

モノイダル圏における左右単位子の一致の証明

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Proof.

以下、 λ_I に恒等射を入れて行きます。

$$\begin{aligned}\lambda_I &= \lambda_I \\ &\circ (\lambda_I \otimes Id_I) \circ (\lambda_I^{-1} \otimes Id_I) \\ &= \lambda_I \\ &\circ (\lambda_I \otimes Id_I) \\ &\circ (\rho_I^{-1} \otimes Id_I) \circ (\rho_I \otimes Id_I) \\ &\circ (\lambda_I^{-1} \otimes Id_I) \\ &= \lambda_I \\ &\circ (\lambda_I \otimes Id_I) \\ &\circ (\rho_I^{-1} \otimes Id_I) \\ &\circ \rho_I^{-1} \circ \rho_I \\ &\circ (\rho_I \otimes Id_I) \\ &\circ (\lambda_I^{-1} \otimes Id_I) \\ &= \lambda_I \\ &\circ (\lambda_I \otimes Id_I) \\ &\circ (\rho_I^{-1} \otimes Id_I) \circ \rho_I^{-1} \circ \rho_I \circ (\rho_I \otimes Id_I) \\ &\circ \alpha_{I,I,I}^{-1} \circ \alpha_{I,I,I} \\ &\circ (\lambda_I^{-1} \otimes Id_I) \\ &= \lambda_I \\ &\circ (\lambda_I \otimes Id_I) \\ &\circ (\rho_I^{-1} \otimes Id_I) \circ \rho_I^{-1} \circ \rho_I \circ (\rho_I \otimes Id_I) \\ &\circ \alpha_{I,I,I}^{-1} \\ &\circ (Id_I \otimes \lambda_I^{-1}) \circ (Id_I \otimes \lambda_I) \\ &\circ \alpha_{I,I,I} \\ &\circ (\lambda_I^{-1} \otimes Id_I)\end{aligned}$$

$$\begin{aligned}
&= \lambda_I \\
&\circ (\lambda_I \otimes Id_I) \\
&\circ (\rho_I^{-1} \otimes Id_I) \circ \rho_I^{-1} \circ \rho_I \circ (\rho_I \otimes Id_I) \\
&\circ \alpha_{I,I,I}^{-1} \circ (Id_I \otimes \lambda_I^{-1}) \\
&\circ (\lambda_I \otimes Id_I) \circ (\lambda_I^{-1} \otimes Id_I) \circ (Id_I \otimes \lambda_I) \circ \alpha_{I,I,I} \\
&\circ (\lambda_I^{-1} \otimes Id_I) \\
&= \lambda_I \\
&\circ (\lambda_I \otimes Id_I) \\
&\circ (\rho_I^{-1} \otimes Id_I) \circ \rho_I^{-1} \circ \rho_I \circ (\rho_I \otimes Id_I) \\
&\circ \alpha_{I,I,I}^{-1} \circ (Id_I \otimes \lambda_I^{-1}) \circ (\lambda_I \otimes Id_I) \\
&\circ \alpha_{I,I,I}^{-1} \circ \alpha_{I,I,I} \\
&\circ (\lambda_I^{-1} \otimes Id_I) \circ (Id_I \otimes \lambda_I) \circ \alpha_{I,I,I} \\
&\circ (\lambda_I^{-1} \otimes Id_I) \\
&= \lambda_I \\
&\circ (\lambda_I \otimes Id_I) \\
&\circ (\rho_I^{-1} \otimes Id_I) \circ \rho_I^{-1} \circ \rho_I \circ (\rho_I \otimes Id_I) \\
&\circ \alpha_{I,I,I}^{-1} \circ (Id_I \otimes \lambda_I^{-1}) \circ (\lambda_I \otimes Id_I) \circ \alpha_{I,I,I}^{-1} \\
&\circ \lambda_{I \otimes I}^{-1} \circ \lambda_{I \otimes I} \\
&\circ \alpha_{I,I,I} \circ (\lambda_I^{-1} \otimes Id_I) \circ (Id_I \otimes \lambda_I) \circ \alpha_{I,I,I} \\
&\circ (\lambda_I^{-1} \otimes Id_I)
\end{aligned}$$

