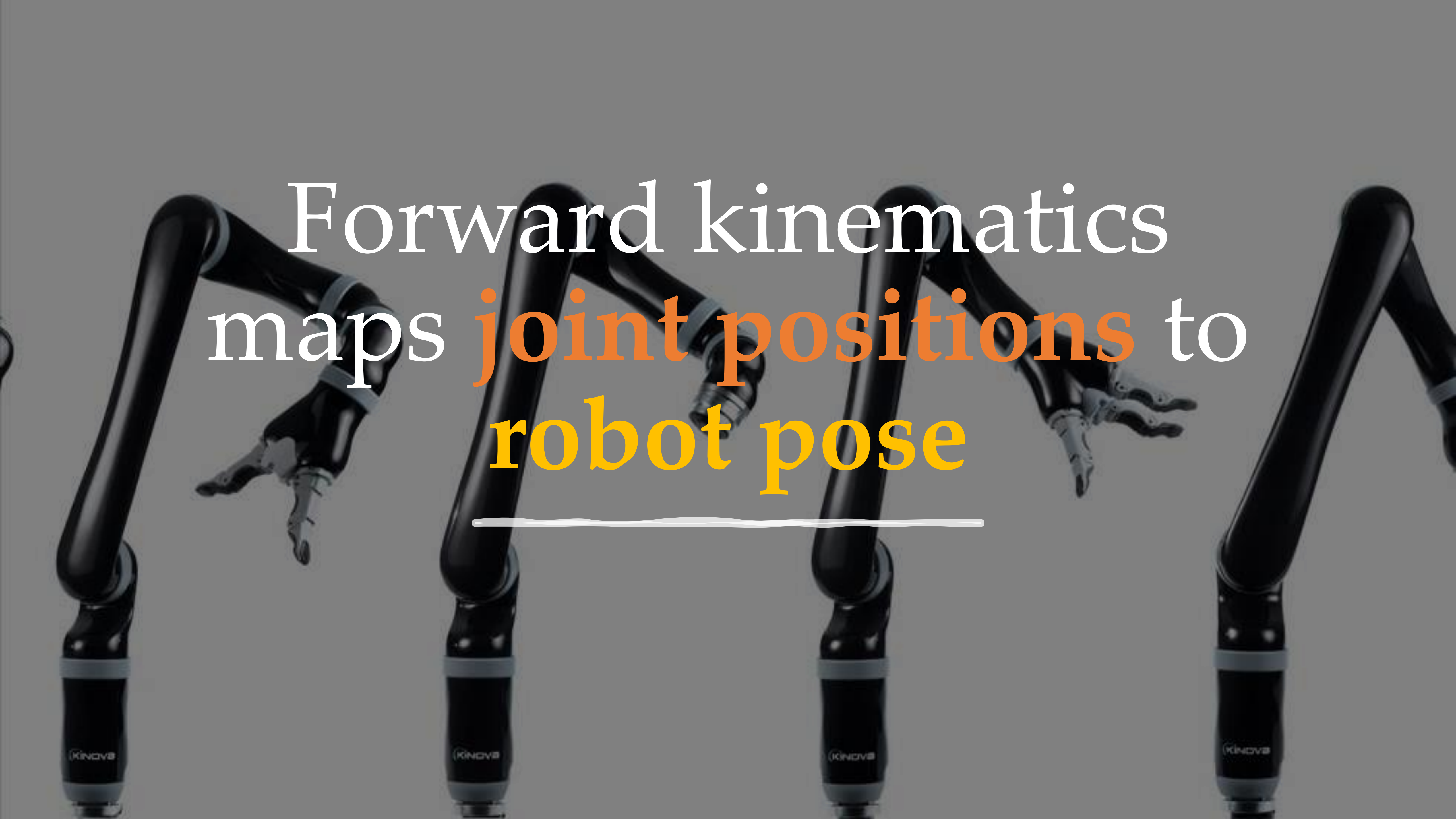
Forward Kinematics: Examples

Reading: Modern Robotics 4.1

This Lecture

- How do I apply the product of exponentials formula?
- Practice forward kinematics with two examples


Forward kinematics
maps joint positions to
robot pose
KINOVA
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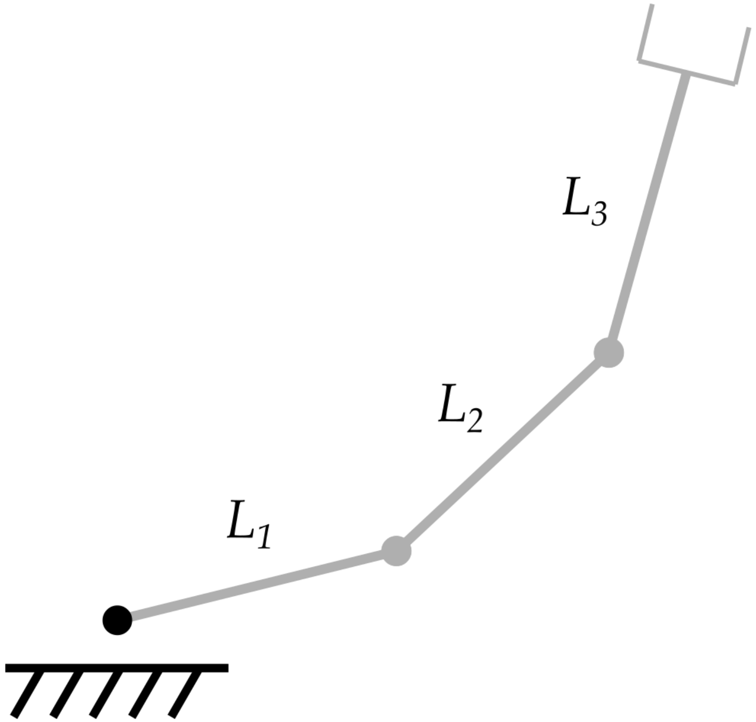
1. Revolute

Three-DoF robot arm.

Given joint values θ, what is the **pose** of the end-effector?



