

Transformations



Reading: Modern Robotics 3.3.1



This Lecture

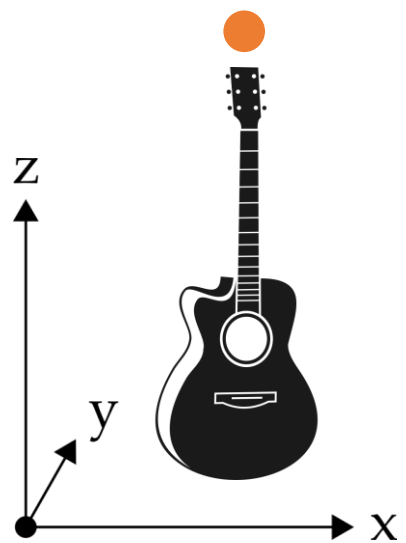


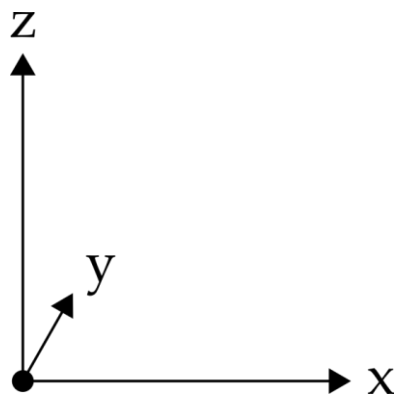
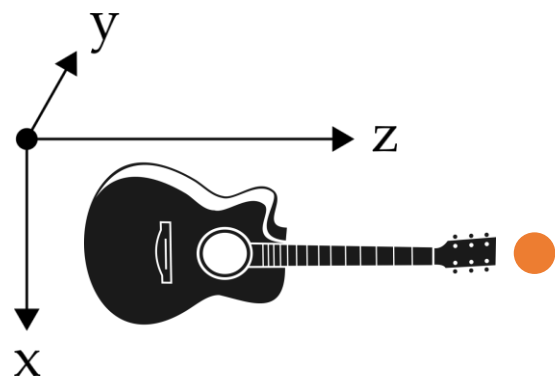
- How do we combine rotation and position?
- How do we capture rigid-body motion?
- What is a transformation matrix?

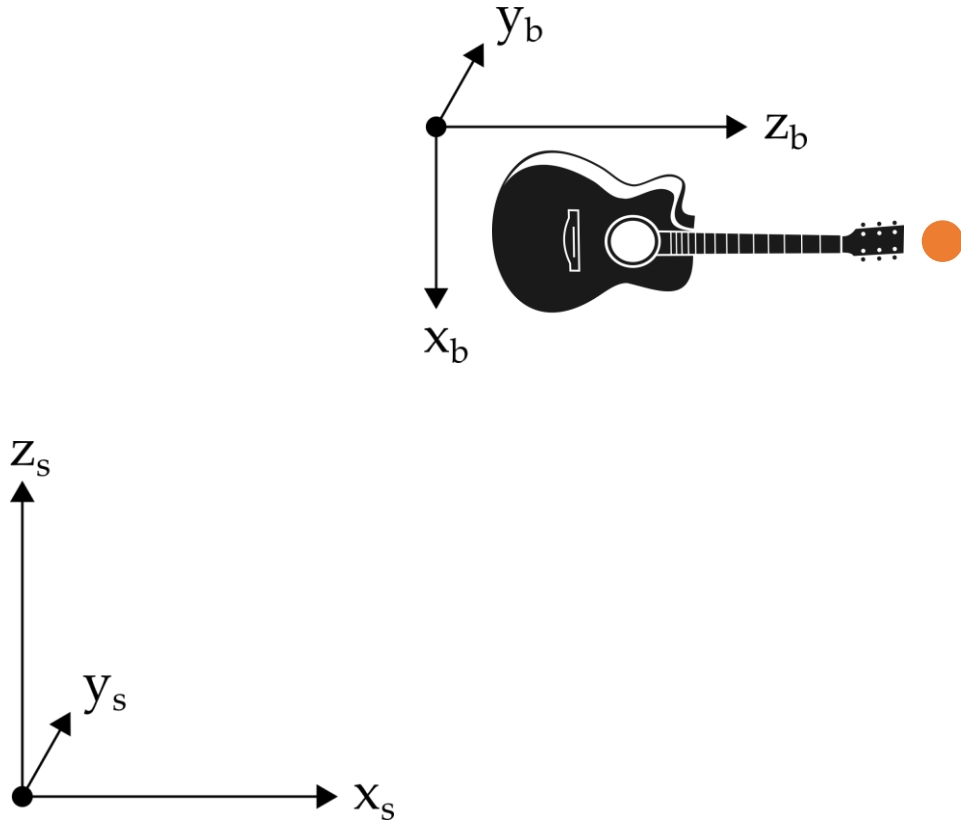
A stack of wooden blocks is arranged in a pyramid shape. The base consists of four blocks in a row. The second row has three blocks, and the third row has two blocks. On top of the two blocks in the third row sits a single red block. The background is a solid blue color.

