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1 Uniform Random Numbers

Let U be a uniform random variable between 0 and 1.

1.1 Generate 10^6 samples of U using a C program and save into a file called uni.dat .

Solution: Download the following files and execute the C program.

```
wget https://github.com/Ritvik-Sai-C/Random_numbers/blob/main/1.1/exrand.c
wget https://github.com/Ritvik-Sai-C/Random_numbers/blob/main/1.1/coeffs.h
```

Use the below command in the terminal to run the code

```
gcc exrand.c -lm
./a.out
```

1.2 Load the uni.dat file into python and plot the empirical CDF of U using the samples in uni.dat. The CDF is defined as

$$F_U(x) = \Pr(U \le x) \tag{1.1}$$

Solution: The graph 0 is obtained by running the below code

```
https://github.com/Ritvik-Sai-C/Random_numbers/blob/main/1.2/uni_cdf.py
```

Run the following command in the terminal to run the code

```
python3 uni_cdf.py
```

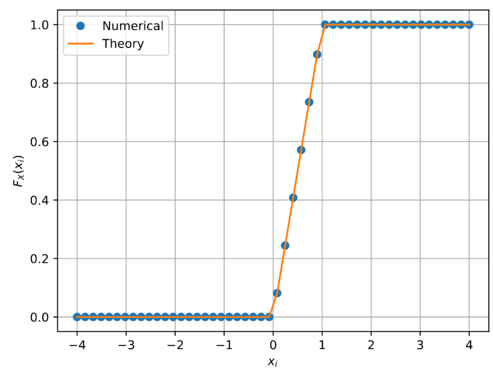

Fig. 0: The CDF of U

1.3 Find a theoretical expression for $F_U(x)$. **Solution:** Since U is an uniform random variable distribution, $P_U(x_i) = P_U(x_j) = k, \forall i, j$ CDF of $P_U(x)$=$F_U(x)$

$$= \int P_U(x)dx \tag{1.2}$$

$$= \int kdx \tag{1.3}$$

$$\text{wkt} \int_0^1 kdx = 1 \tag{1.4}$$

$$\therefore k = 1 \tag{1.5}$$

$$\therefore F_U(x) = x \tag{1.6}$$

1.4 The mean of U is defined as

$$E[U] = \frac{1}{N}\sum_{i=1}^{N} U_i \tag{1.7}$$

and its variance as

$$\text{var}[U] = E[U - E[U]]^2 \tag{1.8}$$

Write a C program to find the mean and variance of U.
Solution:

```
wget https://github.com/
    Ritvik-Sai-C/Random_
    numbers/blob/main/1.4/
    mean_var.c
```

Use below command to run file,

```
gcc mean_var.c -lm
./a.out
```

running the code gives us Mean =0.500007 , Variance =0.083301

1.5 Verify your result theoretically given that

$$E\left[U^k\right] = \int_{-\infty}^{\infty} x^k dF_U(x) \tag{1.9}$$

$$dF_U(x) = dx \tag{1.10}$$

$$\therefore E[U^k] = \int_{-\infty}^{\infty} x^k dx \tag{1.11}$$

$$E[U] = \int_0^1 x dx = \frac{1}{2} \tag{1.12}$$

$$E[U^2] = \int_0^1 x^2 dx = \frac{1}{3} \tag{1.13}$$

$$\because P_X(x) = 0, \forall x \in (1,\infty) \cap (-\infty, 0) \tag{1.14}$$

$$Var(X) = E[U^2] - (E[U])^2 = \frac{1}{3} - \frac{1}{4} = \frac{1}{12} \tag{1.15}$$

2 Central Limit Theorem

2.1 Generate 10^6 samples of the random variable

$$X = \sum_{i=1}^{12} U_i - 6 \tag{2.1}$$

using a C program, where $U_i, i = 1, 2, \ldots, 12$ are a set of independent uniform random variables between 0 and 1 and save in a file called gau.dat
Solution:

```
wget https://github.com/
    Ritvik-Sai-C/Random_
    numbers/blob/main/1.1/
    exrand.c
wget https://github.com/
    Ritvik-Sai-C/Random_
    numbers/blob/main/1.1/
    coeffs.h
```

Running the above codes generates uni.dat and gau.dat file. Use the command

```
gcc exrand.c -lm
.\a.out
```

2.2 Load gau.dat in python and plot the empirical CDF of X using the samples in gau.dat. What properties does a CDF have?
Solution: The CDF of X is plotted in 0,Properties of the CDF:

