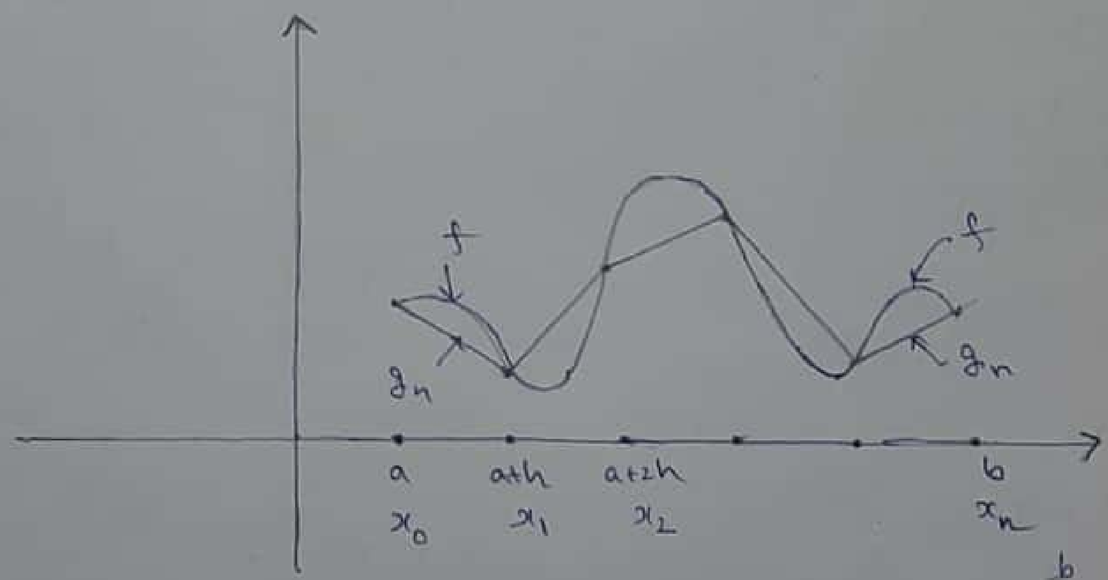
$(a+h, f(a+h)), (a+2h, f(a+2h)), \ldots, (a+nh, f(a+nh))$.
It is clear that $(a+nh, f(a+nh)) = (b, f(b))$ since $a+nh = b$. Let $x_k = a+kh$, $k = 0,1,2,\ldots,n$.
Observe that the equation of g_n on $[x_{k-1}, x_k]$ is the linear interpolation polynomial $P_1(x)$ that interpolates f at x_{k-1} and x_k, $k = 1,\ldots,n$.

Hence $g_n(x) = \dfrac{x - x_k}{x_{k-1} - x_k} f(x_{k-1}) + \dfrac{x - x_{k-1}}{x_k - x_{k-1}} f(x_k)$

$x \in [x_{k-1}, x_k]$, $k = 1, 2, \ldots, n$.



It is reasonable to approximate $\int_a^b f(x)dx$ by $\int_a^b g_n(x)dx$ provided that f is reasonably smooth and n is sufficiently large.

