Density Ratio Spectrum in a Double-Gyre Flow

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1 Double-Gyre Fluid Flow

The double-gyre flow is a simple and very well-studied fluid model consisting of two counter-rotating vortices, where the separatrix between vortices oscillates periodically to emulate wind forcing. The double-gyre flow is characterized by the stream function [1]

$$\psi(x, y, t) = A \sin(\pi f(x, t)) \sin(\pi y), \tag{1}$$

where

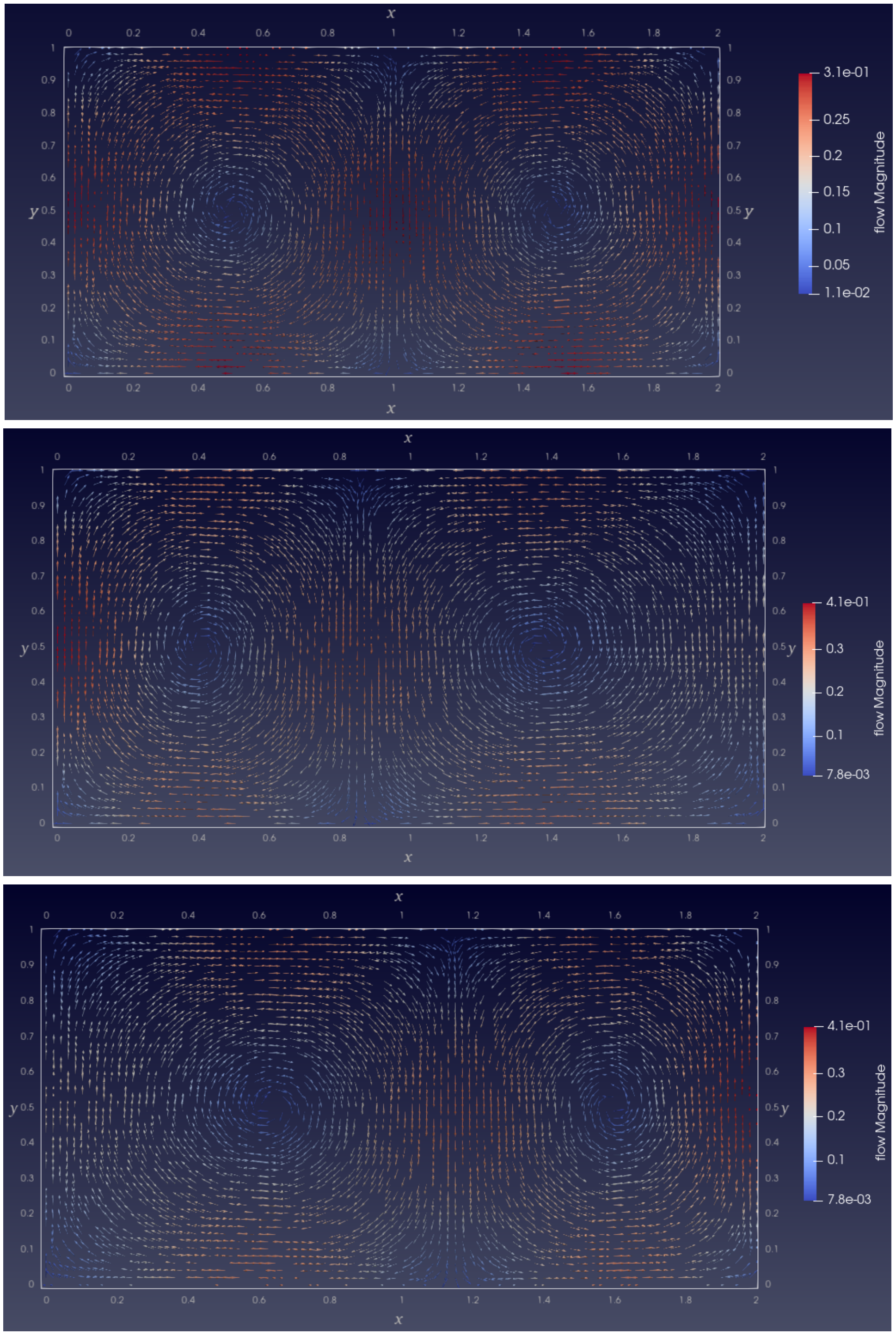
$$\begin{aligned} f(x, t) &= a(t)x^2 + b(t)x, \\ a(t) &= \epsilon \sin(\omega t), \\ b(t) &= 1 - 2\epsilon \sin(\omega t). \end{aligned} \tag{2}$$

Taking derivatives of the streamfunction, the velocity field for the double-gyre model is found to be

$$\begin{aligned} \dot{x} &= -\frac{\partial \psi}{\partial y} = -\pi A \sin(\pi f(x, t)) \cos(\pi y), \\ \dot{y} &= \frac{\partial \psi}{\partial x} = \pi A \cos(\pi f(x, t)) \sin(\pi y) \frac{df}{dx}, \end{aligned} \tag{3}$$

where $\frac{\omega}{2\pi}$ is the frequency of the separatrix oscillation, A is the approximate amplitude of the velocity vectors, and ϵ determines the amplitude of the left-right motion of the separatrix between the two gyres. When $\epsilon = 0$, the flow becomes time-independent, and if $\epsilon \neq 0$, the gyres undergo a periodic expansion and contraction in the x direction [1]. In this work, we focus on the time-dependent flow as it is more interesting and also more relevant to the realistic geophysical flows.

