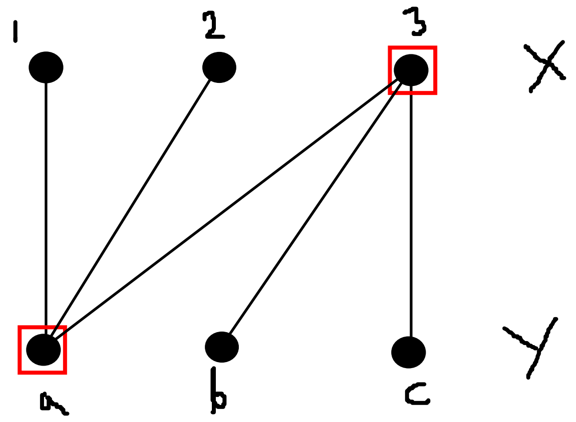
Vertex Cover and Independent Set

Vertex Cover

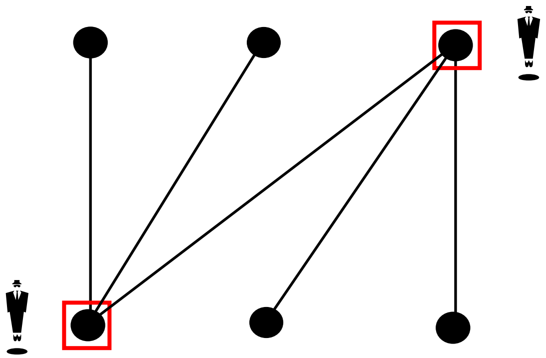
- A *vertex cover* of a graph G is a set $Q \subseteq V(G)$ that contains at least one endpoint of every edge. The vertices in Q cover $E(G)$.



Minimum vertex cover Q={3, a}
M={(1,a),(3,c)}

Vertex Cover

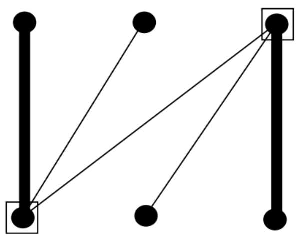
- In a graph that represents a road network (with straight roads and no isolated vertices).
  - Finding a minimum vertex cover = Placing the minimum number of policemen to guard the entire road network.

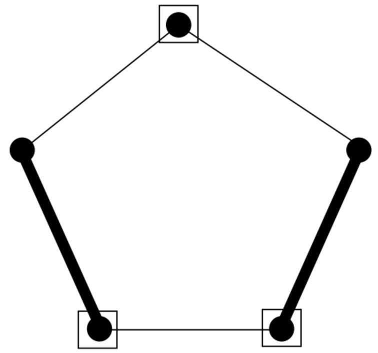


Matchings and Vertex covers

- In the graph G1,
  - We mark a vertex cover of size 2 and show a matching of size 2 in bold.
  - |vertex cover| = | matching|
- As illustrated on the G2, the optimal values differ by 1 for an odd cycle. The difference can be arbitrarily large.
  - |vertex cover| $\geq$ | matching|

G1





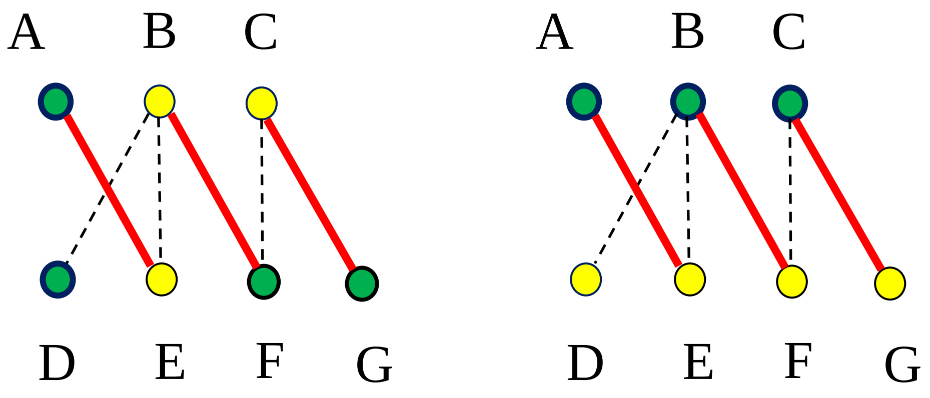
G2

Konig's Theorem: If G is a bipartite graph, then the maximum size of a matching in G equals the minimum size of a vertex cover of G.

