Nishant Bharali
PID: 906641308

Problem Set-1

1.1

$V_1 = [1, 0, 0]^T$
$V_2 = [1, 0, 1]^T$
$V_3 = [0, 1, 1]^T$

Sol^n:- To prove that the 3 vectors above are linearly independent :-

$c_1 V_1 + c_2 V_2 + c_3 V_3 = 0$,

where c_1, c_2, c_3 scalar (constants) should be 0 individually.

$$\therefore c_1 \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix} + c_2 \begin{bmatrix} 1 \\ 0 \\ 1 \end{bmatrix} + c_3 \begin{bmatrix} 0 \\ 1 \\ 1 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$$

$$\Rightarrow \begin{bmatrix} c_1 \\ 0 \\ 0 \end{bmatrix} + \begin{bmatrix} c_2 \\ 0 \\ c_2 \end{bmatrix} + \begin{bmatrix} 0 \\ c_3 \\ c_3 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$$

$\Rightarrow c_1 + c_2 = 0$
$c_3 = 0$
$c_2 + c_3 = 0$

Solving the 3 equations, we get :-

$c_1 = c_2 = c_3 = 0$

$\therefore$ Thus, V_1, V_2 & V_3 are linearly independent.

[1.2] Similarly;

$\Rightarrow c_1 v_1 + c_2 v_2 + c_3 v_3 = 0$ for linear independence

$$\therefore c_1 \begin{bmatrix} 1 \\ 1 \\ 0 \end{bmatrix} + c_2 \begin{bmatrix} 1 \\ 2 \\ 1 \end{bmatrix} + c_3 \begin{bmatrix} 0 \\ 1 \\ 1 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$$

$\Rightarrow$ We get 3 equivalent equations to RHS:

$$\Rightarrow c_1 + c_2 = 0 \quad \text{——— (I)}$$
$$c_1 + 2c_2 + c_3 = 0 \quad \text{——— (II)}$$
$$c_2 + c_3 = 0 \quad \text{——— (III)}$$

Adding eqn. (I) & (III) we get:

$$\Rightarrow c_1 + (c_2 + c_2) + c_3 = c_1 + 2c_2 + c_3 = 0$$

which is the same as eqn. (II);

$$\therefore \begin{bmatrix} 1 & 1 & 0 \\ 1 & 2 & 1 \\ 0 & 1 & 1 \end{bmatrix} \begin{bmatrix} c_1 \\ c_2 \\ c_3 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$$ ... Reducing to Echelon form

$R_2 \rightarrow R_2 - 1$

$R_3 \rightarrow R_3 - R_2$

$$\begin{bmatrix} 1 & 1 & 0 \\ 0 & 1 & 1 \\ 0 & 0 & 0 \end{bmatrix} \begin{bmatrix} c_1 \\ c_2 \\ c_3 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$$

$\therefore$ $$c_1 + c_2 = 0$$
$$c_2 + c_3 = 0$$

On solving,

$\Rightarrow$ $$c_1 - c_3 = 0$$

$\therefore$ We are getting non-zero values for the scalars.

Thus V_1, V_2 & V_3 are linearly dependent.

2| ب

2.1|

(a.) Acc. to Gruebler's equation is

$$DoF = m(N-1-K) + \sum_{i=1}^{K} f_i$$

The robot has 6 links (including the ground) and 7 joints (6 revolute and 1 prismatic joint).

∴ Plugging the value for the robot in 2D-plane :

$$\Rightarrow DoF = 3(6-7-1) + (1+1+ \text{---} +1)$$

$$= 3(-2) + 7$$

$$= (7-6) = 1$$

$$\therefore \boxed{DoF = 1}$$

2.2

(b) Robot has 8 links (1 ground, 1 end-effector, 6 with the prismatic joints) & 9 joints (6 universal joints & 3 prismatic joints) in a 3D plane.