Chasles' Theorem

Any rigid body motion is the **translation** of a designated point & a **rotation** of the whole body about that point.

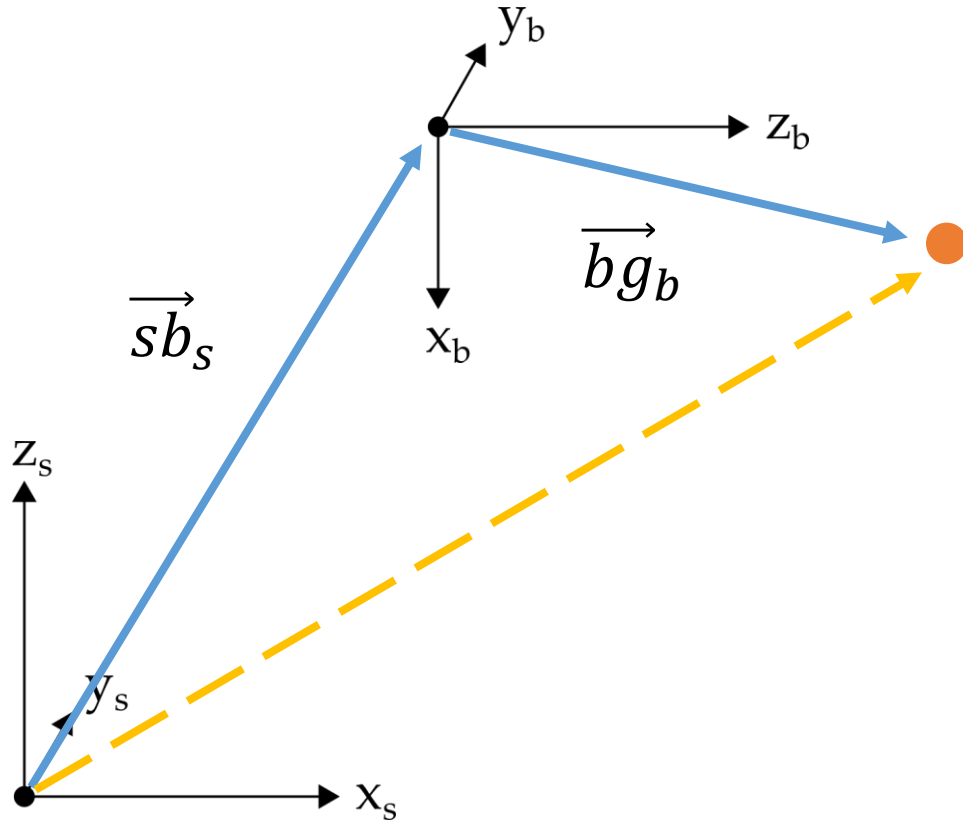






How do we capture this motion?

