Path Planning Challenge

Magicc Lab, Summer 2023

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Abstract

Convex optimization challenge for summer 2023.

1 Introduction

The convex optimization challenge will use the VTOL example from [1], which is describe below for convenience.

In this design study we will explore the control design for a simplified planar version of a quadrotor following a ground target. In particular, we will constrain the dynamics to be in two a dimensional plane. In Cartesian space, the position of the VTOL can be defined by a variable describing the vertical position, one describing the horizontal position, and one angular variable describing the orientation, as shown in 1. The planar vertical take-off and landing (VTOL) system is comprised of a center pod of mass m_c and inertia J_c, a right motor/rotor that is modeled as a point mass m_r that exerts a force f_r at a distance d from the center of mass, and a left motor/rotor that is modeled as a point mass m_ℓ that exerts a force f_ℓ at a distance $-d$ from the center of mass. The position of the center of mass of the planar VTOL system is given by horizontal position z_v and altitude h. The airflow through the rotor creates a change in the direction of flow of air and causes what is called "momentum drag." Momentum drag can be modeled as a viscous drag force that is proportional to the horizontal velocity $\dot{z}_v$. In other words, the drag force is $F_{\text{drag}} = -\mu \dot{z}_v$. The target on the ground will be modeled as an object with position z_t and altitude $h = 0$. We will not explicitly model the dynamics of the target.

Use the following physical parameters: $m_c = 1$ kg, $J_c = 0.0042$ kg m^2, $m_r = 0.25$ kg, $m_\ell = 0.25$ kg, $d = 0.3$ m, $\mu = 0.1$ kg/s, $g = 9.81$ m/s^2.

2 Dynamics

The dynamics are given by

$$\begin{pmatrix} m_c + 2m_r & 0 & 0 \\ 0 & m_c + 2m_r & 0 \\ 0 & 0 & J_c + 2m_r d^2 \end{pmatrix} \begin{pmatrix} \ddot{z} \\ \ddot{h} \\ \ddot{\theta} \end{pmatrix} = \begin{pmatrix} -(f_r + f_\ell)\sin\theta - \mu \dot{z} \\ -(m_c + 2m_r)g + (f_r + f_\ell)\cos\theta \\ d\,(f_r - f_\ell) \end{pmatrix}. \tag{1}$$

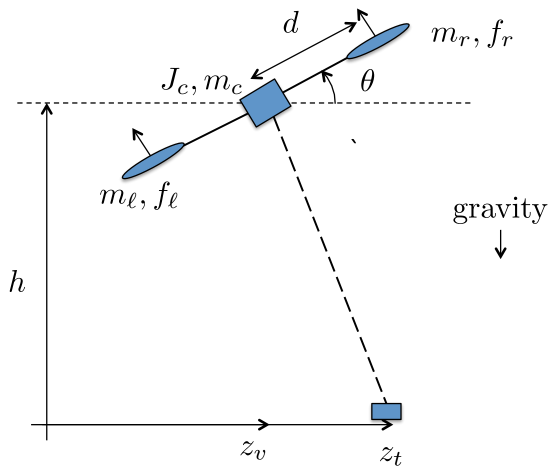

Figure 1: Planar vertical take-off and landing (VTOL)

We will assume that the right and left rotor forces are constrained as follows

$$0 \leq f_r, f_\ell \leq 0.75(m_c + 2m_r)g. \tag{2}$$

If the thrust was limited to $0.5(m_c + 2m_r)g$, then at full throttle, the VTOL could just hover. Therefore, the limit allows some additional control authority, but is somewhat constraining.

3 Linearization

Since f_r and f_ℓ appear in the equations as either $f_r + f_\ell$ or $d(f_r - f_\ell)$, it is easier to think of the inputs as the total force $F \triangleq f_r + f_\ell$, and the torque $\tau \triangleq (f_r - f_\ell)$. Note that since

$$\begin{pmatrix} F \\ \tau \end{pmatrix} = \begin{pmatrix} 1 & 1 \\ d & -d \end{pmatrix} \begin{pmatrix} f_r \\ f_\ell \end{pmatrix},$$

that

$$\begin{pmatrix} f_r \\ f_\ell \end{pmatrix} = \begin{pmatrix} 1 & 1 \\ d & -d \end{pmatrix}^{-1} \begin{pmatrix} F \\ \tau \end{pmatrix} = \begin{pmatrix} \frac{1}{2} & \frac{1}{2d} \\ \frac{1}{2} & -\frac{1}{2d} \end{pmatrix} \begin{pmatrix} F \\ \tau \end{pmatrix}.$$