$N = 8, \ K = 9, \ m = 6$

$$\therefore \ DoF = m(N-1-K) + \sum_{i=1}^{K} f_i$$

$$= 6(8-1-9) + [(6\times2) + (1\times3)]$$

$$= -12 + 15 = 3$$

$$\boxed{DoF = 3}$$

2.3

(c) Here,

$N = 7 \ ; \ K = 8 \ ; \ m = 3.$

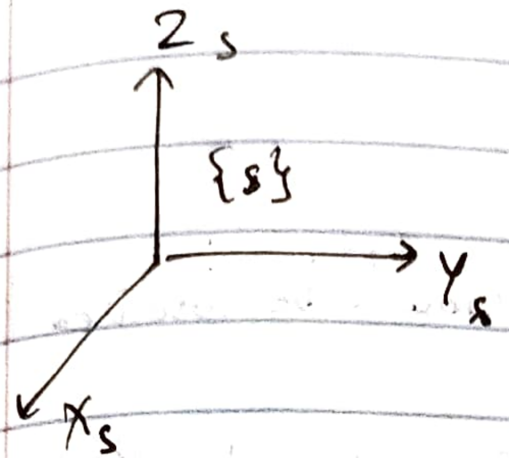
$$\therefore \ DoF = 3(7-1-8) + (8) = (-6) + 8$$

$$\boxed{DoF = 2}$$

2.4

(d) Only 1 actuator is needed to control robot
(a) due to the 2D-space restriction & 1 DoF.

3.11

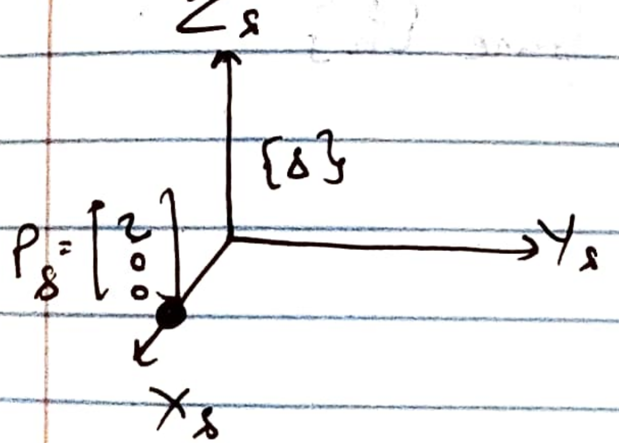


$$R_{sb} = \begin{bmatrix} 0 & 0 & -1 \\ 0 & 1 & 0 \\ 1 & 0 & 0 \end{bmatrix} \; ; \quad P_b = \begin{bmatrix} 0 \\ 0 \\ -2 \end{bmatrix}$$

Sol^n: Using subscript-cancellation; position p in frame {S}:

$$\Rightarrow \quad P_s = R_{sb} \, P_b = \begin{bmatrix} 0 & 0 & -1 \\ 0 & 1 & 0 \\ 1 & 0 & 0 \end{bmatrix} \begin{bmatrix} 0 \\ 0 \\ -2 \end{bmatrix} = \begin{bmatrix} 2 \\ 0 \\ 0 \end{bmatrix}$$

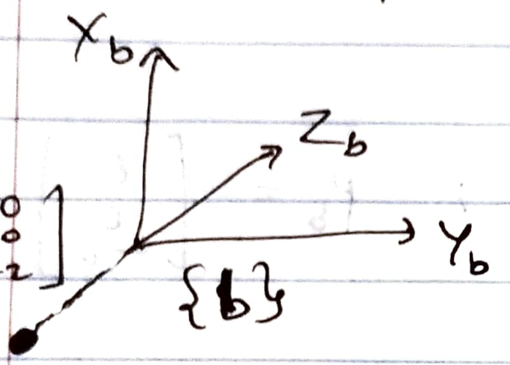
$\therefore$



Now, since P_b is $\begin{bmatrix} 0 \\ 0 \\ -2 \end{bmatrix}$;

Using <u>frame {s}</u> ;

co-ordinate frame {b} would be oriented like ;