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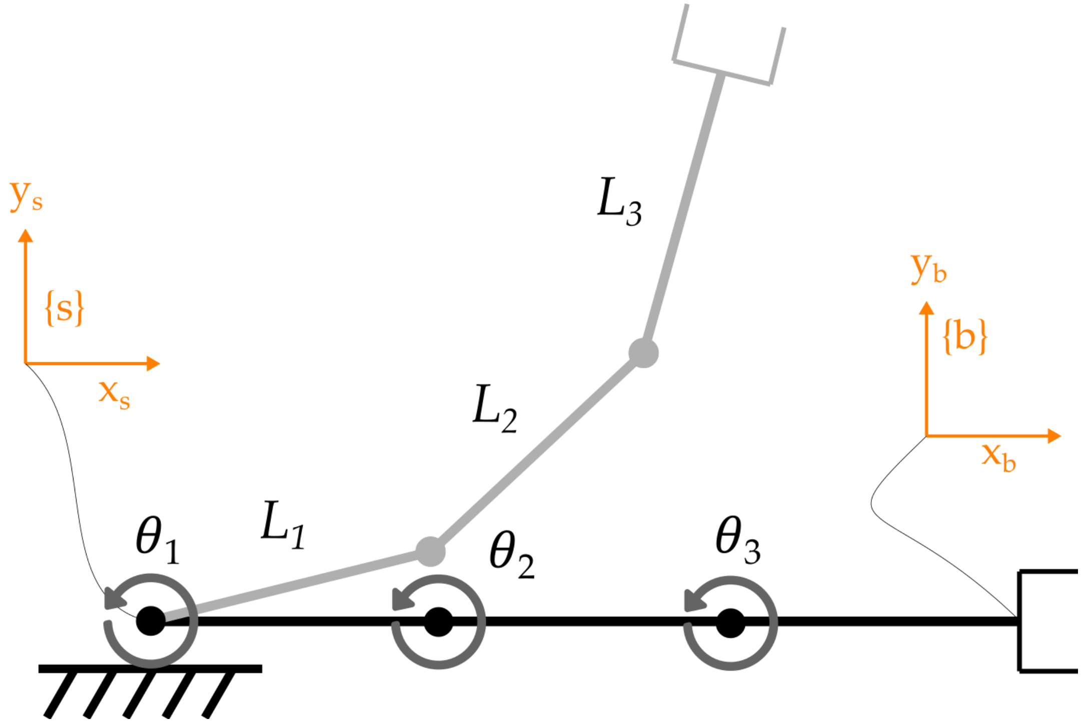
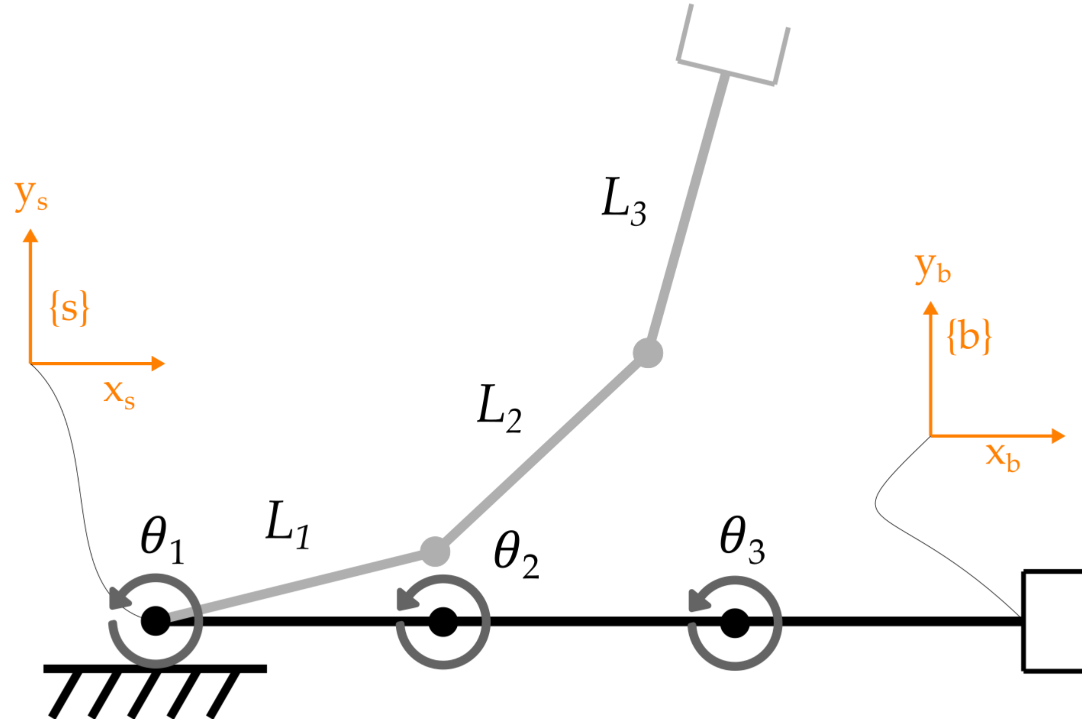


1. Revolute

Three-DoF robot arm.

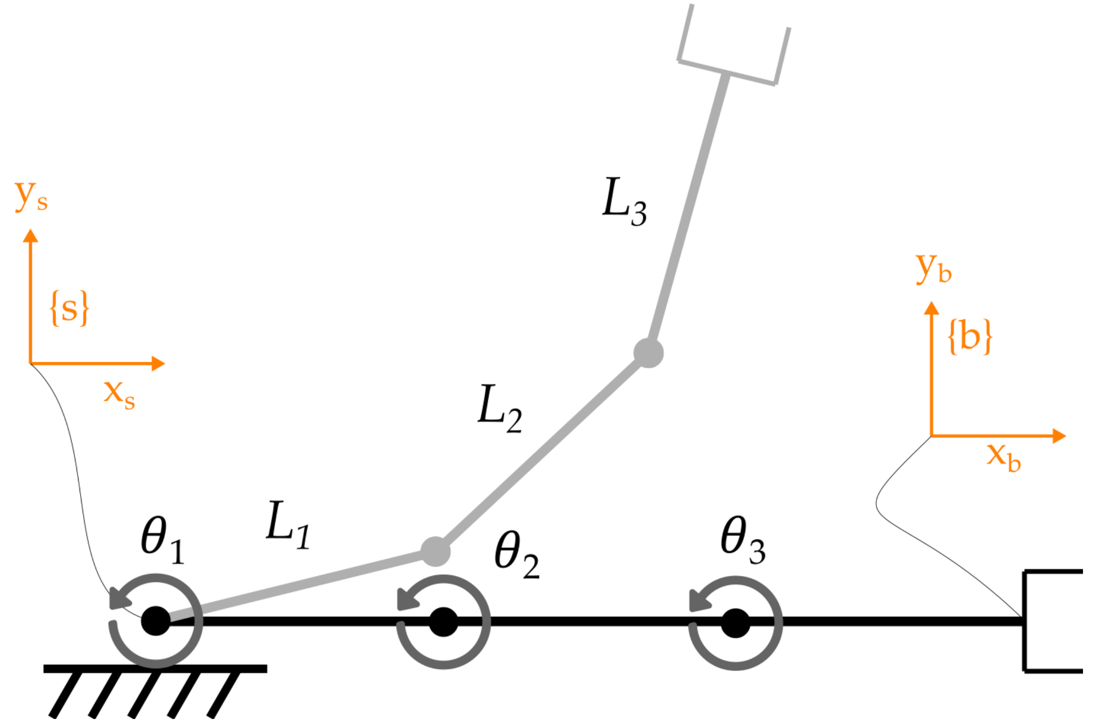
Given joint values θ, what is the **pose** of the end-effector?