Let us compute $\int_{x_{k-1}}^{x_k} g_n(x)dx$. It is clear that

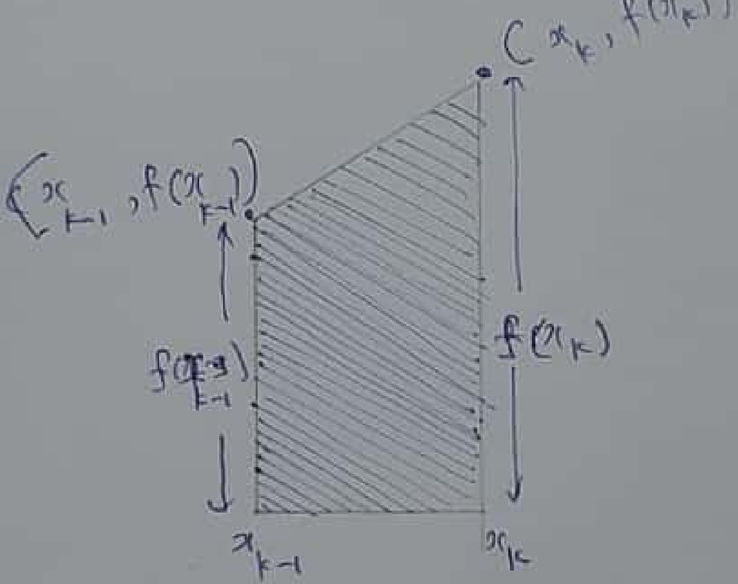
$$\int_{x_{k-1}}^{x_k} g_n(x)dx = \int_{x_{k-1}}^{x_k} \frac{x - x_k}{x_{k-1} - x_k} f(x_{k-1})dx + \int_{x_{k-1}}^{x_k} \frac{x - x_{k-1}}{x_k - x_{k-1}} f(x_k)dx$$

$$= \frac{f(x_{k-1})}{x_{k-1}-x_k} \frac{(x-x_k)^2}{2}\Bigg|_{x_{k-1}}^{x_k} + \frac{f(x_k)}{x_k - x_{k-1}} \frac{(x-x_{k-1})^2}{2}\Bigg|_{x_{k-1}}^{x_k}$$

$$= -\frac{f(x_{k-1})(x_{k-1}-x_k)}{2} + \frac{f(x_k)(x_k - x_{k-1})}{2}$$

$$= \frac{(x_k - x_{k-1})}{2}\left[f(x_{k-1}) + f(x_k)\right]$$

$$= \frac{\left[f(x_k) + f(x_{k-1})\right]}{2}(x_k - x_{k-1})$$



It is clear from the diagram that area of the trapezoid $= \frac{\left[f(x_k) + f(x_{k-1})\right]}{2}(x_k - x_{k-1})$.

Therefore
$$\int_a^b g_n(x)\,dx = \sum_{k=1}^{n} \int_{x_{k-1}}^{x_k} g_n(x)\,dx$$

$$= \sum_{k=1}^{n} \frac{\left[f(x_k) + f(x_{k-1})\right]}{2}(x_k - x_{k-1})$$

$$= \frac{(x_k - x_{k-1})}{2}\left\{f(x_0) + f(x_1) + f(x_1) + f(x_2) + f(x_2) + f(x_3) + \cdots + f(x_{n-1}) + f(x_n)\right\}$$

$$= \frac{h}{2}\left\{f(a) + 2f(a+h) + 2f(a+2h) + \cdots + 2f(a+(n-1)h) + f(b)\right\}$$

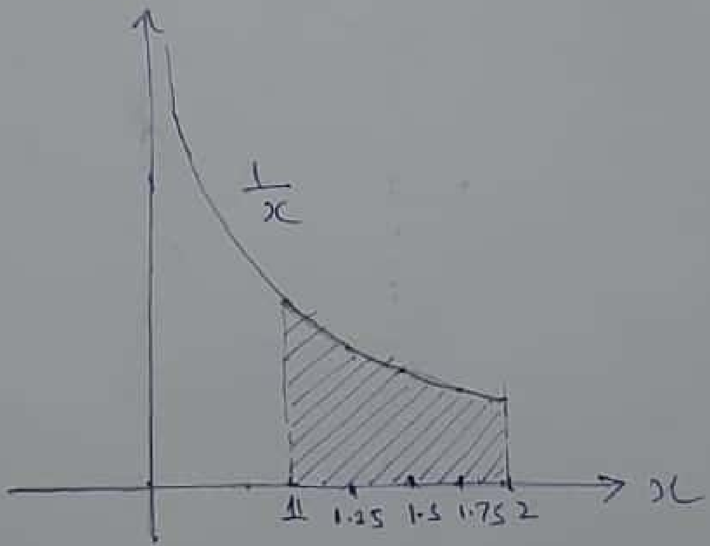
Therefore $\int_a^b f(x)\,dx \simeq h\left\{\frac{f(a)}{2} + \sum_{k=1}^{n-1} f(a+kh) + \frac{f(b)}{2}\right\}$.

