Scientific Computing Lab - End sem

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Abstract

Comparing the analytical and numerical solution for steady state temperatures of a thin rectangular homogeneous thermally conducting plate by Gauss-Seidel iterative methods and doing SOR analysis.

Comparing the analytical and numerical solution of given convection-diffusion equation by using Thomas algorithm .

1 Steady state temperatures of plate

1.1 Obtaining the FDE

The given problem is carefully understood and solved for obtaining an finite differential equation which can be solved using the computational techniques (here) Gauss-Siedel iterative solver.
The hand-written step by step procedure for obtaining the result is shown in the figure 1 below.

1.2 Using the algorithm

Now that we have all the equations required for solving the given steady state problem numerically (i.e equations 1 & 2 in the figure 1), we need to code the problem and obtain the values at steady state. In Gauss-Seidel method we keep on updating all the elements simultaneously. This plays a major role in reducing the number of iterations.

1.3 Code

The code is written in .cpp format which includes the files named 1a.cpp and 1b.cpp the hard copy of which is attached at the end of documentation. 1a.cpp contains the numerical solution with SOR analysis and 1.b has the analytical solution. The output of the code is directed to a text file which will have all the values at the points at the steady state condition.
Note that here a tolerance has been set for stopping the iterations upon which the final values will be printed.
Using 'R' we have plotted a filled contour where we can see the range of the final temperature of the plate The obtained plots are shown below.

1.4 Result and Inferences

When deltax is 0.025 and beta is 1, the number of iterations for a tolerance of 0.0001 are :
2878 for w = 1
2521 for w =2
160 for w = 1.89178
This shows that for a particular omega the number of iterations are least and if you move away from this optimum the number of iterations increases. If omega is very less compared to optimum it takes small steps but converges to some values. If value is much larger then there is a chance that it diverges and getting infinity.
You can also observe that as you increase the value of beta keeping other parameters constant the heat flow reduced.
The numerical solutions are close enough to the analytical solution with little dicrepencies. Note: In all the headings of plots, x refers to deltax.

Since we are keeping the tolerance same so the contour will be mostly same in all the cases.

2 Convection-Diffusion equation

2.1 Obtaining the FDE

Using central differences, we have obtained an finite difference equation which can be solved by Thomas algorithm as it represents solving a tri-diagonal system of equations. The hand-written step by step procedure for obtaining the result is shown in the figure 2 below

2.2 Implementation & Code

Using 'C++' we have obtained the values (code in 2.cpp) which are then using 'R' made the required plots for comparisons. All the codes pertaining to the question are attached at the end of the documentation.

2.3 Result and Inferences

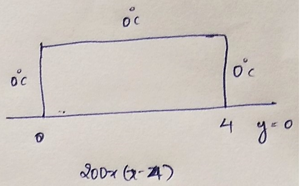
First of all lets talk about oscillatory nature of numerical solution about analytical solution.
For a given Re as delx decreases the oscillations of numerical solution about analytical solution decreases. This is because as delx decreases we get more concentration of points in the grid which leads to a better approximation of the analytical solution
For a given delx as Re increases the oscillations of numerical solution about analytical solution increases.

- X

1.

given limits $0 \le x \le 4$; $0 \le y \le 1$

Let $T(x,y)$ be the temperature fn.



It follows LaPlace's eqn

$$T_{xx} + T_{yy} = 0$$

$$\frac{T_{i+1,j} - 2T_{i,j} + T_{i-1,j}}{\Delta x^2} + O(\Delta x^2) + \frac{T_{i,j+1} - 2T_{i,j} + T_{i,j-1}}{\Delta y^2} + O(\Delta y^2) = 0$$

$$\Rightarrow \frac{T_{i+1,j} - 2T_{i,j} + T_{i-1,j}}{\Delta x^2} + \frac{T_{i,j+1} - 2T_{i,j} + T_{i,j-1}}{\Delta y^2} = 0$$

Take $\beta = \Delta x / \Delta y$, then we get

$$T_{i+1,j} + \beta^2 T_{i,j+1} + T_{i-1,j} + \beta^2 T_{i,j-1} - 2(1+\beta^2) = 0$$

$$\Rightarrow T_{i,j} = \frac{T_{i+1,j} + \beta^2 T_{i,j+1} + T_{i-1,j} + \beta^2 T_{i,j-1}}{2(1+\beta^2)}$$

