Dynamics of Voters Movement in an Election

Nishanta Baral

Montclair State University

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Abstract

In this work, we study the dynamics of voters movement in a election with two major parties using the compartmental model. We use the idea of compartmental model from disease modeling and epidemiology to incorporate neutral, two political parties, and apathetic voters. Then, we study a simplified model with just two compartments with interactions among people of two political parties to study the voters movement. Similarly, we study the steady states, their existing condition, and basic reproduction number.

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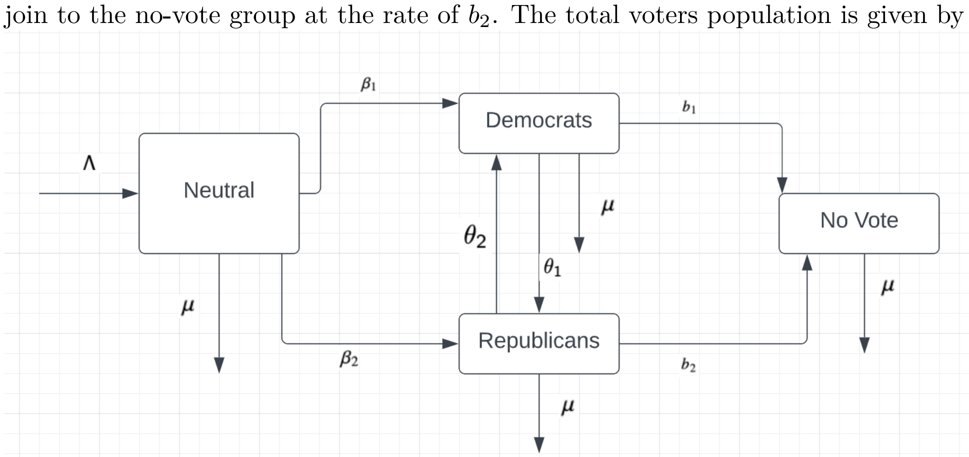
1. Introduction

Compartmental models are a type of mathematical model that divide a population into different groups or compartments based on their characteristics or status. These models can be used to study the dynamics of infectious diseases, population growth, and other phenomena that involve the movement of individuals between different states or compartments. In the context of an election, a compartmental model could be used to represent the movement of voters between different parties or candidate preferences. For example, the model could include compartments for voters who support a particular candidate, voters who are undecided, and voters who are unlikely to vote. To build a compartmental model for voters in an election, we would need to define the compartments and the rules for how individuals move between them. Once we have defined the compartments and the rules for movement between them, we can use the model to simulate the evolution of voter preferences over time and make predictions about the outcome of the election. This can be a useful tool for election strategists, as it can help them understand the factors that drive voter behavior and identify opportunities to influence the outcome of the election.

2. Mathematical model

: This model has been used from pre-existing work that has been referenced in Ref[1]. The flowchart above describe the mathematical model we used for this project. The compartment of neutral group is increased by the rate of people into N at a rate Λ. All compartment are decreased when people exit from the system with a rate μ. The people of neutral group interact with the people from Democrats (X_1) at the rate of β_1. The people of neutral group interact with the people from Republicans (X_2) at the rate of β_2. The people from X_1 join the people to X_2 at the rate of θ_1. The people from X_2 join the people to X_1 at the rate of θ_2. The people from X_1 group join to the no-vote group (A) at the rate of b_1 and the people from X_2 group join to the no-vote group at the rate of b_2. The total voters population is given by $V = N + X_1 + X_2 + A$.



We get out set of governing equations as:

$$\frac{dN}{dt} = \Lambda - \mu N - \frac{\beta_1 N X_1}{V} - \frac{\beta_2 N X_2}{V}$$

$$\frac{dX_1}{dt} = -\mu X_1 - b_1 X_1 - (\theta_1 - \theta_2)\frac{X_1 X_2}{V} + \frac{\beta_1 N X_1}{V}$$

$$\frac{dX_2}{dt} = -\mu X_2 - b_2 X_2 - (\theta_2 - \theta_1)\frac{X_1 X_2}{V} + \frac{\beta_2 N X_2}{V}$$

$$\frac{dA}{dt} = -\mu A + b_1 X_1 + b_2 X_2$$

In this model, population is constant as we assume $V = N + X_1 + X_2 + A$.

We can see that the limiting value from our model is $\frac{\Lambda}{\mu}$ as

$$\frac{dV}{dt} = \frac{dN}{dt} + \frac{dX_1}{dt} + \frac{dX_2}{dt} + \frac{dA}{dt} \implies \frac{dV}{dt} = \Lambda - \mu V$$

This is an easy ordinary first order linear differential equation to solve and we get $V(t) = \frac{\Lambda}{\mu}(1 - e^{-\mu t}) + V(0)e^{-\mu t}$ which as $\lim_{t\to\infty} V = \frac{\Lambda}{\mu}$.

3. Non-Dimensionalization of the model

Firstly, we will non-dimensionalize the model using $\frac{N}{V} = n$, $\frac{X_1}{V} = x_1$, $\frac{X_2}{V} = x_2$ and $\frac{A}{V} = a$. Then, $n + x_1 + x_2 + a = 1$.

$$\frac{dn}{dt} = \mu - \mu n - \beta_1 n x_1 - \beta_2 n x_2 = f_1$$

$$\frac{dx_1}{dt} = -\mu x_1 - b_1 x_1 - (\theta_1 - \theta_2) x_1 x_2 + \beta_1 n x_1 = f_2$$

$$\frac{dx_2}{dt} = -\mu x_2 - b_2 x_2 - (\theta_2 - \theta_1) x_1 x_2 + \beta_2 n x_2 = f_3$$

$$\frac{da}{dt} = -\mu a + b_1 x_1 + b_2 x_2 = f_4$$

4. Equilibrium analysis

To find the equilibrium points and their existing stability condition, we first need to compute the Jacoboian matrix from the set of governing equations which is given as:

