# Mathematics



# NOTES



Naveen Kumar

# LINEAR ALGEBRA

## Matrix :

A set of mn elements can be arranged in a rectangular array of m rows and n columns is known as an mxn matrix.



$$A = \begin{bmatrix} a_{11} & a_{12} & \cdots & a_{1n} \\ a_{21} & a_{22} & \cdots & a_{2n} \\ \vdots & & & \\ a_{m1} & a_{m2} & \cdots & a_{mn} \end{bmatrix}_{m\times n}$$

$m \to$ No. of rows.
$n \to$ No. of columns.

$$A = (a_{ij})_{m\times n} \quad 1 \le i \le m, \quad 1 < j \le n$$



1. Row Matrix -

Matrix having only one row.

eg. -

$$A = \begin{bmatrix} 2 & 3 & 4 \end{bmatrix}_{1\times 3}$$



2. Column Matrix -

Matrix having only one column.

eg. -

$$A = \begin{bmatrix} 4 \\ 2 \\ 1 \end{bmatrix}_{3\times 1}$$

3. Square Matrix -

No. of rows = No. of Columns.

Eg.-

$$A = \begin{bmatrix} 1 & 3 \\ 4 & 2 \end{bmatrix}_{2\times 2}$$

4. Rectangular Matrix-

No. of rows ≠ No. of columns

eg.-

$$A = \begin{bmatrix} 1 & 3 & 5 \\ 2 & 3 & 4 \end{bmatrix}_{2\times 3}$$

$$B = \begin{bmatrix} 1 & 2 & 3 & 4 \\ 2 & 3 & 4 & 5 \end{bmatrix}_{2\times 4}$$

5. Principle Diagonal-

In a square matrix, the digonal which consisting the elements $a_{11}, a_{22}, a_{33}$----- is known as principle diagonal.

eg. -

$$A = \begin{bmatrix} 1 & 5 & 7 \\ 2 & 4 & 6 \\ 1 & 2 & 3 \end{bmatrix}$$

Principle diagonal

6. Uppertringular Matrix -

A square matrix is said to be an upper tringular matrix if all the elements below the principle diagonal are zero.

eg.-

$$A = \begin{bmatrix} 1 & 2 & 3 \\ 0 & 2 & 4 \\ 0 & 0 & 5 \end{bmatrix}$$

7. Lowertringular Matrix-

In a squarematrix if all the elements above the principle diagonal are zero it is known as lower triangular matrix.

Eg-

$$A = \begin{bmatrix} 1 & 0 & 0 \\ 2 & 4 & 0 \\ 1 & 2 & 3 \end{bmatrix}$$

8- Diagonal matrix-

A square matrix is said to be diagonal matrix if it is both upper tringular and lower tringular.

Eg.-

$$A = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 2 & 0 \\ 0 & 0 & 4 \end{bmatrix}$$

$A = [a_{ij}]_{n \times n}$ is said to be diagonal matrix.

If $a_{ij} = 0 \quad i \neq j$

9. Identity matrix (or) Unit Matrix-

A diagonal matrix is said to be identity matrix if all its principal diagonal elements are unity i.e. equal to 1.

eg.-

$$I = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$$

$$I = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

Transpose of a Matrix :-

The transpose of A is denoted by $A^T$ and is obtained by interchanging rows and columns of A.

$$A = \begin{bmatrix} 1 & 2 & 3 \\ 8 & 7 & 9 \end{bmatrix}_{2 \times 3} \qquad A^T = \begin{bmatrix} 1 & 8 \\ 2 & 7 \\ 3 & 9 \end{bmatrix}$$

i- $(A^T)^T = A$

ii- $(A+B)^T = A^T + B^T$

iii- $(AB)^T = B^T A^T$

A square matrix 'A' is said to be-

→ Symmetric matrix if $A^T = A$

→ Skew-Symmetric matrix if $A^T = -A$

→ Orthogonal matrix if $AA^T = A^T A = I$

Conjugte of a Matrix-

The conjugate of a matrix A is denoted by $\bar{A}$ and is obtained by replacing the elements of A by its corresponding conjugates.

Eg.-

$$A = \begin{bmatrix} 1 & 2+i & 3 \\ 1-i & 3+2i & 4 \end{bmatrix}_{2\times3}$$

$$\bar{A} = \begin{bmatrix} 1 & 2-i & 3 \\ 1+i & 3-2i & 4 \end{bmatrix}_{2\times3}$$

Tranjugate of a Matrix-

The tranjugate of a complex matrix A is denoted by $A^\theta$ (or) $A^*$ and is obtained by taking transpose of conjugate of A (or) conjugate of transpose of A.

i.e-

$$A^\theta = \bar{A}^T = \overline{A^T}$$

$$A = \begin{bmatrix} 1 & 2+3i & 1-i \\ 3 & 1+2i & 2-i \end{bmatrix}$$

$$\bar{A} = \begin{bmatrix} 1 & 2-3i & 1+i \\ 3 & 1-2i & 2+i \end{bmatrix}$$

$$A' = \begin{bmatrix} 1 & 3 \\ 2+3i & 1+2i \\ 1-i & 2-i \end{bmatrix}$$

$$\bar{A}^T = \begin{bmatrix} 1 & 3 \\ 2-3i & 1-2i \\ 1+i & 2+i \end{bmatrix} \qquad \overline{A^T} = \begin{bmatrix} 1 & 3 \\ 2-3i & 1-2i \\ 1+i & 2+i \end{bmatrix}$$

- A square matrix A is said to be-

→ Hermitian matrix if $A^\theta = A$

→ Skew-Hermitian matrix if $A^\theta = -A$

→ Unitary matrix if $AA^\theta = A^\theta A = I$

$$A = \begin{bmatrix} 2 & 1+i \\ 1-i & 4 \end{bmatrix} \qquad A = \begin{bmatrix} 0 & 1-i \\ -1-i & 0 \end{bmatrix} \qquad A = \begin{bmatrix} \frac{1+i}{2} & \frac{-1+i}{2} \\ \frac{1+i}{2} & \frac{1-i}{2} \end{bmatrix}$$

Hermitian matrix     Skew-hermitian matrix     unitary Matrix

Idempotent matrix -

A square matrix 'A' is said to be idempotent if $A^2 = A$

eg. -

$$A = \begin{bmatrix} 2 & -2 & -4 \\ -1 & 3 & 4 \\ 1 & -2 & -3 \end{bmatrix}$$

Involutary matrix -

A square matrix 'A' is said to be Involutory if $A^2 = I$

eg. -

$$A = \begin{bmatrix} 4 & 3 & 3 \\ -1 & 0 & -1 \\ -4 & -4 & -3 \end{bmatrix}$$

Nil-potent matrix of Index m -

A square matrix 'A' is said to be nil potent matrix of index m if -

$A^m = 0$ $\quad A^{m-1} \neq 0$

m is a positive integer.

eg. -

$$A = \begin{bmatrix} 2 & 4 \\ -1 & -2 \end{bmatrix}$$

$$A = \begin{bmatrix} 2 & 4 \\ -1 & -2 \end{bmatrix} \begin{bmatrix} 2 & 4 \\ -1 & -2 \end{bmatrix}$$

$$= \begin{bmatrix} 4-4 & 8-8 \\ -2+2 & -4+4 \end{bmatrix}$$

$A \neq 0 \qquad A^2 = 0 = \begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix}$

A is nil-potent of index → 2

eg-

$$A = \begin{bmatrix} 1 & 1 & 3 \\ 5 & 2 & 6 \\ -2 & -1 & -3 \end{bmatrix}$$

$A \neq 0$
$A^2 \neq 0$
$A^3 = 0$

A is nil-potent of index → 3

Determinant Value of a Matrix-

If $A = \begin{bmatrix} \boxed{a \quad b} \\ c \quad d \end{bmatrix}$

$\det A = |A| = ad - bc$

The sum of the products of elements of a row (a column) with their corresponding cofactors is known as determinant value of the matrix.

| | |
|---|---|
| cofactor of a is d | ad |
| cofactor of b is -c | -bc |

$\det A = Ad - bC$

eg-

$$A = \begin{bmatrix} 1 & 1 & 1 \\ 2 & 3 & -1 \\ 1 & 3 & 2 \end{bmatrix}$$

$$\begin{aligned} \det A &= 1(6+3) - 1(4+1) + 1(6-3) \\ &= 1(9) - 1(5) + 1(3) \\ &= 9 - 5 + 3 \\ &= 7 \end{aligned}$$

method 2-

$$U = 3 - 3 + 4 = 4$$

$$\begin{array}{ccccc} 1 & 1 & 1 & 1 & 1 \\ 2 & 3 & -1 & 2 & 3 \\ 1 & 3 & 2 & 1 & 3 \end{array}$$

$$L = 6 - 1 + 6 = 11$$

$$\det A = L - U \Rightarrow 11 - 4 \Rightarrow 7$$

⊙

Singular matrix $\det A = 0$

Non Singular Matrix $\det A \neq 0$

⊙ PROPERTIES :

1. $\det A = \det A^T$

2. $\det I = 1$

3. $\det(AB) = (\det A)(\det B)$

4. $\det(A^{-1}) = \dfrac{1}{\det A}$

5. The determinant value of a tringular matrix or a diagonal matrix is equal to the product of its principle diagonal element.

Eg.-

1. $A = \begin{bmatrix} 1 & 2 \\ 3 & 4 \end{bmatrix}$ $\qquad A^T = \begin{bmatrix} 1 & 3 \\ 2 & 4 \end{bmatrix}$

$\det A = 4-6$
$= -2$

$\det A^T = 4-6$
$= -2$

2. $I = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$

$\det I = 1$

3. $A = \begin{bmatrix} 2 & 0 & 0 \\ 3 & 4 & 0 \\ 5 & 6 & 7 \end{bmatrix}$

$\det A = 56$

4. $A = \begin{bmatrix} 1 & 2 \\ 3 & 4 \end{bmatrix}$ $\qquad B = \begin{bmatrix} 1 & 2 \\ 4 & 1 \end{bmatrix}$

$AB = \begin{bmatrix} 1+8 & 2+2 \\ 3+16 & 6+4 \end{bmatrix}$

$= \begin{bmatrix} 9 & 4 \\ 19 & 10 \end{bmatrix}$

$\det(A) = -2$
$\det(B) = -7$
$\det(AB) = 14$
$(\det A)(\det B) = (-2)(-7)$
$= 14$

5. $A = \begin{bmatrix} 1 & 2 \\ 3 & 4 \end{bmatrix}$ $\qquad \text{Adj } A = \begin{bmatrix} 4 & -2 \\ -3 & 1 \end{bmatrix}$

$A = \begin{bmatrix} a & b \\ c & d \end{bmatrix}$ $\qquad \text{Adj} A = \begin{bmatrix} d & -b \\ -c & a \end{bmatrix}$

$\det A = 4-6 = -2$
$\det A = ad - bc$

$$A^{-1} = \frac{\text{Adj } A}{\det A} = \begin{bmatrix} \frac{4}{-2} & -\frac{2}{2} \\ \frac{-3}{-2} & \frac{1}{-2} \end{bmatrix} = \begin{bmatrix} -2 & 1 \\ \frac{3}{2} & -\frac{1}{2} \end{bmatrix}$$

$$\det A^{-1} = \frac{2}{2} - \frac{3}{2} = \underline{\underline{-\frac{1}{2}}}$$

$$\frac{1}{\det A} = \frac{1}{-2} = \underline{\underline{-\frac{1}{2}}}$$

6. If two rows (columns) of a matrix are identical then the determinant value of the matrix is zero.

Eg. -

$$A = \begin{bmatrix} 1 & 2 & 3 \\ 1 & 5 & 1 \\ 1 & 5 & 1 \end{bmatrix} \qquad \det A = 0$$

7. If each element of a row (column) in a matrix is zero then the value of the determinant is zero.

Eg. -

$$A = \begin{bmatrix} 1 & 3 & 2 \\ 2 & 3 & 1 \\ 0 & 0 & 0 \end{bmatrix} \qquad \det A = 0$$

8. If two rows (columns) of a matrix are interchanged then the numerical value of determinant remains same but changes by its sign.

Eg. -

$$A = \begin{bmatrix} 1 & 2 & 3 \\ 3 & 2 & 1 \\ 2 & 1 & 3 \end{bmatrix} \qquad B = \begin{bmatrix} 1 & 2 & 3 \\ 2 & 1 & 3 \\ 3 & 2 & 1 \end{bmatrix}$$

$$\det B = -\det A$$

$$\det A = 1(6-1) - 2(9-2) + 3(3-4)$$
$$= 5 - 14 - 3$$
$$= -12$$

$$\det B = 1(1-6) - 2(2-9) + 3(4-3)$$
$$= 1(-5) - 2(-7) + 3(1)$$
$$= -5 + 14 + 3$$
$$= 12$$

9. If $A$ is a square matrix of order $n \times n$ and $K$ is a Scalar-

$$|KA| = K^n |A|$$

10. $$A(\text{Adj}A) = |A| I$$

eg.- $A = \begin{bmatrix} a & b \\ c & d \end{bmatrix}$ $\quad \text{Adj}A = \begin{bmatrix} d & -b \\ -c & a \end{bmatrix}$

$$\det A = ad - bc$$

$$A(\text{Adj}A) = \begin{bmatrix} ad-bc & -ab+ab \\ cd-cd & -bc+ad \end{bmatrix}$$

$$= \begin{bmatrix} ad-bc & 0 \\ 0 & ad-bc \end{bmatrix}$$

$$= (ad-bc) \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$$

$$= (\det A)\, \underline{I}$$

## Inverse of a Matrix -

Let $A$ be a non-singular matrix of order $n$. If there exists an another non-singular matrix $B$ of order $n$ such that $AB = BA = I$.
then $B$ is called inverse of $A$.

i.e - $B = A^{-1}$

$\therefore \quad AA^{-1} = A^{-1}A = I$

1. $(A^{-1})^{-1} = A$
2. $(AB)^{-1} = B^{-1}A^{-1}$
3. $(ABC)^{-1} = C^{-1}B^{-1}A^{-1}$
4. $A^{-1} = \dfrac{\text{Adj } A}{\det A}$

## Adjoint of a Matrix -

Let $A$ be a Square matrix of order $n$ the transpose of cofactor matrix of $A$ is known as Adjoint of $A$.

The elements of $A$

The cofactor matrix of $A$ is given by replacing the elements of $A$ by its corresponding co-factors.

Eg -

$$A = \begin{bmatrix} 1 & 1 & 1 \\ 1 & 3 & 2 \\ 2 & 1 & 3 \end{bmatrix}$$

$$\begin{array}{c|cccc} 1 & 1 & 1 & 1 & 1 \\ 1 & 3 & 2 & 1 & 3 \\ 2 & 1 & 3 & 2 & 1 \\ 1 & 1 & 1 & 1 & 1 \\ 1 & 3 & 2 & 1 & 3 \end{array}$$

$$\text{cofactor matrix} = \begin{bmatrix} 7 & 1 & -5 \\ -2 & 1 & 1 \\ -1 & -1 & 2 \end{bmatrix}$$

$$\text{Adj } A = \begin{bmatrix} 7 & -2 & -1 \\ 1 & 1 & -1 \\ -5 & 1 & 2 \end{bmatrix}$$

$$\det A = 1(9-2) - 1(3-4) + 1(1-6)$$
$$= 1(7) - 1(-1) + 1(-5)$$
$$= 7 + 1 - 5$$
$$= 3$$

$$A^{-1} = \frac{\text{Adj } A}{\det A} = \frac{1}{3}\begin{bmatrix} 7 & -2 & -1 \\ 1 & 1 & -1 \\ -5 & 1 & 2 \end{bmatrix}$$

Eg. -

$$A = \begin{bmatrix} 1 & 2 & 1 \\ 3 & 1 & 1 \\ 1 & 3 & 1 \end{bmatrix}$$

$$= \left[\begin{array}{c|cccc} 1 & 2 & 1 & 1 & 2 \\ \hline 3 & 1 & 1 & 3 & 1 \\ 1 & 3 & 1 & 1 & 3 \\ 1 & 2 & 1 & 1 & 2 \\ 3 & 1 & 1 & 3 & 1 \end{array}\right]$$

$$\text{Cofactor matrix} = \begin{bmatrix} -2 & -2 & 8 \\ 1 & 0 & -1 \\ 1 & 2 & -5 \end{bmatrix}$$

$$\text{Adj } A = \begin{bmatrix} -2 & 1 & 1 \\ -2 & 0 & 2 \\ 8 & -1 & -5 \end{bmatrix}$$

RANK OF A MATRIX –

The order of the highest ordered non-vanishing minor is called rank of the matrix.

minor of the matrix –

The determinant value of a square submatrix is called minor of a matrix.

eg. –

$$A = \begin{array}{c} \begin{bmatrix} 1 & 1 & 1 & 2 & 5 & 1 \\ 2 & 3 & 5 & 3 & 2 & 1 \\ 1 & 2 & 1 & 4 & 3 & 1 \end{bmatrix}_{3\times 6} \\ \begin{matrix} C_1 & C_2 & C_3 & \phantom{4} & \phantom{3} & \phantom{1} \end{matrix} \end{array}$$

3x3

$$\begin{vmatrix} 1 & 1 & 1 \\ 2 & 3 & 5 \\ 1 & 2 & 1 \end{vmatrix} \qquad \begin{vmatrix} 1 & 1 & 2 \\ 3 & 5 & 3 \\ 2 & 1 & 4 \end{vmatrix}$$

$$\downarrow$$

$$1(3-10) - 1(2-5) + 1(4-3)$$

$$-7 + 3 + 1 = -3 \neq 0$$

Rank of $A = 3$

1. If $A$ is a non-singular matrix of order $n$ then rank of $A$ is $n$.
2. If $A$ is a singular matrix of order $n$ then rank of $A$ is less than $n$.
3. The rank ~~k~~ of an Identity matrix is its order.
4. The rank of a non-zero matrix is always non-zero.
5. The rank of a zero matrix is always zero.

## System of Solutions of equations -

Homgeniaus

$x + 2y = 0$

$2x + y = 0$

$$\begin{bmatrix} 1 & 2 \\ 2 & 1 \end{bmatrix} \begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$$

$AX = 0$

Non homogeneous

$x + 2y = 3$

$2x + y = 3$

$$\begin{bmatrix} 1 & 2 \\ 2 & 1 \end{bmatrix} \begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} 3 \\ 3 \end{bmatrix}$$

$AX = B$

### ⊙ Homogeneous System -

1. Reduce the matrix A into Echelon form which gives rank of A.
2. Let rank of A be $r$.
   no. of variables $n$.

If $r = n$ then the system $AX = 0$ passes zero solution. [trivial Solution].

If $r < n$ then the system $AX = 0$ passes non-zero solution. [Non-trivial Solution].

Eg. -

$2x + 4y = 0$

$3x + 6y = 0$

$$\begin{bmatrix} 2 & 4 \\ 3 & 6 \end{bmatrix} \begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$$

$$A = \begin{bmatrix} 2 & 4 \\ 3 & 6 \end{bmatrix}$$

$AX = 0$

$r < n$

non-zero solution exists.

The eq$^n$ is $2x + 4y = 0$

$$A = \begin{bmatrix} 2 & 4 \\ 3 & 6 \end{bmatrix}$$

$2R_2 - 3R_1$ :

$$\begin{bmatrix} 2 & 4 \\ 0 & 0 \end{bmatrix}$$

rank of $A = 1$

no. of variables $= 2$

$y = K$ be independent

$2x = -4y$

$$x = \frac{-4K}{2} = -2K$$

$x = -2K$ $\quad$ $y = K$

$K = 1$ $\quad$ $x = -2$ ; $y = 1$

$K = 2$ $\quad$ $x = -4$ ; $y = 2$

Eg. -

$x + 2y = 0$

$2x + 3y = 0$

$$\begin{bmatrix} 1 & 2 \\ 2 & 3 \end{bmatrix} \begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$$

$R_2 - 2R_1$

$$\begin{bmatrix} 1 & 2 \\ 0 & -1 \end{bmatrix} \begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$$

$r = 2$ (rank)

$n = 2$

if $r = n$

zero solution exist.

$x = 0$ ; $y = 0$

NOTE- $|A| = 0$ then $AX = 0$ posses non-zero $Sol^n$

$|A| \neq 0$ then $AX = 0$ posses zero Solution.

## Non- Homogenous System-

$$AX = B \qquad B \neq 0$$

1. Consider the Augmented matrix $[A\ B]$
2. Reduce the augmented matrix into Echelon form which gives rank of $A$ & rank of $[A\ B]$.

Let rank of $A = r$

rank of $[AB] = R$

no. of variables $= n$

If $r \neq R$ then $AX = B$ has no Solution.

If $r = R = n$ then $AX = B$ possesses unique solution.

If $r = R < n$ then $AX = B$ posses infinitely many $sol^n$.

Ques- Solve

$$x - y + 3z = 4$$
$$x + z = 2$$
$$x + y - z = 0$$

$Sol^n$-

$$\begin{bmatrix} 1 & -1 & 3 \\ 1 & 0 & 1 \\ 1 & 1 & -1 \end{bmatrix} \begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} 4 \\ 2 \\ 0 \end{bmatrix}$$

$AX = B \rightarrow$

$$\text{Augmented matrix } [AB] = \left[\begin{array}{ccc|c} 1 & -1 & 3 & 4 \\ 1 & 0 & 1 & 2 \\ 1 & 1 & -1 & 0 \end{array}\right]$$

$R_2 - R_1,\ R_3 - R_1$

$$= \begin{bmatrix} 1 & -1 & 3 & 4 \\ 0 & 1 & -2 & -2 \\ 0 & 2 & -4 & -4 \end{bmatrix}$$

$R_3 - 2R_2 \rightarrow$
$$\left[\begin{array}{ccc|c} 1 & -1 & 3 & 4 \\ 0 & 1 & -2 & -2 \\ 0 & 0 & 0 & 0 \end{array}\right]$$

rank of A $r = 2$
rank of [AB] $R = 2$
no. of variables $n = 3$
$r = R < n$

$Ax = B$ possesses infinitly many solutions.

$$x - y + 3z = 4 \quad — (1)$$
$$y - 2z = -2 \quad — (2)$$

let $z = K$ be independent :
$$y = -2 + 2z = -2 + 2K$$
$$x = 4 + y - 3z = 4 - 2 + 2K - 3K$$
$$= 2 - K$$

$x = 2 - K$ $\quad$ $y = -2 + 2K$ $\quad$ $z = K$

Ques- Rank of $\rightarrow$ $A = \begin{pmatrix} 1 & 1 & 1 \\ 1 & -1 & 0 \\ 1 & 1 & 1 \end{pmatrix}$ is-

a. 0 $\quad$ c. 2
b. 1 $\quad$ d. 3

Sol$^n$- (c).

two rows are common so $r \neq 3$.
$$\begin{vmatrix} 1 & 1 \\ 1 & -1 \end{vmatrix} = -1 - 1 = -2 \neq 0$$

$\Rightarrow 2 \times 2$
$\rho(A) = 2$

Ques- Rank of A →

$$A = \begin{bmatrix} 1 & 2 & 3 & 0 \\ 2 & 4 & 3 & 2 \\ 3 & 2 & 1 & 3 \\ 6 & 8 & 7 & 5 \end{bmatrix}$$

a. 5 c. 3

b. 4 d. 2

Soln- $R_2 - 2R_1$ ; $R_3 - 3R_1$ ; $R_4 - 6R_1$

$$\Rightarrow \begin{bmatrix} 1 & 2 & 3 & 0 \\ 0 & 0 & -3 & 2 \\ 0 & -4 & -8 & 3 \\ 0 & -4 & -11 & 5 \end{bmatrix}$$

$R_2 \longleftrightarrow R_3$

$$\Rightarrow \begin{bmatrix} 1 & 2 & 3 & 0 \\ 0 & -4 & -8 & 3 \\ 0 & 0 & -3 & 2 \\ 0 & -4 & -11 & 5 \end{bmatrix}$$

$R_4 - R_2$

$$\Rightarrow \begin{bmatrix} 1 & 2 & 3 & 0 \\ 0 & -4 & -8 & 3 \\ 0 & 0 & -3 & 2 \\ 0 & 0 & -3 & 2 \end{bmatrix}$$

$R_4 - R_3$

$$\Rightarrow \begin{bmatrix} 1 & 2 & 3 & 0 \\ 0 & -4 & -8 & 3 \\ 0 & 0 & -3 & 2 \\ 0 & 0 & 0 & 0 \end{bmatrix}$$

ANS- (C)

Ques- The rank of $A = \begin{bmatrix} 2 & -1 & 3 \\ 4 & 7 & \lambda \\ 1 & 4 & 5 \end{bmatrix}$ is 2 then $\lambda = ?$

a. $-3$ c. $13$

b. $3$ d. $-13$

Solⁿ-

$$2(35-4\lambda) + 1(20-\lambda) + 3(16-7) = 0$$
$$70 - 8\lambda + 20 - \lambda + 27 = 0$$
$$-9\lambda + 117 = 0$$
$$9\lambda = 117$$
$$\lambda = 13$$

ANS- (C)

Ques- the adjoint of $A = \begin{bmatrix} -1 & -2 & -2 \\ 2 & 1 & -2 \\ 2 & -2 & 1 \end{bmatrix}$ is-

a. $A$ c. $3A^T$

b. $A^T$ d. $-3A^T$

Solⁿ- cofactor →

$$\begin{array}{r|rrrr} -1 & -2 & -2 & -1 & -2 \\ \hline 2 & 1 & -2 & 2 & 1 \\ 2 & -2 & 1 & 2 & -2 \\ -1 & -2 & -2 & -1 & -2 \\ 2 & 1 & -2 & 2 & 1 \end{array}$$

cofactor matrix →

$$\begin{bmatrix} -3 & -6 & -6 \\ 6 & 3 & -6 \\ 6 & -6 & 3 \end{bmatrix}$$

$$\text{adj } A = \begin{bmatrix} -3 & 6 & 6 \\ -6 & 3 & -6 \\ -6 & -6 & 3 \end{bmatrix} = 3\begin{bmatrix} -1 & 2 & 2 \\ -2 & 1 & -2 \\ -2 & -2 & 1 \end{bmatrix} \Rightarrow 3A^T$$

ANS- (C).

Ques- If $A_{3\times3}$ matrix and $|A| = 2$
then $A\,(\text{Adj}\,A) =$

a. $2A$ c. $-2I$

b. $2A^T$ d. $2I$

Sol$^n$- $A\,(\text{Adj}\,A) = |A|\,I$

$A\,(\text{Adj}\,A) = 2I$

ans- (d)

Ques- If $A$ is $3\times3$ matrix & $\det A = 7$ then $\det(2A)^{-1} =$ ---

a. $1/42$ c. $1/56$

b. $\frac{1}{48}$ d. $\frac{7}{2}$

Sol$^n$-
$$\det A^{-1} = \frac{1}{\det A}$$

$$|KA| = K^n\,|A|$$

$$\det(2A)^{-1} = \frac{1}{|2A|} \Rightarrow \frac{1}{2^3\,|A|} \Rightarrow \frac{1}{8\cdot 7} = \frac{1}{56}$$

Ans- (c)

Ques- If $A \rightarrow$

$$A = \begin{bmatrix} 0 & 1 & 0 & 0 \\ 0 & 2 & 0 & 1 \\ 0 & 3 & 2 & 0 \\ 1 & 4 & 1 & 0 \end{bmatrix}_{4\times4}$$

then $\det(A^{-1}) =$

a. $2$ c. $-2$

b. $1/2$ d. $-1/2$

Sol$^n$-

$$\det A = -1\,[\,+1\,(0-2)\,] \Rightarrow 2$$

$$\det A^{-1} = \frac{1}{2}$$

Ans- (b)

Ques- If $A = \begin{bmatrix} 2x & 0 \\ x & x \end{bmatrix}$ & $A^{-1} = \begin{bmatrix} 1 & 0 \\ -1 & 2 \end{bmatrix}$ then $x = ?$

a. 1  c. $\frac{1}{2}$
b. 2  d. $\frac{1}{4}$

Soln- $AA^{-1} = I$

$$\begin{bmatrix} 2x & 0 \\ 0 & 2x \end{bmatrix} = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$$

$2x = 1$

$x = \frac{1}{2}$

Ans→ (C)

Ques- If the product of

$$A = \begin{bmatrix} \cos^2\theta & \cos\theta\sin\theta \\ \cos\theta\sin\theta & \sin^2\theta \end{bmatrix} \quad B = \begin{bmatrix} \cos^2\phi & \cos\phi\sin\phi \\ \cos\phi\sin\phi & \sin^2\phi \end{bmatrix}$$

is null matrix then $\theta$ & $\phi$ differ by-

a. even multiples of $\pi/2$.
b. " " " $\pi$
c. odd multiples " $\pi/2$
d. " " " $\pi$

Soln- $AB = 0$

$\theta = 0 \quad \phi = \frac{\pi}{2}$

$$\begin{array}{cc} A & B \\ \downarrow & \downarrow \end{array}$$
$$\begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix} \begin{bmatrix} 0 & 0 \\ 0 & 1 \end{bmatrix} = \begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix}$$

$\theta = \frac{\pi}{4} \quad \phi = \frac{3\pi}{4}$

$$\begin{bmatrix} \frac{1}{2} & \frac{1}{2} \\ \frac{1}{2} & \frac{1}{2} \end{bmatrix} \begin{bmatrix} \frac{1}{2} & \frac{-1}{2} \\ \frac{-1}{2} & \frac{1}{2} \end{bmatrix} = \begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix}$$

$\theta = 0 \quad \phi = \pi/2$
$\theta = \pi/2 \quad \phi = 3\pi/4$

ANS- (C)

Ques- The system of eqⁿ has

$$x + 2y + 3z = 1$$
$$4x + 5y + 9z = 4$$
$$2x + y + 3z = 2$$

a. a unique Solⁿ  c. an infinitly many solution.
b. no solution  d. none of the above.

Solⁿ-

$$[A\ B] = \begin{bmatrix} 1 & 2 & 3 & 1 \\ 4 & 5 & 9 & 4 \\ 2 & 1 & 3 & 2 \end{bmatrix}$$

$R_2 - 4R_1$, $R_3 - 2R_1 \rightarrow$

$$= \begin{bmatrix} 1 & 2 & 3 & 1 \\ 0 & -3 & -3 & 0 \\ 0 & -3 & 3 & 0 \end{bmatrix}$$

$R_3 - R_2$

$$= \begin{bmatrix} 1 & 2 & 3 & 1 \\ 0 & -3 & -3 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix}$$

rank of $A = 2$
$(AB) = 2$
$n = 3$
$r = R < n$
infinitly many.

ANS → (c)

Ques- If the system of eqⁿs $x + 2y - 3z = 1$
$(2p+1)y + z = 2$
$(p+2)z = 3$

is inconsistant then $p =$ ___

a. 2  c. −2
b. 0  d. $-\frac{1}{2}$

Sol$^n$ - (c)

$$[A\ B] = \begin{bmatrix} 1 & 2 & -3 & 1 \\ 0 & 2p+1 & 1 & 2 \\ 0 & 0 & p+2 & 3 \end{bmatrix}$$

$p+2 \neq 0$ then system will be consistant.
$p+2 = 0$ then system will be inconsistant.
$\hookrightarrow p = -2$

Ques- the existance of System -

$$2x + 3y - z = 6$$
$$x + y + z = \mu$$
$$5x - y + \lambda z = 3$$

depends on -

a. only $\mu$
b. only $\lambda$
c. both $\lambda$ & $\mu$
d. None of these.