- $\Phi(x) = P(Z \le x) = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{x} \exp\left\{-\frac{u^2}{2}\right\} du$
- $\lim_{x\to\infty} \Phi(x) = 1, \quad \lim_{x\to-\infty} \Phi(x) = 0$
- $\Phi(0) = \frac{1}{2}$
- $\Phi(-x) = 1 - \Phi(x)$

2.3 Load gau.dat in python and plot the empirical PDF of X using the samples

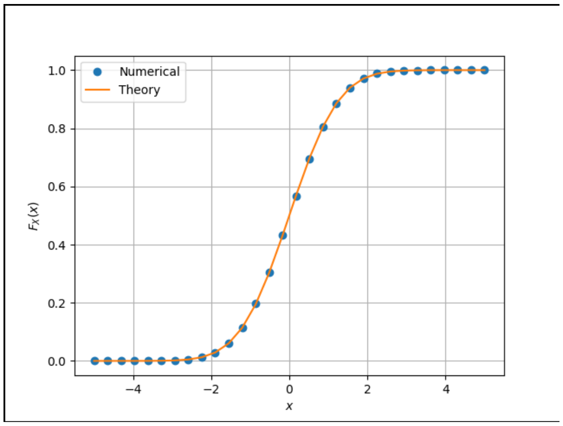

Fig. 0: The CDF of X

in gau.dat. The PDF of X is defined as

$$p_X(x) = \frac{d}{dx} F_X(x) \tag{2.2}$$

What properties does the PDF have?

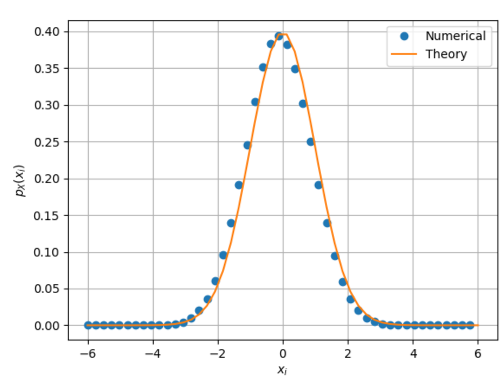

Fig. 0: The PDF of X

Solution: The PDF of X is plotted in 0 using the code below

```
https://github.com/Ritvik
    -Sai-C/Random_numbers/
    blob/main/2.3/gauss_
    pdf.py
```

Use the below command to run the code:

```
python3 pdf.py
```

Properties of PDF:

- PDF is symmetric about $x = 0$
- graph is bell shaped
- mean of graph is situated at the apex point of the bell

2.4 Find the mean and variance of X by writing a C program.
Solution: Running the below code gives Mean = 0.000294 Variance= 0.999560

```
https://github.com/Ritvik
    -Sai-C/Random_numbers/
    blob/main/2.4/mean_var
    (gau).c
```

Command used:

```
gcc mean_var(gau).c -lm
./a.out
```

2.5 Given that

$$p_X(x) = \frac{1}{\sqrt{2\pi}} \exp\left(-\frac{x^2}{2}\right), -\infty < x < \infty, \tag{2.3}$$

repeat the above exercise theoretically.

Given $,p_X(x) = \frac{1}{\sqrt{2\pi}} e^{\frac{-x^2}{2}}$

$$E[x] = \int_{-\infty}^{\infty} x p_X(x) dx \tag{2.4}$$

$$= \int_{-\infty}^{\infty} \frac{1}{\sqrt{2\pi}} x e^{-\frac{-x^2}{2}} \tag{2.5}$$

$$\because x e^{-\frac{-x^2}{2}} \text{is a odd function,} \tag{2.6}$$

$$E[x] = 0$$

$$E[x^2] = \int_{-\infty}^{\infty} x^2 p_X(x) dx \tag{2.7}$$

$$= \int_{-\infty}^{\infty} \frac{1}{\sqrt{2\pi}} x(x e^{-\frac{-x^2}{2}}) dx \tag{2.8}$$