$$\begin{aligned}
&= \lambda_I \\
&\circ (\lambda_I \otimes Id_I) \\
&\circ (\rho_I^{-1} \otimes Id_I) \circ \rho_I^{-1} \circ \rho_I \circ (\rho_I \otimes Id_I) \\
&\circ \alpha_{I,I,I}^{-1} \circ (Id_I \otimes \lambda_I^{-1}) \\
&\circ (\lambda_I \otimes Id_I) \circ \alpha_{I,I,I}^{-1} \circ \lambda_{I \otimes I}^{-1} \\
&\circ \lambda_I^{-1} \circ \lambda_I \\
&\circ \lambda_{I \otimes I} \circ \alpha_{I,I,I} \circ (\lambda_I^{-1} \otimes Id_I) \circ (Id_I \otimes \lambda_I) \circ \alpha_{I,I,I} \\
&\circ (\lambda_I^{-1} \otimes Id_I) \\
&= \lambda_I \\
&\circ (\lambda_I \otimes Id_I) \\
&\circ (\rho_I^{-1} \otimes Id_I) \circ \rho_I^{-1} \circ \rho_I \circ (\rho_I \otimes Id_I) \\
&\circ \alpha_{I,I,I}^{-1} \circ (Id_I \otimes \lambda_I^{-1}) \circ (\lambda_I \otimes Id_I) \\
&\circ \alpha_{I,I,I}^{-1} \circ \lambda_{I \otimes I}^{-1} \circ \lambda_I^{-1} \circ \lambda_I \circ \lambda_{I \otimes I} \circ \alpha_{I,I,I} \circ (\lambda_I^{-1} \otimes Id_I) \circ (Id_I \otimes \lambda_I) \circ \alpha_{I,I,I} \\
&\circ ((\lambda_I \otimes Id_I) \otimes Id_I) \circ ((\lambda_I^{-1} \otimes Id_I) \otimes Id_I) \\
&\circ (\lambda_I^{-1} \otimes Id_I)
\end{aligned}$$

α が自然変換であることから以下の図式が可換。

$$\begin{array}{ccc}
 ((I \otimes I) \otimes I) \otimes I & \xrightarrow{\alpha_{I \otimes I, I, I}} & (I \otimes I) \otimes (I \otimes I) \\
 \downarrow (\lambda_I \otimes Id_I) \otimes Id_I & \circlearrowleft & \downarrow \lambda_I \otimes (Id_I \otimes Id_I) \\
 (I \otimes I) \otimes I & \xrightarrow{\alpha_{I, I, I}} & I \otimes (I \otimes I)
 \end{array}$$

従って以下の等式が成り立つ

$$\lambda_I \otimes (Id_I \otimes Id_I) \circ \alpha_{I, I, I \otimes I} = \alpha_{I, I, I} \circ (\lambda_I \otimes Id_I) \otimes Id_I$$

$$\begin{aligned}
 &= \lambda_I \\
 &\circ (\lambda_I \otimes Id_I) \\
 &\circ (\rho_I^{-1} \otimes Id_I) \circ \rho_I^{-1} \circ \rho_I \circ (\rho_I \otimes Id_I) \\
 &\circ \alpha_{I, I, I}^{-1} \circ (Id_I \otimes \lambda_I^{-1}) \circ (\lambda_I \otimes Id_I) \circ \alpha_{I, I, I}^{-1} \circ \lambda_{I \otimes I}^{-1} \circ \lambda_I^{-1} \\
 &\circ \lambda_I \circ \lambda_{I \otimes I} \circ \alpha_{I, I, I} \circ (\lambda_I^{-1} \otimes Id_I) \circ (Id_I \otimes \lambda_I) \\
 &\circ \alpha_{I, I, I} \circ (\lambda_I \otimes Id_I) \otimes Id_I \\
 &\circ (\lambda_I \otimes (Id_I \otimes Id_I)) \circ \alpha_{I, I, I \otimes I} \\
 &\circ ((\lambda_I^{-1} \otimes Id_I) \otimes Id_I) \\
 &\circ (\lambda_I^{-1} \otimes Id_I)
 \end{aligned}$$

以下の等式が成り立つ

※ λ_I を当てる順番を入れ替えただけ

$$(\lambda_I \otimes Id_I) \circ ((Id_I \otimes Id_I) \otimes \lambda_I) = \lambda_I \otimes \lambda_I = (Id_I \otimes \lambda_I) \circ (\lambda_I \otimes (Id_I \otimes Id_I))$$

$$\begin{aligned}
 &= \lambda_I \\
 &\circ (\lambda_I \otimes Id_I) \\
 &\circ (\rho_I^{-1} \otimes Id_I) \circ \rho_I^{-1} \circ \rho_I \circ (\rho_I \otimes Id_I) \\
 &\circ \alpha_{I, I, I}^{-1} \circ (Id_I \otimes \lambda_I^{-1}) \circ (\lambda_I \otimes Id_I) \circ \alpha_{I, I, I}^{-1} \circ \lambda_{I \otimes I}^{-1} \\
 &\circ \lambda_I^{-1} \circ \lambda_I \circ \lambda_{I \otimes I} \circ \alpha_{I, I, I} \circ (\lambda_I^{-1} \otimes Id_I) \\
 &\circ (Id_I \otimes \lambda_I) \circ (\lambda_I \otimes (Id_I \otimes Id_I)) \\
 &\circ (\lambda_I \otimes Id_I) \circ ((Id_I \otimes Id_I) \otimes \lambda_I) \\
 &\circ \alpha_{I, I, I \otimes I} \\
 &\circ ((\lambda_I^{-1} \otimes Id_I) \otimes Id_I) \\
 &\circ (\lambda_I^{-1} \otimes Id_I)
 \end{aligned}$$

