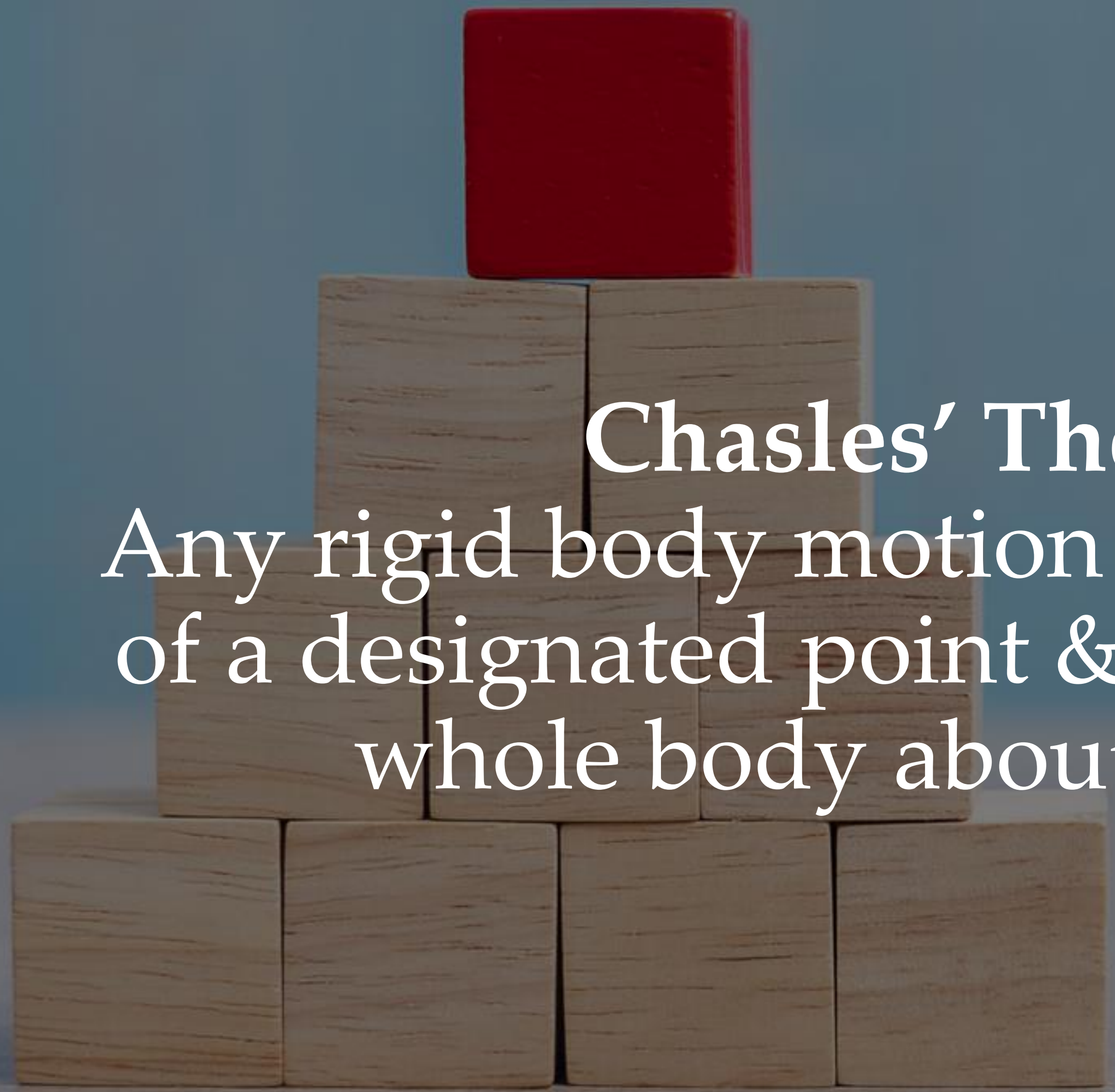
Transformations

Reading: Modern Robotics 3.3.1

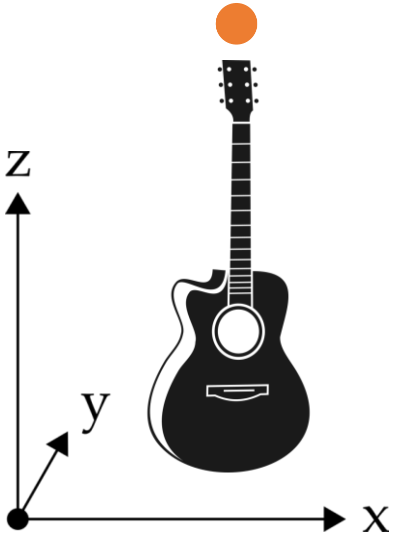
This Lecture

- How do we combine rotation and position?
- How do we capture rigid-body motion?
- What is a transformation matrix?