The above eqn is the FDE

Now we should apply Gauss - Siedel :-

$$T_{i,j}^{k+1} = T_{i,j}^{k} + \Delta T_{i,j}^{k+1} \longrightarrow (1)$$

$$\Delta T_{i,j}^{k+1} = \frac{T_{i+1,j}^{k} + \beta^2 T_{i,j+1}^{k} + T_{i-1,j}^{k+1} + \beta^2 T_{i,j-1}^{k+1} - 2(1+\beta^2) T_{i,j}^{k}}{2(1+\beta^2)}$$

Explanation :-

Basically, you will have a matrix of temperatures

1^{st} row set to $200x(x-4)$

& all others zeros initially and other ends maintained at zero

$$\begin{bmatrix} T_{0,0} & T_{1,0} & T_{2,0} & \cdots & T_{n-1,0} \\ T_{0,1} & T_{1,1} & T_{2,1} & \cdots & T_{n-1,0} \\ \vdots & & & & \\ T_{0,n-1} & T_{1,n-1} & T_{2,n-1} & \cdots & T_{n-1,n-1} \end{bmatrix}$$

Iterations according to (1) & (2) of Gauss - Seidel

Figure 1: Solving for FDE

Figure 2: Stopping criteria and SOR analysis

Stopping Criteria:-

After each iteration of the whole matrix check the maximum $\Delta T_{i,j}$ for all i,j

if $|\max \Delta T_{i,j}| < \varepsilon \longrightarrow$ Tolerance

then stop the iteration. That is our final answer. Plot it & compare plot with analytical plot.

SOR analysis:-

Everything is same as Gauss-Seidel except that you use

$$T_{i,j}^{k+1} = T_{i,j}^{k} + \underbrace{\omega}_{\text{over-relaxation factor}} \Delta T_{i,j}^{k+1}$$

Generally $1 \leq \omega \leq 2$

You will have to experiment with 'w' values and find optimum "w" (i.e for maximum rate of convergence)

Theoretically,

$$\omega_{opt} = 2\left(\frac{1-\sqrt{1-\varepsilon}}{\varepsilon}\right)$$

$$\varepsilon = \left[\frac{\cos(\pi/I) + \beta^2 \cos(\pi/J)}{1+\beta^2}\right]^2$$