Sol$^n$-

$$[AB] = \begin{bmatrix} 2 & 3 & -1 & 6 \\ 1 & 1 & 1 & \mu \\ 5 & -1 & \lambda & 3 \end{bmatrix}$$

$2R_2 - R_1$ ; $2R_3 - 5R_1 \rightarrow$

$$= \begin{bmatrix} 2 & 3 & -1 & 6 \\ 0 & -1 & 3 & 2\mu-6 \\ 0 & -17 & 2\lambda+5 & -24 \end{bmatrix}$$

$R_3 - 17R_2$

$$= \left[\begin{array}{ccc|c} 2 & 3 & -1 & 6 \\ 0 & -1 & 3 & 2\mu-6 \\ 0 & 0 & 2\lambda-46 & -34\mu+78 \end{array}\right]$$

$2\lambda - 46 \neq 0$ then $AX = B$ possesses a Solution.

Ans- (b)

Eigen values & Eigen vectors-

Definition-

Let $A$ be a square matrix of order $n \times n$. $\lambda$ be a scalar.

$A - \lambda I$ is known as charecterstic matrix.

$|A - \lambda I| = 0$ is known as charecterstic equation.

The roots of charecterstic $eq^n$ are called Eigen values.

To each eigen value $\lambda$ there exist a non-zero vector $X$ such that $(A - \lambda I)X = 0$

then $X$ is Eigen vector of matrix $A$ corresponding to eigen value $\lambda$.

Ques- $A = \begin{bmatrix} 4 & 1 \\ 1 & 4 \end{bmatrix}$

$$A - \lambda I = \begin{bmatrix} 4 & 1 \\ 1 & 4 \end{bmatrix} - \begin{bmatrix} \lambda & 0 \\ 0 & \lambda \end{bmatrix} = \begin{bmatrix} 4-\lambda & 1 \\ 1 & 4-\lambda \end{bmatrix}$$

chrecterstic $eq^n$ is $|A - \lambda I| = 0$

$$(4-\lambda)(4-\lambda) - 1 = 0$$

$$16 - 4\lambda - 4\lambda + \lambda^2 - 1 = 0$$

$$\lambda^2 - 8\lambda + 15 = 0$$

$$(\lambda-5)(\lambda-3) = 0$$

$$\lambda = 5, 3$$

Eigen values are 5 & 3.

$\lambda = 5$

consider $(A - 5I)X = 0$

$$\begin{bmatrix} -1 & 1 \\ 1 & -1 \end{bmatrix}_{2\times2} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix}_{2\times1} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}_{2\times1}$$

$R_2 + R_1$

$$\begin{bmatrix} -1 & 1 \\ 0 & 0 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$$

$$-x_1 + x_2 = 0$$

$$-x_1 = -x_2$$
$$x_1 = x_2$$

$x_2 = K \qquad x_1 = K$

If $\quad K=1 \qquad x_1 = 1$

$\qquad\qquad\qquad\qquad x_2 = 1$

$x = \begin{bmatrix} 1 \\ 1 \end{bmatrix}$ is eigen vector of A corresponding $\lambda = 5$

⊙ Properties –

1. The sum of Eigen values of a matrix is equal to the sum of principal diagonal elements.
2. The product of eigen values of a matrix is equal to the determinant value of the matrix.

eg. –
$A = \begin{pmatrix} 4 & 1 \\ 1 & 4 \end{pmatrix}$ eigen values are 5, 3.

→ Sum of eigen values $= 5+3 \Rightarrow 8$

Sum of principle diagonal elements $= 4+4 = 8$

→ Product of eigen values $= (5)(3) = 15$

$\det A = 16-1 = 15$

3. the eigen values of A and $A^T$ are same.

4. the eigen values of a tringular matrix (or) a diagonal matrix are it's principle diagonal elements.

Eg-

$$A = \begin{bmatrix} 2 & 0 & 0 \\ 3 & 5 & 0 \\ 7 & 8 & 9 \end{bmatrix}$$

$$A - \lambda I = \begin{bmatrix} 2-\lambda & 0 & 0 \\ 3 & 5-\lambda & 0 \\ 7 & 8 & 9-\lambda \end{bmatrix}$$

$$|A - \lambda I| = 0$$

$$(2-\lambda)(5-\lambda)(9-\lambda) = 0$$

$$\lambda = 2, 5, 9 \text{ (eigen values).}$$

5. If $\lambda$ is eigen value of A then $\frac{1}{\lambda}$ is eigen value of $A^{-1}$.

6. If $\lambda$ is eigen value of A then $\frac{|A|}{\lambda}$ is eigen value of Adj A.

$$AX = \lambda X \longrightarrow (1)$$

$$A^{-1} AX = A^{-1} \lambda X$$

$$IX = \lambda A^{-1} X$$

$$\frac{1}{\lambda} X = A^{-1} X$$

$$A^{-1} = \frac{Adj A}{|A|}$$

$$\frac{1}{\lambda} X = A^{-1} X$$

$$\frac{1}{\lambda} X = \frac{Adj A}{|A|} X$$

multiply by $|A|$ -

$$\frac{|A|}{\lambda} X = (Adj A) X$$

7. If $\lambda_1, \lambda_2, ---- \lambda_n$ are Eigen values of A then-
$\lambda_1^2, \lambda_2^2, ---- \lambda_n^2$ are eigen values of $A^2$.

8. If $\lambda_1, \lambda_2, ----- \lambda_n$ are Eigen values of A then-
$\lambda_1^m, \lambda_2^m, ---- \lambda_n^m$ are Eigen values of $A^m$.
m is a +ve integer.

$$Ax = \lambda x \longrightarrow (1)$$

$$A^2X = A\lambda X$$
$$= \lambda \underbrace{AX}$$
$$= \lambda\lambda X \Rightarrow \lambda^2 X \quad \text{from (1)}$$
$$A^2X = \lambda^2 X$$

9. If $\lambda_1, \lambda_2, \ldots, \lambda_n$ are Eigen values of A then $K\lambda_1, K\lambda_2, \ldots K\lambda_n$ are Eigen values of KA where K is a scaler.

10. If $\lambda_1, \lambda_2, \ldots \lambda_n$ are eigen values of A then $\lambda_1+K, \lambda_2+K, \ldots \lambda_n+K$ are eigen values of $A+KI$.

11. If $\lambda_1, \lambda_2, \ldots \lambda_n$ are eigen values of A then $\lambda_1-K, \lambda_2-K, \ldots \lambda_n-K$ are eigen values of $A-KI$.

12. The Eigen values of a hermitian matrix are always real.

13. The eign values of a Skew-hermitian matrix are either zero (or) purely imaginary.

$$A = \begin{bmatrix} 0 & 1-i \\ -1-i & 0 \end{bmatrix}$$

$$\bar{A} = \begin{bmatrix} 0 & 1+i \\ -1+i & 0 \end{bmatrix}$$

$$A^{\theta} = \begin{bmatrix} 0 & -1+i \\ 1+i & 0 \end{bmatrix} \Rightarrow -\begin{bmatrix} 0 & 1-i \\ -1-i & 0 \end{bmatrix}$$
$$= -A$$

$$A-\lambda I = \begin{bmatrix} -\lambda & 1-i \\ -1-i & -\lambda \end{bmatrix}$$

Charecteristic eqn is $|A-\lambda I| = 0$

$$\lambda^2 - (-1-i)(1-i) = 0$$
$$\lambda^2 - (-1+i-i+i^2) = 0$$
$$\lambda^2 - (-1-1) = 0$$
$$\lambda^2 + 2 = 0$$
$$\lambda^2 = -2$$
$$\lambda = \pm\sqrt{2}\, i$$
$$\lambda = \sqrt{2}\, i , -\sqrt{2}\, i$$

$$A = \begin{bmatrix} 0 & 4 \\ -4 & 0 \end{bmatrix}$$

$$A-\lambda I = \begin{bmatrix} -\lambda & 4 \\ -4 & -\lambda \end{bmatrix}$$

$$|(A-\lambda I)| = 0$$
$$\lambda^2 + 16 = 0$$
$$\lambda = \pm 4i$$

14. The eign values of Unitary matrix have absolute value: 1.

15. $\lambda$ is eign value of a non-singular matrix if and only if $\lambda \neq 0$

16. If A is a singular matrix then at least one of the eigen values of A is zero.

⊙ CAYLEY- Hamilton Thearem-

Every square matrix satisfies its charecterstic equation.

$$A = \begin{bmatrix} 3 & 4 \\ -4 & -5 \end{bmatrix}$$

Characterstic matrix $A-\lambda I = \begin{bmatrix} 3-\lambda & 4 \\ -4 & -5-\lambda \end{bmatrix}$

every square matrix satisfies its charecterstic equatron.

Ch. $eq^n$ is $|A-\lambda I| = 0$

$$(3-\lambda)(-5-\lambda)+16 = 0$$

$$-15-3\lambda+5\lambda+\lambda^2+16 = 0$$

$$\lambda^2+2\lambda+1 = 0$$

$$A^2+2A+I = 0$$

Ques- The eign values of $A = \begin{bmatrix} 4 & -2 \\ -2 & 1 \end{bmatrix}$ are -

a. 1 & 2 b. 0 & 5 c. 1 & 4 d. None.

Solⁿ- Ans- (b)

Ques- Eign values of $A = \begin{bmatrix} 1 & -6 & -4 \\ 0 & 4 & 2 \\ 0 & -6 & -3 \end{bmatrix}$ are-

a. 0, 1, 2

b. 0, 1, -1

c. 0, 1, 1

d. 1, 2, 3

Ans - (c)

Ques- Eign values of $\begin{bmatrix} 1 & 2 & 3 \\ 0 & 2 & 3 \\ 0 & 0 & 2 \end{bmatrix}$ are-

a. 1, 2, 3 c. 1, 2, 2

b. 1, 2, 1 d. 1, 0, 0

Ans- (c)

Ques- Let P and Q be square matrix such that $PQ = I$ then zero is an eign value of -

a. P but not Q.

b. Q but not P.

c. both P & Q.

d. Neither P nor Q.

Soln- Ans- (d)

$$PQ = I$$

$$|P| \neq 0 \qquad |Q| \neq 0$$

$$AA^T = A^T A = I$$

Ques- If 3 & 6 are Eigen values of A then Eigen values $A^T$ are-

a. $\frac{1}{3}, \frac{1}{6}$ c. 9, 36

b. 3, 6 d. 2, 3

Soln- Ans- (b)

eigen values of A = eigen values of $A^T$

Ques- If 1, -1, 0 are eigen values of A then $\det[I + A^{100}]$ is-

a. 4 c. 8

b. 6 d. 0

Soln- $\lambda_1, \lambda_2, \text{----} A$ then $\lambda_1^m, \lambda_2^m, \lambda_3^m \text{----} A^m$

$\lambda_1, \lambda_2, \text{-----} A$ then $\lambda_1 + K, \lambda_2 + K, \text{-----} A + KI$

$1, -1, 0 \text{------} A$

$(1)^{100}, (-1)^{100}, 0^{100} \text{......} A^{100}$

$1, 1, 0 \text{----} A^{100}$ then $1+1, 1+1, 0+1$, are eign values of $A^{100} + I$.

i.e - 2, 2, 1 ---------- $A^{100} + I$

det $(A^{100} + I) = (2)(2)(1) = \underline{\underline{4}}$

Ques- If 1, 3, 4 are Eigen values of A then Eigen values of Adj A are-

a. 1, 3, 4
b. $1, \frac{1}{3}, \frac{1}{4}$
c. 12, 4, 3
d. -1, -3, -4

Sol$^n$-

$\lambda_1, \lambda_2, \lambda_3$ ---- A

$\frac{|A|}{\lambda_1}, \frac{|A|}{\lambda_2}, \frac{|A|}{\lambda_3}$ ---- Adj A

$\frac{12}{1}, \frac{12}{3}, \frac{12}{4}$

ans - (c)

Ques- Eigen vectors of the matrix $\begin{bmatrix} 1 & -2 \\ -5 & 4 \end{bmatrix}$ are-

a. $[2 \quad -5]^T, [1 \quad 1]^T$
b. $[2 \quad 5]^T, [1 \quad 1]^T$
c. $[2 \quad -5]^T, [1 \quad -1]^T$
d. $[1 \quad 2]^T, [1 \quad 5]^T$

Sol$^n$-

Ch. eq$^n$ is: $|A - \lambda I| = 0$

$(1-\lambda)(4-\lambda) - 10 = 0$

$4 - \lambda - 4\lambda + \lambda^2 - 10 = 0$

$\lambda = 6, -1$

$\lambda = 6$

$(A - \lambda I)X = 0$

$\begin{bmatrix} -5 & -2 \\ -5 & -2 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$

$-5x_1 - 2x_2 = 0$

$\frac{x_1}{2} = \frac{x_2}{-5}$

$\begin{bmatrix} 2 \\ -5 \end{bmatrix}$

$\lambda = -1$

$$\begin{bmatrix} 2 & -2 \\ -5 & 5 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$$

$$2x_1 - 2x_2 = 0$$
$$2x_1 = 2x_2$$
$$\frac{x_1}{1} = \frac{x_2}{1} \qquad \begin{bmatrix} 1 \\ 1 \end{bmatrix}$$

Ans- (a)

Method-2:

$$AX = \lambda X$$

$$\underset{A}{\begin{bmatrix} 1 & -2 \\ -5 & 4 \end{bmatrix}} \underset{X}{\begin{bmatrix} 2 \\ -5 \end{bmatrix}} = \begin{bmatrix} 2+10 \\ -10-20 \end{bmatrix}$$

$$= \begin{bmatrix} 12 \\ -30 \end{bmatrix}$$

$$= 6\begin{bmatrix} 2 \\ -5 \end{bmatrix} = \lambda X$$

Ques- For the matrix $P = \begin{bmatrix} 3 & -2 & 2 \\ 0 & -2 & 1 \\ 0 & 0 & 1 \end{bmatrix}$

one of the eign value is -2.
which one of the following is an eign vector.

a. $\begin{bmatrix} 3 \\ -2 \\ 1 \end{bmatrix}$ b. $\begin{bmatrix} -3 \\ 2 \\ -1 \end{bmatrix}$ c. $\begin{bmatrix} 1 \\ -2 \\ 3 \end{bmatrix}$ d. $\begin{bmatrix} 2 \\ 5 \\ 0 \end{bmatrix}$

Sol$^n$- Ans (d)

$\lambda = -2$ $\qquad (P - \lambda I)X = 0$

$$\begin{bmatrix} 3-\lambda & -2 & 2 \\ 0 & -2-\lambda & 1 \\ 0 & 0 & 1-\lambda \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$$

Since $\lambda = -2$

$$\begin{array}{ccc} 3 & 4 & 2 \\ \downarrow & \downarrow & \downarrow \end{array}$$

$$\begin{bmatrix} 5 & -2 & 2 \\ 0 & 0 & 1 \\ 0 & 0 & 3 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$$

1 & 2 →

$$\begin{array}{cccc} -2 & 2 & 5 & -2 \\ 0 & 1 & 0 & 0 \end{array}$$

$$\frac{x_1}{-2} = \frac{x_2}{-5} = \frac{x_3}{0}$$

$$\frac{x_1}{2} = \frac{x_2}{5} = \frac{x_3}{0}$$

1 & 3 →

$$\begin{array}{cccc} -2 & 2 & 5 & -2 \\ 0 & 3 & 0 & 0 \end{array}$$

$$\frac{x_1}{-6} = \frac{x_2}{-15} = \frac{x}{30}$$

$$\frac{x_1}{2} = \frac{x_2}{5} = \frac{x_3}{0}$$

Ques- Eign values of A are 1 and 3 then $A^3 =$

a. $12 \times A - 13I$

b. $13A - 12I$

c. $12(A-I)$

d. $13(A-I)$

Sol$^n$- Eigen values of A are 1 & 3 →

Ch. eq$^n$ is :

$$(\lambda-1)(\lambda-3) = 0$$

$$\lambda^2 - 4\lambda + 3 = 0$$

By. C-H. Theorem:

$$A^2 - 4A + 3I = 0 \quad -(1)$$

$$A^2 = 4A - 3I$$

multip. with A:

$$A^3 - 4A^2 + 3A = 0$$

$$A^3 = 4A^2 - 3A$$

$= 4(4A-3I) - 3A$

$= 16A - 12I - 3A$

$= 13A - 12I$

Ques- Let the char' of A be $\lambda^2 - \lambda - 1 = 0$ then

a. $A^{-1}$ does not exist

b. $A^{-1}$ exists but can not be determined from the data.

c- $A^{-1} = A + I$

d- $A^{-1} = A - I$

Solⁿ- ans- (d)

$\lambda^2 - \lambda - 1 = 0$

$$\lambda = \frac{1 \pm \sqrt{1+4}}{2} = \frac{1 \pm \sqrt{5}}{2}$$

C-H Theorem:

$|A| \neq 0$

$A^2 - A - I = 0$

by $A^{-1} \Rightarrow$

$A^2 A^{-1} - AA^{-1} - IA^{-1} = 0$

$A - I = A^{-1}$

$A - I = A^{-1}$

# CALCULUS

## Limit –

A function $f(x)$ is said to have a point $x = a$ if : $\underset{x\to a}{Lt}\ f(x)$ exists.

i.e. $\underset{x\to a^-}{Lt}\ f(x) = \underset{x\to a^+}{Lt}\ f(x)$

## L' Hospital :

$$\underset{x\to a}{Lt}\ \frac{f(x)}{g(x)} \quad \text{where } f(a) = g(a) = 0$$

$$= \underset{x\to a}{Lt}\ \frac{f'(x)}{g'(x)}$$

$$= \underset{x\to a}{Lt}\ \frac{f''(x)}{g''(x)}$$

eg. – (i) – $\underset{x\to 0}{Lt}\ \frac{\sin x}{x} = \frac{0}{0}$

L' Hos. → $\underset{x\to 0}{Lt}\ \frac{\cos x}{1} = 1$

(ii) – $\underset{x\to 0}{Lt}\ \frac{(1+x)^n - 1}{x} : \frac{0}{0}$

$\underset{x\to 0}{Lt}\ \frac{n(1+x)^{n-1}}{1} = n$

Ques – $\underset{x\to 0}{Lt}\ \frac{x - \tan x}{x^3} = ----$

a. $\frac{1}{3}$

b. $-\frac{1}{3}$

c. 1

d. 0

Sol$^n$ – $\underset{x\to 0}{Lt}\ \frac{1 - \sec^2 x}{3x^2}$

$\underset{x\to 0}{Lt}\ \frac{-\tan^2 x}{3x^2}$

$\underset{x\to 0}{Lt}\ \left(-\frac{1}{3}\right)\left[\frac{\tan x}{x} \cdot \frac{\tan x}{x}\right]$ ans → (b)

# Some Important Function

1. $\lim_{x\to 0} \frac{\sin x}{x} = 1$

2. $\lim_{x\to 0} \cos x = 1$

3. $\lim_{x\to 0} \frac{\tan x}{x} = 1$

4. $\lim_{x\to 0} \frac{e^x - 1}{x} = \log_e e = 1$

5. $\lim_{x\to 0} \frac{a^x - 1}{x} = \log_e a$

6. $\lim_{x\to a} \frac{x^n - a^n}{x-a} = na^{n-1}$

7. $\lim_{x\to 0} (1+x)^{\frac{1}{x}} = e$

8. $\lim_{x\to \infty} \left(1+\frac{1}{x}\right)^x = e$

9. $\lim_{x\to 0} \frac{(1-\cos mx)}{(1-\cos nx)} = \frac{m^2}{n^2}$

Ques- $\lim_{x\to 1} \frac{x^x - x}{1-x+\log x} = \frac{0}{0}$

a. 2

b. -2

c. 0

d. 1

Solⁿ- $\lim_{x\to 1} \frac{x^x(1+\log x) - 1}{-1+\frac{1}{x}} \quad \frac{0}{0}$

$\lim_{x\to 1} \frac{x^x\left(\frac{1}{x}\right) + (1+\log x)^2 x^x}{-\frac{1}{x^2}}$

$= \frac{1+1}{-1} \Rightarrow \underline{\underline{-2}}$

Solⁿ because- $y = x^x$

$\log y = x\log x$

$\frac{1}{y}\frac{dy}{dx} = x \cdot \frac{1}{x} + \log x$

$\frac{1}{y}\frac{dy}{dx} = 1 + \log x$

Ques- $\lim_{x\to 0} \frac{\tan x - x}{x^2 \tan x} = ---$

a. 1

b. $1/3$

c. $-1/3$

d. 0

Soln- $\lim_{x\to 0} \frac{\tan x - x}{x^3} \cdot \frac{x}{\tan x}$

$\lim_{x\to 0} \frac{\sec^2 x - 1}{3x^2} \cdot \lim_{x\to 0} \frac{x}{\tan x}$

$\lim_{x\to 0} \frac{\tan^2 x}{3x^2}$ (1)

$\frac{1}{3}$ ans - (b)

Ques- $\lim_{x\to 0} \frac{\log(x^2)}{\cot x^2} = ----$

a. 0

b. 1

c. $-1$

d. $\infty$

Soln- $\lim_{x\to 0} \frac{\frac{1}{x^2} \cdot \cancel{2x}}{-\operatorname{cosec}^2(x^2) \cdot \cancel{2x}}$

$\lim_{x\to 0} \frac{\frac{1}{x^2}}{-\frac{1}{\sin^2(x^2)}} = \lim_{x\to 0} - \frac{\sin^2(x^2)}{x^2}$

$= \lim_{x\to 0} - \left[\frac{\sin(x^2)}{x^2}\right] \cdot \sin(x^2)$

$= 0$

$= \lim_{x\to 0} \sin x \cdot \log x \quad (0 \times \infty)$

$= \lim_{x\to 0} \frac{\log x}{\operatorname{cosec} x} \quad \left(\frac{\infty}{\infty}\right)$

$= \lim_{x\to 0} \frac{1/x}{-\operatorname{cosec} x \cdot \cot x}$

$= \lim_{x\to 0} \frac{1/x}{-\frac{1}{\sin x}} \cdot \tan x$

$$= \lim_{x\to 0} -\frac{\sin x}{x}(\tan x) \qquad = 0$$

Ans → (a)

Ques- $\lim_{x\to 0}\left[\frac{1}{e^x-1} - \frac{1}{x}\right]$ $(\infty - \infty)$

Soln-

a. 0 b. 1 c. $\frac{1}{2}$ d. $-\frac{1}{2}$

$$= \lim_{x\to 0} \frac{x-(e^x-1)}{(e^x-1)\,x} \qquad \left(\frac{0}{0}\right)$$

$$= \lim_{x\to 0} \frac{1-e^x}{(e^x-1)+xe^x} = \lim_{x\to 0} \frac{-e^x}{e^x+xe^x+e^x}$$

$$= \lim_{x\to 0} \frac{-e^x}{e^x[1+x+1]} = \frac{-1}{2}$$

Ans → (d)

Ques- $\lim_{x\to 0} x^x$ $\quad 0^0$

a. 0
b. 1
c. $e$
d. $1/e$

Soln- $y = \lim_{x\to 0} x^x$

taking log →

$$\log y = \lim_{x\to 0} x\log x \qquad 0\times\infty$$

$$= \lim_{x\to 0} \frac{\log x}{1/x} \qquad \left(\frac{\infty}{\infty}\right)$$

$$= \lim_{x\to 0} \frac{1/x}{-1/x^2} \Rightarrow \lim_{x\to 0} -\frac{x^2}{x} = 0$$

$\log y = 0$

$y = e^0 = 1$

$y = 1$

Ans. (b)

Ques – $\lim_{x\to 1} x^{\frac{1}{1-x}}$ $\quad 1^{\infty}$

a. $e$

b. $1/e$

c. $1/e^2$

d. $0$

Soln –

$$e^{\lim_{x\to 1} (x-1)\frac{1}{1-x}}$$

$$= e^{\lim_{x\to 1}(-1)}$$

$$= e^{-1} \quad \text{ans} - (b)$$

Rule → when $1^{\infty}$ form then :–

$$\lim_{x\to a} f(x)^{g(x)}$$

$$e^{\lim_{x\to a} [f(x)-1]\, g(x)}$$

⊙ Continuity of a function at a point –

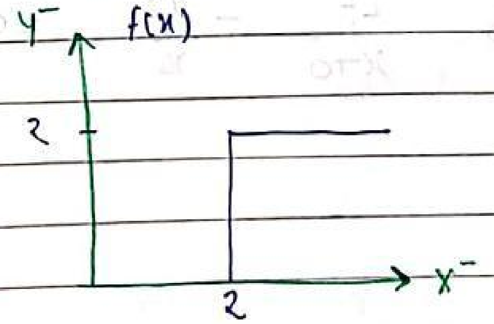
A function $f(x)$ is said to be continuous at a point $x = a$ if –

$$\lim_{x\to a} f(x) = f(a)$$

i.e. –

$$\lim_{x\to a^-} f(x) = \lim_{x\to a^+} f(x) = f(a)$$

L.H.L. = R.H.L. = $f(a)$




L.H.L $\lim_{x\to 2^-} [x] = 1$

R.H.L.

$\lim_{x\to 2^+} [x] = 2$

$f(2) = [2] = 2$

## Derivability of a function at a point –

A function $f(x)$ is said to be differentiable at a point $x = a$.

if $\underset{x\to a}{Lt} \dfrac{f(x) - f(a)}{(x-a)}$ exists finitely.

i.e. – $\underset{x\to a^-}{Lt} \dfrac{f(x) - f(a)}{(x-a)} = \underset{x\to a^+}{Lt} \dfrac{f(x) - f(a)}{(x-a)}$ = finite value.

i.e. – $\underset{h\to 0}{Lt} \dfrac{f(a-h) - f(a)}{-h} = \underset{h\to 0}{Lt} \dfrac{f(a+h) - f(a)}{h}$ = finite value

LHD = RHD

→ Every differentiable function is continuous. the converse need not to be true.

i.e. – If a function is continuous then it may or may not differentiable.

→ Function $f(x)$ is said to be differentiable at a point $x = a$.

if – $\underset{x\to a}{Lt} \dfrac{f(x) - f(a)}{x-a}$ exists finitely.

i.e – $\underset{x\to a^-}{Lt} \dfrac{f(x) - f(a)}{x-a} = \underset{x\to a^+}{Lt} \dfrac{f(x) - f(a)}{(x-a)}$ = finite value

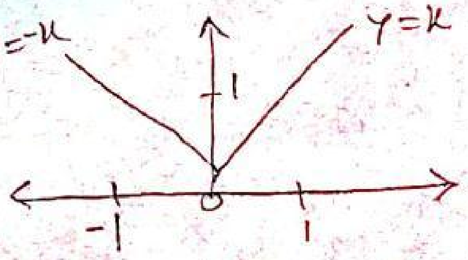
eg – $f(x) = |x|$ is – continuous.

The function is not differentiable at →

$x = 0$

L.H.D → $-1$

R.H.D → $+1$



2- If a function is discontinuous at a point then it is not derivable at that point- 

L.H.D-

$$\lim_{x \to 0^-} \frac{f(x) - f(0)}{x - 0} = \lim_{x \to 0^-} \frac{|x| - |0|}{x - 0}$$

$$\lim_{x \to 0^-} \frac{-x - 0}{x - 0} = -1$$

R.H.D→

$$\lim_{x \to 0^+} \frac{f(x) - f(0)}{x - 0}$$

$$= \lim_{x \to 0^+} \frac{|x| - |0|}{x - 0} \Rightarrow \lim_{x \to 0^+} \frac{x - 0}{x - 0} = 1$$

# Mean Value Theorems

- Rolle's theorem -

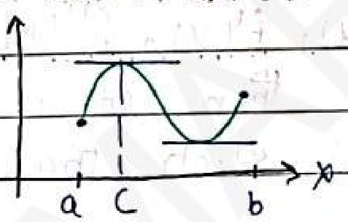
If $f(x)$ is defined on $[a, b]$ such that

(i)- $f(x)$ is continuous on $[a, b]$

(ii)- $f(x)$ is derivable on $(a, b)$

(iii)- $f(a) = f(b)$ then there exist at least one point $c \in (a,b)$ such that -

$$f'(c) = 0$$




- Lagranges mean value theorem -

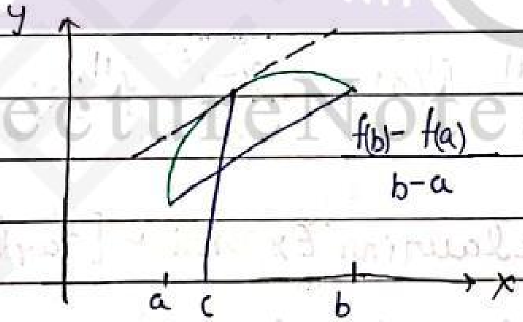
If $f(x)$ is defined on $[a, b]$ such that -

(i)- $f(x)$ is continuous on $[a, b]$

(ii)- $f(x)$ is derivable on $(a, b)$

then there exist at least one point $c \in (a, b)$ such that,

$$f'(c) = \frac{f(b) - f(a)}{b-a}$$




- Cauchy's mean value theorem -

If $f(x)$, $g(x)$ are defined on $[a, b]$ such that -

(i)- $f(x)$, $g(x)$ are continuous on $[a, b]$.

(ii)- $f(x)$, $g(x)$ are derivable on $(a, b)$.

(iii)- $g'(x) \neq 0 \quad \forall \; x \in (a, b)$ then there exist at least one point $c \in (a,b)$ such that -

$$\frac{f'(c)}{g'(c)} = \frac{f(b) - f(a)}{g(b) - g(a)}$$

- **Generalised mean value theorem: [Taylor's theorem] -**

If a function $f(x)$ is defined on $[a, a+h]$ such that

(i)- $f(x), f'(x), f''(x) \ldots f^{n-1}(x)$ are continuous on $[a, a+h]$

(ii)- $f(x), f'(x), f''(x) \ldots f^{n-1}(x)$ are derivable on $(a, a+h)$ then there exists at least one value $\theta \in (0,1)$ such that

$$f(a+h) = f(a) + \frac{h f'(a)}{\lfloor 1} + \frac{h^2 f''(a)}{\lfloor 2} + \frac{h^3}{\lfloor 3} f'''(a) + \ldots + \frac{h^n}{\lfloor n} f^n(a+\theta h)$$

where $\theta \in (0,1)$

put $a+h = x \implies h = x-a$

Taylor's expansion about the point 'a'.

1. $$f(x) = f(a) + \frac{(x-a)}{\lfloor 1} f'(a) + \frac{(x-a)^2}{\lfloor 2} f''(a) + \frac{(x-a)^3}{\lfloor 3} f'''(a) + \ldots$$

put $a = 0$

Maclaurins Expansion [Taylor's expansion about the origin]

2. $$f(x) = f(0) + x f'(0) + \frac{x^2}{\lfloor 2} f''(0) + \frac{x^3}{\lfloor 3} f'''(0) + \ldots$$

Ques- Find c of Rolle's theorem for $F(x) = e^x \sin x$ on $[0, \pi]$ -

a. $\pi/4$ b. $3\pi/4$ c. $\pi$ d. $\pi/3$

Solⁿ- $F'(x) = e^x \cos x + \sin x \cdot e^x$

$\Rightarrow e^x [\cos x + \sin x]$ exist Finitly $\forall\ x \in (0, \pi)$

$f(0) = e^0 \sin 0^\circ = 0$

$F(\pi) = e^\pi \sin \pi = 0$ by Rolle's theorem-

$F(0) = F(\pi)$

$f'(c) = 0$

$c[\cos c + \sin c] = 0$

$\cos c = -\sin c$

$c = \frac{3\pi}{4}$

$\cos \frac{3\pi}{2} = \frac{-1}{\sqrt{2}}$

$\sin \frac{3\pi}{2} = \frac{1}{\sqrt{2}}$

Ques- Rolle's theorem cannot be applied for $F(x) = 2 + (x-1)^{2/3}$ in $[0, 2]$ because-

a. F is not continuous on $[0, 2]$

b. F " " derivable on $(0, 2)$

c. $F(0) \neq F(2)$

d. None of the above.