Proof : Let G be an X, Y-bigraph.

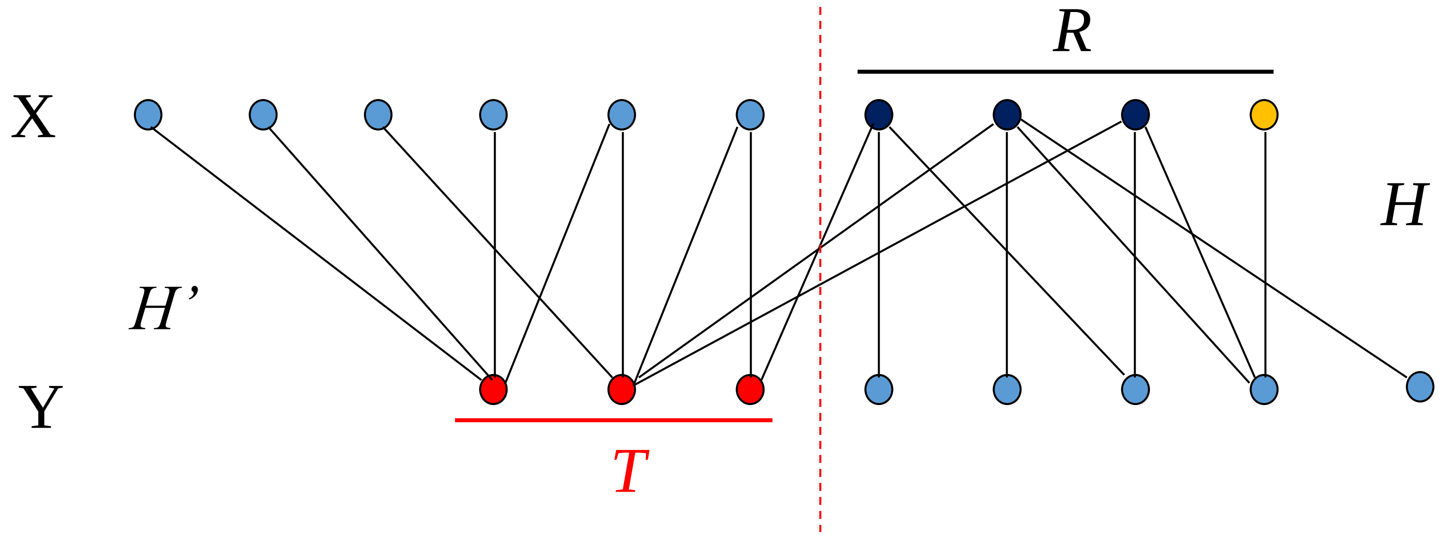
- Since distinct vertices must be used to cover the edges of a matching, $|Q| \geq |M|$ whenever Q is a vertex cover and M is a matching in G.
- Given a smallest vertex cover Q of G, we construct a matching of size $|Q|$ to prove that equality can always be achieved.