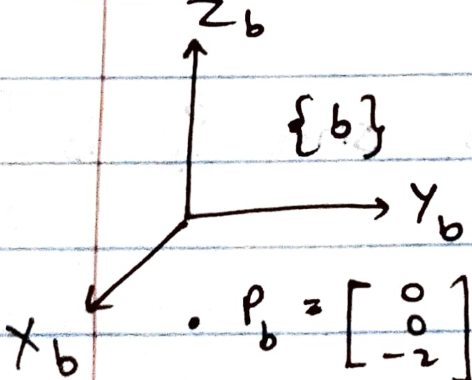


$P_b = \begin{bmatrix} 0 \\ 0 \\ -2 \end{bmatrix}$

[According to Right Hand Rule]

In <u>general</u> ;

<u>frame {b}</u> ; (without in respect to / reference to frame {h})



$P_b = \begin{bmatrix} 0 \\ 0 \\ -2 \end{bmatrix}$

[3.21] Consider a Robotic arm with 3 revolute joints, defining a coordinate frame {A} at base (ground) & {S} at end-effector.

$T \to$ transformation matrix from {A} to {B}:

$T = R_1 R_2 R_3$, where R_1, R_2, R_3 are rotational matrices for each joints' motion.

Presume a cube with edge a positioned at end-effector in frame {S}. Thus, its volume is:

$$V = a^3$$

Transforming this cube to frame {A} using T :-

$$\Rightarrow \quad V' = \det(T) V$$

& since T is just a sequence of rotations, it should preserve volumes, thus:

$$\Rightarrow \quad V' = V$$

$$\therefore \quad \det(T) = \det(R_1 R_2 R_3) = \det(R_1)\det(R_2)\det(R_3)$$
$$= 1$$

And, we know the product of Rotational Matrices ~~are~~ is a Rotational matrix, so each R_x should satisfy :

$$\boxed{\det(R_x) = 1}$$

$\therefore$ For the example of robotic arm above proves that the determinant of any rotational matrix must be +1 to preserve volumes under the transform. If we violate, it would improperly scale volumes, which is impossible for a pure rotation.

$\det(R) = i \cdot (j \times k)$ [i, j, k are unit vectors along x, y, z axes]

$\det(R) = 1$ [positive orientation of 3-axis following right-handed orientation]

$\det |X| = i_2 \cdot (j_2 \times k_2) = -1$

[Here, k points negatively, the cross product $j \times k$ results in a vector pointing in opposite direction compared to right-handed rule.]

$\therefore$ $\det(X) = -1$ for this left-handed coordinate frame.

But, this doesn't represent a valid rotational transform.

$\det(R) = 1$ is needed to preserve volumes and orientations.

Examples of R and X :-

$$R = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$$; Identity matrix

transf. matrix X that flips the orientation by inverting one axis :

$$X = \begin{bmatrix} -1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$$; here inverting the x-axis.

3.3 $\|x\| = \sqrt{x^T x}$ (magnitude of vector x)

→ We know that :

$$y = R_1 R_2 R_3 x \quad \text{———} \text{ (I)}$$

&, we know, for Rotational Matrices

$$\therefore \quad R^T = R^{-1} \quad \text{———} \text{ (II)}$$

Using eq. (I) :

$$\therefore \quad y^T = (R_1 R_2 R_3 x)^T$$

$$y^T = x^T (R_1 R_2 R_3)^T$$

$$\Rightarrow \quad y^T = x^T R_3^T R_2^T R_1^T = x^T R_3^{-1} R_2^{-1} R_1^{-1}$$

Using eqn. (II) :-

& $\|y\| = \sqrt{y^T y}$ (magnitude of vector y)

$$\Rightarrow \|y\| = \sqrt{x^T \cancel{R_3^{-1}} \cancel{R_2^{-1}} \cancel{R_1^{-1}} \cdot \cancel{R_1} \cancel{R_2} \cancel{R_3}\, x}$$

$$\Rightarrow \|y\| = \sqrt{x^T x}$$

$$\Rightarrow \|y\| = \|x\|$$

Which verifies that the magnitude of y is the same as x.