以下は恒等射なので削除

$$(\lambda_I^{-1} \otimes Id_I) \circ (\lambda_I \otimes Id_I)$$

$$\begin{aligned}
&= \lambda_I \\
&\circ (\lambda_I \otimes Id_I) \\
&\circ (\rho_I^{-1} \otimes Id_I) \circ \rho_I^{-1} \circ \rho_I \circ (\rho_I \otimes Id_I) \\
&\circ \alpha_{I,I,I}^{-1} \circ (Id_I \otimes \lambda_I^{-1}) \circ (\lambda_I \otimes Id_I) \circ \alpha_{I,I,I}^{-1} \circ \lambda_{I \otimes I}^{-1} \\
&\circ \lambda_I^{-1} \circ \lambda_I \circ \lambda_{I \otimes I} \circ \alpha_{I,I,I} \\
&\circ (\lambda_I^{-1} \otimes Id_I) \circ (\lambda_I \otimes Id_I) \\
&\circ ((Id_I \otimes Id_I) \otimes \lambda_I) \\
&\circ \alpha_{I,I,I \otimes I} \\
&\circ ((\lambda_I^{-1} \otimes Id_I) \otimes Id_I) \\
&\circ (\lambda_I^{-1} \otimes Id_I)
\end{aligned}$$

α が自然変換であることから以下の図式が可換。

$$\begin{array}{ccc}
(I \otimes I) \otimes (I \otimes I) & \xrightarrow{\alpha_{I,I,I \otimes I}} & I \otimes (I \otimes (I \otimes I)) \\
\downarrow (Id_I \otimes Id_I) \otimes \lambda_I & \circlearrowleft & \downarrow Id_I \otimes (Id_I \otimes \lambda_I) \\
(I \otimes I) \otimes I & \xrightarrow{\alpha_{I,I,I}} & I \otimes (I \otimes I)
\end{array}$$

従って以下の等式が成り立つ

$$(Id_I \otimes (Id_I \otimes \lambda_I)) \circ \alpha_{I,I,I \otimes I} = \alpha_{I,I,I} \circ ((Id_I \otimes Id_I) \otimes \lambda_I)$$

$$\begin{aligned}
&= \lambda_I \\
&\circ (\lambda_I \otimes Id_I) \\
&\circ (\rho_I^{-1} \otimes Id_I) \circ \rho_I^{-1} \circ \rho_I \circ (\rho_I \otimes Id_I) \\
&\circ \alpha_{I,I,I}^{-1} \circ (Id_I \otimes \lambda_I^{-1}) \circ (\lambda_I \otimes Id_I) \circ \alpha_{I,I,I}^{-1} \circ \lambda_{I \otimes I}^{-1} \circ \lambda_I^{-1} \circ \lambda_I \circ \lambda_{I \otimes I} \\
&\circ \alpha_{I,I,I} \circ ((Id_I \otimes Id_I) \otimes \lambda_I) \\
&\circ (Id_I \otimes (Id_I \otimes \lambda_I)) \circ \alpha_{I,I,I \otimes I} \\
&\circ \alpha_{I,I,I \otimes I} \\
&\circ ((\lambda_I^{-1} \otimes Id_I) \otimes Id_I) \\
&\circ (\lambda_I^{-1} \otimes Id_I)
\end{aligned}$$

五角等式より

$$\begin{array}{ccccc}
 & & (I \otimes (I \otimes I)) \otimes I & \xrightarrow{\alpha_{I, I \otimes I, I}} & I \otimes ((I \otimes I) \otimes I) \\
 & \nearrow \alpha_{I, I, I \otimes I} & & & \searrow Id_I \otimes \alpha_{I, I, I} \\
 ((I \otimes I) \otimes I) \otimes I & & \circlearrowleft & & I \otimes (I \otimes (I \otimes I)) \\
 & \searrow \alpha_{I \otimes I, I, I} & & \nearrow \alpha_{I, I, I \otimes I} & \\
 & & (I \otimes I) \otimes (I \otimes I) & &
 \end{array}$$

従って以下の等式が成り立つ

$$\begin{aligned}
 & (Id_I \otimes \alpha_{I, I, I}) \circ \alpha_{I, I \otimes I, I} \circ (\alpha_{I, I, I} \otimes Id_I) = \alpha_{I, I, I \otimes I} \circ \alpha_{I, I, I \otimes I} \\
 & = \lambda_I \\
 & \circ (\lambda_I \otimes Id_I) \\
 & \circ (\rho_I^{-1} \otimes Id_I) \circ \rho_I^{-1} \circ \rho_I \circ (\rho_I \otimes Id_I) \\
 & \circ \alpha_{I, I, I}^{-1} \circ (Id_I \otimes \lambda_I^{-1}) \circ (\lambda_I \otimes Id_I) \circ \alpha_{I, I, I}^{-1} \circ \lambda_{I \otimes I}^{-1} \circ \lambda_I^{-1} \circ \lambda_I \circ \lambda_{I \otimes I} \\
 & \circ (Id_I \otimes (Id_I \otimes \lambda_I)) \\
 & \circ \alpha_{I, I, I \otimes I} \circ \alpha_{I, I, I \otimes I} \\
 & \circ (Id_I \otimes \alpha_{I, I, I}) \circ \alpha_{I, I \otimes I, I} \circ (\alpha_{I, I, I} \otimes Id_I) \\
 & \circ ((\lambda_I^{-1} \otimes Id_I) \otimes Id_I) \\
 & \circ (\lambda_I^{-1} \otimes Id_I)
 \end{aligned}$$