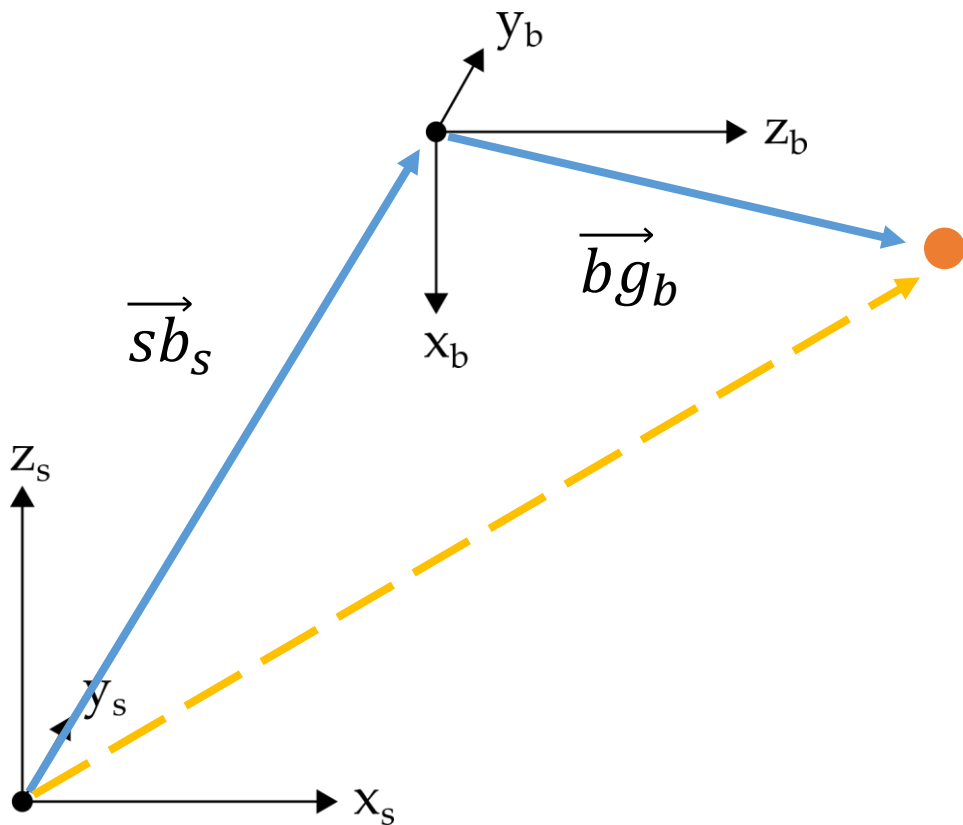
What is the position of the guitar in our original frame $\{s\}$?



You are given:

- $\vec{bg_b}$ position of guitar in $\{b\}$
- $\vec{sb_s}$ position of $\{b\}$ in $\{s\}$
- R_{sb} orientation of $\{b\}$ relative to $\{s\}$

What is p_{sg} the position of the guitar in our original frame $\{s\}$?



