Random Numbers

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Contents

1 Uniform Random Numbers

Let U be a uniform random variable between 0 and 1.

1.1 Generate 10^6 samples of U using a C program and save into a file called uni.dat .
Solution: Download the following files and execute the C program.

```
wget https://github.com/Ritvik−Sai−C/
    Random_numbers/blob/main/1.1/exrand.c
wget https://github.com/Ritvik−Sai−C/
    Random_numbers/blob/main/1.1/coeffs.h
```

Use the below command in the terminal to run the code

```
gcc exrand.c −lm
./a.out
```

1.2 Load the uni.dat file into python and plot the empirical CDF of U using the samples in uni.dat. The CDF is defined as

$$F_U(x) = \Pr(U \leq x) \tag{1.1}$$

Solution: The graph 1.2 is obtained by running the below code

```
https://github.com/Ritvik−Sai−C/
    Random_numbers/blob/main/1.2/uni_cdf.
    py
```

Run the following command in the terminal to run the code.

```
python3 uni_cdf.py
```

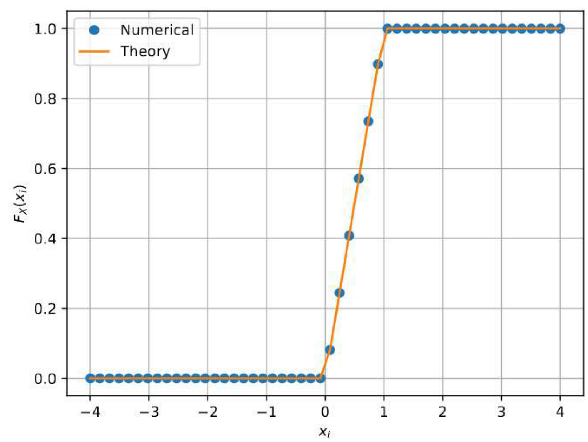

Fig. 1.2: The CDF of U

1.3 Find a theoretical expression for $F_U(x)$. **Solution:** Since U is an uniform random variable distribution, $P_U(x_i) = P_U(x_j) = k, \forall i, j$ CDF of $P_U(x)$=$F_U(x)$

$$= \int P_U(x)dx \tag{1.2}$$

$$= \int kdx \tag{1.3}$$

$$\text{wkt} \int_0^1 kdx = 1 \tag{1.4}$$

$$\therefore k = 1 \tag{1.5}$$

$$\therefore F_U(x) = x \tag{1.6}$$

1.4 The mean of U is defined as

$$E[U] = \frac{1}{N}\sum_{i=1}^{N} U_i \tag{1.7}$$

and its variance as

$$\text{var}[U] = E[U - E[U]]^2 \tag{1.8}$$

Write a C program to find the mean and variance of U.
Solution:

```
wget https://github.com/Ritvik-Sai-C/
    Random_numbers/blob/main/1.4/
    mean_var.c
```

Use below command to run file,

```
gcc mean_var.c -lm
./a.out
```

running the code gives us Mean =0.500137 , Variance =0.083251

1.5 Verify your result theoretically given that

$$E\left[U^k\right] = \int_{-\infty}^{\infty} x^k dF_U(x) \tag{1.9}$$

$$dF_U(x) = dx \tag{1.10}$$

$$\therefore E[U^k] = \int_{-\infty}^{\infty} x^k dx \tag{1.11}$$

$$E[U] = \int_0^1 x dx = \frac{1}{2} \tag{1.12}$$

$$E[U^2] = \int_0^1 x^2 dx = \frac{1}{3} \tag{1.13}$$

$$\because P_X(x) = 0, \forall x \in (1,\infty) \cap (-\infty, 0) \tag{1.14}$$

$$Var(X) = E[U^2] - (E[U])^2 = \frac{1}{3} - \frac{1}{4} = \frac{1}{12} \tag{1.15}$$

2 Central Limit Theorem

2.1 Generate 10^6 samples of the random variable

$$X = \sum_{i=1}^{12} U_i - 6 \tag{2.1}$$

using a C program, where $U_i, i = 1, 2, \ldots, 12$ are a set of independent uniform random variables between 0 and 1 and save in a file called gau.dat
Solution:

```
wget https://github.com/Ritvik-Sai-C/
    Random_numbers/blob/main/1.1/exrand.c
wget https://github.com/Ritvik-Sai-C/
    Random_numbers/blob/main/1.1/coeffs.h
```

