モノイダル圏における左右単位律子の一致の証明

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1　モノイダル圏

以下の５つの物 (テンソル積, 単位対象, 結合律子, 左単位律子, 右単位律子) が存在し、下記の二つの等式 (三角等式, 五角等式) を満たす圏をモノイダル圏と言う。

- テンソル積 $\otimes$
  関手 $\otimes : C \times C \to C$
- 単位対象 I
  $I \in Ob(C)$
- 結合律子 α
  自然同型 α
  $(A \otimes B) \otimes C \xrightarrow{\alpha_{A,B,C}} A \otimes (B \otimes C)$
- 左単位律子 λ
  自然同型 λ
  $I \otimes A \xrightarrow{\lambda_A} A$
- 右単位律子 ρ
  自然同型 ρ
  $A \otimes I \xrightarrow{\rho_A} A$
- 三角等式
  任意の A,B に対し以下の図式が可換

$$\begin{tikzcd}
(A \otimes I) \otimes B \arrow[rr, "\alpha_{A,I,B}"] \arrow[dr, "(\rho_A \otimes Id_B)"'] \arrow[rr, phantom, shift right=8ex, "\circlearrowleft"] & & A \otimes (I \otimes B) \arrow[dl, "(Id_A \otimes \lambda_B)"] \\
& A \otimes B &
\end{tikzcd}$$

- 五角等式

任意の A,B,C,D に対し以下の図式が可換

$$
\begin{tikzcd}
 & (A \otimes (B \otimes C)) \otimes D \arrow[rr, "\alpha_{A,B\otimes C,D}"] & & A \otimes ((B \otimes C) \otimes D) \arrow[dr, "Id_A \otimes \alpha_{B,C,D}"] & \\
((A \otimes B) \otimes C) \otimes D \arrow[ur, "\alpha_{A,B,C} \otimes Id_D"] \arrow[drr, "\alpha_{A\otimes B,C,D}"'] \arrow[rrrr, phantom, "\circlearrowleft"] & & & & A \otimes (B \otimes (C \otimes D)) \\
 & & (A \otimes B) \otimes (C \otimes D) \arrow[urr, "\alpha_{A,B,C\otimes D}"'] & &
\end{tikzcd}
$$

2 証明

Proof.

以下、λ_I(左単位律子) に恒等射を入れて行きます。

$$
\begin{aligned}
\lambda_I = {} & \lambda_I \\
& \circ (\lambda_I \otimes Id_I) \circ (\lambda_I^{-1} \otimes Id_I) \\
& = \lambda_I \\
& \circ (\lambda_I \otimes Id_I) \\
& \circ (\rho_I^{-1} \otimes Id_I) \circ (\rho_I \otimes Id_I) \\
& \circ (\lambda_I^{-1} \otimes Id_I) \\
& = \lambda_I \\
& \circ (\lambda_I \otimes Id_I) \\
& \circ (\rho_I^{-1} \otimes Id_I) \\
& \circ \rho_I^{-1} \circ \rho_I \\
& \circ (\rho_I \otimes Id_I) \\
& \circ (\lambda_I^{-1} \otimes Id_I) \\
& = \lambda_I \\
& \circ (\lambda_I \otimes Id_I) \\
& \circ (\rho_I^{-1} \otimes Id_I) \circ \rho_I^{-1} \circ \rho_I \circ (\rho_I \otimes Id_I) \\
& \circ \alpha_{I,I,I}^{-1} \circ \alpha_{I,I,I} \\
& \circ (\lambda_I^{-1} \otimes Id_I) \\
& = \lambda_I \\
& \circ (\lambda_I \otimes Id_I) \\
& \circ (\rho_I^{-1} \otimes Id_I) \circ \rho_I^{-1} \circ \rho_I \circ (\rho_I \otimes Id_I) \\
& \circ \alpha_{I,I,I}^{-1} \\
& \circ (Id_I \otimes \lambda_I^{-1}) \circ (Id_I \otimes \lambda_I) \\
& \circ \alpha_{I,I,I} \\
& \circ (\lambda_I^{-1} \otimes Id_I)
\end{aligned}
$$

$$
\begin{aligned}
&= \lambda_I \\
&\circ (\lambda_I \otimes Id_I) \\
&\circ (\rho_I^{-1} \otimes Id_I) \circ \rho_I^{-1} \circ \rho_I \circ (\rho_I \otimes Id_I) \\
&\circ \alpha_{I,I,I}^{-1} \circ (Id_I \otimes \lambda_I^{-1}) \\
&\circ \color{red}{(\lambda_I \otimes Id_I) \circ (\lambda_I^{-1} \otimes Id_I)} \circ (Id_I \otimes \lambda_I) \circ \alpha_{I,I,I} \\
&\circ (\lambda_I^{-1} \otimes Id_I) \\
&= \lambda_I \\
&\circ (\lambda_I \otimes Id_I) \\
&\circ (\rho_I^{-1} \otimes Id_I) \circ \rho_I^{-1} \circ \rho_I \circ (\rho_I \otimes Id_I) \\
&\circ \alpha_{I,I,I}^{-1} \circ (Id_I \otimes \lambda_I^{-1}) \circ (\lambda_I \otimes Id_I) \\
&\circ \color{red}{\alpha_{I,I,I}^{-1} \circ \alpha_{I,I,I}} \\
&\circ (\lambda_I^{-1} \otimes Id_I) \circ (Id_I \otimes \lambda_I) \circ \alpha_{I,I,I} \\
&\circ (\lambda_I^{-1} \otimes Id_I) \\
&= \lambda_I \\
&\circ (\lambda_I \otimes Id_I) \\
&\circ (\rho_I^{-1} \otimes Id_I) \circ \rho_I^{-1} \circ \rho_I \circ (\rho_I \otimes Id_I) \\
&\circ \alpha_{I,I,I}^{-1} \circ (Id_I \otimes \lambda_I^{-1}) \circ (\lambda_I \otimes Id_I) \circ \alpha_{I,I,I}^{-1} \\
&\circ \color{red}{\lambda_{I\otimes I}^{-1} \circ \lambda_{I\otimes I}} \\
&\circ \alpha_{I,I,I} \circ (\lambda_I^{-1} \otimes Id_I) \circ (Id_I \otimes \lambda_I) \circ \alpha_{I,I,I} \\
&\circ (\lambda_I^{-1} \otimes Id_I)
\end{aligned}
$$