λ が自然変換であることから以下の図式が可換。

$$\begin{array}{ccc}
 I \otimes (I \otimes (I \otimes I)) & \xrightarrow{\lambda_{I \otimes (I \otimes I)}} & I \otimes (I \otimes I) \\
 \downarrow Id_I \otimes (Id_I \otimes \lambda_I) & \circlearrowleft & \downarrow Id_I \otimes \lambda_I \\
 I \otimes (I \otimes I) & \xrightarrow{\lambda_{I \otimes I}} & I \otimes I
 \end{array}$$

従って以下の等式が成り立つ

$$(Id_I \otimes \lambda_I) \circ \lambda_{I \otimes (I \otimes I)} = \lambda_{I \otimes I} \circ (Id_I \otimes (Id_I \otimes \lambda_I))$$

$$\begin{aligned}
 &= \lambda_I \\
 &\circ (\lambda_I \otimes Id_I) \\
 &\circ (\rho_I^{-1} \otimes Id_I) \circ \rho_I^{-1} \circ \rho_I \circ (\rho_I \otimes Id_I) \\
 &\circ \alpha_{I,I,I}^{-1} \circ (Id_I \otimes \lambda_I^{-1}) \circ (\lambda_I \otimes Id_I) \circ \alpha_{I,I,I}^{-1} \circ \lambda_{I \otimes I}^{-1} \circ \lambda_I^{-1} \circ \lambda_I \\
 &\circ \lambda_{I \otimes I} \circ (Id_I \otimes (Id_I \otimes \lambda_I)) \\
 &\circ (Id_I \otimes \lambda_I) \circ \lambda_{I \otimes (I \otimes I)} \\
 &\circ (Id_I \otimes \alpha_{I,I,I}) \circ \alpha_{I,I \otimes I,I} \circ (\alpha_{I,I,I} \otimes Id_I) \\
 &\circ ((\lambda_I^{-1} \otimes Id_I) \otimes Id_I) \\
 &\circ (\lambda_I^{-1} \otimes Id_I)
 \end{aligned}$$

λ が自然変換であることから以下の図式が可換。

$$\begin{array}{ccc}
 I \otimes (I \otimes I) & \xrightarrow{\lambda_{I \otimes I}} & I \otimes I \\
 \downarrow Id_I \otimes \lambda_I & \circlearrowright & \downarrow \lambda_I \\
 I \otimes I & \xrightarrow{\lambda_I} & I
 \end{array}$$

従って以下の等式が成り立つ

$$\lambda_I \circ \lambda_{I \otimes I} = \lambda_I \circ (Id_I \otimes \lambda_I)$$

$$\begin{aligned}
 &= \lambda_I \\
 &\circ (\lambda_I \otimes Id_I) \\
 &\circ (\rho_I^{-1} \otimes Id_I) \circ \rho_I^{-1} \circ \rho_I \circ (\rho_I \otimes Id_I) \\
 &\circ \alpha_{I,I,I}^{-1} \circ (Id_I \otimes \lambda_I^{-1}) \circ (\lambda_I \otimes Id_I) \circ \alpha_{I,I,I}^{-1} \circ \lambda_{I \otimes I}^{-1} \circ \lambda_I^{-1} \\
 &\circ \lambda_I \circ (Id_I \otimes \lambda_I) \\
 &\circ \lambda_I \circ \lambda_{I \otimes I} \\
 &\circ \lambda_{I \otimes (I \otimes I)} \\
 &\circ (Id_I \otimes \alpha_{I,I,I}) \circ \alpha_{I,I \otimes I,I} \circ (\alpha_{I,I,I} \otimes Id_I) \\
 &\circ ((\lambda_I^{-1} \otimes Id_I) \otimes Id_I) \\
 &\circ (\lambda_I^{-1} \otimes Id_I)
 \end{aligned}$$

以下は恒等射なので削除

$$\lambda_{I \otimes I}^{-1} \circ \lambda_I^{-1} \circ \lambda_I \circ \lambda_{I \otimes I}$$

$$\begin{aligned}
 &= \lambda_I \\
 &\circ (\lambda_I \otimes Id_I) \\
 &\circ (\rho_I^{-1} \otimes Id_I) \circ \rho_I^{-1} \circ \rho_I \circ (\rho_I \otimes Id_I) \\
 &\circ \alpha_{I,I,I}^{-1} \circ (Id_I \otimes \lambda_I^{-1}) \circ (\lambda_I \otimes Id_I) \circ \alpha_{I,I,I}^{-1} \\
 &\circ \lambda_{I \otimes I}^{-1} \circ \lambda_I^{-1} \circ \lambda_I \circ \lambda_{I \otimes I} \\
 &\circ \lambda_{I \otimes (I \otimes I)} \\
 &\circ (Id_I \otimes \alpha_{I,I,I}) \circ \alpha_{I,I \otimes I,I} \circ (\alpha_{I,I,I} \otimes Id_I) \\
 &\circ ((\lambda_I^{-1} \otimes Id_I) \otimes Id_I) \\
 &\circ (\lambda_I^{-1} \otimes Id_I)
 \end{aligned}$$