Running the above codes generates uni.dat and gau.dat file. Use the command

```
gcc exrand.c -lm
.\a.out
```

2.2 Load gau.dat in python and plot the empirical CDF of X using the samples in gau.dat. What properties does a CDF have?
Solution: The CDF of X is plotted in 2.2,Properties of the CDF:

- $\Phi(x) = P(Z \le x) = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{x} \exp\left\{-\frac{u^2}{2}\right\} du$
- $\lim_{x\to\infty} \Phi(x) = 1, \quad \lim_{x\to-\infty} \Phi(x) = 0$
- $\Phi(0) = \frac{1}{2}$
- $\Phi(-x) = 1 - \Phi(x)$

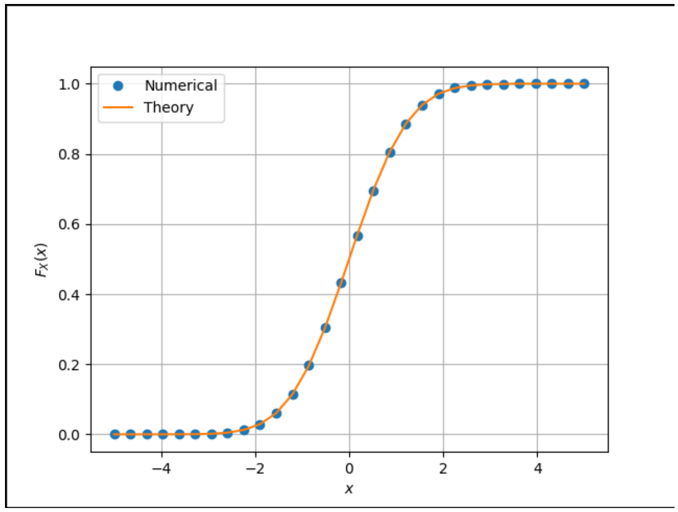

Fig. 2.2: The CDF of X

2.3 Load gau.dat in python and plot the empirical PDF of X using the samples in gau.dat. The PDF of X is defined as

$$p_X(x) = \frac{d}{dx} F_X(x) \tag{2.2}$$

What properties does the PDF have?

Solution: The PDF of X is plotted in 2.3 using the code below

```
https://github.com/Ritvik-Sai-C/
    Random_numbers/blob/main/2.3/
    gauss_pdf.py
```

Use the below command to run the code:

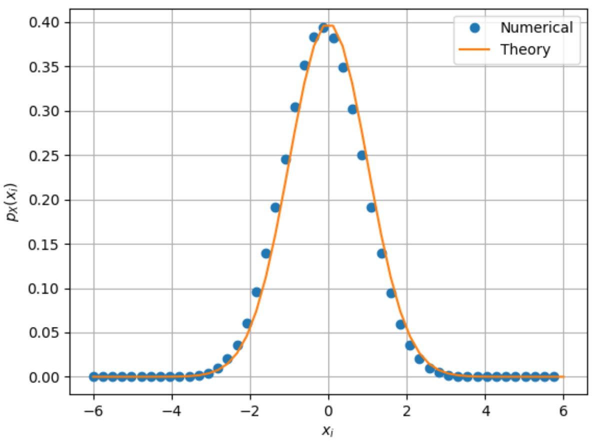

Fig. 2.3: The PDF of X

```
python3 pdf.py
```

Properties of PDF:

- PDF is symmetric about $x = 0$
- graph is bell shaped
- mean of graph is situated at the apex point of the bell

2.4 Find the mean and variance of X by writing a C program.
Solution: Running the below code gives Mean = -0.000417 Variance= 0.999902

```
https://github.com/Ritvik−Sai−C/
    Random_numbers/blob/main/2.4/
    mean_var(gau).c
```

Command used:

```
gcc mean_var(gau).c −lm
./a.out
```

2.5 Given that

$$p_X(x) = \frac{1}{\sqrt{2\pi}} \exp\left(-\frac{x^2}{2}\right), -\infty < x < \infty, \quad (2.3)$$

repeat the above exercise theoretically.