$= \lambda_I$

$\circ (\lambda_I \otimes Id_I)$

$\circ (\rho_I^{-1} \otimes Id_I) \circ \rho_I^{-1} \circ \rho_I \circ (\rho_I \otimes Id_I)$

$\circ \alpha_{I,I,I}^{-1} \circ (Id_I \otimes \lambda_I^{-1})$

$\circ (\lambda_I \otimes Id_I) \circ \alpha_{I,I,I}^{-1} \circ \lambda_{I\otimes I}^{-1}$

$\circ \lambda_I^{-1} \circ \lambda_I$

$\circ \lambda_{I\otimes I} \circ \alpha_{I,I,I} \circ (\lambda_I^{-1} \otimes Id_I) \circ (Id_I \otimes \lambda_I) \circ \alpha_{I,I,I}$

$\circ (\lambda_I^{-1} \otimes Id_I)$

$= \lambda_I$

$\circ (\lambda_I \otimes Id_I)$

$\circ (\rho_I^{-1} \otimes Id_I) \circ \rho_I^{-1} \circ \rho_I \circ (\rho_I \otimes Id_I)$

$\circ \alpha_{I,I,I}^{-1} \circ (Id_I \otimes \lambda_I^{-1}) \circ (\lambda_I \otimes Id_I)$

$\circ \alpha_{I,I,I}^{-1} \circ \lambda_{I\otimes I}^{-1} \circ \lambda_I^{-1} \circ \lambda_I \circ \lambda_{I\otimes I} \circ \alpha_{I,I,I} \circ (\lambda_I^{-1} \otimes Id_I) \circ (Id_I \otimes \lambda_I) \circ \alpha_{I,I,I}$

$\circ ((\lambda_I \otimes Id_I) \otimes Id_I) \circ ((\lambda_I^{-1} \otimes Id_I) \otimes Id_I)$

$\circ (\lambda_I^{-1} \otimes Id_I)$

α が自然変換であることから以下の図式が可換。

$$\begin{tikzcd}
((I \otimes I) \otimes I) \otimes I \arrow[r, "\alpha_{I\otimes I,I,I}"] \arrow[d, "(\lambda_I \otimes Id_I)\otimes Id_I"'] \arrow[dr, phantom, "\circlearrowleft"] & (I \otimes I) \otimes (I \otimes I) \arrow[d, "\lambda_I \otimes (Id_I \otimes Id_I)"] \\
(I \otimes I) \otimes I \arrow[r, "\alpha_{I,I,I}"'] & I \otimes (I \otimes I)
\end{tikzcd}$$

従って以下の等式が成り立つ

$$\lambda_I \otimes (Id_I \otimes Id_I) \circ \alpha_{I,I,I\otimes I} = \alpha_{I,I,I} \circ (\lambda_I \otimes Id_I) \otimes Id_I$$

$$\begin{aligned}
&= \lambda_I \\
&\circ (\lambda_I \otimes Id_I) \\
&\circ (\rho_I^{-1} \otimes Id_I) \circ \rho_I^{-1} \circ \rho_I \circ (\rho_I \otimes Id_I) \\
&\circ \alpha_{I,I,I}^{-1} \circ (Id_I \otimes \lambda_I^{-1}) \circ (\lambda_I \otimes Id_I) \circ \alpha_{I,I,I}^{-1} \circ \lambda_{I\otimes I}^{-1} \circ \lambda_I^{-1} \\
&\circ \lambda_I \circ \lambda_{I\otimes I} \circ \alpha_{I,I,I} \circ (\lambda_I^{-1} \otimes Id_I) \circ (Id_I \otimes \lambda_I) \\
&\circ \cancel{\alpha_{I,I,I} \circ (\lambda_I \otimes Id_I) \otimes Id_I} \\
&\circ (\lambda_I \otimes (Id_I \otimes Id_I)) \circ \alpha_{I,I,I\otimes I} \\
&\circ ((\lambda_I^{-1} \otimes Id_I) \otimes Id_I) \\
&\circ (\lambda_I^{-1} \otimes Id_I)
\end{aligned}$$

