SCS 2210 - Last Year (2019) Final Exam Paper.
Solutions.

Q1). (a) Suppose to the contrary that there exists a composite three-digit number n with no prime factors less than or equal to 31. By the fundamental theorem of arithmatic, n can be written uniquely as a product of primes, with the prime factors in the product written in nondecreasing order. So, let $n = p_1 \cdot p_2 \cdot p_3 \cdots p_m$, where $p_1, p_2, \ldots, p_m$ are primes such that $p_1 \le p_2 \le p_3 \le \cdots \le p_m$ and $m \ge 2$. Note that $m \ge 2$ because otherwise, i.e., if $m = 1$ (since $m \in \mathbb{N}$), then n is prime as p_1 is prime, which is not the case as n is a composite number. Now, since n has no prime factors less than or equal to 31, $p_i \ge 37$ for each $i = 1, 2, \ldots, m$. Thus $n = p_1 \cdot p_2 \cdots p_m \ge p_1 \cdot p_2 \ge 37 \cdot 37 = 1369$. This is a contradiction as n is a three digit number.

Therefore, every composite three-digit number must have a prime factor less than or equal to 31.

Q2). (a) WTS (want to show) $41 \mid (2^{20} - 1)$.

Note that $2^5 \equiv -9 \pmod{41}$. Thus, $(2^5)^4 \equiv (-9)^4 = 9^4 \pmod{41}$.
Since $9^2 \equiv -1 \pmod{41}$, $9^4 = (9^2)^2 \equiv (-1)^2 = 1 \pmod{41}$. Now, because $2^{20} \equiv 9^4 \pmod{41}$ and $9^4 \equiv 1 \pmod{41}$, it follows that $2^{20} \equiv 1 \pmod{41}$.
Therefore, $41 \mid (2^{20} - 1)$.

(b). Note that

$3672 = 2 \cdot 1566 + 540$, so $\gcd(3672, 1566) = \gcd(1566, 540)$
$1566 = 2 \cdot 540 + 486$, so $\gcd(1566, 540) = \gcd(540, 486)$
$540 = 1 \cdot 486 + 54$, so $\gcd(540, 486) = \gcd(486, 54)$
$486 = 9 \cdot 54 + 0$, so $\gcd(486, 54) = \gcd(54, 0) = 54$.

Thus, $\gcd(3672, 1566) = 54$ (the last nonzero remainder).

Now, from the next-to-last equation,

$54 = 540 - 1\cdot 486$ — $*$

From the second equation $486 = 1566 - 2\cdot 540$. — $**$

Substituting $486 = 1566 - 2\cdot 540$ in $*$ gives

$54 = 540 - 1\cdot(1566 - 2\cdot 540) = 3\cdot 540 - 1\cdot 1566$ — $***$

From the first equation, $540 = 3672 - 2\cdot 1566$

Substituting $540 = 3672 - 2\cdot 1566$ in $***$ gives

$54 = 3\cdot(3672 - 2\cdot 1566) - 1\cdot 1566 = 3\cdot 3672 - 7\cdot 1566.$

So, $\gcd(3672, 1566) = 54 = 3672\cdot(3) + 1566\cdot(-7).$

(c). see Tutorial 3, problem 12.

03). (a).

(i) There are 4 non-vegetable toppings and 6 vegetable toppings.

The number of different pizzas that can be ordered is equal to

$^{10}C_0 + {}^{10}C_1 + {}^{10}C_2 + \cdots + {}^{10}C_{10} = (1+1)^{10} = 2^{10} = 1024.$

(ii) The number of different pizzas that contain no vegetable topping is equal to

$^{4}C_0 + {}^{4}C_1 + {}^{4}C_2 + {}^{4}C_3 + {}^{4}C_4 = (1+1)^4 = 2^4 = 16.$

(iii) The number of different pizzas that contain at most one non-vegetable topping = the number of different pizzas that contain no non-vegetable topping + ~~the number of different pizzas~~ that contain exactly one non-vegetable topping $= ({}^6C_0 + {}^6C_1 + \cdots + {}^6C_6) + (4\cdot{}^6C_0 + 4\cdot{}^6C_1 + \cdots + 4\cdot{}^6C_6) =$

$5\,({}^6C_0 + {}^6C_1 + \cdots + {}^6C_6) = 5\cdot(1+1)^6 = 5\cdot 2^6 = 5\cdot 64 = 320.$

(b). According to the problem flags are available in 4 colors, namely yellow, blue, maroon and red.

(i). The number of ways of arranging three colored flags without repetition is equal to $4 \times 3 \times 2 = 24$.