$$J = \begin{bmatrix} \frac{\partial f_1}{\partial n} & \frac{\partial f_1}{\partial x_1} & \frac{\partial f_1}{\partial x_2} & \frac{\partial f_1}{\partial a} \\ \frac{\partial f_2}{\partial n} & \frac{\partial f_2}{\partial x_1} & \frac{\partial f_2}{\partial x_2} & \frac{\partial f_2}{\partial a} \\ \frac{\partial f_3}{\partial n} & \frac{\partial f_3}{\partial x_1} & \frac{\partial f_3}{\partial x_2} & \frac{\partial f_3}{\partial a} \\ \frac{\partial f_4}{\partial n} & \frac{\partial f_4}{\partial x_1} & \frac{\partial f_4}{\partial x_2} & \frac{\partial f_4}{\partial a} \end{bmatrix}$$ Therefore, the Jacobian matrix of our model is given as:

$$J = \begin{bmatrix} -\mu - \beta_1 x_1 - \beta_2 x_2 & -\beta_1 n & -\beta_2 n & 0 \\ -\beta_1 x_1 & -\mu - b_1 - (\theta_1 - \theta_2)x_2 + \beta_1 n & (\theta_1 - \theta_2)x_1 & 0 \\ \beta_2 x_2 & (\theta_2 - \theta_1)x_2 & -\mu - b_2 - (\theta_2 - \theta_1)x_1 + \beta_2 n & 0 \\ 0 & b_1 & b_2 & -\mu \end{bmatrix}$$

We have four equilibrium conditions. We get the first equilibrium condition when both $x_1 = 0$ and $x_2 = 0$ which implies that there is no support for either of the parties. We get second and third equilibrium condition when either $x_1 = 0$ and $x_2 \neq 0$ or $x_1 \neq 0$ and $x_2 = 0$. This implies there is support for only one of the two political parties. We get the final equilibrium point when both $x_1 \neq 0$ and $x_2 \neq 0$. This implies when there is support for both political parties and this is the most realistic scenario in daily life. We will study all four cases below.

1. When $x_1 = 0$ and $x_2 = 0$

   $(n, x_1, x_2, a) = (1, 0, 0, 0)$

   $$J = \begin{bmatrix} -\mu & -\beta_1 & -\beta_2 & 0 \\ 0 & -\mu - b_1 + \beta_1 & 0 & 0 \\ 0 & 0 & -\mu - b_2 + \beta_2 & 0 \\ 0 & b_1 & b_2 & -\mu \end{bmatrix}$$

   Now we need to calculate it's eigenvalues.

   $$(-\mu - \lambda) \begin{bmatrix} -\mu - b_1 + \beta_1 - \lambda & 0 & 0 \\ 0 & -\mu - b_2 + \beta_2 - \lambda & 0 \\ b_1 & b_2 & -\mu - \lambda \end{bmatrix} = 0$$

   $(-\mu - \lambda)(-\mu - b_1 + \beta_1 - \lambda)(-\mu - b_2 + \beta_2 - \lambda)(-\mu - \lambda) = 0$

   $\lambda_{1,2} = -\mu$, $\lambda_3 = -\mu - b_1 + \beta_1$ and $\lambda_4 = -\mu - b_2 + \beta_2$

   $\lambda_{1,2}$ are negative. $\lambda_3 < 0$ iff $-\mu - b_1 + \beta_1 < 0 \implies \beta_1 < \mu + b_1 \implies \frac{\beta_1}{\mu + b_1} < 1$. Similarly, $\lambda_4 < 0$ iff $\frac{\beta_2}{\mu + b_2} < 1$.

   Therefore, $R_0 = \max(\frac{\beta_1}{\mu + b_1}, \frac{\beta_2}{\mu + b_2})$. This is the voters free equilibrium state.

2. When $x_1 = 0$ and $x_2 \neq 0$.

Solve $\frac{dx_2}{dt} = 0. \implies -\mu x_2 - b_2 x_2 - (\theta_2 - \theta_1)x_1 x_2 + \beta_2 n x_2 = 0$

$-\mu x_2 - b_2 x_2 + \beta_2 n x_2 = 0 \implies -\mu - b_2 + \beta_2 n = 0 \implies n = \frac{\mu + b_2}{\beta_2}$

Solve $\frac{dn}{dt} = 0$ with $x_1 = 0$ and $n = \frac{\mu + b_2}{\beta_2}$.

$\mu - \mu n - \beta_1 n x_1 - \beta_2 n x_2 = 0 \implies \mu - \frac{\mu(\mu + b_2)}{\beta_2} - \frac{\beta_2(\mu + b_2)}{\beta_2} x_2 = 0$

$\mu - \frac{\mu(\mu + b_2)}{\beta_2} = \frac{\beta_2(\mu + b_2)}{\beta_2} x_2 \implies x_2 = \frac{\mu\beta_2 - \mu(\mu + b_2)}{\beta_2(\mu + b_2)} \implies x_2 = \frac{\mu(\beta_2 - \mu - b_2)}{\beta_2(\mu + b_2)}$

Solve $\frac{da}{dt} = 0$

$-\mu a + b_1 x_1 + b_2 x_2 = 0 \implies \mu a = b_2 \frac{\mu(\beta_2 - \mu - b_2)}{\beta_2(\mu + b_2)} \implies a = \frac{b_2(\beta_2 - \mu - b_2)}{\beta_2(\mu + b_2)}$

Therefore $(n, x_1, x_2, a) = (\frac{\mu + b_2}{\beta_2}, 0, \frac{\mu(\beta_2 - \mu - b_2)}{\beta_2(\mu + b_2)}, \frac{b_2(\beta_2 - \mu - b_2)}{\beta_2(\mu + b_2)})$

$$J = \begin{bmatrix} -\mu - \frac{\mu(\beta_2 - \mu - b_2)}{\mu + b_2} & -\beta_1 \frac{\mu + b_2}{\beta_2} & -(\mu + b_2) & 0 \\ 0 & -\mu - b_1 - (\theta_1 - \theta_2)\frac{\mu(\beta_2 - \mu - b_2)}{\beta_2(\mu + b_2)} + \beta_1 \frac{\mu + b_2}{\beta_2} & 0 & 0 \\ \frac{\mu(\beta_2 - \mu - b_2)}{\mu + b_2} & (\theta_2 - \theta_1)\frac{\mu(\beta_2 - \mu - b_2)}{\beta_2(\mu + b_2)} & 0 & 0 \\ 0 & b_1 & b_2 & -\mu \end{bmatrix}$$

