モノイダル圏における左右単位律子の一致の証明

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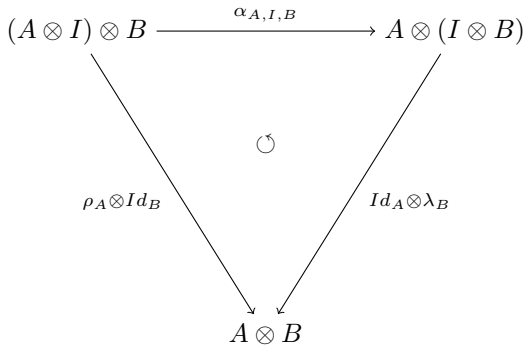
1　モノイダル圏

以下の５つの物 (テンソル積, 単位対象, 結合律子, 左単位律子, 右単位律子) が存在し、下記の二つの等式 (三角等式, 五角等式) を満たす圏をモノイダル圏と言う。

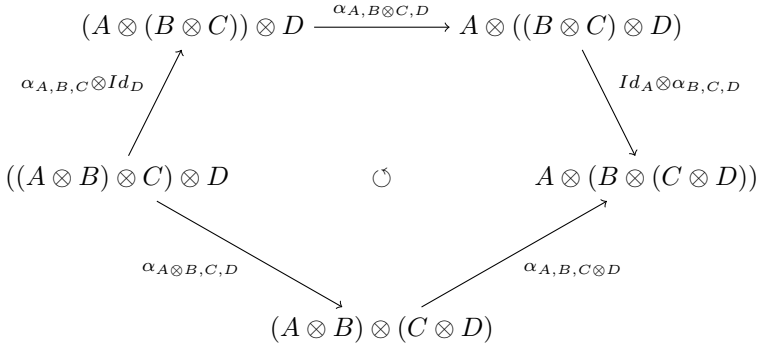
- テンソル積 $\otimes$
  関手 $\otimes : C \times C \to C$
- 単位対象 I
  $I \in Ob(C)$
- 結合律子 α
  自然同型 α
  $(A \otimes B) \otimes C \xrightarrow{\alpha_{A,B,C}} A \otimes (B \otimes C)$
- 左単位律子 λ
  自然同型 λ
  $I \otimes A \xrightarrow{\lambda_A} A$
- 右単位律子 ρ
  自然同型 ρ
  $A \otimes I \xrightarrow{\rho_A} A$
- 三角等式
  任意の A,B に対し以下の図式が可換



つまり以下の等式が成り立つ

$$\rho_A \otimes Id_B = (Id_A \otimes \lambda_B) \circ (\alpha_{A,I,B})$$

- 五角等式
  任意の A,B,C,D に対し以下の図式が可換



つまり以下の等式が成り立つ

$$(Id_A \otimes \alpha_{B,C,D}) \circ \alpha_{A,B\otimes C,D} \circ (\alpha_{A,B,C} \otimes Id_D) = \alpha_{A,B,C\otimes D} \circ \alpha_{A\otimes B,C,D}$$

2　証明

Proof.

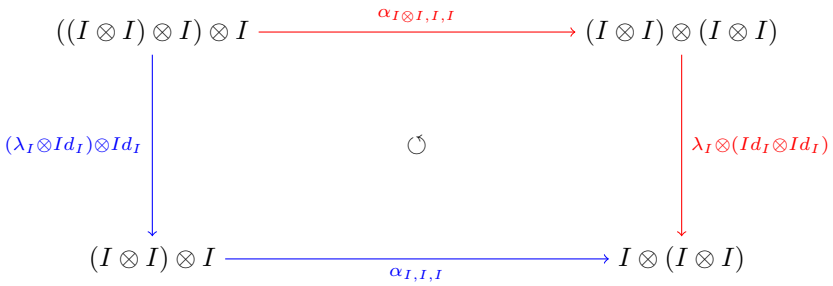
以下、λ_I(左単位律子) に恒等射を入れて行きます。

$$
\begin{aligned}
&\lambda_I = \lambda_I \\
&\quad \circ (\lambda_I \otimes Id_I) \circ (\lambda_I^{-1} \otimes Id_I) \\
&\quad = \lambda_I \\
&\quad \circ (\lambda_I \otimes Id_I) \\
&\quad \circ (\rho_I^{-1} \otimes Id_I) \circ (\rho_I \otimes Id_I) \\
&\quad \circ (\lambda_I^{-1} \otimes Id_I) \\
&\quad = \lambda_I \\
&\quad \circ (\lambda_I \otimes Id_I) \\
&\quad \circ (\rho_I^{-1} \otimes Id_I) \\
&\quad \circ \rho_I^{-1} \circ \rho_I \\
&\quad \circ (\rho_I \otimes Id_I) \\
&\quad \circ (\lambda_I^{-1} \otimes Id_I) \\
&\quad = \lambda_I \\
&\quad \circ (\lambda_I \otimes Id_I) \\
&\quad \circ (\rho_I^{-1} \otimes Id_I) \circ \rho_I^{-1} \circ \rho_I \circ (\rho_I \otimes Id_I) \\
&\quad \circ \alpha_{I,I,I}^{-1} \circ \alpha_{I,I,I} \\
&\quad \circ (\lambda_I^{-1} \otimes Id_I) \\
&\quad = \lambda_I \\
&\quad \circ (\lambda_I \otimes Id_I) \\
&\quad \circ (\rho_I^{-1} \otimes Id_I) \circ \rho_I^{-1} \circ \rho_I \circ (\rho_I \otimes Id_I) \\
&\quad \circ \alpha_{I,I,I}^{-1} \\
&\quad \circ (Id_I \otimes \lambda_I^{-1}) \circ (Id_I \otimes \lambda_I) \\
&\quad \circ \alpha_{I,I,I} \\
&\quad \circ (\lambda_I^{-1} \otimes Id_I)
\end{aligned}
$$