Example.

Use the trapezoidal approximation with $n=4$ to approximate $\log_e 2 = \int_1^2 \frac{1}{x}\,dx$.

Solution: It is given that $n=4$, $a=1$, $b=2$.

Then $h = \frac{b-a}{n} = \frac{2-1}{4} = 0.25$. Clearly $f(x) = \frac{1}{x}$.



By trapezoidal rule

$$\int_1^2 \frac{1}{x}\,dx \simeq \frac{1}{4}\left\{\frac{f(1)}{2} + f(1.25) + f(1.5) + f(1.75) + \frac{f(2)}{2}\right\}$$

$$= \frac{1}{4}\left\{\frac{1}{2} + \frac{1}{1.25} + \frac{1}{1.5} + \frac{1}{1.75} + \frac{1}{4}\right\}$$

$$= \frac{1}{4}\left\{0.5 + 0.8 + 0.6667 + 0.5714 + 0.25\right\}$$

$$= \frac{1}{4}(2.7881) = 0.697025.$$

$(\log_e 2 = 0.6931.)$

Error Estimate in Trapezoidal Rule

Recall that when f has a $n+1$-st cts derivative, for each $x \in (x_0, x_n)$, there exists $t_x \in (x_0, x_n)$ such that

$$e_n(x) = f(x) - p_n(x) = (x-x_0)(x-x_1)\cdots(x-x_n)\frac{f^{(n+1)}(t_x)}{(n+1)!},$$

where p_n is the interpolating polynomial that interpolates f at $x_0, x_1, x_2, \ldots, x_n$.

Let us use this result to estimate the error in approximating $\int_{x_{k-1}}^{x_k} f(x)\,dx$ by $\int_{x_{k-1}}^{x_k} p_1(x)\,dx$.

On the interval $[x_{k-1}, x_k]$, there exists $t_k(x)$ such that the error

$$e(x) = f(x) - p_1(x) = f(x) - \left\{ \frac{x - x_k}{x_{k-1} - x_k} . f(x_{k-1}) + \frac{x - x_{k-1}}{x_k - x_{k-1}} . f(x_k) \right\}$$

$$= (x - x_{k-1})(x - x_k)\frac{f^{(2)}(t_k(x))}{2!}.$$

Thus the error on the interval $[x_{k-1}, x_k]$

$$= \int_{x_{k-1}}^{x_k} (x - x_{k-1})(x - x_k)\frac{f^{(2)}(t_k(x))}{2}\,dx.$$

Observe that for each $x \in [x_{k-1}, x_k]$,

$(x - x_{k-1})(x - x_k) \leq 0$. Since $f^{(2)}(t_k(x))$ is continuous on $[x_{k-1}, x_k]$, there exists $t_k \in [x_{k-1}, x_k]$

such that $\int_{x_{k-1}}^{x_k} (x-x_{k-1})(x-x_k) \frac{f^{(2)}(t_k(x))}{2} dx$

$$= \frac{f^{(2)}(\bar{t}_k)}{2} \int_{x_{k-1}}^{x_k} (x-x_{k-1})(x-x_k)\,dx$$

[Mean-Value Theorem of integral calculus].

Set $v = x - x_{k-1}$. Then $dv = dx$.

When $x = x_{k-1}$, $v=0$ and $x = x_k$, $v = x_k - x_{k-1} = h$.

Then $\frac{f^{(2)}(\bar{t}_k)}{2} \int_{x_{k-1}}^{x_k} (x-x_{k-1})(x-x_k)\,dx$

$$= \frac{f^{(2)}(\bar{t}_k)}{2} \int_0^h v(v-h)\,dv$$

$$= \frac{f^{(2)}(\bar{t}_k)}{2} \int_0^h (v^2 - vh)\,dv$$

$$= \frac{f^{(2)}(\bar{t}_k)}{2} \left(\frac{1}{3}v^3 - \frac{1}{2}v^2 h\right)\Big|_0^h$$

$$= \frac{f^{(2)}(\bar{t}_k)}{2} \left(\frac{1}{3}h^3 - \frac{1}{2}h^3\right)$$

$$= -\frac{h^3 f^{(2)}(\bar{t}_k)}{12}.$$

Therefore $\int_{x_{k-1}}^{x_1} f(x)\,dx = \frac{h[f(x_{k-1}) + f(x_k)]}{2} - \frac{h^3}{12} f(\bar{t}_k)$