Now to find its eigenvalues, expanding along the fourth column,

$$(-\mu - \lambda) \begin{bmatrix} -\mu - \frac{\mu(\beta_2 - \mu - b_2)}{\mu + b_2} - \lambda & -\beta_1 \frac{\mu + b_2}{\beta_2} & -(\mu + b_2) \\ 0 & -\mu - b_1 - (\theta_1 - \theta_2)\frac{\mu(\beta_2 - \mu - b_2)}{\beta_2(\mu + b_2)} + \beta_1 \frac{\mu + b_2}{\beta_2} - \lambda & 0 \\ \frac{\mu(\beta_2 - \mu - b_2)}{\mu + b_2} & (\theta_2 - \theta_1)\frac{\mu(\beta_2 - \mu - b_2)}{\beta_2(\mu + b_2)} & -\lambda \end{bmatrix} = 0$$

Let $-\mu - \frac{\mu(\beta_2 - \mu - b_2)}{\mu + b_2} - \lambda = A$,

$-\mu - b_1 - (\theta_1 - \theta_2)\frac{\mu(\beta_2 - \mu - b_2)}{\beta_2(\mu + b_2)} + \beta_1 \frac{\mu + b_2}{\beta_2} - \lambda = B$, and

$\frac{\mu(\beta_2 - \mu - b_2)}{\mu + b_2} = C$

$$(-\mu-\lambda)\begin{bmatrix} A & -\beta_1\frac{\mu+b_2}{\beta_2} & -(\mu+b_2) \\ 0 & B & 0 \\ C & (\theta_2-\theta_1)\frac{\mu(\beta_2-\mu-b_2)}{\beta_2(\mu+b_2)} & -\lambda \end{bmatrix} = 0$$

$$(-\mu-\lambda)[-(\mu+b_2)(-BC)-\lambda AB] = 0$$

$$(-\mu-\lambda)B((\mu+b_2)C-\lambda A) = 0$$

$$\lambda_1 = -\mu$$

$$B = 0 \implies -\mu - b_1 - (\theta_1-\theta_2)\frac{\mu(\beta_2-\mu-b_2)}{\beta_2(\mu+b_2)} + \beta_1\frac{\mu+b_2}{\beta_2} - \lambda_2 = 0$$

$$\lambda_2 = -\mu - b_1 - (\theta_1-\theta_2)\frac{\mu(\beta_2-\mu-b_2)}{\beta_2(\mu+b_2)} + \beta_1\frac{\mu+b_2}{\beta_2}$$

$$\lambda_2 = \frac{-\mu\beta_2(\mu+b_2)-b_1\beta_2(\mu+b_2)-(\theta_1-\theta_2)\mu(\beta_2-\mu-b_2)+\beta_1(\mu+b_2)^2}{\beta_2(\mu+\beta_2)}$$

$$(\mu+b_2)C-\lambda A = 0 \implies \mu(\beta_2-\mu-b_2)-\lambda[-\mu-\frac{\mu(\beta_2-\mu-b_2)}{\mu+b_2}-\lambda] = 0$$

$$\lambda^2+\lambda(\mu+\frac{\mu(\beta_2-\mu-b_2)}{\mu+b_2})+\mu(\beta_2-\mu-b_2) = 0$$

$$\lambda^2+\lambda(\frac{\mu\beta_2}{\mu+b_2})+\mu(\beta_2-\mu-b_2) = 0$$

$$\lambda_{3,4} = \frac{-\frac{\mu\beta_2}{\mu+b_2}\pm\sqrt{(\frac{\mu\beta_2}{\mu+b_2})^2-4\mu(\beta_2-\mu-b_2)}}{2}$$

$$\lambda_{3,4} = \frac{-\mu\beta_2\pm\sqrt{(\mu\beta_2)^2-4\mu(\mu+\beta_2)^2(\beta_2-\mu-b_2)}}{2(\mu+b_2)}$$

This equilibrium is locally asymptotically stable if $-\mu\beta_2(\mu+b_2)-b_1\beta_2(\mu+b_2)-(\theta_1-\theta_2)\mu(\beta_2-\mu-b_2)+\beta_1(\mu+b_2)^2 < 0$ and $\frac{\mu\beta_2}{\sqrt{(\mu\beta_2)^2-4\mu(\mu+\beta_2)^2(\beta_2-\mu-b_2)}} < 1$
This is the equilibrium state when there is support for only Republicans in the system.

3. When $x_1 \neq 0$ and $x_2 = 0$.

Solve $\frac{dx_1}{dt} = 0. \implies -\mu x_1 - b_1x_1 - (\theta_1-\theta_2)x_1x_2 + \beta_1nx_1 = 0$

$$-\mu x_1 - b_1x_1 + \beta_1nx_1 = 0 \implies -\mu - b_1 + \beta_1 n = 0 \implies n = \frac{\mu+b_1}{\beta_1}$$

Solve $\frac{dn}{dt} = 0$ with $x_2 = 0$ and $n = \frac{\mu+b_1}{\beta_1}$.

$$\mu - \mu n - \beta_1nx_1 - \beta_2nx_2 = 0 \implies \mu - \frac{\mu(\mu+b_1)}{\beta_1} - \frac{\beta_1(\mu+b_1)}{\beta_1}x_1 = 0$$

$$\mu - \frac{\mu(\mu+b_1)}{\beta_1} = \frac{\beta_1(\mu+b_1)}{\beta_1}x_1 \implies x_1 = \frac{\mu\beta_1-\mu(\mu+b_1)}{\beta_1(\mu+b_1)} \implies x_1 = \frac{\mu(\beta_1-\mu-b_1)}{\beta_1(\mu+b_1)}$$

Solve $\frac{da}{dt} = 0$

$$-\mu a + b_1x_1 + b_2x_2 = 0 \implies \mu a = b_1\frac{\mu(\beta_1-\mu-b_1)}{\beta_1(\mu+b_1)} \implies a = \frac{b_1(\beta_1-\mu-b_1)}{\beta_1(\mu+b_1)}$$