$$T(\theta) = e^{[S_1]\theta_1} e^{[S_2]\theta_2} e^{[S_3]\theta_3} M$$



1. Revolute

Step 1. $M = T_{sb}$ when the robot is in home position



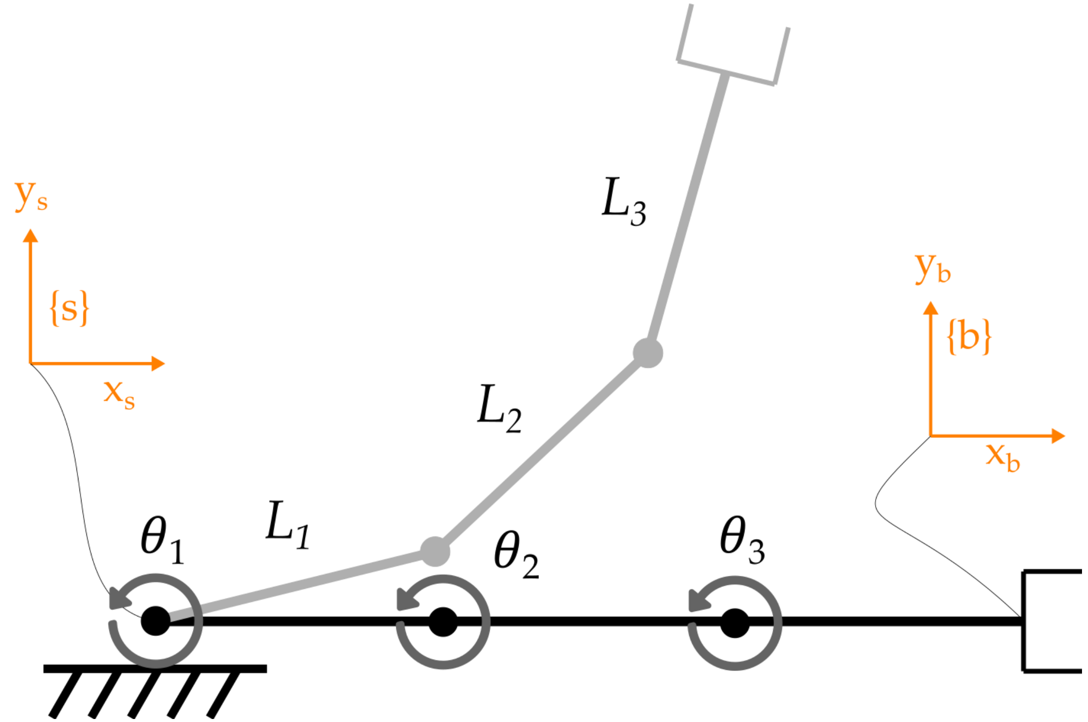
1. Revolute

Step 1. $M = T_{sb}$ when the robot is in home position

$$M = \begin{bmatrix} 1 & 0 & 0 & L_1 + L_2 + L_3 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ & 0 & & 1 \end{bmatrix}$$

$R = I$ since $\{b\}$ is aligned with $\{s\}$

$\{b\}$ is $L_1 + L_2 + L_3$ units along the x_s axis

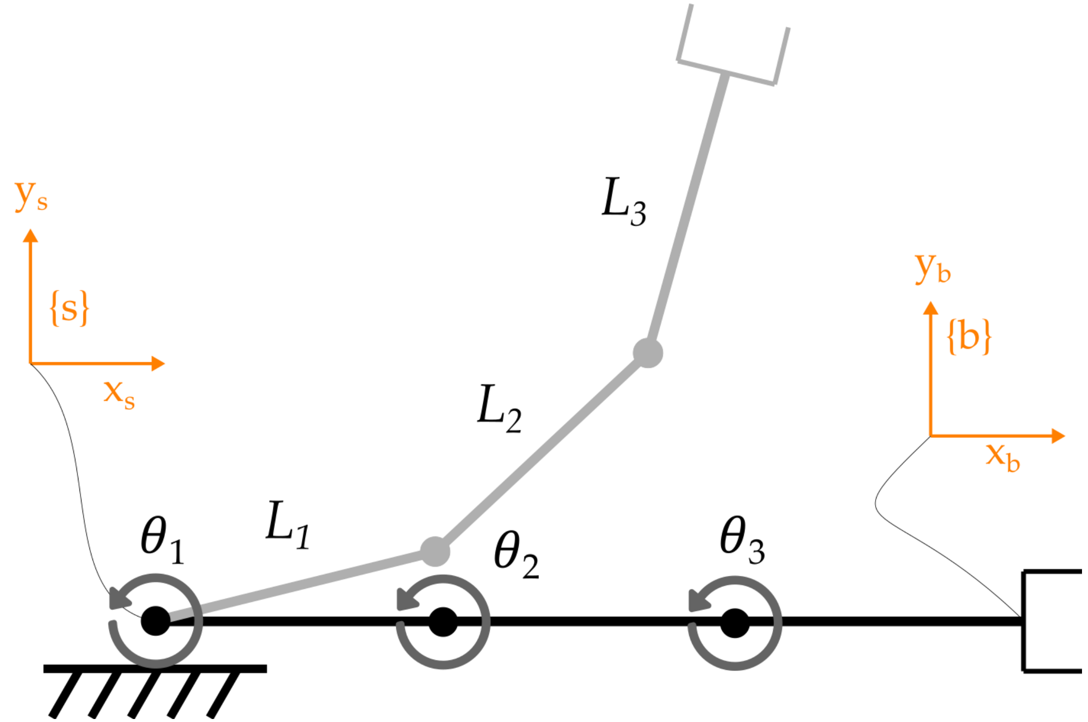


1. Revolute

Step 2. S_i is the screw for the i-th joint when the robot is in home position

$$S = \begin{bmatrix} \omega_s \\ -\omega_s \times q \end{bmatrix}$$

- ω_s is unit vector in the direction of the axis of positive rotation
- q is vector from $\{s\}$ to the joint axis



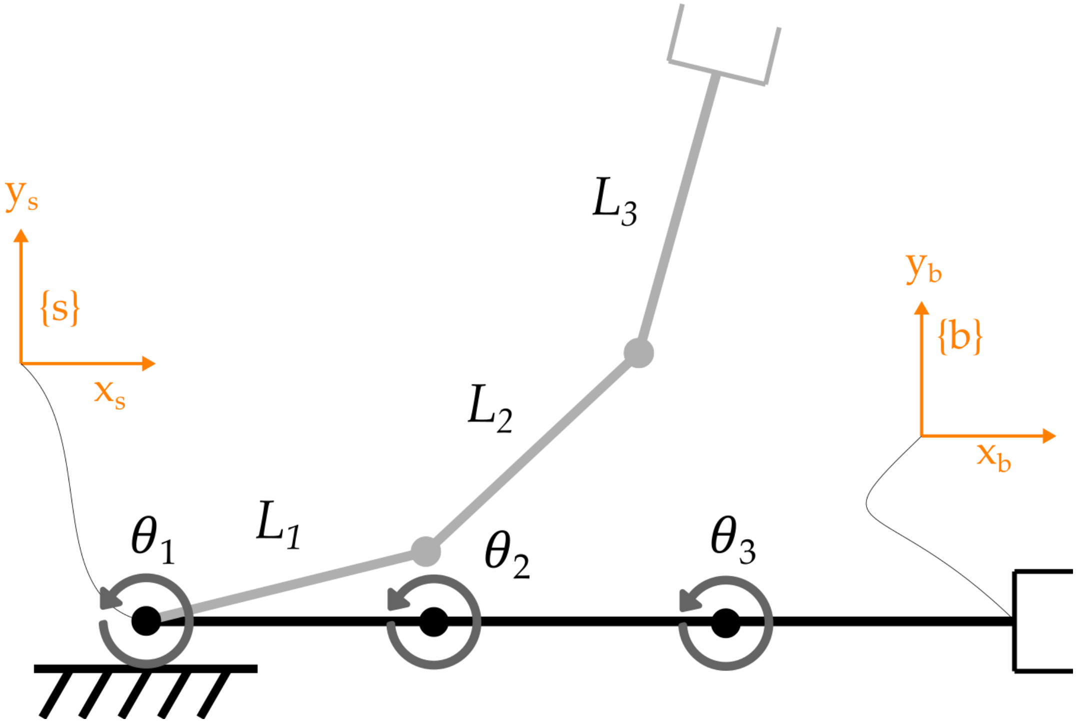
1. Revolute

Step 2. S_i is the screw for the i-th joint when the robot is in home position

$$S = \begin{bmatrix} \omega_s \\ -\omega_s \times q \end{bmatrix}$$

$$\omega_{s1} = \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix} \quad \underline{q_1 = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}} \quad S_1 = \begin{bmatrix} 0 \\ 0 \\ 1 \\ 0 \\ 0 \\ 0 \end{bmatrix}$$