Green: Vertex cover
Red: Matching
$|Q| \geq |M|$



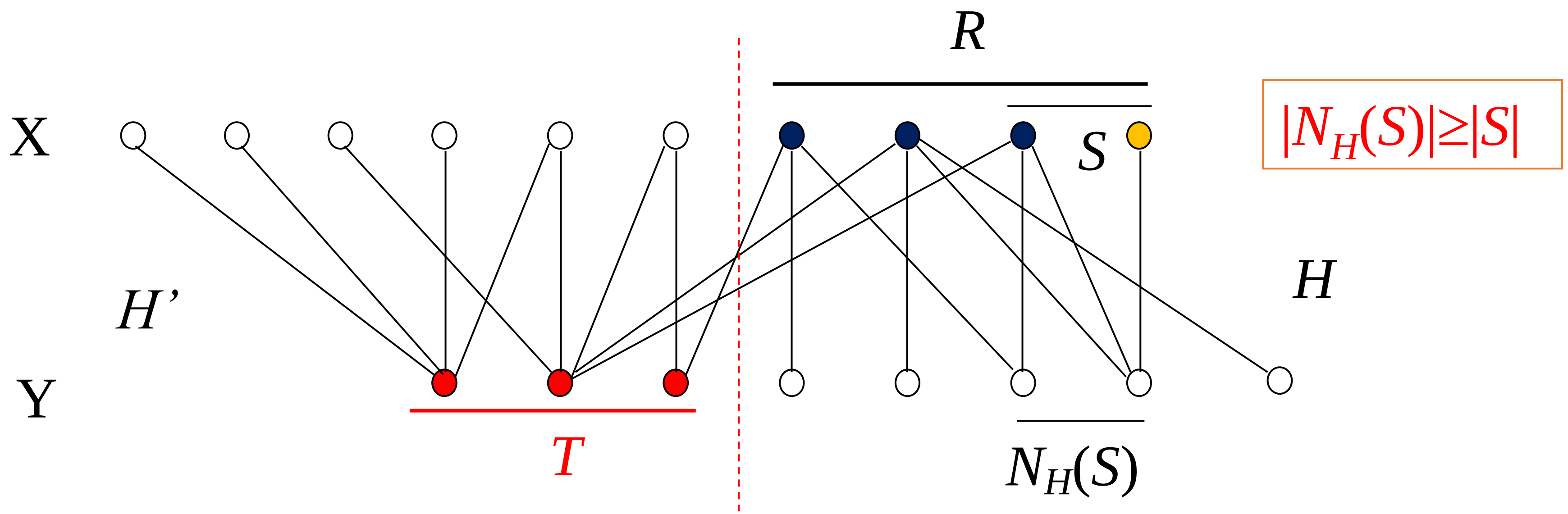
Theorem Continue

- Partition Q by letting $R=Q\cap X$ and $T=Q\cap Y$.
  - Let H and H' be the subgraphs of G induced by $R\cup(Y\text{-}T)$ and $T\cup(X\text{-}R)$, respectively.
  - We use Hall's Theorem to show that H has a matching that saturates R into $Y\text{-}T$ and H' has a matching that saturated T.
  - Since H and H' are disjoint, the two matchings together form a matching of size $|Q|$ in G.