Therefore $(n, x_1, x_2, a) = (\frac{\mu+b_1}{\beta_1}, \frac{\mu(\beta_1-\mu-b_1)}{\beta_1(\mu+b_1)}, 0, \frac{b_1(\beta_1-\mu-b_1)}{\beta_1(\mu+b_1)})$

$$J = \begin{bmatrix} -\mu - \frac{\mu(\beta_1-\mu-b_1)}{\mu+b_1} & -(\mu+b_1) & -\beta_2\frac{\mu+b_1}{\beta_1} & 0 \\ -\frac{\mu(\beta_1-\mu-b_1)}{\mu+b_1} & 0 & (\theta_1-\theta_2)\frac{\mu(\beta_1-\mu-b_1)}{\beta_1(\mu+b_1} & 0 \\ 0 & 0 & -\mu-b_2-(\theta_2-\theta_1)\frac{\mu(\beta_1-\mu-b_1)}{\beta_1(\mu+b_1)} + \beta_2\frac{\mu+b_1}{\beta_1} & 0 \\ 0 & b_1 & b_2 & -\mu \end{bmatrix}$$

Now to find its eigenvalues, expanding along the fourth column,

$$(-\mu-\lambda)\begin{bmatrix} -\mu - \frac{\mu(\beta_1-\mu-b_1)}{\mu+b_1} - \lambda & -(\mu+b_1) & -\beta_2\frac{\mu+b_1}{\beta_1} \\ -\frac{\mu(\beta_1-\mu-b_1)}{\mu+b_1} & -\lambda & (\theta_1-\theta_2)\frac{\mu(\beta_1-\mu-b_1)}{\beta_1(\mu+b_1} \\ 0 & 0 & -\mu-b_2-(\theta_2-\theta_1)\frac{\mu(\beta_1-\mu-b_1)}{\beta_1(\mu+b_1)} + \beta_2\frac{\mu+b_1}{\beta_1} - \lambda \end{bmatrix} = 0$$

Let $A = -\mu - \frac{\mu(\beta_1-\mu-b_1)}{\mu+b_1} - \lambda$

$B = \frac{\mu(\beta_1-\mu-b_1)}{\mu+b_1}$

$C = -\mu - b_2 - (\theta_2-\theta_1)\frac{\mu(\beta_1-\mu-b_1)}{\beta_1(\mu+b_1)} + \beta_2\frac{\mu+b_1}{\beta_1} - \lambda$

$$(-\mu-\lambda)\begin{bmatrix} A & -(\mu+b_1) & -\beta_2\frac{\mu+b_1}{\beta_1} \\ -B & -\lambda & (\theta_1-\theta_2)\frac{\mu(\beta_1-\mu-b_1)}{\beta_1(\mu+b_1} \\ 0 & 0 & C \end{bmatrix} = 0$$

$(-\mu-\lambda)[-\lambda AC + B \times -(\mu+b_1) \times C] = 0$

$(-\mu-\lambda)C[-\lambda(-\mu - \frac{\mu(\beta_1-\mu-b_1)}{\mu+b_1} - \lambda) - \mu(\beta_1-\mu-b_1)] = 0$

$(-\mu-\lambda)C[\lambda^2 + (\mu + \frac{\mu(\beta_1-\mu-b_1)}{\mu+b_1})\lambda - \mu(\beta_1-\mu-b_1)] = 0$

$\lambda_1 = -\mu$

$C = 0 \implies -\mu - b_2 - (\theta_2-\theta_1)\frac{\mu(\beta_1-\mu-b_1)}{\beta_1(\mu+b_1)} + \beta_2\frac{\mu+b_1}{\beta_1} - \lambda_2 = 0$

$\lambda_2 = -\mu - b_2 - (\theta_2-\theta_1)\frac{\mu(\beta_1-\mu-b_1)}{\beta_1(\mu+b_1)} + \beta_2\frac{\mu+b_1}{\beta_1}$

$\lambda_2 = \frac{-\mu\beta_1(\mu+b_1) - b_2\beta_1(\mu+b_1) - (\theta_2-\theta_1)\mu(\beta_1-\mu-b_1) + \beta_2(\mu+b_1)^2}{\beta_1(\mu+b_1)}$

$\lambda^2 + (\mu + \frac{\mu(\beta_1-\mu-b_1)}{\mu+b_1})\lambda - \mu(\beta_1-\mu-b_1) = 0$

$\lambda^2 + (\frac{\mu\beta_1}{\mu+b_1})\lambda - \mu(\beta_1-\mu-b_1) = 0$

$\lambda_{3,4} = \frac{-\frac{\mu\beta_1}{\mu+b_1} \pm \sqrt{(\frac{\mu\beta_1}{\mu+b_1})^2 - 4\mu(\beta_1-\mu-b_1)}}{2}$

$\lambda_{3,4} = \frac{-\mu\beta_1 \pm \sqrt{(\mu\beta_1)^2 - 4\mu(\mu+\beta_1)^2(\beta_1-\mu-b_1)}}{2(\mu+b_1)}$

This equilibrium is locally asymptotically stable if $-\mu\beta_1(\mu+b_1) - b_2\beta_1(\mu+b_1) - (\theta_2-\theta_1)\mu(\beta_1-\mu-b_1) + \beta_2(\mu+b_1)^2 < 0$ and $\frac{\mu\beta_1}{\sqrt{(\mu\beta_1)^2-4\mu(\mu+\beta_1)^2(\beta_1-\mu-b_1)}} < 1$.
This is the equilibrium state when there is support for only Democrats in the system.