$$
\begin{aligned}
&= \lambda_I \\
&\circ (\lambda_I \otimes Id_I) \\
&\circ (\rho_I^{-1} \otimes Id_I) \circ \rho_I^{-1} \circ \rho_I \circ (\rho_I \otimes Id_I) \\
&\circ \alpha_{I,I,I}^{-1} \circ (Id_I \otimes \lambda_I^{-1}) \\
&\circ (\lambda_I \otimes Id_I) \circ (\lambda_I^{-1} \otimes Id_I) \circ (Id_I \otimes \lambda_I) \circ \alpha_{I,I,I} \\
&\circ (\lambda_I^{-1} \otimes Id_I) \\
&= \lambda_I \\
&\circ (\lambda_I \otimes Id_I) \\
&\circ (\rho_I^{-1} \otimes Id_I) \circ \rho_I^{-1} \circ \rho_I \circ (\rho_I \otimes Id_I) \\
&\circ \alpha_{I,I,I}^{-1} \circ (Id_I \otimes \lambda_I^{-1}) \circ (\lambda_I \otimes Id_I) \\
&\circ \alpha_{I,I,I}^{-1} \circ \alpha_{I,I,I} \\
&\circ (\lambda_I^{-1} \otimes Id_I) \circ (Id_I \otimes \lambda_I) \circ \alpha_{I,I,I} \\
&\circ (\lambda_I^{-1} \otimes Id_I) \\
&= \lambda_I \\
&\circ (\lambda_I \otimes Id_I) \\
&\circ (\rho_I^{-1} \otimes Id_I) \circ \rho_I^{-1} \circ \rho_I \circ (\rho_I \otimes Id_I) \\
&\circ \alpha_{I,I,I}^{-1} \circ (Id_I \otimes \lambda_I^{-1}) \circ (\lambda_I \otimes Id_I) \circ \alpha_{I,I,I}^{-1} \\
&\circ \lambda_{I\otimes I}^{-1} \circ \lambda_{I\otimes I} \\
&\circ \alpha_{I,I,I} \circ (\lambda_I^{-1} \otimes Id_I) \circ (Id_I \otimes \lambda_I) \circ \alpha_{I,I,I} \\
&\circ (\lambda_I^{-1} \otimes Id_I)
\end{aligned}
$$

$$
\begin{aligned}
&= \lambda_I \\
&\circ (\lambda_I \otimes Id_I) \\
&\circ (\rho_I^{-1} \otimes Id_I) \circ \rho_I^{-1} \circ \rho_I \circ (\rho_I \otimes Id_I) \\
&\circ \alpha_{I,I,I}^{-1} \circ (Id_I \otimes \lambda_I^{-1}) \\
&\circ (\lambda_I \otimes Id_I) \circ \alpha_{I,I,I}^{-1} \circ \lambda_{I\otimes I}^{-1} \\
&\circ \lambda_I^{-1} \circ \lambda_I \\
&\circ \lambda_{I\otimes I} \circ \alpha_{I,I,I} \circ (\lambda_I^{-1} \otimes Id_I) \circ (Id_I \otimes \lambda_I) \circ \alpha_{I,I,I} \\
&\circ (\lambda_I^{-1} \otimes Id_I) \\
&= \lambda_I \\
&\circ (\lambda_I \otimes Id_I) \\
&\circ (\rho_I^{-1} \otimes Id_I) \circ \rho_I^{-1} \circ \rho_I \circ (\rho_I \otimes Id_I) \\
&\circ \alpha_{I,I,I}^{-1} \circ (Id_I \otimes \lambda_I^{-1}) \circ (\lambda_I \otimes Id_I) \\
&\circ \alpha_{I,I,I}^{-1} \circ \lambda_{I\otimes I}^{-1} \circ \lambda_I^{-1} \circ \lambda_I \circ \lambda_{I\otimes I} \circ \alpha_{I,I,I} \circ (\lambda_I^{-1} \otimes Id_I) \circ (Id_I \otimes \lambda_I) \circ \alpha_{I,I,I} \\
&\circ ((\lambda_I \otimes Id_I) \otimes Id_I) \circ ((\lambda_I^{-1} \otimes Id_I) \otimes Id_I) \\
&\circ (\lambda_I^{-1} \otimes Id_I)
\end{aligned}
$$

α が自然変換であることから以下の図式が可換。



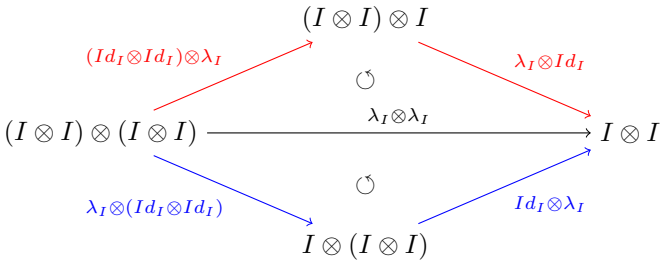
従って以下の等式が成り立つ

$$\lambda_I \otimes (Id_I \otimes Id_I) \circ \alpha_{I,I,I\otimes I} = \alpha_{I,I,I} \circ (\lambda_I \otimes Id_I) \otimes Id_I$$

$$
\begin{aligned}
&= \lambda_I \\
&\circ (\lambda_I \otimes Id_I) \\
&\circ (\rho_I^{-1} \otimes Id_I) \circ \rho_I^{-1} \circ \rho_I \circ (\rho_I \otimes Id_I) \\
&\circ \alpha_{I,I,I}^{-1} \circ (Id_I \otimes \lambda_I^{-1}) \circ (\lambda_I \otimes Id_I) \circ \alpha_{I,I,I}^{-1} \circ \lambda_{I\otimes I}^{-1} \circ \lambda_I^{-1} \\
&\circ \lambda_I \circ \lambda_{I\otimes I} \circ \alpha_{I,I,I} \circ (\lambda_I^{-1} \otimes Id_I) \circ (Id_I \otimes \lambda_I) \\
&\circ \ \cancel{\alpha_{I,I,I} \circ (\lambda_I \otimes Id_I) \otimes Id_I} \\
&\circ (\lambda_I \otimes (Id_I \otimes Id_I)) \circ \alpha_{I,I,I\otimes I} \\
&\circ ((\lambda_I^{-1} \otimes Id_I) \otimes Id_I) \\
&\circ (\lambda_I^{-1} \otimes Id_I)
\end{aligned}
$$

以下の等式が成り立つ
※交換則 $(a \otimes b) \circ (c \otimes d) = (a \circ c) \otimes (b \circ d)$ を使用