Using integration by parts:

$$= x \int x e^{-\frac{-x^2}{2}} dx - \int \frac{d(x)}{dx} \int x e^{-\frac{-x^2}{2}} dx \tag{2.9}$$

$$I = \int x e^{-\frac{-x^2}{2}} \tag{2.10}$$

$$\text{Let} \frac{x^2}{2} = t \tag{2.11}$$

$$\implies x dx = dt \tag{2.12}$$

$$\implies = \int e^{-t} dt = -e^{-t} + c \tag{2.13}$$

$$\therefore \int x e^{-\frac{-x^2}{2}} = -e^{-\frac{-x^2}{2}} + c \tag{2.14}$$

Using (2.14) in (2.9)

$$= -x e^{-\frac{-x^2}{2}} + \int e^{-\frac{-x^2}{2}} dx \tag{2.15}$$

$$\text{Also,} \int_{-\infty}^{\infty} e^{-\frac{-x^2}{2}} dx = \sqrt{2\pi} \tag{2.16}$$

$$\therefore \text{substituting limits we get,} E[x^2] = 1 \tag{2.17}$$

$$Var(X) = E[x^2] - (E[x])^2 = 1 - 0 \tag{2.18}$$

3 From Uniform to Other

3.1 Generate samples of

$$V = -2\ln(1 - U) \tag{3.1}$$

and plot its CDF.

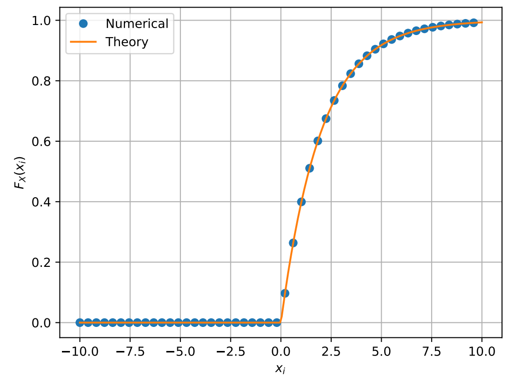

Fig. 0: CDF for (3)

Solution: Running the below code generates samples of V from file uni.dat(U).

```
https://github.com/Ritvik
    -Sai-C/Random_numbers/
    blob/main/3.1/V.py
```

Use the below command in the terminal to run the code:

```
python3 V.py
```

Now these samples are used to plot (0) by running the below code,

```
https://github.com/Ritvik
    -Sai-C/Random_numbers/
    blob/main/3.1/V_cdf.py
```

Use the below command to run the code:

```
python3 V_cdf.py
```

3.2 Find a theoretical expression for $F_V(x)$.

$$F_V(x) = P(V \le x) \quad (3.2)$$
$$= P(-2ln(1-U) \le x) \quad (3.3)$$
$$= P(1 - e^{\frac{-x}{2}} \ge U) \quad (3.4)$$
$$P(U < x) = \int_0^x dx = x \quad (3.5)$$
$$\therefore P(1 - e^{\frac{-x}{2}} \ge U) = 1 - e^{\frac{-x}{2}}, \forall x \ge 0 \quad (3.6)$$

4 Triangular Distribution

4.1 Generate

$$T = U_1 + U_2 \quad (4.1)$$

Solution: Run the below code to generate T.dat

```
https://github.com/Ritvik
    -Sai-C/Random_numbers/
    blob/main/4.1/T_gen_
    dat.c
```

Run the command below in the terminal

```
gcc T_gen_dat.c -lm
./a.out
```

4.2 Find the CDF of T.

$$F_T(t) = P(T < t) \quad (4.2)$$
$$= P(U_1 + U_2 < t) \quad (4.3)$$

we know that $0 \le U_1 \le 1$ and $0 \le U_2 \le 1$
$\therefore 0 \le U_1 + U_2 \le 2$, so
$\forall t > 2, P(U_1 + U_2 < t) = 1$
$\forall t < 0, P(U_1 + U_2 < t) = 0$
for $0 \le t \le 2$ let us split it into 2 cases, for $0 \le t \le 1$ and $1 < t \le 2$

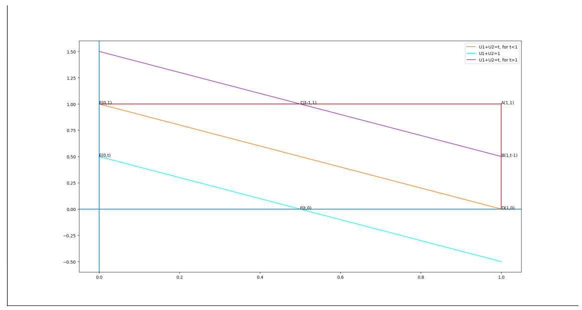

Fig. 0: Plot

The above figure is produced by the following code

```
https://github.com/Ritvik
    -Sai-C/Random_numbers/
    blob/main/4.2/T_plot.
    py
```