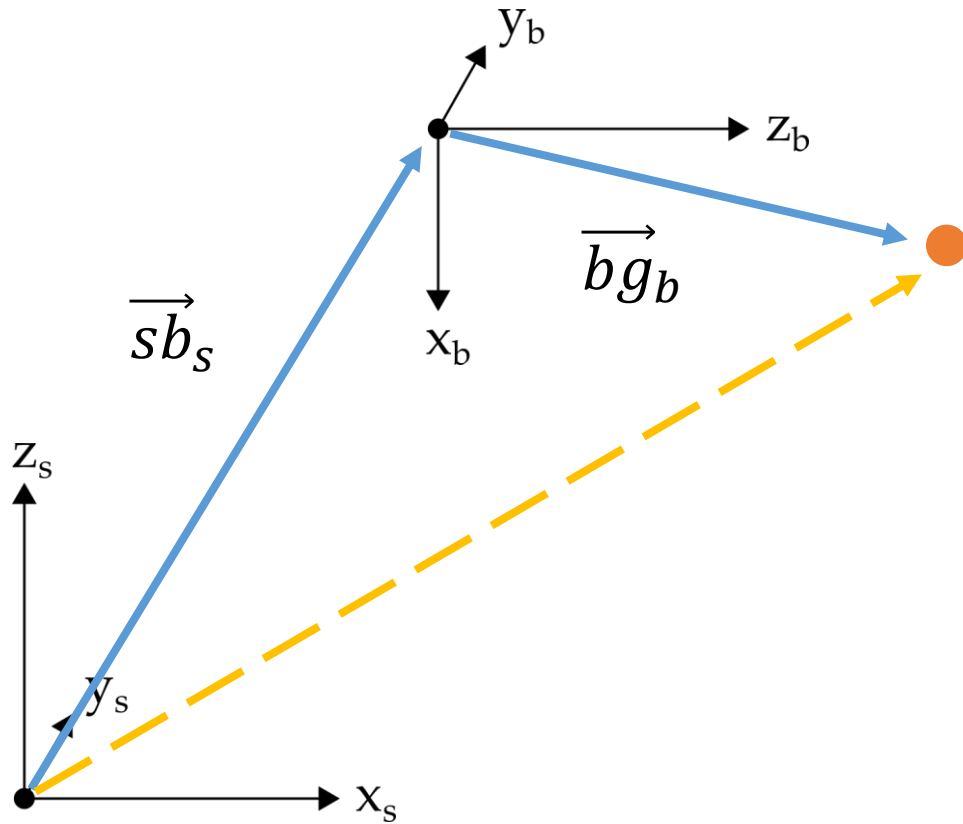
Naïve Solution

Add the vectors in their current frames?

$$p_{sg} = \vec{sg}_s = \vec{bg_b} + \vec{sb_s}$$

$$p_{sg} = \begin{bmatrix} 4 \\ 0 \\ 10 \end{bmatrix}$$

$$R_{sb} = \begin{bmatrix} 0 & 0 & 1 \\ 0 & 1 & 0 \\ -1 & 0 & 0 \end{bmatrix} \quad \vec{sb_s} = \begin{bmatrix} 3 \\ 0 \\ 7 \end{bmatrix} \quad \vec{bg_b} = \begin{bmatrix} 1 \\ 0 \\ 3 \end{bmatrix}$$



$$R_{sb} = \begin{bmatrix} 0 & 0 & 1 \\ 0 & 1 & 0 \\ -1 & 0 & 0 \end{bmatrix} \quad \vec{s}b_s = \begin{bmatrix} 3 \\ 0 \\ 7 \end{bmatrix} \quad \vec{b}g_b = \begin{bmatrix} 1 \\ 0 \\ 3 \end{bmatrix}$$

Correct Solution

Put all vectors in frame {s} then add

$$p_{sg} = R_{sb} \vec{b}g_b + \vec{s}b_s$$

↑
[b] to [g] in frame {s}

↑
{s} to {b} in frame {s}

$$p_{sg} = \begin{bmatrix} 6 \\ 0 \\ 6 \end{bmatrix}$$

Transformation Matrix

We can write any rigid-body motion as a matrix:

$$p_{sg} = R_{sb}p_{bg} + p_{sb}$$

$$\begin{bmatrix} p_{sg} \\ 1 \end{bmatrix} = \begin{bmatrix} & \\ & \end{bmatrix} \begin{bmatrix} p_{bg} \\ 1 \end{bmatrix}$$



New position



Original position

Transformation Matrix

We can write any rigid-body motion as a matrix:

$$p_{sg} = R_{sb}p_{bg} + p_{sb}$$

$$\begin{bmatrix} p_{sg} \\ 1 \end{bmatrix} = \begin{bmatrix} \mathbf{R}_{sb} & \mathbf{p}_{sb} \\ \mathbf{0} & \mathbf{1} \end{bmatrix} \begin{bmatrix} p_{bg} \\ 1 \end{bmatrix}$$


Transformation Matrix

We can write any rigid-body motion as a matrix:

$$p_{sg} = R_{sb}p_{bg} + p_{sb}$$

$$\begin{bmatrix} p_{sg} \\ 1 \end{bmatrix} = \underbrace{\begin{bmatrix} \mathbf{R}_{sb} & \mathbf{p}_{sb} \\ \mathbf{0} & \mathbf{1} \end{bmatrix}} \begin{bmatrix} p_{bg} \\ 1 \end{bmatrix}$$

This matrix is called a transformation matrix

Transformation Matrix

A (homogeneous) transformation matrix T is a 4×4 matrix where:

$$T = \begin{bmatrix} R & p \\ 0 & 1 \end{bmatrix} = \begin{bmatrix} r_{11} & r_{12} & r_{13} & p_1 \\ r_{21} & r_{22} & r_{23} & p_2 \\ r_{31} & r_{32} & r_{33} & p_3 \\ 0 & 0 & 0 & 1 \end{bmatrix}$$

Transformation T **combines** a rotation matrix R and a position vector p

A stack of wooden blocks is shown on a light-colored surface. The stack consists of four layers: the bottom layer has four blocks, the second layer has three blocks, the third layer has two blocks, and the top layer has one red block. The text "What are the properties of transformation matrices?" is overlaid in white serif font on the right side of the stack.