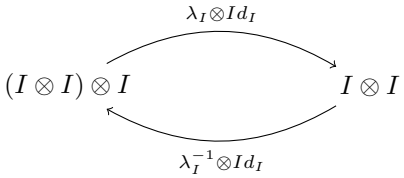
$$(\lambda_I \otimes Id_I) \circ ((Id_I \otimes Id_I) \otimes \lambda_I) = \lambda_I \otimes \lambda_I = (Id_I \otimes \lambda_I) \circ (\lambda_I \otimes (Id_I \otimes Id_I))$$



$$
\begin{aligned}
&= \lambda_I \\
&\circ (\lambda_I \otimes Id_I) \\
&\circ (\rho_I^{-1} \otimes Id_I) \circ \rho_I^{-1} \circ \rho_I \circ (\rho_I \otimes Id_I) \\
&\circ \alpha_{I,I,I}^{-1} \circ (Id_I \otimes \lambda_I^{-1}) \circ (\lambda_I \otimes Id_I) \circ \alpha_{I,I,I}^{-1} \circ \lambda_{I\otimes I}^{-1} \\
&\circ \lambda_I^{-1} \circ \lambda_I \circ \lambda_{I\otimes I} \circ \alpha_{I,I,I} \circ (\lambda_I^{-1} \otimes Id_I) \\
&\circ \cancel{(Id_I \otimes \lambda_I) \circ (\lambda_I \otimes (Id_I \otimes Id_I))} \\
&\circ (\lambda_I \otimes Id_I) \circ ((Id_I \otimes Id_I) \otimes \lambda_I) \\
&\circ \alpha_{I,I,I\otimes I} \\
&\circ ((\lambda_I^{-1} \otimes Id_I) \otimes Id_I) \\
&\circ (\lambda_I^{-1} \otimes Id_I)
\end{aligned}
$$

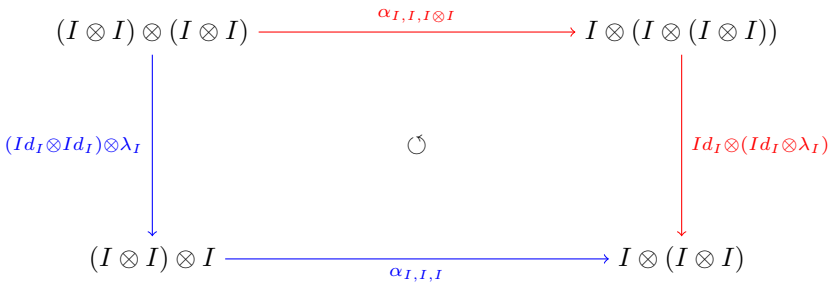
以下は恒等射なので削除

$(\lambda_I^{-1} \otimes Id_I) \circ (\lambda_I \otimes Id_I)$



$= \lambda_I$
$\circ (\lambda_I \otimes Id_I)$
$\circ (\rho_I^{-1} \otimes Id_I) \circ \rho_I^{-1} \circ \rho_I \circ (\rho_I \otimes Id_I)$
$\circ \alpha_{I,I,I}^{-1} \circ (Id_I \otimes \lambda_I^{-1}) \circ (\lambda_I \otimes Id_I) \circ \alpha_{I,I,I}^{-1} \circ \lambda_{I\otimes I}^{-1}$
$\circ \lambda_I^{-1} \circ \lambda_I \circ \lambda_{I\otimes I} \circ \alpha_{I,I,I}$
$\circ$ ~~$(\lambda_I^{-1} \otimes Id_I) \circ (\lambda_I \otimes Id_I)$~~
$\circ ((Id_I \otimes Id_I) \otimes \lambda_I)$
$\circ \alpha_{I,I,I\otimes I}$
$\circ ((\lambda_I^{-1} \otimes Id_I) \otimes Id_I)$
$\circ (\lambda_I^{-1} \otimes Id_I)$

α が自然変換であることから以下の図式が可換。

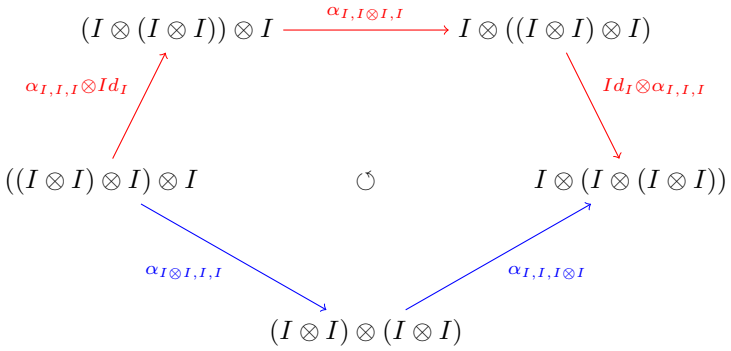


従って以下の等式が成り立つ

$$(Id_I \otimes (Id_I \otimes \lambda_I)) \circ \alpha_{I,I,I\otimes I} = \alpha_{I,I,I} \circ ((Id_I \otimes Id_I) \otimes \lambda_I)$$

$$
\begin{aligned}
&= \lambda_I \\
&\circ (\lambda_I \otimes Id_I) \\
&\circ (\rho_I^{-1} \otimes Id_I) \circ \rho_I^{-1} \circ \rho_I \circ (\rho_I \otimes Id_I) \\
&\circ \alpha_{I,I,I}^{-1} \circ (Id_I \otimes \lambda_I^{-1}) \circ (\lambda_I \otimes Id_I) \circ \alpha_{I,I,I}^{-1} \circ \lambda_{I\otimes I}^{-1} \circ \lambda_I^{-1} \circ \lambda_I \circ \lambda_{I\otimes I} \\
&\circ \ \cancel{\alpha_{I,I,I} \circ ((Id_I \otimes Id_I) \otimes \lambda_I)} \\
&\circ (Id_I \otimes (Id_I \otimes \lambda_I)) \circ \alpha_{I,I,I\otimes I} \\
&\circ \alpha_{I,I,I\otimes I} \\
&\circ ((\lambda_I^{-1} \otimes Id_I) \otimes Id_I) \\
&\circ (\lambda_I^{-1} \otimes Id_I)
\end{aligned}
$$