Therefore,

$$\begin{aligned} f_r &= \frac{1}{2}F + \frac{1}{2d}\tau \\ f_\ell &= \frac{1}{2}F - \frac{1}{2d}\tau. \end{aligned}$$

Using the expression $F = f_r + f_l$ and $\tau = d(f_r - f_l)$, the equations of motion are given by

$$\begin{pmatrix} m_c + 2m_r & 0 & 0 \\ 0 & m_c + 2m_r & 0 \\ 0 & 0 & J_c + 2m_r d^2 \end{pmatrix} \begin{pmatrix} \ddot{z} \\ \ddot{h} \\ \ddot{\theta} \end{pmatrix} = \begin{pmatrix} -F\sin\theta - \mu\dot{z} \\ -(m_c + 2m_r)g + F\cos\theta \\ \tau \end{pmatrix}.$$

At equilibria when $\dot{z} = \ddot{z} = \dot{h} = \ddot{h} = \dot{\theta} = \ddot{\theta} = 0$ we have

$$-F_e \sin\theta_e = 0 \tag{3}$$

$$-(m_c + 2m_r)\, g + F_e \cos\theta_e = 0 \tag{4}$$

$$\tau_e = 0. \tag{5}$$

From Equation (3), either $\theta_e = 0$ or $F_e = 0$, since $F_e = 0$ would make Equation (4) impossible to satisfy, we conclude that $\theta_e = 0$. From (3), $F_e = (m_c + 2m_r)\, g$, which makes sense since that is the force required to keep the VTOL in place.

To linearize, define

$$\begin{aligned} \tilde{\theta} &\triangleq \theta - \theta_e & \Rightarrow \qquad \theta &= \theta_e + \tilde{\theta} \\ \tilde{F} &\triangleq F - F_e & \Rightarrow \qquad F &= F_e + \tilde{F} \end{aligned}$$

to get Therefore, the linear equations of motion are

$$\begin{aligned} (m_c + 2m_r)\, \ddot{\tilde{z}} + \mu\dot{\tilde{z}} &= -F_e\tilde{\theta} \\ (m_c + 2m_r)\, \ddot{\tilde{h}} &= \tilde{F} \\ \left(J_c + 2m_r d^2\right) \ddot{\tilde{\theta}} &= \tilde{\tau}. \end{aligned}$$

Since the two nonlinearities are in the equations of motion cannot simultaneously be eliminated by correct choice of the single input F, feedback linearization is not a viable option for this system.

4 State Space Equations

Defining the state as

$$x \triangleq \begin{pmatrix} z & h & \theta & \dot{z} & \dot{h} & \dot{\theta} \end{pmatrix},$$

we get that

$$\dot{x} = Ax + Bu$$

where

$$u = \begin{pmatrix} \tilde{F} \\ \tau \end{pmatrix}$$

$$A = \begin{pmatrix} 0 & 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 \\ 0 & 0 & -\frac{F_e}{m_c+2m_r} & -\frac{\mu}{m_c+2m_r} & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \end{pmatrix}$$

$$B = \begin{pmatrix} 0 & 0 \\ 0 & 0 \\ 0 & 0 \\ 0 & 0 \\ \frac{1}{m_c+2m_r} & 0 \\ 0 & \frac{1}{J_c+2m_r d^2} \end{pmatrix}$$

5 Trajectory Following

Assume that the trajectory generator produces a reference trajectory $(x^r(t), u^r(t))$, and define the reference trajectory error as

$$\tilde{x} \triangleq x - x^r, \qquad \tilde{u} \triangleq u - u^r,$$

then

$$\begin{aligned} \dot{\tilde{x}} &= \dot{x} - \dot{x}^r \\ &= Ax + Bu - \dot{x}^r + (Ax^r + Bu^r) - (Ax^r + Bu^r) \\ &= A\tilde{x} + B\tilde{u} + (Ax^r + Bu^r - \dot{x}^r). \end{aligned}$$

Note that if the reference trajectory is *feasible* in the sense that the reference trajectory satisfies the dynamics, i.e.,

$$\dot{x}^r = Ax_r + Bu^r,$$

then

$$\dot{\tilde{x}} = A\tilde{x} + B\tilde{u}.$$

Otherwise, the term $d \triangleq Ax^r + Bu^r - \dot{x}^r$ can be thought of as a disturbance acting on the system.