Chasles' Theorem

Any rigid body motion is the **translation** of a designated point & a **rotation** of the whole body about that point.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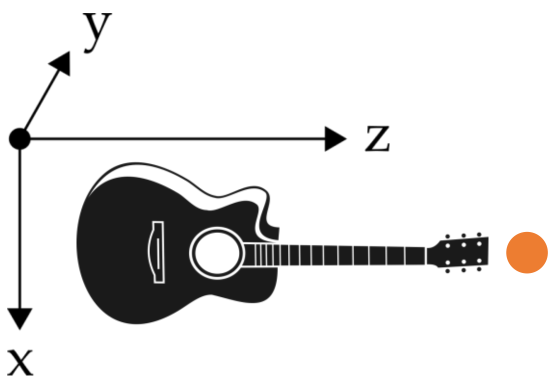
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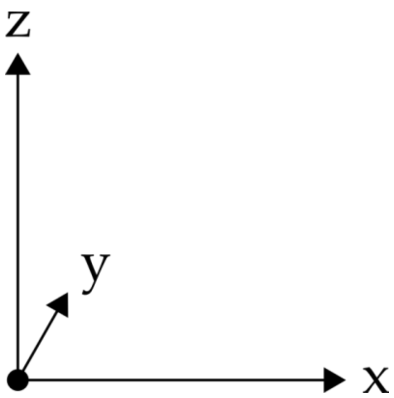
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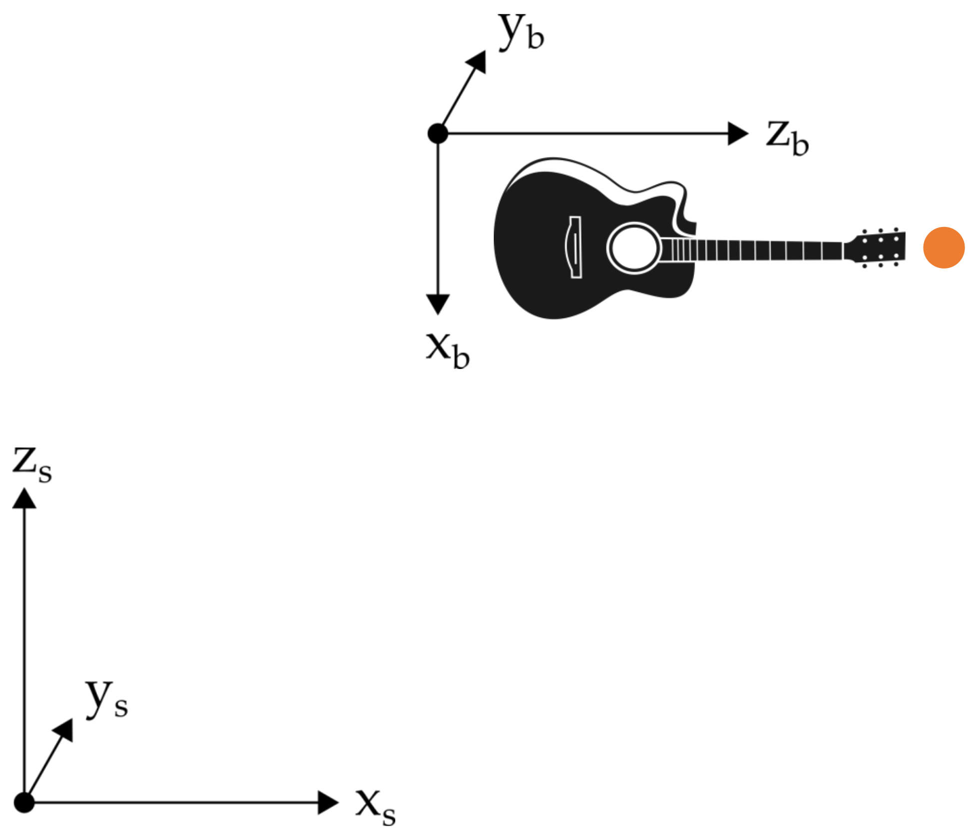
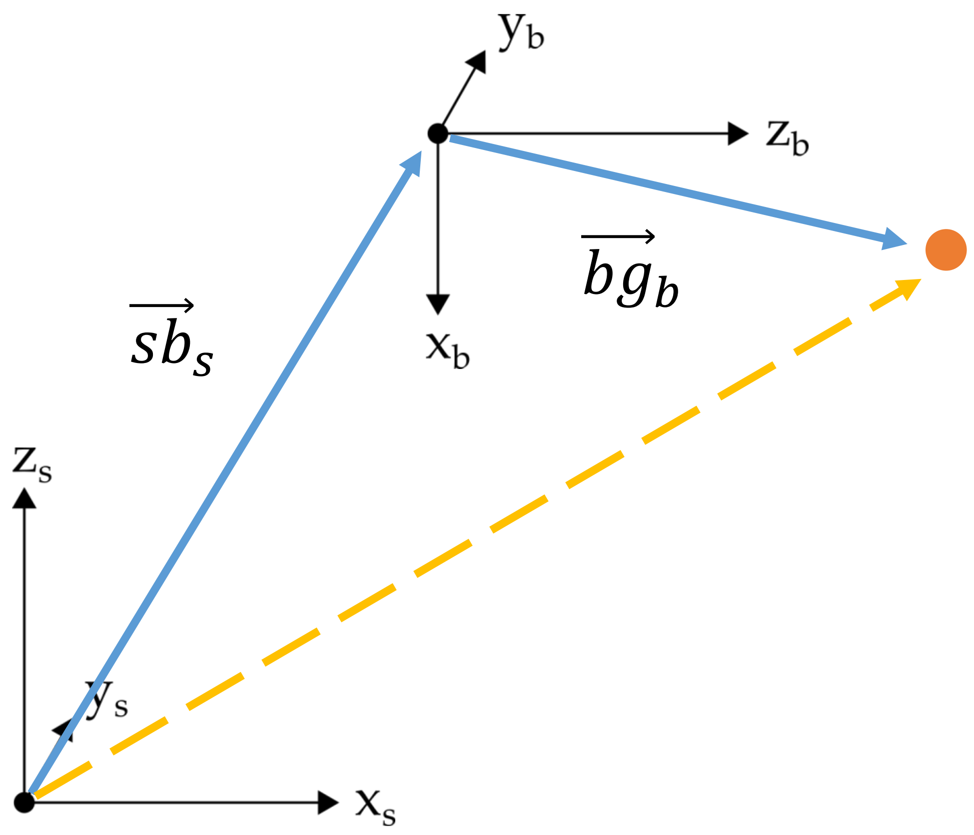


