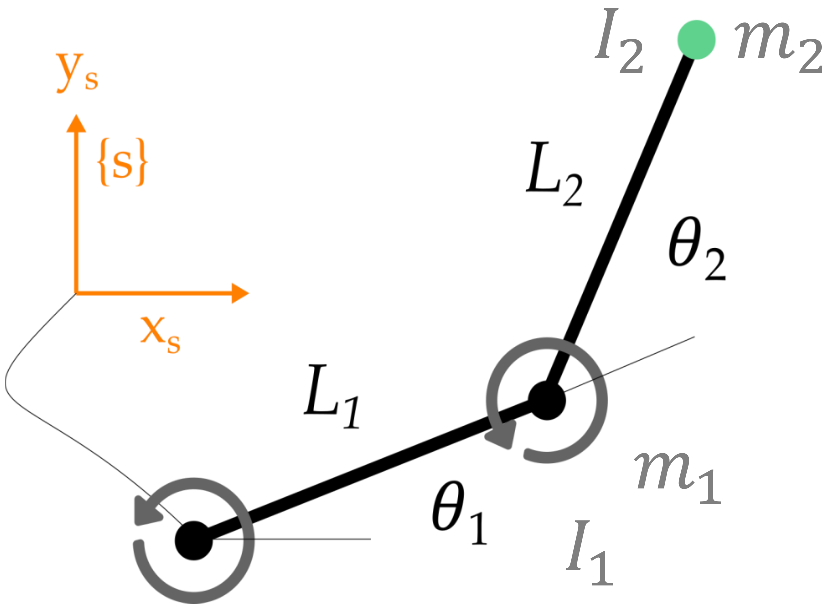
Mass matrix is symmetric, positive definite

Mass Matrix

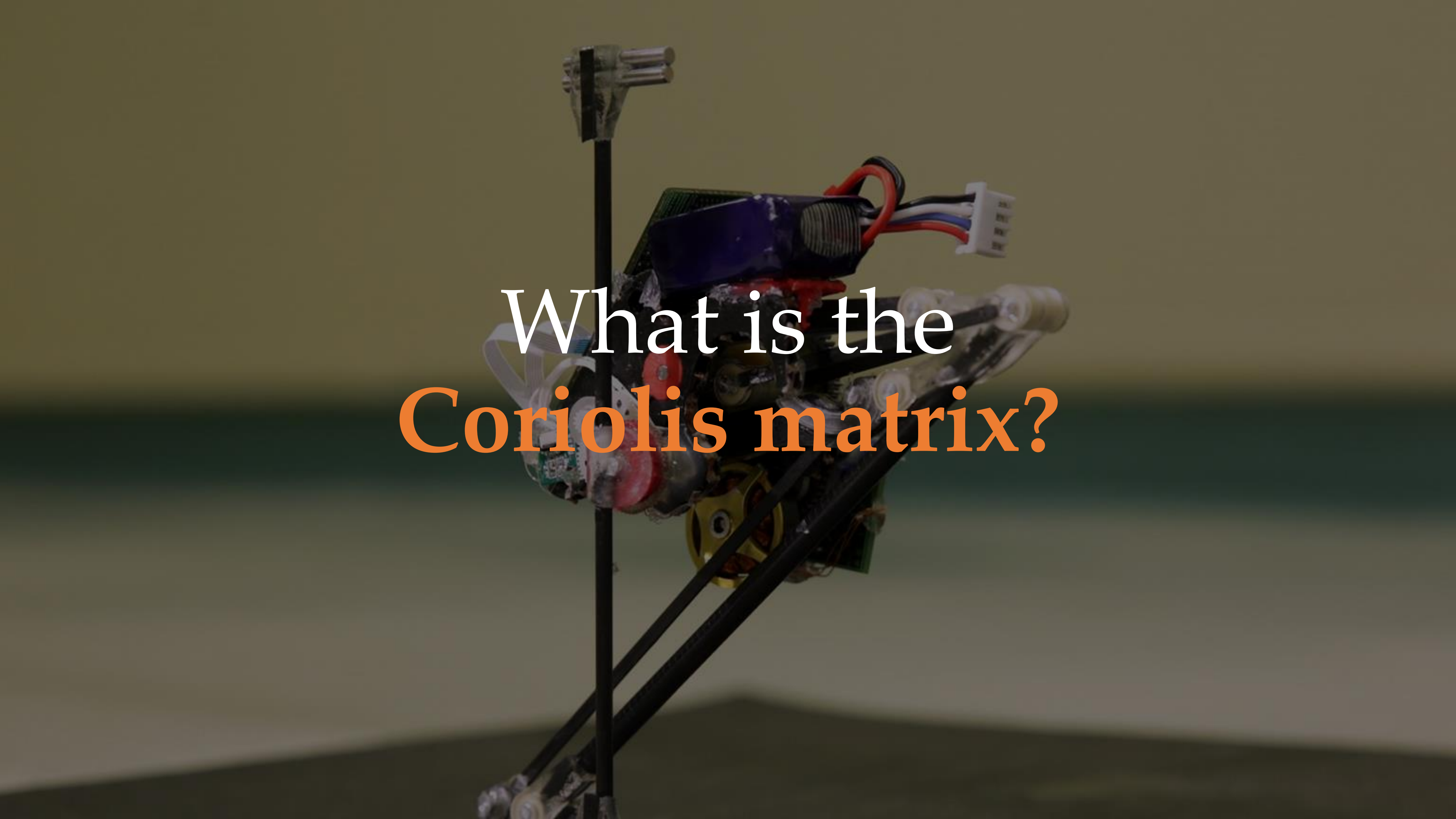
Summing the kinetic energy for both links:

$$K(\theta, \dot{\theta}) = \frac{1}{2}\dot{\theta}^T M \dot{\theta}$$

$$M = \begin{bmatrix} m_1 L_1^2 + m_2(L_1^2 + L_2^2 + 2L_1 L_2 c_2) + I_1 + I_2 & m_2(L_2^2 + L_1 L_2 c_2) + I_2 \\ m_2(L_2^2 + L_1 L_2 c_2) + I_2 & m_2 L_2^2 + I_2 \end{bmatrix}$$



What is the **Coriolis matrix?**



Coriolis Matrix

We get the $n \times n$ Coriolis matrix from **mass matrix**

$$M(\theta) = \begin{bmatrix} m_{11} & m_{12} & \\ m_{21} & m_{22} & \\ & & \ddots \end{bmatrix}$$

Start by getting kinetic energy and finding M
Here m_{ij} is the ij^{th} element of matrix M

Coriolis Matrix

We get the $n \times n$ Coriolis matrix from **mass matrix**

$$C(\theta, \dot{\theta}) = \begin{bmatrix} c_{11} & c_{12} & \\ c_{21} & c_{22} & \\ & & \ddots \end{bmatrix}$$

$$c_{kj} = \sum_{i=1}^{n} \frac{1}{2} \left\{ \frac{\partial m_{kj}}{\partial \theta_i} + \frac{\partial m_{ki}}{\partial \theta_j} - \frac{\partial m_{ij}}{\partial \theta_k} \right\} \dot{\theta}_i$$

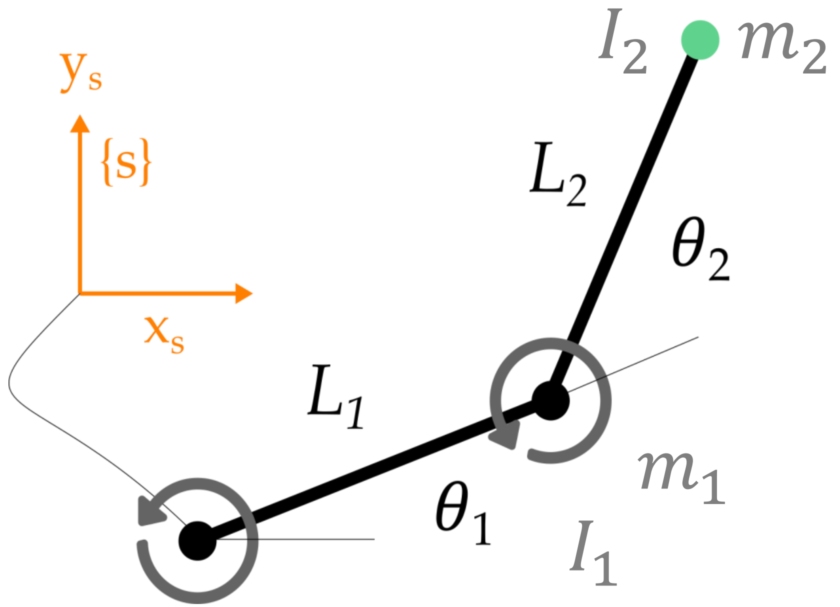
c_{kj} is the kj^{th} element of the Corioilis matrix C