以下の等式が成り立つ
※ λ_I を当てる順番を入れ替えただけ

$$(\lambda_I \otimes Id_I) \circ ((Id_I \otimes Id_I) \otimes \lambda_I) = \lambda_I \otimes \lambda_I = (Id_I \otimes \lambda_I) \circ (\lambda_I \otimes (Id_I \otimes Id_I))$$

$$\begin{aligned}
&= \lambda_I \\
&\circ (\lambda_I \otimes Id_I) \\
&\circ (\rho_I^{-1} \otimes Id_I) \circ \rho_I^{-1} \circ \rho_I \circ (\rho_I \otimes Id_I) \\
&\circ \alpha_{I,I,I}^{-1} \circ (Id_I \otimes \lambda_I^{-1}) \circ (\lambda_I \otimes Id_I) \circ \alpha_{I,I,I}^{-1} \circ \lambda_{I\otimes I}^{-1} \\
&\circ \lambda_I^{-1} \circ \lambda_I \circ \lambda_{I\otimes I} \circ \alpha_{I,I,I} \circ (\lambda_I^{-1} \otimes Id_I) \\
&\circ \cancel{(Id_I \otimes \lambda_I) \circ (\lambda_I \otimes (Id_I \otimes Id_I))} \\
&\circ (\lambda_I \otimes Id_I) \circ ((Id_I \otimes Id_I) \otimes \lambda_I) \\
&\circ \alpha_{I,I,I\otimes I} \\
&\circ ((\lambda_I^{-1} \otimes Id_I) \otimes Id_I) \\
&\circ (\lambda_I^{-1} \otimes Id_I)
\end{aligned}$$

以下は恒等射なので削除

$$(\lambda_I^{-1} \otimes Id_I) \circ (\lambda_I \otimes Id_I)$$

$$
\begin{aligned}
&= \lambda_I \\
&\circ (\lambda_I \otimes Id_I) \\
&\circ (\rho_I^{-1} \otimes Id_I) \circ \rho_I^{-1} \circ \rho_I \circ (\rho_I \otimes Id_I) \\
&\circ \alpha_{I,I,I}^{-1} \circ (Id_I \otimes \lambda_I^{-1}) \circ (\lambda_I \otimes Id_I) \circ \alpha_{I,I,I}^{-1} \circ \lambda_{I\otimes I}^{-1} \\
&\circ \lambda_I^{-1} \circ \lambda_I \circ \lambda_{I\otimes I} \circ \alpha_{I,I,I} \\
&\circ \cancel{(\lambda_I^{-1} \otimes Id_I) \circ (\lambda_I \otimes Id_I)} \\
&\circ ((Id_I \otimes Id_I) \otimes \lambda_I) \\
&\circ \alpha_{I,I,I\otimes I} \\
&\circ ((\lambda_I^{-1} \otimes Id_I) \otimes Id_I) \\
&\circ (\lambda_I^{-1} \otimes Id_I)
\end{aligned}
$$

α が自然変換であることから以下の図式が可換。

$$
\begin{tikzcd}
(I \otimes I) \otimes (I \otimes I) \arrow[r, "\alpha_{I,I,I\otimes I}"] \arrow[d, "(Id_I \otimes Id_I)\otimes \lambda_I"'] \arrow[dr, phantom, "\circlearrowleft"] & I \otimes (I \otimes (I \otimes I)) \arrow[d, "Id_I \otimes (Id_I \otimes \lambda_I)"] \\
(I \otimes I) \otimes I \arrow[r, "\alpha_{I,I,I}"'] & I \otimes (I \otimes I)
\end{tikzcd}
$$

従って以下の等式が成り立つ

$$(Id_I \otimes (Id_I \otimes \lambda_I)) \circ \alpha_{I,I,I\otimes I} = \alpha_{I,I,I} \circ ((Id_I \otimes Id_I) \otimes \lambda_I)$$

$$
\begin{aligned}
&= \lambda_I \\
&\circ (\lambda_I \otimes Id_I) \\
&\circ (\rho_I^{-1} \otimes Id_I) \circ \rho_I^{-1} \circ \rho_I \circ (\rho_I \otimes Id_I) \\
&\circ \alpha_{I,I,I}^{-1} \circ (Id_I \otimes \lambda_I^{-1}) \circ (\lambda_I \otimes Id_I) \circ \alpha_{I,I,I}^{-1} \circ \lambda_{I\otimes I}^{-1} \circ \lambda_I^{-1} \circ \lambda_I \circ \lambda_{I\otimes I} \\
&\circ \cancel{\alpha_{I,I,I} \circ ((Id_I \otimes Id_I) \otimes \lambda_I)} \\
&\circ (Id_I \otimes (Id_I \otimes \lambda_I)) \circ \alpha_{I,I,I\otimes I} \\
&\circ \alpha_{I,I,I\otimes I} \\
&\circ ((\lambda_I^{-1} \otimes Id_I) \otimes Id_I) \\
&\circ (\lambda_I^{-1} \otimes Id_I)
\end{aligned}
$$

五角等式より

$$\begin{tikzcd}
 & (I \otimes (I \otimes I)) \otimes I \arrow[rr, "\alpha_{I,I\otimes I,I}"] & & I \otimes ((I \otimes I) \otimes I) \arrow[dr, "Id_I \otimes \alpha_{I,I,I}"] & \\
((I \otimes I) \otimes I) \otimes I \arrow[ur, "\alpha_{I,I,I} \otimes Id_I"] \arrow[drr, "\alpha_{I\otimes I,I,I}"'] & & \circlearrowleft & & I \otimes (I \otimes (I \otimes I)) \\
 & & (I \otimes I) \otimes (I \otimes I) \arrow[urr, "\alpha_{I,I,I\otimes I}"'] & &
\end{tikzcd}$$