五角等式より以下の図式が可換と言える。

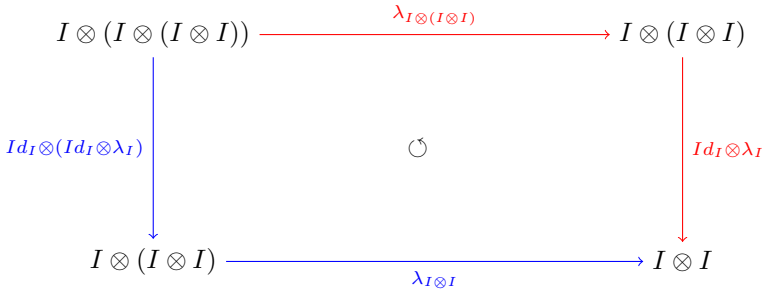


従って以下の等式が成り立つ

$$(Id_I \otimes \alpha_{I,I,I}) \circ \alpha_{I,I\otimes I,I} \circ (\alpha_{I,I,I} \otimes Id_I) = \alpha_{I,I,I\otimes I} \circ \alpha_{I\otimes I,I,I}$$

$$
\begin{aligned}
&= \lambda_I \\
&\circ (\lambda_I \otimes Id_I) \\
&\circ (\rho_I^{-1} \otimes Id_I) \circ \rho_I^{-1} \circ \rho_I \circ (\rho_I \otimes Id_I) \\
&\circ \alpha_{I,I,I}^{-1} \circ (Id_I \otimes \lambda_I^{-1}) \circ (\lambda_I \otimes Id_I) \circ \alpha_{I,I,I}^{-1} \circ \lambda_{I\otimes I}^{-1} \circ \lambda_I^{-1} \circ \lambda_I \circ \lambda_{I\otimes I} \\
&\circ (Id_I \otimes (Id_I \otimes \lambda_I)) \\
&\circ \ \cancel{\alpha_{I,I,I\otimes I} \circ \alpha_{I,I,I\otimes I}} \\
&\circ (Id_I \otimes \alpha_{I,I,I}) \circ \alpha_{I,I\otimes I,I} \circ (\alpha_{I,I,I} \otimes Id_I) \\
&\circ ((\lambda_I^{-1} \otimes Id_I) \otimes Id_I) \\
&\circ (\lambda_I^{-1} \otimes Id_I)
\end{aligned}
$$

λが自然変換であることから以下の図式が可換。

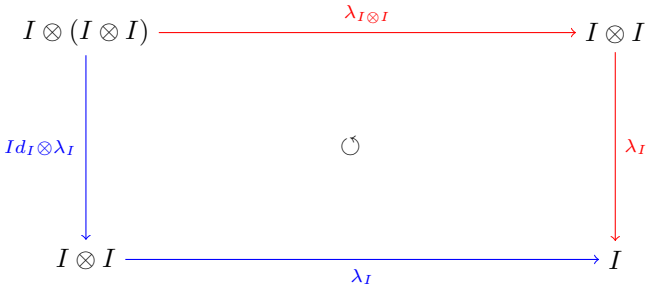


従って以下の等式が成り立つ

$$(Id_I \otimes \lambda_I) \circ \lambda_{I\otimes(I\otimes I)} = \lambda_{I\otimes I} \circ (Id_I \otimes (Id_I \otimes \lambda_I))$$

$$
\begin{aligned}
&= \lambda_I \\
&\circ (\lambda_I \otimes Id_I) \\
&\circ (\rho_I^{-1} \otimes Id_I) \circ \rho_I^{-1} \circ \rho_I \circ (\rho_I \otimes Id_I) \\
&\circ \alpha_{I,I,I}^{-1} \circ (Id_I \otimes \lambda_I^{-1}) \circ (\lambda_I \otimes Id_I) \circ \alpha_{I,I,I}^{-1} \circ \lambda_{I\otimes I}^{-1} \circ \lambda_I^{-1} \circ \lambda_I \\
&\circ \ \cancel{\lambda_{I\otimes I} \circ (Id_I \otimes (Id_I \otimes \lambda_I))} \\
&\circ (Id_I \otimes \lambda_I) \circ \lambda_{I\otimes(I\otimes I)} \\
&\circ (Id_I \otimes \alpha_{I,I,I}) \circ \alpha_{I,I\otimes I,I} \circ (\alpha_{I,I,I} \otimes Id_I) \\
&\circ ((\lambda_I^{-1} \otimes Id_I) \otimes Id_I) \\
&\circ (\lambda_I^{-1} \otimes Id_I)
\end{aligned}
$$

λが自然変換であることから以下の図式が可換。



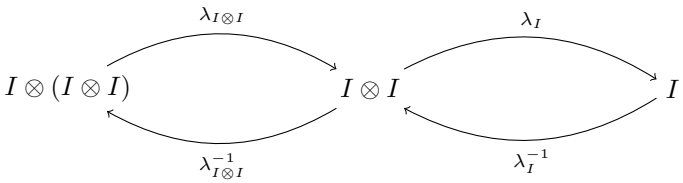
従って以下の等式が成り立つ

$$\lambda_I \circ \lambda_{I\otimes I} = \lambda_I \circ (Id_I \otimes \lambda_I)$$

$= \lambda_I$
$\circ (\lambda_I \otimes Id_I)$
$\circ (\rho_I^{-1} \otimes Id_I) \circ \rho_I^{-1} \circ \rho_I \circ (\rho_I \otimes Id_I)$
$\circ \alpha_{I,I,I}^{-1} \circ (Id_I \otimes \lambda_I^{-1}) \circ (\lambda_I \otimes Id_I) \circ \alpha_{I,I,I}^{-1} \circ \lambda_{I\otimes I}^{-1} \circ \lambda_I^{-1}$
$\circ$ ~~$\lambda_I \circ (Id_I \otimes \lambda_I)$~~
$\circ \lambda_I \circ \lambda_{I\otimes I}$
$\circ \lambda_{I\otimes(I\otimes I)}$
$\circ (Id_I \otimes \alpha_{I,I,I}) \circ \alpha_{I,I\otimes I,I} \circ (\alpha_{I,I,I} \otimes Id_I)$
$\circ ((\lambda_I^{-1} \otimes Id_I) \otimes Id_I)$
$\circ (\lambda_I^{-1} \otimes Id_I)$

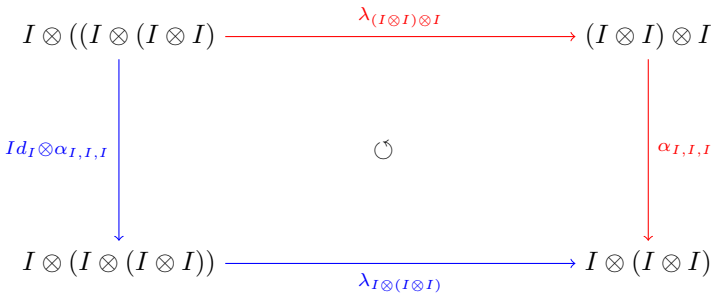
以下は恒等射なので削除

$\lambda_{I\otimes I}^{-1} \circ \lambda_I^{-1} \circ \lambda_I \circ \lambda_{I\otimes I}$