Following the integrator design described in Chapter 12 of [1], we desire to add integrators on position z and altitude h. Define

$$C = \begin{pmatrix} 1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 & 0 \end{pmatrix}$$

and the integrator state

$$x_I = \int_{-\infty}^{t} C\tilde{x}(\tau)d\tau,$$

which implies that

$$\dot{x}_I = C\tilde{x},$$

or

$$\frac{d}{dt}\begin{pmatrix}\tilde{x}\\ x_I\end{pmatrix} = \begin{pmatrix}A & 0\\ C & 0\end{pmatrix}\begin{pmatrix}\tilde{x}\\ x_I\end{pmatrix} + \begin{pmatrix}B\\ 0\end{pmatrix}\tilde{u}.$$

Defining

$$A_1 \triangleq \begin{pmatrix}A & 0\\ C & 0\end{pmatrix}, \qquad B_1 \triangleq \begin{pmatrix}B\\ 0\end{pmatrix}, \qquad K_1 \triangleq \begin{pmatrix}K & K_I\end{pmatrix},$$

we find K_1 using `K_1 = place(A_1, B_1, p_1)`, where p_1 are the desired poles. The trajectory following controller is given by

$$\begin{aligned}
&\tilde{u} = -\begin{pmatrix}K & K_I\end{pmatrix}\begin{pmatrix}\tilde{x}\\ x_I\end{pmatrix}\\
\Longrightarrow\ &\tilde{u} = -K\tilde{x} - K_I\int_{-\infty}^{t} C\tilde{x}(\tau)d\tau\\
\Longrightarrow\ &u = u^r - K(x - x^r) - K_I\int_{-\infty}^{t} C(x - x^r)d\tau\\
\Longrightarrow\ &\begin{pmatrix}F\\ \tau\end{pmatrix} = \begin{pmatrix}F_e\\ 0\end{pmatrix} + u^r - K(x - x^r) - K_I\int_{-\infty}^{t} C(x - x^r)d\tau\\
\Longrightarrow\ &\begin{pmatrix}f_\ell\\ f_r\end{pmatrix} = \begin{pmatrix}\frac{1}{2} & -\frac{1}{2d}\\ \frac{1}{2} & \frac{1}{2d}\end{pmatrix}\left[\begin{pmatrix}F_e\\ 0\end{pmatrix} + u^r - K(x - x^r) - K_I\int_{-\infty}^{t} C(x - x^r)d\tau\right].
\end{aligned}$$

6 Landing Constraints

The objective is to minimize the time t_f required to land the VTOL from initial state

$$x(t_0) = \begin{pmatrix}z(t_0) & h(t_0) & \theta(t_0) & \dot{z}(t_0) & \dot{h}(t_0) & \dot{\theta}(t_0)\end{pmatrix}^\top,$$

to the final state

$$x(t_f) = \begin{pmatrix}z(t_f) & \\ h(t_f) & \\ \theta(t_f) & \\ \dot{z}(t_f) & \\ \dot{h}(t_f) & \dot{\theta}(t_f)\end{pmatrix} = \begin{pmatrix}0 & \\ 0 & \\ 0 & \\ 0 & \\ -0.1 & 0\end{pmatrix}$$

while satisfying the state constraints

$$\begin{aligned}
m_1^\top(C_c x - p_1) \geq 0\\
m_2^\top(C_c x - p_2) \geq 0,
\end{aligned}$$

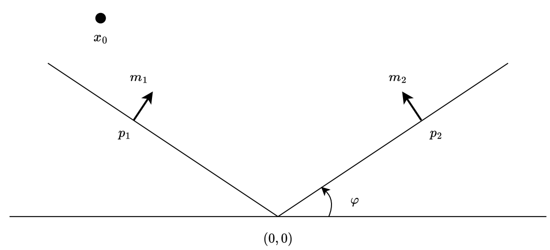

Figure 2: State constraints

where

$$
\begin{aligned}
C_c &= \begin{pmatrix} 1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 & 0 \end{pmatrix} \\
m_1 &= \begin{pmatrix} \cos\varphi & \sin\varphi \end{pmatrix}^\top \\
m_2 &= \begin{pmatrix} \cos\varphi & -\sin\varphi \end{pmatrix}^\top \\
p_1 &= \begin{pmatrix} 0 & 0 \end{pmatrix}^\top \\
p_2 &= \begin{pmatrix} 0 & 0 \end{pmatrix}^\top ,
\end{aligned}
$$

where $\varphi = 30\pi/180$. The constraints are shown schematically in Figure 2.

The planned trajectory must also satisfy the input constraints given in Equation (2).

References

[1] R. W. Beard, T. W. McLain, C. Peterson, and M. Killpack, *Introduction to Feedback Control Using Design Studies.* Amazon, 2016.