1st joint axis is through $\{s\}$



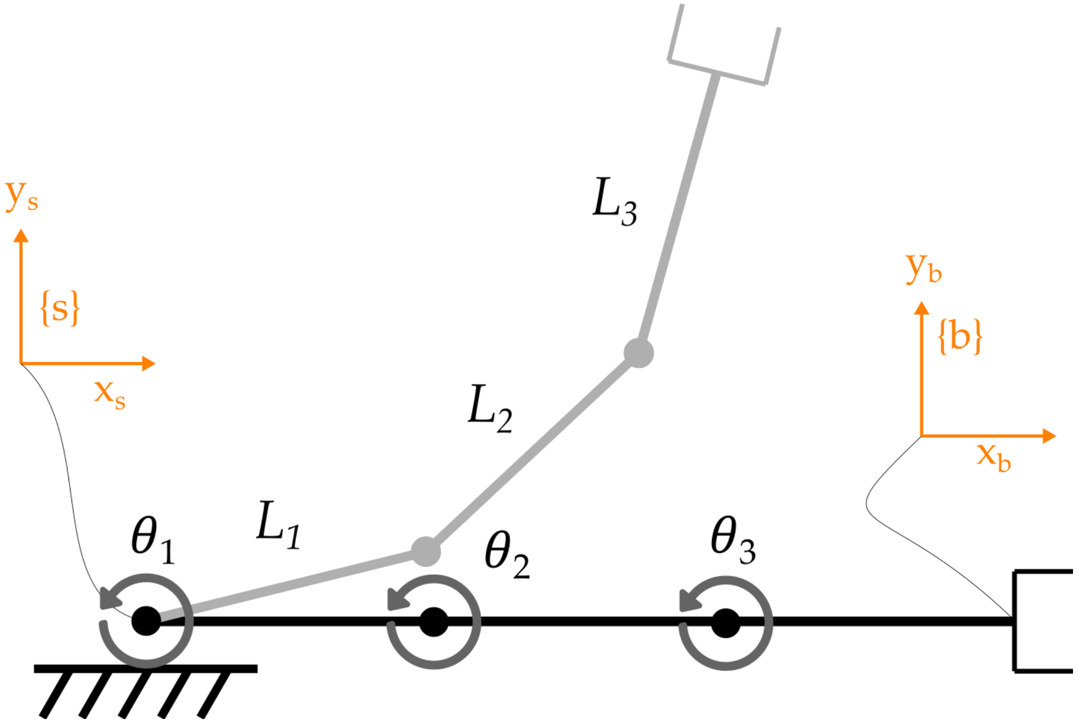
1. Revolute

Step 2. S_i is the screw for the i-th joint when the robot is in home position

$$S = \begin{bmatrix} \omega_s \\ -\omega_s \times q \end{bmatrix}$$

$$\omega_{s2} = \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix} \quad q_2 = \begin{bmatrix} L_1 \\ 0 \\ 0 \end{bmatrix} \quad S_2 = \begin{bmatrix} 0 \\ 0 \\ 1 \\ 0 \\ -L_1 \\ 0 \end{bmatrix}$$

2nd joint axis is L_1 units along x_s



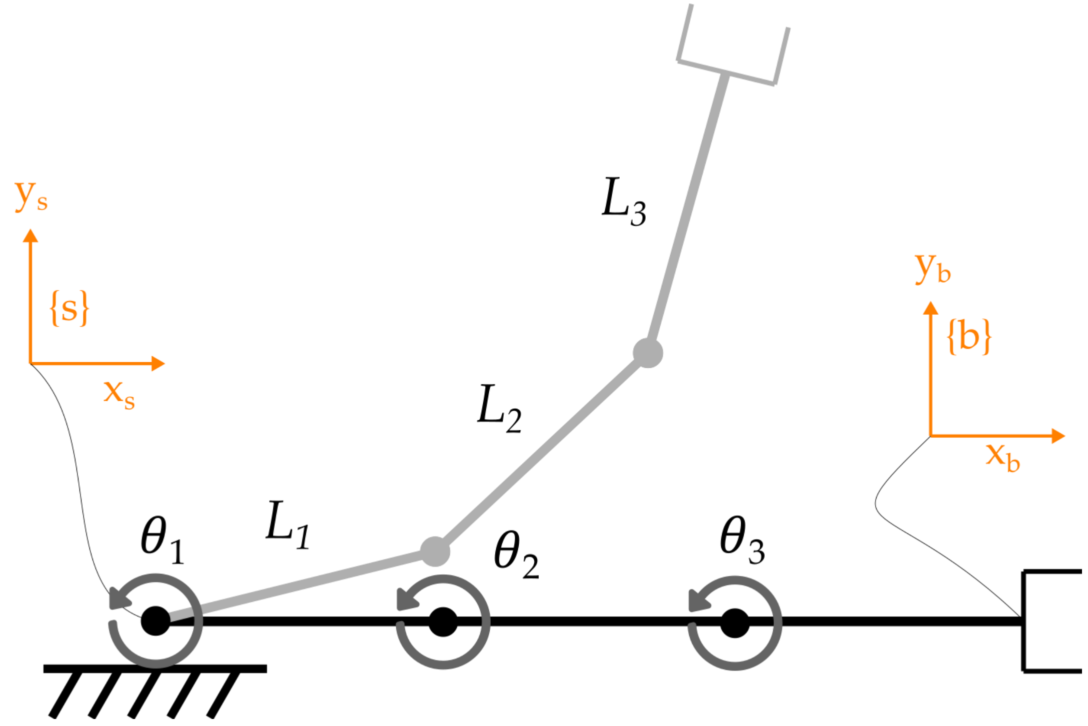
1. Revolute

Step 2. S_i is the screw for the i-th joint when the robot is in home position

$$S = \begin{bmatrix} \omega_s \\ -\omega_s \times q \end{bmatrix}$$

$$\omega_{s3} = \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix} \quad q_3 = \begin{bmatrix} L_1 + L_2 \\ 0 \\ 0 \end{bmatrix} \quad S_3 = \begin{bmatrix} 0 \\ 0 \\ 1 \\ 0 \\ -L_1 - L_2 \\ 0 \end{bmatrix}$$