$= \lambda_I$
$\circ\, (\lambda_I \otimes Id_I)$
$\circ\, (\rho_I^{-1} \otimes Id_I) \circ \rho_I^{-1} \circ \rho_I \circ (\rho_I \otimes Id_I)$
$\circ\, \alpha_{I,I,I}^{-1} \circ (Id_I \otimes \lambda_I^{-1}) \circ (\lambda_I \otimes Id_I) \circ \alpha_{I,I,I}^{-1}$
$\circ\, \cancel{\lambda_{I\otimes I}^{-1} \circ \lambda_I^{-1} \circ \lambda_I \circ \lambda_{I\otimes I}}$
$\circ\, \lambda_{I\otimes(I\otimes I)}$
$\circ\, (Id_I \otimes \alpha_{I,I,I}) \circ \alpha_{I,I\otimes I,I} \circ (\alpha_{I,I,I} \otimes Id_I)$
$\circ\, ((\lambda_I^{-1} \otimes Id_I) \otimes Id_I)$
$\circ\, (\lambda_I^{-1} \otimes Id_I)$

λが自然変換であることから以下の図式が可換。



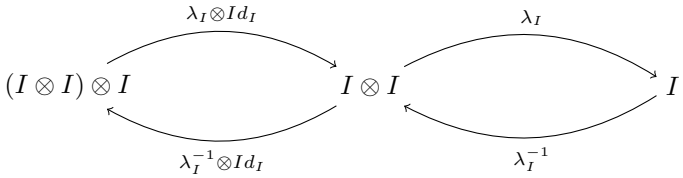
従って以下の等式が成り立つ

$$\alpha_{I,I,I} \circ \lambda_{(I\otimes I)\otimes I} = \lambda_{I\otimes(I\otimes I)} \circ (Id_I \otimes \alpha_{I,I,I})$$

$$
\begin{aligned}
&= \lambda_I \\
&\circ (\lambda_I \otimes Id_I) \\
&\circ (\rho_I^{-1} \otimes Id_I) \circ \rho_I^{-1} \circ \rho_I \circ (\rho_I \otimes Id_I) \\
&\circ \alpha_{I,I,I}^{-1} \circ (Id_I \otimes \lambda_I^{-1}) \circ (\lambda_I \otimes Id_I) \circ \alpha_{I,I,I}^{-1} \\
&\circ \cancel{\lambda_{I\otimes(I\otimes I)} \circ (Id_I \otimes \alpha_{I,I,I})} \\
&\circ \alpha_{I,I,I} \circ \lambda_{(I\otimes I)\otimes I} \\
&\circ \alpha_{I,I\otimes I,I} \circ (\alpha_{I,I,I} \otimes Id_I) \\
&\circ ((\lambda_I^{-1} \otimes Id_I) \otimes Id_I) \\
&\circ (\lambda_I^{-1} \otimes Id_I)
\end{aligned}
$$

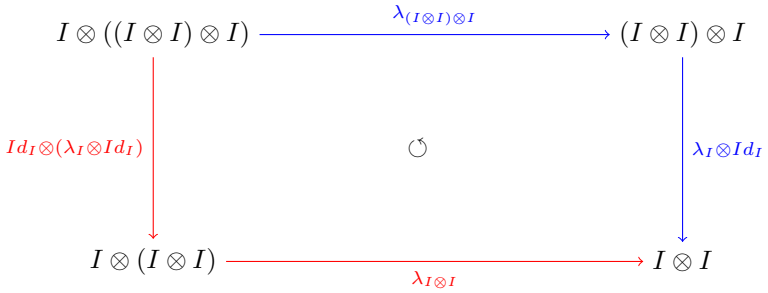
以下の恒等射を挿入

$$(\lambda_I^{-1} \otimes Id_I) \circ \lambda_I^{-1} \circ \lambda_I \circ (\lambda_I \otimes Id_I)$$



$$
\begin{aligned}
&= \lambda_I \\
&\circ (\lambda_I \otimes Id_I) \\
&\circ (\rho_I^{-1} \otimes Id_I) \circ \rho_I^{-1} \circ \rho_I \circ (\rho_I \otimes Id_I) \\
&\circ \alpha_{I,I,I}^{-1} \circ (Id_I \otimes \lambda_I^{-1}) \circ (\lambda_I \otimes Id_I) \circ \alpha_{I,I,I}^{-1} \\
&\circ \alpha_{I,I,I} \\
&\circ (\lambda_I^{-1} \otimes Id_I) \circ \lambda_I^{-1} \circ \lambda_I \circ (\lambda_I \otimes Id_I) \\
&\circ \lambda_{(I \otimes I) \otimes I} \\
&\circ \alpha_{I,I \otimes I,I} \circ (\alpha_{I,I,I} \otimes Id_I) \\
&\circ ((\lambda_I^{-1} \otimes Id_I) \otimes Id_I) \\
&\circ (\lambda_I^{-1} \otimes Id_I)
\end{aligned}
$$

λが自然変換であることから以下の図式が可換。

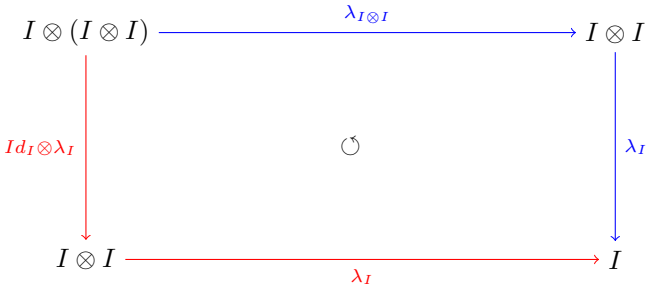


従って以下の等式が成り立つ

$$\lambda_{I\otimes I} \circ (Id_I \otimes (\lambda_I \otimes Id_I)) = (\lambda_I \otimes Id_I) \circ \lambda_{(I\otimes I)\otimes I}$$