I = no. of spatial increments in x-direction

J = no. of spatial increments in y-direction

Figure 3: Analytical solution of steady state

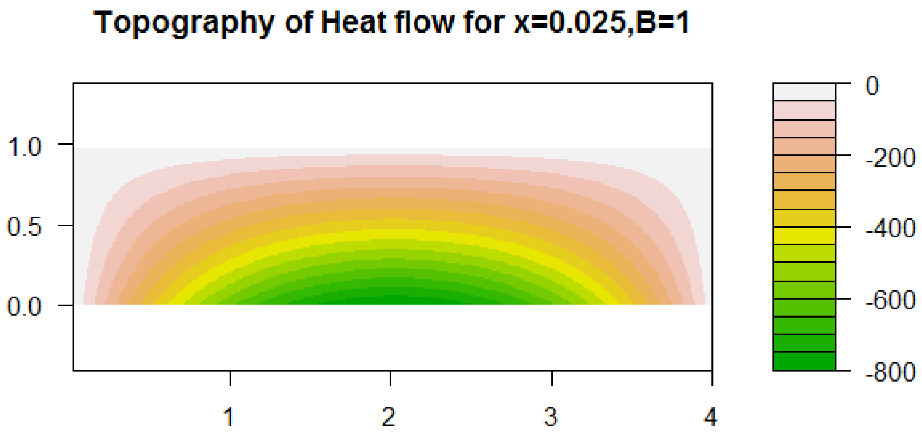


Figure 4: Numerical solution

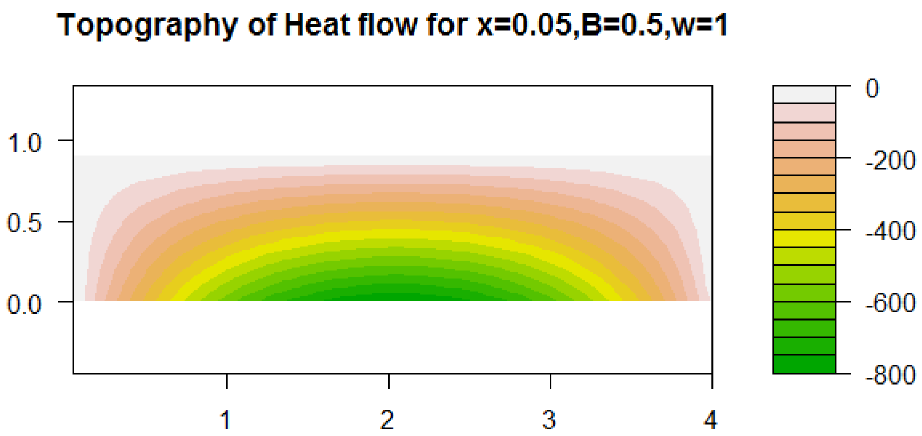


Figure 5: Numerical solution

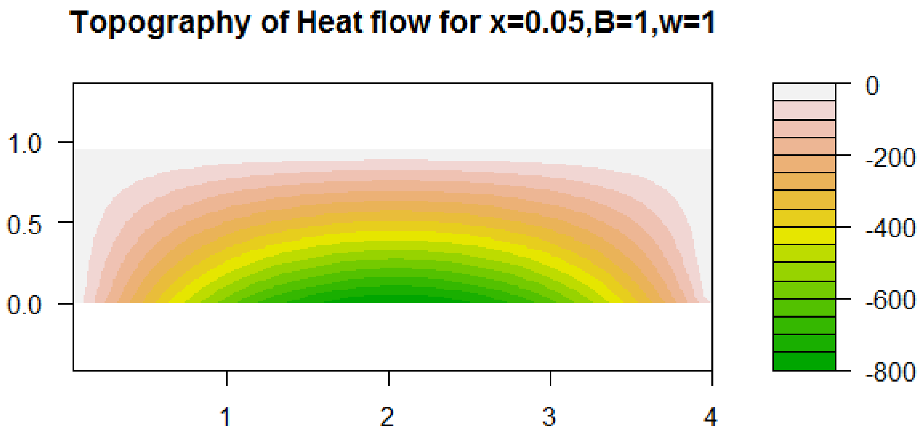


Figure 6: Numerical solution