What are the properties of
transformation matrices?

Properties

Product of two transformation matrices is a transformation matrix

Proof

Let T_1 and T_2 be two transformation matrices, and $T_3 = T_1 T_2$

$$T_3 = \begin{bmatrix} R_1 & p_1 \\ 0 & 1 \end{bmatrix} \begin{bmatrix} R_2 & p_2 \\ 0 & 1 \end{bmatrix}$$

$$T_3 = \begin{bmatrix} R_1 R_2 & R_1 p_2 + p_1 \\ 0 & 1 \end{bmatrix}$$

Properties

Inverse of a transformation matrix is:

$$T^{-1} = \begin{bmatrix} R^T & -R^T p \\ 0 & 1 \end{bmatrix}$$

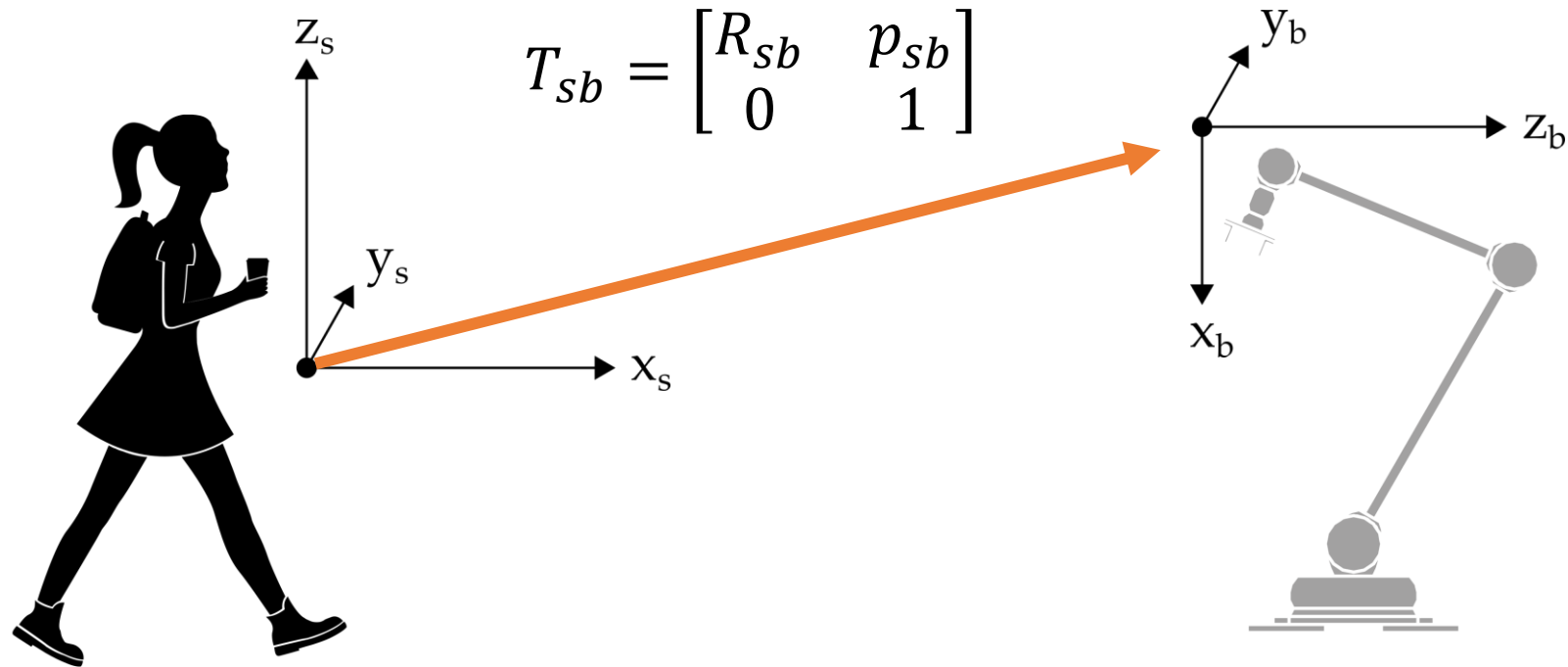
Proof

We need to show that $T^{-1}T = I$

$$T^{-1}T = \begin{bmatrix} R^T & -R^T p \\ 0 & 1 \end{bmatrix} \begin{bmatrix} R & p \\ 0 & 1 \end{bmatrix} = \begin{bmatrix} R^T R & R^T p - R^T p \\ 0 & 1 \end{bmatrix}$$