4. When both $x_1, x_2 \neq 0$.

$\frac{dx_1}{dt} = 0 \implies -\mu x_1 - b_1 x_1 - (\theta_1-\theta_2)x_1 x_2 + \beta_1 n x_1 = 0$

$-\mu - b_1 - (\theta_1-\theta_2)x_2 + \beta_1 n = 0$

Similarly, $\frac{dx_2}{dt} = 0 \implies -\mu - b_2 - (\theta_2-\theta_1)x_1 + \beta_2 n = 0$

Isolating x_1 and x_2, we get,

$x_1 = \frac{-\mu-b_2+\beta_2 n}{\theta_2-\theta_1}$ and $x_2 = \frac{-\mu-b_1+\beta_1 n}{\theta_1-\theta_2}$

Now set: $\frac{dn}{dt} = 0$

$\mu - \mu n - \beta_1 n x_1 - \beta_2 n x_2 = 0$

$\mu - \mu n - \beta_1 n \frac{-\mu-b_2+\beta_2 n}{\theta_2-\theta_1} - \beta_2 n \frac{-\mu-b_1+\beta_1 n}{\theta_1-\theta_2} = 0$

$\mu(\theta_2-\theta_1) - \mu n(\theta_2-\theta_1) + \beta_1 n\mu + \beta_1 n b_2 - \beta_1\beta_2 n^2 - \beta_2 n\mu - b_1\beta_2 n + \beta_1\beta_2 n^2 = 0$

$\mu(\theta_2-\theta_1) = \mu n(\theta_2-\theta_1) - \beta_1 n\mu - \beta_1 n b_2 + \beta_2 n\mu + b_1\beta_2 n$

$n = \frac{\mu(\theta_2-\theta_1)}{\mu\theta_2-\mu\theta_1-\mu\beta_1+\mu\beta_2-\beta_1 b_2+b_1\beta_2}$

Now substitute the value of n into $-\mu - b_2 - (\theta_2-\theta_1)x_1 + \beta_2 n = 0$ to get x_1

$(\theta_2-\theta_1)x_1 = -\mu - b_2 + \beta_2 n$

$(\theta_2-\theta_1)x_1 = -\mu - b_2 + \beta_2 \frac{\mu(\theta_2-\theta_1)}{\mu\theta_2-\mu\theta_1-\mu\beta_1+\mu\beta_2-\beta_1 b_2+b_1\beta_2}$

$(\theta_2-\theta_1)x_1 = \frac{-\mu^2\theta_2+\mu^2\theta_1+\mu^2\beta_1-\mu^2\beta_2+\mu\beta_1 b_2-\mu b_1\beta_2-b_2\mu\theta_2+b_2\mu\theta_1+b_2\mu\beta_1-b_2\mu\beta_2+\beta_1 b_2^2-b_1 b_2\beta_2+\mu\beta_2\theta_2-\mu\beta_2\theta_1}{\mu\theta_2-\mu\theta_1-\mu\beta_1+\mu\beta_2-\beta_1 b_2+b_1\beta_2}$

$x_1 = \frac{\mu^2(\theta_1-\theta_2+\beta_1-\beta_2)+\mu(2\beta_1 b_2-b_1\beta_2-b_2\theta_2+b_2\theta_1-b_2\beta_2+\beta_2\theta_2-\beta_2\theta_1)+\beta_1 b_2^2-b_1 b_2\beta_2}{(\theta_2-\theta_1)(\mu\theta_2-\mu\theta_1-\mu\beta_1+\mu\beta_2-\beta_1 b_2+b_1\beta_2)}$

Now substitute the value of n into $-\mu - b_1 - (\theta_1-\theta_2)x_2 + \beta_1 n = 0$ to get x_2

$(\theta_2-\theta_1)x_2 = \mu + b_1 - \beta_1 n$

$(\theta_2-\theta_1)x_2 = \mu + b_1 - \beta_1 \frac{\mu(\theta_2-\theta_1)}{\mu\theta_2-\mu\theta_1-\mu\beta_1+\mu\beta_2-\beta_1 b_2+b_1\beta_2}$

$(\theta_2-\theta_1)x_2 = \frac{\mu^2\theta_2-\mu^2\theta_1-\mu^2\beta_1+\mu^2\beta_2-\mu\beta_1 b_2+\mu b_1\beta_2+b_1\mu\theta_2-b_1\mu\theta_1-b_1\mu\beta_1+b_1\mu\beta_2-\beta_1 b_1 b_2+b_1^2\beta_2-\mu\beta_1\theta_2+\mu\beta_1\theta_1}{\mu\theta_2-\mu\theta_1-\mu\beta_1+\mu\beta_2-\beta_1 b_2+b_1\beta_2}$

$x_2 = \frac{\mu^2(\theta_2-\theta_1-\beta_1+\beta_2)+\mu(-\beta_1 b_2+2b_1\beta_2+b_1\theta_2-b_1\theta_1-b_1\beta_1-\beta_1\theta_2+\beta_1\theta_1)-\beta_1 b_1 b_2+b_1^2\beta_2}{(\theta_2-\theta_1)(\mu\theta_2-\mu\theta_1-\mu\beta_1+\mu\beta_2-\beta_1 b_2+b_1\beta_2)}$

Now solve $\frac{da}{dt} = 0 \implies -\mu a + b_1 x_1 + b_2 x_2 = 0$ to get the value of a.

$a = \frac{b_1x_1+b_2x_2}{\mu}$

$a = \frac{(\mu b_1-\mu b_2)(\theta_1-\theta_2+\beta_1-\beta_2)+(b_1b_2\beta_1+b_1b_2\beta_2-b_1^2\beta_2-b_2^2\beta_1-b_1\theta_1\beta_2+b_2\beta_1\theta_1+b_1\beta_2\theta_2-b_2\beta_1\theta_2)}{(\theta_2-\theta_1)(\mu\theta_2-\mu\theta_1-\mu\beta_1+\mu\beta_2-\beta_1b_2+b_1\beta_2)}$

Therefore,

$$(n, x_1, x_2, a) = (\frac{\mu(\theta_2-\theta_1)^2}{D}, \frac{A}{D}, \frac{B}{D}, \frac{C}{D})$$

$A = \mu^2(\theta_1 - \theta_2 + \beta_1 - \beta_2) + \mu(2\beta_1b_2 - b_1\beta_2 - b_2\theta_2 + b_2\theta_1 - b_2\beta_2 + \beta_2\theta_2 - \beta_2\theta_1) + \beta_1b_2^2 - b_1b_2\beta_2$

$B = \mu^2(\theta_2 - \theta_1 - \beta_1 + \beta_2) + \mu(-\beta_1b_2 + 2b_1\beta_2 + b_1\theta_2 - b_1\theta_1 - b_1\beta_1 - \beta_1\theta_2 + \beta_1\theta_1) - \beta_1b_1b_2 + b_1^2\beta_2$