Solⁿ- $F'(x) = 0 + \frac{2}{3}(x-1)^{\frac{2}{3}-1}$

$\Rightarrow \frac{2}{3}(x-1)^{-1/3}$

$= \frac{2}{3} \frac{1}{(x-1)^{1/3}}$

F(x) is not derivable at $x = 1 \in (0, 2)$

since $F'(x) = \infty$ at $x = 1$

NOTE-

Algebric function with positive rational power is always continuous.

eg.- $2 + (x-1)^{2/3}$.

$F(0) = 2 + (0-1)^{2/3} = 3$
$f(2) = 2 + (2-1)^{2/3} = 3$

$F(0) = F(2)$ [So third option is not correct].
only (b) option is satisfing.

Ques- Find c of Rolle's theorem for $f(x) = x^2 - 6x + 8$ in [2, 4].

a. 2.2 b. 3 c. 2.75 d. None.

Solⁿ-
$F(x) = x^2 - 6x + 8$
$F'(x) = 2x - 6$
$F(2) = 4 - 12 + 8$
$= 0$
$F(4) = 16 - 24 + 8$
$= 0$
$f'(c) = 0$
$2c - 6 = 0$
$c = 3$
Ans → (b)

Ques- Find c of LMVT for the $f(x) = \sqrt{x^2-4}$ on $[2,4]$

a. 2.5 b. 3 c. $\sqrt{6}$ d. $2\sqrt{6}$

Sol$^n$-

$$f'(x) = \frac{1}{2\sqrt{x^2-4}} \cdot 2x = \frac{x}{\sqrt{x^2-4}}$$ exists finitly.

by LMVT $\forall\ x = (2,4)$

$$\cancel{f'(x)} = \frac{f(4)-f(2)}{4-2}$$

$$\frac{c}{\sqrt{c^2-4}} = \frac{\sqrt{16-4} - 0}{2}$$

$$\frac{c}{\sqrt{c^2-4}} = \frac{\sqrt{12}}{2}$$

$$4c^2 = 12c^2 - 48$$

$$8c^2 = 48$$

$$c^2 = 6$$

$$c = \sqrt{6} \in (2,4)$$

ans - (C)

Ques- Find c of LMVT for $f(x) = lx^2 + m(x) + n$ in $[a,b]$

a. $2ab/(a+b)$ b. $\sqrt{ab}$ c. $(a+b)/2$ d. $\frac{(b-a)}{2}$

Sol$^n$- $f'(x) = 2lx + m$

$f(x)$ exists finitly $\forall\ x\ (a,b)$

$$f'(c) = \frac{f(b)-f(a)}{b-a}$$

$$2lc + m = \frac{(lb^2+mb+n) - (la^2+ma+n)}{b-a}$$

$$2lc + m = \frac{l(b^2-a^2) + m(b-a)}{b-a}$$

$$2lc + m = l(b+a) + m$$

$$2lc = l(b+a)$$

$$c = \frac{b+a}{2} \in (a,b)$$

ANS - (C)

Ques- Find c of CMVT for $f(x) = x^2$, $g(x) = x^3$ in $[1,2]$

a. $\frac{14}{9}$ b. 1.5 c. 1.475 d. None.

Sol$^n$- $f'(x) = 2x$ $g'(x) = 3x^2$

CMVT-

$$\frac{f'(c)}{g'(c)} = \frac{f(2)-f(1)}{g(2)-g(1)}$$

$$\frac{2c}{3c^2} = \frac{4-1}{8-1}$$

$$\frac{2}{3c} = \frac{3}{7}$$

$$9c = 14$$

$$c = \frac{14}{9}$$

Ans-(a)

Ques- Find c of CMVT for-

$f(x) = \frac{1}{x^2}$ $g(x) = \frac{1}{x}$ on $[a,b]$

a. $\frac{a+b}{2}$ b. $\sqrt{ab}$ c. $\frac{2ab}{a+b}$ d. $\frac{b-a}{2}$

Sol$^n$- $g(x) = \frac{1}{x}$ on $[a,b]$

$f'(x) = \frac{-2}{x^3}$ $g'(x) = \frac{-1}{x^2}$

exist finitely $\forall x \in (a,b)$

$$\frac{f'(c)}{g'(c)} = \frac{f(b)-f(a)}{g(b)-g(a)}$$

$$\frac{-2/c^3}{-1/c^2} = \frac{\frac{1}{b^2} - \frac{1}{a^2}}{\frac{1}{b} - \frac{1}{a}}$$

$$-\frac{2}{c^3} \times \frac{c^2}{-1} = \frac{a^2-b^2}{a^2b^2} \times \frac{ab}{a-b}$$

$$\frac{2}{c} = \frac{a+b}{ab}$$

$$\frac{c}{2} = \frac{ab}{a+b}$$

$$c = \frac{2ab}{a+b} \in (a, b)$$

Ans- (c).

Ques- Tan $x$ in terms of $x$ is —

a. $x + \frac{x^3}{\lfloor 3} + \text{------}$

b. $x - \frac{x^3}{\lfloor 3} + \text{------}$

c. $x + \frac{x^3}{3} + \text{------}$

d. $x - \frac{x^3}{3} + \text{------}$

Solⁿ-

$$f(x) = f(0) + x f'(0) + \frac{x^2}{\lfloor 2} f''(0) + \frac{x^3}{\lfloor 3} f'''(0) + \text{-----}$$

$$\tan x = 0 + x(1) + \frac{x^2}{\lfloor 2}(0) + \frac{x^3}{\lfloor 3}(2) + \text{-----}$$

$$\tan x = x + \frac{x^3}{3} + \text{-----}$$

$f(x) = \tan \qquad f(0) = \tan 0 = 0$

$f'(x) = \sec^2 x \qquad f'(0) = \sec^2$

$f''(x) = 2\sec x . \sec x \tan x = 2\sec^2 x \tan x$

$f'''(x) = 2\sec^2 x + \tan x . 4\sec x . \sec x \tan x$

$f'''(0) = 2 + 0 = 2$

Ans – (c)

Ques- For the function $e^{-x}$, the linear approximation around $x=2$ is-

a. $(3-x)\,e^{-2}$ b. $(1-x)$ c. $[3+2\sqrt{2}-(1+\sqrt{2})x]\,e^{-2}$ d. $e^{-2}$

Soln-

$f(x) = e^{-x}$ $f(2) = e^{-2}$

$f'(x) = -e^{-x}$ $f'(2) = -e^{-2}$

$f''(x) = e^{-x}$ $f''(2) = e^{-2}$

$\vdots$ $\vdots$

$$f(x) = f(2) + (x-2)\,f'(2) + \frac{(x-2)^2}{\lfloor 2}\,f''(2) + \text{-----}$$

$$e^{-x} = F(2) + (x-2)\,(-e^{-2}) + \text{-----}$$

linear approximation is-

$$e^{-x} = e^{-2} + (x-2)\,(-e^{-2})$$

$$e^{-x} = e^{-2}[1+(x-2)(-1)]$$

$$= e^{-2}\,[1-x+2]$$

$$= e^{-2}\,[3-x)$$

Ans- (a)

# PARTIAL DIFFERENTIATION

$u = f(x, y)$

$$p = \frac{du}{dx} = u_x$$

$$q = \frac{du}{dy} = u_y$$

$$r = \frac{d^2u}{dx^2} = u_{xx}$$

$$s = \frac{d^2u}{dx\,dy} \text{ (or) } \frac{d^2u}{dy\,dx} = u_{x,y}$$

$$t = \frac{d^2u}{dy^2} = u_{yy}.$$

Homogeneous Function -

A function $f(x, y)$ is said to be homogenious function of degree $n$ if $f(kx, ky) = k^n f(x, y)$.

Eg.

$u = \frac{x^2+y^2}{x^3+y^3}$ is homogenous function of degree $-1$

→

$$u(kx, ky) = \frac{(kx)^2 + (ky)^2}{(kx)^3 + (ky)^3} = \frac{k^2x^2 + k^2y^2}{k^3x^3 + k^3y^3} = \frac{k^2}{k^3}\frac{(x^2+y^2)}{(x^3+y^3)}$$

$$= k^{2-3}\frac{(x^2+y^2)}{(x^3+y^3)}$$

$$= k^{-1}u$$

Euler's theorem -

If $f(x, y)$ is a homogenious function of degree $n$ then -

(i) - $$x\frac{du}{dx} + y\frac{du}{dy} = nu$$

(ii) - $$x^2\frac{d^2u}{dx^2} + 2xy\frac{d^2u}{dx\,dy} + y^2\frac{d^2u}{dy^2} = n(n-1)u$$

Eg- If $u(x,y) = x\sin\left(\frac{y}{x}\right)$ then-

(i)- $x\frac{du}{dx} + y\frac{du}{dy} = \underline{\quad} \; nu = 1\cdot u = u$

(ii)- $x^2\frac{d^2u}{dx^2} + 2xy\frac{d^2u}{dx\,dy} + y^2\frac{d^2u}{dy^2} = n(n-1)u$

$= 1(1-1)u \Rightarrow 0$

$$u(kx, ky) = kx\sin\left(\frac{ky}{kx}\right)$$

$$= kx\sin\left(\frac{y}{x}\right)$$

degree $= ku(x,y)$

NOTE-

If $u = f(x,y)$ is not a homogenious function but $F(u)$ is a homogenious function of degree $n$ then-

(i)- $x\frac{du}{dx} + y\frac{du}{dy} = n\left[\frac{F(u)}{F'(u)}\right]$

(ii)- $x^2\frac{d^2u}{dx^2} + 2xy\frac{d^2u}{dx\,dy} + y^2\frac{d^2u}{dy^2} = G(u)[G'(u)-1]$

where $G(u) = n\left[\frac{F(u)}{F'(u)}\right]$

eg.- If $u = \log\left(\frac{x^4+y^4}{x-y}\right)$ then-

(i)- $x\frac{du}{dx} + y\frac{du}{dy} = 3\left[\frac{e^u}{d(e^u)}\right] = 3\frac{e^u}{e^u} = \underline{\underline{3}} \rightarrow g(u)$

(ii)- $x^2\frac{d^2u}{dx^2} + 2xy\frac{d^2u}{dx\,dy} + y^2\frac{d^2u}{dy^2} = g(u)[g'(u)-1]$

$= 3[0-1] = \underline{\underline{-3}}$

$e^u = \frac{x^4+y^4}{x-y}$ is homogenuous function of degree 3.

eg- $u = \sin^{-1}\left(\dfrac{x^3+y^3}{x-y}\right)$

$$F(u) \leftarrow \sin u = \frac{x^3+y^3}{x-y}$$

$$x \cdot \frac{d}{dx}\{F(u)\} + y\frac{d}{dy}\{F(u)\} = n \cdot F(u)$$

$$x F'(u)\frac{du}{dx} + y \cdot F'(u)\frac{du}{dy} = n \cdot F(u)$$

$$x\frac{du}{dx} + y\frac{du}{dy} = n\frac{F(u)}{F'(u)}$$

$$x\frac{d}{dx}(\sin u) + y\frac{d}{dy}(\sin u) = 2\sin y$$

$$x\cos u\frac{du}{dx} + y\cos u\frac{du}{dy} = 2\sin u$$

$$x\frac{du}{dx} + y\frac{du}{dy} = 2\frac{\sin u}{\cos u} = 2\tan u.$$

Total Derivative-

If $u = f(x,y)$ and $x = \phi(t)$, $y = \psi(t)$ then the total derivative of $u$ with respect to $t$ is given by

$$\frac{du}{dt} = \frac{du}{dx}\cdot\frac{dx}{dt} + \frac{du}{dy}\cdot\frac{dy}{dt}$$

$$\begin{array}{ccc} & u & \\ \swarrow & & \searrow \\ x & & y \end{array}$$

$$\frac{du}{dx} = \frac{du}{dx} + \frac{du}{dy}\cdot\frac{dy}{dx}$$

Eg.- $u = x^2 y$ and $x^3 + y^3 = 3xy$

then $\frac{du}{dx} = ?$

Soln- $\frac{\partial u}{\partial x} = 2xy$

$\frac{\partial u}{\partial y} = x^2$

$x^3 + y^3 - 3xy = 0$

$$\frac{dy}{dx} = -\frac{\phi_x}{\phi_y} = \frac{-(3x^2 - 3y)}{3y^2 - 3x} = -\frac{(x^2 - y)}{y^2 - x}$$

$$\frac{du}{dx} = \frac{\partial u}{\partial x} + \frac{\partial u}{\partial y} \cdot \frac{dy}{dx}$$

$$= 2xy + x^2 \left[ -\frac{(x^2 - y)}{y^2 - x} \right]$$

$$= 2xy - x^2 \left[ \frac{x^2 - y}{y^2 - x} \right]$$

directly-

$$3x^2 + 3y^2 \frac{dy}{dx} - 3\left[x \frac{dy}{dx} + y\right] = 0$$

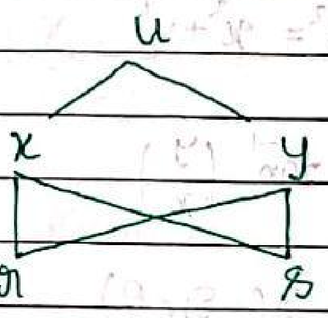
$$3x^2 - 3y + 3y^2 \frac{dy}{dx} - 3x \frac{dy}{dx} = 0$$

$$\frac{dy}{dx} = \frac{y - x^2}{y^2 - x}$$

⊙ If $u = f(x, y)$ $x = \phi_1(r, s)$ $y = \phi_2(r, s)$ then,

$$\frac{\partial u}{\partial r} = \frac{\partial u}{\partial x} \cdot \frac{\partial x}{\partial r} + \frac{\partial u}{\partial y} \cdot \frac{\partial y}{\partial r}$$

$$\frac{\partial u}{\partial s} = \frac{\partial u}{\partial x} \cdot \frac{\partial x}{\partial s} + \frac{\partial u}{\partial y} \cdot \frac{\partial y}{\partial s}$$




$$u_r = u_x x_r + u_y y_r$$
$$u_s = u_x x_s + u_y y_s$$

Ques- If $u = f(x, y)$ ; $x = r+s$ ; $y = r-s$ then-

$$u_r + u_s =$$

a. $2u_x$ c. $-2u_x$

b. $2u_y$ d. $-2u_y$

Solⁿ-

$$u_r = u_x(1) + u_y(1)$$
$$u_r = u_x + u_y \longrightarrow (1)$$

$$u_s = u_x x_s + u_y u_s$$
$$= u_x(1) + u_y(-1)$$
$$u_s = u_x - u_y \longrightarrow (2)$$

(1) + (2)

$$u_r + u_s = u_x + \cancel{u_y} + u_x - \cancel{u_y}$$
$$= 2u_x$$

Ans- (a)

⊙ JACOBIAN :

If $u = f_1(x, y)$ $v = f_2(x, y)$ then the Jacobian of u and v with respect to x and y is given by :

$$J\left(\frac{u, v}{x, y}\right) = \begin{vmatrix} \frac{du}{dx} & \frac{du}{dy} \\ \frac{dv}{dx} & \frac{dv}{dy} \end{vmatrix}$$

eg.- $x = r\cos\theta \quad y = r\sin\theta \quad r^2 = x^2+y^2$

$\tan\theta = \frac{y}{x} \quad \theta = \tan^{-1}\left(\frac{y}{x}\right)$

Solⁿ- $\frac{dx}{dr} = \cos\theta \quad \frac{dx}{d\theta} = r(-\sin\theta)$

$\frac{dy}{dr} = \sin\theta \quad \frac{dy}{d\theta} = r\cos\theta$

$$J\left(\frac{x,y}{r,\theta}\right) = \begin{vmatrix} \frac{dx}{dr} & \frac{dx}{d\theta} \\ \frac{dy}{dr} & \frac{dy}{d\theta} \end{vmatrix}$$

$$= \begin{vmatrix} \cos\theta & -r\sin\theta \\ \sin\theta & r\cos\theta \end{vmatrix} = r\cos^2\theta + r\sin^2\theta$$

$$= \underline{\underline{r}}$$

$r^2 = x^2+y^2$

diff. wrt $x$-

$2r\frac{dr}{dx} = 2x$

$\frac{dr}{dx} = \frac{x}{r}$

$2r\frac{dr}{dy} = 2y$

$\frac{dr}{dy} = \frac{y}{r}$

$\tan\theta = \frac{y}{x}$

$\theta = \tan^{-1}\left(\frac{y}{x}\right)$

$\frac{d\theta}{dx} = \frac{1}{1+\frac{y^2}{x^2}}\left(\frac{-y}{x^2}\right)$

$= \frac{x^2}{x^2+y^2}\left(\frac{-y}{x^2}\right) = \frac{-y}{x^2+y^2}$

$$J\left(\frac{r,\theta}{x,y}\right) = \begin{vmatrix} r_x & r_y \\ \theta_x & \theta_y \end{vmatrix}$$

$= \frac{x^2}{r(x^2+y^2)} + \frac{y^2}{r(x^2+y^2)} = \frac{x^2+y^2}{r(x^2+y^2)}$

$= \underline{\underline{\frac{1}{r}}}$

$\frac{d\theta}{dy} = \frac{1}{1+\frac{y^2}{x^2}}\left(\frac{1}{x}\right)$

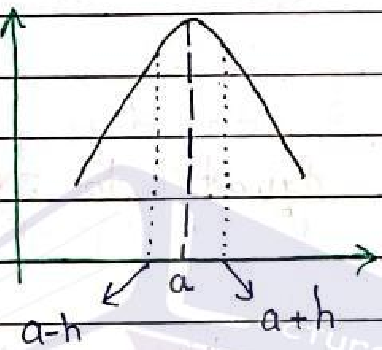
$= \frac{x^2}{x^2+y^2}\left(\frac{1}{x}\right) \Rightarrow \frac{x}{x^2+y^2}$

$$J\left(\frac{x,y}{r,\theta}\right) \cdot J\left(\frac{r,\theta}{x,y}\right) = 1$$

## ⊙ MAXIMAA & MINIMA –

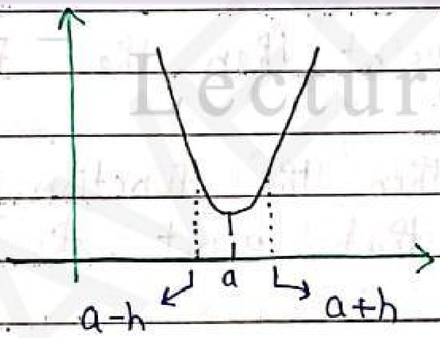
Definition –

A function $f(x)$ is said to be maximum at a point $x = a$ if $f(a) \geq f(a+h)$ for all values of $h$.



Definition –

A function $f(x)$ is said to be minimum at a point $x = a$ if $f(a) \leq f(a+h)$ for all values of $h$.



Ques-(i) Find $\frac{dy}{dx}$ and then equate to zero to get $y = f(x)$

Stationary values of $x$.

(ii) – Find the value of $\frac{d^2y}{dx^2}$ at each stationary point.

a. If $\frac{d^2y}{dx^2} > 0$ at a point then the function is minimum at that point.

b- If $\frac{d^2y}{dx^2} < 0$ at a point then the function is maximum at that point.

c- If $\frac{d^2y}{dx^2} = 0$ at a point then the function is neither minimum nor maximum at that point. the point is known as saddle point.

Ques- $u = f(x,y)$

1. Find $\frac{du}{dx}$ and $\frac{du}{dy}$, then equate to zero to get stationary point.
2. Find $r = \frac{\partial^2 u}{\partial x^2}$, $s = \frac{\partial^2 u}{\partial x \partial y}$, $t = \frac{\partial^2 u}{\partial y^2}$ at each stationary point.

(a)- If $rt - s^2 > 0$, $r > 0$ at a point then the function is minimum at that point.

(b)- If $rt - s^2 > 0$, $r < 0$ at a point then the function is maximum at that point.

(c)- If $rt - s^2 < 0$ at a point then the function is neither minimum nor maximum at that point. the point is known as saddle point.

(d)- If $rt - s^2 = 0$ at a point then no condition can be drawn.

Ques- The function $f(x) = 2x^3 - 9x^2 - 24x - 20$ has a maxima at $x = ?$

a. 4 b. -1 c. 2 d. -3

Soln-

$$f'(x) = 6x^2 - 18x - 24$$

$$f'(x) = 12x - 8$$

$$f''(-1) = 12(-1) - 18$$

$$= \underline{-30}$$

at stationary points-

$$f'(x) = 0$$

$$6x^2 - 18x - 24 = 0$$

$$x^2 - 3x - 4 = 0$$

$$(x-4)(x+1) = 0$$

$$x = 4, -1$$

Ans - (b)

Ques- The function $f(x) = (1-x)^2 e^x$ has maxima at $\underline{x=?}$

a. 1 b. -1 c. 2 d. -2

Soln-

$$f'(x) = (1-x)^2 e^x + e^x[2(1-x)(-1)]$$

$$= e^x[(1-x)^2 - 2(1-x)]$$

$$= e^x(1+x^2-2x-2+2x)$$

$$= e^x(x^2-1)$$

$$f''(x) = e^x(2x) + (x^2-1)e^x$$

$$= e^x[2x + x^2 - 1]$$

at stationary points-

$$f'(x) = 0$$

$$e^x(x^2-1) = 0$$

$$x^2 = 1$$

$$x = \pm 1$$

$x = 1$ $\quad f''(1) = e[2+1-1] = 2e > 0$

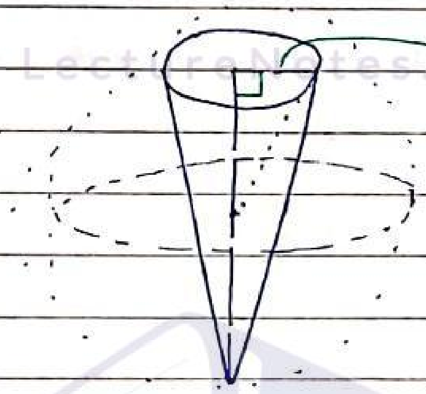
$x = -1$ $\quad f''(-1) = e^{-1}[-2+1-1] = \frac{-2}{e} < 0$

ans- (a).

Ques- A right circular cone is inscribed in a sphere of radius 1m, has a maximum volume with a height of -

a. 3m
b. 1m
c. $\frac{2\sqrt{5}}{3}$ m
d. $\frac{4}{3}$ m

Soln -




$$r = \sqrt{1-(h-1)^2} = \sqrt{1-(h^2-2h+1)} = \sqrt{2h-h^2}$$

volume of the cone -

$$V = \frac{1}{3}\pi r^2 h = \frac{1}{3}\pi h(2h-h^2) = \frac{\pi}{3}(2h^2-h^3)$$

$$\frac{dV}{dh} = \frac{\pi}{3}(4h-3h^2)$$

$$\frac{d^2V}{dh^2} = \frac{\pi}{3}(4-6h)$$

at stationary point →

$$\frac{dV}{dh} = 0$$

$$\frac{\pi}{3}(4h-3h^2) = 0$$

$$4h-3h^2 = 0$$

$$h(4-3h) = 0$$

$$h = 0 \quad ; \quad h = 4/3.$$

$h=0 \quad \frac{d^2V}{dh^2} = \frac{\pi}{3}(4) \quad > 0 \quad \rightarrow \quad V_{min}$

$h=\frac{4}{3} \quad \frac{d^2V}{dh^2} = \frac{\pi}{3}\left(4-\frac{24}{3}\right) = \frac{-4\pi}{3} < 0 \quad V_{max}$

Ans- (d)

Ques- $f(x,y) = x^3 + y^3 - 3xy$ has-

a. Maxima at (1,1)  c. A Saddle point (1,1)
b. Minima at (1,1)  d. No conclusion at (1,1)

Solⁿ-

$\frac{du}{dx} = 3x^2 - 3y \qquad \frac{du}{dy} = 3y^2 - 3x$

$r = \frac{d^2u}{dx^2} = 6x \qquad s = \frac{d^2u}{dx\,dy} = -3$

$t = \frac{d^2u}{dy^2} = 6y$

at (1,1)-

$r = 6 \qquad s = -3 \qquad t = 6$

$r = 6 > 0$

$rt - s^2 = 36 - (-3)^2 = 36 - 9 = 27 > 0$

Ans- (b)

Ques- At $(\sqrt{2}, -\sqrt{2})$, $f(x,y) = x^4 + y^4 - 2x^2 + 4xy - 2y^2$ has-

a. Minima  c. Saddle point.
b. Maxima  d. None.

Solⁿ- $\frac{du}{dx} = 4x^3 - 4x + 4y \qquad \frac{du}{dy} = ~~4x~~$

$= 4y^3 + 4x - 4y$

$r = \frac{d^2u}{dx^2} = 12x^2 - 4 \qquad s = \frac{d^2u}{dx\,dy} = 4 \qquad t = \frac{d^2u}{dy^2} = 12y^2 - 4$

At $(\sqrt{2}, -\sqrt{2})$

$r = 12(2) - 4 = 20 > 0$

$s = 4$

$t = 12(2) - 4 = 20$

$rt - s^2 = (20)(20) - (4)^2$
$= 400 - 16$
$= 384 > 0$

Ans - (a).

Ques- A rectangular box open at the Top is to have a volume 32 CC find the dimensions of the box requiring least material for it's construction-

a. 2, 2, 8
b. 4, 4, 2
c. 16, 1, 1
d. 8, 8, $\frac{1}{2}$

Solⁿ-

$$xyz = 32$$

$$z = \frac{32}{xy}$$

$$S = xy + 2yz + 2zx$$

$$= xy + 2y\left(\frac{32}{xy}\right) + 2x\left(\frac{32}{xy}\right)$$

$$= xy + \frac{64}{x} + \frac{64}{y}$$

$$\frac{dS}{dx} = y - \frac{64}{x^2} \qquad \frac{dS}{dy} = x - \frac{64}{y^2}$$

$$r = \frac{d^2S}{dx^2} = \frac{128}{x^3} \qquad s = \frac{d^2S}{dx\,dy} = 1$$

$$t = \frac{d^2S}{dy^2} = \frac{128}{y^3}$$

At Stationary point-

$$\frac{dS}{dx} = 0 \qquad \frac{dS}{dy} = 0$$

$$y - \frac{64}{x^2} = 0 \qquad x - \frac{64}{y^2} = 0$$

$$y = \frac{64}{x^2} \text{ --- (i)} \qquad x = \frac{64}{y^2} \text{ --- (ii)}$$

Sub (i) in (ii) →

$$x = \frac{64}{(64)^2/x^4} = \frac{x^4}{64}$$

$$64x = x^4$$

$$x^3 = 64$$

$x = 4$

$y = 4$

$z = 2$

at $(4,4)$ →

$$r = \frac{128}{64} = 2 > 0$$

$$s = 1$$

$$t = \frac{128}{64} = 2$$

$$rt - s^2 = (2)(2) - (1)^2 = 4 - 1 = 3 > 0$$

$rt - s^2 > 0 \quad r > 0 \quad S$ is min.

Ans (b)

# INTEGRATION

$\int x^n dx = \frac{x^{n+1}}{n+1}, \quad n \neq -1$

$\int \frac{1}{x} dx = \log x$

$\int e^x dx = e^x$

$\int a^x dx = \frac{a^x}{\log a}$

$\int \sin x \, dx = -\cos x$

$\int \cos x \, dx = \sin x$

$\int \sec^2 x \, dx = \tan x$

$\int \operatorname{cosec}^2 x \, dx = -\cot x$

$\int \sec x \tan x \, dx = \sec x$

$\int \operatorname{cosec} x \cot x \, dx = -\operatorname{cosec} x$

$\int \frac{f'(x)}{f(x)} dx = \log (f(x))$

$\int \tan x \, dx = \log (\sec x)$

$\int \cot x \, dx = \log (\sin x)$

$\int \sec x \, dx = \log (\sec x + \tan x)$

$\int \operatorname{cosec} x \, dx = \log (\operatorname{cosec} x - \cot x)$

$\int \frac{1}{1+x^2} dx = \tan^{-1} x$ (or) $-\cot^{-1} x$

$\int \frac{1}{\sqrt{1-x^2}} dx = \sin^{-1} x$ (or) $-\cos^{-1} x$

$\int \frac{1}{|x| \sqrt{x^2-1}} dx = \sec^{-1} x$ (or) $-\operatorname{cosec}^{-1} x$

$$\int \frac{1}{a^2+x^2}\,dx = \frac{1}{a}\tan^{-1}\left(\frac{x}{a}\right)$$

$$\int \frac{1}{a^2-x^2}\,dx = \frac{1}{2a}\log\frac{(a+x)}{(a-x)}$$

$$\int \frac{1}{x^2-a^2}\,dx = \frac{1}{2a}\log\left(\frac{x-a}{x+a}\right)$$

$$\int \frac{1}{\sqrt{a^2+x^2}}\,dx = \sinh^{-1}\left(\frac{x}{a}\right)$$

$$\int \frac{1}{\sqrt{a^2-x^2}}\,dx = \sin^{-1}\left(\frac{x}{a}\right)$$

$$\int \frac{1}{\sqrt{x^2-a^2}}\,dx = \cosh^{-1}\left(\frac{x}{a}\right)$$

$$\int [f(x)]^n f'(x)\,dx = \frac{[f(x)]^{n+1}}{n+1}$$

$$\int e^{ax}\cos bx\,dx = \frac{e^{ax}}{a^2+b^2}(a\cos bx + b\sin bx)$$

$$\int e^{ax}\sin bx\,dx = \frac{e^{ax}}{a^2+b^2}(a\sin bx - b\cos bx)$$

$$\int \sqrt{a^2+x^2}\,dx = \frac{x}{2}\sqrt{a^2+x^2} + \frac{a^2}{2}\sinh^{-1}\left(\frac{x}{a}\right)$$

$$\int \sqrt{a^2-x^2}\,dx = \frac{x}{2}\sqrt{a^2-x^2} + \frac{a^2}{2}\sin^{-1}\left(\frac{x}{a}\right)$$

$$\int \sqrt{x^2-a^2}\,dx = \frac{x}{2}\sqrt{x^2-a^2} - \frac{a^2}{2}\cosh^{-1}\left(\frac{x}{a}\right)$$

$$\int u\,dv = uv - \int v\,du$$

(Integration by parts).

ILATE →

I → Inverse trignometric
L → logarithmic function.
A → Algebric function.
T → Trignometric
E → Exponential.

Definite Integrals

1. $\int_a^b f(x)\,dx = \int_a^b f(t) \cdot dt$

2. $\int_a^b f(x)\,dx = -\int_b^a f(x)\,dx$

3. If $a < c < b$ then →

$$\int_a^b f(x)\,dx = \int_a^c f(x)\,dx + \int_c^b f(x)\,dx$$

4. $$\int_{-a}^{a} f(x)\,dx = \begin{cases} 2\int_0^a f(x)\,dx & \text{if } f(x) \text{ is even fun}^n \\ 0 & \text{if } f(x) \text{ is odd function} \end{cases}$$

even function ⇒ $f(-x) = f(x)$
odd function ⇒ $f(-x) = -f(x)$

5. $$\int_0^{2a} f(x)\,dx = \begin{cases} 2\int_0^a f(x)\,dx & \text{if } f(2a-x) = f(x) \\ 0 & \text{if } f(2a-x) = -f(x) \end{cases}$$

6- $\int_0^a f(x)\,dx = \int_0^a f(a-x)\,dx$

7. $$\int_0^a \frac{f(x)}{f(x)+f(a-x)}\,dx = \frac{a}{2}$$

8. $$\int_0^{na} f(x)\,dx = n\int_0^a f(x)\,dx \quad \text{if } f(x+a) = f(x)$$

i.e $f(x)$ is a periodic function with period $a$.