Run the following command in the terminal to run the code

```
python3 T_plot.py
```

From Fig (0)

$$P(U_1 + U_2 < t, 0 \le t \le 1) = \frac{\Delta(EOF)}{\Delta(AEOD)} \tag{4.4}$$

$$= \frac{t^2}{2} \tag{4.5}$$

$$P(U_1 + U_2 < t, 1 \le t \le 2) = \frac{\Delta(ABC)}{\Delta(AEOD)} \tag{4.6}$$

$$= 1 - \frac{(2-t)^2}{2} \tag{4.7}$$

$$\therefore F_T(t) = P(U_1 + U_2 < t) = \begin{cases} 0 & t < 0 \\ \frac{t^2}{2} & 0 \le t \le 1 \\ 1 - \frac{(2-t)^2}{2} & 1 < t \le 2 \\ 1 & \end{cases} \tag{4.8}$$

4.3 Find the PDF of T.
Solution:

$$P_T(t) = \frac{d(F_T(t))}{dt} \tag{4.9}$$

$$\therefore P_T(t) = \begin{cases} 0 & t < 0 \\ t & 0 \le t \le 1 \\ 2 - t & 0 < t \le 2 \\ 0 & t > 2 \end{cases} \tag{4.10}$$

4.4 Find the theoretical expressions for the PDF and CDF of T.
Solution:

$$P_T(t) = \begin{cases} 0 & t < 0 \\ t & 0 \le t \le 1 \\ 2 - t & 0 < t \le 2 \\ 0 & t > 2 \end{cases} \tag{4.11}$$

$$F_T(t) = \begin{cases} 0 & t < 0 \\ \frac{t^2}{2} & 0 \le t \le 1 \\ 1 - \frac{(2-t)^2}{2} & 1 < t \le 2 \\ 1 & t > 2 \end{cases} \tag{4.12}$$

4.5 Verify your results through a plot.
Solution: Run the below code to get

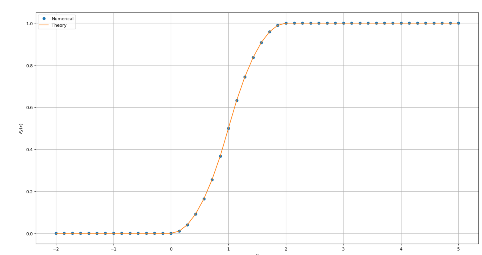

Fig. 0: CDF for (4)

the cdf

```
https://github.com/Ritvik
  -Sai-C/Random_numbers/
  blob/main/4.5/T_cdf.py
```

Use the following command in the terminal to run the code

```
python3 T_cdf.py
```

Run the below code to get the pdf

```
https://github.com/Ritvik
  -Sai-C/Random_numbers/
  blob/main/4.5/T_cdf.py
```

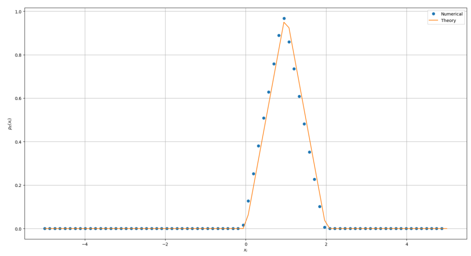

Fig. 0: PDF for (4)

Use the following command in the terminal to run the code

```
python3 T_pdf.py
```

5 Maximul Likelihood

5.1 Generate equiprobable $X \in \{1, -1\}$.
Solution: Run the below code,

```
https://github.com/Ritvik
    -Sai-C/Random_numbers/
    blob/main/5.1/
    bernoulli.c
```

Use the below command in the terminal to run the code

```
gcc bernoulli.c -lm
./a.out
```

5.2 Generate

$$Y = AX + N, \tag{5.1}$$

where $A = 5$ dB, and $N \sim \mathcal{N}(0, 1)$.
Solution: Run the below code for generating samples of Y,

```
https://github.com/Ritvik
    -Sai-C/Random_numbers/
    blob/main/5.2/Ygen.c
```

Use the below command in the terminal to run the code

```
gcc Ygen.c -lm
./a.out
```