Coriolis Matrix

Given M, find the Coriolis matrix:

$$M = \begin{bmatrix} m_1 L_1^2 + m_2(L_1^2 + L_2^2 + 2L_1 L_2 c_2) + I_1 + I_2 & m_2(L_2^2 + L_1 L_2 c_2) + I_2 \\ m_2(L_2^2 + L_1 L_2 c_2) + I_2 & m_2 L_2^2 + I_2 \end{bmatrix}$$

$$c_{kj} = \sum_{i=1}^{n} \frac{1}{2} \left\{ \frac{\partial m_{kj}}{\partial \theta_i} + \frac{\partial m_{ki}}{\partial \theta_j} - \frac{\partial m_{ij}}{\partial \theta_k} \right\} \dot{\theta}_i$$



Coriolis Matrix

Given M, find the Coriolis matrix:

$$M = \begin{bmatrix} m_1 L_1^2 + m_2(L_1^2 + L_2^2 + 2L_1L_2c_2) + I_1 + I_2 & m_2(L_2^2 + L_1L_2c_2) + I_2 \\ m_2(L_2^2 + L_1L_2c_2) + I_2 & m_2L_2^2 + I_2 \end{bmatrix}$$

$$c_{11} = \frac{1}{2}\left\{\frac{\partial m_{11}}{\partial \theta_1} + \frac{\partial m_{11}}{\partial \theta_1} - \frac{\partial m_{11}}{\partial \theta_1}\right\}\dot{\theta}_1 + \frac{1}{2}\left\{\frac{\partial m_{11}}{\partial \theta_2} + \frac{\partial m_{12}}{\partial \theta_1} - \frac{\partial m_{21}}{\partial \theta_1}\right\}\dot{\theta}_2 = -m_2L_1L_2s_2\dot{\theta}_2$$

M does not depend on θ_1, so **these terms** are zero

Only non-zero term

Coriolis Matrix

Given M, find the Coriolis matrix:

$$M = \begin{bmatrix} m_1 L_1^2 + m_2(L_1^2 + L_2^2 + 2L_1 L_2 c_2) + I_1 + I_2 & m_2(L_2^2 + L_1 L_2 c_2) + I_2 \\ m_2(L_2^2 + L_1 L_2 c_2) + I_2 & m_2 L_2^2 + I_2 \end{bmatrix}$$

$$c_{21} = \frac{1}{2}\left\{\frac{\partial m_{21}}{\partial \theta_1} + \frac{\partial m_{21}}{\partial \theta_1} - \underbrace{\frac{\partial m_{11}}{\partial \theta_2}}_{\text{Only non-zero term}}\right\}\dot{\theta}_1 + \frac{1}{2}\left\{\underbrace{\frac{\partial m_{21}}{\partial \theta_2}}_{} + \frac{\partial m_{22}}{\partial \theta_1} - \underbrace{\frac{\partial m_{21}}{\partial \theta_2}}_{}\right\}\dot{\theta}_2 = m_2 L_1 L_2 s_2 \dot{\theta}_1$$

These two terms cancel

Coriolis Matrix

Given M, find the Coriolis matrix:

$$M = \begin{bmatrix} m_1L_1^2 + m_2(L_1^2 + L_2^2 + 2L_1L_2c_2) + I_1 + I_2 & m_2(L_2^2 + L_1L_2c_2) + I_2 \\ m_2(L_2^2 + L_1L_2c_2) + I_2 & m_2L_2^2 + I_2 \end{bmatrix}$$

$$c_{12} = \frac{1}{2}\left\{\frac{\partial m_{12}}{\partial \theta_1} + \underline{\frac{\partial m_{11}}{\partial \theta_2}} - \frac{\partial m_{12}}{\partial \theta_1}\right\}\dot{\theta}_1 + \frac{1}{2}\left\{\underline{\frac{\partial m_{12}}{\partial \theta_2} + \frac{\partial m_{12}}{\partial \theta_2}} - \frac{\partial m_{22}}{\partial \theta_1}\right\}\dot{\theta}_2 = -m_2L_1L_2s_2(\dot{\theta}_1 + \dot{\theta}_2)$$

