

モノイダル圏における左右単位律子の一致の証明

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1 モノイダル圏

以下の 5 つの物 (テンソル積, 単位対象, 結合律子, 左単位律子, 右単位律子) が存在し、下記の二つの等式 (三角等式, 五角等式) を満たす圏をモノイダル圏と言う。

- テンソル積 \otimes
関手 $\otimes : C \times C \rightarrow C$
- 単位対象 I
 $I \in \text{Ob}(C)$
- 結合律子 α
自然同型 α
$$(A \otimes B) \otimes C \xrightarrow{\alpha_{A,B,C}} A \otimes (B \otimes C)$$
- 左単位律子 λ
自然同型 λ
$$I \otimes A \xrightarrow{\lambda_A} A$$
- 右単位律子 ρ
自然同型 ρ
$$A \otimes I \xrightarrow{\rho_A} A$$
- 三角等式
任意の A, B に対し以下の図式が可換

$$\begin{array}{ccc}
(A \otimes I) \otimes B & \xrightarrow{\alpha_{A,I,B}} & A \otimes (I \otimes B) \\
& \searrow \rho_A \otimes Id_B & \swarrow Id_A \otimes \lambda_B \\
& A \otimes B &
\end{array}$$

つまり以下の等式が成り立つ

$$\rho_A \otimes Id_B = (Id_A \otimes \lambda_B) \circ (\alpha_{A,I,B})$$

- 五角等式

任意の A,B,C,D に対し以下の図式が可換

$$\begin{array}{ccccc}
& (A \otimes (B \otimes C)) \otimes D & \xrightarrow{\alpha_{A,B \otimes C,D}} & A \otimes ((B \otimes C) \otimes D) & \\
\alpha_{A,B,C} \otimes Id_D \nearrow & & & & \searrow Id_A \otimes \alpha_{B,C,D} \\
((A \otimes B) \otimes C) \otimes D & & \circlearrowleft & & A \otimes (B \otimes (C \otimes D)) \\
& \searrow \alpha_{A \otimes B,C,D} & & \nearrow \alpha_{A,B,C \otimes D} & \\
& (A \otimes B) \otimes (C \otimes D) & & &
\end{array}$$

つまり以下の等式が成り立つ

$$(Id_A \otimes \alpha_{B,C,D}) \circ \alpha_{A,B \otimes C,D} \circ (\alpha_{A,B,C} \otimes Id_D) = \alpha_{A,B,C \otimes D} \circ \alpha_{A \otimes B,C,D}$$

2 証明

Proof.

以下、 λ_I (左単位律子) に恒等射を入れて行きます。

$$\begin{aligned}
\lambda_I &= \lambda_I \\
&\circ (\lambda_I \otimes Id_I) \circ (\lambda_I^{-1} \otimes Id_I) \\
&= \lambda_I \\
&\circ (\lambda_I \otimes Id_I) \\
&\circ (\rho_I^{-1} \otimes Id_I) \circ (\rho_I \otimes Id_I) \\
&\circ (\lambda_I^{-1} \otimes Id_I) \\
&= \lambda_I \\
&\circ (\lambda_I \otimes Id_I) \\
&\circ (\rho_I^{-1} \otimes Id_I) \\
&\circ \rho_I^{-1} \circ \rho_I \\
&\circ (\rho_I \otimes Id_I) \\
&\circ (\lambda_I^{-1} \otimes Id_I) \\
&= \lambda_I \\
&\circ (\lambda_I \otimes Id_I) \\
&\circ (\rho_I^{-1} \otimes Id_I) \circ \rho_I^{-1} \circ \rho_I \circ (\rho_I \otimes Id_I) \\
&\circ \alpha_{I,I,I}^{-1} \circ \alpha_{I,I,I} \\
&\circ (\lambda_I^{-1} \otimes Id_I) \\
&= \lambda_I \\
&\circ (\lambda_I \otimes Id_I) \\
&\circ (\rho_I^{-1} \otimes Id_I) \circ \rho_I^{-1} \circ \rho_I \circ (\rho_I \otimes Id_I) \\
&\circ \alpha_{I,I,I}^{-1} \\
&\circ (Id_I \otimes \lambda_I^{-1}) \circ (Id_I \otimes \lambda_I) \\
&\circ \alpha_{I,I,I} \\
&\circ (\lambda_I^{-1} \otimes Id_I)
\end{aligned}$$