従って以下の等式が成り立つ

$$(Id_I \otimes \alpha_{I,I,I}) \circ \alpha_{I,I\otimes I,I} \circ (\alpha_{I,I,I} \otimes Id_I) = \alpha_{I,I,I\otimes I} \circ \alpha_{I,I,I\otimes I}$$

$= \lambda_I$

$\circ (\lambda_I \otimes Id_I)$

$\circ (\rho_I^{-1} \otimes Id_I) \circ \rho_I^{-1} \circ \rho_I \circ (\rho_I \otimes Id_I)$

$\circ \alpha_{I,I,I}^{-1} \circ (Id_I \otimes \lambda_I^{-1}) \circ (\lambda_I \otimes Id_I) \circ \alpha_{I,I,I}^{-1} \circ \lambda_{I\otimes I}^{-1} \circ \lambda_I^{-1} \circ \lambda_I \circ \lambda_{I\otimes I}$

$\circ (Id_I \otimes (Id_I \otimes \lambda_I))$

$\circ \ \cancel{\alpha_{I,I,I\otimes I} \circ \alpha_{I,I,I\otimes I}}$

$\circ (Id_I \otimes \alpha_{I,I,I}) \circ \alpha_{I,I\otimes I,I} \circ (\alpha_{I,I,I} \otimes Id_I)$

$\circ ((\lambda_I^{-1} \otimes Id_I) \otimes Id_I)$

$\circ (\lambda_I^{-1} \otimes Id_I)$

λ が自然変換であることから以下の図式が可換。

$$\begin{tikzcd}
I \otimes (I \otimes (I \otimes I)) \arrow[r, "\lambda_{I\otimes(I\otimes I)}"] \arrow[d, "Id_I\otimes(Id_I\otimes\lambda_I)"'] \arrow[dr, phantom, "\circlearrowleft"] & I \otimes (I \otimes I) \arrow[d, "Id_I\otimes\lambda_I"] \\
I \otimes (I \otimes I) \arrow[r, "\lambda_{I\otimes I}"'] & I \otimes I
\end{tikzcd}$$

従って以下の等式が成り立つ

$$(Id_I \otimes \lambda_I) \circ \lambda_{I\otimes(I\otimes I)} = \lambda_{I\otimes I} \circ (Id_I \otimes (Id_I \otimes \lambda_I))$$

$$\begin{aligned}
&= \lambda_I \\
&\circ (\lambda_I \otimes Id_I) \\
&\circ (\rho_I^{-1} \otimes Id_I) \circ \rho_I^{-1} \circ \rho_I \circ (\rho_I \otimes Id_I) \\
&\circ \alpha_{I,I,I}^{-1} \circ (Id_I \otimes \lambda_I^{-1}) \circ (\lambda_I \otimes Id_I) \circ \alpha_{I,I,I}^{-1} \circ \lambda_{I\otimes I}^{-1} \circ \lambda_I^{-1} \circ \lambda_I \\
&\circ \cancel{\lambda_{I\otimes I} \circ (Id_I \otimes (Id_I \otimes \lambda_I))} \\
&\circ (Id_I \otimes \lambda_I) \circ \lambda_{I\otimes(I\otimes I)} \\
&\circ (Id_I \otimes \alpha_{I,I,I}) \circ \alpha_{I,I\otimes I,I} \circ (\alpha_{I,I,I} \otimes Id_I) \\
&\circ ((\lambda_I^{-1} \otimes Id_I) \otimes Id_I) \\
&\circ (\lambda_I^{-1} \otimes Id_I)
\end{aligned}$$

λ が自然変換であることから以下の図式が可換。

$$
\begin{tikzcd}
I \otimes (I \otimes I) \arrow[rr, "\lambda_{I \otimes I}"] \arrow[d, "Id_I \otimes \lambda_I"'] \arrow[drr, phantom, "\circlearrowleft"] & & I \otimes I \arrow[d, "\lambda_I"] \\
I \otimes I \arrow[rr, "\lambda_I"'] & & I
\end{tikzcd}
$$

従って以下の等式が成り立つ

$$\lambda_I \circ \lambda_{I \otimes I} = \lambda_I \circ (Id_I \otimes \lambda_I)$$

$= \lambda_I$

$\circ (\lambda_I \otimes Id_I)$

$\circ (\rho_I^{-1} \otimes Id_I) \circ \rho_I^{-1} \circ \rho_I \circ (\rho_I \otimes Id_I)$

$\circ \alpha_{I,I,I}^{-1} \circ (Id_I \otimes \lambda_I^{-1}) \circ (\lambda_I \otimes Id_I) \circ \alpha_{I,I,I}^{-1} \circ \lambda_{I \otimes I}^{-1} \circ \lambda_I^{-1}$

$\circ \cancel{\lambda_I \circ (Id_I \otimes \lambda_I)}$

$\circ \lambda_I \circ \lambda_{I \otimes I}$

$\circ \lambda_{I \otimes (I \otimes I)}$

$\circ (Id_I \otimes \alpha_{I,I,I}) \circ \alpha_{I,I \otimes I,I} \circ (\alpha_{I,I,I} \otimes Id_I)$