9. $$\int_0^{\pi/2} \sin^n x\,dx = \int_0^{\pi/2} \cos^n x\,dx$$

$$\Rightarrow \begin{cases} \left(\frac{n-1}{n}\right)\left(\frac{n-3}{n-2}\right)\left(\frac{n-5}{n-4}\right) \cdots\cdots \frac{1}{2}\cdot\frac{\pi}{2} & \text{if } n \text{ is even} \\ \left(\frac{n-1}{n}\right)\left(\frac{n-3}{n-2}\right)\left(\frac{n-5}{n-4}\right) \cdots\cdots \frac{2}{3}\cdot 1 & \text{if } n \text{ is odd.} \end{cases}$$

10. $$\int_0^{\frac{\pi}{2}} \sin^m x \cos^n x\,dx = \frac{[(m-1)(m-3)(m-5)\cdots(2\text{ or }1)][(n-1)(n-3)\cdots(2\text{ or }1)]}{(m+n)(m+n-2)(m+n-4)\cdots\cdots(2\text{ or }1)} K$$

NUMERICALS

Ques- $\int_{-1}^{1} \frac{x^2 \sin x}{x^4+1}\,dx = ?$

a. 0 c. 1
b. 2 d. None

Solⁿ- $f(x) = \frac{x^2 \sin x}{x^4+1}$

$f(-x) = \frac{-x^2 \sin x}{x^4+1} = -f(x)$

$f(x)$ is odd function → $\therefore \int_{-1}^{1} f(x)\,dx = \underline{\underline{0}}$ (a)

Ques- $\int_0^1 x(1-x)^5\,dx$

a. $1/48$ b. $1/42$ c. $1/56$ d. $1/13$

Solⁿ- Let $1-x=t$, $-dx=dt$, $dx=-dt$ / $x=0 \to t=1$, $x=1 \to t=0$

$$\int_1^0 (1-t)t^5(-dt) \Rightarrow \int_0^1 (t^5+t^6)\,dt = \left(\frac{t^6}{6} - \frac{t^7}{7}\right)\Big|_0^1 = \frac{1}{42} \quad \text{(b)}$$

Ques- $\int_{-\pi/2}^{\pi/2} x^{10} \log \left(\frac{1+\sin x}{1-\sin x}\right) dx$

a. 0 b. 2 c. 1 d. $\pi$

Soln-

$$f(-x) = (-x)^{10} \log \left[\frac{1+\sin(-x)}{1-\sin(-x)}\right]$$

$$= x^{10} \log \frac{1-\sin x}{1+\sin x}$$

$$= x^{10} \log \left(\frac{1+\sin x}{1-\sin x}\right)^{-1}$$

$$= -x^{10} \log \left(\frac{1+\sin x}{1-\sin x}\right)$$

$$= -f(x)$$

Ans → (a)

Ques- $\int_{-1}^{1} |x| \, dx =$

a. 0 b. $\frac{1}{2}$ c. 1 d. 2

Soln-

$$2\int_0^1 |x| \, dx$$

$$= 2\int_0^1 x \, dx$$

$$= 2\left[\frac{x^2}{2}\right]_0^1 \Rightarrow \frac{2}{2} = 1$$

Ans - 1 (c)

Ques- $\int_0^{\pi/2} \frac{1}{1+\sqrt{\cot x}} dx =$

a. 0 b. $\pi/2$ c. $\pi/4$ d. $\pi$

Soln- Property → $\int_0^a \frac{f(x)}{f(x)+f(a-x)} = \frac{a}{2}$

$$= \int_0^{\frac{\pi}{2}} \frac{1}{1+\frac{\sqrt{\cos x}}{\sqrt{\sin x}}} dx$$

$$= \int_0^{\pi/2} \frac{\sqrt{\sin x}}{\sqrt{\sin x} + \sqrt{\cos x}} dx \Rightarrow \frac{\pi/2}{2} \Rightarrow \frac{\pi}{4}$$ ans → (C)

Ques-

$$\int_0^{\frac{\pi}{4}} \log(1+\tan x)\, dx =$$

a. 0　　b. $\frac{\pi}{2}\log 2$　　c. $\frac{\pi}{4}\log 2$　　d. $\frac{\pi}{8}\log 2$

Sol^n^-

$$\int_0^a f(x)\,dx = \int_0^a f(a-x)\,dx$$

$$= \int_0^{\frac{\pi}{4}} \log\left[1+\tan\left(\frac{\pi}{4}-x\right)\right] dx$$

$$= \int_0^{\frac{\pi}{4}} \log\left[1+\frac{\tan\frac{\pi}{4} - \tan x}{1+\tan\frac{\pi}{4}\tan x}\right] dx$$

$$= \int_0^{\frac{\pi}{4}} \log\left(1+\frac{1-\tan x}{1+\tan x}\right) dx$$

$$= \int_0^{\pi/4} \log\left[\frac{1+\tan x + 1 - \tan x}{1+\tan x}\right] dx$$

$$= \int_0^{\pi/4} \log\left(\frac{2}{1+\tan x}\right) dx$$

$$= \int_0^{\pi/4} \log 2\, dx - \int_0^{\pi/4} \log(1+\tan x)\, dx \Rightarrow (\log 2)(x)\Big|_0^{\frac{\pi}{4}} \quad — I$$

$$I = \log_2\left(\frac{\pi}{4}-0\right) - I$$

$$2I = \frac{\pi}{4}\log 2$$

$$\boxed{I = \frac{\pi}{8}\log 2}$$

Ans- (d)

Ques-

$$I = \int_0^{\pi} x \sin^6 x \cos^4 x\, dx =$$

a. $\frac{3\pi^2}{256}$　　b. $\frac{3\pi^2}{128}$　　c. $\frac{3\pi^2}{512}$　　d. 0

Sol^n^- $$= \int_0^{\pi} (\pi - x) \sin^6(\pi-x) \cos^4(\pi-x)\, dx$$

$$I = \int_0^{\pi} \pi \cdot \sin^6 x \cdot \cos^4 x\, dx - \int_0^{\pi} x \sin^6 x \cos^4 x\, dx$$

Property → $\int_0^{2a} f(x)\,dx = 2\int_0^{a} f(x)\,dx$

if $f(2a-x) = f(x)$

$$I = \pi 2 \cdot \int_0^{\frac{\pi}{2}} \sin^6 x \cdot \cos^4 x \, dx - I$$

$$2I = 2\pi \left[ \frac{5\cdot 3\cdot 1 \times 3\cdot 1}{10\cdot 8\cdot 6 \times 4\cdot 2} \cdot \frac{\pi}{2} \right]$$

$$I = \frac{3\pi^2}{512}$$

Ans - (c)

Ques- $\int_0^{\pi} \sin^3 x \cos^5 x \, dx =$

a. $0$
b. $\frac{3\pi}{128}$
c. $\frac{3\pi}{256}$
d. $\frac{5\pi}{128}$

Sol$^n$- $\sin(\pi - x) = \sin x$ $\quad \cos(\pi - x) = -\cos x$

$$f(x) = \sin^3 x \cos^5 x$$

$$\begin{aligned} \& \; f(\pi - x) &= \sin^3(\pi - x) \cos^5(\pi - x) \\ &= (\sin x)^3 (-\cos x)^5 \\ &= -\sin^3 x \cos^5 x \\ &= -f(x) \end{aligned}$$

Ans → (a)

Ques- $\int_0^{\pi} \frac{1}{a^2\cos^2 x + b^2 \sin^2 x} \, dx =$

a. $0$
b. $\frac{\pi}{ab}$
c. $\pi ab$
d. $\frac{\pi}{a^2+b^2}$

Sol$^n$- $2\int_0^{\frac{\pi}{2}} \frac{1}{a^2\cos^2 x + b^2 \sin^2 x} \, dx$

$$= 2\int_0^{\pi/2} \frac{\sec^2 x}{a^2 + b^2 \tan^2 x} \cdot dx \Rightarrow \frac{2}{b^2} \int_0^{\frac{\pi}{2}} \frac{\sec^2 x}{\left(\frac{a}{b}\right)^2 + \tan^2 x} \, dx$$

$$= \frac{2}{b^2} \int_0^{\infty} \frac{1}{\left(\frac{a}{b}\right)^2 + t^2} \cdot dt$$

Let $\tan x = t$ ; $\sec^2 x\, dx = dt$ ; $x=0$ $t=0$ $x=\pi/2$ $t=\infty$

$$= \frac{2}{b^2} \frac{1}{a/b} \left[\tan^{-1}\left(\frac{t}{a/b}\right)\right]_0^{\infty}$$

$$= \frac{2}{b^2} \cdot \frac{b}{a} \left[\tan^{-1}\infty - \tan^{-1} 0\right]$$

$\int \frac{1}{a^2+x^2} dx = \frac{1}{a} \tan^{-1}\frac{x}{a}$

$$= \frac{2}{ab} \cdot \left(\frac{\pi}{2}\right) \Rightarrow \frac{\pi}{ab}$$

Ans- (b)

Ques- $\int_{-\pi}^{\pi} \sin^4 x\, dx =$

a. $\pi/4$ b. $\pi/2$ c. $3\pi/4$ d. $0$

Solⁿ-

$$\Rightarrow 2\int_0^{\pi} \sin^4 x\, dx \Rightarrow 4\int_0^{\pi/2} \sin^4 x \cdot dx \Rightarrow 4\left[\frac{3}{4} \cdot \frac{1}{2} \cdot \frac{\pi}{2}\right]$$

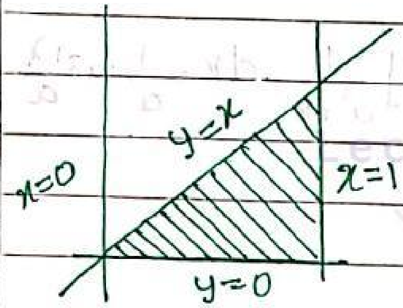
$$= \frac{3\pi}{4}$$

ANS → (c)

~~Ques~~

# Multiple Integrals

Definite Integral →

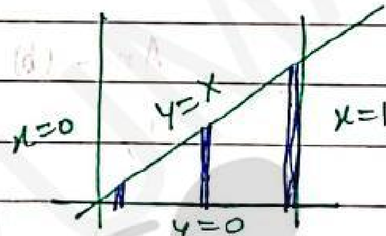



$\Rightarrow \int_0^1 x\,dx$

$\Rightarrow \left(\frac{x^2}{2}\right)_0^1 \Rightarrow \frac{1}{2}$

vertical strip -

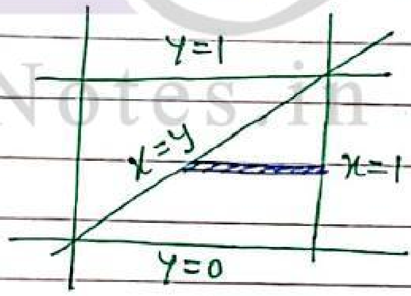
$$\int_{x=0}^{1}\int_0^x dy\,dx$$




$$= \int_0^1 \left[\int_0^x dy\right] dx \Rightarrow \int_0^1 [y]\Big|_0^x dx \Rightarrow \int_0^1 x\,dx \Rightarrow \frac{x^2}{2}\Big|_0^1 = \frac{1}{2}$$

Horizontal Strip -

$$\int_0^1 \int_y^1 dx\,dy$$

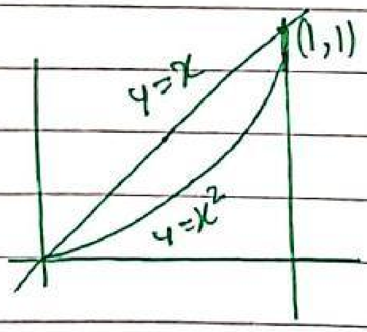



$$= \int_0^1 [x]\Big|_y^1 dy$$

$$= \int_0^1 [1-y]\,dy$$

$$= \left(y - \frac{y^2}{2}\right)\Big|_0^1 = 1 - \frac{1}{2} = \frac{1}{2}$$

Multiple Integral →

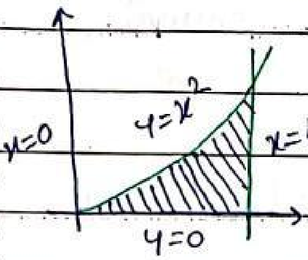


Ques- $\iint_R e^{y/x}\, dy\, dx$ where R is region bounded by - $y=0$, $y=x^2$ in b/w $x=0$ ; $x=4$

a. $e^4-7$ b. $3e^4-7$ c. $e^4-4$ d. $3e^4-8$

Solⁿ-

$$\int_0^4 \int_0^{x^2} e^{y/x}\, dy.dx = \int_0^4 \left[ \frac{e^{y/x}}{1/x} \right]\Bigg|_0^{x^2} dx = \int_0^4 x e^{y/x}\Big|_0^{x^2} dx$$



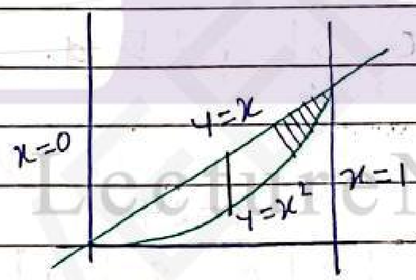

$$= \int_0^4 (xe^x - x)\, dx \Rightarrow \left(xe^x - e^x - \frac{x^2}{2}\right)\Big|_0^4$$

$$= (4e^4 - e^4 - 8) - (0-1-0) \Rightarrow 3e^4 - 7$$

Ques- $\iint_R (x^2y + xy^2)\, dx\, dy$ over the area bounded by $y=x^2$ & $y=x$

a. $\frac{3}{28}$ b. $\frac{3}{56}$ c. $\frac{3}{14}$ d. $\frac{3}{4}$

Solⁿ-




$$= \int_0^1 \int_{x^2}^{x} (x^2y + xy^2)\, dy\, dx$$

$$= \int_0^1 \left[ x^2\left(\frac{y^2}{2}\right) + x\left(\frac{y^3}{3}\right) \right]\Bigg|_{x^2}^{x} dx$$

$$= \int_0^1 \left[ \frac{x^2}{2}(x^2 - x^4) + \frac{x}{3}(x^3 - x^6) \right] dx$$

$$= \int_0^1 \left[ \frac{x^4}{2} - \frac{x^6}{2} + \frac{x^4}{3} - \frac{x^7}{3} \right] dx$$

$$= \left[ \frac{x^5}{10} - \frac{x^7}{14} + \frac{x^5}{15} - \frac{x^8}{24} \right]\Bigg|_0^1 \Rightarrow \frac{1}{10} + \frac{1}{15} - \frac{1}{14} - \frac{1}{24}$$

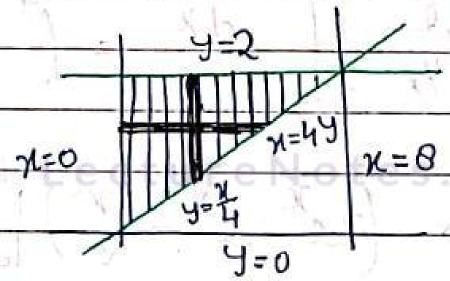
$$= \frac{1}{6} - \frac{1}{14} - \frac{1}{24} = \frac{3}{56}$$

Ans- (b)

Ques- By changing the order of integration $\int_0^8 \int_{x/4}^{2} f(x,y)\,dy\,dx$ leads to $I = \int_r^s \int_p^q f(x,y)\,dx\,dy$ then $q =$

a. $16y^2$ b. $4y$ c. $x$ d. $8$

Sol$^n$-



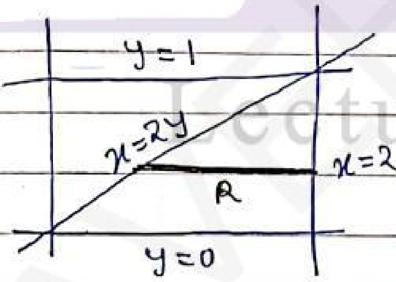

$$\int_{y=0}^{y=2} \int_0^{4y} f(x,y)\,dx\,dy$$

Ans- (b)

Ques- $\iint_R e^{x^2}\,dx\,dy$ where R is given by $0 \le y \le 1$, $2y \le x \le 2$

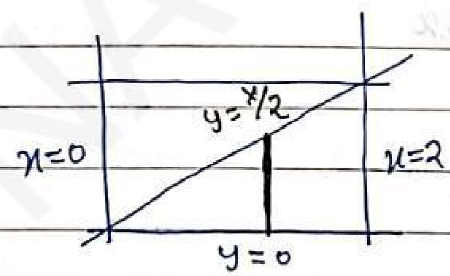
a. $e^4 - 1$ b. $\frac{e^4-1}{4}$ c. $\frac{e^4}{4} - 1$ d. $e^4$

Sol$^n$-




$x = 2y$

| X | 0 | 2 | 4 |
|---|---|---|---|
| y | 0 | 1 | 2 |




$$\int_0^2 \int_0^{x/2} e^{x^2}\,dy\,dx$$

$$= \int_0^2 \left[ e^{x^2} (y) \Big|_0^{x/2} \right] dx$$

$$= \int_0^2 e^{x^2} \left(\frac{x}{2}\right) dx \Rightarrow \int_0^4 \left[e^t\right]_0^4 = \frac{1}{4}\left[e^4 - e^0\right] = \frac{1}{4}\left(e^4 - 1\right)$$

Let $x^2 = t$, $2x\,dx = dt$ | $x\,dx = \frac{dt}{2}$ | $x = 0$, $t = 0$; $x = 2$, $t = 4$

Ans - (b)

# VECTOR CALCULUS

Gradient → vector
Divergent → Scalar
Curl → Vector
Green's → Line & Surface in plane (2 d)
Gauss → Surface & volume
Stokes → Line & Surface in space. (3d)

vector differential operator:

$$\nabla = \bar{i}\frac{\partial}{\partial x} + \bar{j}\frac{\partial}{\partial y} + \bar{k}\frac{\partial}{\partial z}$$

Gradient-

The gradient of scalar function $\phi(x,y,z)$ is given by-

$$\nabla\phi = \bar{i}\frac{\partial\phi}{\partial x} + \bar{j}\frac{\partial\phi}{\partial y} + \bar{k}\frac{\partial\phi}{\partial z}$$

unit normal vector is given by $\dfrac{\nabla\phi}{|\nabla\phi|}$

Directional derivative-

The directional derivative of Scalar point function $\phi(x,y,z)$ at a point P in the direction of $\bar{a}$ is given by $\nabla\phi$ at P $\cdot \dfrac{\bar{a}}{|\bar{a}|}$

i.e. The component of $\nabla\phi$ at point P in the direction o $\bar{a}$

Divergent of vector Point function-

Let $\bar{F} = F_1\bar{i} + F_2\bar{j} + F_3\bar{k}$ be a vector point function.

$$\operatorname{div} \bar{F} = \nabla \cdot \bar{F} = \left(\bar{i}\frac{\partial}{\partial x} + \bar{j}\frac{\partial}{\partial y} + \bar{k}\frac{\partial}{\partial z}\right)\cdot\left(F_1\bar{i} + F_2\bar{j} + F_3\bar{k}\right)$$

$$= \frac{\partial}{\partial x}(F_1) + \frac{\partial}{\partial y}(F_2) + \frac{\partial}{\partial z}(F_3)$$

$\operatorname{div} \bar{F} = 0 \iff \bar{F}$ is a solenoidal vector.

Curl of a vector point function-

Let $\bar{F} = F_1\bar{i} + F_2\bar{j} + F_3\bar{k}$ be a vector point funⁿ

$$\operatorname{curl} \bar{F} = \nabla \times \bar{F}$$

$$= \begin{vmatrix} \bar{i} & \bar{j} & \bar{k} \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ F_1 & F_2 & F_3 \end{vmatrix}$$

$$= \bar{i}\left(\frac{\partial F_3}{\partial y} - \frac{\partial F_2}{\partial z}\right) + \bar{j}\left[\frac{\partial F_3}{\partial x} - \frac{\partial F_1}{\partial z}\right] + \bar{k}\left[\frac{\partial F_2}{\partial x} - \frac{\partial F_1}{\partial y}\right]$$

$\operatorname{curl} \bar{F} = 0 \iff \bar{F}$ is irrotational.

Note -

If $\bar{r} = x\bar{i} + y\bar{j} + z\bar{k}$ then -

a. $$\nabla(f(r)) = \frac{f'(r)}{r}\,\bar{r}$$

b. $$\nabla^2[f(r)] = f''(r) + \frac{2}{r}f'(r)$$

$$\nabla\cdot\nabla[f(r)] = \nabla^2[f(r)]$$

eg-

$f(r) = \log r \qquad \bar{r} = x\bar{i} + y\bar{j} + z\bar{k} \qquad |r| = \sqrt{x^2+y^2+z^2}$

$r^2 = x^2+y^2+z^2$

$$\nabla[f(r)] = \bar{i}\frac{\partial}{\partial x}[f(r)] + \bar{j}\frac{\partial}{\partial y}[f(r)] + \bar{k}\frac{\partial}{\partial z}[f(r)]$$

$$\nabla[\log r] = \bar{i}\frac{\partial}{\partial x}(\log r) + \bar{j}\frac{\partial}{\partial y}(\log r) + \bar{k}\frac{\partial}{\partial z}(\log r)$$

$$= \bar{i}\,\frac{1}{r}\,\frac{dr}{dx} + \bar{j}\,\frac{1}{r}\,\frac{dr}{dy} + \bar{k}\,\frac{1}{r}\,\frac{dr}{dz}$$

$$= \bar{i}\,\frac{x}{r^2} + \bar{j}\,\frac{y}{r^2} + \bar{k}\,\frac{z}{r^2}$$

$$= \frac{1}{r^2}\left[x\bar{i} + y\bar{j} + z\bar{k}\right]$$

$\Rightarrow$ $f(r) = \log r$

$$f'(r) = \frac{1}{r}$$

$$\nabla(\log r) = \frac{1/r}{r}\,\bar{r} = \frac{1}{r^2}\,\bar{r}$$

Ques- $\nabla^2(\log r) = ?$

Solⁿ-

$$f(r) = \log r$$

$$f'(r) = \frac{1}{r}$$

$$f''(r) = -\frac{1}{r^2}$$

$$\nabla^2[f(r)] = f''(r) + \frac{2}{r}\,f'(r)$$

$$\nabla^2(\log r) = \frac{-1}{r^2} + \frac{2}{r}\left(\frac{1}{r}\right)$$

$$= \frac{-1}{r^2} + \frac{2}{r^2} \Rightarrow \frac{1}{r^2}$$

Ques- The unit normal vector to the surface $\phi = x^3 - xyz + z^3 - 1$ at $(1,1,1)$ is -

Solⁿ- Unit Normal vector to $\phi$ at P $= \dfrac{\nabla\phi}{|\nabla\phi|}$ at point P.

$$\text{grad } \phi = \nabla\phi = \bar{i}\,\frac{d\phi}{dx} + \bar{j}\,\frac{d\phi}{dy} + \bar{k}\,\frac{d\phi}{dz}$$

$= \bar{i}\,(3x^2 + yz) + \bar{j}\,(-xz) + \bar{k}\,(-xy + 3z^2)$

$\nabla\phi_{at(1,1)} = \bar{i}\,(3-1) + \bar{j}\,(-1) + \bar{k}\,(-1+3)$

$= 2\bar{i} - \bar{j} + 2\bar{k}$

$|\nabla\phi_{at(1,1,1)}| = \sqrt{4+1+4} = 3$

unit normal vector is -

$$\frac{\nabla\phi}{|\nabla\phi|} = \frac{2\bar{i} - \bar{j} + 2\bar{k}}{3}$$

Ques - The directional derivative of $\phi = xyz$ at $(1,1,1)$ in the direction of $\bar{i} + \bar{j} + \bar{k}$ is -

a. 3  b. $\sqrt{3}$  c. $\frac{1}{\sqrt{3}}$  d. 0

Sol$^n$ -

Directional derivative of $\phi$ at $P$ in the direction of $\bar{a}$ is - $\nabla\phi_{at\,P} \cdot \frac{\bar{a}}{|\bar{a}|}$

$\phi = xyz$

$$\text{grad}\,\phi = \nabla\phi = \bar{i}\frac{\partial\phi}{\partial x} + \bar{j}\frac{\partial\phi}{\partial y} + \bar{k}\frac{\partial\phi}{\partial z}$$

$= \bar{i}(yz) + \bar{j}\,(xz) + \bar{k}\,(xy)$

$\text{grad}\,\phi_{at(1,1,1)} = \bar{i} + \bar{j} + \bar{k}$

The directional derivative of $\phi$ at $(1,1,1)$ in the direction of $\bar{a} = \bar{i}, \bar{j}, \bar{k}$ is -

$$\nabla\phi_{at(1,1,1)} \cdot \frac{\bar{a}}{|\bar{a}|}$$

$$= (\bar{i} + \bar{j} + \bar{k}) \cdot \frac{\bar{i} + \bar{j} + \bar{k}}{\sqrt{1+1+1}}$$

$$= \frac{1+1+1}{\sqrt{3}} \Rightarrow \frac{3}{\sqrt{3}} = \sqrt{3} \quad \underline{(b)}$$

Ques- The angle of intersection at the point $(2,-1,2)$ of the surface $\underbrace{x^2+y^2+z^2=a}_{\phi}$ and $\underbrace{z=x^2+y^2-3}_{f}$ is —

a. $\cos^{-1}\left(\frac{1}{3\sqrt{21}}\right)$ b. $\cos^{-1}\left(\frac{8}{3\sqrt{21}}\right)$ c. $30^\circ$ d. $0$

Sol$^n$—

$$\nabla\phi = \bar{i}\frac{\partial\phi}{\partial x} + \bar{j}\frac{\partial\phi}{\partial y} + \bar{k}\frac{\partial\phi}{\partial z}$$

$$= \bar{i}(2x) + \bar{j}(2y) + \bar{k}(2z)$$

$$\nabla\phi_{at\,(2,-1,2)} = \underbrace{4\bar{i} - 2\bar{j} + 4\bar{k}}_{\bar{a}\text{ say}}$$

$$\Delta f = \bar{i}\frac{\partial f}{\partial x} + \bar{j}\frac{\partial f}{\partial y} + \bar{k}\frac{\partial f}{\partial z}$$

$$= \bar{i}(2x) + \bar{j}(2y) - \bar{k}$$

$$\nabla f_{at\,(2,-1,2)} = \underbrace{4\bar{i} - 2\bar{j} - \bar{k}}_{\bar{b}\text{ say}}$$

$$\cos\theta = \frac{\bar{a}\cdot\bar{b}}{|\bar{a}||\bar{b}|} = \frac{(4\bar{i}-2\bar{j}+4\bar{k})\cdot(4\bar{i}-2j+k)}{\sqrt{16+4+16}\;\sqrt{16+4+1}}$$

$$= \frac{16+4-4}{6\sqrt{21}} \Rightarrow \frac{16}{6\sqrt{21}} \Rightarrow \frac{8}{3\sqrt{21}}$$

Ans → (b).

Ques- If $\bar{r} = x\bar{i} + y\bar{j} + z\bar{k}$ then $\nabla(r) =$ ____

a. $0$ b. $\bar{r}$ c. $\frac{\bar{r}}{r}$ d. $-\frac{\bar{r}}{r^3}$

Sol$^n$— $f(r) = r$ $f'(r) = 1$

$$\nabla[f(r)] = \frac{f'(r)}{r}\bar{r}$$

$$\nabla(r) = \frac{1}{r}\bar{r} \Rightarrow \frac{\bar{r}}{r}$$

Ques- If $\bar{r} = x\bar{i} + y\bar{j} + z\bar{k}$ then → div $\bar{r}$ = ____

a. 0 b. 1 c. 3 d. 2

Solⁿ-

$$\text{div } \bar{F} = \frac{\partial F_1}{\partial x} + \frac{\partial F_2}{\partial y} + \frac{\partial F_3}{\partial z}$$

$$\text{div } \bar{r} = \frac{\partial}{\partial x}(x) + \frac{\partial}{\partial y}(y) + \frac{\partial}{\partial z}(z)$$

$$= 1+1+1 \quad = 3$$

Ques- If $\bar{f} = (x+3y)\bar{i} + (y-2z)\bar{j} + (x+az)\bar{k}$ is a solenoidal vector then $a =$

a. 2 b. -2 c. 0 d. 4

Solⁿ-

$$\text{div } \bar{F} = \frac{\partial F_1}{\partial x} + \frac{\partial F_2}{\partial y} + \frac{\partial F_3}{\partial z}$$

$$= \frac{\partial}{\partial x}(x+3y) + \frac{\partial}{\partial y}(y-2z) + \frac{\partial}{\partial z}(x+az)$$

$\bar{F}$ is solenoidal

$$\therefore \text{div } \bar{F} = 0$$

$$1+1+a = 0$$

$$a = -2 \qquad \text{Ans- (b)}$$

Ques- If $\bar{r} = x\bar{i} + y\bar{j} + z\bar{k}$ then $\nabla^2(r^3) =$ ____

a. 0 b. $9r$ c. $3r$ d. $12r$

Solⁿ-

$f(r) = r^3 \qquad f'(r) = 3r^2 \qquad f''(r) = 6r$

$$\nabla^2[f(r)] = f''(r) + \frac{2}{r}f'(r)$$

$$\nabla^2(r^3) = 6r + \frac{2}{r}(3r^2)$$

$$= 6r + 6r$$

$$= \underline{\underline{12r}}$$

Ans → (d)

Ques- If $\bar{F} = (x+2y+az)\,\bar{i} + (bx-3y-z)\,\bar{j} + (4x+cy+2z)\,\bar{k}$ is irrotational then a, b, c are -

a. 4, 2, 1

b. 4, -2, 1

c. 4, 2, -1

d. -4, -2, -1

Sol$^n$- $\bar{F}$ is irrotational

$$\nabla \times \bar{F} = 0$$

$$\begin{vmatrix} \bar{i} & \bar{j} & \bar{k} \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ (x+2y+az) & (bx-3y-z) & (4x+cy+2z) \end{vmatrix} = 0$$

$$\bar{i}\,(c+1) - \bar{j}\,(4-a) + \bar{k}\,(b-2) = 0$$

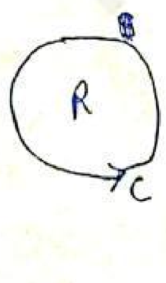
$c = -1$ $\quad a = 4$ $\quad b = 2$

Ans- (c)

<u>Green's Theorem</u>-

If P and Q are two differentiable functions of x and y in the region R bounded by a closed curve C, then

$$\int_C P\,dx + Q\,dy = \iint_R \left( \frac{\partial Q}{\partial x} - \frac{\partial P}{\partial y} \right) dx\,dy$$



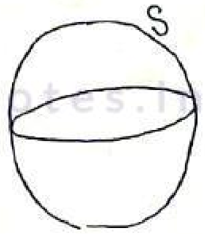

Relation b/w line integral and surface integral when the surface is defined as a plane.