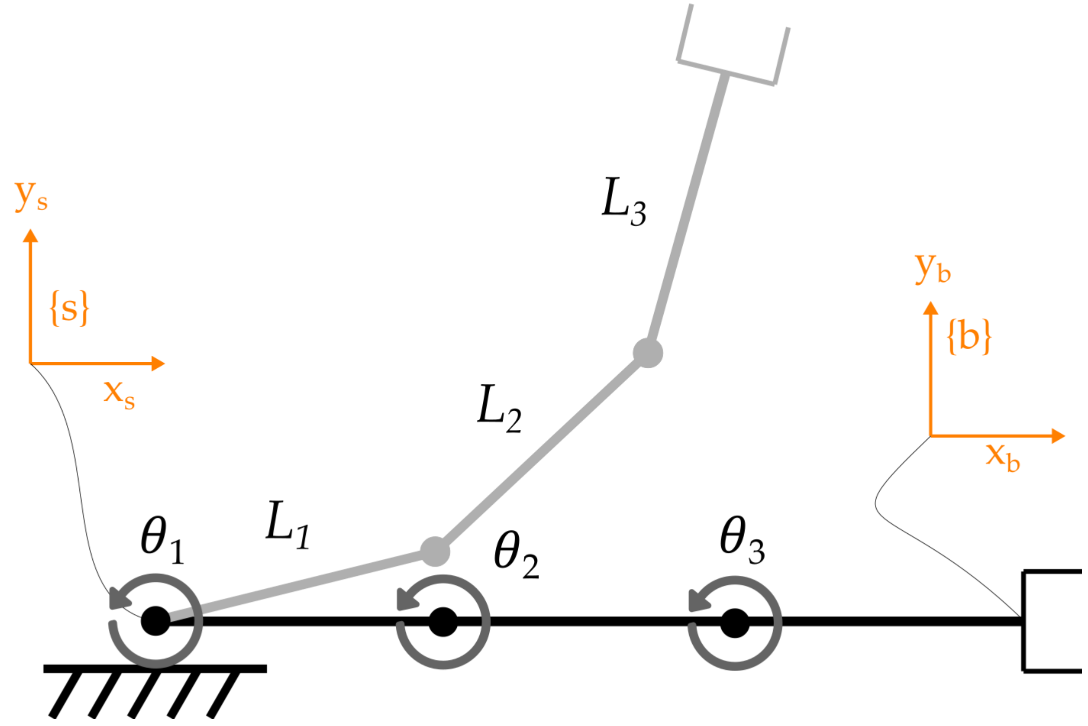
3rd joint axis is $L_1 + L_2$ units along x_s



1. Revolute

Step 3. Use our formula to get $T(\theta)$

$$T(\theta) = e^{[S_1]\theta_1} e^{[S_2]\theta_2} e^{[S_3]\theta_3} M$$



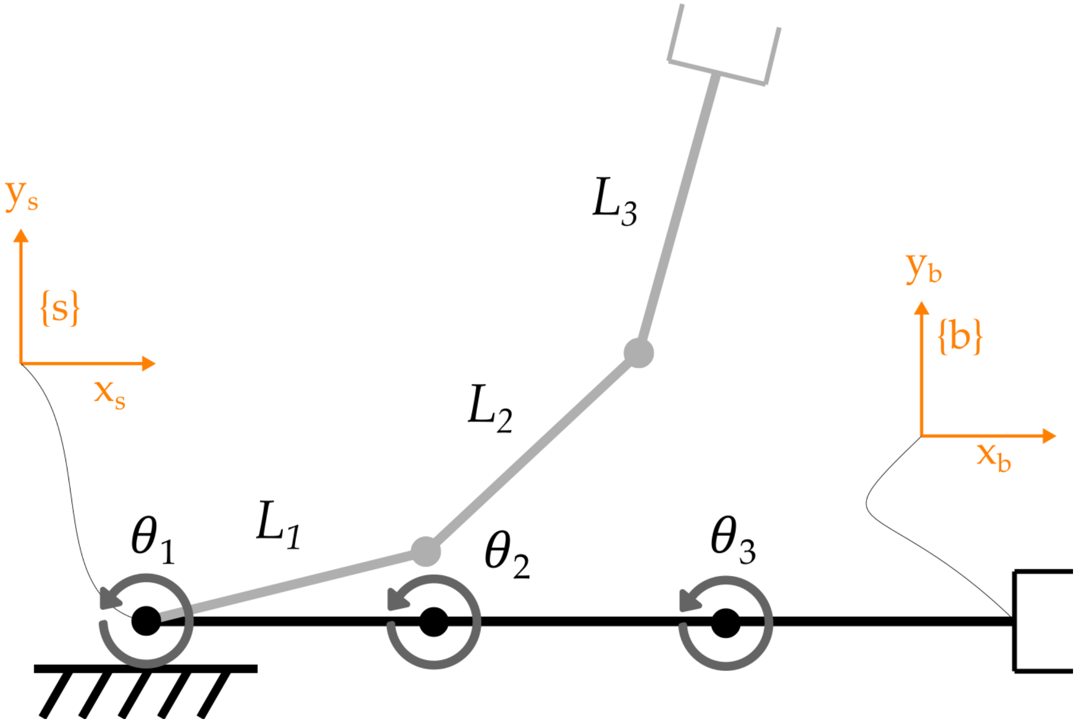
```
syms theta1 theta2 theta3 L1 L2 L3 real

M = [eye(3), [L1+L2+L3; 0; 0]; 0 0 0 1];
S1 = [0; 0; 1; 0; 0; 0];
S2 = [0; 0; 1; 0; -L1; 0];
S3 = [0; 0; 1; 0; -L1-L2; 0];
T = expm(bracket(S1)*theta1) * ...
        expm(bracket(S2)*theta2) * ...
            expm(bracket(S3)*theta3) * M;
simplify(T)

function S_matrix = bracket(S)
    S_matrix = [0 -S(3) S(2) S(4);
                S(3) 0 -S(1) S(5);
                -S(2) S(1) 0 S(6); 0 0 0 0];
end
```

1. Revolute

Step 3. Use our formula to get $T(\theta)$

$$T(\theta) = \begin{bmatrix} c_{123} & -s_{123} & 0 & L_1 c_1 + L_2 c_{12} + L_3 c_{123} \\ s_{123} & c_{123} & 0 & L_1 s_1 + L_2 s_{12} + L_3 s_{123} \\ 0 & 0 & 1 & 0 \\ & 0 & & 1 \end{bmatrix}$$

$c_1 = \cos\theta_1$
$s_1 = \sin\theta_1$
$c_{12} = \cos(\theta_1 + \theta_2)$
$s_{12} = \sin(\theta_1 + \theta_2)$