The double-gyre flow is usually studied on the rectangular spatial domain of $\Omega : [0 \times 2] \times [0 \times 1]$. For the time-independent case, one finds a steady flow with two counter-rotating gyres. One of the time-independent gyres is located on part of the domain at $\Omega_1 : [0 \times 1] \times [0 \times 1]$, while the second time-independent gyre is located on the part of the domain at $\Omega_2 : [1 \times 2] \times [0 \times 1]$. In the time-dependent case, the two gyres are separated by a heteroclinic manifold which connects the equilibrium points at $(1, 0)$ and $(1, 1)$. Thus, the separatrix acts as a barrier to particle transport. Figure 1 shows snapshots of the time-dependent velocity field at three different time instances which highlight the periodic left-right motion of the separatrix. In the time-independent case, the velocity field is steady and always has the form shown in Fig. 1 (a).



a

b

c

Figure 1. Velocity field of the time-dependent double-gyre flow that demonstrates the periodic expansion and contraction of the gyres. The velocity fields are shown (top) at $t = 0$, with the separatrix located in the middle at $x = 1$, (middle) at $t = 2$, with the separatrix located in the left half of the domain, and (bottom) at $t = 7$, with the separatrix located in the right half of the domain

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2 Dynamics of Inertial Particles

To understand the dynamics of inertial particles in a double-gyre flow model, we explore the preferential concentration of inertial particles using the Maxey-Riley (MR) equation [2], given as

$$
\begin{aligned}
m_p \dot{v} = m_f \frac{D}{Dt} u(r(t), t) - \frac{1}{2} m_f \frac{d}{dt} [v - u(r(t), t) - \frac{1}{10} a^2 \nabla^2 u(r(t), t)] \\
-6\pi a \mu X(t) + (m_p - m_f) g - 6\pi a^2 \mu \int_0^t d\tau \frac{\frac{dX(\tau)}{d\tau}}{\sqrt{\pi v (t-\tau)}},
\end{aligned}
\tag{4}
$$

with

$$X(t) = v(t) - u(r(t), t) - \frac{1}{6} a^2 \nabla^2 u.$$

Equation (4) is valid for two-dimensional (2D) or three-dimensional (3D) flows, where $r(t)$ denotes the position of a spherical particle at time t, $v(t) = \dot{r}(t)$ is the corresponding velocity of the particle, m_p is the mass of the inertial particle, m_f is the mass of the fluid displaced by the particle, $u(r(t), t)$ is the velocity of the fluid at the location $r(t)$ and time t, μ is the viscosity of the underlying fluid, a is the radius of the particle, and g is the acceleration due to gravity. In Eq. (4), the derivative Du/Dt is the material derivative, and d/dt is the usual total derivative.

We non-dimensionalize the MR equation given by Eq. (4) using the velocity scale, U, and the length scale, L, of the external flow as shown in [3] and the non-dimensional MR equation as follows:

$$r\ddot{(t)} = \frac{1}{St}(u(r(t), t) - r\dot{(t)}) + \frac{3}{2} R \frac{Du}{Dt} \tag{5}$$

where

$$St^{-1} = \frac{6\pi a \mu L}{\left(m_p + \frac{1}{2} m_F\right) U}, \quad R = \frac{m_f}{m_p + \frac{1}{2} m_f} \tag{6}$$

In Eq. (5), St is the Stokes number, defined as the ratio of the characteristic time of a particle to a characteristic time of the flow, and R is the density ratio parameter. If $R = 2/3$, the particles have the same density as that of the carrier fluid, and we refer to those particles as neutrally buoyant. If $R > 2/3$, then the particles are lighter than the carrier fluid, and we refer to them as bubbles. Similarly, particles with $R < 2/3$ are denser than the carrier fluid, and we refer to them as aerosols [3].

Figure 2 shows the particle aggregation of aerosols, neutrally buoyant particles, and bubbles at time $t = 15$ after advecting the inertial particles using the MR equation given by Eq. (5) for $St = 0.5$. We see that aerosols ($R = 0$) aggregate toward the maximal FTLE ridges while bubbles ($R = 1$) are repelled from the maximal FTLE ridges. Instead, bubbles form clusters in the center of the gyres. Neutrally buoyant particles are however evenly spread out throughout our domain.