5.3 Plot Y using a scatter plot.
Solution:

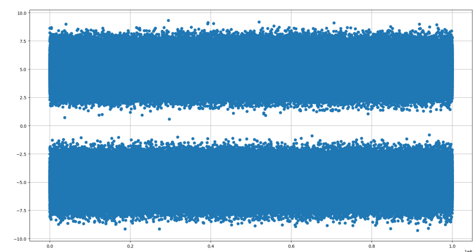

Fig. 0: plot for (5.3)

Run the following code to generate the scatter plot

```
https://github.com/Ritvik
    -Sai-C/Random_numbers/
    blob/main/5.3/Yplot.py
```

Use the below command to run the code,

```
python3 Yplot.py
```

5.4 Guess how to estimate X from Y.
Solution: if the received signal is greater than 0, then the receiver assumes s_1 was transmitted.
if the received signal is less than or equal to 0, then the receiver assumes s_0 was transmitted, where s_0 and s_1 are cases of $X = 1$and $X = -1$ respectively where threshold 0 is taken to be

the decision boundary.

$$y > 0 \implies s_1 \tag{5.2}$$
$$y \le 0 \implies s_0 \tag{5.3}$$

5.5 Find

$$P_{e|0} = \Pr\left(\hat{X} = -1 | X = 1\right) \tag{5.4}$$

and

$$P_{e|1} = \Pr\left(\hat{X} = 1 | X = -1\right) \tag{5.5}$$

Solution: Here s_1 and s_2 are equally probable ie, $p(s_1) = p(s_0) = \frac{1}{2}$

$$Q(x) = \frac{1}{\sqrt{2\pi}} \int_x^{\infty} e^{\frac{-x^2}{2}} dx \tag{5.6}$$

$$p(e|s_1) = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{0} e^{-\frac{(y-A)^2}{2}} dy$$
$$= Q(A) \tag{5.7}$$

$$p(e|s_0) = \frac{1}{\sqrt{2\pi}} \int_{0}^{\infty} e^{-\frac{(y+A)^2}{2}} dy$$
$$= Q(A) \tag{5.8}$$

5.6 Find P_e assuming that X has equiprobable symbols.
Solution: Total probability of bit error:

$$P_e = p(s_1)p(e|s_1) + p(s_0)p(e|s_0) \tag{5.9}$$

$$= \frac{1}{2}[Q(A) + Q(A)] \tag{5.10}$$

$$\because p(s_1) = p(s_0) = \frac{1}{2}, \text{X has equiprobable symbols} \tag{5.11}$$

$$= Q(A)$$

5.7 Verify by plotting the theoretical P_e with respect to A from 0 to 10 dB.

Solution:

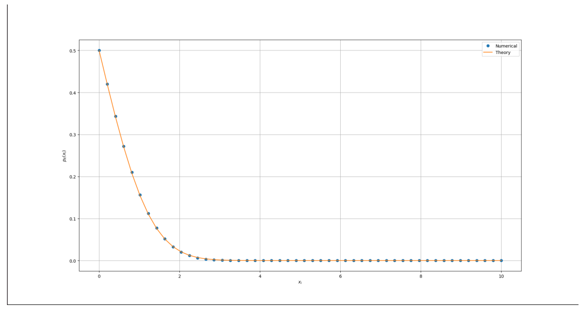

Fig. 0: plot for (5.7)

```
https://github.com/Ritvik
  -Sai-C/Random_numbers/
  blob/main/5.7/Pplot.py
```

Use the below command to run the code,

```
python3 Pplot.py
```

5.8 Now, consider a threshold δ while estimating X from Y. $= Q(5)$; $A = 5$ Find the value of δ that minimizes the theoretical P_e.
Solution: Threshold=δ,

$$y > \delta \implies s_1 \quad (5.12)$$
$$y \le \delta \implies s_0 \quad (5.13)$$
$$p(e|s_1) = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\delta} e^{-\frac{(y-A)^2}{2}} dy \quad (5.14)$$
$$p(e|s_0) = \frac{1}{\sqrt{2\pi}} \int_{\delta}^{\infty} e^{-\frac{(y+A)^2}{2}} dy$$
$$P_e = \frac{1}{2\sqrt{2\pi}} (\int_{-\infty}^{\delta} e^{-\frac{(y-A)^2}{2}} dy + \int_{\delta}^{\infty} e^{-\frac{(y+A)^2}{2}} dy)$$
$$P_e = \frac{Q(\delta + A) + Q(A - \delta)}{2} \quad (5.15)$$
$$P_e = f(\delta) \quad (5.16)$$
$$\text{to minimize} P_e, \frac{d(f(\delta))}{d\delta} = 0 \text{ and} f\text{''}(\delta) > 0 \quad (5.17)$$
$$e^{\frac{-(A-\delta)^2}{2}} - e^{\frac{-(A+\delta)^2}{2}} = 0 \quad (5.18)$$
$$\therefore A - \delta = A + \delta, \implies \delta = 0 \quad (5.19)$$
$$f\text{''}(\delta) = k((A - \delta)e^{\frac{-(A-\delta)^2}{2}} + (A + \delta)e^{\frac{-(A+\delta)^2}{2}}) > 0 \quad (5.20)$$