$$
\begin{aligned}
&= \lambda_I \\
&\circ (\lambda_I \otimes Id_I) \\
&\circ (\rho_I^{-1} \otimes Id_I) \circ \rho_I^{-1} \circ \rho_I \circ (\rho_I \otimes Id_I) \\
&\circ \alpha_{I,I,I}^{-1} \circ (Id_I \otimes \lambda_I^{-1}) \circ (\lambda_I \otimes Id_I) \circ \alpha_{I,I,I}^{-1} \\
&\circ \alpha_{I,I,I} \\
&\circ (\lambda_I^{-1} \otimes Id_I) \circ \lambda_I^{-1} \circ \lambda_I \\
&\circ \cancel{(\lambda_I \otimes Id_I) \circ \lambda_{(I\otimes I)\otimes I}} \\
&\circ \lambda_{I\otimes I} \circ (Id_I \otimes (\lambda_I \otimes Id_I)) \\
&\circ \alpha_{I,I\otimes I,I} \circ (\alpha_{I,I,I} \otimes Id_I) \\
&\circ ((\lambda_I^{-1} \otimes Id_I) \otimes Id_I) \\
&\circ (\lambda_I^{-1} \otimes Id_I)
\end{aligned}
$$

λ が自然変換であることから以下の図式が可換。



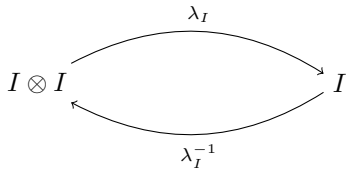
従って以下の等式が成り立つ

$$\lambda_I \circ (Id_I \otimes \lambda_I) = \lambda_I \circ \lambda_{I \otimes I}$$

$$
\begin{aligned}
&= \lambda_I \\
&\circ (\lambda_I \otimes Id_I) \\
&\circ (\rho_I^{-1} \otimes Id_I) \circ \rho_I^{-1} \circ \rho_I \circ (\rho_I \otimes Id_I) \\
&\circ \alpha_{I,I,I}^{-1} \circ (Id_I \otimes \lambda_I^{-1}) \circ (\lambda_I \otimes Id_I) \circ \alpha_{I,I,I}^{-1} \\
&\circ \alpha_{I,I,I} \\
&\circ (\lambda_I^{-1} \otimes I) \circ \lambda_I^{-1} \\
&\circ \ \cancel{\lambda_I \circ \lambda_{I \otimes I}} \\
&\circ \lambda_I \circ (Id_I \otimes \lambda_I) \\
&\circ (Id_I \otimes (\lambda_I \otimes I)) \\
&\circ \alpha_{I,I \otimes I,I} \circ (\alpha_{I,I,I} \otimes Id_I) \\
&\circ ((\lambda_I^{-1} \otimes Id_I) \otimes Id_I) \\
&\circ (\lambda_I^{-1} \otimes Id_I)
\end{aligned}
$$

以下は恒等射なので削除

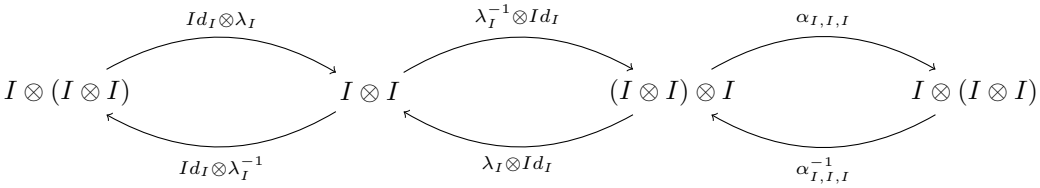
$\lambda_I^{-1} \circ \lambda_I$



$= \lambda_I$
$\circ (\lambda_I \otimes Id_I)$
$\circ (\rho_I^{-1} \otimes Id_I) \circ \rho_I^{-1} \circ \rho_I \circ (\rho_I \otimes Id_I)$
$\circ \alpha_{I,I,I}^{-1} \circ (Id_I \otimes \lambda_I^{-1}) \circ (\lambda_I \otimes Id_I) \circ \alpha_{I,I,I}^{-1}$
$\circ \alpha_{I,I,I}$
$\circ (\lambda_I^{-1} \otimes I)$
$\circ$ ~~$\lambda_I^{-1} \circ \lambda_I$~~
$\circ (Id_I \otimes \lambda_I)$
$\circ (Id_I \otimes (\lambda_I \otimes I))$
$\circ \alpha_{I,I\otimes I,I} \circ (\alpha_{I,I,I} \otimes Id_I)$
$\circ ((\lambda_I^{-1} \otimes Id_I) \otimes Id_I)$
$\circ (\lambda_I^{-1} \otimes Id_I)$

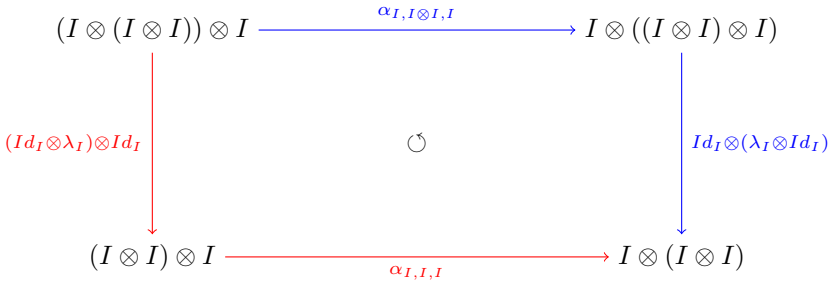
以下は恒等射なので削除

$$(Id_I \otimes \lambda_I^{-1}) \circ (\lambda_I \otimes Id_I) \circ \alpha_{I,I,I}^{-1} \circ \alpha_{I,I,I} \circ (\lambda_I^{-1} \otimes Id_I) \circ (Id_I \otimes \lambda_I)$$



$$
\begin{aligned}
&= \lambda_I \\
&\circ (\lambda_I \otimes Id_I) \\
&\circ (\rho_I^{-1} \otimes Id_I) \circ \rho_I^{-1} \circ \rho_I \circ (\rho_I \otimes Id_I) \\
&\circ \alpha_{I,I,I}^{-1} \\
&\circ \ \cancel{(Id_I \otimes \lambda_I^{-1}) \circ (\lambda_I \otimes Id_I) \circ \alpha_{I,I,I}^{-1} \circ \alpha_{I,I,I} \circ (\lambda_I^{-1} \otimes Id_I) \circ (Id_I \otimes \lambda_I)} \\
&\circ (Id_I \otimes (\lambda_I \otimes Id_I)) \\
&\circ \alpha_{I,I\otimes I,I} \circ (\alpha_{I,I,I} \otimes Id_I) \\
&\circ ((\lambda_I^{-1} \otimes Id_I) \otimes Id_I) \\
&\circ (\lambda_I^{-1} \otimes Id_I)
\end{aligned}
$$