Gauss divergence theorem-

If $\bar{F} = F_1 \bar{i} + F_2 \bar{j} + F_3 \bar{k}$

is a differential vector point function on a closed surface S enclosing volume V then

$$\iint_S \bar{F} \cdot \hat{n}\, ds = \iiint_V \text{div}\, \bar{F}\, dV$$






Stoke's theorem-

If $\bar{F} = F_1 \bar{i} + F_2 \bar{j} + F_3 \bar{k}$ is a differentiable vector point function on a open surface S bounded by closed curve c then,

$$\int_c \bar{F} \cdot d\bar{r} = \iint_S \text{curl}\, \bar{F} \cdot \hat{n}\, ds.$$

Ques- Evaluate $\int_c \bar{F} \cdot d\bar{r}$ where $\bar{F} = 3x^2 \bar{i} + (2xz - y)\bar{j} + z\bar{k}$ along straigh line joining $(0,0,0)$ & $(2,1,3)$ is-

a. 14

b. 16

c. 18

d. 20



Soln-
$$\bar{F} = 3x^2 \bar{i} + (2xz - y)\bar{j} + z\bar{k}$$

$$d\bar{r} = dx\bar{i} + dy\bar{j} + dz\bar{k}$$

$$\bar{F} \cdot d\bar{r} = 3x^2 dx + (2xz - y)dy + z\, dz$$

eqn of line-

$$\frac{x - x_1}{x_2 - x_1} = \frac{y - y_1}{y_2 - y_1} = \frac{z - z_1}{z_2 - z_1}$$

$$\frac{x - 0}{2 - 0} = \frac{y - 0}{1 - 0} = \frac{z - 0}{3 - 0}$$

$(0,0,0) \quad (2,1,3)$

$x=2t$

$x=0$
$2t=0$ $\quad t=1$

$$\frac{x}{2}=\frac{y}{1}=\frac{z}{3}=t \quad (say)$$

$x=2t \qquad y=t \qquad z=3t$

$dx=2dt \qquad dy=dt \qquad dz=3dt$

$$\bar{F} = 3x^2\,\bar{i} + (2xz-y)\,\bar{j} + z\,\bar{k}$$

$$d\bar{r} = dx\,\bar{i} + dy\,\bar{j} + dz\,\bar{k}$$

$$\bar{F}\,d\bar{r} = 3x^2\,dx + (2xz-y)\,dy + z\,dz$$

$$\int_c \bar{F}.d\bar{r} = \int 3(2t)^2.2dt + [2(2t)(3t)-t]\,dt + (3t)\,3dt$$

$$= \int (24t^2+12t^2-t+9t)\,dt$$

$$= \int_0^1 (36t^2+8t)\,dt \Rightarrow \left[36\left(\frac{t^3}{3}\right)+8\left[\frac{t^2}{2}\right]\right]\Big|_0^1$$

$$= \frac{36}{3}+\frac{8}{2} \Rightarrow 12+4$$

$$= 16$$

Ques- If $\bar{f} = x^2 i + xy\,\bar{j}$ then evaluate $\int_c \bar{f}\cdot d\bar{r}$ where c is a straight line joining two points $(0,0)$ & $(1,1)$

a. $3/2$   c. $1/3$

b. $2/3$   d. $0$

Solⁿ- $(0,0) \quad (1,1)$

$y=x$   $x$ varies from 0 to 1

$dy=dx$   $y$ varies from 0 to 1.

$$F = x^2\,\bar{i} + xy\,\bar{j}$$

$$d\bar{r} = dx\,\bar{i} + dy\,\bar{j}$$

$$\bar{F}\cdot d\bar{r} = x^2\,dx + xy\,dy$$

$\longrightarrow$ on C   $y=x$, $dy=dx$

$$\int_c \bar{F}\,d\bar{r} = \int_c x^2\,dx + x(x)\,dx$$

$$= \int_c (x^2+x^2)\, dx = \int_0^1 2x^2\, dx = \frac{2}{3}$$ Ans-(b)

Ques- If $\bar{F} = yz\,\bar{i} + zx\,\bar{j} + xy\,\bar{k}$ and S is the part of $x^2+y^2+z^2=1$ in the first octant then

$$\iint_S \bar{F}\cdot\hat{n}\, dS =$$

a. $3/2$ c. $3/4$

b. $3/8$ d. $1/8$

Soln

$$\iint_S \bar{F}\cdot\hat{n}\, dS = \iint_S F_1 \cos\alpha\, dS + \iint_S \cos\beta\, dS + \iint_S F_3 \cos\gamma\, dS$$

$$= \iint_S F_1\, \underset{x=0}{\underset{\uparrow}{dy\,dz}} + \iint_S F_2\, \underset{y=0}{\underset{\uparrow}{dx\,dz}} + \iint_S F_3\, \underset{z=0}{\underset{\uparrow}{dx\,dy}}$$

$$= \int_0^1 \int_0^{\sqrt{1-y^2}} yz\, dy\,dz + \int_0^1 \int_0^{\sqrt{1-x^2}} zx\, dx\,dz + \int_0^1 \int_0^{\sqrt{1-x^2}} xy\, dy\,dx$$

$$= \int_0^1 y\left(\frac{z^2}{2}\right)\Big|_0^{\sqrt{1-y^2}} dy + \int_0^1 x\left(\frac{z^2}{2}\right)\Big|_0^{\sqrt{1-x^2}} dx + \int_0^1 x\left(\frac{y^2}{2}\right)\Big|_0^{\sqrt{1-x^2}} dx$$

$$= \int_0^1 y\,\frac{(1-y^2)}{2}\, dy + \int_0^1 x\,\frac{(1-x^2)}{2}\, dx + \int_0^1 x\,\frac{(1-x^2)}{2}\, dx$$

$$= \frac{1}{2}\left[\frac{y^2}{2} - \frac{y^4}{4}\right]\Big|_0^1 + \frac{1}{2}\left(\frac{x^2}{2} - \frac{x^4}{4}\right)\Big|_0^1 + \frac{1}{2}\left(\frac{x^2}{2} - \frac{x^4}{4}\right)\Big|_0^1$$

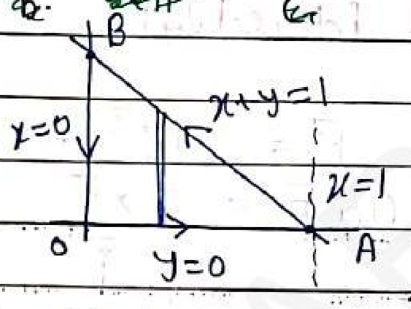
$$= \frac{1}{2}\left(\frac{1}{2} - \frac{1}{4}\right) + \frac{1}{2}\left(\frac{1}{2} - \frac{1}{4}\right) + \frac{1}{2}\left(\frac{1}{2} - \frac{1}{4}\right)$$

$$= \frac{1}{8} + \frac{1}{8} + \frac{1}{8} = \frac{3}{8}$$

Ques- $\int_C (3x^2 - 8y^2)\,dx + (4y - 6xy)\,dy$
where c is boundary of $x=0$; $y=0$; $x+y=1$.

Soln- $P = 3x^2 - 8y^2$
$Q = 4y - 6xy$

$\frac{\partial P}{\partial y} = -16y \qquad \frac{\partial Q}{\partial x} = -6y$




$$\int P\,dx + Q\,dy = \iint_R \left(\frac{\partial Q}{\partial x} - \frac{\partial P}{\partial y}\right) dy\,dx$$

$$= \int_0^1 \int_{y=0}^{y=1-x} (-6y + 16y)\,dy\,dx$$

$$= \int_0^1 \left[\int_0^{1-x} 10y\,dy\right] dx$$

$$= \int_0^1 10\left(\frac{y^2}{2}\right)\Big|_0^{1-x} dx \Rightarrow \int_0^1 5(1-x)^2\,dx$$

$$= 5\int_0^1 (1 + x^2 - 2x)\,dx \Rightarrow 5\left[x + \frac{x^3}{3} - 2\frac{x^2}{2}\right]\Big|_0^1 = \frac{5}{3}$$

Ques- If S is a closed surface $x^2+y^2+z^2 = 4$ and $\bar{r} = x\bar{i} + y\bar{j} + z\bar{k}$ then $\iint_S \bar{r}\cdot\hat{n}\,ds = ?$

a. $8\pi$ b. $32\pi$ c. $16\pi$ d. $4\pi$

Soln-

$$\iint_S \vec{F}\cdot\hat{n}\,ds = \iiint_V \operatorname{div}\vec{F}\,dV$$

$$\iint_S \bar{r}\cdot\hat{n}\,ds = \iiint_V \operatorname{div}\bar{r}\,dV \Rightarrow \iiint_V 3\,dV$$

$$= 3V \Rightarrow \cancel{3}\left(\frac{32\pi}{\cancel{3}}\right)$$

$$= 32\pi$$

because- $\bar{r} = x\bar{i} + y\bar{j} + z\bar{k}$

$$\operatorname{div}\bar{r} = \nabla\cdot\bar{r}$$

$$= \frac{d}{dx}(x) + \frac{d}{dy}(y) + \frac{d}{dz}(z)$$

$$= 1+1+1 \Rightarrow 3$$

Ques- $\int_C yz\,dx + xz\,dy + xy\,dz$ where $x^2+y^2=4$ & $z=0$

a. 4 b. 2 c. 3 d- 0

Ans- (d). [directly]

$$\bar{F} = yz\,\bar{i} + xz\,\bar{j} + xy\,\bar{k}$$

$$\text{curl } \bar{F} = \nabla \times \bar{F}$$

$$= \begin{vmatrix} \bar{i} & \bar{j} & \bar{k} \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ yz & xz & xy \end{vmatrix}$$

$$= \bar{i}\,[x-x] - \bar{j}\,[y-y] + \bar{k}\,[z-z]$$

$$= \underline{\underline{0}}$$

# DIFFERENTIAL EQUATIONS

Ordinary Differential Equation | Partial Differential Equations.

eg. $\dfrac{d^2y}{dx^2} + 4\dfrac{dy}{dx} + y = 0$ | eg. $\dfrac{d^2u}{dx^2} + \dfrac{d^2u}{dy^2} = 0$

Order of differential $eq^n$ -

The order of a differential $eq^n$ is the order of the highest ordered derivative occuring in the differential $eq^n$.

eg. → order

$$\frac{d^3y}{dx^3} - 3\frac{d^2y}{dx^2} + 3\frac{dy}{dx} - y = 0$$

Order is → 3

Degree of a differential $eq^n$ -

The degree of a differential $eq^n$ is the degree of the highest ordered derivative occuring in the differential $eq^n$ when the derivatives are free from fractional powers.

eg →

$$\frac{d^2y}{dx^2} = \sqrt{1 + \left(\frac{dy}{dx}\right)^3}$$

Squaring on both sides -

$$\left(\frac{d^2y}{dx^2}\right)^2 = 1 + \left(\frac{dy}{dx}\right)^3$$

→ degree.

→ order.

Formation of differential Equations –

1. Write the equation of the give curve.
2. Differentiate the above eq^n as no. of times as no. of arbitrary constants.
3. Eliminate arbitrary constants from the given eq^ns by using the eq^ns obtained in step 1 & step 2.

eq^n – Find the diff. eq^n of the family of order →

$$x^2+y^2+2gx=0 \rightarrow (1)$$

diff. (1) w.r.t. $x$ –

$$2x+2y\frac{dy}{dx}+2g=0$$

$$x+y\frac{dy}{dx}+g=0$$

$$g=-x-y\frac{dy}{dx} \rightarrow (ii)$$

Sub (2) in (1) →

$$x^2+y^2+2x\left(-x-y\frac{dy}{dx}\right)=0$$

$$x^2+y^2-2x^2-2xy\frac{dy}{dx}=0$$

$$y^2-x^2-2xy\frac{dy}{dx}=0$$

which is the required diff. eq^n.

Ques – Find the diff. eq^n of the family of curves → $xy=c^2$

Sol^n – given eq^n is $xy=c^2 \rightarrow (1)$

diff. (1) w.r.t. $x$ –

$$x\frac{dy}{dx}+y=0 \quad — (2)$$

which is the required diff. eq^n.

eg.- Find diff $eq^n$ of the fimily of curves $y = Ax^2 + Bx$.

Sol$^n$- given $eq^n$ is $y = Ax^2 + Bx \rightarrow (1)$

$y_1 = 2Ax + B \rightarrow (2)$

$y_2 = 2A + 0 \rightarrow (3)$

First method is substitution method.

Second Method -

The differential $eq^n$ is $\begin{vmatrix} y & x^2 & x \\ y_1 & 2x & 1 \\ y_2 & 2 & 0 \end{vmatrix} = 0$

$\Rightarrow y(0-2) - x^2(0-y_2) + x(2y_1 - 2xy_2) = 0$

$-2y + x^2y_2 + 2xy_1 - 2x^2y_2 = 0$

$x^2y_2 - 2xy_1 + 2y = 0 \rightarrow$ is the required $eq^n$.

Ques- The diff. $eq^n$ of $\rightarrow$ $y = Ae^{2x} + Be^{-3x}$

Sol$^n$- roots are $2, -3$

diff. $eq^n$ is $\rightarrow$

$(D-2)(D+3)y = 0$

$(D^2 - 2D + 3D - 6)y = 0$

$(D^2 + D - 6)y = 0$

+ only when given $eq^n$ consist exponential function.

① $y = Ae^{2x} + Be^{3x} + 5$

$= Ae^{2x} + Be^{3x} + 5e^{0x}$

$\Rightarrow$ Solution of diffrential $eq^n$ -

$$\frac{d^2y}{dx^2} + 4\frac{dy}{dx} + 4y = 0$$

Let $\frac{d}{dx} = D$

$(D^2 - 4D + 4)y = 0$

$\frac{d^2}{dx^2} = D^2$ Aux. $eq^n$ is $\rightarrow$ $m^2 - 4m + 4 = 0$

$m = 2, 2$ CF is $\underline{(C_1 + C_2x)e^{2x}}$

| roots of auxilary $eq^n$ | corresponding complementry $fn^n$ |
| --- | --- |
| Real and Distinct $m_1, m_2, m_3$ | $C_1 e^{m_1 x} + C_2 e^{m_2 x} + C_3 e^{m_3 x}$ |
| Real and equal $m, m, m$ | $(C_1 + C_2 x + C_3 x^2) e^{mx}$ |
| Conjugate complex pairs. $\alpha \pm i\beta$ | $e^{\alpha x}[C_1 \cos\beta x + C_2 \sin\beta x]$ |

| | |
| --- | --- |
| $m_1, m_2, m, m$ | $C_1 e^{m_1 x} + C_2 e^{m_2 x} + (C_3 + C_4 x) e^{mx}$ |
| $\alpha \pm i\beta, m, m$ | $e^{\alpha x}(C_1 \cos\beta x + C_2 \sin\beta x) + (C_3 + C_4 x) e^{mx}$ |
| $\alpha \pm i\beta, \alpha \pm i\beta$ | $e^{\alpha x}[(C_1 + C_2 x)\cos\beta x + (C_3 + C_4 x)\sin\beta x]$ |

eg.- $(D^4 - 16) y = 0$

Auxilary $eq^n$ is

$$m^4 - 16 = 0$$

$$(m^2-4)(m^2+4) = 0$$

$$m^2 - 4 = 0 \qquad m^2 + 4 = 0$$

$$m = \pm 2, \quad \pm 2i$$

CF is $y = C_1 e^{-2x} + C_2 e^{2x} + e^{0x}[C_3 \cos 2x + C_4 \sin 2x]$

$$y = C_1 e^{-2x} + C_2 e^{2x} + C_3 \cos 2x + C_4 \sin 2x$$

eg.- $\frac{d^2y}{dx^2} - 6\frac{dy}{dx} + 9y = 0$

$$(D^2 - 6D + 9)y = 0$$

AE is:

$$m^2 - 6m + 9 = 0$$

$$(m-3)^2 = 0$$

$$m = 3, 3 \quad \text{real and equal.}$$

CF is $y = (C_1 + C_2 x)e^{3x}$

Ques- $(D^2 - 3D + 2)y = e^{4x}$

Sol$^n$- AE is:

$$m^2 - 3m + 2 = 0$$

$$(m-2)(m-1) = 0$$

$$m = 2, 1$$

CF is $C_1 e^{2x} + C_2 e^{x}$

$$P.I = \frac{1}{D^2 - 3D + 2} e^{4x} \qquad D = 4$$

$$= \frac{1}{4^2 - 3(4) + 2} e^{4x}$$

$$= \frac{1}{16 - 12 + 2} e^{4x} \Rightarrow \frac{1}{6} e^{4x} \Rightarrow \frac{e^{4x}}{6}$$

G.S. = CF + PI

$$y = C_1 e^{2x} + C_2 e^{x} + \frac{e^{4x}}{6}$$

Ques- $(D^2 - 3D + 2)y = e^{2x}$

Sol$^n$- CF is $C_1 e^{2x} + C_2 e^{x}$

$$P.I = \frac{1}{D^2 - 3D + 2} e^{2x} \qquad D = 2$$

[Case Failure]

$= x \cdot \frac{1}{2D-3} e^{2x}$ [derivative of den.]

$(D=2)$

$= x \cdot \frac{1}{2(2)-3} e^{2x}$

$= x e^{2x}$

Ques- Solve $(D^2 - 6D + 9)y = e^{3x}$

Sol$^n$- AE is:

$m^2 - 6m + 9 = 0$

$m = (3, 3)$

CF is $\rightarrow$ $(c_1 + c_2 x) e^{3x}$

P.I. is $\rightarrow$

$= \frac{1}{D^2 - 6D + 9} \cdot e^{3x}$

$D = 3$ case of failure

$= x \cdot \frac{1}{2D-6} \cdot e^{3x}$

$D = 3$ case of failure.

$= x^2 \cdot \frac{1}{2} e^{3x}$

$= \frac{x^2 e^{3x}}{2}$

⊙ $\sin \beta x$ or $\cos \beta x$ :

$f(D^2) y = \cos \beta x$

$P.I. = \frac{1}{f(D^2)} \cdot \cos \beta x$

$D^2 = -(\beta^2)$

Ques- Solve $(D^2+9)\,y = \cos 2x$

Solⁿ- AE is $m^2+9=0$

$m = \pm 3i$

CF is $e^{0x}(C_1\cos 3x + C_2\sin 3x)$

$= C_1\cos 3x + C_2 \sin 3x$

$P.I. = \frac{1}{D^2+9}\cos 2x$ $\qquad D^2 = -(2^2) = -4$

$= \frac{1}{-4+9}\cos 2x$

$= \frac{\cos 2x}{5}$

$G.S. = C.F. + P.I.$

$y = C_1\cos 3x + C_2\sin 3x + \frac{\cos 2x}{3}$

Ques- Solve $(D^2+4)y = \sin 2x$

Solⁿ- AE is $m^2+4=0$

$m = \pm 2i$

CF is $C_1\cos 2x + C_2\sin 2x$

$PI = \frac{1}{D^2+4}\sin 2x$ $\qquad D^2 = -(2^2) = -4$

$= x\frac{1}{2D}\sin 2x$ $\qquad$ case of failure.

$= \frac{x}{2}\cdot\frac{1}{D}(\sin 2x)$

$= \frac{x}{2}\int \sin 2x\, dx$

$= \frac{x}{2}\left(\frac{-\cos 2x}{2}\right) \Rightarrow \frac{-x\cos 2x}{4}$

Same → $PI = \frac{1}{D^2+4}\sin 2x \Rightarrow x\frac{1}{2D}\sin 2x \Rightarrow \frac{x}{2}\frac{D}{D^2}(\sin 2x)$

$= \frac{x}{2}\frac{D}{(-4)}(\sin 2x) \Rightarrow \frac{-x}{8}(2\cos 2x) \Rightarrow -\frac{x\cos 2x}{4}$

Ques- $(D^2-2D+1)y = \cos 2x$

Sol$^n$- AE is $m^2-2m+1=0$

$m = 1, 1$

CF is $(C_1+C_2x)e^x$

$$P.I. = \frac{1}{D^2-2D+1}\cos 2x \qquad D^2 = -(2^2) = -4$$

$$= \frac{1}{-4-2D+1}\cos 2x$$

$$= \frac{1}{-3-2D}\cos 2x$$

$$= \frac{1}{-(3+2D)}\cos 2x \Rightarrow -\frac{(3-2D)}{(3+2D)(3-2D)}\cos 2x$$

$$\Rightarrow \frac{-3+2D}{9-4D^2}\cos 2x \Rightarrow \frac{-3+2D}{9-4(-4)}\cos 2x$$

$$\Rightarrow \frac{-3\cos 2x + 2D(\cos 2x)}{9+16}$$

$$\Rightarrow \frac{-3\cos 2x + 2(-2\sin 2x)}{25} \Rightarrow \frac{-3}{25}\cos 2x - \frac{4}{25}\sin 2x$$

G.S. = CF + PI

$$y = (C_1+C_2x)e^x - \frac{3}{25}\cos 2x - \frac{4}{25}\sin 2x$$

$$\boxed{\begin{aligned}(1+t)^{-1} &= 1-t+t^2-t^3+\text{-------}\\ (1-t)^{-1} &= 1+t+t^2+t^3+\text{-------}\end{aligned}}$$

Ques- $(D^2-3D+2)y = x^2$

Sol$^n$- AE is $m^2-3m+2=0$

$m = 1, 2$

CF is $C_1e^x + C_2e^{2x}$

$$P.I = \frac{1}{D^2-3D+2}x^2 \Rightarrow \frac{1}{2-3D+D^2}x^2$$

$$= \frac{1}{2\left[1-\left(\frac{3D-D^2}{2}\right)\right]}x^2$$

$$= \frac{1}{2}\left[1-\left(\frac{3D}{2}-\frac{D^2}{2}\right)\right]^{-1}x^2$$

$$= \frac{1}{2}\left\{1+\left(\frac{3D}{2}-\frac{D^2}{2}\right)+\left(\frac{3D}{2}-\frac{D^2}{2}\right)^2+------\right\}x^2$$

$$= \frac{1}{2}\left\{1+\frac{3D}{2}-\frac{D^2}{2}+\frac{9D^2}{4}\right\}x^2$$

$D(x^2)=2x$
$D^2(x)^2=2$
$D^3(x^2)=0$

$$= \frac{1}{2}\left\{x^2+\frac{3}{2}D(x^2)-\frac{D^2(x^2)}{2}+\frac{9}{4}D^2(x^2)\right\}$$

$$= \frac{1}{2}\left\{x^2+\frac{3}{2}(2x)-\frac{2}{2}+\frac{9}{4}(2)\right\}$$

$$= \frac{1}{2}\left\{x^2+3x-1+\frac{9}{2}\right\}$$

$$\Rightarrow \frac{1}{2}\left\{x^2+3x+\frac{7}{2}\right\}$$

⊙ $e^{\alpha x}V$

$V = \sin\beta x$ (or) $\cos\beta x$ (or) $x^K$

Ques-Form ⇒ $f(D)y = e^{\alpha x}V$

Solⁿ-

$$P.I. = \frac{1}{f(D)}e^{\alpha x}V \qquad D = D+\alpha$$

$$= e^{\alpha x}\frac{1}{f(D+\alpha)}V$$

Ques- $(D^2+4D+4)y = e^{2x}\cdot x^2$

AE is $m^2+4m+4=0$

$m=-2,-2$

CF is $(C_1 + C_2 x) e^{-2x}$

$$P.I. = \frac{1}{D^2+4D+4} e^{-2x} \cdot x^2$$

$D = D-2$

$$= e^{-2x} \frac{1}{(D-2)^2+4(D-2)+4} \cdot x^2$$

$$= e^{-2x} \frac{1}{D^2-4D+4+4D-8+4} x^2$$

$$= e^{-2x} \cdot \frac{1}{D^2} \cdot x^2$$

$$= e^{-2x} \cdot \frac{1}{D} \cdot \frac{1}{D} (x^2) \Rightarrow e^{-2x} \cdot \frac{1}{D} \int x^2 \, dx$$

$$\Rightarrow e^{-2x} \int \frac{x^3}{3} dx \Rightarrow e^{-2x} \cdot \frac{1}{3} \cdot \frac{x^4}{4} \Rightarrow \frac{e^{-2x} x^4}{12}$$

GS = CF + PI.

⊙ $x \sin\beta x$ (or) $x \cos\beta x$

$$e^{i\theta} = \cos\theta + i\sin\theta$$

$\cos\theta$ = Real part of $e^{i\theta}$

$\sin\theta$ = I. P. of $e^{i\theta}$

$$P.I. = \frac{1}{f(D)} \cdot x \sin\beta x$$

$$= IP \text{ of } \frac{1}{f(D)} \cdot x e^{i\beta x}$$

$$= IP \text{ of } e^{i\beta x} \frac{1}{f(D+i\beta)} x$$

$D = D + i\beta$

$$\boxed{\frac{1}{f(D)} xV = x \cdot \frac{1}{f(D)} V - \frac{f'(D)}{[f(D)]^2} V}$$

Ques- $(D^2+4)y = x\sin 3x$

Sol$^n$- AE $m^2+4=0$

$m = \pm 2i$

CF is $C_1\cos 2x + C_2 \sin 2x$

$$P.I. = \frac{1}{D^2+4} x\sin 3x = x.\frac{1}{D^2+4}\sin 3x - \frac{2D}{(D^2+4)^2}\sin 3x$$

put→

$D^2 = -(3^2) = -9$

$$= x.\frac{1}{(-9+4)}\sin 3x - \frac{2D}{(-9+4)^2}\sin 3x$$

$$= \frac{-x\sin 3x}{5} - \frac{2D}{25}(\sin 3x)$$

$$= \frac{-x\sin 3x}{5} - \frac{2}{25}(3\cos 3x) \Rightarrow \frac{-x\sin 3x}{5} - \frac{6\cos 3x}{25}$$

## ⊙ Variation of Parameters-

$f(D)y = R(x)$

$CF = C_1 u(x) + C_2 v(x)$

$P.I. = A\,v(x) + B\,v(x)$

$$\boxed{A = -\int \frac{v(x)R(x)}{u\frac{dv}{dx} - v\frac{du}{dx}}}$$

$$\boxed{B = \int \frac{u(x)R(x)\,dx}{u\frac{dv}{dx} - v\frac{du}{dx}}}$$

eg.- $(D^2+1)y = \operatorname{Cosec} x$

AE is

$$m^2+1 = 0$$
$$m^2 = -1$$
$$m = \pm i$$

CF is $C_1 \cos x + C_2 \sin x$

$u(x) = \cos x$ $\quad$ $v(x) = \sin x$

$\frac{dy}{dx} = -\sin x$ $\quad$ $\frac{dv}{dx} = \cos x$

$$u\frac{dv}{dx} - v\frac{du}{dx} = \cos^2 x + \sin^2 x \Rightarrow \underline{1}$$

$$A = -\int \frac{v(x)R(x)\,dx}{u\frac{dv}{dx} - v\frac{du}{dx}} = \int \frac{\cos x \cdot \operatorname{cosec} x}{1}\,dx$$

$$\Rightarrow \int \cot x\,dx = \log(\sin x)$$

$$P.I = Av + Bv$$
$$= -x\cos x + \sin x \log(\sin x)$$

$$G.S = CF + PI$$

⊙ <u>Cauchy's Euler eqn's</u> -

$$x^2\frac{d^2y}{dx^2} - 3x\frac{dy}{dx} - 5y = \sin(\log x)$$

$$x = e^z \Longleftrightarrow z = \log x$$

$$x\frac{d}{dx} = \theta = \frac{d}{dz}$$

$$x^2 \frac{d^2}{dx^2} = \theta(\theta-1)$$

$$x^3 \frac{d^3}{dx^3} = \theta(\theta-1)(\theta-2)$$

$$[\theta(\theta-1) - 3\theta - 5]\, y = \sin Z$$

$$[\theta^2 - \theta - 3\theta - 5]\, y = \sin Z$$

$$(\theta^2 - 4\theta - 5)\, y = \sin Z$$

AE is -

$$m^2 - 4m - 5 = 0$$

$$(m-5).(m+1) = 0$$

$$m = 5, -1$$

CF is → $C_1 e^{5Z} + C_2 e^{-Z}$

$$\text{P.I.} = \frac{1}{\theta^2 - 4\theta - 5} \sin Z$$

$$= \frac{1}{-1-4\theta-5} \sin Z \qquad \theta^2 = -1$$

$$= \frac{1}{-6-4\theta} \sin Z$$

$$= \frac{1}{-(6+4\theta)} \sin Z \Rightarrow \frac{6-4\theta}{-[(6+4\theta)(6-4\theta)]} \sin Z$$

$$= \frac{-(6-4\theta)}{36-16\theta^2} \sin Z \Rightarrow \frac{(-6+4\theta)\sin Z}{36-16(-1)} \qquad \theta^2 = -1$$

$$= \frac{-6\sin Z + 4\theta(\sin Z)}{36+16}$$

$$= \frac{-6\sin Z + 4\cos Z}{52}$$

G.S. = CF + PI

$$y = C_1 e^{5Z} + C_2 e^{-Z} - \frac{6\sin Z}{52} + \frac{4\cos Z}{52}$$

$$= C_1 (e^Z)^5 + C_2 (e^Z)^{-1} - \frac{3\sin Z}{26} + \frac{1}{13}\cos Z$$

$$= c_1 x^5 + c_2 x^{-1} - \frac{3 \sin(\log x)}{26} + \frac{1}{13} \cos(\log x)$$

Since $x = e^z \Leftrightarrow z = \log x$

$\Rightarrow$ First Order Differential Equations -

$$\frac{dy}{dx} = \frac{f(x)}{g(y)} \quad [\text{or}] \quad f(x) \cdot g(y) \quad [\text{or}] \quad \frac{g(y)}{f(x)}$$

eg. -

$$\frac{dy}{dx} = e^{x-y} + x^2 e^{-y}$$

$$\frac{dy}{dx} = e^x \cdot e^{-y} + x^2 e^{-y}$$

$$= (e^x + x^2)\, e^{-y}$$

$$\frac{dy}{e^{-y}} = (e^x + x^2)\, dx$$

$$e^y\, dy = (e^x + x^2)\, dx$$

$$\int e^y\, dy = \int (e^x + x^2)\, dx$$

$$e^y = e^x + \frac{x^3}{3} + c$$

Ques - Solve $\rightarrow$ $\frac{dy}{dx} = 1 + x + y + xy$

Solⁿ -

$$\frac{dy}{dx} = (1+x)(1+y)$$

$$\frac{dy}{(1+y)} = (1+x)\, dx$$

$$\int \frac{dy}{1+y} = \int (1+x)\, dx$$

$$\log(1+y) = x + \frac{x^2}{2} + c$$

Ques- $\frac{dy}{dx} = (4x+y+1)^2$ —(i)

Sol$^n$- Let→ $4x+y+1 = z$ —(ii)

diff. 2 w.r.t $x$

$$4 + \frac{dy}{dx} = \frac{dz}{dx}$$

$$\frac{dy}{dx} = \frac{dz}{dx} - 4 \quad \text{—(iii)}$$

Sut 2 & 3 in 1 →

$$\frac{dz}{dx} - 4 = z^2$$

$$\frac{dz}{dx} = z^2 + 4$$

$$\frac{dz}{z^2+4} = dx$$

$$\int \frac{dz}{z^2+4} = \int dx$$

$$\frac{1}{2}\tan^{-1}\left(\frac{z}{2}\right) = x + c$$

$$\frac{1}{2}\tan^{-1}\left(\frac{4x+y+1}{2}\right) = x + c$$

Ques- Solve $\frac{dy}{dx} = \frac{x+y+1}{2x+2y+3}$

Sol$^n$- $\frac{dy}{dx} = \frac{(x+y)+1}{2(x+y)+3}$ —(1)