$\circ ((\lambda_I^{-1} \otimes Id_I) \otimes Id_I)$

$\circ (\lambda_I^{-1} \otimes Id_I)$

以下は恒等射なので削除

$$\lambda_{I \otimes I}^{-1} \circ \lambda_I^{-1} \circ \lambda_I \circ \lambda_{I \otimes I}$$

$= \lambda_I$

$\circ (\lambda_I \otimes Id_I)$

$\circ (\rho_I^{-1} \otimes Id_I) \circ \rho_I^{-1} \circ \rho_I \circ (\rho_I \otimes Id_I)$

$\circ \alpha_{I,I,I}^{-1} \circ (Id_I \otimes \lambda_I^{-1}) \circ (\lambda_I \otimes Id_I) \circ \alpha_{I,I,I}^{-1}$

$\circ \cancel{\lambda_{I \otimes I}^{-1} \circ \lambda_I^{-1} \circ \lambda_I \circ \lambda_{I \otimes I}}$

$\circ \lambda_{I \otimes (I \otimes I)}$

$\circ (Id_I \otimes \alpha_{I,I,I}) \circ \alpha_{I,I \otimes I,I} \circ (\alpha_{I,I,I} \otimes Id_I)$

$\circ ((\lambda_I^{-1} \otimes Id_I) \otimes Id_I)$

$\circ (\lambda_I^{-1} \otimes Id_I)$

λ が自然変換であることから以下の図式が可換。

$$
\begin{tikzcd}
I \otimes ((I \otimes (I \otimes I) \arrow[r, "\lambda_{(I\otimes I)\otimes I}"] \arrow[d, "Id_I \otimes \alpha_{I,I,I}"'] \arrow[dr, phantom, "\circlearrowleft"] & (I \otimes I) \otimes I \arrow[d, "\alpha_{I,I,I}"] \\
I \otimes (I \otimes (I \otimes I)) \arrow[r, "\lambda_{I\otimes(I\otimes I)}"'] & I \otimes (I \otimes I)
\end{tikzcd}
$$

従って以下の等式が成り立つ

$$\alpha_{I,I,I} \circ \lambda_{(I\otimes I)\otimes I} = \lambda_{I\otimes(I\otimes I)} \circ (Id_I \otimes \alpha_{I,I,I})$$

$= \lambda_I$
$\circ\, (\lambda_I \otimes Id_I)$
$\circ\, (\rho_I^{-1} \otimes Id_I) \circ \rho_I^{-1} \circ \rho_I \circ (\rho_I \otimes Id_I)$
$\circ\, \alpha_{I,I,I}^{-1} \circ (Id_I \otimes \lambda_I^{-1}) \circ (\lambda_I \otimes Id_I) \circ \alpha_{I,I,I}^{-1}$
$\circ\, \cancel{\lambda_{I\otimes(I\otimes I)} \circ (Id_I \otimes \alpha_{I,I,I})}$
$\circ\, \alpha_{I,I,I} \circ \lambda_{(I\otimes I)\otimes I}$
$\circ\, \alpha_{I,I\otimes I,I} \circ (\alpha_{I,I,I} \otimes Id_I)$
$\circ\, ((\lambda_I^{-1} \otimes Id_I) \otimes Id_I)$
$\circ\, (\lambda_I^{-1} \otimes Id_I)$

以下の恒等射を挿入

$$(\lambda_I^{-1} \otimes I) \circ \lambda_I^{-1} \circ \lambda_I \circ (\lambda_I \otimes I)$$

$= \lambda_I$
$\circ\, (\lambda_I \otimes Id_I)$
$\circ\, (\rho_I^{-1} \otimes Id_I) \circ \rho_I^{-1} \circ \rho_I \circ (\rho_I \otimes Id_I)$
$\circ\, \alpha_{I,I,I}^{-1} \circ (Id_I \otimes \lambda_I^{-1}) \circ (\lambda_I \otimes Id_I) \circ \alpha_{I,I,I}^{-1}$
$\circ\, \alpha_{I,I,I}$
$\circ\, (\lambda_I^{-1} \otimes I) \circ \lambda_I^{-1} \circ \lambda_I \circ (\lambda_I \otimes I)$
$\circ\, \lambda_{(I\otimes I)\otimes I}$
$\circ\, \alpha_{I,I\otimes I,I} \circ (\alpha_{I,I,I} \otimes Id_I)$
$\circ\, ((\lambda_I^{-1} \otimes Id_I) \otimes Id_I)$
$\circ\, (\lambda_I^{-1} \otimes Id_I)$

λ が自然変換であることから以下の図式が可換。

$$\begin{tikzcd}
I \otimes ((I \otimes I) \otimes I) \arrow[r, "\lambda_{(I\otimes I)\otimes I}"] \arrow[d, "Id_I \otimes (\lambda_I \otimes I)"'] \arrow[dr, phantom, "\circlearrowleft"] & (I \otimes I) \otimes I \arrow[d, "\lambda_I \otimes I"] \\
I \otimes (I \otimes I) \arrow[r, "\lambda_{I\otimes I}"'] & I \otimes I
\end{tikzcd}$$