$C = (\mu b_1 - \mu b_2)(\theta_1 - \theta_2 + \beta_1 - \beta_2) + (b_1b_2\beta_1 + b_1b_2\beta_2 - b_1^2\beta_2 - b_2^2\beta_1 - b_1\theta_1\beta_2 + b_2\beta_1\theta_1 + b_1\beta_2\theta_2 - b_2\beta_1\theta_2)$

$D = (\theta_2 - \theta_1)(\mu\theta_2 - \mu\theta_1 - \mu\beta_1 + \mu\beta_2 - \beta_1b_2 + b_1\beta_2)$

Now to find check the conditions for stability and find the basic reproductive number, we need to plug the value of (n, x_1, x_2, a) into the Jacobian matrix

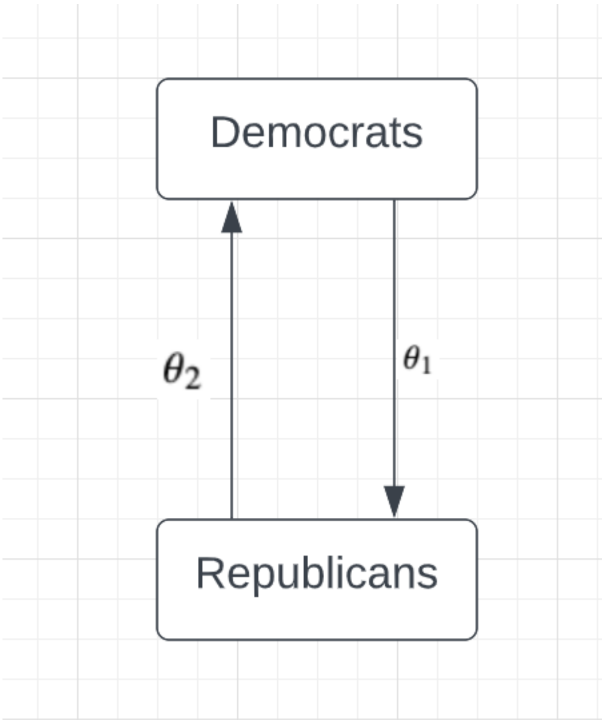
$$J = \begin{bmatrix} -\mu - \beta_1x_1 - \beta_2x_2 & -\beta_1n & -\beta_2n & 0 \\ -\beta_1x_1 & -\mu - b_1 - (\theta_1-\theta_2)x_2 + \beta_1n & (\theta_1-\theta_2)x_1 & 0 \\ \beta_2x_2 & (\theta_2-\theta_1)x_2 & -\mu - b_2 - (\theta_2-\theta_1)x_1 + \beta_2n & 0 \\ 0 & b_1 & b_2 & -\mu \end{bmatrix}$$

and calculate its eigenvalues.

This task seems algebraically really complicated, but could be done numerically. This is the equilibrium state when there is support for both Democrats and Republicans in the system.

5. Simplied model with two compartments

Since we saw above the complications of theoretically analyzing the model with multiple compartments, we will analyze a compartmental model using just two compartments: Democrats and Republicans. This model was proposed by Thomas Lux in The Economic Journal of 1995 titled "Herd Behaviour, Bubbles and Crashes", which is embedded in Ref[2].



Let θ_1 be the rate in which people change from Democrats to Republicans and let θ_2 be the rate in which people change from Republicans to Democrats.

Then we have the system of equations,

$\frac{dD}{dt} = \theta_2 R - \theta_1 D$
$\frac{dR}{dt} = \theta_1 D - \theta_2 R$

Now we will non-dimensionalize the model and change it into a system of differential equation with only one variable.

Let $D + R = 2m$ and define $X = \frac{D-R}{2m}$ which can also be said as imbalance towards Democrats.

$X' = \frac{D'-R'}{2m} = \frac{\theta_2 R - \theta_1 D - \theta_1 D + \theta_2 R}{2m} = \frac{1}{m}(\theta_2 R - \theta_1 D)$

Next step is to define θ_1 and θ_2. Let $\theta_1 = re^{-aX}$ and $\theta_2 = re^{aX}$ which can also be termed as bandwagon effect. This bandwagon effect is created by the newspapers, media outlets, social media, and so on.

$1 + X = \frac{2m}{2m} + \frac{D-R}{2m} = \frac{2m+D-R}{2m} = \frac{D}{m}$. Similalrly, we can show that $1 - X = \frac{R}{m}$

So, we finally have our model, a system of equations with just single variables,

$$X' = (1 - X)re^{ax} - (1 + X)re^{-ax}$$

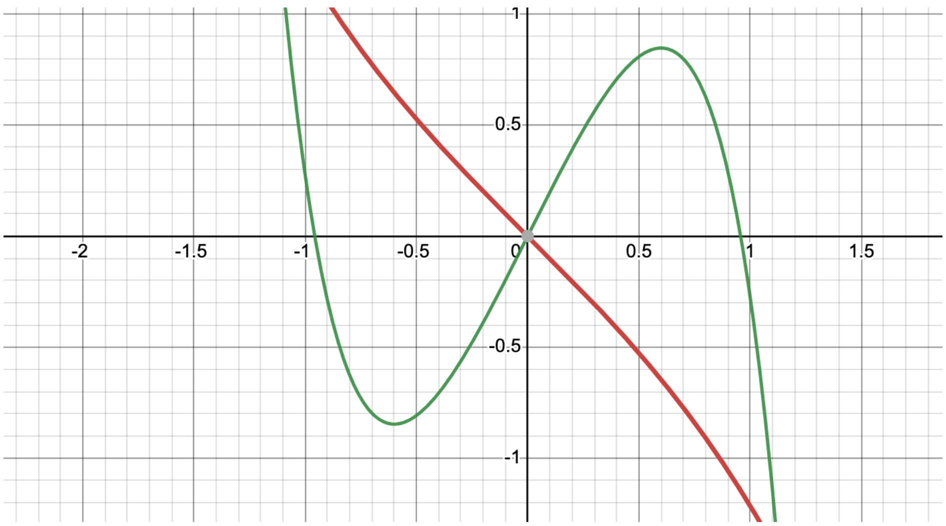
6. Equilibrium analysis, Bifurcation, and Phase Portraits

6.1 Equilibrium analysis

We perform the equilibrium analysis by setting $X' = 0$.

$$X' = 0 \implies (1-X)re^{ax} - (1+X)re^{-ax} = 0 \implies (1-X)e^{ax} = (1+X)e^{-ax}$$

$x = 0$ is the only equilibrium point and it is a stable equilibrium point when $a < 1$. When $a > 1$, we will get a pair of two stable equilibrium points and $x = 0$ becomes an unstable equilibrium point.