Hence $\int_a^b f(x)\,dx = \sum_{k=1}^{n} \int_{x_{k-1}}^{x_k} f(x)\,dx = \sum_{k=1}^{n} \left\{ \frac{h[f(x_{k-1}) + f(x_k)]}{2} - \frac{h^3}{12} f^{(2)}(\bar{t}_k) \right\}$

$$= h\left\{\frac{f(a)}{2} + f(x_1) + f(x_2) + \cdots + f(x_{n-1}) + \frac{f(b)}{2}\right\}$$

$$- \frac{h^3}{12}\left\{f^{(2)}(\bar{t}_1) + f^{(2)}(\bar{t}_2) + \cdots + f^{(2)}(\bar{t}_n)\right\}$$

It is given that $f^{(2)}$ is continuous on $[a,b]$. Thus there exist m, M such that $m = \min\{f^{(2)}(t) : t \in [a,b]\}$, $M = \max\{f^{(2)}(t) : t \in [a,b]\}$.

Also there exist $t_m, t_M \in [a,b]$ such that

$$m = f^{(2)}(t_m) \text{ and } M = f^{(2)}(t_M).$$

Observe that $m \le f^{(2)}(\bar{t}_k) \le M, \quad k = 1, 2, \dots, n.$

Thus $$\sum_{k=1}^{n} m \le \sum_{k=1}^{n} f^{(2)}(\bar{t}_k) \le \sum_{k=1}^{n} M.$$

So $$nm \le \sum_{k=1}^{n} f^{(2)}(\bar{t}_k) \le nM.$$

Hence $$m \le \sum_{k=1}^{n} \frac{f^{(2)}(\bar{t}_k)}{n} \le M.$$

Then by Intermediate Value Theorem for continuous functions, there exist $t \in [a,b]$ such that $$\sum_{k=1}^{n} \frac{f^{(2)}(\bar{t}_k)}{n} = f^{(2)}(t).$$

Then $$\int_a^b f(x)\,dx = h\left\{\frac{f(a)}{2} + f(x_1) + f(x_2) + \dots + f(x_{n-1}) + \frac{f(b)}{2}\right\} - \frac{h^3}{12} n f^{(2)}(t).$$

$$= h\left\{\frac{f(a)}{2} + f(x_1) + f(x_2) + \dots + f(x_{n-1}) + \frac{f(b)}{2}\right\} - \frac{(b-a)}{12} h^2 f^{(2)}(t).$$

In particular the error E_n in trapezoidal rule is given by $E_n = -\frac{(b-a)h^2}{12} f^{(2)}(t)$ for some $t \in [a,b]$.

where $h = \frac{b-a}{n} = x_k - x_{k-1}, \quad k = 1, 2, \dots, n.$

Estimate in the nth trapezoidal rule is usually denoted by $T_n(f)$.

Thus $\boxed{\int_a^b f(x)dx = T_n(f) - \frac{(b-a)h^2}{12} f^{(2)}(t)}$ for some $t \in [a,b]$. Notice that we know the existence of t and we don't know t.

Theorem: Suppose $f, f^{(1)}, f^{(2)}$ be continuous on $[a,b]$. Set $B_2 = \max\{|f^{(2)}(x)| : x \in [a,b]\}$. Then

$$\left|\int_a^b f(x)dx - T_n(f)\right| \leq \frac{(b-a)h^2}{12} B_2.$$

This inequality can be written in the form

$$\left|\int_a^b f(x)dx - T_n(f)\right| \leq \frac{(b-a)^3}{12n^2} B_2 \quad \text{Since } h = \frac{b-a}{n}.$$