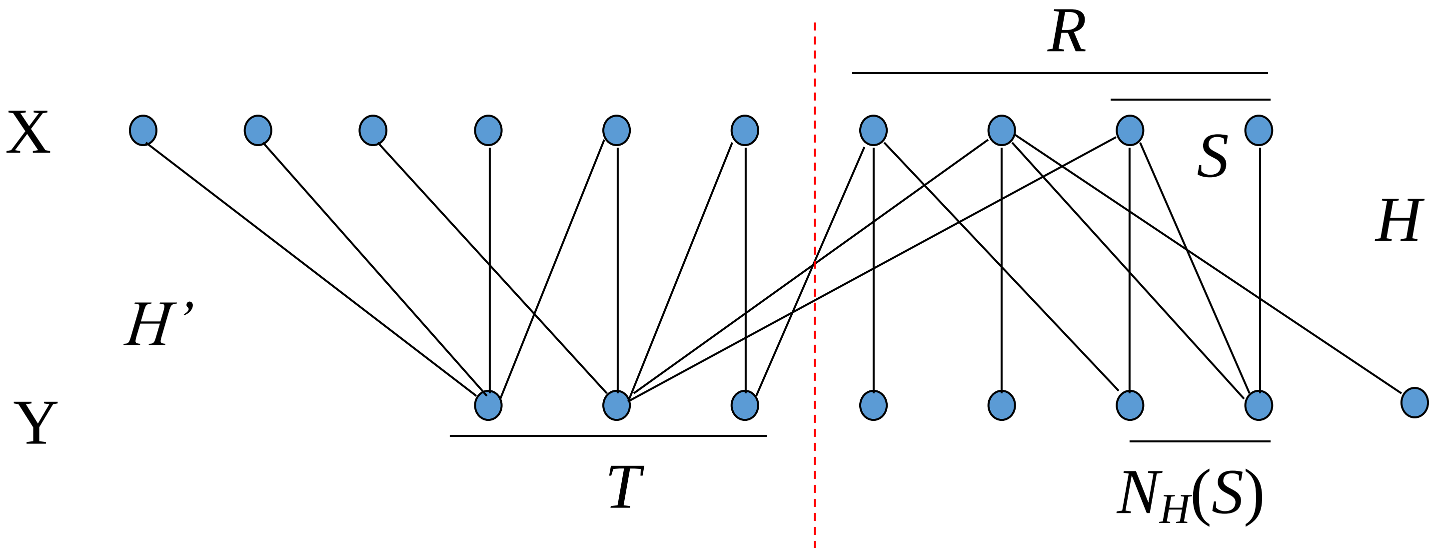
Theorem Continue

- Since $R \cup T$ is a vertex cover, G has no edge from Y-T to X-R.
  - Otherwise, an edge between Y-T to X-R is not covered
- For each $S \subseteq R$, we consider $N_H(S)$, which is contained in Y-T. If $|N_H(S)|<|S|$, then we can substitute $N_H(S)$ for S in Q to obtain a smaller vertex cover, since $N_H(S)$ cover all edges incident to S that are not covered by T.



Theorem continue

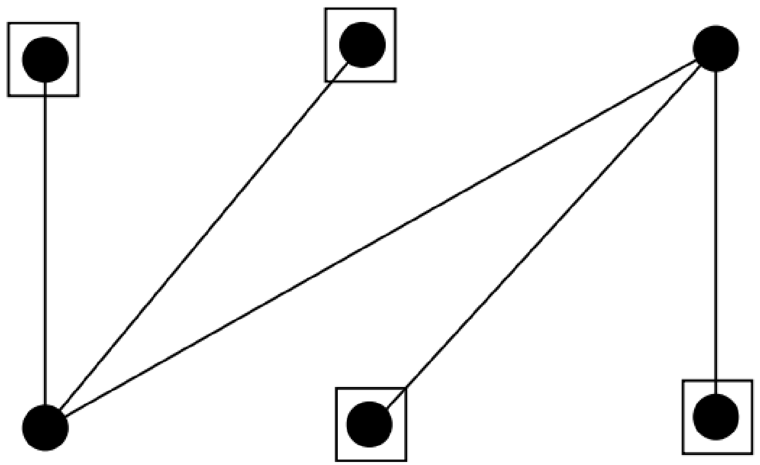
- The minimality of *Q* thus yields Hall's Condition in *H*, and hence *H* has a matching that saturates *R*. Applying the same argument to *H'* yields the matching that saturates *T*.



A matching exists in *H*' of size $|T|$ A matching exists in *H* of size $|R|$

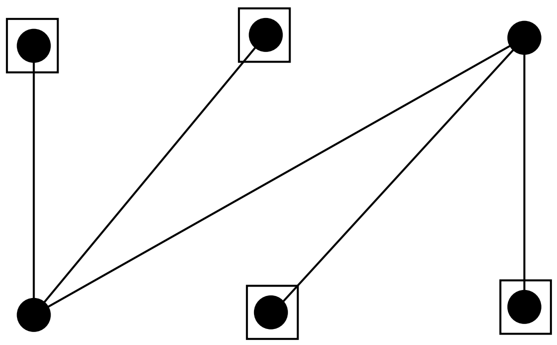
Independent set

- An independent set $S \subseteq V(G)$ such that no two vertices in S are adjacent(i.e. no two vertices in S are connected by an edge).