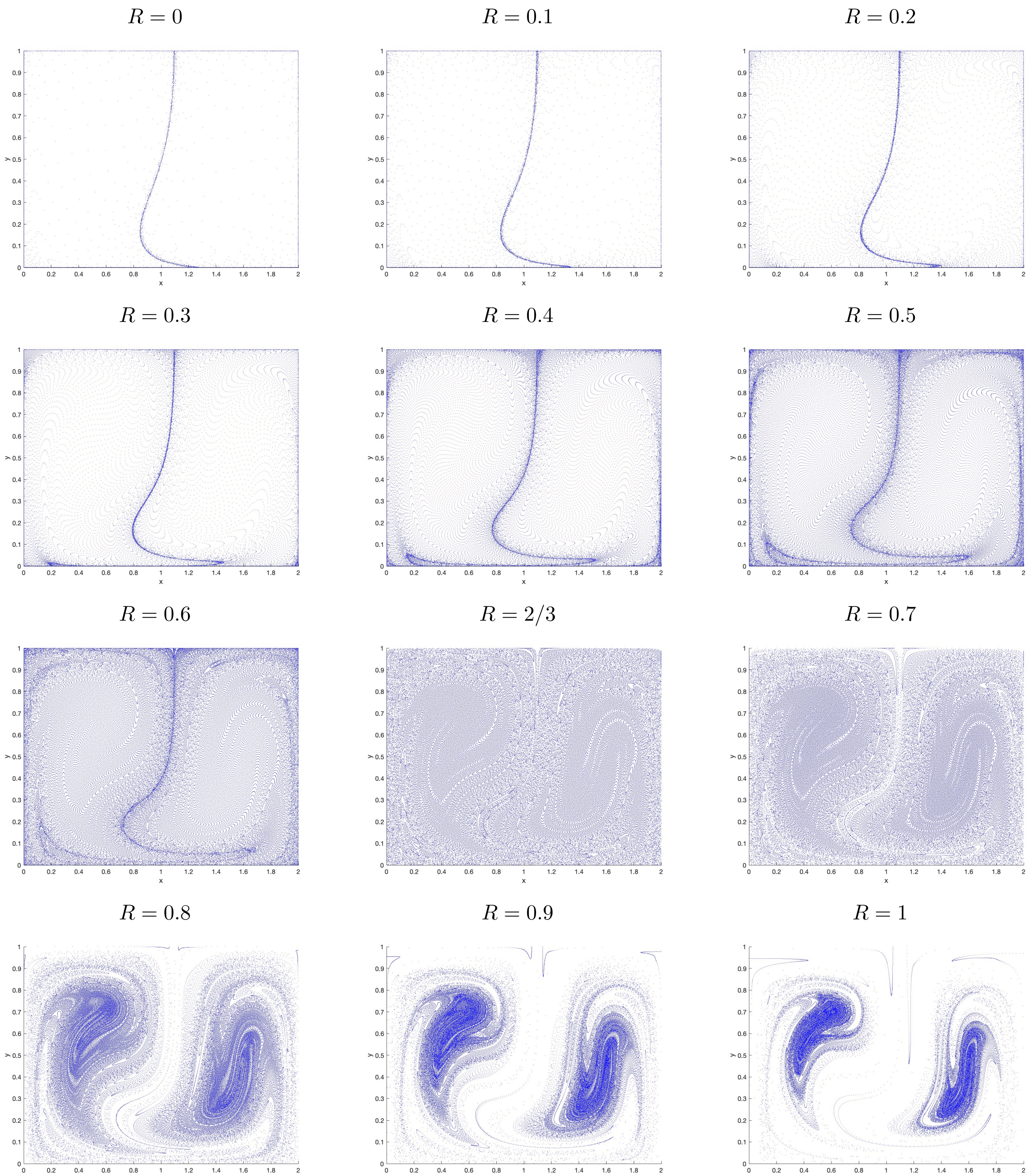

Figure 2. Preferential aggregation for inertial particles at $t = 15$ for $St = 0.5$ and varying density ratios. A uniform grid of 500×250 inertial particles were initialized at time $t = 0$ throughout the domain of the double-gyre flow. The parameter values for the double-gyre flow used in the computation are $A = 0.1$, $\omega = 6\pi/10$, $\epsilon = 0.25$.

References

[1] E. Forgoston, L. Billings, P. Yecko, and I. B. Schwartz, "Set-based corral control in stochastic dynamical systems: Making almost invariant sets more invariant," *Chaos: An Interdisciplinary Journal of Nonlinear Science*, vol. 21, no. 1, p. 013116, 2011.

[2] M. R. Maxey and J. J. Riley, "Equation of motion for a small rigid sphere in a nonuniform flow," *The Physics of Fluids*, vol. 26, no. 4, pp. 883–889, 1983.

[3] M. Sudharsan, S. L. Brunton, and J. J. Riley, "Lagrangian coherent structures and inertial particle dynamics," *Physical Review E*, vol. 93, no. 3, p. 033108, 2016.