Proof: Follows from

$$\int_a^b f(x)dx - T_n(f) = -\frac{(b-a)h^2}{12} f^{(2)}(t).$$

Indeed
$$\left|\int_a^b f(x)dx - T_n(f)\right| = \left|-\frac{(b-a)h^2}{12} f^{(2)}(t)\right|$$
$$= \frac{(b-a)h^2}{12} |f^{(2)}(t)|$$
$$\leq \frac{(b-a)h^2}{12} \max\{|f^{(2)}(x)| : x \in [a,b]\}$$
$$= \frac{(b-a)h^2}{12} B_2.$$

Example. Prove that $0.0013 = \frac{1}{768} \le T_4 - \log_e 2 \le \frac{1}{96} < 0.0105$.

Solution: Let $f(x) = \frac{1}{x}$. Then $\int_1^2 f(x)dx = \ln 2$.

From the above work, there exist $t \in [1,2]$ such that $\int_1^2 f(x)dx - T_4(f) = -\frac{(2-1)(0.25)^2}{12} f^{(2)}(t)$.

Observe that $f(x) = \frac{1}{x}$, $f^{(1)}(x) = \frac{-1}{x^2}$ and $f^{(2)}(x) = \frac{2}{x^3}$, for each $x \in [1, 2]$.

Thus $\int_1^2 f(x)dx - T_4(f) = -\frac{1}{16 \cdot 12} \cdot \frac{2}{t^3} = -\frac{1}{96t^3}$

Hence $T_4(f) - \int_1^2 f(x)dx = \frac{1}{96t^3}$

Clearly, $1 \le t \le 2$.

Thus $96 \le 96t^3 \le 96.8 = 768$

Hence $\frac{1}{768} \le \frac{1}{96t^3} \le \frac{1}{96}$

Therefore $0.0013 < \frac{1}{768} \le T_4(f) - \log_e 2 \le \frac{1}{96} < 0.0105$

(Notice that $\frac{1}{96} = 0.0104166 < 0.0105$)

Example. Let $f(x) = e^{-x^2}$, $x \in [0,1]$.

Show that $\left| T_8(f) - \int_0^1 e^{-x^2}dx \right| \le \frac{1}{384} \le 0.003$

and $\left| T_{16}(f) - \int_0^1 e^{-x^2}dx \right| \le \frac{1}{1536} < 0.00066$.

Solution: Let $f(x) = e^{-x^2}$, $x \in [0,1]$.

Then $f^{(1)}(x) = -2x e^{-x^2}$ and

$$f^{(2)}(x) = -2e^{-x^2} + (-2x)(-2x)e^{-x^2}$$
$$= 2e^{-x^2}(2x^2-1).$$

Now let $x \in [0,1]$. Then

$$|f^{(2)}(x)| = |2e^{-x^2}(2x^2-1)|$$
$$= \frac{2|(2x^2-1)|}{e^{x^2}}$$

It is clear that $2x^2-1 \geq 0$ if and only if $2x^2 \geq 1$

if and only if $x \geq \frac{1}{\sqrt{2}} = \frac{\sqrt{2}}{2}$

Then $|2x^2-1| = \begin{cases} 2x^2-1, & x \in [\frac{\sqrt{2}}{2}, 1] \\ 1-2x^2, & x \in [0, \frac{\sqrt{2}}{2}]. \end{cases}$

Now let $x \in [\frac{\sqrt{2}}{2}, 1]$.

Then $|f^{(2)}(x)| = \frac{2(2x^2-1)}{e^{x^2}} \leq \frac{2(2x^2-1)}{1+x^2}$.

Since $0 < x < 1$, we have $0 < x^2 < 1$.

Then $x^2 < 2$. So $2x^2-1 < 1+x^2$

then $\frac{2(2x^2-1)}{1+x^2} < 2$. Hence $|f^{(2)}(x)| \leq 2$.

Finally let $x \in [0, \frac{\sqrt{2}}{2}]$. Then $|f^{(2)}(x)| = \frac{2(1-2x^2)}{e^{x^2}}$

Hence $|f^2(x)| < \frac{2}{e^{x^2}} \leq 2$. Thus for each

$x \in [0,1]$, $|f^2(x)| \leq 2$. Thus $B_2 \leq 2$.