α が自然変換であることから以下の図式が可換。



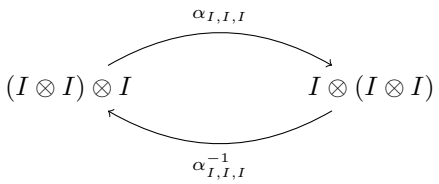
従って以下の等式が成り立つ

$$\alpha_{I,I,I} \circ ((Id_I \otimes \lambda_I) \otimes Id_I) = (Id_I \otimes (\lambda_I \otimes Id_I)) \circ \alpha_{I,I\otimes I,I}$$

$$\begin{aligned}
&= \lambda_I \\
&\circ (\lambda_I \otimes Id_I) \\
&\circ (\rho_I^{-1} \otimes Id_I) \circ \rho_I^{-1} \circ \rho_I \circ (\rho_I \otimes Id_I) \\
&\circ \alpha_{I,I,I}^{-1} \\
&\circ \cancel{(Id_I \otimes (\lambda_I \otimes Id_I)) \circ \alpha_{I,I\otimes I,I}} \\
&\circ \alpha_{I,I,I} \circ ((Id_I \otimes \lambda_I) \otimes Id_I) \\
&\circ (\alpha_{I,I,I} \otimes Id_I) \\
&\circ ((\lambda_I^{-1} \otimes Id_I) \otimes Id_I) \\
&\circ (\lambda_I^{-1} \otimes Id_I)
\end{aligned}$$

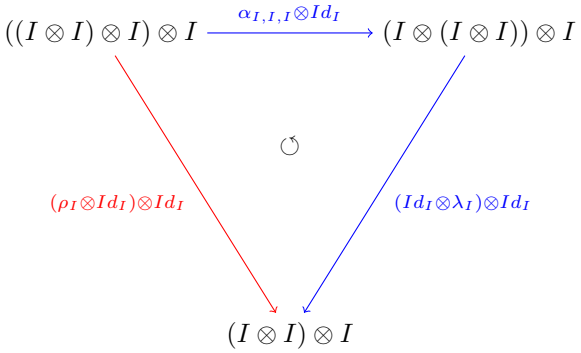
以下は恒等射なので削除

$$\alpha^{-1}_{I,I,I} \circ \alpha_{I,I,I}$$



$$
\begin{aligned}
&= \lambda_I \\
&\circ (\lambda_I \otimes Id_I) \\
&\circ (\rho_I^{-1} \otimes Id_I) \circ \rho_I^{-1} \circ \rho_I \circ (\rho_I \otimes Id_I) \\
&\circ \ \cancel{\alpha^{-1}_{I,I,I} \circ \alpha_{I,I,I}} \\
&\circ ((Id_I \otimes \lambda_I) \otimes Id_I) \\
&\circ (\alpha_{I,I,I} \otimes Id_I) \\
&\circ ((\lambda_I^{-1} \otimes Id_I) \otimes Id_I) \\
&\circ (\lambda_I^{-1} \otimes Id_I)
\end{aligned}
$$

三角等式より以下の図式が可換と言える。

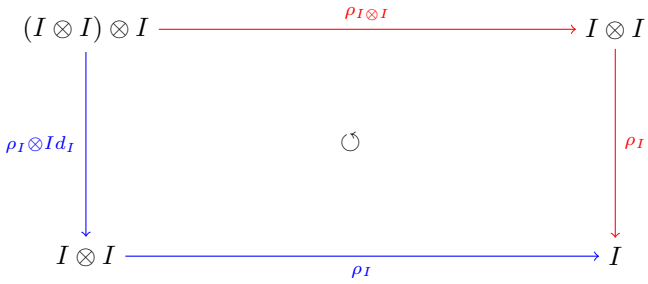


従って以下の等式が成り立つ

$$(\rho_I \otimes Id_I) \otimes Id_I = ((Id_I \otimes \lambda_I) \otimes Id_I) \circ (\alpha_{I,I,I} \otimes Id_I)$$

$$\begin{aligned}
&= \lambda_I \\
&\circ (\lambda_I \otimes Id_I) \\
&\circ (\rho_I^{-1} \otimes Id_I) \circ \rho_I^{-1} \circ \rho_I \circ (\rho_I \otimes Id_I) \\
&\circ \cancel{((Id_I \otimes \lambda_I) \otimes Id_I) \circ (\alpha_{I,I,I} \otimes Id_I)} \\
&\circ ((\rho_I \otimes Id_I) \otimes Id_I) \\
&\circ ((\lambda_I^{-1} \otimes Id_I) \otimes Id_I) \\
&\circ (\lambda_I^{-1} \otimes Id_I)
\end{aligned}$$

ρ が自然変換であることから以下の図式が可換。

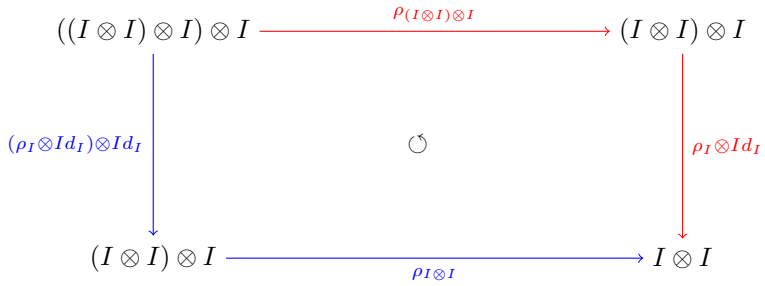


従って以下の等式が成り立つ

$$\rho_I \circ \rho_{I\otimes I} = \rho_I \circ (\rho_I \otimes Id_I)$$

$$\begin{aligned}
&= \lambda_I \\
&\circ (\lambda_I \otimes Id_I) \\
&\circ (\rho_I^{-1} \otimes Id_I) \circ \rho_I^{-1} \\
&\circ \ \cancel{\rho_I \circ (\rho_I \otimes Id_I)} \\
&\circ \rho_I \circ \rho_{I\otimes I} \\
&\circ ((\rho_I \otimes Id_I) \otimes Id_I) \\
&\circ ((\lambda_I^{-1} \otimes Id_I) \otimes Id_I) \\
&\circ (\lambda_I^{-1} \otimes Id_I)
\end{aligned}$$