3.4) let's assume 2 Rot. Matrices R_1 & R_2 in a 2D space :

R_1 with a $\theta = (\pi/2)$:

$$R_1 = \begin{bmatrix} \cos(\pi/2) & -\sin(\pi/2) \\ \sin(\pi/2) & \cos(\pi/2) \end{bmatrix} = \begin{bmatrix} 0 & -1 \\ 1 & 0 \end{bmatrix}$$

&, R_2 with $\theta = \pi$:

$$R_2 = \begin{bmatrix} \cos\pi & -\sin\pi \\ \sin\pi & \cos\pi \end{bmatrix} = \begin{bmatrix} -1 & 0 \\ 0 & -1 \end{bmatrix}$$

Commutative property says :-

$\Rightarrow R_1 R_2 = R_2 R_1$

LHS ⇁

$$\therefore \begin{bmatrix} 0 & -1 \\ 1 & 0 \end{bmatrix} \begin{bmatrix} -1 & 0 \\ 0 & -1 \end{bmatrix} = \begin{bmatrix} 0 & 1 \\ -1 & 0 \end{bmatrix} = R_1 R_2$$

RHS ↳

$$\begin{bmatrix} -1 & 0 \\ 0 & -1 \end{bmatrix} \begin{bmatrix} 0 & -1 \\ 1 & 0 \end{bmatrix} = \begin{bmatrix} 0 & 1 \\ -1 & 0 \end{bmatrix} = R_2 R_1$$

Thus, proving :

$R_1 R_2 = R_2 R_1$; R_1 & R_2 are commutative.
for the provided case.

for a <u>general case</u> :

$$R_{\theta_1} R_{\theta_2} = \begin{bmatrix} \cos\theta_1 \cos\theta_2 - \sin\theta_1 \sin\theta_2 & -\sin\theta_2 \cos\theta_1 - \sin\theta_1 \cos\theta_2 \\ \sin\theta_1 \cos\theta_2 - \cos\theta_1 \sin\theta_2 & -\sin\theta_1 \sin\theta_2 + \cos\theta_1 \cos\theta_2 \end{bmatrix}$$

which is equal to $R_{\theta_2} R_{\theta_1}$ calculated, thus proving the commutative property of Rotational Matrices in the space.

```python
import numpy as np
# Define the angles of rotation
theta_z1 = np.pi / 4  # π/4 radians
theta_y = -np.pi / 3  # -π/3 radians
theta_z2 = np.pi / 2  # π/2 radians

# Create the rotation matrices
R1 = np.array([[np.cos(theta_z1), -np.sin(theta_z1), 0],
               [np.sin(theta_z1), np.cos(theta_z1), 0],
               [0, 0, 1]])

R2 = np.array([[np.cos(theta_y), 0, np.sin(theta_y)],
               [0, 1, 0],
               [-np.sin(theta_y), 0, np.cos(theta_y)]])

R3 = np.array([[np.cos(theta_z2), -np.sin(theta_z2), 0],
               [np.sin(theta_z2), np.cos(theta_z2), 0],
               [0, 0, 1]])

# Multiply the rotation matrices
R = np.dot(R1, np.dot(R2, R3))

# Multiply the transpose of R to R itself to prove R is a rotational matrix as the result is an identity matrix
result_prove_rotational = R.T @ R
determinant = np.linalg.det(R)
R_inverse = np.linalg.inv(R)

print("Below is the rotational matrix R: ")
print("\n", R)
print("\nDeterminant of rotational matrix R:")
print("\n", determinant)
print("\nR is a rotational matrix as result is an identity matrix after multiplying R with R transpose: ")
print("\n", result_prove_rotational)
print("\n R.T is the transpose of R to equate it with R inverse below: ")
print("\n", R.T)
print("\n R_inverse is the inverse of R to equate it with R transpose above: ")
print("\n", R_inverse)
```

```
Below is the rotational matrix R:

 [[-7.07106781e-01 -3.53553391e-01 -6.12372436e-01]
 [ 7.07106781e-01 -3.53553391e-01 -6.12372436e-01]
 [ 5.30287619e-17 -8.66025404e-01  5.00000000e-01]]

Determinant of rotational matrix R:

 1.0

R is a rotational matrix as result is an identity matrix after multiplying R with R transpose:

 [[1.00000000e+00 1.46601644e-17 1.01628368e-17]
 [1.46601644e-17 1.00000000e+00 1.48741681e-17]
 [1.01628368e-17 1.48741681e-17 1.00000000e+00]]

 R.T is the transpose of R to equate it with R inverse below:

 [[-7.07106781e-01  7.07106781e-01  5.30287619e-17]
 [-3.53553391e-01 -3.53553391e-01 -8.66025404e-01]
 [-6.12372436e-01 -6.12372436e-01  5.00000000e-01]]

 R_inverse is the inverse of R to equate it with R transpose above:

 [[-7.07106781e-01  7.07106781e-01 -7.85046229e-17]
 [-3.53553391e-01 -3.53553391e-01 -8.66025404e-01]
 [-6.12372436e-01 -6.12372436e-01  5.00000000e-01]]
```