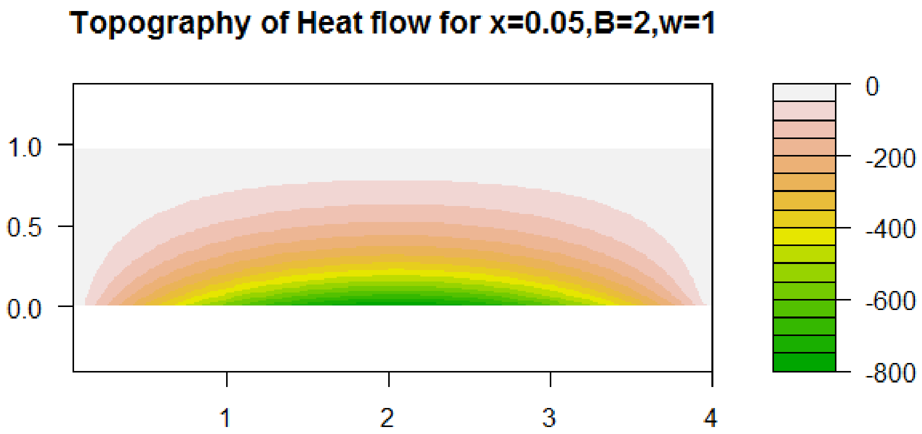


Figure 7: Numerical solution

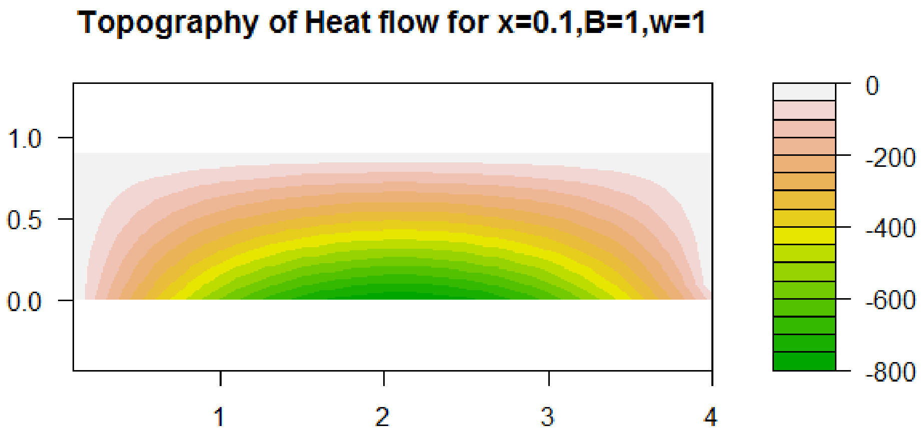


Figure 8: Numerical solution

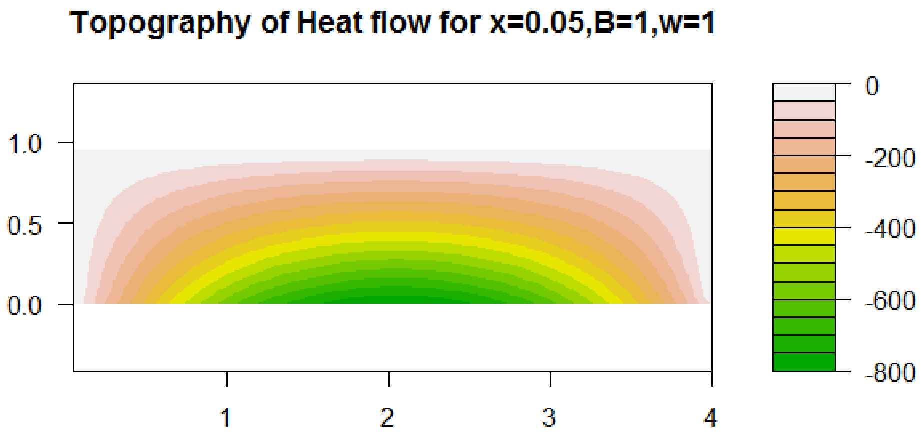


Figure 9: Numerical solution

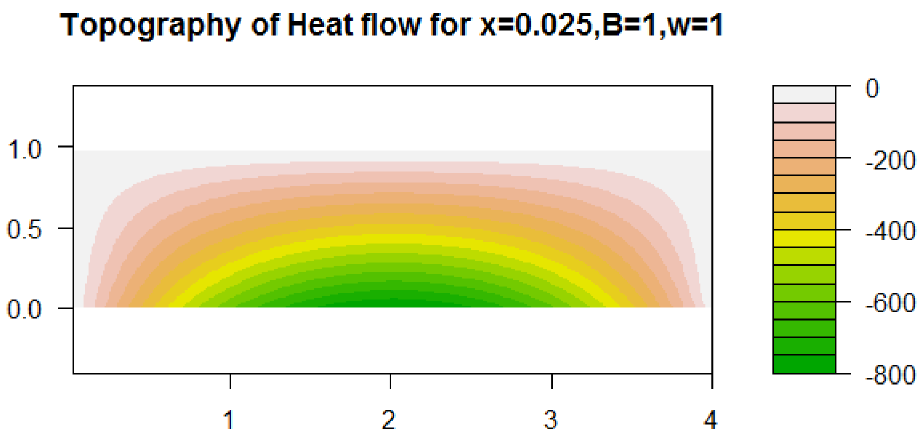


Figure 10: Solving for FDE

2. $$-\frac{\partial^2 u}{\partial x^2} + Re\frac{\partial u}{\partial x} = 0 \qquad 0 \leq x \leq 1$$

