Series

Consider the sequence $\langle x_n \rangle$. Let us try to compute the sum of all the terms of the sequence $\langle x_n \rangle$ in the given order starting from the first term x_1. That is let us try to compute the sum $x_1 + x_2 + x_3 + \cdots$.

Let us denote the sum of the first n terms of the sequence $\langle x_n \rangle$ by S_n. S_n is called the n th partial sum. Thus $S_n = x_1 + x_2 + x_3 + \cdots + x_n$. Hence $S_1 = x_1$, $S_2 = x_1 + x_2$, $S_3 = x_1 + x_2 + x_3$, $\ldots$ $S_{1000} = x_1 + x_2 + x_3 + \cdots + x_{1000}$. We usually use the notation $\sum_{k=1}^{n} x_k$ for S_n. Thus $S_n = \sum_{k=1}^{n} x_k$. So $S_1 = \sum_{k=1}^{1} x_k$, $S_2 = \sum_{k=1}^{2} x_k$, $S_3 = \sum_{k=1}^{3} x_k$. It seems that we want to compute $S_\infty = \sum_{k=1}^{\infty} x_k$. However, ∞ is not a positive integer. Thus S_∞ cannot be computed. Regardless of how large n you consider, after computing S_n, there are infinitely many terms x_k to be added in the sequence $\langle x_k \rangle$.

$$x_1 + x_2 + x_3 + \cdots + x_n + \underbrace{x_{n+1} + x_{n+2} + \cdots}_{\text{infinitely many terms.}}$$

very large n

Hence it is impossible to add all the terms $x_1, x_2, x_3, \ldots, x_n, x_{n+1}, \ldots$. Therefore, if possible, some number is needed which is some what like $x_1+x_2+x_3+\cdots$. You may see that if s_n has a limit l as n tends to infinity, this limit l would be a good candidate for our need. Indeed it is how we define.

Definition. Let $\langle x_n \rangle$ be a sequence and l be a real number. We say that $\sum_{k=1}^{\infty} x_k = l$ if $\lim_n s_n = l$. In this case we say that $\sum_{n=1}^{\infty} x_n$ converges or $\sum_{n=1}^{\infty} x_n$ is convergent. Also we say $\sum_{k=1}^{\infty} x_k$ converges to l, and write $\sum_{n=1}^{\infty} x_n = l$.

Remark: When we say $\sum_{k=1}^{3} k = 6$ means $1+2+3=6$. To the contrary, when we say $\sum_{n=1}^{\infty} x_n = l$, it does not mean $x_1+x_2+x_3+\cdots = l$. What it means is $\lim_{n\to\infty} (x_1+x_2+\cdots+x_n) = l$.

Definition. Let $\langle x_n \rangle$ be a sequence. We say that $\sum_{n=1}^{\infty} x_n$ diverges or $\sum_{n=1}^{\infty} x_n$ is divergent if $\sum_{n=1}^{\infty} x_n$ is not convergent.

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Remark: $\sum_{k=1}^{\infty} x_k$ is called the series. The series $\sum_{k=1}^{\infty} x_k$ converges means the sequence $\langle S_n \rangle$ of partial sums converges. The series $\sum_{n=1}^{\infty} x_n$ converges to l means the sequence $\langle S_n \rangle$ converges to l, i.e. $\lim_n S_n = l$.

Example. Show that the series, $\sum_{n=1}^{\infty} \frac{1}{2^n}$ converges.

Find $\sum_{n=1}^{\infty} \frac{1}{2^n}$.

Solution: Let $n \in \mathbb{N}$.

Then $S_n = \frac{1}{2} + \frac{1}{2^2} + \frac{1}{2^3} + \cdots + \frac{1}{2^n}$.

Therefore $2S_n = 1 + \frac{1}{2} + \frac{1}{2^2} + \cdots + \frac{1}{2^{n-1}}$.

Hence $S_n = 1 - \frac{1}{2^n}$. Then for each $n \in \mathbb{N}$, $S_n = 1 - \frac{1}{2^n}$. It is clear that $\lim_n \left(1 - \frac{1}{2^n}\right) = 1 - \lim_n \frac{1}{2^n}$ $= 1 - 0 = 1$. Thus the series $\sum_{n=1}^{\infty} \frac{1}{2^n}$ converges and $\sum_{n=1}^{\infty} \frac{1}{2^n} = 1$.

Note: From the above work, it is clear that S_n will never be equal to 1. However, for very very large n, S_n will be very very close to 1.

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Indeed, you can find n_0 so that for each $n > n_0$, S_n is as close to l as you desired.

Remark: Did you notice that $\sum_{k=1}^{\infty} x_k$, $\sum_{n=1}^{\infty} x_n$, $\sum_{r=1}^{\infty} x_r$ all mean the same thing.