Only non-zero terms

Coriolis Matrix

Given M, find the Coriolis matrix:

$$M = \begin{bmatrix} m_1L_1^2 + m_2(L_1^2 + L_2^2 + 2L_1L_2c_2) + I_1 + I_2 & m_2(L_2^2 + L_1L_2c_2) + I_2 \\ m_2(L_2^2 + L_1L_2c_2) + I_2 & m_2L_2^2 + I_2 \end{bmatrix}$$

$$c_{22} = \frac{1}{2}\left\{\frac{\partial m_{22}}{\partial \theta_1} + \underbrace{\frac{\partial m_{21}}{\partial \theta_2} - \frac{\partial m_{12}}{\partial \theta_2}}\right\}\dot{\theta}_1 + \frac{1}{2}\left\{\underbrace{\frac{\partial m_{22}}{\partial \theta_2} + \frac{\partial m_{22}}{\partial \theta_2} - \frac{\partial m_{22}}{\partial \theta_2}}\right\}\dot{\theta}_2 = 0$$

These terms cancel because $m_{21} = m_{12}$

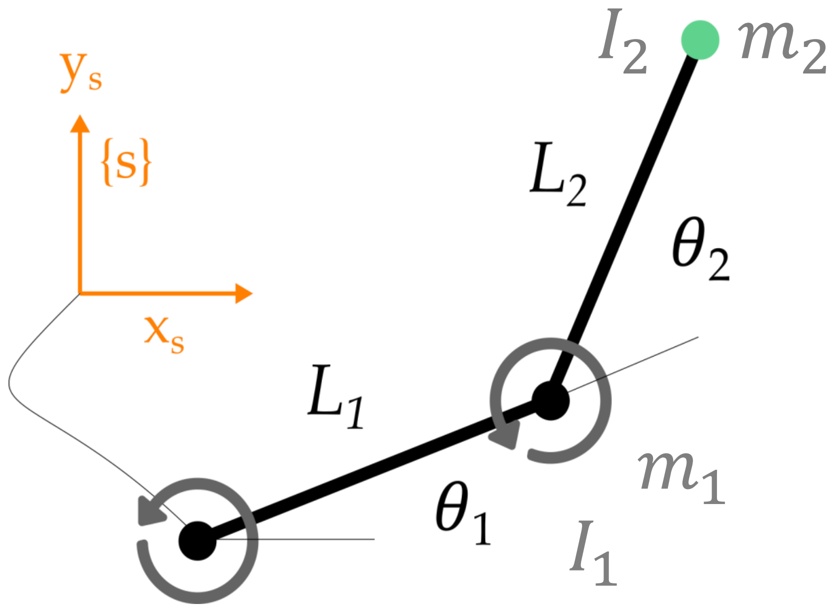
m_{22} does not depend on θ

Coriolis Matrix

Given M, find the Coriolis matrix:

$$c_{kj} = \sum_{i=1}^{n} \frac{1}{2} \left\{ \frac{\partial m_{kj}}{\partial \theta_i} + \frac{\partial m_{ki}}{\partial \theta_j} - \frac{\partial m_{ij}}{\partial \theta_k} \right\} \dot{\theta}_i$$

$$C = \begin{bmatrix} -m_2 L_1 L_2 s_2 \dot{\theta}_2 & -m_2 L_1 L_2 s_2 (\dot{\theta}_1 + \dot{\theta}_2) \\ m_2 L_1 L_2 s_2 \dot{\theta}_1 & 0 \end{bmatrix}$$



What is the **gravity vector?**

Gravity Vector

We get the $n \times 1$ gravity vector from **potential energy**

$$g(\theta) = \begin{bmatrix} g_1 \\ g_2 \\ \vdots \end{bmatrix}, \qquad g_i = \frac{\partial P(\theta)}{\partial \theta_i}$$

$P(\theta)$ is the total potential energy of the arm,
and we take the partial derivative for each joint