How do we capture this motion?

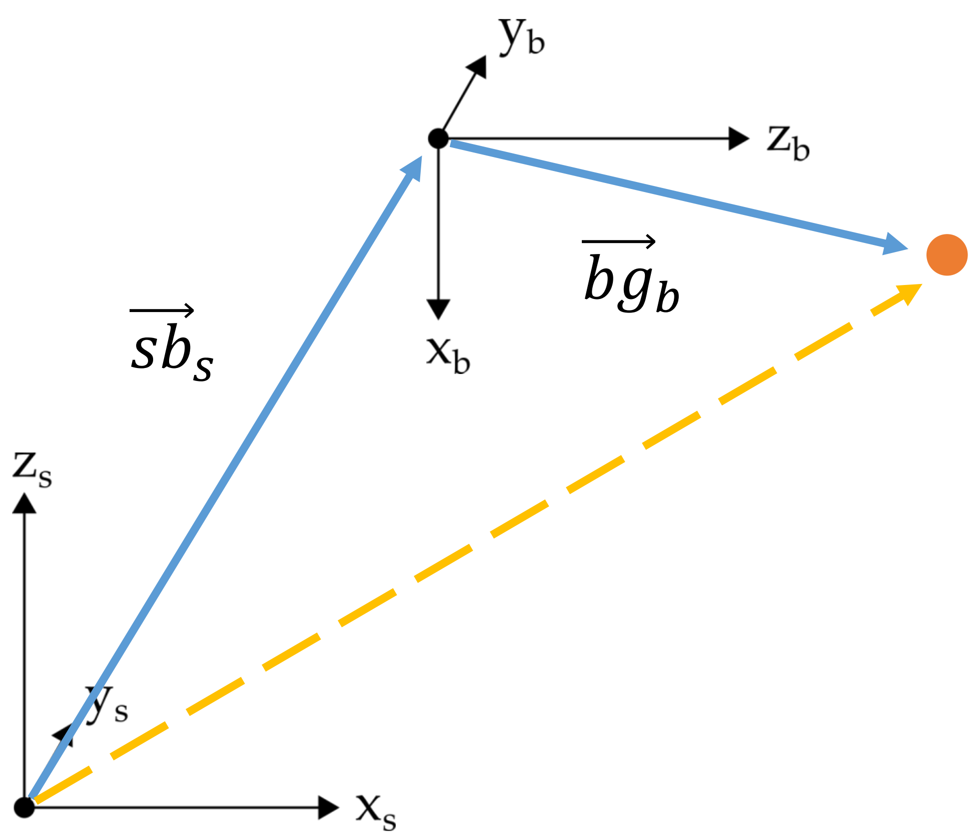
What is the position of the guitar in our original frame $\{s\}$?