As we can see from the graph above, the red graph is at $a = 0$ and the green graph is at $a = 2$. We have two stable equilibrium points and an unstable equilibrium point when $a > 1$ and only one stable equilibrium point when $a < 1$.

6.2 Bifurcation

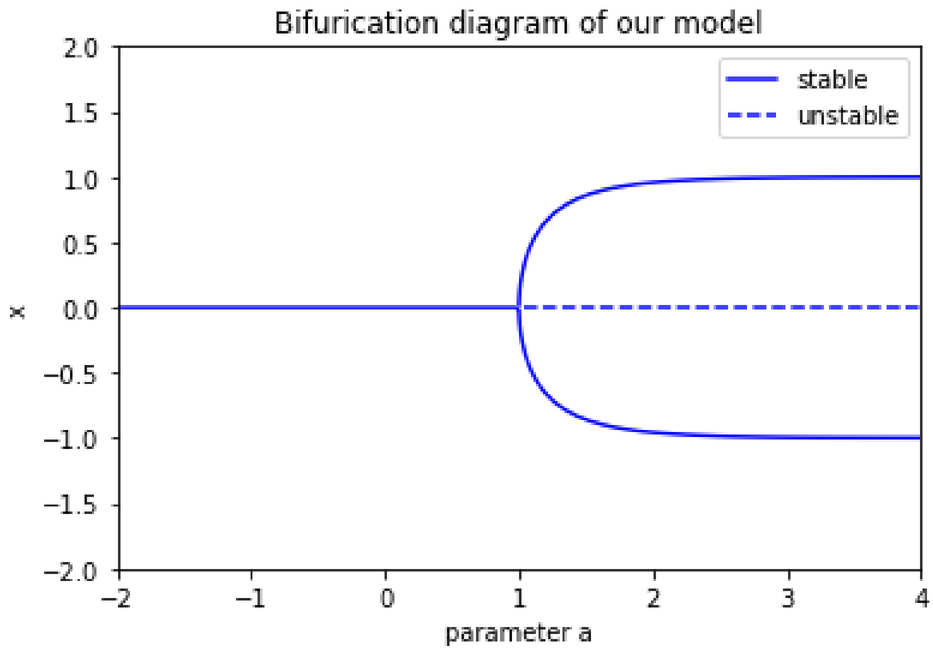
:

According to Wolfram-Alpha the link to which is listed in Ref[3], let $f : \mathbb{R} \times \mathbb{R} \to \mathbb{R}$ be a one-parameter family of C^3 maps satisfying

$$f(-x,\mu) = -f(x,\mu), \quad \left.\frac{\partial f}{\partial x}\right|_{\mu=0,x=0} = 0, \left.\frac{\partial^2 f}{\partial x \partial \mu}\right|_{\mu=0,x=0} > 0, \quad \left.\frac{\partial^3 f}{\partial x^3}\right|_{\mu=0,x=0} < 0$$

although condition (1) can actually be relaxed slightly. Then there are intervals having a single stable fixed point and three fixed points (two of which are stable and one of which is unstable). This type of bifurcation is called a pitchfork bifurcation.

Since we have a stable equilibrium point at $x = 0$ until $a < 1$ and we get a pair of stable equilibrium point and an unstable equilibrium point at $x = 0$, we see the pitchfork bifurcation in our model which is shown in figure below.



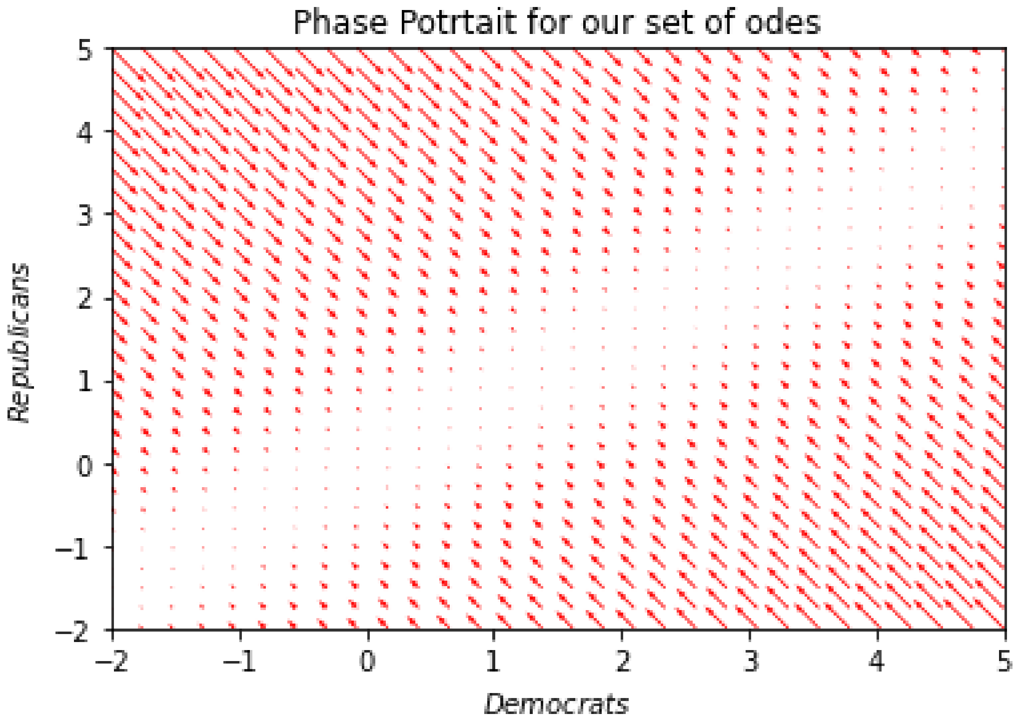
6.3 Phase Portrait

We can create a phase portrait for our system of equations. Recall our set of governing equations:

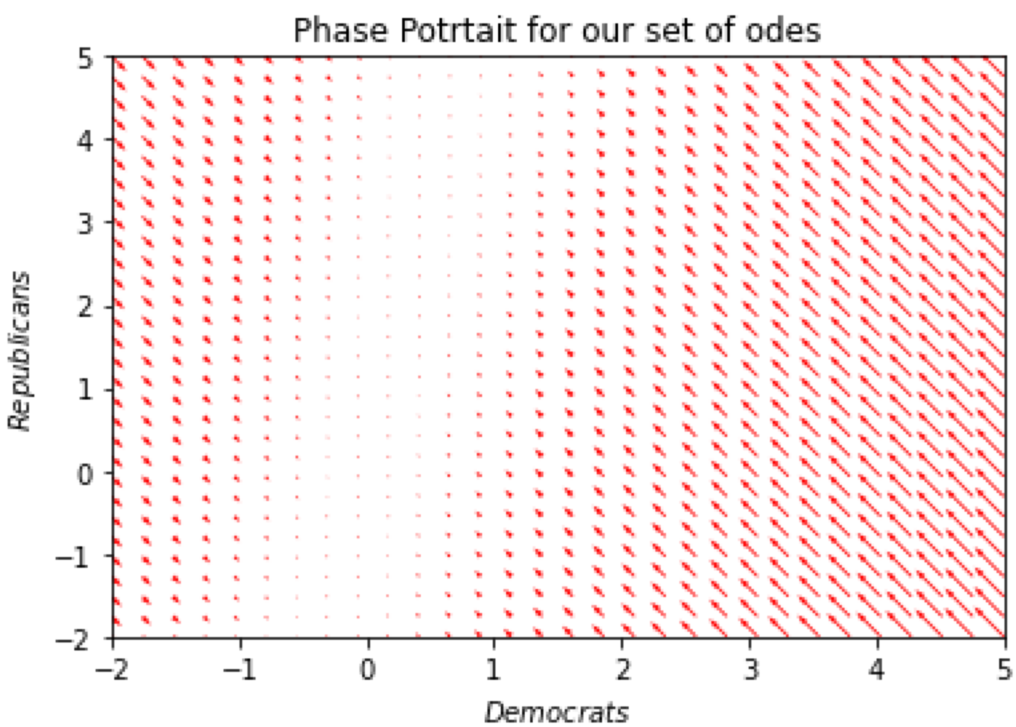
$\frac{dD}{dt} = \theta_2 R - \theta_1 D$

$\frac{dR}{dt} = \theta_1 D - \theta_2 R$

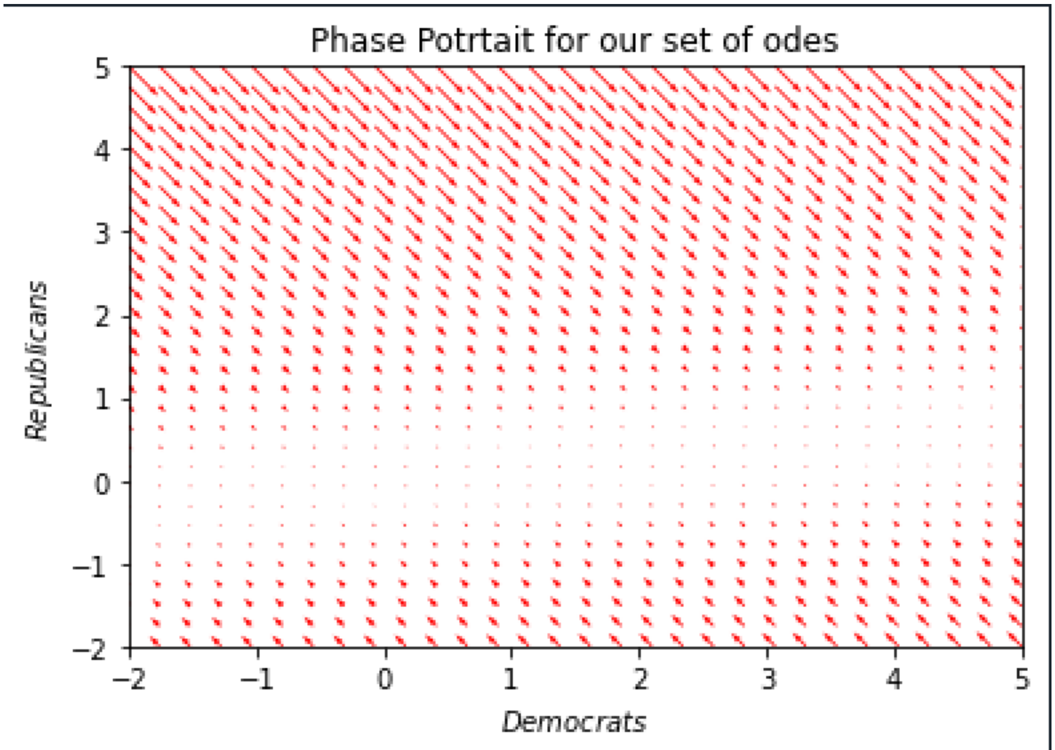
We will create phase portraits for different scenarios. Let us assume 6% people are changing from Democrats to Republicans, and 8% people are changing from Republicans to Democrats. Then, $\theta_1 = 0.06$ and $\theta_2 = 0.08$. Then we get the phase portrait as follows:



Let us assume 60% people are changing from Democrats to Republicans, and 8% people are changing from Republicans to Democrats. Then, $\theta_1 = 0.6$ and $\theta_2 = 0.08$. Then we get the phase portrait as follows:



Let us assume 6% people are changing from Democrats to Republicans, and 50% people are changing from Republicans to Democrats. Then, $\theta_1 = 0.06$ and $\theta_2 = 0.5$. Then we get the phase portrait as follows:



7. Conclusion

Thus in this project we studied the compartmental model using four compartments and a simplified two compartmental model. One limitation of this study is we do not have data to fit into the model for four compartmental model. The next extension of this project to have actual data and do parameter and model fitting.

8. References

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- https://mathworld.wolfram.com/PitchforkBifurcation.html