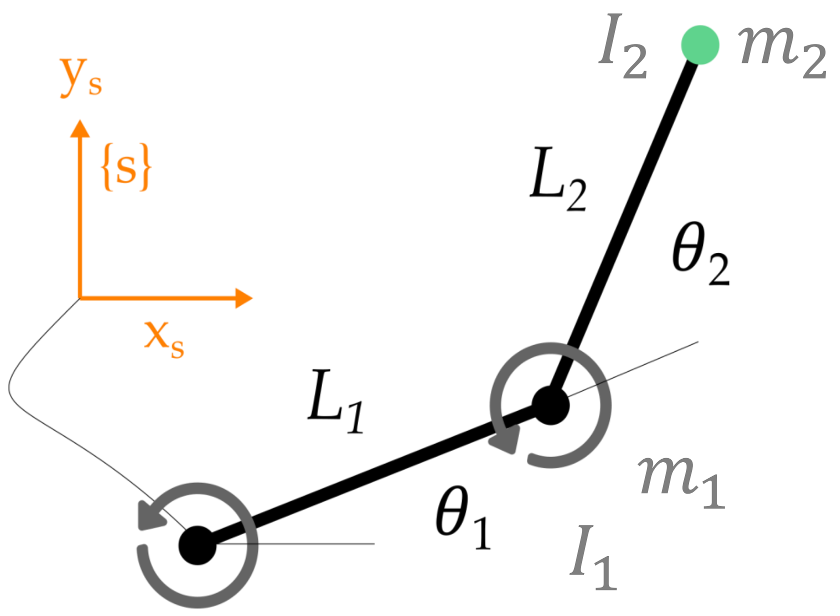
Gravity Vector

Given P, find the gravity vector:

$$P(\theta) = g(m_1 + m_2)L_1 s_1 + g m_2 L_2 s_{12}$$

g is acceleration due to gravity

$$g_i = \frac{\partial P(\theta)}{\partial \theta_i}$$

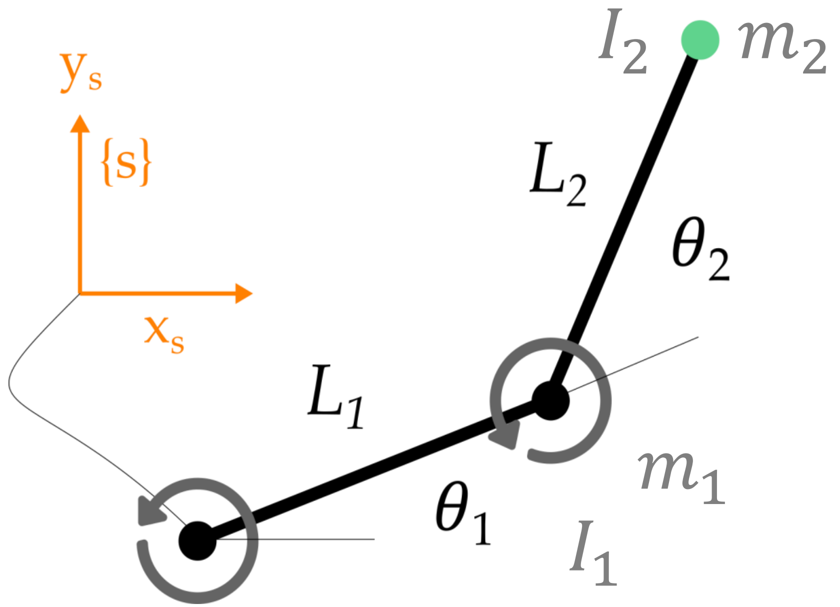


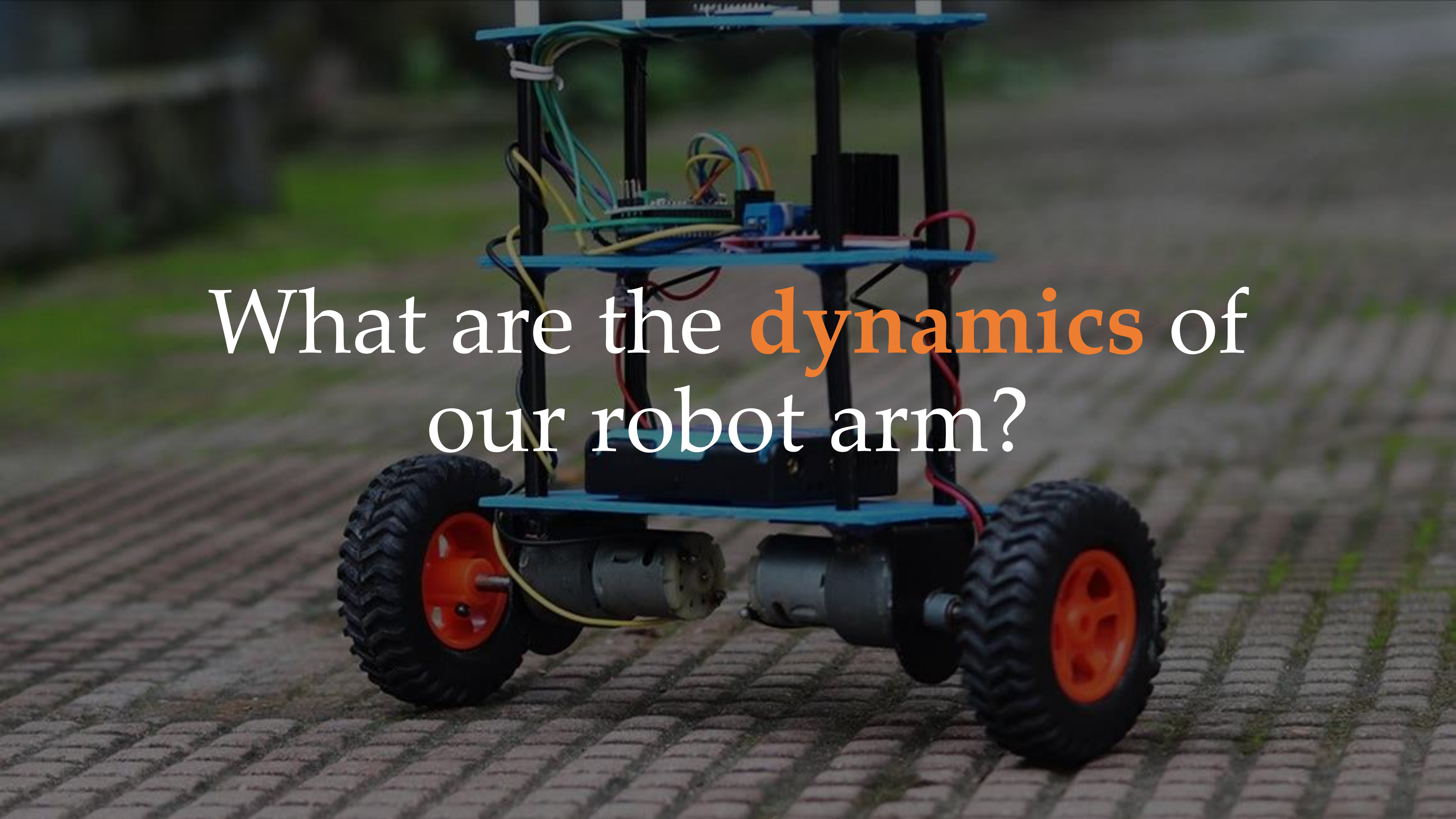
Gravity Vector

Given P, find the gravity vector:

$$P(\theta) = g(m_1 + m_2)L_1 s_1 + g m_2 L_2 s_{12}$$

$$g(\theta) = \begin{bmatrix} g(m_1 + m_2)L_1 c_1 + g m_2 L_2 c_{12} \\ g m_2 L_2 c_{12} \end{bmatrix}$$




What are the dynamics of
our robot arm?

Dynamics

Plug the mass matrix, Coriolis matrix, and gravity vector into the **dynamics**:

$$\tau = M(\theta)\ddot{\theta} + C\left(\theta, \dot{\theta}\right)\dot{\theta} + g(\theta)$$