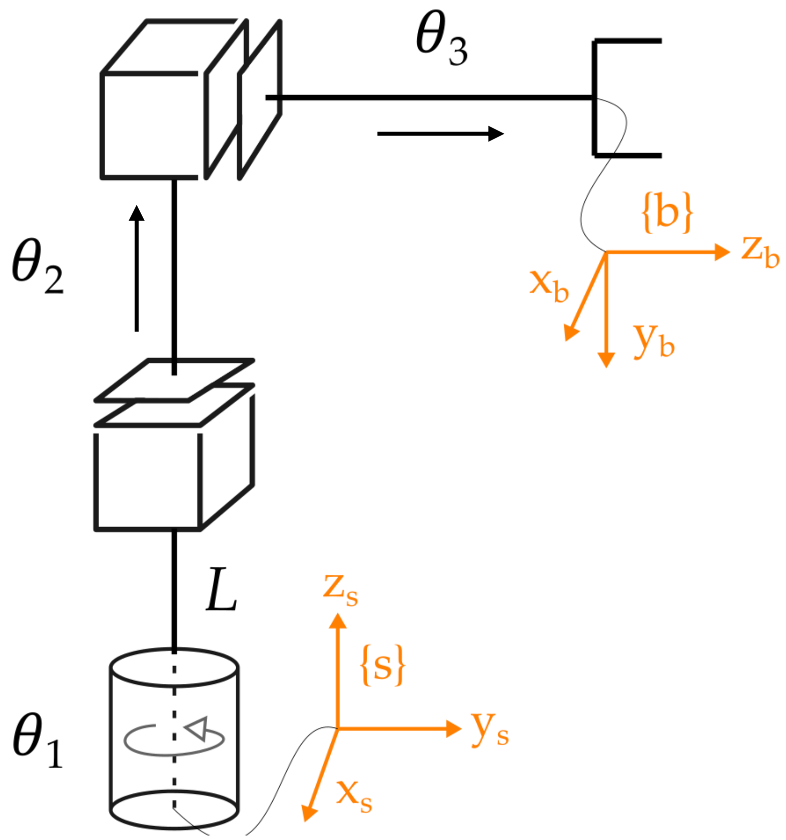


Let's do one more example

2. Prismatic

Three-DoF robot arm.

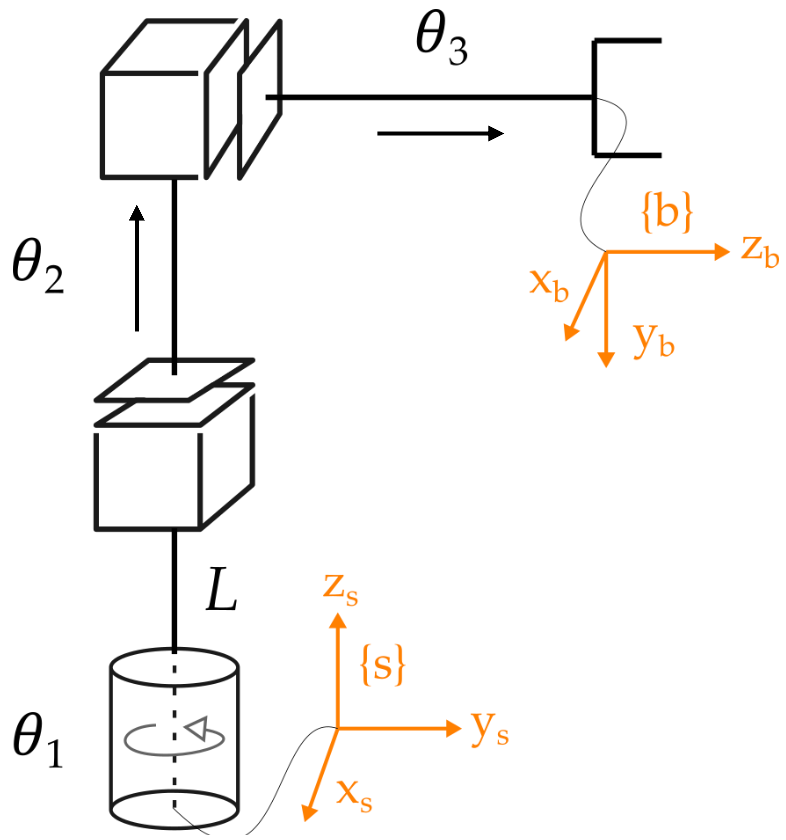
Given joint values θ, what is the **pose** of the end-effector?



2. Prismatic

Step 1. $M = T_{sb}$ when the robot is in home position

Hint: M should *never* include θ.
We find $M = T(0)$ when $\theta = 0$.



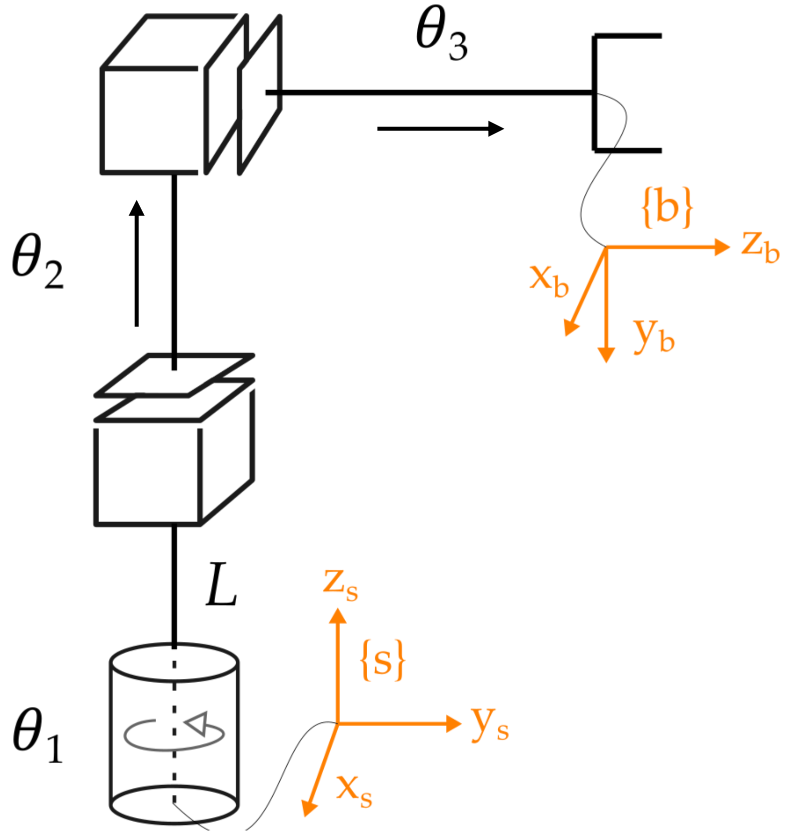
2. Prismatic

Step 1. $M = T_{sb}$ when the robot is in home position

$$M = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & -1 & 0 & L \\ & 0 & & 1 \end{bmatrix}$$