5.1 $R_{dh} \rightarrow$ Orientation of human in the drone's frame $\{d\}$

Using subscript - cancellation :

$\Rightarrow \quad R_{dh} = R_{db}\,(R_{bh})^T \qquad \left[\text{Here, } (R_{bh})^T = R_{hb}\right]$

$$\therefore \quad R_{dh} = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1/2 & -\sqrt{3}/2 \\ 0 & \sqrt{3}/2 & 1/2 \end{bmatrix} \begin{bmatrix} 1/\sqrt{2} & -1/\sqrt{2} & 0 \\ 1/\sqrt{2} & 1/\sqrt{2} & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

$$\Rightarrow \quad R_{dh} = \begin{bmatrix} 1/\sqrt{2} & -1/\sqrt{2} & 0 \\ \frac{1}{2\sqrt{2}} & \frac{1}{2\sqrt{2}} & -\sqrt{3}/2 \\ \frac{\sqrt{3}}{2\sqrt{2}} & \frac{\sqrt{3}}{2\sqrt{2}} & 1/2 \end{bmatrix}$$

5.2. Orientation of drone in the human's frame $\{h\}$:

$\because$ We know that $R_{ab}^{\;T} = R_{ba}$;

$$\therefore \quad R_{hd} = (R_{dh})^T = \begin{bmatrix} \frac{1}{\sqrt{2}} & \frac{1}{2\sqrt{2}} & \frac{\sqrt{3}}{2\sqrt{2}} \\ -\frac{1}{\sqrt{2}} & \frac{1}{2\sqrt{2}} & \frac{\sqrt{3}}{2\sqrt{2}} \\ 0 & -\frac{\sqrt{3}}{2\sqrt{2}} & \frac{1}{2} \end{bmatrix}$$

5.3 Given,

$p_h = \overrightarrow{hb_h} = [1, 2, 0]^T$

& $p_d = \overrightarrow{db_d} = [2, 0, -2]^T$

Therefore, to get p_{hd} → position of drone in frame h.

$$\Rightarrow \boxed{p_{hd} = (R_{hd})^T p_d + p_h} \quad (p_{hd} = R_{hd}\, p_d + p_h)$$

$$= \begin{bmatrix} 1/\sqrt{2} & -1/\sqrt{2} & 0 \\ 1/2\sqrt{2} & 1/2\sqrt{2} & -\sqrt{3}/2 \\ \sqrt{3}/2\sqrt{2} & \frac{\sqrt{3}}{2\sqrt{2}} & \frac{1}{2} \end{bmatrix} \begin{bmatrix} 2 \\ 0 \\ -2 \end{bmatrix} + \begin{bmatrix} 1 \\ 2 \\ 0 \end{bmatrix}$$

$$p_{hd} = \begin{bmatrix} \sqrt{2} \\ \frac{1}{\sqrt{2}} + \sqrt{3} \\ \frac{\sqrt{3}}{\sqrt{2}} - 1 \end{bmatrix} + \begin{bmatrix} 1 \\ 2 \\ 0 \end{bmatrix} = \begin{bmatrix} \sqrt{2} + 1 \\ \frac{1}{\sqrt{2}} + \sqrt{3} + 2 \\ \frac{\sqrt{3}}{\sqrt{2}} - 1 \end{bmatrix}$$

$$\therefore \boxed{p_{hd} = \begin{bmatrix} 2.414 \\ 4.439 \\ 1.225 \end{bmatrix}}$$

Position of drone in frame $\{h\}$