5.9 Repeat the above exercise when

$$p_X(0) = p \quad (5.21)$$

Solution: $p_X(0) = p$

$$\implies p_X(1) = 1 - p$$
$$P_e = pP(e|s_0) + (1 - p)P(e|s_1) \quad (5.22)$$
$$= pQ(A + \delta) + (1 - p)Q(A - \delta) \quad (5.23)$$
$$\frac{d(P_e)}{d(\delta)} = 0 \quad (5.24)$$
$$\implies e^{\frac{(A+\delta)^2-(A-\delta)^2}{2}} = \frac{p}{1 - p} \quad (5.25)$$
$$\therefore \delta = \frac{1}{2A} log(\frac{p}{1 - p}) \quad (5.26)$$
$$\frac{d(P_e)}{d(\delta)} \quad at \quad \delta + \epsilon > 0 \quad (5.27)$$
$$\frac{d(P_e)}{d(\delta)} \quad at \quad \delta - \epsilon < 0$$
$$\therefore \delta = \frac{1}{2A} log\left(\frac{p}{1 - p}\right) \longrightarrow minima$$
$$A = 5 \implies \delta = \frac{1}{10} log\left(\frac{p}{1 - p}\right) \quad (5.28)$$

5.10 Repeat the above exercise using the MAP criterion.

Solution:

$$P_{X|Y}\left(x|y\right)\Big|_{X=1} = \frac{P(Y=y|X=1)P(X=1)}{P(Y=y)} \tag{5.29}$$

$$\begin{aligned}P(Y=y) &= P(Y=y|X=1)P(X=1)\\ &+P(Y=y|X=-1)P(X=-1)\end{aligned} \tag{5.30}$$

$$P(Y=y|X=1)P(X=1) = pP(Y=A+N) \tag{5.31}$$

$$= p\left(\frac{1}{\sqrt{2\pi}}e^{\frac{-(y-A)^2}{2}}\right) \tag{5.32}$$

$$\therefore P_{X|Y}(x|y)\Big|_{X=1} = \frac{p\left(\frac{1}{\sqrt{2\pi}}e^{\frac{-(y-A)^2}{2}}\right)}{P(Y=y)} \tag{5.33}$$

$$P_{X|Y}(x|y)\Big|_{X=-1} = \frac{P(Y=y|X=-1)P(X=-1)}{P(Y=y)} \tag{5.34}$$

$$P(Y=y|X=-1)P(X=-1) = (1-p)P(Y=-A+N)$$

$$= (1-p)\left(\frac{1}{\sqrt{2\pi}}e^{\frac{-(y+A)^2}{2}}\right) \tag{5.35}$$

$$\therefore P_{X|Y}(x|y)\Big|_{X=-1} = \frac{(1-p)\left(\frac{1}{\sqrt{2\pi}}e^{\frac{-(y+A)^2}{2}}\right)}{P(Y=y)} \tag{5.36}$$

Now comparing $a = P_{X|Y}(x|y)\Big|_{X=-1}$ and $b = P_{X|Y}(x|y)\Big|_{X=1}$, if $a > b, X = -1$ is more likely ,$a < b, X = 1$ is more likely.

$pe^{\frac{-(y-A)^2}{2}} \gtrless (1-p)e^{\frac{-(y+A)^2}{2}}$

$\implies e^{2Ay} \gtrless \frac{1-p}{p}$

$\implies y \gtrless \frac{1}{2A}log\left(\frac{1-p}{p}\right)$

$\delta = \frac{1}{2A}log\left(\frac{1-p}{p}\right)$

$y > \delta \implies$ X=1 is more likely

$y < \delta \implies$ X=-1 is more likely