y_b aligned with $-z_s$
z_b aligned with y_s

$\{b\}$ is L units up along the z_s axis



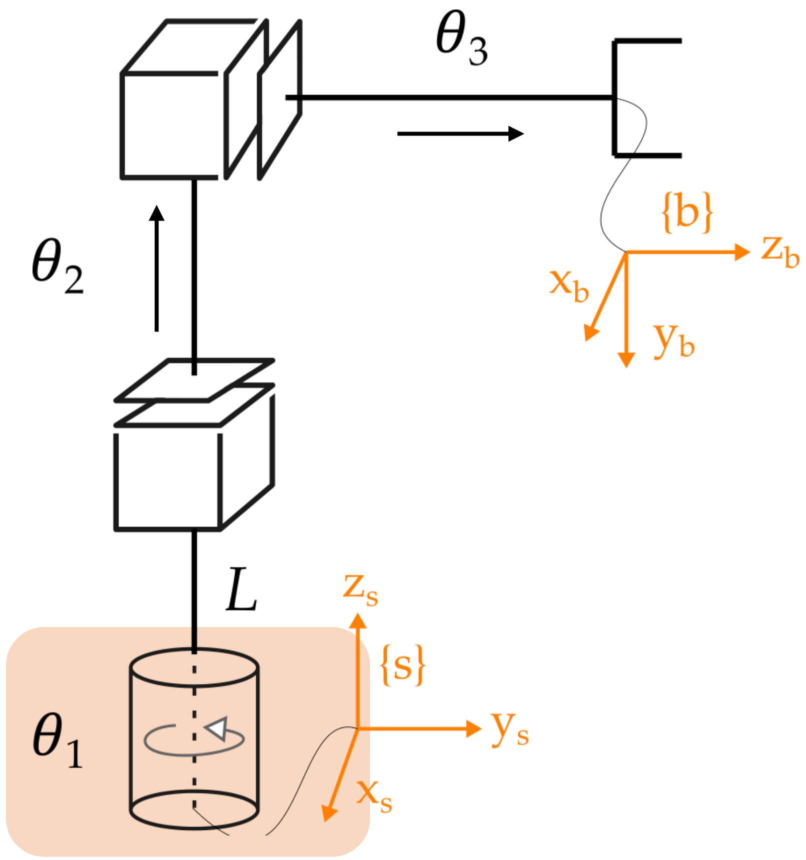
2. Prismatic

Step 2. S_i is the screw for the i-th joint when the robot is in home position

$$S = \begin{bmatrix} \omega_s \\ -\omega_s \times q \end{bmatrix}$$

$$\omega_{s1} = \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix} \quad \underline{q_1 = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}} \quad S_1 = \begin{bmatrix} 0 \\ 0 \\ 1 \\ 0 \\ 0 \\ 0 \end{bmatrix}$$

1st joint axis is through $\{s\}$

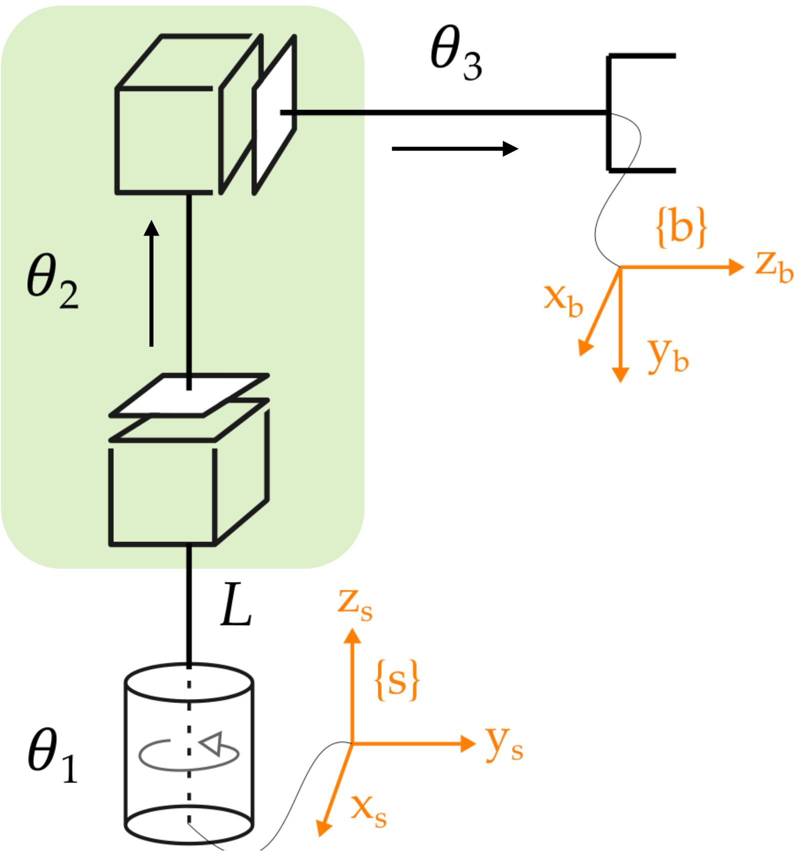


2. Prismatic

Step 2. S_i is the screw for the i-th joint when the robot is in home position

$$S = \begin{bmatrix} 0 \\ \boxed{v_s} \end{bmatrix}$$

- v_s is unit vector in the direction of positive translation



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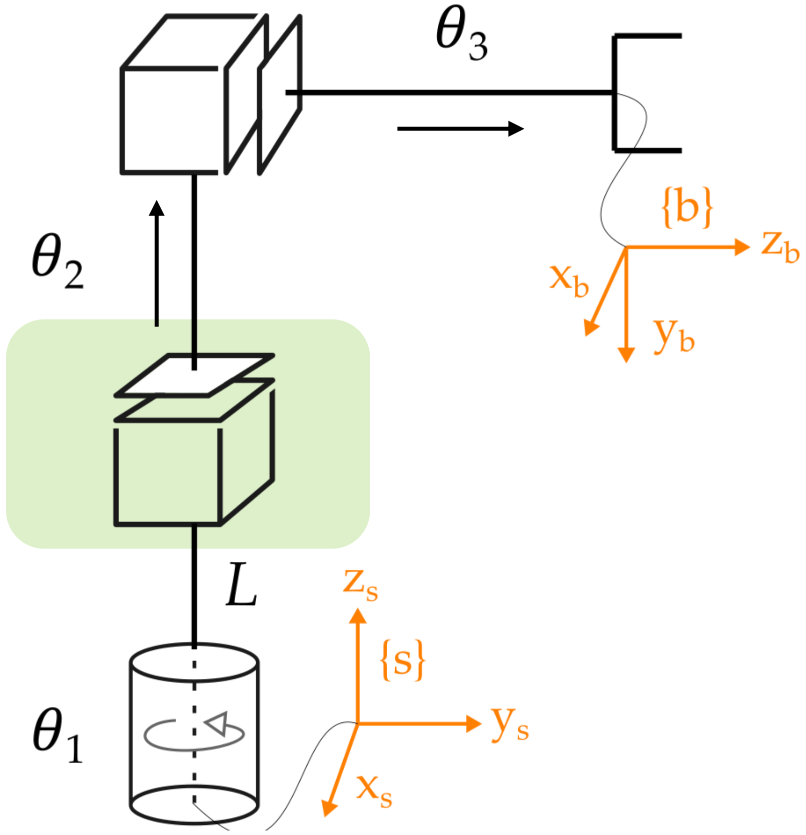
Step 2. S_i is the screw for the i-th joint when the robot is in home position

$$S = \begin{bmatrix} 0 \\ v_s \end{bmatrix}$$

$$v_{s2} = \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix}, \qquad S_2 = \begin{bmatrix} 0 \\ 0 \\ 0 \\ 0 \\ 0 \\ 1 \end{bmatrix}$$