You are given:

- $\overrightarrow{bg}_b$ position of guitar in $\{b\}$
- $\overrightarrow{sb}_s$ position of $\{b\}$ in $\{s\}$
- R_{sb} orientation of {b} relative to {s}

What is p_{sg} the position of the guitar in our original frame $\{s\}$?



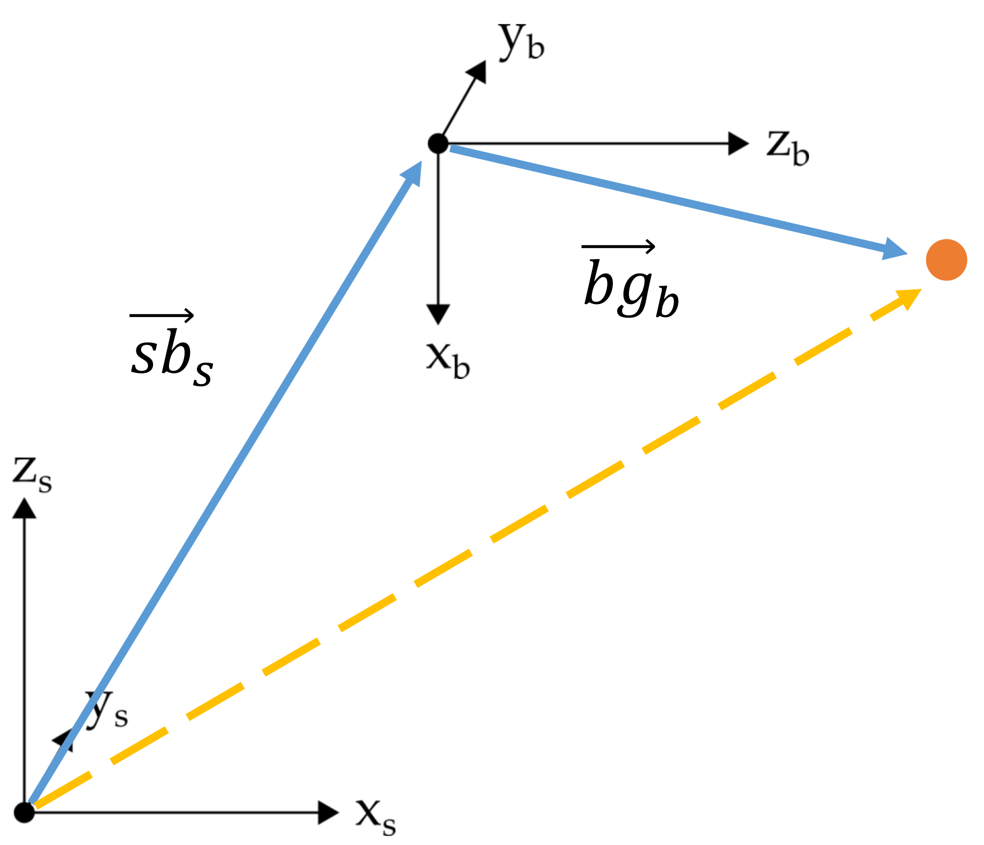
$$R_{sb} = \begin{bmatrix} 0 & 0 & 1 \\ 0 & 1 & 0 \\ -1 & 0 & 0 \end{bmatrix} \quad \overrightarrow{sb}_s = \begin{bmatrix} 3 \\ 0 \\ 7 \end{bmatrix} \quad \overrightarrow{bg}_b = \begin{bmatrix} 1 \\ 0 \\ 3 \end{bmatrix}$$

Naïve Solution

Add the vectors in their current frames?

$$p_{sg} = \overrightarrow{sg}_s = \boldsymbol{\overrightarrow{bg}_b} + \overrightarrow{sb}_s$$

$$p_{sg} = \begin{bmatrix} 4 \\ 0 \\ 10 \end{bmatrix}$$



$$R_{sb} = \begin{bmatrix} 0 & 0 & 1 \\ 0 & 1 & 0 \\ -1 & 0 & 0 \end{bmatrix} \quad \overrightarrow{sb}_s = \begin{bmatrix} 3 \\ 0 \\ 7 \end{bmatrix} \quad \overrightarrow{bg}_b = \begin{bmatrix} 1 \\ 0 \\ 3 \end{bmatrix}$$