従って以下の等式が成り立つ

$$\lambda_{I\otimes I} \circ (Id_I \otimes (\lambda_I \otimes I)) = (\lambda_I \otimes I) \circ \lambda_{(I\otimes I)\otimes I}$$

$$\begin{aligned}
&= \lambda_I \\
&\circ (\lambda_I \otimes Id_I) \\
&\circ (\rho_I^{-1} \otimes Id_I) \circ \rho_I^{-1} \circ \rho_I \circ (\rho_I \otimes Id_I) \\
&\circ \alpha_{I,I,I}^{-1} \circ (Id_I \otimes \lambda_I^{-1}) \circ (\lambda_I \otimes Id_I) \circ \alpha_{I,I,I}^{-1} \\
&\circ \alpha_{I,I,I} \\
&\circ (\lambda_I^{-1} \otimes I) \circ \lambda_I^{-1} \circ \lambda_I \\
&\circ \cancel{(\lambda_I \otimes I) \circ \lambda_{(I\otimes I)\otimes I}} \\
&\circ \lambda_{I\otimes I} \circ (Id_I \otimes (\lambda_I \otimes I)) \\
&\circ \alpha_{I,I\otimes I,I} \circ (\alpha_{I,I,I} \otimes Id_I) \\
&\circ ((\lambda_I^{-1} \otimes Id_I) \otimes Id_I) \\
&\circ (\lambda_I^{-1} \otimes Id_I)
\end{aligned}$$

λ が自然変換であることから以下の図式が可換。

$$\begin{tikzcd}
I \otimes (I \otimes I) \arrow[rr, "\lambda_{I\otimes I}"] \arrow[d, "Id_I \otimes \lambda_I"'] & & I \otimes I \arrow[d, "\lambda_I"] \\
I \otimes I \arrow[rr, "\lambda_I"'] & \circlearrowleft & I
\end{tikzcd}$$

従って以下の等式が成り立つ

$$\lambda_I \circ (Id_I \otimes \lambda_I) = \lambda_I \circ \lambda_{I\otimes I}$$

$$\begin{aligned}
&= \lambda_I \\
&\circ (\lambda_I \otimes Id_I) \\
&\circ (\rho_I^{-1} \otimes Id_I) \circ \rho_I^{-1} \circ \rho_I \circ (\rho_I \otimes Id_I) \\
&\circ \alpha_{I,I,I}^{-1} \circ (Id_I \otimes \lambda_I^{-1}) \circ (\lambda_I \otimes Id_I) \circ \alpha_{I,I,I}^{-1} \\
&\circ \alpha_{I,I,I} \\
&\circ (\lambda_I^{-1} \otimes I) \circ \lambda_I^{-1} \\
&\circ \ \cancel{\lambda_I \circ \lambda_{I\otimes I}} \\
&\circ \lambda_I \circ (Id_I \otimes \lambda_I) \\
&\circ (Id_I \otimes (\lambda_I \otimes I)) \\
&\circ \alpha_{I,I\otimes I,I} \circ (\alpha_{I,I,I} \otimes Id_I) \\
&\circ ((\lambda_I^{-1} \otimes Id_I) \otimes Id_I) \\
&\circ (\lambda_I^{-1} \otimes Id_I)
\end{aligned}$$

以下は恒等射なので削除

$$\lambda_I^{-1} \circ \lambda_I$$

$$\begin{aligned}
&= \lambda_I \\
&\circ (\lambda_I \otimes Id_I) \\
&\circ (\rho_I^{-1} \otimes Id_I) \circ \rho_I^{-1} \circ \rho_I \circ (\rho_I \otimes Id_I) \\
&\circ \alpha_{I,I,I}^{-1} \circ (Id_I \otimes \lambda_I^{-1}) \circ (\lambda_I \otimes Id_I) \circ \alpha_{I,I,I}^{-1} \\
&\circ \alpha_{I,I,I} \\
&\circ (\lambda_I^{-1} \otimes I) \\
&\circ \ \cancel{\lambda_I^{-1} \circ \lambda_I} \\
&\circ (Id_I \otimes \lambda_I) \\
&\circ (Id_I \otimes (\lambda_I \otimes I)) \\
&\circ \alpha_{I,I\otimes I,I} \circ (\alpha_{I,I,I} \otimes Id_I) \\
&\circ ((\lambda_I^{-1} \otimes Id_I) \otimes Id_I) \\
&\circ (\lambda_I^{-1} \otimes Id_I)
\end{aligned}$$

以下は恒等射なので削除

$$(Id_I \otimes \lambda_I^{-1}) \circ (\lambda_I \otimes Id_I) \circ \alpha_{I,I,I}^{-1} \circ \alpha_{I,I,I} \circ (\lambda_I^{-1} \otimes Id_I) \circ (Id_I \otimes \lambda_I)$$

$$\begin{aligned}
&= \lambda_I \\
&\circ (\lambda_I \otimes Id_I) \\
&\circ (\rho_I^{-1} \otimes Id_I) \circ \rho_I^{-1} \circ \rho_I \circ (\rho_I \otimes Id_I) \\
&\circ \alpha_{I,I,I}^{-1} \\
&\circ \ \cancel{(Id_I \otimes \lambda_I^{-1}) \circ (\lambda_I \otimes Id_I) \circ \alpha_{I,I,I}^{-1} \circ \alpha_{I,I,I} \circ (\lambda_I^{-1} \otimes Id_I) \circ (Id_I \otimes \lambda_I)} \\
&\circ (Id_I \otimes (\lambda_I \otimes Id_I)) \\
&\circ \alpha_{I,I\otimes I,I} \circ (\alpha_{I,I,I} \otimes Id_I) \\
&\circ ((\lambda_I^{-1} \otimes Id_I) \otimes Id_I) \\
&\circ (\lambda_I^{-1} \otimes Id_I)
\end{aligned}$$