ρ が自然変換であることから以下の図式が可換。



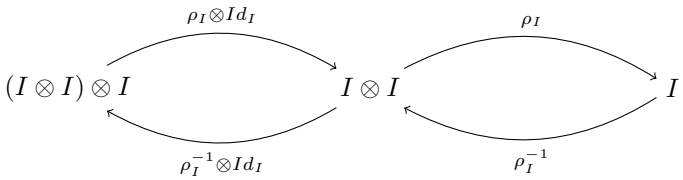
従って以下の等式が成り立つ

$$(\rho_I \otimes Id_I) \circ \rho_{(I\otimes I)\otimes I} = \rho_{I\otimes I} \circ ((\rho_I \otimes Id_I) \otimes Id_I)$$

$$\begin{aligned}
&= \lambda_I \\
&\circ (\lambda_I \otimes Id_I) \\
&\circ (\rho_I^{-1} \otimes Id_I) \\
&\circ \rho_I^{-1} \\
&\circ \rho_I \\
&\circ \cancel{\rho_{I\otimes I} \circ ((\rho_I \otimes Id_I) \otimes Id_I)} \\
&\circ (\rho_I \otimes Id_I) \circ \rho_{(I\otimes I)\otimes I} \\
&\circ ((\lambda_I^{-1} \otimes Id_I) \otimes Id_I) \\
&\circ (\lambda_I^{-1} \otimes Id_I)
\end{aligned}$$

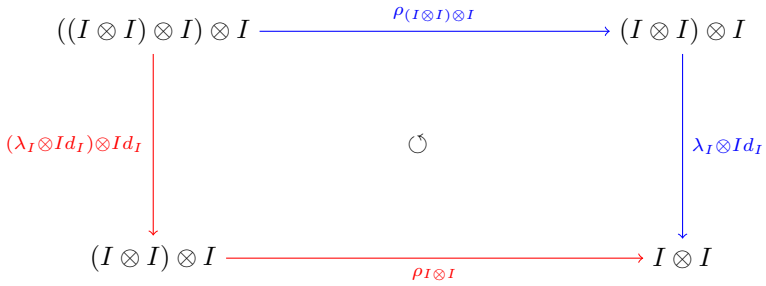
以下は恒等射なので削除

$$(\rho_I^{-1} \otimes Id_I) \circ \rho_I^{-1} \circ \rho_I \circ (\rho_I \otimes Id_I)$$



$$\begin{aligned}
&= \lambda_I \\
&\circ (\lambda_I \otimes Id_I) \\
&\circ \cancel{(\rho_I^{-1} \otimes Id_I) \circ \rho_I^{-1} \circ \rho_I \circ (\rho_I \otimes Id_I)} \\
&\circ \rho_{(I\otimes I)\otimes I} \\
&\circ ((\lambda_I^{-1} \otimes Id_I) \otimes Id_I) \\
&\circ (\lambda_I^{-1} \otimes Id_I)
\end{aligned}$$

ρ が自然変換であることから以下の図式が可換。



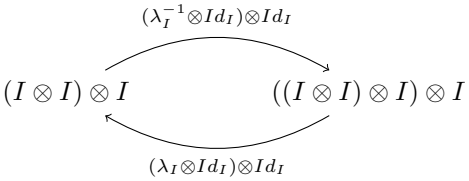
従って以下の等式が成り立つ

$$\rho_{I\otimes I} \circ ((\lambda_I \otimes Id_I) \otimes Id_I) = (\lambda_I \otimes Id_I) \circ \rho_{(I\otimes I)\otimes I}$$

$$\begin{aligned}
&= \lambda_I \\
&\circ \cancel{(\lambda_I \otimes Id_I) \circ \rho_{(I\otimes I)\otimes I}} \\
&\circ \rho_{I\otimes I} \circ ((\lambda_I \otimes Id_I) \otimes Id_I) \\
&\circ ((\lambda_I^{-1} \otimes Id_I) \otimes Id_I) \\
&\circ (\lambda_I^{-1} \otimes Id_I)
\end{aligned}$$

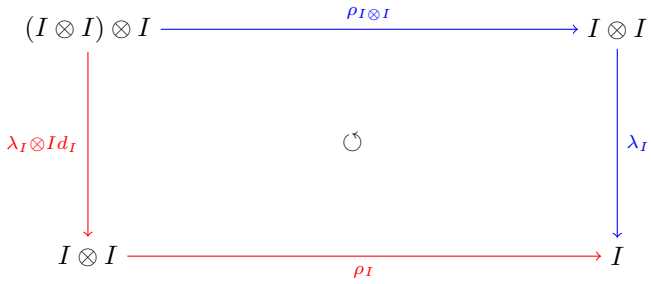
以下は恒等射なので削除

$$((\lambda_I \otimes Id_I) \otimes Id_I) \circ ((\lambda_I^{-1} \otimes Id_I) \otimes Id_I)$$



$$\begin{aligned}
&= \lambda_I \\
&\circ \rho_{I\otimes I} \\
&\circ \cancel{((\lambda_I \otimes Id_I) \otimes Id_I) \circ ((\lambda_I^{-1} \otimes Id_I) \otimes Id_I)} \\
&\circ (\lambda_I^{-1} \otimes Id_I)
\end{aligned}$$

ρ が自然変換であることから以下の図式が可換。



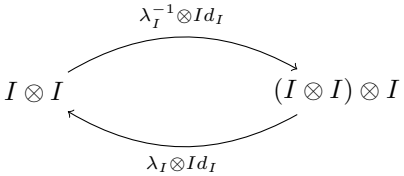
従って以下の等式が成り立つ

$$\rho_I \circ (\lambda_I \otimes Id_I) = \lambda_I \circ \rho_{I\otimes I}$$

$$= \cancel{\lambda_I \circ \rho_{I\otimes I}}$$
$$\rho_I \circ (\lambda_I \otimes Id_I)$$
$$\circ (\lambda_I^{-1} \otimes Id_I)$$

以下は恒等射なので削除

$$(\lambda_I \otimes Id_I) \circ (\lambda_I^{-1} \otimes Id_I)$$



$$= \rho_I \circ \cancel{(\lambda_I \otimes Id_I) \circ (\lambda_I^{-1} \otimes Id_I)}$$
$$= \rho_I (\text{右単位律子})$$

□