Assignment 4

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Outline

Problem

Using the central limit theorem. show that for large n:

$$\frac{c^n}{(n-1)!}x^{n-1}e^{-cx} \approx \frac{c}{\sqrt{2\pi n}}e^{-(cx-n)^2/2n} \tag{1}$$

Solution

Erlang Density Function

For $N = 0, 1, 2, 3, \ldots$ and constant $a > 0$

$$f_X(x) = \frac{a^N}{(N-1)!} a^{N-1} e^{-ax} \tag{2}$$

$$\bar{X} = \frac{N}{a} \qquad \sigma^2 = \frac{N}{a^2} \tag{3}$$

$$\Phi_X(\omega) = \left(\frac{a}{a - j\omega}\right)^N \tag{4}$$

Solution

Characteristics function and Normality

The characteristic function of a random vector is by definition the function:

$$\Phi(\Omega) = E(e^{j\Omega \mathbf{X}'}) = E(e^{j(\omega_1 x_1 + \omega_2 x_2 + \ldots + \omega_n x_n)}) = \Phi(j\Omega) \tag{5}$$

where,

$$\mathbf{X} = [x_1, \ldots, x_n] \quad \Omega = [\omega_1, \ldots, \omega_n] \tag{6}$$

Let $z = x_1 + x_2 + \ldots + x_n$ be a random variable
Now, If the random variables x_i are independent with respective densities $f_i(x_i)$, then

$$E(e^{j(\omega_1 x_1 + \omega_2 x_2 + \ldots + \omega_n x_n)}) = E(e^{j\omega_1 x_1}) \ldots E(e^{j\omega_n x_n}) \tag{7}$$

Solution

Hence,

$$\Phi_z(\omega) = E(e^{j(\omega_1 x_1 + \omega_2 x_2 + \ldots + \omega_n x_n)}) = \Phi_1(\omega) \ldots \Phi_n(\omega) \tag{8}$$

where $\Phi_i(\omega)$ is the chacteristic function of x_i
On Applying the convolution theorem for Fourier transform, we obtain

$$f_z(z) = f_1(z) * \ldots * f_n(z) \tag{9}$$

Central Limit Theorem

Given n independent random variable x_i, we form their sum

$$x = x_1 + \ldots + x_n \tag{10}$$

This is a random variable with mean $\eta = \eta_1 + \ldots + \eta_n$ and variance $\sigma^2 = \sigma_1^2 + \ldots + \sigma_n^2$

Solution

The Central Limit Theorem (CLT) states that under certain general conditions, the distribution $F(x)$ of x approaches a normal distribution with the same mean and variance as n increases.

$$F(x) \simeq G\left(\frac{x-\eta}{\sigma}\right) \tag{11}$$

Futhermore, if the random variables x_i are of continous type, the density $f(x)$ of x approaches a normal density.

$$f(x) \simeq \frac{1}{\sigma\sqrt{2\pi}} e^{-(x-\eta)^2/2\sigma^2} \tag{12}$$

Soltuion

Let $x_1, \ldots, x_n$ be some i.i.d random variables with exponential density function $\lambda e^{-\lambda x}$ and characteristic function $\Phi_{x_i}(\omega)$. Consider a random variable $X = x_1 + \ldots + x_n$.

$$\Phi_{x_i}(\omega) = \frac{a}{a - j\omega} \tag{13}$$

Since from equation (9), we have,

$$\Phi_X(\omega) = \Phi_{x_1}(\omega) \ldots \Phi_{x_n}(\omega) = \left(\frac{a}{a - j\omega}\right)^n \tag{14}$$

On applying Convolution Theorem, we obtain

$$f_X(x) = \frac{a^n}{(n-1)!} a^{n-1} e^{-ax} \quad \textbf{(Erlang Density Function)} \tag{15}$$

Solution

On Applying Central Limit Theorem on random variable X (For some Large Value of n). we get

$$f_X(x) = \frac{1}{\sqrt{2\pi\sigma^2}} e^{-(x-\eta)^2/2\sigma^2} \tag{16}$$

So On Comparing equation (16) and (17), we conclude:
Erlang Density Function approaches Normal Density Curve with same mean and same variance at large value of n.

$$\frac{c^n}{(n-1)!} c^{n-1} e^{-cx} \approx \frac{1}{\sqrt{2\pi\frac{n}{c^2}}} e^{(x-\frac{n}{c})^2/2\frac{n}{c^2}} = \frac{c}{\sqrt{2\pi n}} e^{-(cx-n)^2/2n} \tag{17}$$

Hence Proved.