Correct Solution

Put all vectors in frame {s} then add

$$p_{sg} = \underbrace{R_{sb}\overrightarrow{bg}_b}_{\text{\{b\} to \{g\} in frame \{s\}}} + \underbrace{\overrightarrow{sb}_s}_{\text{\{s\} to \{b\} in frame \{s\}}}$$

$$p_{sg} = \begin{bmatrix} 6 \\ 0 \\ 6 \end{bmatrix}$$

Transformation Matrix

We can write any rigid-body motion as a matrix:

$$p_{sg} = R_{sb}p_{bg} + p_{sb}$$

$$\begin{bmatrix} p_{sg} \\ 1 \end{bmatrix} = \begin{bmatrix} \quad & \quad \end{bmatrix} \begin{bmatrix} p_{bg} \\ 1 \end{bmatrix}$$

New position

Original position

Transformation Matrix

We can write any rigid-body motion as a matrix:

$$p_{sg} = R_{sb} p_{bg} + p_{sb}$$

$$\begin{bmatrix} p_{sg} \\ 1 \end{bmatrix} = \begin{bmatrix} \boldsymbol{R_{sb}} & \boldsymbol{p_{sb}} \\ \boldsymbol{0} & \boldsymbol{1} \end{bmatrix} \begin{bmatrix} p_{bg} \\ 1 \end{bmatrix}$$

Transformation Matrix

We can write any rigid-body motion as a matrix:

$$p_{sg} = R_{sb}p_{bg} + p_{sb}$$

$$\begin{bmatrix} p_{sg} \\ 1 \end{bmatrix} = \underbrace{\begin{bmatrix} \boldsymbol{R_{sb}} & \boldsymbol{p_{sb}} \\ \boldsymbol{0} & \boldsymbol{1} \end{bmatrix}}\begin{bmatrix} p_{bg} \\ 1 \end{bmatrix}$$

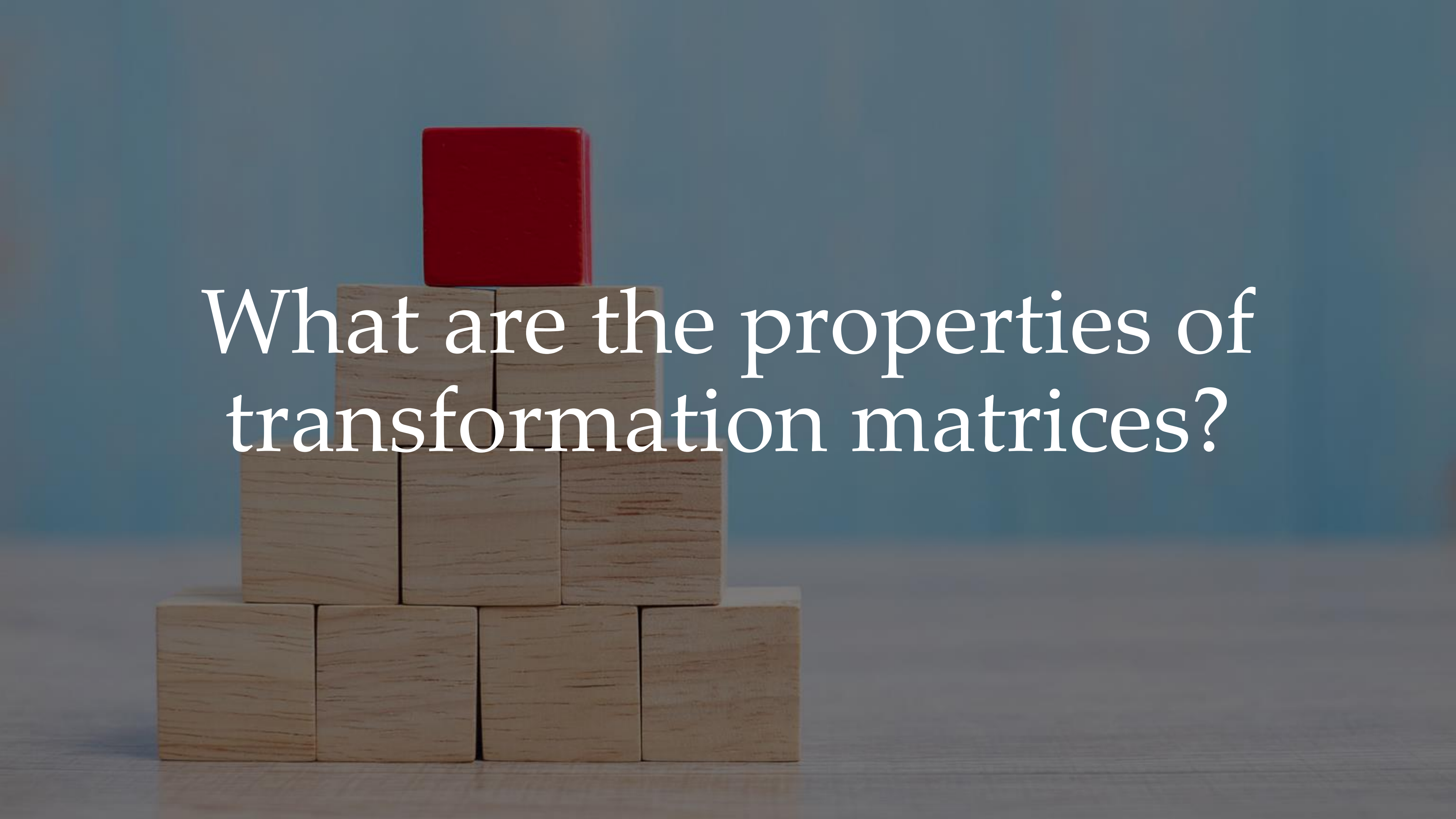
This matrix is called a transformation matrix