Let $x+y = z$ —(2)

diff w.r.t $x$ →

$$1 + \frac{dy}{dx} = \frac{dz}{dx}$$

$$\frac{dy}{dx} = \frac{dz}{dx} - 1 \quad \text{—(3)}$$

Sub. (2) & (3) in (1) →

$$\frac{dz}{dx} - 1 = \frac{z+1}{2z+3}$$

$$\frac{dz}{dx} = \frac{z+1}{2z+3} + 1$$

$$\frac{dz}{dx} = \frac{z+1+2z+3}{2z+3}$$

$$\frac{dz}{dx} = \frac{3z+4}{2z+3}$$

$$\frac{2z+3}{3z+4}\,dz = dx$$

$$\int \frac{2z+3}{3z+4}\,dz = \int dx$$

$$\int \left(\frac{2}{3} + \frac{1/3}{3z+4}\right) dz = \int dx$$

$$\frac{2}{3}\int dz + \frac{1}{3}\int \frac{1}{3z+4}\,dz = \int dx$$

$$\frac{2}{3}\int dz + \frac{1}{3\cdot 3}\int \frac{3}{3z+4}\,dz = \int dx$$

$$\frac{2}{3}(z) + \frac{1}{9}\log(3z+4) = x + c$$

$$3z+4\,\overline{)\,2z+3\,}\left(\frac{2}{3}\right.$$
$$\underline{2z + \frac{8}{3}}$$
$$3 - \frac{8}{3} \Rightarrow \frac{1}{3}$$

Since

$$z = x+y$$

$$\frac{2}{3}(x+y) + \frac{1}{9}\log(3x+3y+4) = x + C$$

$$\frac{11}{9} \qquad 9\,\overline{)\,11\,}(1$$
$$\underline{9}$$
$$(2)$$

$$\frac{11}{9} = 1 + \frac{2}{9}$$

Ques-Form ⇒ $\dfrac{dy}{dx} = \dfrac{f_1(x,y)}{f_2(x,y)}$

$f_1(x,y)$ & $f_2(x,y)$ are homogeneous functions of same degree then put $y = vx$

$$\boxed{\frac{dy}{dx} = v + x\frac{dv}{dx}}$$

Ques- $\dfrac{dy}{dx} = \dfrac{y}{x} + \tan\left(\dfrac{y}{x}\right) \longrightarrow (1)$

Solⁿ- put $y = vx \longrightarrow (2)$

$$\frac{dy}{dx} = v + x\frac{dv}{dx} \longrightarrow (3)$$

Sub(2) & (3) in(1) →

$$v + x\frac{dv}{dx} = \frac{vx}{x} + \tan\left(\frac{vx}{x}\right)$$

$$v + x\frac{dv}{dx} = v + \tan v$$

$$x\frac{dv}{dx} = \tan v$$

$$\frac{dv}{\tan v} = \frac{dx}{x}$$

$$\int \cot v \cdot dv = \int \frac{dx}{x}$$

$$\log(\sin v) = \log x + \log c$$

$$\log(\sin v) = \log(xc)$$

$$\sin v = xc$$

Since $v = \dfrac{y}{x}$

$$\sin\left(\frac{y}{x}\right) = xc$$

⊙ Ques - Form:- $\dfrac{dx}{dy} = \dfrac{f_1(x,y)}{f_2(x,y)}$

$f_1(x,y)$ & $f_2(x,y)$ are homogeneous functions of same degree then put $x = vy$

$$\boxed{\frac{dx}{dy} = v + y\frac{dv}{dy}}$$

Ques - $$(1+e^{x/y})\,dx + \left(1-\frac{x}{y}\right)e^{x/y}\,dy = 0$$

Sol$^n$ -

$$(1+e^{x/y})\,dx = -\left(1-\frac{x}{y}\right)e^{x/y}\,dy$$

$$\frac{dx}{dy} = \frac{-\left(1-\frac{x}{y}\right)e^{x/y}}{1+e^{x/y}} \longrightarrow (1)$$

put $x = vy \longrightarrow (2)$

$$\frac{dx}{dy} = v + y\frac{dv}{dy} \longrightarrow (3)$$

Sub (2) & (3) in (1) →

$$v + y\frac{dv}{dy} = \frac{-(1-v)e^{v}}{(1+e^{v})}$$

$$v + y\frac{dv}{dy} = \frac{(-1+v)e^{v}}{1+e^{v}}$$

$$y\frac{dv}{dy} = \frac{-e^{v}+ve^{v}}{1+e^{v}} - v = \frac{-e^{v}+ve^{v}-v-ve^{v}}{1+e^{v}}$$

$$y\frac{dv}{dy} = -\frac{(v+e^{v})}{1+e^{v}}$$

$$\frac{1+e^{v}}{v+e^{v}}\,dv = -\frac{dy}{y}$$

Since $v = \dfrac{x}{y}$

$$\frac{x}{y} + e^{x/y} = \frac{c}{y}$$

# Linear differential equations -

def. →

A differential $eq^n$ is said to be L.D.E. if the dependent variable and its differential coefficient occuring with first degree and they are not multiplied together.

$$\frac{dy}{dx} + Py = Q$$

P & Q are functions of $x$

$$IF = e^{\int P\,dx}$$

Solution is →

$$y(IF) = \int Q\,(IF)\,dx + C$$

Ques - Solve $\frac{dy}{dx} + y\cot x = \sin 2x$

Sol$^n$ -

$P = \cot x \qquad Q = \sin 2x$

$$IF = e^{\int \cot x\,dx} = e^{\log(\sin x)} = \sin x$$

Solution is → $y(IF) = \int Q\,(IF)\,dx$

$$y\sin x = \int \sin 2x \cdot \sin x\,dx$$

$$= \int 2\sin^2 x \cos x\,dx$$

$$= 2\left(\frac{\sin^3 x}{3}\right) + C$$

⊙ $\sin 2x = 2\sin x \cos x$

$$\int [f(x)]^n f'(x)\,dx = \frac{[f(x)]^{n+1}}{n+1}$$

$$\frac{dx}{dy} + Px = Q$$

P & Q functions of y →

$$IF = e^{\int P dy}$$

Solution is →

$$x(IF) = \int Q(IF)\, dy$$

Ques- $(x+2y^3)\frac{dy}{dx} = y$

Soln- $x + 2y^3 = y\frac{dx}{dy}$

$$y\frac{dx}{dy} = x + 2y^3$$

$$\frac{dx}{dy} = \frac{x}{y} + 2y^2$$

$$\frac{dx}{dy} - \frac{x}{y} = 2y^2 \longrightarrow (1)$$

$P = \frac{-1}{y}$ $\quad Q = 2y^2$

$$IF = e^{\int P dy} = e^{\int -\frac{1}{y} dy} \Rightarrow e^{-\log y} \Rightarrow e^{\log y^{-1}} \Rightarrow y^{-1} \Rightarrow \frac{1}{y}$$

Solution is →

$$x(IF) = \int Q(IF)\, dy$$

$$x\left(\frac{1}{y}\right) = \int 2y^2\left(\frac{1}{y}\right) dy$$

$$\frac{x}{y} = \int 2y\, dy$$

$$\frac{x}{y} = 2\left(\frac{y^2}{2}\right) + c$$

$$\frac{x}{y} = \underline{\underline{y^2 + c}}$$

Ques- $\frac{dy}{dx} + Py = Qy^n \longrightarrow (1)$

Soln- divide by $y^n \rightarrow$

$$\frac{1}{y^n}\frac{dy}{dx} + Py^{1-n} = Q$$

let

$$y^{1-n} = z \quad — (3)$$

diff (3) w.r.t $x$ we get

$$(1-n)\, y^{1-n-1} \frac{dy}{dx} = \frac{dz}{dx}$$

$$(1-n)\, y^{-n} \frac{dy}{dx} = \frac{dz}{dx}$$

$$y^{-n} \frac{dy}{dx} = \frac{1}{1-n}\frac{dz}{dx} \quad — (4)$$

Substituting (3) & (4) in (2) $\rightarrow$

$$\frac{1}{1-n}\frac{dz}{dx} + Pz = Q$$

$$\frac{dz}{dx} + (1-n)Pz = (1-n)Q \quad — (5)$$

which is linear in $z \rightarrow$

Solution is -

$$z(IF) = \int (1-n)Q\,(IF)\,dx$$

$$IF = e^{\int (1-n)P\,dx}$$

Ques- Solve

$$\frac{dy}{dx} = xy + x^3y^2$$

Sol$^n$ → $\frac{dy}{dx} - xy = x^3 y^2 \longrightarrow (1)$

divide by $y^2$ →

$$\frac{1}{y^2}\frac{dy}{dx} - x\cdot\frac{1}{y} = x^3 \longrightarrow (2)$$

Let $-\frac{1}{y} = z \quad \text{———} (3)$

$$\frac{1}{y^2}\frac{dy}{dx} = \frac{dz}{dx} \longrightarrow (4)$$

Substituting (3) & (4) in (2) →

$$\frac{dz}{dx} + xz = x^3 \longrightarrow (5)$$

$$I.F. = e^{\int x\,dx} = e^{x^2/2}$$

$$P = x \qquad Q = x^3$$

Solution is →

$$z(IF) = \int Q(IF)\,dx$$

$$z\,e^{x^2/2} = \int x^3 \left(e^{x^2/2}\right) dx \qquad \frac{x^2}{2} = t$$

$$z\,e^{x^2/2} = \int x^2 e^{x^2/2} \cdot x\,dx \qquad \frac{2x\,dx}{2} = dt$$

Since $z = -\frac{1}{y}$

$$= \int 2t\,e^t\,dt \qquad x\,dx = dt$$

$$= 2\left(te^t - e^t\right) + c$$

$$-\frac{1}{y}\,e^{x^2/2} = 2\left[\frac{x^2}{2}e^{x^2/2} - e^{x^2/2}\right] + c$$

AE → expon

AT

↓ algebraic ↓ trigonometric

$+t \qquad 1 \qquad +0$

$e^t \searrow e^t \searrow e^t$

$te^t - e^t$

# EXACT DIFFERENTIAL EQUATION

$M(x,y)\,dx + N(x,y)\,dy = 0$ is said to be an exact differential eq$^n$ if $\frac{\partial M}{\partial y} = \frac{\partial N}{\partial x}$

Solution is →

$$\int M\,dx + \int (\text{Terms in } N \text{ free from } x)\,dy = C$$

1. M & N are homogenious functions of same degree & $Mx + Ny$ is a single term then

$$IF = \frac{1}{Mx + Ny}$$

2. M & N are not homogenious but →

$$M = y f_1(x,y)$$
$$N = x f_2(x,y) \;;$$

$Mx + Ny$ is single term then →

$$IF = \frac{1}{Mx - Ny}$$

3.

$M \qquad N$

$\frac{\partial M}{\partial y} \qquad \frac{\partial N}{\partial x}$

If $\dfrac{\frac{\partial M}{\partial y} - \frac{\partial N}{\partial x}}{N} = f(x)$ then I.F. $= e^{\int f(x)\,dx}$

4. If $\dfrac{\frac{\partial N}{\partial x} - \frac{\partial M}{\partial y}}{M} = f(y)$ then I.F. $= e^{\int f(y)\,dy}$

$M \qquad N$

$\frac{\partial M}{\partial y} \qquad \frac{\partial N}{\partial x}$

Ques- $(y^4+2y)\,dx + (xy^3+2y^4-4x)\,dy = 0$

Sol$^n$-

$M = y^4+2y$

$N = xy^3+2y^4-4x$

$\frac{\partial M}{\partial y} = 4y^3+2 \qquad \frac{\partial N}{\partial x} = y^3-4$

$\frac{\partial M}{\partial y} \neq \frac{\partial N}{\partial x}$

$$\frac{\frac{\partial N}{\partial x}-\frac{\partial M}{\partial y}}{M} = \frac{y^3-4-4y^3-2}{y^4+2y} = \frac{-3y^3-6}{y^4+2y}$$

$$= \frac{-3\cancel{(y^3+2)}}{y\cancel{(y^3+2)}} = \frac{-3}{y}$$

IF $\rightarrow e^{\int\frac{-3}{y}dy} = e^{-3\log y} = y^{-3} = \frac{1}{y^3}$

multiply (1) with IF $\rightarrow$

$$\left(\frac{y^4+2y}{y^3}\right)dx + \left(\frac{xy^3+2y^4-4x}{y^3}\right)dy = 0$$

$$\left(y+\frac{2}{y^2}\right)dx + \left(x+2y-\frac{4x}{y^3}\right)dy = 0 \quad \text{---(2)}$$

$M_1 = y+\frac{2}{y^2} \qquad N_1 = x+2y-\frac{4x}{y^3}$

$\frac{\partial M_1}{\partial y} = 1-\frac{4}{y^3} \qquad \frac{\partial N_1}{\partial x} = 1-\frac{4}{y^3}$

Sol$^n$ (2) is exact $\rightarrow$

$\int M_1\,dx + \int(\text{Terms in } N_1 \text{ Free From } x)\,dy = C$

$$\int\left(y+\frac{2}{y^2}\right)dx+\int 2y\,dy=c$$

$$\left(y+\frac{2}{y^2}\right)x+2\left(\frac{y^2}{2}\right)=c$$

$$\left(y+\frac{2}{y^2}\right)x+y^2=C$$

Ques- $y^2dx+(x^2-xy)\,dy=0$

Sol$^n$-

$M=y^2 \qquad N=x^2-xy$

$\frac{\partial M}{\partial y}=2y \qquad \frac{\partial N}{\partial x}=2x-y$

$$\frac{\partial M}{\partial y}\neq\frac{\partial N}{\partial x}$$

$$Mx+Ny=\cancel{y^2x}+x^2y-\cancel{xy^2}=x^2y$$

$$I.F.=\frac{1}{Mx+Ny}=\frac{1}{x^2y}$$

Ques- $(x^2y^2+xy+1)\,y\,dx+(x^2y^2-xy+1)\,x\,dy=0$

Sol$^n$-

$M=x^2y^3+xy^2+y \qquad N=x^3y^2-x^2y+x$

$\frac{\partial M}{\partial y}=3x^2y^2+2xy+1 \qquad \frac{\partial N}{\partial x}=3x^2y^2-2xy+1$

$$Mx=\cancel{x^3y^3}+x^2y^2+\cancel{xy}$$
$$Ny=\cancel{x^3y^3}-x^2y^2+\cancel{xy}$$

$$Mx-Ny=2x^2y^2$$

$$I.F.=\frac{1}{2x^2y^2}$$

Ques- $\dfrac{dy}{dx} = \dfrac{x^3 + y^3}{xy^2}$

Sol$^n$-

$$(x^3+y^3)\,dx - xy^2\,dy = 0$$

$M = x^3+y^3$ $\quad\quad$ $N = -xy^2$

$\dfrac{dM}{dy} = 3y^2$ $\quad\quad$ $\dfrac{dN}{dx} = -y^2$

$Mx = x^4 + y^3 x$

$Ny = -xy^3$

$Mx + Ny = x^4$

$$\frac{\frac{dM}{dy} - \frac{dN}{dx}}{N} \Rightarrow \frac{3y^2-(-y^2)}{-xy^2} = \frac{4y^2}{-xy^2}$$

$$= \frac{-4}{x}$$

$\text{I.F} = \dfrac{1}{Mx+Ny}$

$= \dfrac{1}{x^4}$

$$\text{I.F.} = e^{\int -\frac{4}{x}\,dx} \Rightarrow x^{-4} = \frac{1}{x^4}$$

## ORTOGONAL DIFFERENTIAL EQUATIONS

If two straight lines are perpendicular to each other then-

$$m_1 . m_2 = -1$$

$m_1, m_2 \rightarrow$ Slope.

$y = f(x)$

→ Write the equation of the given family.

→ Find differential $eq^n$ of the given family by eliminating arbitrary constants.

→ Replace $\frac{dy}{dx}$ by $-\frac{dx}{dy}$ to get differential $eq^n$ of orthogonal family.

→ Solve the diff. $eq^n$ obtained in STEP-3 to get the required orthogonal trajactory.

Ques- Find the orthogonal trajactory of the family of curves of the form $xy = c^2$.

Sol$^n$-

The $eq^n$ of the given family is

$$xy = c^2 \quad \text{—(I)}$$

diff. (I) w.r.t. $x$ -

$$x\frac{dy}{dx} + y = 0 \quad \text{—(II)}$$

- which is differential $eq^n$ of $eq^n$ (I).

Replacing $\frac{dy}{dx}$ by $-\frac{dx}{dy}$, we will get diff. $eq^n$ of orthogonal family.

$$x\left(-\frac{dx}{dy}\right) + y = 0$$

$$-x\frac{dx}{dy} = -y$$

| | |
|---|---|
| $-x\,dx = -y\,dy$ | $\int x\,dx = \int y\,dy$ |
| $x\,dx = y\,dy$ | $\frac{x^2}{2} = \frac{y^2}{2} + K$ |

$$\frac{x^2}{2} - \frac{y^2}{2} = K$$

$$x^2 - y^2 = 2K$$

Ques- Find orthogonal trajectory of family of curves of the form $r^n = a^n \sin n\theta$.

Soln-

The given eqn is $r^n = a^n \sin n\theta$ —(I)

Take log on both sides-

$$\log r^n = \log a^n + \log(\sin n\theta)$$

$$n \log r = n \log a + \log(\sin n\theta) \quad \text{—(II)}$$

differentiate (2) w.r.t. $\theta$ -

$$n \frac{1}{r} \frac{dr}{d\theta} = 0 + \frac{1}{\sin n\theta} \cdot n\cos n\theta$$

$$\frac{1}{r} \frac{dr}{d\theta} = \frac{\cos n\theta}{\sin n\theta} \quad \text{—(III)}$$

Replace $\frac{1}{r}\frac{dr}{d\theta}$ by $-r.\frac{d\theta}{dr}$ -

$$-r \frac{d\theta}{dr} = \frac{\cos n\theta}{\sin n\theta} \rightarrow$$

$$\int \frac{\sin n\theta}{\cos n\theta} \cdot d\theta = \int -\frac{dr}{r}$$

$$\int \tan n\theta \, d\theta = -\int \frac{dr}{r}$$

$$\frac{\log(\sec n\theta)}{n} = -\log r + \log c$$

$$\log(\sec n\theta)^{\frac{1}{n}} = \log\left(\frac{c}{r}\right)$$

$$\frac{c}{r} = [\operatorname{Sec} n\theta]^{1/n}$$

$$\frac{c^n}{r^n} = \operatorname{Sec} n\theta$$

$$r^n \operatorname{Sec} n\theta = c^n$$



To find the roots of the equation-

1- Every third degree equation must contain atleast one real root.

2- If sum of all the coefficients is zero then $m=1$ satisfies the equation.

3- Sum of the coefficients of the odd power = Sum of the coefficients of even power then $m=-1$ satisfies the $eq^n$

Ques- $m^3 - 3m + 2 = 0$

$Sol^n$- $1-3+2=0$ so $m=1$ satisfies-

| | | | | |
|---|---|---|---|---|
| | 1 | 0 | -3 | 2 |
| m=1 | 0 | 1 | 1 | -2 |
| | 1 | 1 | -2 | 0 |
| m=1 | 0 | 1 | 2 | |
| | 1 | 2 | 0 | |

$m+2=0$

$m=-2$



Ques-

1- $(D^3 - 4D^2 + 5D - 2)y = 0$ Ans- 1, 1, 2

2- $(D^4 - 2D^3 + 2D - 1)y = 0$ 1, 1, 1, -1

3- $(D^3 + 6D^2 + 11D + 6)y = 0$ -1, -2, -3

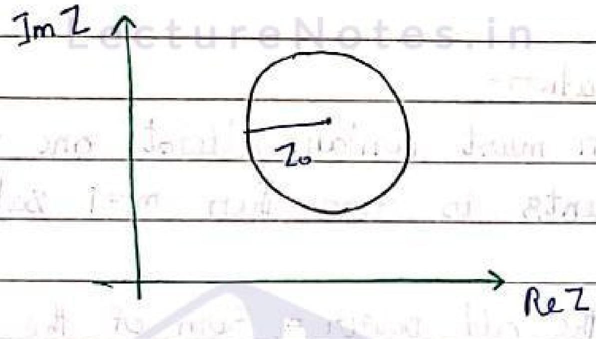
4- $(D^4 - D^3 - 9D^2 - 11D - 4)y = 0$ -1, -1, -1, 4

5- $(D^2 - 3D + 4)y = 0$ $\left(C_1 \cos\frac{\sqrt{3}}{2}x + C_2 \sin\frac{\sqrt{3}}{2}x\right) e^{\frac{3}{2}x}$

# COMPLEX VARIABLES

## Analytic Function-

A function $f(z)$ is said to be analytic at the point $z = z_0$ if it is differential every where in the neighbourhood of $z_0$.



$$f(z) = u(x,y) + i\,v(x,y)$$

C-R eqⁿs-

1. $u_x = v_y$ ; $u_y = -v_x$

2. $u_x, u_y, v_x, v_y$ are continuous functions.

## Harmonic Function-

A function $f(x,y)$ is said to be harmonic function if-

$$\frac{d^2f}{dx^2} + \frac{d^2f}{dy^2} = 0$$

$u(x,y) = x^2 - y^2$

$u_x = 2x$ | $u_y = -2y$

$u_{xx} = 2$ | $u_{yy} = -2$

$u_{xx} + u_{yy} = 2 - 2 = 0$

$u_{xx} + u_{yy} = 0$ ← Laplace eqⁿ

$v(x,y) = 2xy$

$v_x = 2y$ $\quad v_y = 2x$

$v_{xx} = 0$ $\quad v_{yy} = 0$

$v_{xx} + v_{yy} = 0$

$u$ & $v$ are harmonic.

$$w = f(z) = z^2$$

$$u + iv = (x^2 - y^2) + i(2xy)$$

$$v(x,y) = 2xy$$

$$m_2 = \frac{dy}{dx} = \frac{-v_x}{v_y} = \frac{-2y}{2x} = \frac{-y}{x}$$

$$m_1 \cdot m_2 = \frac{x}{y}\left(\frac{-y}{x}\right) = \underline{\underline{-1}}$$

Note-

$f(z) = u + iv$ is analytic then 'u' is known as harmonic conjugate of v, 'v' is known as harmonic conjugate of u.

$$u_x = v_y$$
$$u_y = -v_x$$

⊙ MILNE - THOMSON METHOD-

• $u(x,y)$ is given-

1- $u_x$ & $u_y$

2- We know that $f'(z) = u_x + iv_x$

$$f'(z) = u_x - iu_y$$

3- $f(z) = \int f'(z)\,dz = \int (u_x - iu_y)\Big|_{\substack{x=z \\ y=0}} dz$

by Milne-Thomson method-

put $x = z$
$y = 0$

Ex- $u(x,y) = x^2 - y^2$

Soln- $u_x = 2x$ $u_y = -2y$

$$f'(z) = u_x + i v_x$$
$$= u_x - i u_y$$
$$= 2x + i2y$$

put $x = z$ & $y = 0$

$$f'(z) = 2z + i(0) = 2z$$
$$f(z) = \int f'(z)\,dz = \int 2z\,dz = 2\left(\frac{z^2}{2}\right) + c$$
$$= z^2 + c$$

$$f(z) = z^2 + c$$

· $u(x,y)$ is given -

1- Find $u_x$ & $v_y$

2- We know that $f'(z) = u_x + i v_x$
$$= v_y + i v_x$$

3- $f(z) = \int f'(z)\,dz = \int \left. v_y + i v_x \right|_{\substack{x=z \\ y=0}} dz$

by Milne-Thomson put $x = z$ $y = 0$

Ex- $v(x,y) = 2xy$

Soln- $v_x = 2y$ $v_y = 2x$

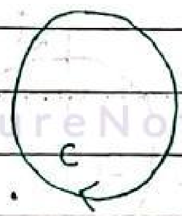
$$f'(z) = u_x + i v_x$$
$$= v_y + i v_x$$
$$= 2x + i2y$$

put $x = z$ & $y = 0$

$$f'(z) = 2z + i(0) = 2z$$
$$f(z) = \int f'(z)\,dz$$
$$= \int 2z\,dz$$
$$f(z) = z^2 + c$$

## Cauchy's Integral Theorem-

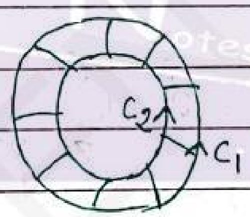
If $f(z)$ is analytic with in and on a simple closed curve $c$ then $\int_c f(z)\,dz = 0$




## Cauchy's Integral theorem in doubly connected region-

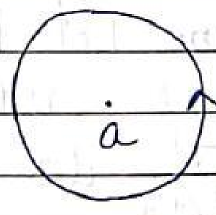
If $f(z)$ is analytic with in and on a double connected region bounded by $C_1$ & $C_2$ then

$$\int_{C_1} f(z)\,dz = \int_{C_2} f(z)\,dz$$




$$f(a) = \frac{1}{2\pi i}\int_c \frac{f(z)}{z-a}\,dz$$

i.e.- $$\int_c \frac{f(z)}{z-a}\,dz = 2\pi i\, f(a)$$




NOTE-

$$\int_c \frac{f(z)}{(z-a)^2}\,dz = 2\pi i\, f'(a)$$

$$\int_c \frac{f(z)}{(z-a)^3}\,dz = \frac{2\pi i}{\lfloor 2}\, f''(a)$$

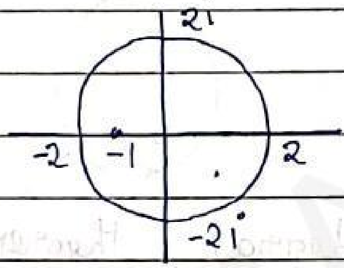
$$\int_c \frac{f(z)}{(z-a)^{n+1}}\,dz = \frac{2\pi i}{\lfloor n}\, f^{n}(a)$$

Ques- $\int_c \frac{e^{2z}}{(z+1)^4} dz$ where $c$ is $|z| = 2$

a. $2\pi i e^{-1}$ c. $\frac{2\pi i}{3} e^{-2}$

b. $\frac{8\pi i}{3} e^{-2}$ d. $0$

Sol$^n$-




$$\int_c \frac{e^{2z}}{(z+1)^4} dz = \int_c \frac{e^{2z}}{[z-(-1)]^4} dz$$

$$= \frac{2\pi i}{\lfloor 3} f'''(-1)$$

$$= \frac{2\pi i}{\lfloor 3} 2^3 (e^{2z})\Big|_{z=-1}$$

$$= \frac{8\pi i}{3} e^{-2} \quad \text{ans-(b)}$$

$z=-1$ lies inside $|z| = 2$

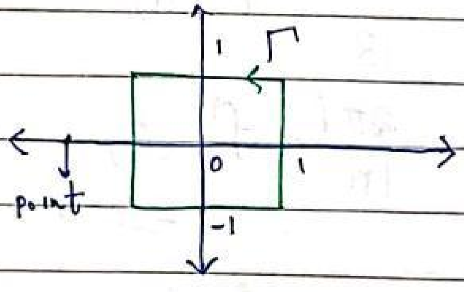
$f(z) = e^{2z}$ $\quad f''(z) = 2^2 e^{2z}$

$f'(z) = 2e^{2z}$ $\quad f'''(z) = 2^3 e^{2z}$

Ques- Let $\Gamma$ denote the boundary of the square whose sides are $x = \pm 1$, $y = \pm 1$ then value of $\int_\Gamma \frac{z^2}{2z+3} dz$ is -

a. $\frac{\pi i}{4}$ b. $2\pi i$ c. $-2\pi i$ d. $0$

Sol$^n$- $\int_\Gamma \frac{z^2}{2z+3} dz = \int \frac{z^2}{2(z+\frac{3}{2})} dz = 0$




$z \to -\frac{3}{2}$ lies outside the square $\Gamma$

Ques- If an analytic function $f(z) = u + iv$ where $i = \sqrt{-1}$. If $u = xy$ then $v$ should be -

(i)- $\frac{(x+y)^2}{2} + K$  (iii)- $\frac{(y^2-x^2)}{2} + K$

(ii)- $\frac{(x^2-y^2)}{2} + K$  (iv)- $\frac{(x-y)^2}{2} + K$

Soln-

$$u = xy$$

$$u_x = y \qquad u_y = x$$

$$f'(z) = u_x + i v_x$$

$$= u_x - i u_y$$

$$= y - ix$$

$$\Rightarrow v_x = -u_y$$

put $x = z$ ; $y = 0$  $f'(z) = 0 - iz = -iz$

$$f(z) = \int -iz\, dz = -i \frac{z^2}{2}$$

$$f(z) = -\frac{i}{2}(x+iy)^2 + K$$

$$u + iv = -\frac{i}{2}(x^2 - y^2 + 2ixy) + K$$

$$= -\frac{i}{2}(x^2-y^2) + \frac{2xy}{2} + K$$

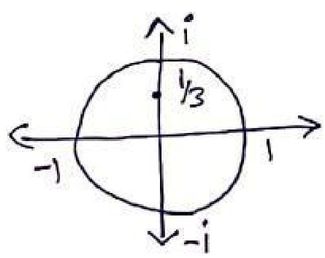
$$= \frac{1}{2}(y^2-x^2)i + xy + K$$

$$v = \frac{y^2-x^2}{2} + K \qquad \text{Ans-(iii)}$$

Ques- $\oint_c \frac{z^3-6}{3z-i} dz$ is ___ where c is $|z| = 1$

Soln-

$$\int_c \frac{z^3-6}{3(z-\frac{i}{3})} dz = \int_c \frac{\frac{z^3-6}{3}}{z-\frac{i}{3}} dz$$




$$= 2\pi i\, f\left(\frac{i}{3}\right) \quad \text{where } f(z) = \frac{z^3-6}{3}$$

$$= \frac{2\pi i}{1}\left[\frac{\left(\frac{i}{3}\right)^3 - 6}{3}\right]$$

$$= \frac{2\pi i}{3}\left(\frac{i^3}{27} - 6\right)$$

$$= \frac{2\pi}{81} - 4\pi i$$

$z = \frac{i}{3}$ lies inside

$|z| = 1$

$$\int_c \frac{f(z)}{z-a}\,dz = 2\pi i f(a)$$

Ques- The value of $\oint_c \frac{1}{1+z^2}\,dz$. C is $\left|z - \frac{i}{2}\right| = 1$

a. $2\pi i$ b. $\pi$ c. $-2\pi i$ d. $\tan^{-1} z$

Soln-

$$\int_c \frac{1}{z^2+1}\,dz = \int_c \frac{1}{(z-i)(z+i)}\,dz$$

$$= \int_c \frac{1/z+i}{z-i}\,dz$$

$$= 2\pi i f(i)$$

where- $f(z) = \frac{1}{z+i}$

$$= 2\pi i\left(\frac{1}{i+i}\right) \Rightarrow \pi$$

ANS-b

$$\left|x + iy - \frac{i}{2}\right| = 1$$

$$\left|x + \left(y - \frac{1}{2}\right)i\right| = 1$$

$$x^2 + \left(y - \frac{1}{2}\right)^2 = 1$$

$z = i$ lies inside

$z = -i$ lies outside

TAYLOR'S SERIES-

If a function $f(z)$ is analytic at all points inside circle c with centre at a and radius r then-

$$f(z) = f(a) + (z-a)f'(a) + \frac{(z-a)^2}{\lfloor 2}f''(a) + \frac{(z-a)^3}{\lfloor 3}f'''(a) + \text{-----}$$

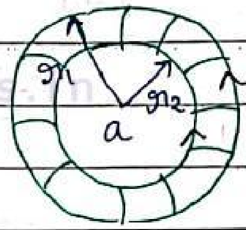
LAURENT'S SERIES-

If a function is analytic inside and on the boundry of a ring shaped region bounded two ~~orders~~ circles $c_1$ & $c_2$ with centre at a and radii $r_1$ & $r_2$ $(r_1 > r_2)$ then-

$$f(z) = a_0 + a_1(z-a) + a_2(z-a)^2 + a_3(z-a)^3 + \cdots + a_{-1}(z-a)^{-1} + a_{-2}(z-a)^{-2} + \cdots$$

Residue of $f(z)$ at $z=a$

$$a_n = \frac{1}{2\pi i}\int_{c_1} \frac{f(z)}{(z-a)^{n+1}}\,dz \qquad a_{-n} = \frac{1}{2\pi i}\int_{c_2} \frac{f(z)}{(z-a)^{-n+1}}\,dz$$




$$f(z) = \frac{1}{(z-3)(z-2)} = \frac{A}{z-3} + \frac{B}{z-2}$$

Residue $f(z) = 1$ at $z = 3$    res. $f(z) = -1$, $z=2$

$$= \frac{1}{z-3} + \frac{-1}{z-2}$$

Singular Point -

A point at which the function $f(z)$ fails to be analytic is known as singular point.

ex-

$$f(z) = \frac{1}{(z-3)(z-2)^2}$$ $z = 3, 2$ are singular points

Pole of order m-

A point $z=a$ is said to be a pole of order m if:

$$\lim_{z\to a} (z-a)^m \cdot f(z) \neq 0$$

if $m=1$ then it is known as simple pole.

ex- $$f(z) = \frac{z}{(z+2)(z-3)^4}$$

$z=-2$ is a simple pole

$z=3$ is a pole of order 4.