(ii). The number of ways of choosing three colored flags with repetition from the collection =

number of ways of choosing three colored flags in three different colors +

number of ways of choosing three colored flags in two different colors +

number of ways of choosing three colored flags in only one color.

$$= {}^4C_3 + {}^4C_1 \times {}^3C_1 + {}^4C_1 = 4 + 4\cdot 3 + 4 = 20.$$

04). (a). 3, 5, 11, 21, 43, 85, ... Let x_n denote the n^{th} term of the sequence.
Then, $x_7 = 2\cdot 43 + 85 = 171$ and $x_8 = 2\cdot 85 + 171 = 341$.
Observe that for each $n \geq 3$, $x_n = 2x_{n-2} + x_{n-1}$ and $x_1 = 3$, $x_2 = 5$.

The characteristic equation corresponding to this recurrence relation is $r^2 - r - 2 = 0$. So, $r^2 - r - 2 = 0$ if and only if $(r-2)(r+1) = 0$ if and only if $r = 2$ or $r = -1$. Hence, characteristic roots are $r = 2$ and $r = -1$.
Therefore, the general solution of the recurrence relation is,
$x_n = \alpha \cdot 2^n + \beta(-1)^n$, where α, β are arbitrary constants, $n \in \mathbb{N}$.

Now, when $n = 1$, $2\alpha - \beta = 3$ and when $n = 2$, $4\alpha + \beta = 5$. Solving these two equations gives $\alpha = 4/3$ and $\beta = -1/3$.

So, for each $n \in \mathbb{N}$, $x_n = 4/3 \cdot 2^n - 1/3 (-1)^n$.

(b). a_n = the number of $1 \times n$ tile designs that you can make using 1×1 squares available in 4 colors and 1×2 dominoes available in 5 colors.

(i). Observe that there are two different ways to start tiling. First, we can start with a 1×1 square tile. Second, we can start with a 1×2 domino. Suppose we start with a 1×1 square tile of a particular color. Then, the number of $1 \times n$ tile designs that can make is equal to a_{n-1}. Because 1×1 tiles are available in 4 colors, the total number of $1 \times n$ tile designs that can make starting with a 1×1 square tile is equal to $4 \cdot a_{n-1}$.

Now suppose we start with a 1×2 domino of a particular color. Then, the number of $1 \times n$ tile designs that can make is equal to a_{n-2}. Because 1×2 dominoes are available in 5 colors, the total number of $1 \times n$ tile designs that can make starting with a 1×2 domino is equal to $5 \cdot a_{n-2}$.

Therefore, $\boxed{a_n = 4a_{n-1} + 5 \cdot a_{n-2},}$ for each $n \geq 3$.

Notice that when $n=1$, $a_1 = 4$. (In this case, i.e. when $n=1$, we have to use only 1×1 tiles and since they are available in 4 colors, the tiling can be done in 4 ways).

When $n=2$, you can use either 1×1 tiles only or 1×2 dominoes only, but not both, for the tiling. If we use only 1×1 tiles, then the number of tile designs is equal to $4 \times 4 = 16$ and if we use only 1×2 dominoes, then the number of tile designs is equal to 5. Thus, $a_2 = 16 + 5 = 21$.

So, the initial conditions are $a_1 = 4$ and $a_2 = 21$.

(ii) $a_1 = 4$, $a_2 = 21$, $a_3 = 4 \cdot a_2 + 5 \cdot a_1 = 4 \cdot 21 + 5 \cdot 4 = 104$, $a_4 = 4 \cdot a_3 + 5 \cdot a_2 =$
$4 \cdot 104 + 5 \cdot 21 = 521$, $a_5 = 4 \cdot a_4 + 5 \cdot a_3 = 4 \cdot 521 + 5 \cdot 104 = 2604$,

$a_6 = 4 \cdot a_5 + 5 \cdot a_4 = 4 \cdot 2604 + 5 \cdot 521 = 13021$.

The recurrence relation is $a_n = 4a_{n-1} + 5a_{n-2}$ with $a_1 = 4$ and $a_2 = 21$.

The characteristic equation corresponding to the recurrence relation is $r^2 - 4r - 5 = 0$. Now, $r^2 - 4r - 5 = 0$ iff. $(r-5)(r+1) = 0$ iff $r = 5$ or $r = -1$.
Therefore, the general solution of the recurrence relation is,

$a_n = \alpha \cdot 5^n + \beta(-1)^n$, where $\alpha, \beta \in \mathbb{R}$.
Now, when $n=1$, $5\alpha - \beta = 4$ and when $n=2$, $25\alpha + \beta = 21$.
Solving these two equations gives $\alpha = 5/6$ and $\beta = 1/6$.

So, $a_n = 5/6 \cdot 5^n + 1/6 \cdot (-1)^n$ for $n \in \mathbb{N}$.

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