Positive translation along z_s



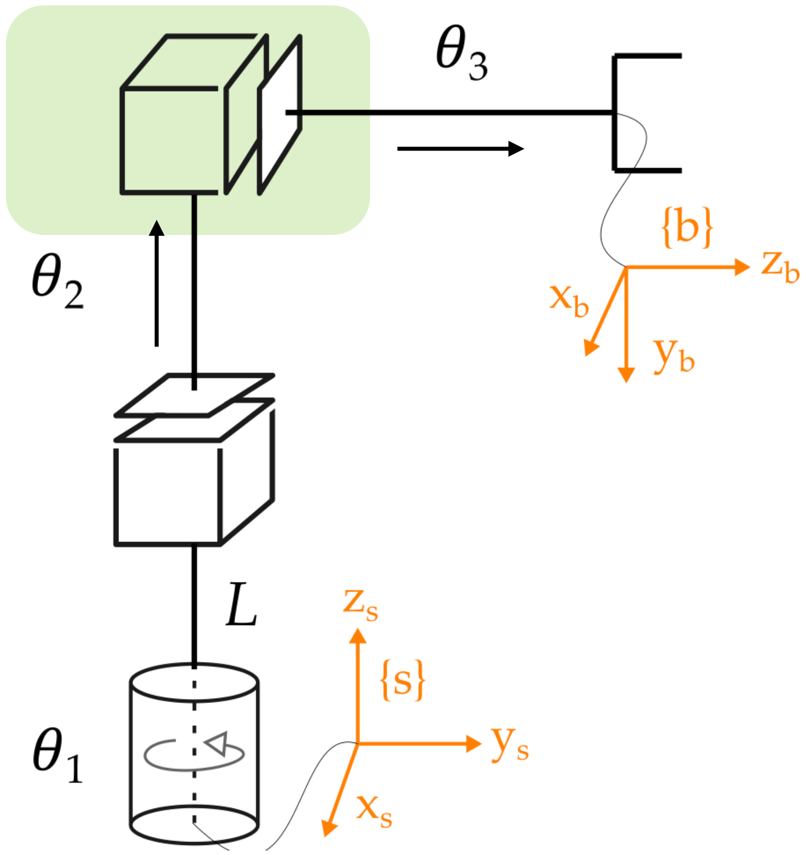
2. Prismatic

Step 2. S_i is the screw for the i-th joint when the robot is in home position

$$S = \begin{bmatrix} 0 \\ v_s \end{bmatrix}$$

$$v_{s3} = \begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix}, \qquad S_3 = \begin{bmatrix} 0 \\ 0 \\ 0 \\ 0 \\ 1 \\ 0 \end{bmatrix}$$

Positive translation along y_s



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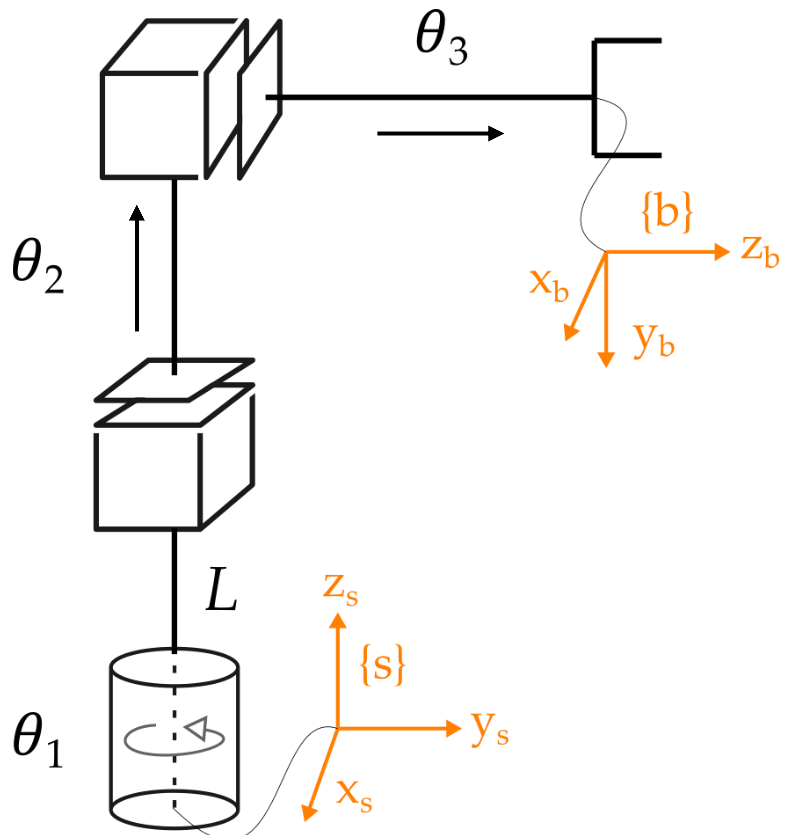

```matlab
syms theta1 theta2 theta3 L real

M = [1 0 0 0; 0 0 1 0; 0 -1 0 L; 0 0 0 1];
S1 = [0; 0; 1; 0; 0; 0];
S2 = [0; 0; 0; 0; 0; 1];
S3 = [0; 0; 0; 0; 1; 0];
T = expm(bracket(S1)*theta1) * ...
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end
```
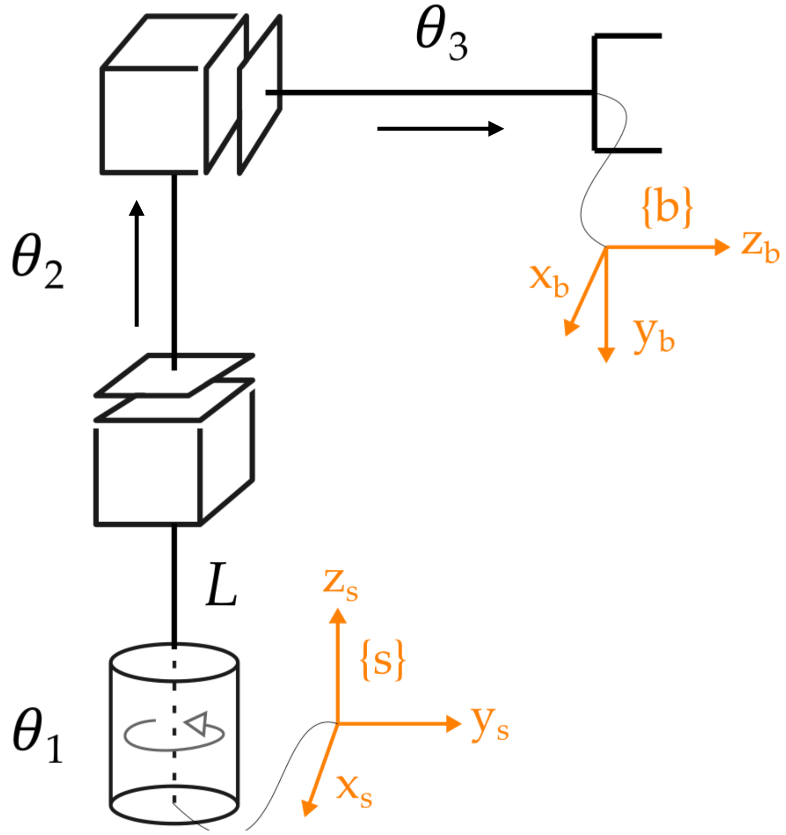

2. Prismatic

Step 3. Use our formula to get $T(\theta)$

$$T(\theta) = \begin{bmatrix} c_1 & 0 & -s_1 & -\theta_3 s_1 \\ s_1 & 0 & c_1 & \theta_3 c_1 \\ 0 & -1 & 0 & L + \theta_2 \\ & 0 & & 1 \end{bmatrix}$$

$c_1 = \cos\theta_1$
$s_1 = \sin\theta_1$



This Lecture

- How do I apply the product of exponentials formula?
- Practice forward kinematics with two examples

Next Lecture

- Wrap-up forward kinematics with one more example