Using central differences, we get

$$\frac{-(u_{i+1} - 2u_i + u_{i-1})}{(\Delta x)^2} + Re\frac{(u_{i+1} - u_{i-1})}{2(\Delta x)} = 0$$

$$\Rightarrow -u_{i+1} + 2u_i - u_{i-1} + Re \cdot \frac{\Delta x}{2}(u_{i+1} - u_{i-1}) = 0$$

let, $Re \cdot \frac{\Delta x}{2} = t$

$$-u_{i+1} + 2u_i - u_{i-1} + u_{i+1} - tu_{i-1} = 0$$

$$\Rightarrow (t-1)u_{i+1} + 2u_i - (t+1)u_{i-1} = 0 \longrightarrow (1)$$

Solve eqn (1) using Thomas algorithm

Note that in eqn (1), $t = Re \cdot \frac{(\Delta x)}{2} \longrightarrow (2)$

"Thomas algo returns a vector $(u_0, u_1, u_2 \ldots u_n)$"

We need to Plot this vector and compare with plot of analytical soln given in question.

Figure 11: For Re=10

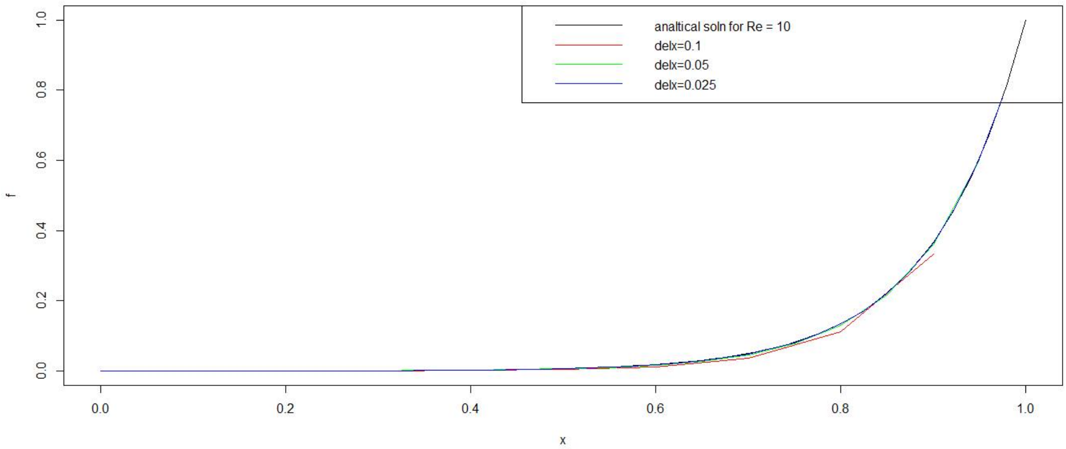


Figure 12: For Re=50

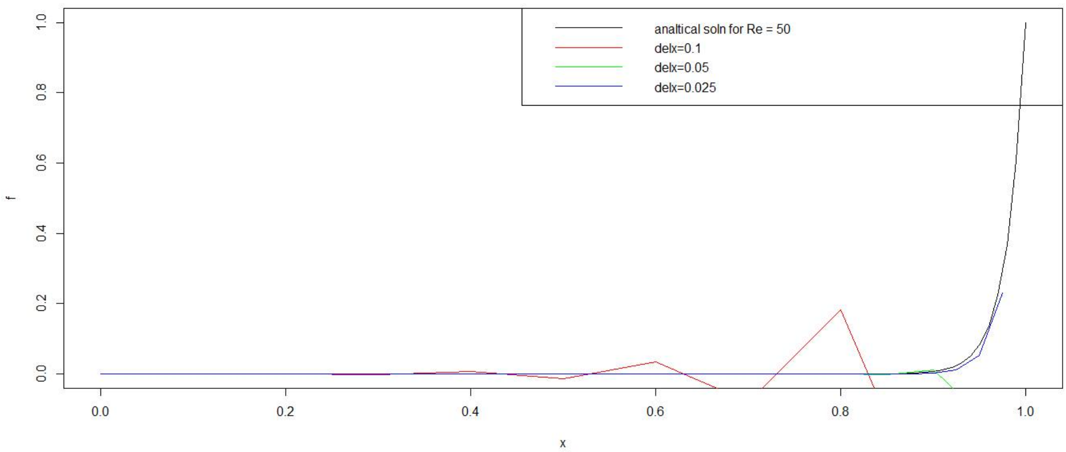


Figure 13: For Re=100