α が自然変換であることから以下の図式が可換。

$$\begin{tikzcd}
(I \otimes (I \otimes I)) \otimes I \arrow[r, "\alpha_{I,I\otimes I,I}"] \arrow[d, "(Id_I \otimes \lambda_I) \otimes Id_I"'] \arrow[dr, phantom, "\circlearrowleft"] & I \otimes ((I \otimes I) \otimes I) \arrow[d, "Id_I \otimes (\lambda_I \otimes Id_I)"] \\
(I \otimes I) \otimes I \arrow[r, "\alpha_{I,I,I}"'] & I \otimes (I \otimes I)
\end{tikzcd}$$

従って以下の等式が成り立つ

$$\alpha_{I,I,I} \circ ((Id_I \otimes \lambda_I) \otimes Id_I) = (Id_I \otimes (\lambda_I \otimes Id_I)) \circ \alpha_{I,I\otimes I,I}$$

$= \lambda_I$
$\circ (\lambda_I \otimes Id_I)$
$\circ (\rho_I^{-1} \otimes Id_I) \circ \rho_I^{-1} \circ \rho_I \circ (\rho_I \otimes Id_I)$
$\circ \alpha^{-1}_{I,I,I}$
$\circ \cancel{(Id_I \otimes (\lambda_I \otimes Id_I)) \circ \alpha_{I,I\otimes I,I}}$
$\circ \alpha_{I,I,I} \circ ((Id_I \otimes \lambda_I) \otimes Id_I)$
$\circ (\alpha_{I,I,I} \otimes Id_I)$
$\circ ((\lambda_I^{-1} \otimes Id_I) \otimes Id_I)$
$\circ (\lambda_I^{-1} \otimes Id_I)$

以下は恒等射なので削除

$$\alpha^{-1}_{I,I,I} \circ \alpha_{I,I,I}$$

$= \lambda_I$
$\circ (\lambda_I \otimes Id_I)$
$\circ (\rho_I^{-1} \otimes Id_I) \circ \rho_I^{-1} \circ \rho_I \circ (\rho_I \otimes Id_I)$
$\circ \cancel{\alpha^{-1}_{I,I,I} \circ \alpha_{I,I,I}}$
$\circ ((Id_I \otimes \lambda_I) \otimes Id_I)$
$\circ (\alpha_{I,I,I} \otimes Id_I)$
$\circ ((\lambda_I^{-1} \otimes Id_I) \otimes Id_I)$
$\circ (\lambda_I^{-1} \otimes Id_I)$

三角等式より

$$
\begin{tikzcd}
((I \otimes I) \otimes I) \otimes I \arrow[rr, "\alpha_{I,I,I} \otimes Id_I"] \arrow[dr, "(\rho_I \otimes Id_I) \otimes Id_I"'] & \circlearrowleft & (I \otimes (I \otimes I)) \otimes I \arrow[dl, "(Id_I \otimes \lambda_I) \otimes Id_I"] \\
& (I \otimes I) \otimes I &
\end{tikzcd}
$$

従って以下の等式が成り立つ

$$(\rho_I \otimes Id_I) \otimes Id_I = ((Id_I \otimes \lambda_I) \otimes Id_I) \circ (\alpha_{I,I,I} \otimes Id_I)$$

$$
\begin{aligned}
&= \lambda_I \\
&\circ (\lambda_I \otimes Id_I) \\
&\circ (\rho_I^{-1} \otimes Id_I) \circ \rho_I^{-1} \circ \rho_I \circ (\rho_I \otimes Id_I) \\
&\circ \cancel{((Id_I \otimes \lambda_I) \otimes Id_I) \circ (\alpha_{I,I,I} \otimes Id_I)} \\
&\circ ((\rho_I \otimes Id_I) \otimes Id_I) \\
&\circ ((\lambda_I^{-1} \otimes Id_I) \otimes Id_I) \\
&\circ (\lambda_I^{-1} \otimes Id_I)
\end{aligned}
$$

ρ が自然変換であることから以下の図式が可換。

$$\begin{tikzcd}
(I \otimes I) \otimes I \arrow[r, "\rho_{I \otimes I}"] \arrow[d, "\rho_I \otimes Id_I"'] \arrow[dr, phantom, "\circlearrowleft"] & I \otimes I \arrow[d, "\rho_I"] \\
I \otimes I \arrow[r, "\rho_I"'] & I
\end{tikzcd}$$