- $\boldsymbol{M}$ is $n \times n$ **mass matrix**
- $\boldsymbol{C}$ is $n \times n$ Coriolis matrix
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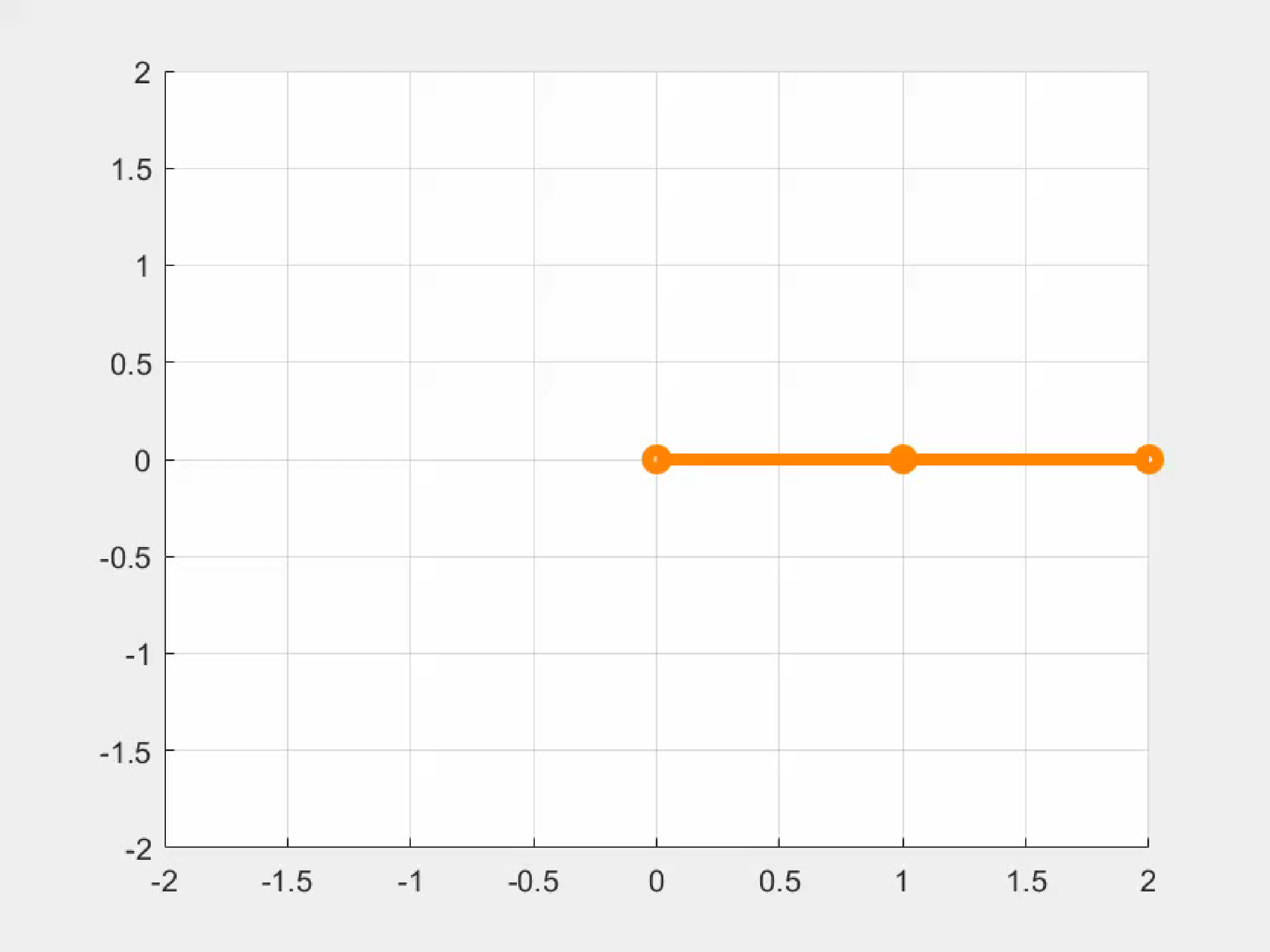
Dynamics

To **simulate**, solve for acceleration then integrate to get velocity and position:

$$\ddot{\theta}^{t+1} = M(\theta^t)^{-1}\left(\tau - C(\theta^t, \dot{\theta}^t)\dot{\theta}^t - g(\theta^t)\right)$$

$$\dot{\theta}^{t+1} = \dot{\theta}^t + \Delta T \cdot \ddot{\theta}^t$$
$$\theta^{t+1} = \theta^t + \Delta T \cdot \dot{\theta}^t$$