λ が自然変換であることから以下の図式が可換。

$$\begin{array}{ccc}
 I \otimes ((I \otimes (I \otimes I)) & \xrightarrow{\lambda_{(I \otimes I) \otimes I}} & (I \otimes I) \otimes I \\
 \downarrow Id_I \otimes \alpha_{I,I,I} & \circlearrowright & \downarrow \alpha_{I,I,I} \\
 I \otimes (I \otimes (I \otimes I)) & \xrightarrow{\lambda_{I \otimes (I \otimes I)}} & I \otimes (I \otimes I)
 \end{array}$$

従って以下の等式が成り立つ

$$\begin{aligned}
 \alpha_{I,I,I} \circ \lambda_{(I \otimes I) \otimes I} &= \lambda_{I \otimes (I \otimes I)} \circ (Id_I \otimes \alpha_{I,I,I}) \\
 &= \lambda_I \\
 &\circ (\lambda_I \otimes Id_I) \\
 &\circ (\rho_I^{-1} \otimes Id_I) \circ \rho_I^{-1} \circ \rho_I \circ (\rho_I \otimes Id_I) \\
 &\circ \alpha_{I,I,I}^{-1} \circ (Id_I \otimes \lambda_I^{-1}) \circ (\lambda_I \otimes Id_I) \circ \alpha_{I,I,I}^{-1} \\
 &\circ \lambda_{I \otimes (I \otimes I)} \circ (Id_I \otimes \alpha_{I,I,I}) \\
 &\circ \alpha_{I,I,I} \circ \lambda_{(I \otimes I) \otimes I} \\
 &\circ \alpha_{I,I \otimes I,I} \circ (\alpha_{I,I,I} \otimes Id_I) \\
 &\circ ((\lambda_I^{-1} \otimes Id_I) \otimes Id_I) \\
 &\circ (\lambda_I^{-1} \otimes Id_I)
 \end{aligned}$$

以下の恒等射を挿入

$$\begin{aligned}
 &(\lambda_I^{-1} \otimes I) \circ \lambda_I^{-1} \circ \lambda_I \circ (\lambda_I \otimes I) \\
 &= \lambda_I \\
 &\circ (\lambda_I \otimes Id_I) \\
 &\circ (\rho_I^{-1} \otimes Id_I) \circ \rho_I^{-1} \circ \rho_I \circ (\rho_I \otimes Id_I) \\
 &\circ \alpha_{I,I,I}^{-1} \circ (Id_I \otimes \lambda_I^{-1}) \circ (\lambda_I \otimes Id_I) \circ \alpha_{I,I,I}^{-1} \\
 &\circ \alpha_{I,I,I} \\
 &\circ (\lambda_I^{-1} \otimes I) \circ \lambda_I^{-1} \circ \lambda_I \circ (\lambda_I \otimes I) \\
 &\circ \lambda_{(I \otimes I) \otimes I} \\
 &\circ \alpha_{I,I \otimes I,I} \circ (\alpha_{I,I,I} \otimes Id_I) \\
 &\circ ((\lambda_I^{-1} \otimes Id_I) \otimes Id_I) \\
 &\circ (\lambda_I^{-1} \otimes Id_I)
 \end{aligned}$$

λ が自然変換であることから以下の図式が可換。

$$\begin{array}{ccc}
 I \otimes ((I \otimes I) \otimes I) & \xrightarrow{\lambda_{(I \otimes I) \otimes I}} & (I \otimes I) \otimes I \\
 \downarrow Id_I \otimes (\lambda_I \otimes I) & \circlearrowleft & \downarrow \lambda_I \otimes I \\
 I \otimes (I \otimes I) & \xrightarrow{\lambda_{I \otimes I}} & I \otimes I
 \end{array}$$

従って以下の等式が成り立つ

$$\begin{aligned}
 \lambda_{I \otimes I} \circ (Id_I \otimes (\lambda_I \otimes I)) &= (\lambda_I \otimes I) \circ \lambda_{(I \otimes I) \otimes I} \\
 &= \lambda_I \\
 &\circ (\lambda_I \otimes Id_I) \\
 &\circ (\rho_I^{-1} \otimes Id_I) \circ \rho_I^{-1} \circ \rho_I \circ (\rho_I \otimes Id_I) \\
 &\circ \alpha_{I,I,I}^{-1} \circ (Id_I \otimes \lambda_I^{-1}) \circ (\lambda_I \otimes Id_I) \circ \alpha_{I,I,I}^{-1} \\
 &\circ \alpha_{I,I,I} \\
 &\circ (\lambda_I^{-1} \otimes I) \circ \lambda_I^{-1} \circ \lambda_I \\
 &\circ (\lambda_I \otimes I) \circ \lambda_{(I \otimes I) \otimes I} \\
 &\circ \lambda_{I \otimes I} \circ (Id_I \otimes (\lambda_I \otimes I)) \\
 &\circ \alpha_{I,I \otimes I,I} \circ (\alpha_{I,I,I} \otimes Id_I) \\
 &\circ ((\lambda_I^{-1} \otimes Id_I) \otimes Id_I) \\
 &\circ (\lambda_I^{-1} \otimes Id_I)
 \end{aligned}$$