Transformation Matrix

A (homogeneous) transformation matrix T is a 4×4 matrix where:

$$T = \begin{bmatrix} R & p \\ 0 & 1 \end{bmatrix} = \begin{bmatrix} r_{11} & r_{12} & r_{13} & p_1 \\ r_{21} & r_{22} & r_{23} & p_2 \\ r_{31} & r_{32} & r_{33} & p_3 \\ 0 & 0 & 0 & 1 \end{bmatrix}$$

Transformation T **combines** a rotation matrix R and a position vector p



What are the properties of transformation matrices?

Properties

Product of two transformation matrices is a transformation matrix

Proof

Let T_1 and T_2 be two transformation matrices, and $T_3 = T_1 T_2$

$$T_3 = \begin{bmatrix} R_1 & p_1 \\ 0 & 1 \end{bmatrix} \begin{bmatrix} R_2 & p_2 \\ 0 & 1 \end{bmatrix}$$

$$T_3 = \begin{bmatrix} R_1 R_2 & R_1 p_2 + p_1 \\ 0 & 1 \end{bmatrix}$$

Properties

Inverse of a transformation matrix is:

$$T^{-1} = \begin{bmatrix} R^T & -R^T p \\ 0 & 1 \end{bmatrix}$$

Proof

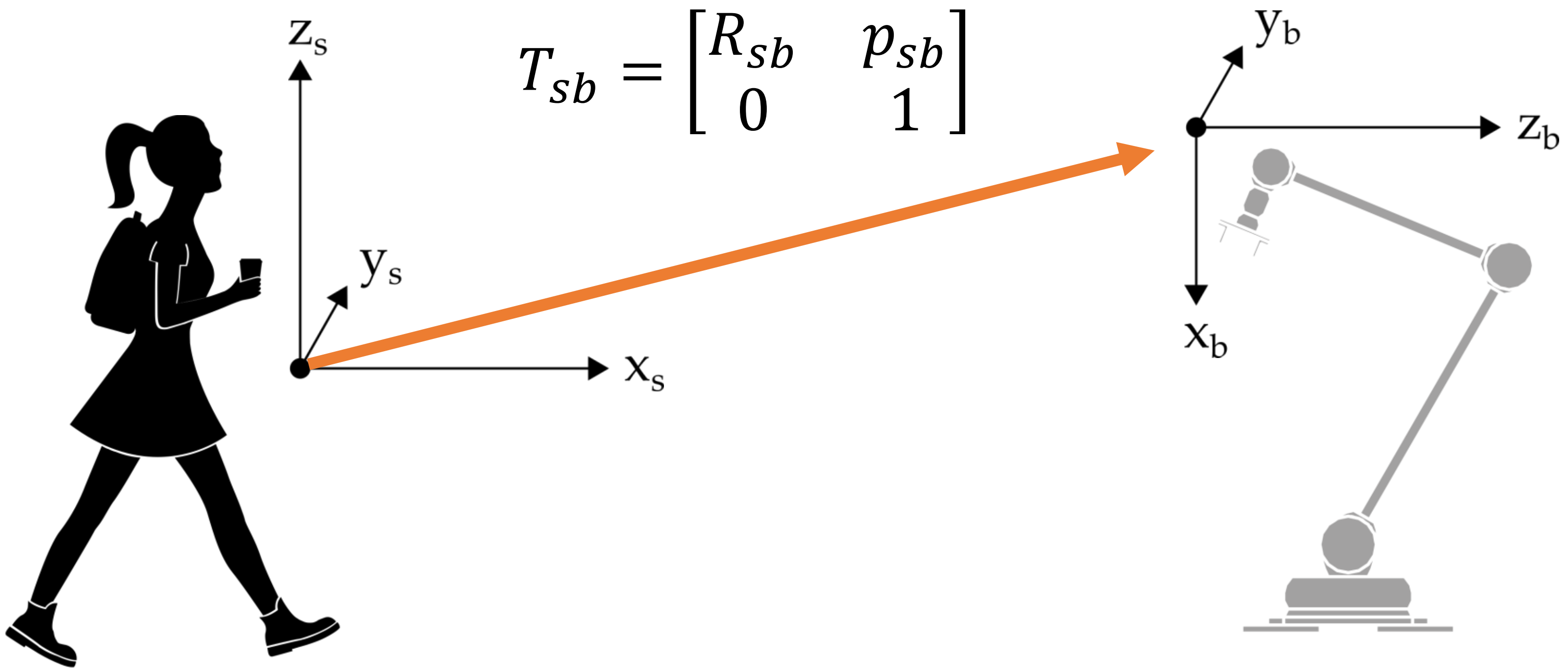
We need to show that $T^{-1}T = I$

$$T^{-1}T = \begin{bmatrix} R^T & -R^T p \\ 0 & 1 \end{bmatrix} \begin{bmatrix} R & p \\ 0 & 1 \end{bmatrix} = \begin{bmatrix} R^T R & R^T p - R^T p \\ 0 & 1 \end{bmatrix}$$

Properties

Inverse switches the frame of reference

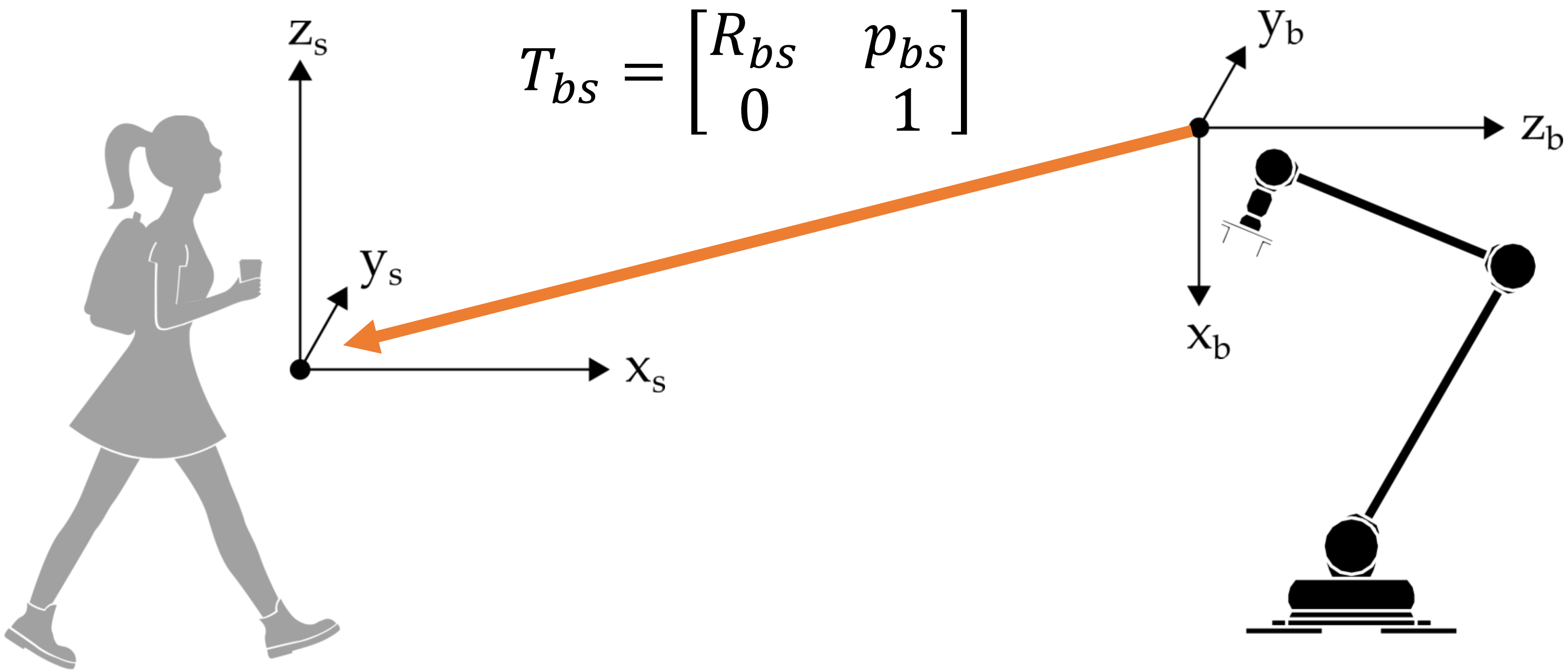
$$T_{sb} = \begin{bmatrix} R_{sb} & p_{sb} \\ 0 & 1 \end{bmatrix}$$