Dynamics



Dynamics

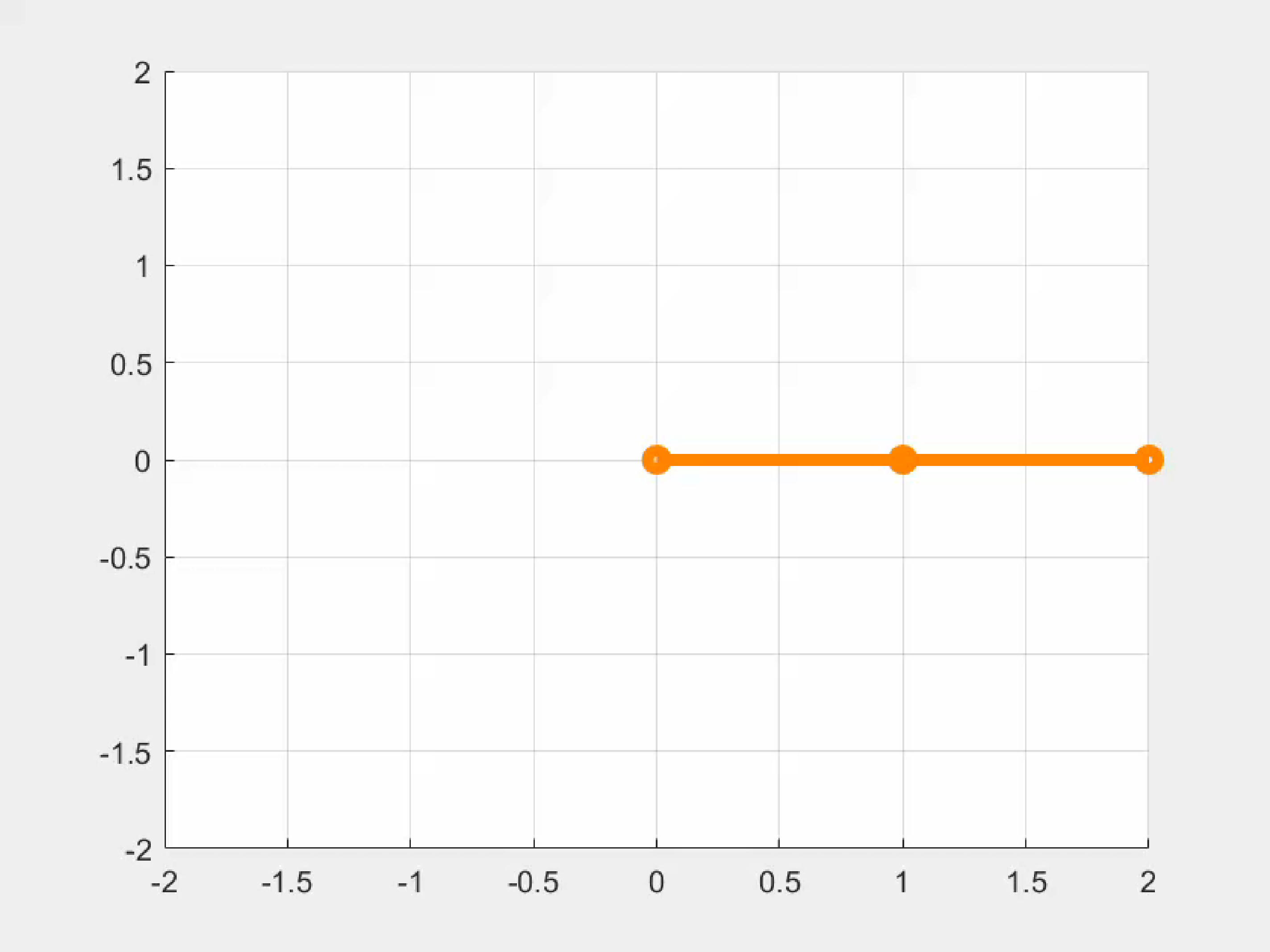
You can incorporate friction into the model using an additional term:

$$\tau = M(\theta)\ddot{\theta} + C\big(\theta, \dot{\theta}\big)\dot{\theta} + \underline{B\dot{\theta}} + g(\theta)$$

- $\boldsymbol{B}$ is $n \times n$ **friction matrix**
- $\boldsymbol{B}$ is often a diagonal matrix with positive terms
- This choice captures viscous friction

$$B = \begin{bmatrix} b_1 & 0 & 0 \\ 0 & b_2 & 0 \\ 0 & 0 & \ddots \end{bmatrix}$$

Dynamics



This Lecture

- What are the equations of motion of a robot arm?
- What are the mass matrix, Coriolis matrix, and gravity vector?
- How do we obtain these terms from kinetic and potential energy?
- How can we simulate these dynamics?

Next Lecture

- Practicing dynamics with a new example