Example. Discuss the convergence of the series

$$\sum_{n=1}^{\infty} \frac{1}{n(n+1)}.$$

Solution: Let $n \in \mathbb{N}$. Observe that

$$S_n = \sum_{k=1}^{n} \frac{1}{k(k+1)} = \sum_{k=1}^{n} \left(\frac{1}{k} - \frac{1}{k+1} \right) \text{ (partial fractions)}$$

$$= \frac{1}{1} - \frac{1}{2} + \frac{1}{2} - \frac{1}{3} + \frac{1}{3} - \frac{1}{4} + \cdots + \frac{1}{n} - \frac{1}{n+1} = 1 - \frac{1}{n+1}.$$

It is clear that $\lim_n \left(1 - \frac{1}{n+1}\right) = 1 - \lim_n \frac{1}{n+1} = 1 - 0 = 1.$

Hence $\lim_n S_n = 1$.

Thus the series $\sum_{n=1}^{\infty} \frac{1}{n(n+1)}$ converges and

$$\sum_{n=1}^{\infty} \frac{1}{n(n+1)} = 1.$$

Note. Notice that although $\sum_{n=1}^{\infty} \frac{1}{n(n+1)} = 1$, for each $n \in \mathbb{N}$, $\sum_{k=1}^{n} \frac{1}{k(k+1)} \neq 1$.

Example. Discuss the convergence of the series

$$\sum_{n=1}^{\infty} (-1)^n.$$

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Solution: Let $n \in \mathbb{N}$. It is clear that

if n is even, $S_n = (-1)+1+(-1)+1+\cdots+(-1)+1 = 0$.

and if n is odd, $S_n = (-1)+1+(-1)+1+\cdots+(-1)+1+(-1) = -1$.

$$\text{Thus } S_n = \begin{cases} 0, & \text{if } n \text{ is even} \\ -1, & \text{if } n \text{ is odd} \end{cases}$$

So $\langle S_n \rangle = (-1, 0, -1, 0, -1, 0, \ldots, -1, 0, \ldots)$

Clearly $\langle S_n \rangle$ does not converge.

Thus $\sum_{n=1}^{\infty} (-1)^n$ diverges.

Theorem. Suppose the series $\sum_{n=1}^{\infty} x_n$ converges. Then the sequence $\langle x_n \rangle$ converges and $\lim_n x_n = 0$.

Proof: Since $\sum_{n=1}^{\infty} x_n$ converges, there exists $l \in \mathbb{R}$ such that $\lim_{n\to\infty} S_n = l$. Hence $\lim_{n\to\infty} S_{n-1} = l$.

$$\text{Thus } 0 = l - l = \lim_{n\to\infty} S_n - \lim_{n\to\infty} S_{n-1}$$

$$= \lim_{n\to\infty}(x_1+x_2+\cdots+x_n) - \lim_{n\to\infty}(x_1+x_2+\cdots+x_{n-1})$$

$$= \lim_{n\to\infty}(x_1+x_2+\cdots+x_n) - (x_1+x_2+\cdots+x_{n-1})$$

$$= \lim_{n\to\infty} x_n$$

That is $\langle x_n \rangle$ converges and $\lim_n x_n = 0$.

Theorem: Suppose $\langle x_n \rangle$ diverges or $\lim_n x_n \neq 0$. Then $\sum_{n=1}^{\infty}$ diverges.

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Proof: This is the contrapositive of the previous theorem.

Examples: (i) $\sum_{n=1}^{\infty} (-1)^n$ diverges because $\langle (-1)^n \rangle$ diverges.

(ii) $\sum_{n=1}^{\infty} \frac{n+1}{n+2}$ diverges because $\lim_{n\to\infty} \frac{n+1}{n+2} = 1 \neq 0$.

Definition. $\sum_{n=1}^{\infty} \frac{1}{n}$ is called the Harmonic series.

Let $p \in \mathbb{R}$. The series $\sum_{n=1}^{\infty} \frac{1}{n^p}$ is called a p-series.

Theorem (p-series test).

$\sum_{n=1}^{\infty} \frac{1}{n^p}$ converges if and only if $p > 1$.

Corollary. The Harmonic series diverges.

Proof. Follows from p series test since $p = 1$.

Example. $\sum_{n=1}^{\infty} \frac{1}{n^2}$ converges by p series test since $p = 2 > 1$.

Remark. Can you believe that $\sum_{n=1}^{\infty} \frac{1}{n} = \infty$

and $\sum_{n=1}^{\infty} \frac{1}{n^2}$ is a real number (< 2).

Comparison Test: Let $\langle a_n \rangle$, $\langle b_n \rangle$ be sequences of nonnegative terms such that for each $n \in \mathbb{N}$, $0 \le a_n \le b_n$. Then

(i) If $\sum_{n=1}^{\infty} b_n$ converges then $\sum_{n=1}^{\infty} a_n$ converges.

(ii) If $\sum_{n=1}^{\infty} a_n$ diverges then $\sum_{n=1}^{\infty} b_n$ diverges.

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