Thus $\left| T_8(f) - \int_0^1 e^{-x^2}dx \right| \le \frac{(1-0)^3}{12 \cdot 8^2} \cdot 2 = \frac{1}{384} \le .003$

Also $\left| T_{16}(f) - \int_0^1 e^{-x^2}dx \right| \le \frac{(1-0)^3}{12 \cdot 16^2} \cdot 2 = \frac{1}{1536} \le 0.00066$

Example. Approximate $\int_0^\pi \sin x \, dx$ using trapezoidal rule by dividing the range of integration into 6 equal parts. Find the actual value of $\int_0^\pi \sin x \, dx$. Calculate the percentage error.

Solution:
Let $f(x) = \sin x$

x	0	$\pi/6$	$\pi/3$	$\pi/2$	$\frac{2\pi}{3}$	$5\pi/6$	π
$f(x)$	0	0.5	0.866	1	0.866	0.5	0

Thus $T_6(f) = \frac{\pi}{6}\left\{ \frac{f(0)}{2} + f(\pi/6) + f(\pi/3) + f(\pi/2) + f(\frac{2\pi}{3}) + f(\frac{5\pi}{6}) + \frac{f(\pi)}{2} \right\}$

$= \frac{\pi}{6}\left\{ 0 + 0.5 + 0.866 + 1 + 0.866 + 0.5 + 0 \right\}$

$= \frac{\pi}{6} \times 3.732 = \pi \times 0.622 = 1.9540$

$\int_0^\pi \sin x \, dx = -\cos x \Big|_0^\pi = -\cos\pi - (-\cos 0) = -(-1) - (-1)$
$= 2.$

Thus the percentage of the error $= \frac{2 - 1.9540}{2} \times 100$

$= 2.3\%$

Example. Evaluate the integral $\int_0^1 \frac{1}{1+x^2}\,dx$ using trapezoidal rule by dividing the interval [0,1] into 4 equal parts. Hence compute the approximate value of π.

Solution: Let $f(x) = \frac{1}{1+x^2}$.

First let us tabulate the function as.

x	0	1/4	1/2	3/4	1
$f(x)$	1	0.9412	0.8000	0.6400	0.5000

Thus $T_4(f) = \frac{1}{4}\left\{\frac{1}{2} + (0.9412 + 0.8000 + 0.6400) + \frac{0.5}{2}\right\}$

$= \frac{1}{4} \times 3.1312 = 0.7828$

It is clear that $\int_0^1 \frac{1}{1+x^2}\,dx = \tan^{-1}x\Big|_0^1 = \tan^{-1}1 - \tan^{-1}0$

$= \frac{\pi}{4} - 0 = \frac{\pi}{4}$.

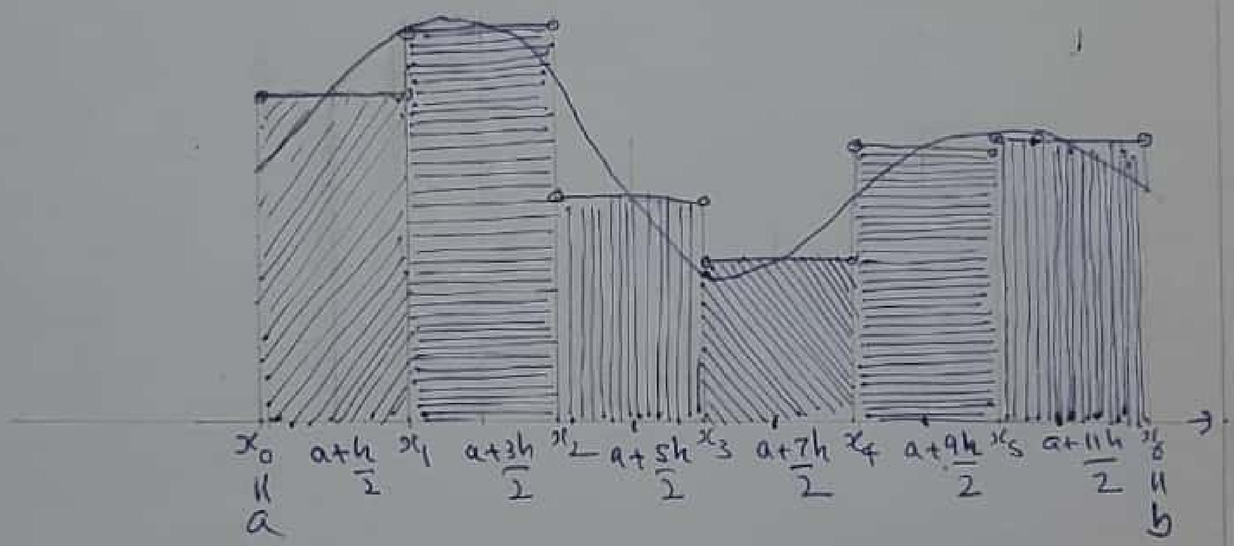
Thus $\frac{\pi}{4} \simeq 0.7828$.