従って以下の等式が成り立つ

$$\rho_I \circ \rho_{I \otimes I} = \rho_I \circ (\rho_I \otimes Id_I)$$

$= \lambda_I$

$\circ (\lambda_I \otimes Id_I)$

$\circ (\rho_I^{-1} \otimes Id_I) \circ \rho_I^{-1}$

$\circ \ \cancel{\rho_I \circ (\rho_I \otimes Id_I)}$

$\circ \ \rho_I \circ \rho_{I \otimes I}$

$\circ ((\rho_I \otimes Id_I) \otimes Id_I)$

$\circ ((\lambda_I^{-1} \otimes Id_I) \otimes Id_I)$

$\circ (\lambda_I^{-1} \otimes Id_I)$

ρ が自然変換であることから以下の図式が可換。

$$
\begin{tikzcd}
((I \otimes I) \otimes I) \otimes I \arrow[r, "\rho_{(I\otimes I)\otimes I}"] \arrow[d, "(\rho_I \otimes Id_I)\otimes Id_I"'] \arrow[dr, phantom, "\circlearrowleft"] & (I \otimes I) \otimes I \arrow[d, "\rho_I \otimes Id_I"] \\
(I \otimes I) \otimes I \arrow[r, "\rho_{I\otimes I}"'] & I \otimes I
\end{tikzcd}
$$

従って以下の等式が成り立つ

$$(\rho_I \otimes Id_I) \circ \rho_{(I\otimes I)\otimes I} = \rho_{I\otimes I} \circ ((\rho_I \otimes Id_I) \otimes Id_I)$$

$$
\begin{aligned}
&= \lambda_I \\
&\circ (\lambda_I \otimes Id_I) \\
&\circ (\rho_I^{-1} \otimes Id_I) \\
&\circ \rho_I^{-1} \\
&\circ \rho_I \\
&\circ \cancel{\rho_{I\otimes I} \circ ((\rho_I \otimes Id_I) \otimes Id_I)} \\
&\circ (\rho_I \otimes Id_I) \circ \rho_{(I\otimes I)\otimes I} \\
&\circ ((\lambda_I^{-1} \otimes Id_I) \otimes Id_I) \\
&\circ (\lambda_I^{-1} \otimes Id_I)
\end{aligned}
$$

以下は恒等射なので削除

$$(\rho_I^{-1} \otimes Id_I) \circ \rho_I^{-1} \circ \rho_I \circ (\rho_I \otimes Id_I)$$

$$
\begin{aligned}
&= \lambda_I \\
&\circ (\lambda_I \otimes Id_I) \\
&\circ \cancel{(\rho_I^{-1} \otimes Id_I) \circ \rho_I^{-1} \circ \rho_I \circ (\rho_I \otimes Id_I)} \\
&\circ \rho_{(I\otimes I)\otimes I} \\
&\circ ((\lambda_I^{-1} \otimes Id_I) \otimes Id_I) \\
&\circ (\lambda_I^{-1} \otimes Id_I)
\end{aligned}
$$

ρ が自然変換であることから以下の図式が可換。

$$\begin{tikzcd}
((I \otimes I) \otimes I) \otimes I \arrow[r, "\rho_{(I\otimes I)\otimes I}"] \arrow[d, "(\lambda_I \otimes Id_I)\otimes Id_I"'] \arrow[dr, phantom, "\circlearrowleft"] & (I \otimes I) \otimes I \arrow[d, "\lambda_I \otimes Id_I"] \\
(I \otimes I) \otimes I \arrow[r, "\rho_{I\otimes I}"'] & I \otimes I
\end{tikzcd}$$

従って以下の等式が成り立つ

$$\rho_{I\otimes I} \circ ((\lambda_I \otimes Id_I) \otimes Id_I) = (\lambda_I \otimes Id_I) \circ \rho_{(I\otimes I)\otimes I}$$

$= \lambda_I$
$\circ\ \cancel{(\lambda_I \otimes Id_I) \circ \rho_{(I\otimes I)\otimes I}}$
$\circ\, \rho_{I\otimes I} \circ ((\lambda_I \otimes Id_I) \otimes Id_I)$
$\circ\, ((\lambda_I^{-1} \otimes Id_I) \otimes Id_I)$
$\circ\, (\lambda_I^{-1} \otimes Id_I)$

以下は恒等射なので削除

$$((\lambda_I \otimes Id_I) \otimes Id_I) \circ ((\lambda_I^{-1} \otimes Id_I) \otimes Id_I)$$

$= \lambda_I$
$\circ\, \rho_{I\otimes I}$
$\circ\ \cancel{((\lambda_I \otimes Id_I) \otimes Id_I) \circ ((\lambda_I^{-1} \otimes Id_I) \otimes Id_I)}$
$\circ\, (\lambda_I^{-1} \otimes Id_I)$

ρ が自然変換であることから以下の図式が可換。

$$\begin{tikzcd}
(I \otimes I) \otimes I \arrow[r, "\rho_{I \otimes I}"] \arrow[d, "\lambda_I \otimes Id_I"'] \arrow[dr, phantom, "\circlearrowleft"] & I \otimes I \arrow[d, "\lambda_I"] \\
I \otimes I \arrow[r, "\rho_I"'] & I
\end{tikzcd}$$

従って以下の等式が成り立つ

$$\rho_I \circ (\lambda_I \otimes Id_I) = \lambda_I \circ \rho_{I \otimes I}$$

$$= \cancel{\lambda_I \circ \rho_{I \otimes I}}$$
$$\rho_I \circ (\lambda_I \otimes Id_I)$$
$$\circ (\lambda_I^{-1} \otimes Id_I)$$

以下は恒等射なので削除

$$(\lambda_I \otimes Id_I) \circ (\lambda_I^{-1} \otimes Id_I)$$

$$= \rho_I \circ \cancel{(\lambda_I \otimes Id_I) \circ (\lambda_I^{-1} \otimes Id_I)}$$
$= \rho_I$(右単位律子)

□