Given ,$p_X(x) = \frac{1}{\sqrt{2\pi}} e^{\frac{-x^2}{2}}$

$$E[x] = \int_{-\infty}^{\infty} x p_X(x) dx \quad (2.4)$$

$$= \int_{-\infty}^{\infty} \frac{1}{\sqrt{2\pi}} x e^{-\frac{-x^2}{2}} \quad (2.5)$$

$$\because x e^{-\frac{-x^2}{2}} \text{is a odd function,} \quad (2.6)$$

$$E[x] = 0$$

$$E[x^2] = \int_{-\infty}^{\infty} x^2 p_X(x) dx \quad (2.7)$$

$$= \int_{-\infty}^{\infty} \frac{1}{\sqrt{2\pi}} x(x e^{-\frac{-x^2}{2}}) dx \quad (2.8)$$

Using integration by parts:

$$= x \int x e^{-\frac{-x^2}{2}} dx - \int \frac{d(x)}{dx} \int x e^{-\frac{-x^2}{2}} dx \quad (2.9)$$

$$I = \int x e^{-\frac{-x^2}{2}} \quad (2.10)$$

$$\text{Let} \frac{x^2}{2} = t \quad (2.11)$$

$$\implies x dx = dt \quad (2.12)$$

$$\implies = \int e^{-t} dt = -e^{-t} + c \quad (2.13)$$

$$\therefore \int x e^{-\frac{-x^2}{2}} = -e^{-\frac{-x^2}{2}} + c \quad (2.14)$$

Using (2.14) in (2.9)

$$= -x e^{-\frac{-x^2}{2}} + \int e^{-\frac{-x^2}{2}} dx \quad (2.15)$$

$$\text{Also,} \int_{-\infty}^{\infty} e^{-\frac{-x^2}{2}} dx = \sqrt{2\pi} \quad (2.16)$$

$$\therefore \text{substituting limits we get,} E[x^2] = 1 \quad (2.17)$$

$$Var(X) = E[x^2] - (E[x])^2 = 1 - 0 \quad (2.18)$$

3 From Uniform to Other

3.1 Generate samples of

$$V = -2 \ln (1 - U) \quad (3.1)$$

and plot its CDF.

Solution: Running the below code generates samples of V from file uni.dat(U).

```
https://github.com/Ritvik−Sai−C/
    Random_numbers/blob/main/3.1/V.py
```

Use the below command in the terminal to run the code:

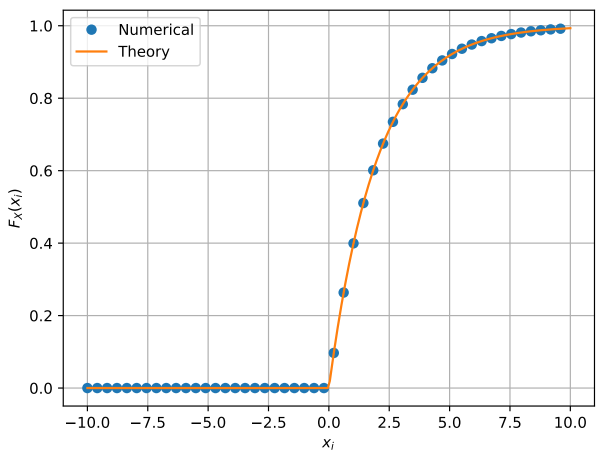

Fig. 3.1: CDF for (3)

```
python3 V.py
```

Now these samples are used to plot (3.1) by running the below code,

```
https://github.com/Ritvik−Sai−C/
    Random_numbers/blob/main/3.1/V_cdf.
    py
```

Use the below command to run the code:

```
python3 V_cdf.py
```

3.2 Find a theoretical expression for $F_V(x)$.

$$F_V(x) = P(V \leq x) \tag{3.2}$$

$$= P(-2ln(1-U) \leq x) \tag{3.3}$$

$$= P(1 - e^{\frac{-x}{2}} \geq U) \tag{3.4}$$

$$P(U < x) = \int_0^x dx = x \tag{3.5}$$

$$\therefore P(1 - e^{\frac{-x}{2}} \geq U) = 1 - e^{\frac{-x}{2}}, \forall x \geq 0 \tag{3.6}$$