λ が自然変換であることから以下の図式が可換。

$$\begin{array}{ccc}
 I \otimes (I \otimes I) & \xrightarrow{\lambda_{I \otimes I}} & I \otimes I \\
 \downarrow Id_I \otimes \lambda_I & \circlearrowleft & \downarrow \lambda_I \\
 I \otimes I & \xrightarrow{\lambda_I} & I
 \end{array}$$

従って以下の等式が成り立つ

$$\lambda_I \circ (Id_I \otimes \lambda_I) = \lambda_I \circ \lambda_{I \otimes I}$$

$$\begin{aligned}
 &= \lambda_I \\
 &\circ (\lambda_I \otimes Id_I) \\
 &\circ (\rho_I^{-1} \otimes Id_I) \circ \rho_I^{-1} \circ \rho_I \circ (\rho_I \otimes Id_I) \\
 &\circ \alpha_{I,I,I}^{-1} \circ (Id_I \otimes \lambda_I^{-1}) \circ (\lambda_I \otimes Id_I) \circ \alpha_{I,I,I}^{-1} \\
 &\circ \alpha_{I,I,I} \\
 &\circ (\lambda_I^{-1} \otimes I) \circ \lambda_I^{-1} \\
 &\circ \lambda_I \circ \lambda_{I \otimes I} \\
 &\circ \lambda_I \circ (Id_I \otimes \lambda_I) \\
 &\circ (Id_I \otimes (\lambda_I \otimes I)) \\
 &\circ \alpha_{I,I \otimes I,I} \circ (\alpha_{I,I,I} \otimes Id_I) \\
 &\circ ((\lambda_I^{-1} \otimes Id_I) \otimes Id_I) \\
 &\circ (\lambda_I^{-1} \otimes Id_I)
 \end{aligned}$$

以下は恒等射なので削除

$$\lambda_I^{-1} \circ \lambda_I$$

$$\begin{aligned}
&= \lambda_I \\
&\circ (\lambda_I \otimes Id_I) \\
&\circ (\rho_I^{-1} \otimes Id_I) \circ \rho_I^{-1} \circ \rho_I \circ (\rho_I \otimes Id_I) \\
&\circ \alpha_{I,I,I}^{-1} \circ (Id_I \otimes \lambda_I^{-1}) \circ (\lambda_I \otimes Id_I) \circ \alpha_{I,I,I}^{-1} \\
&\circ \alpha_{I,I,I} \\
&\circ (\lambda_I^{-1} \otimes I) \\
&\circ \lambda_I^{-1} \circ \lambda_I \\
&\circ (Id_I \otimes \lambda_I) \\
&\circ (Id_I \otimes (\lambda_I \otimes I)) \\
&\circ \alpha_{I,I \otimes I,I} \circ (\alpha_{I,I,I} \otimes Id_I) \\
&\circ ((\lambda_I^{-1} \otimes Id_I) \otimes Id_I) \\
&\circ (\lambda_I^{-1} \otimes Id_I)
\end{aligned}$$

以下は恒等射なので削除

$$(Id_I \otimes \lambda_I^{-1}) \circ (\lambda_I \otimes Id_I) \circ \alpha_{I,I,I}^{-1} \circ \alpha_{I,I,I} \circ (\lambda_I^{-1} \otimes Id_I) \circ (Id_I \otimes \lambda_I)$$

$$\begin{aligned}
&= \lambda_I \\
&\circ (\lambda_I \otimes Id_I) \\
&\circ (\rho_I^{-1} \otimes Id_I) \circ \rho_I^{-1} \circ \rho_I \circ (\rho_I \otimes Id_I) \\
&\circ \alpha_{I,I,I}^{-1} \\
&\circ \cancel{(Id_I \otimes \lambda_I^{-1}) \circ (\lambda_I \otimes Id_I) \circ \alpha_{I,I,I}^{-1} \circ \alpha_{I,I,I} \circ (\lambda_I^{-1} \otimes Id_I) \circ (Id_I \otimes \lambda_I)} \\
&\circ (Id_I \otimes (\lambda_I \otimes Id_I)) \\
&\circ \alpha_{I,I \otimes I,I} \circ (\alpha_{I,I,I} \otimes Id_I) \\
&\circ ((\lambda_I^{-1} \otimes Id_I) \otimes Id_I) \\
&\circ (\lambda_I^{-1} \otimes Id_I)
\end{aligned}$$

α が自然変換であることから以下の図式が可換。

$$\begin{array}{ccc}
 (I \otimes (I \otimes I)) \otimes I & \xrightarrow{\alpha_{I, I \otimes I, I}} & I \otimes ((I \otimes I) \otimes I) \\
 \downarrow (Id_I \otimes \lambda_I) \otimes Id_I & \circlearrowleft & \downarrow Id_I \otimes (\lambda_I \otimes Id_I) \\
 (I \otimes I) \otimes I & \xrightarrow{\alpha_{I, I, I}} & I \otimes (I \otimes I)
 \end{array}$$