Calculation of residues -

1. $z=a$ is a simple pole -

$$\underset{z=a}{\text{Residue of}}\ f(z) = \underset{z\to a}{Lt}\ (z-a)\cdot f(z)$$

ex- $f(z) = \dfrac{z+2}{(z-3)(z-1)}$ $z=3,1$ are simple poles.

$$\underset{z=3}{Res}\ f(z) = \underset{z\to 3}{Lt}\ \cancel{(z-3)}\ \frac{(z+2)}{\cancel{(z-3)}(z-1)} = \frac{3+2}{3-1} = \frac{5}{2}$$

$$\underset{z=1}{Res}\ f(z) = \underset{z\to 1}{Lt}\ \cancel{(z-1)}\ \frac{(z+2)}{(z-3)\cancel{(z-1)}} = \frac{1+2}{1-3} = -\frac{3}{2}$$

$$f(z) = \frac{\phi(z)}{\psi(z)} \quad \text{where } \phi(a) \neq 0,\ \psi(a) = 0$$

$$\underset{z=a}{Res}\ f(z) = \underset{z=a}{Res}\ \frac{\phi(z)}{\psi(z)} = \underset{z\to a}{Lt}\ \frac{\phi(a)}{\psi'(a)}$$

ex-

$f(z) = \cot z$ at $z=0$

$f(z) = \dfrac{\cos z}{\sin z}$ $\quad \cos 0 = 1 \neq 0$, $\sin 0 = 0$

$$\underset{z=0}{Res}\ f(z) = \underset{z=0}{Res}\ \frac{\cos z}{\sin z} = \underset{z\to 0}{Lt}\ \frac{\cos z}{d(\sin z)} = \underset{z\to 0}{Lt}\ \frac{\cos z}{\cos z}$$

$$= 1$$

2- $z=a$ is a pole of order $m$ -

$$\underset{z=a}{\text{Residue of } f(z)} = \lim_{z\to a} \frac{1}{\lfloor(m-1)} \frac{d^{m-1}}{dz^{m-1}} \left[(z-a)^m \cdot f(z)\right]$$

Ex-

$$f(z) = \frac{z-1}{(z+2)^2(z+1)} \qquad z=-2 \text{ is a pole of order } 2$$

$$\underset{z=-2}{\text{Res } f(z)} = \lim_{z\to -2} \frac{1}{\lfloor(2-1)} \frac{d^{2-1}}{dz^{2-1}} \left[(z+2)^2 \cdot \frac{z-1}{(z+2)^2(z+1)}\right]$$

$$= \lim_{z\to -2} \frac{d}{dz}\left(\frac{z-1}{z+1}\right)$$

$$= \lim_{z\to -2} \frac{(z+1)-(z-1)}{(z+1)^2}$$

$$= \lim_{z\to -2} \frac{2}{(z+1)^2} \Rightarrow \frac{2}{(-2+1)^2} = 2$$

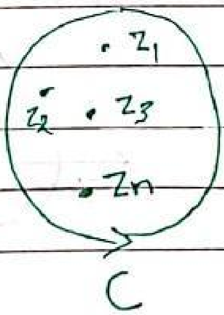


◎ Cauchy's Residue Theorem -

If $f(z)$ is analytic inside and on the closed curve $C$ at all points except at the singular points $z_1, z_2, z_3, \ldots\ldots z_n$ then -

$$\int_C f(z)\, dz = 2\pi i\,[\text{Sum of residues at its poles}]$$

$$= 2\pi i\left[\underset{z=z_1}{\text{Res } f(z)} + \underset{z=z_2}{\text{Res } f(z)} + \ldots\ldots + \underset{z=z_n}{\text{Res } f(z)}\right]$$




C

Ex- $\int_c \frac{1}{z^2+4}\, dz$ where $c$ is $|z-i| = 2$

a. $\pi/2$ b. $-\pi/2$ c. $\pi i/2$ d. $-\pi i/2$

Sol$^n$- $|z-i| = 2$

$z^2+4 = 0$

$|x+iy-i| = 2$ $z = \pm 2i$

$|x+(y-1)i| = 2$

$x^2 + (y-1)^2 = 4$

$z = 2i$ Lies inside

$z = -2i$ Lies outside

$\underset{z=-2i}{\text{Res}}\ f(z) = 0$

$\underset{z=2i}{\text{Res}}\ f(z) = \underset{z \to 2i}{Lt}\ (z-2i)\frac{1}{z^2+4} = \underset{z \to 2i}{Lt}\ \cancel{(z-2i)} \frac{1}{(z+2i)\cancel{(z-2i)}}$

$= \frac{1}{2i+2i} = \frac{1}{4i}$

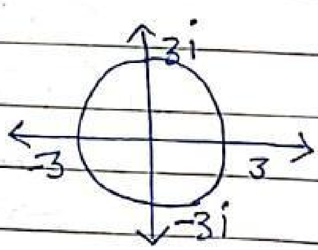
$\int_c \frac{1}{z^2+4}\, dz = 2\pi i\left[0 + \frac{1}{4i}\right] = \frac{2\pi i}{4 i} = \underline{\pi/2}$ ANS-(a)

Ex- $\int_c \frac{\sin \pi z^2 + \cos \pi z^2}{(z-4)(z-2)}\, dz$ where $c$ is $|z| = 3$

a. $2\pi i$ b. $-2\pi i$ c. $\pi 2 i$ d. $-\pi i$

Sol$^n$- $z=4$ lies outside

$z=2$ lies inside




$\underset{z=4}{\text{Res}}\ f(z) = 0$

$$\operatorname{Res}_{Z=2} f(z) = \underset{z\to 2}{Lt} \; (z-2) \cdot \frac{\sin\pi z^2 + \cos\pi z^2}{(z-4)(z-2)} = \frac{\sin 4\pi + \cos 4\pi}{(2-4)} = \frac{-1}{2}$$

$$\int_C \frac{\sin\pi z^2 + \cos\pi z^2}{(z-4)(z-2)} = 2\pi i\left(-\frac{1}{2} + 0\right)$$

$$= -\pi i \quad \text{Ans-d}$$

Ques- The residue of $\frac{\cos z}{z^8}$ at $z=0$ is -

a. 0 b. $\frac{1}{\lfloor 7}$ c. $\frac{-1}{\lfloor 7}$ d. None

Soln-

$$f(z) = \frac{\cos z}{z^8}$$

$z=0$ is a pole of order 8.

$$\operatorname{Res}_{Z=0} f(z) = \underset{z\to 0}{Lt} \frac{1}{\lfloor 7} \frac{d^7}{dz^7}\left[(z-0)^8 \cdot \frac{\cos z}{z^8}\right]$$

$$= \underset{z\to 0}{Lt} \frac{1}{\lfloor 7} \frac{d^{\overset{\text{odd}}{7}}}{dz^7} (\cos z) = 0 \quad \text{Ans-a}$$

Ques- The taylor Series expansion of $f(z) = \frac{1}{z+1}$ about $z=1$ is -

(i) $-\frac{1}{2}\left[1 - \frac{1}{2}(z-1) + \frac{1}{2^2}(z-1)^2 - - - - -\right]$

(ii) $\frac{1}{2}\left[1 - \frac{1}{2}(z-1) + \frac{1}{2^2}(z-1)^2 - - - - -\right]$

(iii) $-\frac{1}{2}\left[1 + \frac{1}{2}(z-1) + \frac{1}{2^2}(z-1)^2 - - - - -\right]$

(iv) None

Soln- about $z=1$

$$z-1 = \omega \Rightarrow z = \omega + 1$$

$$f(z) = \frac{1}{z+1} = \frac{1}{\omega+1+1} = \frac{1}{2+\omega} = \frac{1}{2}\,\frac{1}{\left(1+\frac{\omega}{2}\right)} = \frac{1}{2}\left(1+\frac{\omega}{2}\right)^{-1}$$

$$= \frac{1}{2}\left[1 - \frac{\omega}{2} + \left(\frac{\omega}{2}\right)^2 - \left(\frac{\omega}{2}\right)^3 + \cdots\right]$$

$z = a$

$$f(z) = f(a) + (z-a) f'(a) + \cdots$$

$$= \frac{1}{2}\left(1 - \left(\frac{z-1}{2}\right) + \frac{(z-1)^2}{2^2} - \frac{(z-1)^3}{2^3} + \cdots\right)$$

Ans - b

Ques- In the laurents expansion of $f(z) = \frac{1}{z-1} - \frac{1}{z-2}$ valid in the region $|z| > 2$ the coefficient of $\frac{1}{z^2}$ is -

a. -1 b. 0 c. 1 d. 2

Soln-

$$f(z) = \frac{1}{z-1} - \frac{1}{z-2}$$

$|z| > 2$

$2 < |z|$

$\frac{2}{|z|} < 1$

$$= \frac{1}{z(1-\frac{1}{z})} - \frac{1}{z(1-\frac{2}{z})}$$

$$= \frac{1}{z}\left(1 - \frac{1}{z}\right)^{-1} - \frac{1}{z}\left(1 - \frac{2}{z}\right)^{-1}$$

$$= \frac{1}{z}\left(1 + \frac{1}{z} + \frac{1}{z^2} + \frac{1}{z^3} + \cdots\right) - \frac{1}{z}\left(1 + \frac{2}{z} + \frac{4}{z^2} + \cdots\right)$$

$$= \frac{1}{z} + \frac{1}{z^2} + \frac{1}{z^3} + \frac{1}{z^4} + \cdots - \frac{1}{z} - \frac{2}{z^2} - \frac{4}{z^3} - \cdots$$

$$= \frac{-1}{z^2} - \frac{3}{z^3} - \frac{7}{z^4} - \cdots$$

Ans - (a)

→ If $1 < |z| < 2$ given -

$1 < |z|$ $|z| < 2$

$\frac{1}{z} < 1$ $\frac{z}{2} < 1$

$$f(z) = \frac{1}{z-1} - \frac{1}{z-2} \Rightarrow \frac{1}{z-1} + \frac{1}{2-z} \Rightarrow \frac{1}{z(1-\frac{1}{z})} + \frac{1}{2(1-\frac{z}{2})}$$

# PROBABILITY

**Mutually Exclusive-**
Two events A & B are said to be mutually exclusive if $P(A\cap B)=0$

**Equally likely-**
Two events A & B are said to be equally likely if there is no preference to one than the other.

**Exhaustive-**
All possible outcomes in a random experiment are Exhaustive.

**Independent-**
Two events A & B are said to be independent if $P(A\cap B) = P(A)\cdot P(B)$

**Conditional Probability-**
The conditional probability of an event B, given A denoted by

$$P\left(\frac{B}{A}\right) = \frac{P(B\cap A)}{P(A)}, \quad P(A)>0$$

Similarly-

$$P\left(\frac{A}{B}\right) = \frac{P(A\cap B)}{P(B)}, \quad P(B)>0$$

Ex- $P(Ch) = 10\%$
$P(Ph) = 12\%$
$P(Ph\cap Ch) = 8\%$

$$P(Ch/Ph) = \frac{P(Ch\cap Ph)}{P(Ph)} = \frac{8/100}{12/100} = \frac{2}{3}$$

• Additive theorem of probability -

1- If A & B are any two events then -

$$P(A\cup B) = P(A) + P(B) - P(A\cap B)$$

2- If A, B, C are three events then $P(A\cup B\cup C) \Rightarrow$

$$P(A\cup B\cup C) = P(A) + P(B) + P(C) - P(A\cap B) - P(B\cap C) - P(C\cap A) + P(A\cap B\cap C)$$

3- If A & B are independent events then -

(i)- $P(A\cap B^c) = P(A) \cdot P(B^c)$

(ii)- $P(A^c \cap B^c) = P(A^c) \cdot P(B^c)$

4. If A & B are any two events then -

(i)- $P(A\cap B^c) = P(A) - P(A\cap B)$

(ii)- $P(A^c \cap B^c) = P[(A\cup B)^c]$ } D' margan laws.

(iii)- $P(A^c \cup B^c) = P[(A\cap B)^c]$

Ques- A No. is selected at random from first 200 Natural No. the probability that the no. is divisible by 6 or 8 - a. 1/3 b. 1/4 c. 1/5 d. 2/3

Solⁿ- A - No. divisible by 6

B - " " by 8

$A\cap B$ - " " by 6 & 8

$A\cup B$ - " " by 6 or 8

$$P(A\cup B) = P(A) + P(B) - P(A\cap B)$$

$$= \frac{33}{200} + \frac{25}{200} - \frac{8}{200}$$

$$= \frac{50}{200} = \frac{1}{4}$$ Ans-(b)

$6\overline{)200}(33$ ; 18 ; 20 ; 18 ; (2)

$8\overline{)200}(25$ ; 16 ; 40 ; 40 ; 0

$24\overline{)200}(8$ ; 192 ; 8

Ques- A coin is weighted so that $P(H) = \frac{2}{3}$ & $P(T) = \frac{1}{3}$. If head occurs, then a no. is selected at random from 1 to 9. If tail occurs, then a no. is selected at random from 1 to 5. Find probability that an even number is selected.

(i)- $\frac{67}{145}$ (ii)- $\frac{58}{135}$ (iii)- $\frac{74}{157}$ (iv)- $\frac{43}{142}$

Soln-

Even → H ($2/3$) → 1 to 9 ($4/9$)

Even → T ($1/3$) → 1 to 5 ($2/5$)

$$P(\text{even}) = \frac{2}{3} \cdot \frac{4}{9} + \frac{1}{3} \cdot \frac{2}{5} \Rightarrow \frac{8}{27} + \frac{2}{15} \Rightarrow \frac{58}{135} \text{ -(ii)}$$

Ques- Three horses A, B, C are in a race. A is twice likely to win as B. B is twice likely to win as C. What is probability that B or C wins.

a. $2/7$ b. $3/7$ c. $4/7$ d. $6/7$

Soln-

$C \rightarrow x \longrightarrow 1/7$

$B \rightarrow 2x \longrightarrow 2/7$

$A \rightarrow 4x \longrightarrow 4/7$

$7x$

$7x = 1$

$x = \frac{1}{7}$ ans-(b)

A or C wins means→ $\frac{1}{7} + \frac{2}{7} = \frac{3}{7}$

Ques- Two independent events are A & B are such that $P(A \cap B) = \frac{1}{6}$, $P(A^c \cap B^c) = \frac{1}{3}$, then $P(A) =$

A. $\frac{1}{2}$ B. $\frac{2}{3}$ C. $\frac{1}{4}$ D. $\frac{3}{4}$

Sol$^n$– $P(A^c \cap B^c) = P(A^c) \cdot P(B^c)$

$= \{1 - P(A)\}.\{1 - P(B)\}$

$= 1 - P(A) - P(B) + P(A).P(B)$

$\frac{1}{3} = 1 - P(A) - P(B) + \frac{1}{6}$

$P(A) + P(B) = 1 + \frac{1}{6} - \frac{1}{3} = \frac{6+1-2}{6} = \frac{5}{6}$

$P(A) + P(B) = \frac{5}{6}$ ---(i)

$P(A \cap B) = P(A).P(B) = \frac{1}{6}$ ---(ii)

from (2) $P(B) = \frac{1}{6P(A)}$

$P(A) + \frac{1}{6P(A)} = \frac{5}{6}$

$\frac{6[P(A)]^2 + 1}{6P(A)} = \frac{5}{6}$

$6[P(A)]^2 + 1 = 5P(A)$

$6(P(A))^2 - 5P(A) + 1 = 0$

$(3P(A) - 1)(2P(A) - 1) = 0$

$P(A) = \frac{1}{3}$ $P(A) = \frac{1}{2}$

Ques- The probability of occurance of an event A is 0.7. The probability no-occurance of an event B is 0.45. the probability of at least one of A & B non occurance is 0.6. The probability that at least one of A & B occurs is-

a. 0.4 b. 0.6 c. 1 d. 0.85

Sol$^n$– $P(A) = 0.7$ $P(B^c) = 0.45$

$\therefore P(B) = 1 - P(B^c) = 1 - 0.45 = 0.55$

$P(A^c \cup B^c) = 0.6$

$P[(A \cap B)^c] = 0.6$

$P[A \cap B] = 1 - 0.6 = 0.4$

$$P(A \cup B) = P(A) + P(B) - P(A \cap B) \Rightarrow 0.7 + 0.55 - 0.4$$
$$= \underline{0.85}$$

Ques- A gambler has 4 coins in his pocket two coins are double headed. One is double tailed one is normal. The gambler takes a coin at random, open his eyes and see that the upperface is a head. what is the probability that the lower face is also a head?

a. $5/8$ b. $2/3$ c. $4/5$ d. $\frac{1}{2}$.

Soln-

$$P(L/U) = \frac{P(L \cap U)}{P(U)} = \frac{2/4}{5/8} = \frac{2}{4} \times \frac{8}{5} = \frac{4}{5} \quad \underline{(C)}$$

HH HH
TT 8 faces
TH 5 faces - H
3 faces - T.

Ques- The probability that at least one of the events A & B occurs is 0.6. If A & B occur simultaneously with probability 0.2. then -

$$P(\bar{A}) + P(\bar{B}) = \_\_\_\_$$

a. 0.4 b. 0.8 c. 1.2 d. 1.4

Soln-

$$P(A \cup B) = 0.6$$
$$P(A \cap B) = 0.2$$
$$P(\bar{A}) + P(\bar{B}) = 1 - P(A) + 1 - P(B) \Rightarrow 2 - [P(A) + P(B)]$$
$$\Rightarrow 2 - (0.6 + 0.2) \Rightarrow \underline{1.2} \ (C)$$

Ques- There are two identical locks with two identical keys and the keys are amoung the six keys which a person carries in his pocket. In a ahurry he drops one key somewhere. then the probability that the lock can still opened by drawing one key at random

18- a. $\frac{1}{3}$ b. $\frac{5}{6}$ c. $\frac{1}{30}$ d. $\frac{1}{12}$

Solⁿ-

Open the identical lock { droping non identical key $(4/6)$ — $\frac{2}{5}$ ; droping an identical key $(2/6)$ — $\frac{1}{5}$ }

$$= \frac{4}{6} \cdot \frac{2}{5} + \frac{2}{6} \cdot \frac{1}{5}$$

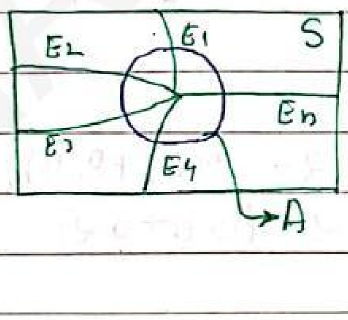
$$= \frac{1}{3} \quad \underline{\underline{(a)}}$$

BAYS THEOREM-

If $E_1, E_2, E_3 --- E_n$ are $n$ mutually exclusive events in sample space $S$ And $A$ is any arbitrary event in $S$ then-

$$P(E_i/A) = \frac{P(E_i) \cdot P(A/E_i)}{\sum_{i=1}^{n} P(E_i) \cdot P(A/E_i)}$$

$$= \frac{P(E_i) \cdot P(A/E_i)}{P(E_1) \cdot P(A/E_1) + P(E_2) \cdot P(A/E_2) + ---- + P(E_n) \cdot P(A/E_n)}$$



Ques- Three machines A, B, C produce 50%, 30%, 20% of the total no. of products respectively. the percentage of defective output of these machines 3%, 4%, 5% respectively. If a product is selected at random and is found to be defective then the probability that it is produced by M/B is -

a. $\frac{15}{37}$ b. $\frac{16}{37}$ c. $\frac{12}{37}$ d. $\frac{14}{37}$

Soln -

$$= \frac{\left(\frac{30}{100}\right)\left(\frac{4}{100}\right)}{\left(\frac{50}{100}\right)\left(\frac{3}{100}\right) + \frac{30}{100}\left(\frac{4}{100}\right) + \frac{20}{100}\left(\frac{5}{100}\right)}$$

$$= \frac{120}{150+120+100} = \frac{12}{37} \quad \underline{\underline{c}}$$

# Random Variables

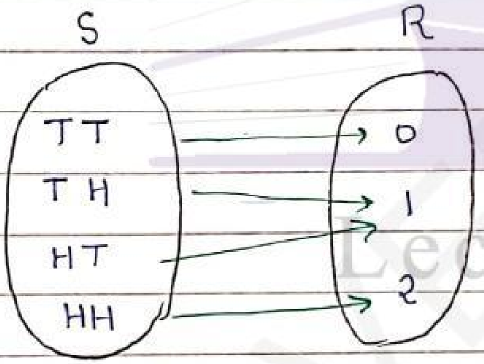
- Discrete
  - probability mass functions
- Continuous
  - probability density Functions.

A random variable is random function whose domain is sample space of random experiment whose range is subset of real no. system.

• Tossing of two coins -

$X : S \to R$

$X$ : no. of heads




| $X$ | 0 | 1 | 2 |
|---|---|---|---|
| $P(x)$ | $1/4$ | $2/4$ | $1/4$ |

Discrete
↳ probability mass functions -

$P(X = x)$

1. $P(x_i) \geq 0$
2. $\sum_{i=1}^{n} P(x_i) = 1$

• A random variable is said to be continuous R.V. if it takes values in an interval.

## Mean (or) Mathematical Expectation-

The mean of a random variable $X$ is given by -

$$\text{mean of } X = E(X) = \begin{cases} \sum_i x_i \cdot P(x_i) & \text{if } X \text{ is discrete R.V.} \\ \int_{-\infty}^{\infty} x f(x)\, dx & \text{if } X \text{ is continuous R.V.} \end{cases}$$

Properties-

1- $E(ax+b) = aE(x) + b$, where $a$ & $b$ are constant.

2- $E(K) = K$, where $K$ is constant.

## Variance-

The variance of a random variable $X$ is given by -

$$\text{Var } X = E[(X-\mu)^2] = \begin{cases} \sum_i (x_i-\mu)^2 \cdot P(x_i) & \text{if } X \text{ is discrete. where } \mu = E(X) \\ \int_{-\infty}^{\infty} (x-\mu)^2 f(x)\, dx & \text{if } X \text{ is continuous. where } \mu = E(X) \end{cases}$$

Alter-

$$\text{Var} X = E(X^2) - (E(X))^2$$

Note-

1. Standard deviation $= \sqrt{\text{variance}}$

2. $\text{var}(k) = 0$ where $K$ is constant

3. $\text{var}(aX) = a^2 \text{Var} X$

4. $\text{Var}(aX+b) = a^2 \text{var} X$

## Co-Variance-

The co-variance of two random variables $X$ and $Y$ is given by

$$\text{Cov}(X,Y) = E(XY) - E(X)E(Y)$$

Ques- A R.V. X has the following distribution- then $P(3 < X \leq 6) = ?$

| X | 0 | 1 | 2 | 3 | 4 | 5 | 6 |
|---|---|---|---|---|---|---|---|
| P(x) | K | 3K | 5K | 7K | 9K | 11K | 13K |

Solⁿ- then

$$P(3 < X \leq 6) = P(4) + P(5) + P(6) = 33K$$

$49k = 1$

$k = 1/49$

$$= \frac{33}{49}$$

Ques- A R.V. X has probability function $f(x) = \left(\frac{1}{2}\right)^x$ $x = 1, 2, 3, ----$ then mean of $x$ is -

a. 1 b. 2 c. 3 d. 4

Solⁿ-

| X | 1 | 2 | 3 | ---- |
|---|---|---|---|---|
| P(x) | $\frac{1}{2}$ | $\left(\frac{1}{2}\right)^2$ | $\left(\frac{1}{2}\right)^3$ | ---- |

mean of $x = \sum x_i P(x_i)$

$x = \frac{1}{2} \Rightarrow \quad \mu = 1\left(\frac{1}{2}\right) + 2\left(\frac{1}{2}\right)^2 + 3\left(\frac{1}{2}\right)^3 + ---$

$$\mu = x + 2x^2 + 3x^2 + ---- \quad -(1)$$

$$\mu x = x^2 + 2x^3 + 3x^4 + ----- \quad -(2)$$

$$\mu - \mu x = x + x^2 + x^3 + -----$$

$$\mu(1-x) = \frac{x}{1-x}$$

$$\mu = \frac{x}{(1-x)^2} = \frac{1/2}{(1-1/2)^2} \Rightarrow \frac{4}{2} = 2 \quad \text{Ans - (b)}$$

Ques- A random variable X has the probability function

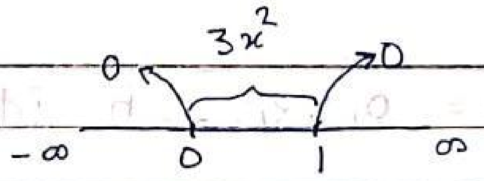
$$f(x) = 3x^2 \quad ; \quad 0 \leq x \leq 1$$

find a such that-

$$P(X > a) = 0.05$$

a. $\left(\frac{1}{20}\right)^{1/3}$ b. $\left(\frac{19}{20}\right)^{1/3}$ c. $\left(\frac{1}{2}\right)^{1/3}$ d. None.

Sol$^n$-




$$\int_a^1 f(x)\,dx = 0.05$$

$$\int_a^1 3x^2\,dx = 0.05$$

$$3\cdot\left(\frac{x^3}{3}\right)\Big|_a^1 = 0.05$$

$$1-a^3 = 0.05$$

$$a^3 = 1-0.05$$

$$a^3 = 0.95$$

$$a = (0.95)^{1/3}$$

Ques- The p.d.f. of $x$ is given by

$$f(x) = \begin{cases} x/2 & 0 \le x \le 2 \\ 0 & \text{otherwise} \end{cases}$$

the mean & variance of $x$ are-

a. $\frac{2}{3}, \frac{4}{9}$ b- $\frac{4}{3}, \frac{4}{9}$ c- $\frac{2}{3}, \frac{2}{9}$ d- $\frac{4}{3}, \frac{2}{9}$

Sol$^n$-

$$\text{mean}\, E(x) = \int_{-\infty}^{\infty} x\, f(x)\,dx \Rightarrow \int_0^2 x\cdot\left(\frac{x}{2}\right)dx \Rightarrow \frac{1}{2}\left(\frac{x^3}{3}\right)\Big|_0^2$$

$$= \frac{1}{6}(8-0) = \underline{\underline{\frac{4}{3}}}$$

$$E(x^2) = \int_{-\infty}^{\infty} x^2 f(x)\,dx$$

$$= \int_0^2 x^2\left(\frac{x}{2}\right)dx$$

$$= \frac{1}{2}\left(\frac{x^4}{4}\right)\Big|_0^2 \Rightarrow \frac{1}{8}(16-0) = \underline{\underline{2}}$$

Ans-(d)

Ques-

◉ BINOMIAL DISTRIBUTION- $X = 0, 1, 2, \ldots n$ [discrete].

$$P(X) = {}^nC_x \, p^x \, q^{n-x}$$

$x \rightarrow$ random variable
$n \rightarrow$ no. of trials.
$p \rightarrow$ Probability of success of r.v.
$q \rightarrow$ Probability of failure of r.v.