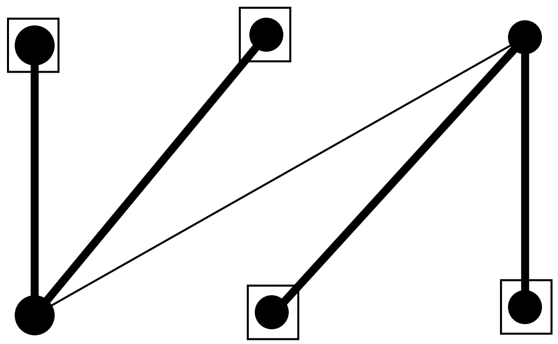
Independent sets and covers

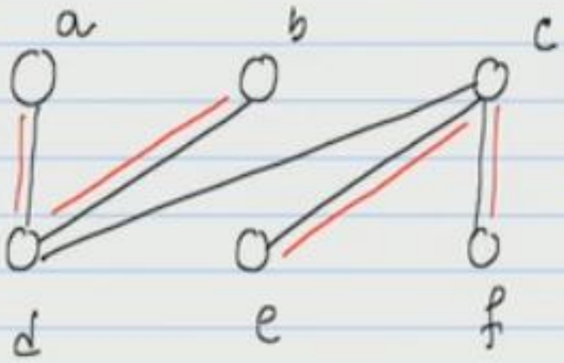
- The independence number of a graph is the maximum size of an independent set of vertices.
- The independence number of a bipartite graph does *not* always equal the size of a partite set.
  - In the graph bellow, both partite sets have size 3, but we have marked an independent set of size 4.



Edge cover

- An ***edge cover*** of *G* is a set *L* of edges such that every vertex of *G* is incident to some edge of *L*.
  - The four bold edges in the following graph form an edge cover.
  - Only graphs without isolated vertices have edge cove

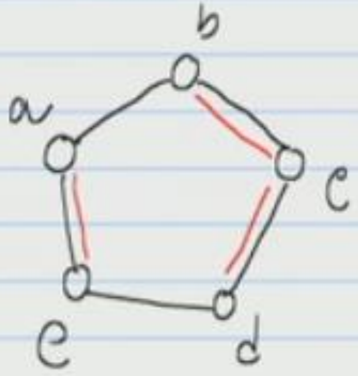




$L = \{ad, bd, ec, fc\}$

min edge cover

$M = \{ad, cf\}$



$L = \{ae, bc, dc\}$

$M = \{ab, ed\}$

Definitions

- For the optimal sizes of the sets of the independence and covering problems we have defined, we use the notation below.
  - Maximum size of independent set $\alpha(G)$
  - Maximum size of matching $\alpha'(G)$
  - Minimum size of vertex cover $\beta(G)$
  - Minimum size of edge cover $\beta'(G)$

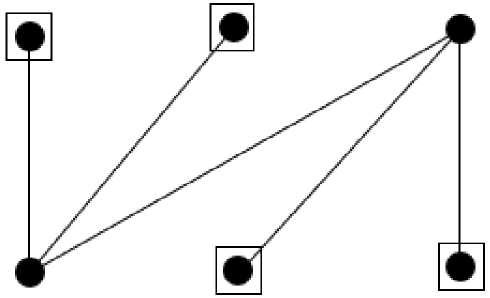
Theorem: In a graph G, $S \subseteq V(G)$ is an independent set if and only if $\bar{S}$ is a vertex cover, and hence

$\alpha(G) + \beta(G) = n(G)$.

Proof: ==> If S is a maximum independent set, then every edge is incident to at most one vertex of S. This implies every edge is incident to at least one vertex of $\bar{S}$. So $\bar{S}$ is cover all edges and $\bar{S}$ is a vertex cover.

- <== if $\bar{S}$ is am minimum vertex cover; $\bar{S}$ covers all the edges, then there are no edges joining vertices of S . So S is an independent Set.

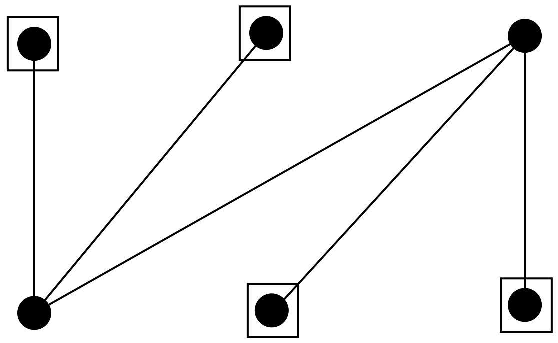
Independent set



Theorem: In a graph G, $S \subseteq V(G)$ is an independent set if and only if $\bar{S}$ is a vertex cover, and hence $\alpha(G) + \beta(G) = n(G)$.

Proof : continued

- Hence every maximum independent set is the complement of a minimum vertex cover, and $\alpha(G)+\beta(G)=n(G)$



$\alpha(G) = 4$

$\beta(G) = 2$

$n(G) = 6$

Theorem: If G is a graph without isolated vertices, then $\alpha'(G)+ \beta'(G) =n(G)$.