従って以下の等式が成り立つ

$$\begin{aligned}
 \alpha_{I, I, I} \circ ((Id_I \otimes \lambda_I) \otimes Id_I) &= (Id_I \otimes (\lambda_I \otimes Id_I)) \circ \alpha_{I, I \otimes I, I} \\
 &= \lambda_I \\
 &\circ (\lambda_I \otimes Id_I) \\
 &\circ (\rho_I^{-1} \otimes Id_I) \circ \rho_I^{-1} \circ \rho_I \circ (\rho_I \otimes Id_I) \\
 &\circ \alpha_{I, I, I}^{-1} \\
 &\circ \cancel{(Id_I \otimes (\lambda_I \otimes Id_I)) \circ \alpha_{I, I \otimes I, I}} \\
 &\circ \alpha_{I, I, I} \circ ((Id_I \otimes \lambda_I) \otimes Id_I) \\
 &\circ (\alpha_{I, I, I} \otimes Id_I) \\
 &\circ ((\lambda_I^{-1} \otimes Id_I) \otimes Id_I) \\
 &\circ (\lambda_I^{-1} \otimes Id_I)
 \end{aligned}$$

以下は恒等射なので削除

$$\alpha_{I, I, I}^{-1} \circ \alpha_{I, I, I}$$

$$\begin{aligned}
 &= \lambda_I \\
 &\circ (\lambda_I \otimes Id_I) \\
 &\circ (\rho_I^{-1} \otimes Id_I) \circ \rho_I^{-1} \circ \rho_I \circ (\rho_I \otimes Id_I) \\
 &\circ \cancel{\alpha_{I, I, I}^{-1} \circ \alpha_{I, I, I}} \\
 &\circ ((Id_I \otimes \lambda_I) \otimes Id_I) \\
 &\circ (\alpha_{I, I, I} \otimes Id_I) \\
 &\circ ((\lambda_I^{-1} \otimes Id_I) \otimes Id_I) \\
 &\circ (\lambda_I^{-1} \otimes Id_I)
 \end{aligned}$$

三角等式より

$$\begin{array}{ccc}
 ((I \otimes I) \otimes I) \otimes I & \xrightarrow{\alpha_{I,I,I} \otimes Id_I} & (I \otimes (I \otimes I)) \otimes I \\
 \searrow (\rho_I \otimes Id_I) \otimes Id_I & \circlearrowleft & \swarrow (Id_I \otimes \lambda_I) \otimes Id_I \\
 & (I \otimes I) \otimes I &
 \end{array}$$

従って以下の等式が成り立つ

$$\begin{aligned}
 (\rho_I \otimes Id_I) \otimes Id_I &= ((Id_I \otimes \lambda_I) \otimes Id_I) \circ (\alpha_{I,I,I} \otimes Id_I) \\
 &= \lambda_I \\
 &\circ (\lambda_I \otimes Id_I) \\
 &\circ (\rho_I^{-1} \otimes Id_I) \circ \rho_I^{-1} \circ \rho_I \circ (\rho_I \otimes Id_I) \\
 &\circ ((Id_I \otimes \lambda_I) \otimes Id_I) \circ (\alpha_{I,I,I} \otimes Id_I) \\
 &\circ ((\rho_I \otimes Id_I) \otimes Id_I) \\
 &\circ ((\lambda_I^{-1} \otimes Id_I) \otimes Id_I) \\
 &\circ (\lambda_I^{-1} \otimes Id_I)
 \end{aligned}$$

ρ が自然変換であることから以下の図式が可換。

$$\begin{array}{ccc}
 (I \otimes I) \otimes I & \xrightarrow{\rho_{I \otimes I}} & I \otimes I \\
 \downarrow \rho_I \otimes Id_I & \circlearrowright & \downarrow \rho_I \\
 I \otimes I & \xrightarrow{\rho_I} & I
 \end{array}$$

従って以下の等式が成り立つ

$$\rho_I \circ \rho_{I \otimes I} = \rho_I \circ (\rho_I \otimes Id_I)$$

$$= \lambda_I$$

$$\circ (\lambda_I \otimes Id_I)$$

$$\circ (\rho_I^{-1} \otimes Id_I) \circ \rho_I^{-1}$$

$$\circ \rho_I \circ (\rho_I \otimes Id_I)$$

$$\circ \rho_I \circ \rho_{I \otimes I}$$

$$\circ ((\rho_I \otimes Id_I) \otimes Id_I)$$

$$\circ ((\lambda_I^{-1} \otimes Id_I) \otimes Id_I)$$

$$\circ (\lambda_I^{-1} \otimes Id_I)$$

ρ が自然変換であることから以下の図式が可換。

$$\begin{array}{ccc}
 ((I \otimes I) \otimes I) \otimes I & \xrightarrow{\rho_{(I \otimes I) \otimes I}} & (I \otimes I) \otimes I \\
 \downarrow (\rho_I \otimes Id_I) \otimes Id_I & \circlearrowleft & \downarrow \rho_I \otimes Id_I \\
 (I \otimes I) \otimes I & \xrightarrow{\rho_{I \otimes I}} & I \otimes I
 \end{array}$$