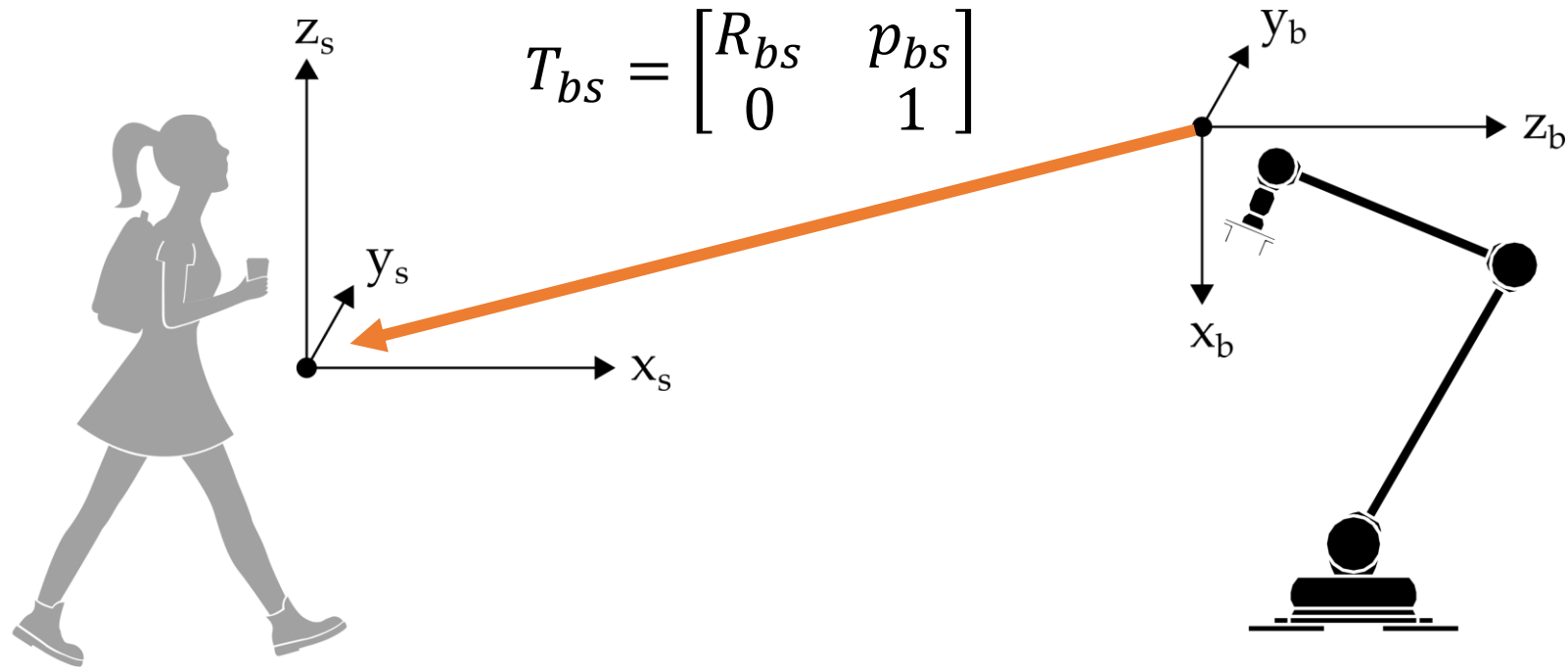
Properties

Inverse switches the frame of reference



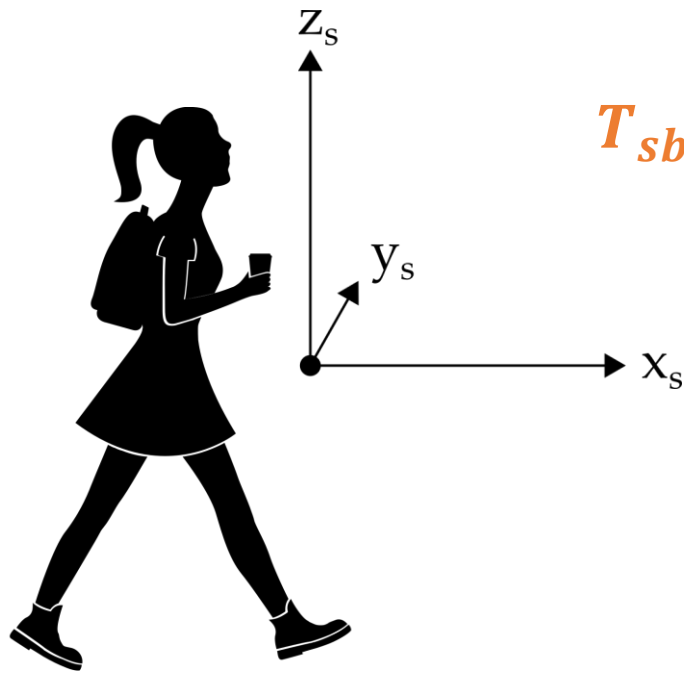
Properties

Inverse switches the frame of reference

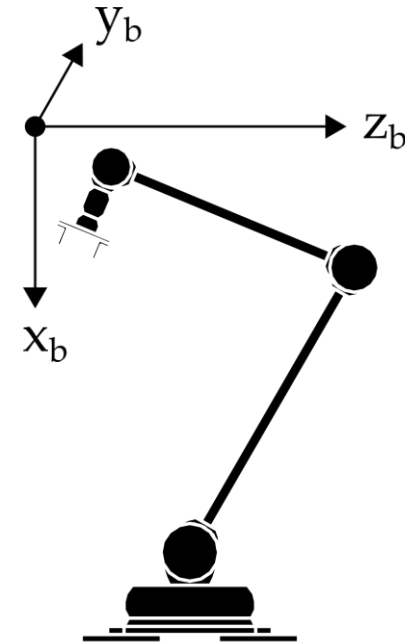


Properties

Inverse switches the frame of reference



$$T_{sb}^{-1} = T_{bs}$$



Properties

Inverse switches the frame of reference

Proof

We need to show that $T_{sb}^{-1} = T_{bs}$

$$T_{sb}^{-1} = \begin{bmatrix} R_{sb}^T & -R_{sb}^T p_{sb} \\ 0 & 1 \end{bmatrix}$$

Properties

Inverse switches the frame of reference

Proof

We need to show that $T_{sb}^{-1} = T_{bs}$

$$T_{sb}^{-1} = \begin{bmatrix} R_{sb}^T & -R_{sb}^T p_{sb} \\ 0 & 1 \end{bmatrix} = \begin{bmatrix} R_{bs} & R_{bs}(-p_{sb}) \\ 0 & 1 \end{bmatrix}$$



Vector from {b} to {s} in frame {s}

Properties

Inverse switches the frame of reference

Proof

We need to show that $T_{sb}^{-1} = T_{bs}$

$$T_{sb}^{-1} = \begin{bmatrix} R_{sb}^T & -R_{sb}^T p_{sb} \\ 0 & 1 \end{bmatrix} = \begin{bmatrix} R_{bs} & R_{bs}(-p_{sb}) \\ 0 & 1 \end{bmatrix} = \begin{bmatrix} R_{bs} & p_{bs} \\ 0 & 1 \end{bmatrix} = T_{bs}$$

This Lecture



- How do we combine rotation and position?
- How do we capture rigid-body motion?
- What is a transformation matrix?

Next Lecture



- How can we use transformation matrices?