$p + q = 1$

mean $= np$

variance $= npq$

Note-
n is small.
$[p = 0.4,\ p = 0.2]$

◉ POISSON DISTRIBUTION- $X = 0, 1, 2, \ldots \infty$ [discrete].

$$P(X) = \frac{e^{-\lambda} \lambda^x}{\lfloor x}$$

$x \rightarrow$ random variable.
$\lambda \rightarrow$ parameter.

mean = variance $= \lambda$

Note-
n is large.
p is very small. $[p = 0.002,\ p = 0.04]$

Ques- Out of 2000 families with 4 children each, howmany families would you expect to have at least one boy?

a. 1250 b. 1500 c. 1875 d. 1825

Sol$^n$- $X =$ no. of boys
$p =$ probability of getting a boy $= 1/2$
$q =$ probability of getting a girl $= 1/2$
$n = 4$

$$P(X) = {}^nC_x\, P^x\, q^{n-x}$$

$$P(X \geq 1) = 1 - P(X=0)$$
$$= 1 - {}^nC_0\, p^0\, q^{n-0}$$
$$= 1 - {}^4C_0 \left(\tfrac{1}{2}\right)^0 \left(\tfrac{1}{2}\right)^4 = 1 - \frac{1}{16} = 15/16$$

no. of Families having at least one boy $= 2000\left(\frac{15}{16}\right) = 1875$
(C) Ans

Ques - If 20% of the bolts produced by a machine are defective. determine probability that out of 4 bolts choosen, the no. of defective bolts is less than 2 -

a- $\frac{27}{64}$  b- $\frac{81}{256}$  c- $\frac{27}{256}$  d- $\frac{512}{625}$

Sol$^n$-
$X \rightarrow$ No. of bolts.
$P \rightarrow$ Probability of $X = 20\% = 20/100 = \frac{1}{5}$
$q \rightarrow 1 - P = 1 - \frac{1}{5} = 4/5$
$n = 4$

$$P(x) = {}^nC_x\, P^x\, q^{n-x} \Rightarrow$$
$$P(X<2) = P(X=0) + P(X=1)$$
$$= {}^4C_0\, p^0\, q^{4-0} + {}^4C_1\, p^1\, q^{4-1}$$
$$= {}^4C_0 \left(\frac{4}{5}\right)^4 + \left(\frac{4}{5}\right)^4 \Rightarrow \frac{256}{625} + \frac{256}{625} = \frac{512}{625}$$

Ans - (d)

Ques - For a binomial distribution $n=10$, $P = 0.6$ then $E(x^2) =$  (a) - 30  (b) - 38  (c) - 38.4  (d) - 8

Sol$^n$-
$n = 10$
$p = 0.6$
$q = 1 - P = 0.4$
mean $E(X) = nP = 10(0.6) = 6$

variance $= npq = 10(0.6)(0.4) = 2.4$

$$\text{var } X = E(x^2) - (E(X))^2$$
$$E(x^2) = \text{Var } X + (E(x))^2$$
$$= 2.4 + (6)^2 \Rightarrow 2.4 + 36 \Rightarrow 38.4 \quad \text{Ans - (c)}$$

Ques- A R.V. X follows poisson distribution such that -

$$2P(X=2) = P(X=1) + 2P(X=0)$$

then var X is —

a. $3/2$ b. $2$ c. $1/2$ d. $1$

Sol$^n$- $2P(X=2) = P(X=1) + 2P(X=0)$

$$2 \cdot \frac{e^{-\lambda}\lambda^2}{\lfloor 2} = \frac{e^{-\lambda}\lambda^1}{\lfloor 1} + 2\frac{e^{-\lambda}\lambda^0}{\lfloor 0}$$

$$e^{-\lambda}\lambda^2 = e^{-\lambda}\lambda + 2e^{-\lambda}$$
$$\lambda^2 = \lambda + 2$$
$$\lambda^2 - \lambda - 2 = 0$$
$$(\lambda-2)(\lambda+1) = 0$$
$$\underline{\underline{\lambda = 2}}, -1 \quad \text{Ans- (b)}$$

Ques- The average incomming call rate is 4 per min The probability that there are not more than three calls is -

a. $\frac{71}{3}e^{-4}$ b. $\frac{71}{3}e^{-6}$ c. $13e^{-4}$ d. $13e^{-6}$

Sol$^n$- $\lambda = 4$

X: No. of incomming calls.

$$P(X \le 3) = P(X=0) + P(X=1) + P(X=2) + P(X=3)$$
$$= \frac{e^{-\lambda}\lambda^0}{\lfloor 0} + \frac{e^{-\lambda}\lambda^1}{\lfloor 1} + \frac{e^{-\lambda}\lambda^2}{\lfloor 2} + \frac{e^{-\lambda}\lambda^3}{\lfloor 3}$$
$$= e^{-\lambda}\left[1 + \lambda + \frac{\lambda^2}{2} + \frac{\lambda^3}{6}\right]$$

$= e^{-4}\left[1+4+8+\frac{32}{3}\right]$

$= \frac{71}{3} e^{-4}$

## Normal Distribution-

Probability density function $f(x) = \frac{1}{\sigma\sqrt{2\pi}} e^{-\frac{\left(\frac{x-\mu}{\sigma}\right)^2}{2}}$; $-\infty \le x < \infty$

mean of N.D. $= \mu$

variance of N.D. $= \sigma^2$

## Standard Normal distribution-

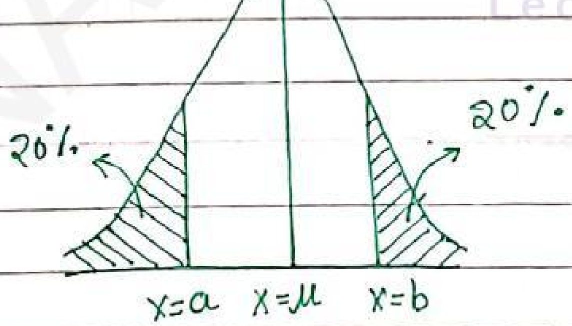
probability density function-

$\phi(z) = \frac{1}{\sigma\sqrt{2\pi}} e^{-z^2/2}$ $\quad -\infty < z < \infty$

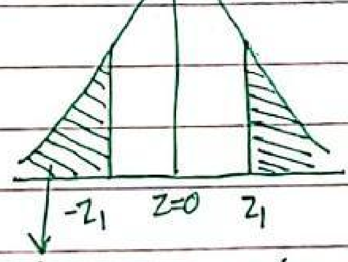
mean $z = 0$

var $z = 1$

Standard Normal variable $z = \frac{x-\mu}{\sigma}$




$P(X<a) = P(X>b)$




$P(Z<-z_1) = P(Z>z_1)$

Ques- X has → mean = 1 , variance = 4

y has → mean = −1 , variance is unknown.

x & y follows Normal distribution.

If $P(X \le -1) = P(Y \ge 2)$ then the S.D. of Y is-

a. 3 b. 2 c. $\sqrt{2}$ d- 1

Sol$^n$-

$$P(X \le -1) = P(Y \ge 2)$$ ← mean & Sd of Y

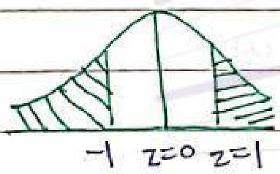
mean & sd of X

$$P\left(\frac{X-\mu}{\sigma} \le \frac{-1-\mu}{\sigma}\right) = P\left(\frac{Y-\mu}{\sigma} \ge \frac{2-\mu}{\sigma}\right)$$

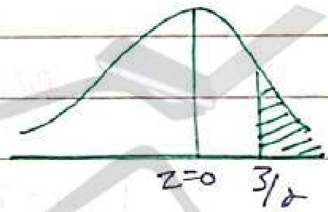
$$Z = \frac{x-\mu}{\sigma}$$

$$P\left(\frac{X-1}{2} \le \frac{-1-1}{2}\right) = P\left(\frac{Y-(-1)}{\sigma} \ge \frac{2-(-1)}{\sigma}\right)$$

$$P(Z \le -1) = P\left(Z \ge \frac{3}{\sigma}\right)$$




$$P(Z \ge 1) = P\left(Z > \frac{3}{\sigma}\right)$$




$$1 = \frac{3}{\sigma}$$

$$\sigma = 3$$

## ⊙ UNIFORM DISTRIBUTION -

The probability density function -

$f(x) = \frac{1}{\beta - \alpha}$ if $X$ follows uniform distribution in $[\alpha, \beta]$.

mean $= \frac{\alpha + \beta}{2}$

variance $= \frac{(\beta - \alpha)^2}{12}$

Ques- If $X$ follows uniform distribution in $[0,10]$ then the probability density function $f(x) = \frac{1}{10-0} = \frac{1}{10}$

1- $P(3 \le X \le 6)$

2- $P(X \ge 6)$

3- $P(X \le 4)$

Solⁿ- 1. $P(3 \le X \le 6) = \int_3^6 f(x)\,dx = \int_3^6 \frac{1}{10}\,dx = \frac{1}{10}\,(x)\Big|_3^6 = \frac{6-3}{10} = \frac{3}{10}$

2. $P(X \ge 6) = \int_6^{10} f(x)\,dx = \int_6^{10} \frac{1}{10}\,dx = \frac{4}{10}$

3- $P(X \le 4) = \int_0^4 f(x)\,dx = \int_0^4 \frac{1}{10}\,dx = \frac{4-0}{10} = \frac{4}{10}$

Ques-

## ⊙ CORRELATION & REGRESSION -

Correlation Coefficient $r = \frac{\mathrm{cov}(x,y)}{\sqrt{\mathrm{Var}\,x}\sqrt{\mathrm{Var}\,y}} = \frac{\mathrm{cov}(x,y)}{(\text{S.D. of } X)(\text{S.D. of } Y)}$

$= \frac{E(XY) - E(X)E(Y)}{\sqrt{E(x^2) - [E(x)]^2}\sqrt{E(y^2) - [E(y)]^2}}$

1- The regression line of $y$ on $x$ is given by -

$y - \bar{y} = r \cdot \frac{\sigma_y}{\sigma_x}(x - \bar{x})$

$$b_{yx} = r \cdot \frac{\sigma_y}{\sigma_x}$$

$$y - \bar{y} = b_{yx}(x - \bar{x})$$

where, $r$ = coefficient of correlation.
$\sigma_y$ = S.D. of Y.
$\sigma_x$ = S.D. of X.
$\bar{X}$ = mean of X
$\bar{Y}$ = mean of Y

Ques- If $\Sigma x_i = 15$, $\Sigma y_i = 36$, $\Sigma x_i y_i = 110$ & $n=5$ then $Cov(X,Y) = ?$

a. 0.15  b. 0.25  c. 0.3  d. 0.4

Soln-

$$Cov(X,Y) = E(XY) - E(X)E(Y)$$

$$= \frac{\Sigma x_i y_i}{n} - \left(\frac{\Sigma x_i}{n}\right)\left(\frac{\Sigma y_i}{n}\right)$$

$$= \left(\frac{110}{5}\right) - \left(\frac{15}{5}\right)\left(\frac{36}{5}\right) = 0.4$$ Ans-(d)

Ques- If $Cov(X,Y) = -12.4$, $var\ X = 3.12$, $Var\ Y = 100$ then the coefficient of correlation is-

a. -0.70  b. -0.35  c. -0.90  d. -0.45

Soln-

$$r = \frac{Cov(X,Y)}{\sqrt{Var X}\sqrt{Var Y}} \Rightarrow \frac{-12.4}{\sqrt{3.12}\sqrt{100}} \Rightarrow -0.70$$

Ans-(a)

Ques- Given that $\Sigma x_i = 24$, $\Sigma y_i = 44$, $\Sigma x_i y_i = 360$, $\Sigma x_i^2 = 184$, $\Sigma y_i^2 = 574$ & $n=4$ then-

1- The regression coefficient $b_{yx}$ =

a. 1.07  c. 3.25
b. 2.40  d. 4.50

2- $b_{xy}$ is -

a. 1.07  c. 3.25
b. 2.40  d. 4.50

Solⁿ-

$$b_{yx} = r\,\frac{\sigma_y}{\sigma_x} \qquad\qquad b_{xy} = r\,\frac{\sigma_x}{\sigma_y}$$

$$= \frac{Cov(XY)}{\sigma_X\ \sigma_Y}\cdot\frac{\sigma_y}{\sigma_X} \qquad\qquad = \frac{Cov(X,Y)}{\sigma_X\cdot\sigma_Y}\times\frac{\sigma_X}{\sigma_Y}$$

$$= \frac{Cov(X,Y)}{\sigma_X^2} \qquad\qquad = \frac{Cov(X,Y)}{\sigma_Y^2}$$

$$= \frac{Cov(X,Y)}{Var X} \qquad\qquad = \frac{Cov(X,Y)}{Var Y}$$

$$Var\,Y = E(Y^2) - [E(Y)]^2 \qquad\qquad Cov(X,Y) = E(XY) - E(X)E(Y)$$

$$= \frac{\Sigma y_i^2}{n} - \left(\frac{\Sigma y_i}{n}\right)^2 \qquad\qquad = \frac{\Sigma x_i y_i}{n} - \left(\frac{\Sigma x_i}{n}\right)\left(\frac{\Sigma y_i}{n}\right)$$

$$= \frac{574}{4} - \left(\frac{44}{4}\right)^2 \qquad\qquad = \frac{360}{4} - \frac{24}{4}\times\frac{44}{4}$$

$$= 22.5 \qquad\qquad = 24$$

$$b_{yx} = \frac{Cov(X,Y)}{Var X} \qquad\qquad b_{xy} = \frac{Cov(X,Y)}{Var Y}$$

$$= \frac{24}{10} = 2.4 \qquad\qquad = \frac{24}{22.5} = 1.07$$

ANS - (b) ANS - (a)

⊙ $b_{yx} = r\cdot\frac{\sigma_y}{\sigma_x}$ $\qquad b_{xy} = r\cdot\frac{\sigma_x}{\sigma_y}$

$$b_{yx}\cdot b_{xy} = r\cdot\frac{\cancel{\sigma_y}}{\cancel{\sigma_x}}\cdot r\,\frac{\cancel{\sigma_x}}{\cancel{\sigma_y}}$$

$$r^2 = b_{yx}\cdot b_{xy}$$

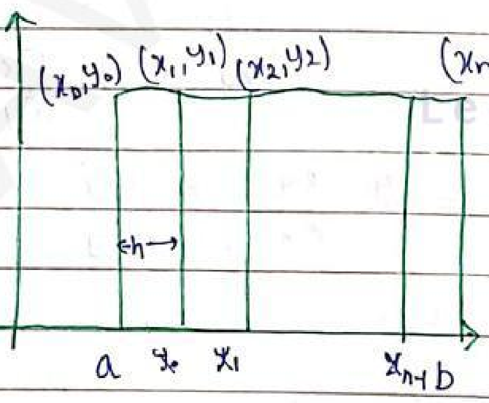
$$\boxed{r = \sqrt{b_{xy}\cdot b_{yx}}}$$

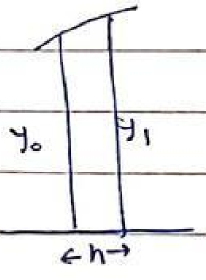
# NUMERICAL METHODS

- Numerical Solutions to equations
    - → Bisection
    - → False Position (Regular - Falsi)
    - → N-R method.
    - → Successive Approximation.
- Numerical Solutions to diff. $eq^n$.
    - → Euler's method
    - → R-K method
- Numerical Integration
    - → Trapezoidal
    - → Simpson's $\frac{1}{3}$ rule.
    - → Simpson's $\frac{3}{8}$ rule.

⊙ Numerical Integration-

· Trapizoidal method-




$$\int_{x_0}^{x_1} y\,dx = \frac{h}{2}(y_0+y_1)$$

$$\int_{x_1}^{x_2} y\,dx = \frac{h}{2}(y_1+y_2)$$

$$\int_{x_2}^{x_3} y\,dx = \frac{h}{2}(y_2+y_3)$$

$$\vdots \qquad\qquad \vdots$$

$$\int_{x_{n-1}}^{x_n} y\,dx = \frac{h}{2}(y_{n-1}+y_n)$$

$$\int_{x_0}^{x_n} y\,dx = \frac{h}{2}\left[(y_0+y_n)+2(y_1+y_2+\dots+y_{n-1})\right]$$ which is trapezoidal rule.

· Trapazoidal Rule-

$$\int_{x_0}^{x_n} y\,dx = \frac{h}{2}\left[(y_0+y_n)+2(y_1+y_2+y_3+\dots+y_{n-1})\right]$$

· Simpson's $\frac{1}{3}$ Rule-

$$\int_{x_0}^{x_n} y\,dx = \frac{h}{3}\left[(y_0+y_n)+4(y_1+y_3+y_5+\dots)+2(y_2+y_4+y_6+\dots)\right]$$

· Simpson's $\frac{3}{8}$ Rule-

$$\int_{x_0}^{x_n} y\,dx = \frac{3h}{8}\left[(y_0+y_n)+3(y_1+y_2+y_4+y_5+\dots)+2(y_3+y_6+y_9+\dots)\right]$$

Ques- A curve passing through the points given by the following table-

| X | 1 | 2 | 3 | 4 | 5 |
|---|---|---|---|---|---|
| Y | 10 | 50 | 70 | 80 | 100 |
| | $y_0$ | $y_1$ | $y_2$ | $y_3$ | $y_4$ |

by Trapezoidal rule, the area bounded by the curve, $\overline{x}$ axis & lines $x=1$ & $x=5$ is-

a. 255

b. 275

c. 305

d. 310

Soln -
$$\int_1^5 y\,dx = \frac{h}{2}\left[(y_0+y_4) + 2(y_1+y_2+y_3)\right]$$

$$= \frac{1}{2}\left[(10+100)+2(50+70+80)\right]$$

$$= 255$$

Ans- (a)

5 - points : 4 interval



Ques- Using Simpson $\frac{1}{3}$ rule, the value of $\int_1^3 f(x)\,dx$ for the following data-

| X | 1 | 1.5 | 2 | 2.5 | 3 |
|---|---|---|---|---|---|
| Y | 2.1 | 2.4 | 2.2 | 2.8 | 3 |
| | $y_0$ | $y_1$ | $y_2$ | $y_3$ | $y_4$ |

a. 4.975 c. 11.1

b. 5.05 d. 55.5

Soln -
$$\int_{x_0}^{x_n} y\,dx = \frac{h}{3}\left[(y_0+y_n)+4(y_1+y_2+---)+2(y_2+y_2+y_6+---)\right]$$

$$\int_1^3 y\,dx = \frac{h}{3}\left[(y_0+y_4)+4(y_1+y_3)+2y_2\right]$$

$$\int_1^3 y\,dx = \frac{0.5}{3}\left[(2.1+3)+4(2.4+2.8)+2(2.2)\right]$$

$$= \underline{\underline{5.05}}$$ Ans- (b)

Ques- By Simpsons $\frac{3}{8}$ rule $\int_{-3}^{3} x^4\,dx$ by taking 6 sub intervals-

a. 96 b. 98 c. 99 d. 100

Soln -

$a=-3$ $b=3$ $n=6$ $h = \frac{b-a}{n} = \frac{6}{6} = 1$

$f(x) = x^4$

| X | -3 | -2 | -1 | 0 | 1 | 2 | 3 |
|---|---|---|---|---|---|---|---|
| Y | 81 | 16 | 1 | 0 | 1 | 16 | 81 |
| | $y_0$ | $y_1$ | $y_2$ | $y_3$ | $y_4$ | $y_5$ | $y_6$ |

Simpson's $\frac{3}{8}$ rule-

$$\int_{x_0}^{x_n} y\,dx = \frac{3h}{8}\left[(y_0+y_n) + 3(y_1+y_2+y_3+\ldots) + 2(y_3+y_6+\ldots)\right]$$

$$\int_{-3}^{3} x^4\,dx = \frac{3h}{8}\left[(y_0+y_6) + 3(y_1+y_2+y_4+y_5) + 2y_3\right]$$

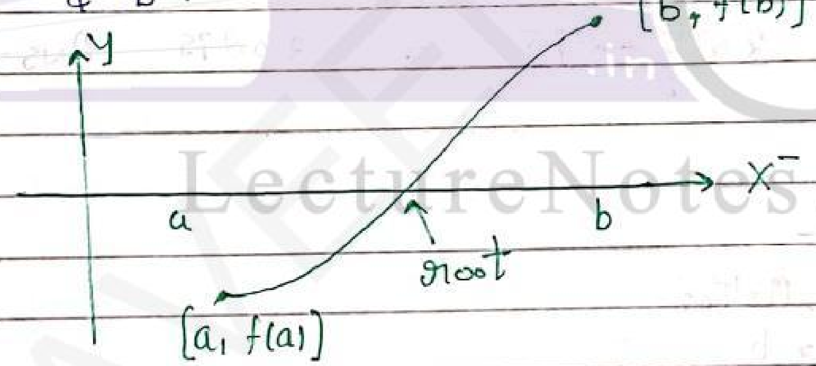
$$= \frac{3}{8}\cancel{\left[(81+81) + 3(16+1+1+16) + 2(0)\right]}$$

$$= 99 \qquad \text{ANS -(c)}$$

~~Ques~~

· Bisection Method-

If $f(x)$ is continuous on $[a, b]$ and $f(a), f(b)$ have apposite sign then at least one root lies in b/w a & b.




Ques- The root of the eq$^n$ $x^3-4x-9=0$ by bisection method in four stages in the interval $[2, 3]$ is -

a. 2.7065
b. 2.6875
c. 2.75
d. None.

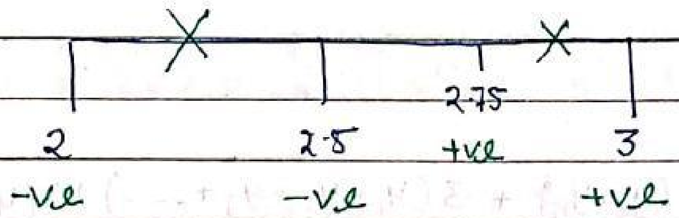
Sol$^n$-

$f(x) = x^3 - 4x - 9$

$a=2 \qquad b=3$

$f(2) = -9 < 0 \qquad f(3) = 6 > 0$

root lies b/w 2 & 3




$$x_0 = \frac{2+3}{2} = 2.5$$

$f(x_0) = f(2.5) = -3.375 < 0$

root lies b/w 2.5 & 3

$$x_1 = \frac{2.5+3}{2} = 2.75$$

$f(2.75) = 0.796 > 0$

root lies b/w 2.5 & 2.75

$$x_2 = \frac{2.5+2.75}{2} = 2.625$$

$f(2.625) = -1.412 < 0$

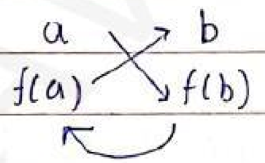
root lies b/w 2.625 & 2.75

$$x_3 = \frac{2.625+2.75}{2} = 2.6875 \quad \text{Ans - (b)}$$



- False Position method-

(or) - Regula Falsi method-




$$\boxed{x = \frac{a f(b) - b f(a)}{f(b) - f(a)}}$$

Ques- Find root of the eq$^n$ $2x - \log_{10} x = 7$ by regula falsi method correct to 3 decimals.

a. 3.683     c. 3.790

b. 3.789     d. 3.788

Sol$^n$- From the four options we can write-

$a = 3 \qquad b = 4$

$f(x) = 2x - \log_{10} x - 7$

$a = 3$ $\quad$ $b = 4$

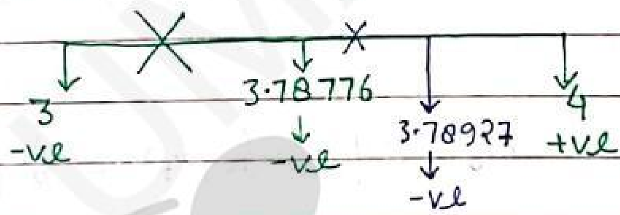
$f(a) = -1.4771$ $\quad$ $f(b) = 0.39794$

$$x_0 = \frac{3(0.39794) - 4(-1.4771)}{0.39794 + 1.4771}$$

$= 3.78776$

$f(x_0) = -0.0028624542$

root lies b/w 3.78776 & 4




$a = 3.78776$ $\quad$ $b = 4$

$f(a) = -0.00286245442$ $\quad$ $f(b) = 0.39794$

$$x_1 = \frac{3.78776(0.39794) - 4(-0.00286245442)}{0.39794 + 0.00286245442}$$

$= 3.78927577$

$f(x_1) = -4.6728313 \times 10^{-6}$

$a =$ $\quad$ $b =$

$f(a) =$ $\quad$ $f(b) =$

$$x_2 = \frac{af(b) - bf(a)}{f(b) - f(a)}$$

Ans- (b)

- Successive Approximation method-

$f(x) = 0$

$x = \phi(x)$

$x_0 \longrightarrow$ $x_1 = \phi(x_0)$

$x_2 = \phi(x_1)$

$x_3 = \phi(x_2)$

$\vdots$

Ques- Find real root of the $eq^n$ $2x - \log_{10} x = 7$ in the interval [3,4].

a. 3.786 b. 3.788 c. 3.789 d. 3.79

Sol$^n$-
$$2x = 7 + \log_{10} x$$
$$x = \frac{1}{2}\left[7 + \log_{10} x\right] \qquad x_0 = 3$$

$$x_1 = \frac{1}{2}\left[7 + \log_{10} x_0\right] = \frac{1}{2}\left(7 + \log_{10} 3\right) = 3.7\overset{3}{\wedge}856062$$

$$x_2 = \frac{1}{2}\left[7 + \log x_1\right] = \frac{1}{2}\left[7 + \log_{10} 3.783856062\right]$$
$$= 3.78635221$$

$$x_3 = \frac{1}{2}\left[7 + \log x_2\right] = \frac{1}{2}\left[7 + \log(3.78635221)\right] = 3.789110505$$

$$x_4 = \frac{1}{2}\left[7 + \log x_3\right] = \frac{1}{2}\left[7 + \log(3.789110505)\right] = 3.789268636$$

ANS- $\underline{\underline{c}}$



• Newton- Raphson method-

$$x_{i+1} = x_i - \frac{f(x_i)}{f'(x_i)} \qquad i = 0, 1, 2, \ldots$$

$$x_1 = x_0 - \frac{f(x_0)}{f'(x_0)}$$

$$x_2 = x_1 - \frac{f(x_1)}{f'(x_1)}$$

$$\vdots$$

Ques- An iterative Scheme is given by $x_{n+1} = \frac{1}{5}\left[16 - \frac{12}{x_n}\right]$.

Such a scheme, with suitable $x_0$ will-

a. not converge.

b. converge to 1.6.

c. converge to 1.8.

d. converge to 2.

Soln-

$$x_{n+1} = x_n = \alpha$$

$$\alpha = \frac{1}{5}\left(16 - \frac{12}{\alpha}\right)$$

$$\alpha = \frac{1}{5}\left(\frac{16\alpha - 12}{\alpha}\right)$$

$$5\alpha^2 = 16\alpha - 12$$

$$5\alpha^2 - 16\alpha + 12 = 0$$

$$5\alpha^2 - 10\alpha - 6\alpha + 12 = 0$$

$$(\alpha-2)(5\alpha-6) = 0$$

$$\alpha = 2, \ 6/5$$

$\alpha$ gets two values it can not converge at two points so it not converge. ANS-(a)

Ques- $x_0 = 1$, one step of N-R method. Solving the eqn $x^3 + 3x - 7 = 0$ gives the next value $x_1$ as-

a. 0.5

b. 1.406

c. 1.5

d. 2

Soln-

$$x^3 + 3x - 7 = 0$$

$$f(x) = x^3 + 3x - 7$$

$$f'(x) = 3x^2 + 3$$

$$x_0 = 1$$

$$f(x_0) = 1 + 3 - 7 = -3$$

$$f'(x_0) = 3 + 3 = 6$$

$$x_1 = x_0 - \frac{f(x_0)}{f'(x_0)} = 1 - \left(\frac{-3}{6}\right)$$

$$= 1 + \frac{1}{2} \Rightarrow 1.5$$

Ans - (C)

Ques- The following eq$^n$ needs to be numerically solved by N-R method-

$$x^3 + 4x - 9 = 0$$

The iterative eq$^n$ is -

a. $x_{k+1} = \dfrac{2x_k^3 + 9}{3x_k^2 + 4}$  c. $x_{k+1} = x_k - 3x_k^2 + 4$

b. $x_{k+1} = \dfrac{3x_k^2 + 4}{2x_k^2 + 9}$  d. $x_{k+1} = \dfrac{4x_k^2 + 3}{9x_k^2 + 2}$

Sol$^n$-

$$f(x) = x^3 + 4x - 9$$

$$f'(x) = 3x^2 + 4$$

$$x_{k+1} = x_k - \frac{f(x_k)}{f'(x_k)} \Rightarrow x_k - \left(\frac{x_k^3 + 4x_k - 9}{3x_k^2 + 4}\right)$$

$$= \frac{3x_k^3 + 4x_k - x_k^3 - 4x_k + 9}{3x_k^2 + 4}$$

$$= \underline{\underline{\frac{2x_k^3 + 9}{3x_k^2 + 4}}}$$



- Eulers Method-

$$\frac{dy}{dx} = f(x, y) \qquad y(x_0) = y_0$$

$y = y_0$ when $x = x_0$

$$y_{i+1} = y_i + h\,f(x_i, y_i)$$

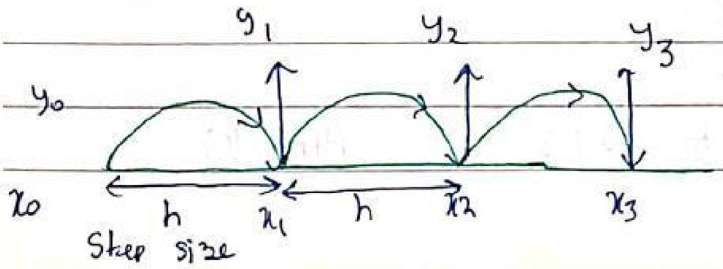
where $x_{i+1} = x_i + h$

(or) $x_0 + (i+1)h$

i.e.-

$$y_1 = y_0 + h\,f(x_0, y_0), \quad x_1 = x_0 + h$$

$$y_2 = y_1 + h\,f(x_1, y_1), \quad x_2 = x_1 + h$$



Ques- $f(x,y) = x+y$ $\quad y(0)=1 \quad y=1$ when $x=0$

$y_0 = 1 \quad , \quad x_0 = 0$

find $y(0.2)$ taking $h=0.1$

Sol$^n$- by Eulers method-

$$x_1 = x_0 + h = 0+0.1 = .1$$

$$y_1 = y_0 + h f(x_0, y_0)$$
$$= y_0 + h(x_0 + y_0)$$
$$= 1 + (0.1)(0+1)$$
$$= 1.1$$

$y(0.1) = 1.1$

$$x_2 = x_1 + h = 0.1 + 0.1 = 0.2$$

$$y_2 = y_1 + h f(x_1, y_1)$$
$$= y_1 + h(x_1 + y_1)$$
$$= 1.1 + 0.1(0.1 + 1.1)$$
$$= 1.2$$

$y(0.2) = 1.22$

- Runge-kutta $4^{th}$ order method-

$$y_{i+1} = y_i + K \quad , \quad x_{i+1} = x_i + h$$

where-

$$K = \frac{1}{6}(K_1 + 2K_2 + 2K_3 + K_4)$$

$$K_1 = h f(x_i, y_i)$$

$$K_2 = h f\left(x_i + \frac{h}{2}, y_i + \frac{K}{2}\right)$$

$$K_3 = h f\left(x_i + \frac{h}{2}, y_i + \frac{K_2}{2}\right)$$

$$K_4 = h f(x_i + h, y_i + K_3)$$

STEPS-

1- $K_1 = h\, f(x_0, y_0)$

2- $K_2 = h\, f\left(x_0 + \frac{h}{2}, y_0 + \frac{K_1}{2}\right)$ ← once asked

3- $K_3 = h\, f\left(x_0 + \frac{h}{2}, y_0 + \frac{K_2}{2}\right)$

4- $K_4 = h\, f(x_0 + h, y_0 + K_3)$

5- $K = \frac{1}{6}(K_1 + 2K_2 + 2K_3 + K_4)$

6- $y_1 = y_0 + K$ & $x_1 = x_0 + h$

Ques- $\frac{dy}{dx} = x+y$ $\quad y(0) = 1$ find $y(0.1)$ by taking $h = 0.1$

$f(x,y) = x+y \quad x_0 = 0 \quad y_0 = 1 \quad h = 0.1$

Soln -

1- $K_1 = h\, f(x_0, y_0)$

$= h(x_0 + y_0) = (0.1)(0+1) = 0.1$

2- $K_2 = h\, f\left(x_0 + \frac{h}{2}, y_0 + \frac{K_1}{2}\right)$

$= h\left(x_0 + \frac{h}{2} + y_0 + \frac{K_1}{2}\right)$

$= 0.1\left(0 + \frac{0.1}{2} + 1 + \frac{0.1}{2}\right)$

$= 0.11$

3- $K_3 = h\, f\left(x_0 + \frac{h}{2}, y_0 + \frac{K_2}{2}\right)$

$= h\left(x_0 + \frac{h}{2}, y_0 + \frac{K_2}{2}\right)$

$= (0.1)\left(0 + \frac{0.1}{2} + 1 + \frac{0.11}{2}\right) = 0.1105$

4- $K_4 = hf(x_0+h, y_0+k_3)$

$= h(x_0+h+y_0+k_3)$

$=(0.1)(0+0.1+1+0.1105)$

$= 0.12105$

5- $K = \frac{1}{6}[K_1+2K_2+2K_3+K_4]$

$= \frac{1}{6}[0.1+2(0.11)+2(0.1105)+0.12105]$

$= 0.1103$

6- $x_1 = x_0+h = 0+.1 = 0.1$

$y_1 = y_0+k = 1.1103$

$\therefore y(0.1) = 1.1103$

→ Unit vector perpendicular to both a and b→

$$n = \frac{a \times b}{|a \times b|}$$

→ Area of $\Delta OAB = \frac{1}{2} |\vec{a} \times \vec{b}|$

→ The component of F parallel to D is-

$$= \frac{\vec{F} \cdot \vec{G}}{|G^2|} G$$

→ The vector component of F perpendicular to G is-

$$= \vec{F} - \frac{\vec{F} \cdot \vec{G}}{|G^2|} \vec{G}$$