Automata Theory
Assignment #1
(Sketch of Solution)
Due: March 17, 2008

1. (10 pts) Show that
$$S_1 \cup S_2 - (S_1 \cap \overline{S_2}) = S_2.$$

   Answer:
$$\begin{aligned} & S_1 \cup S_2 - (S_1 \cap \overline{S_2}) \\ = \; & (S_1 \cup S_2) \cap (\overline{S_1} \cup S_2) \\ = \; & ((S_1 \cup S_2) \cap \overline{S_1}) \cup (((S_1 \cup S_2) \cap S_2) \\ = \; & (\phi \cup S_2 \cap \overline{S_1}) \cup S_2 \\ = \; & (S_2 \cap \overline{S_1}) \cup S_2 \\ = \; & S_2 \end{aligned}$$

2. (10 pts) Prove that for all $n \geq 4$ the inequality $2^n < n!$ holds.

   Answer:
   When $n = 4$, $2^4 = 16 < 4! = 24$.
   Assume that when $n = k$ the inequality still holds, *i.e.*, $2^k < k!$
   We now consider the case of $n = k + 1$ as below.
$$\begin{aligned} 2^{k+1} & = 2^k * 2 \\ & < k! * 2 \text{ (using the inductive hypothesis)} \\ & < (k+1)! \text{ (since} k \geq 4). \end{aligned}$$

   According to the induction, the proof is done.

3. (20 pts) Prove or disprove the following statements.
   (a)The sum of a rational and an irrational number must be irrational.
   (b)The sum of two positive irrational numbers must be irrational.
   (c)The product of a rational and an irrational number must be irrational.

   Answer:

   (a) Let x be rational and y be irrational. Suppose that $x + y = z$ is rational. Then, $y = z - x$(rational) since the difference of two rational numbers is rational. So, y is rational which contradicts that y is irrational. Hence, z is irrational.
   (b) Let $x = 2 - \sqrt{2}$ and $y = \sqrt{2}$ be two irrational numbers. Then, $x + y = 2$ is not irrational according to the proposition. However, this contradicts to the fact that 2 is rational. So, the proposition is not true.
   (c) Consider that $x = 0$ and $y = \sqrt{2}$ where y is irrational. But, $x * y = 0$ is not irrational. Hence, the propositional is not true.

4. (10 pts) Are there languages for which $\overline{L^*} = (\overline{L})^*$.

   Answer: $\overline{L^*}$ contains λ but $(\overline{L})^*$ does not contain it. So, the answer is no.

5. (20 pts) Find grammars for $\sum$=$\{a, b\}$ that generate the sets of
   (a)all strings with exactly one a.
   (b)all strings with at least one a.
   (c)all strings with no more than one a.
   (d)all strings with at least three a's.
   In each case, give convincing arguments that the grammar you give does indeed generate the indicated language.

   Answer: (a) $S \to AaA,\ A \to bA|\lambda$
   (b) $S \to AaA,\ A \to aA|bA|\lambda$
   (c) $S \to A|AaA|AaAaA|AaAaAaA,\ A \to bA|\lambda$
   (d) $S \to AaAaAaA,\ A \to aA|bA|\lambda$

6. (10 pts) Give a verbal description of the language generated by

$$S \to aSb|bSa|a.$$

   Answer:
   The string in this language starts with a and ends with b, starts with b and ends with a, or is a. Besides, the amount of a's in a string is one more than amount of b's. If S does not produce a, then S produces with one a and one b at the same time.

7. (20 pts) Give a grammar that generates all real constants in C.

   Answer:
   $num \to (sign)(numbers)|(sign)(numbers).(numbers)$
   $sign = [+|-]$
   $numbers = [0|1|2|3|4|5|6|7|8|9]^*$