$\alpha'(G)$: Maximum size of matching $\beta'(G)$: Minimum size of edge cover

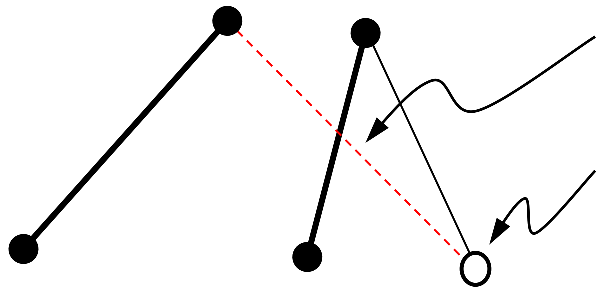
Proof:

- From a maximum matching M, we will construct an edge cover of size $n(G)-|M|$. (see next page)
  - Since a smallest edge cover is no bigger than this cover, this will imply that $\beta'(G)\leq n(G)-\alpha'(G)$.
- Also, from a minimum edge cover L, we will construct a matching of size $n(G)-|L|$. (see next page)
  - Since a largest matching is no smaller than this matching, this will imply that $\alpha'(G)\geq n(G)-\beta'(G)$.
- These two inequalities complete the proof. (detail)➔

Theorem Continue

- Given M, we construct an edge cover of size $n(G)$-$|M|$.
  - Add one edge incident to each unsaturated vertex to M.
  - We have used one edge for each vertex, except that each edge of M takes care of two vertices,
  - So the total size of this edge cover is $n(G)$-$|M|$, as desired.

    ➔ $\beta'(G) \le n(G) - \alpha'(G) \Rightarrow \beta'(G) + \alpha'(G) \le n(G)$.



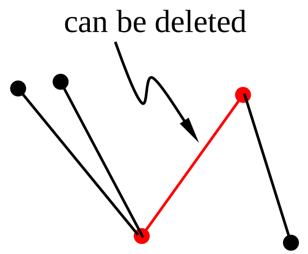
Select one edge for each unsaturated vertex

$|(\text{unsaturated Vertices})| = n(G)\text{-}2|M|$

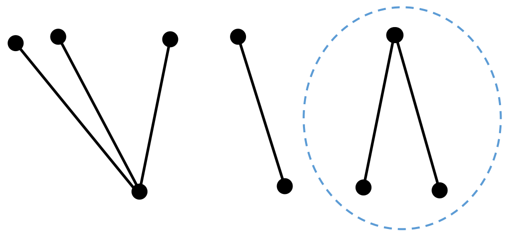
Theorem Continue

- Given a minimum edge cover *L*, we construct a matching of size $n(G)-|L|$.
  - If both ends of an edge *e* belong to edges in *L* other than *e*, then $e \notin L$, since $L-\{e\}$ is also an edge cover.
  - Hence each component formed by edges of *L* has at most one vertex of degree exceeding 1 and is a star (a tree with at most one non-leaf).

Note: Theorem: In a forest with v vertices and k components, the number of edges are v-k



Impossible case

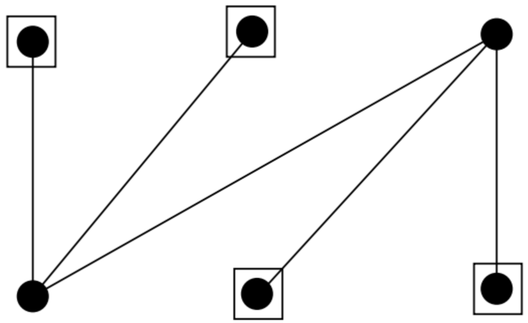


One component

Theorem Continue

- Let k be the number of these components.
- Since L has one edge for each non-central vertex in each star, we have $|L|=n(G)-k$.
- We form a matching M of size $k=n(G)-|L|$ by choosing one edge from each star in L. $\Rightarrow \alpha'(G)\geq n(G)-\beta'(G)$.

Ex.



(König [1916] If G is a bipartite graph with no isolated vertices then $\alpha(G) = \beta'(G)$.

- Proof ?