Properties

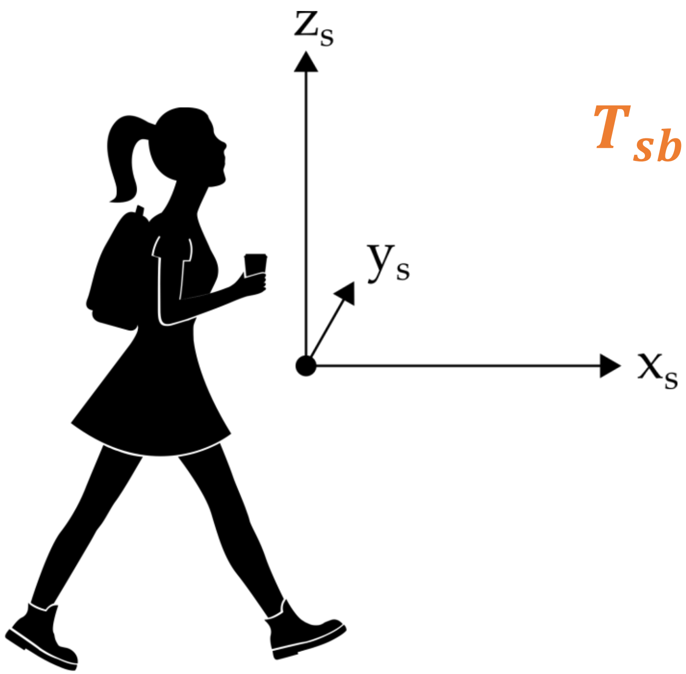
Inverse switches the frame of reference

$$T_{bs} = \begin{bmatrix} R_{bs} & p_{bs} \\ 0 & 1 \end{bmatrix}$$

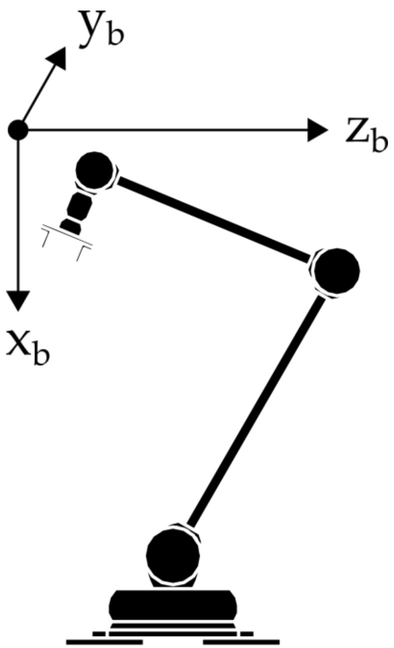


Properties

Inverse switches the frame of reference



$$T_{sb}^{-1} = T_{bs}$$



Properties

Inverse switches the frame of reference

Proof

We need to show that ${T_{sb}}^{-1} = T_{bs}$

$${T_{sb}}^{-1} = \begin{bmatrix} {R_{sb}}^T & -{R_{sb}}^T p_{sb} \\ 0 & 1 \end{bmatrix}$$

Properties

Inverse switches the frame of reference

Proof

We need to show that ${T_{sb}}^{-1} = T_{bs}$

$$
{T_{sb}}^{-1} = \begin{bmatrix} {R_{sb}}^{T} & -{R_{sb}}^{T} p_{sb} \\ 0 & 1 \end{bmatrix} = \begin{bmatrix} R_{bs} & R_{bs}(-\boldsymbol{p}_{sb}) \\ 0 & 1 \end{bmatrix}
$$

Vector from {b} to {s} in frame {s}

Properties

Inverse switches the frame of reference

Proof

We need to show that ${T_{sb}}^{-1} = T_{bs}$

$$ {T_{sb}}^{-1} = \begin{bmatrix} {R_{sb}}^T & -{R_{sb}}^T p_{sb} \\ 0 & 1 \end{bmatrix} = \begin{bmatrix} R_{bs} & R_{bs}(-p_{sb}) \\ 0 & 1 \end{bmatrix} = \begin{bmatrix} R_{bs} & p_{bs} \\ 0 & 1 \end{bmatrix} = T_{bs} $$

This Lecture

- How do we combine rotation and position?
- How do we capture rigid-body motion?
- What is a transformation matrix?

Next Lecture

- How can we use transformation matrices?