Hence $\pi \simeq 0.7828 \times 4 = 3.1312$.

The Midpoint Rule

This is the rectangular rule briefly mentioned earlier. Unlike in trapezoidal rule, in midpoint rule f is approximated by a piecewise constant (step) function. As before let $P_n = \{a, a+h, a+2h, \ldots, a+(n-1)h, b\}$ where $a+nh = b$ and $n \in \mathbb{N}$. Here, $f, f', f^{(2)}$ are continuous on $[a,b]$. Consider the step function $g_n : [a,b] \to \mathbb{R}$ given by $g_n(x) = f\left[a + \left(k - \frac{1}{2}\right)h\right]$, $x \in [a+(k-1)h, a+kh]$

$k = 1, 2, \ldots, n$.



Observe for $x \in (x_0, x_1)$, $g_n(x) = f\left(a + \frac{h}{2}\right)$, $x \in (x_1, x_2)$, $g_n(x) = f\left(a + \frac{3h}{2}\right)$, for $x \in (x_2, x_3)$, $g_n(x) = f\left(a + \frac{5h}{2}\right)$ etc. Clearly g_n is not continuous and

$$\int_{x_0}^{x_n} g_n(x)\,dx = \int_a^b g_n(x)\,dx = h\left\{f\left(a+\frac{h}{2}\right) + f\left(a + \frac{3h}{2}\right) + \cdots + f\left(a + \frac{(2n-1)h}{2}\right)\right\}$$
$$= M_n(f).$$

Now consider $\int_{x_{k-1}}^{x_k} f(x)\,dx$. Let $m_k = \frac{x_k + x_{k-1}}{2}$.

So $m_k = \frac{a + kh + a + (k-1)h}{2} = a + \left(k - \frac{1}{2}\right)h$,

$k = 1, 2, \ldots, n$.

Set $\phi_k(t) = \int_{m_k - t}^{m_k + t} [f(x) - f(m_k)]\,dx, \quad t \in [0, \frac{h}{2}]$

$k = 1, 2, \ldots, n.$

Notice that $m_k - \frac{h}{2} = a + (k-1)h = x_{k-1}$ and

$m_k + \frac{h}{2} = a + kh = x_k.$

Clearly $\phi_k(0) = 0.$

Observe that $\phi_k(t) = \int_{m_k - t}^{m_k + t} f(x)\,dx - f(m_k)2t,$

$= \int_{m_k - t}^{m_k} f(x)\,dx + \int_{m_k}^{m_k + t} f(x)\,dx - f(m_k)2t$

$= \int_{m_k}^{m_k + t} f(x)\,dx - \int_{m_k}^{m_k - t} f(x)\,dx - f(m_k)2t$

Thus $\phi_k'(t) = f(m_k + t) + f(m_k - t) - 2f(m_k).$

Thus $\phi_k'(0) = 0.$

Notice $\phi_k^{(2)}(t) = f'(m_k + t) - f'(m_k - t).$ Since f' is continuous, by Mean Value theorem for differentiable functions there exist $m_{k,t} \in (m_k - t, m_k + t]$ such that $f'(m_k + t) - f'(m_k - t) = f^{(2)}(m_{k,t})2t.$