従って以下の等式が成り立つ

$$\begin{aligned}
 (\rho_I \otimes Id_I) \circ \rho_{(I \otimes I) \otimes I} &= \rho_{I \otimes I} \circ ((\rho_I \otimes Id_I) \otimes Id_I) \\
 &= \lambda_I \\
 &\circ (\lambda_I \otimes Id_I) \\
 &\circ (\rho_I^{-1} \otimes Id_I) \\
 &\circ \rho_I^{-1} \\
 &\circ \rho_I \\
 &\circ \cancel{\rho_{I \otimes I} \circ ((\rho_I \otimes Id_I) \otimes Id_I)} \\
 &\circ (\rho_I \otimes Id_I) \circ \rho_{(I \otimes I) \otimes I} \\
 &\circ ((\lambda_I^{-1} \otimes Id_I) \otimes Id_I) \\
 &\circ (\lambda_I^{-1} \otimes Id_I)
 \end{aligned}$$

以下は恒等射なので削除

$$\begin{aligned}
 &(\rho_I^{-1} \otimes Id_I) \circ \rho_I^{-1} \circ \rho_I \circ (\rho_I \otimes Id_I) \\
 &= \lambda_I \\
 &\circ (\lambda_I \otimes Id_I) \\
 &\circ \cancel{(\rho_I^{-1} \otimes Id_I) \circ \rho_I^{-1} \circ \rho_I \circ (\rho_I \otimes Id_I)} \\
 &\circ \rho_{(I \otimes I) \otimes I} \\
 &\circ ((\lambda_I^{-1} \otimes Id_I) \otimes Id_I) \\
 &\circ (\lambda_I^{-1} \otimes Id_I)
 \end{aligned}$$

ρ が自然変換であることから以下の図式が可換。

$$\begin{array}{ccc}
 ((I \otimes I) \otimes I) \otimes I & \xrightarrow{\rho_{(I \otimes I) \otimes I}} & (I \otimes I) \otimes I \\
 \downarrow (\lambda_I \otimes Id_I) \otimes Id_I & \circlearrowleft & \downarrow \lambda_I \otimes Id_I \\
 (I \otimes I) \otimes I & \xrightarrow{\rho_{I \otimes I}} & I \otimes I
 \end{array}$$

従って以下の等式が成り立つ

$$\begin{aligned}
 \rho_{I \otimes I} \circ ((\lambda_I \otimes Id_I) \otimes Id_I) &= (\lambda_I \otimes Id_I) \circ \rho_{(I \otimes I) \otimes I} \\
 &= \lambda_I \\
 &\circ (\lambda_I \otimes Id_I) \circ \rho_{(I \otimes I) \otimes I} \\
 &\circ \rho_{I \otimes I} \circ ((\lambda_I \otimes Id_I) \otimes Id_I) \\
 &\circ ((\lambda_I^{-1} \otimes Id_I) \otimes Id_I) \\
 &\circ (\lambda_I^{-1} \otimes Id_I)
 \end{aligned}$$

以下は恒等射なので削除

$$\begin{aligned}
 &((\lambda_I \otimes Id_I) \otimes Id_I) \circ ((\lambda_I^{-1} \otimes Id_I) \otimes Id_I) \\
 &= \lambda_I \\
 &\circ \rho_{I \otimes I} \\
 &\circ ((\lambda_I \otimes Id_I) \otimes Id_I) \circ ((\lambda_I^{-1} \otimes Id_I) \otimes Id_I) \\
 &\circ (\lambda_I^{-1} \otimes Id_I)
 \end{aligned}$$

ρ が自然変換であることから以下の図式が可換。

$$\begin{array}{ccc}
 (I \otimes I) \otimes I & \xrightarrow{\rho_{I \otimes I}} & I \otimes I \\
 \lambda_I \otimes Id_I \downarrow & \circlearrowleft & \downarrow \lambda_I \\
 I \otimes I & \xrightarrow{\rho_I} & I
 \end{array}$$

従って以下の等式が成り立つ

$$\begin{aligned}
 \rho_I \circ (\lambda_I \otimes Id_I) &= \lambda_I \circ \rho_{I \otimes I} \\
 &= \lambda_I \circ \rho_{I \otimes I} \\
 \rho_I \circ (\lambda_I \otimes Id_I) & \\
 \circ (\lambda_I^{-1} \otimes Id_I) &
 \end{aligned}$$

以下は恒等射なので削除

$$\begin{aligned}
 &(\lambda_I \otimes Id_I) \circ (\lambda_I^{-1} \otimes Id_I) \\
 &= \rho_I \circ (\lambda_I \otimes Id_I) \circ (\lambda_I^{-1} \otimes Id_I) \\
 &= \rho_I
 \end{aligned}$$

□