Hence $\phi_k^{(2)}(t) = f^{(2)}(m_{k,t})2t.$

Set $A = \min\{f^{(2)}(x) : x \in [a,b]\}$ and

$B = \max\{f^{(2)}(x) : x \in [a,b]\}.$

As a result, $2tA \leq \phi_k^{(2)}(t) \leq 2tB$, $t \in [0, \frac{h}{2}]$,
$k = 1, 2, \ldots, n$,

Hence $\int_0^x 2tA\,dt \leq \int_0^x \phi_k^{(2)}(t)\,dt \leq \int_0^x 2tB\,dt$

$x^2 A \leq \phi_k'(x) \leq x^2 B$ since $\phi_k'(0) = 0$.

Thus $\int_0^t x^2 A\,dx \leq \int_0^t \phi_k'(x)\,dx \leq \int_0^t x^2 B\,dx$,

Hence $\frac{1}{3} A t^3 \leq \phi_k(t) \leq \frac{1}{3} B t^3$ since $\phi_k(0) = 0$.

Thus $\frac{1}{3} A t^3 \leq \phi_k(t) \leq \frac{1}{3} B t^3$ for $t \in [0, \frac{h}{2}]$, $k = 1, 2, \ldots, n$.

In particular $\frac{1}{3} \cdot A \left(\frac{h}{2}\right)^3 \leq \phi_k\left(\frac{h}{2}\right) \leq \frac{1}{3} \cdot B \left(\frac{h}{2}\right)^3$

That is $\frac{1}{24} A h^3 \leq \phi_k\left(\frac{h}{2}\right) \leq \frac{1}{24} B h^3$, $k = 1, 2, \ldots, n$

Notice that $\sum_{k=1}^{n} \phi_k\left(\frac{h}{2}\right) = \sum_{k=1}^{n} \int_{m_k - h/2}^{m_k + h/2} [f(u) - f(m_k)]\,dt$

$$= \sum_{k=1}^{n} \int_{x_{k-1}}^{x_k} f(u)\,du - \sum_{k=1}^{n} f(m_k) \cdot h$$

$$= \int_a^b f(x)\,dx - M_n(f),$$

Hence $\frac{1}{24} A h^3 n \leq \int_a^b f(x)\,dx - M_n(f) \leq \frac{1}{24} B h^3 n$.

Theorem: Suppose $f, f', f^{(2)}$ are continuous on $[a,b]$. Let $M_n(f)$ be the nth midpoint approximation to $\int_a^b f(x)\,dx$. Then there is a point $\gamma \in [a,b]$ such that $\int_a^b f(x)\,dx - M_n(f) = \frac{(b-a)h^2}{24} f^{(2)}(\gamma)$.

Proof: We have already seen that

$$\frac{1}{24} A h^3 n \le \int_a^b f(x)\,dx - M_n(f) \le \frac{1}{24} B h^3 n.$$

Since $f^{(2)}$ is continuous on $[a,b]$, there exists $\gamma \in [a,b]$ such that

$$\int_a^b f(x)\,dx - M_n(f) = \frac{1}{24} f^{(2)}(\gamma) h^3 n.$$
$$= \frac{1}{24} h^2 (b-a) f^{(2)}(\gamma)$$
$$= \frac{(b-a)h^2}{24} f^{(2)}(\gamma).$$

Corollary: Suppose $f, f', f^{(2)}$ are continuous on $[a,b]$ and let $B_2 = \max\{|f^{(2)}(x)| : x \in [a,b]\}$. Then $\left| M_n(f) - \int_a^b f(x)\,dx \right| \le \frac{(b-a)h^2}{24} B_2$.

Proof. Observe that

$$\left| M_n(f) - \int_a^b f(x)\,dx \right| = \frac{(b-a)h^2}{24} |f^{(2)}(\gamma)| \le \frac{(b-a)h^2}{24} B_2.$$