$$\begin{aligned}
&= \lambda_I \\
&\circ (\lambda_I \otimes Id_I) \\
&\circ (\rho_I^{-1} \otimes Id_I) \circ \rho_I^{-1} \circ \rho_I \circ (\rho_I \otimes Id_I) \\
&\circ \alpha_{I,I,I}^{-1} \circ (Id_I \otimes \lambda_I^{-1}) \\
&\circ (\lambda_I \otimes Id_I) \circ (\lambda_I^{-1} \otimes Id_I) \circ (Id_I \otimes \lambda_I) \circ \alpha_{I,I,I} \\
&\circ (\lambda_I^{-1} \otimes Id_I) \\
&= \lambda_I \\
&\circ (\lambda_I \otimes Id_I) \\
&\circ (\rho_I^{-1} \otimes Id_I) \circ \rho_I^{-1} \circ \rho_I \circ (\rho_I \otimes Id_I) \\
&\circ \alpha_{I,I,I}^{-1} \circ (Id_I \otimes \lambda_I^{-1}) \circ (\lambda_I \otimes Id_I) \\
&\circ \alpha_{I,I,I}^{-1} \circ \alpha_{I,I,I} \\
&\circ (\lambda_I^{-1} \otimes Id_I) \circ (Id_I \otimes \lambda_I) \circ \alpha_{I,I,I} \\
&\circ (\lambda_I^{-1} \otimes Id_I) \\
&= \lambda_I \\
&\circ (\lambda_I \otimes Id_I) \\
&\circ (\rho_I^{-1} \otimes Id_I) \circ \rho_I^{-1} \circ \rho_I \circ (\rho_I \otimes Id_I) \\
&\circ \alpha_{I,I,I}^{-1} \circ (Id_I \otimes \lambda_I^{-1}) \circ (\lambda_I \otimes Id_I) \circ \alpha_{I,I,I}^{-1} \\
&\circ \lambda_{I \otimes I}^{-1} \circ \lambda_{I \otimes I} \\
&\circ \alpha_{I,I,I} \circ (\lambda_I^{-1} \otimes Id_I) \circ (Id_I \otimes \lambda_I) \circ \alpha_{I,I,I} \\
&\circ (\lambda_I^{-1} \otimes Id_I)
\end{aligned}$$

$$\begin{aligned}
&= \lambda_I \\
&\circ (\lambda_I \otimes Id_I) \\
&\circ (\rho_I^{-1} \otimes Id_I) \circ \rho_I^{-1} \circ \rho_I \circ (\rho_I \otimes Id_I) \\
&\circ \alpha_{I,I,I}^{-1} \circ (Id_I \otimes \lambda_I^{-1}) \\
&\circ (\lambda_I \otimes Id_I) \circ \alpha_{I,I,I}^{-1} \circ \lambda_{I \otimes I}^{-1} \\
&\circ \lambda_I^{-1} \circ \lambda_I \\
&\circ \lambda_{I \otimes I} \circ \alpha_{I,I,I} \circ (\lambda_I^{-1} \otimes Id_I) \circ (Id_I \otimes \lambda_I) \circ \alpha_{I,I,I} \\
&\circ (\lambda_I^{-1} \otimes Id_I) \\
&= \lambda_I \\
&\circ (\lambda_I \otimes Id_I) \\
&\circ (\rho_I^{-1} \otimes Id_I) \circ \rho_I^{-1} \circ \rho_I \circ (\rho_I \otimes Id_I) \\
&\circ \alpha_{I,I,I}^{-1} \circ (Id_I \otimes \lambda_I^{-1}) \circ (\lambda_I \otimes Id_I) \\
&\circ \alpha_{I,I,I}^{-1} \circ \lambda_{I \otimes I}^{-1} \circ \lambda_I^{-1} \circ \lambda_I \circ \lambda_{I \otimes I} \circ \alpha_{I,I,I} \circ (\lambda_I^{-1} \otimes Id_I) \circ (Id_I \otimes \lambda_I) \circ \alpha_{I,I,I} \\
&\circ ((\lambda_I \otimes Id_I) \otimes Id_I) \circ ((\lambda_I^{-1} \otimes Id_I) \otimes Id_I) \\
&\circ (\lambda_I^{-1} \otimes Id_I)
\end{aligned}$$

α が自然変換であることから以下の図式が可換。

$$\begin{array}{ccc}
 ((I \otimes I) \otimes I) \otimes I & \xrightarrow{\alpha_{I \otimes I, I, I}} & (I \otimes I) \otimes (I \otimes I) \\
 \downarrow (\lambda_I \otimes Id_I) \otimes Id_I & \circlearrowleft & \downarrow \lambda_I \otimes (Id_I \otimes Id_I) \\
 (I \otimes I) \otimes I & \xrightarrow{\alpha_{I, I, I}} & I \otimes (I \otimes I)
 \end{array}$$

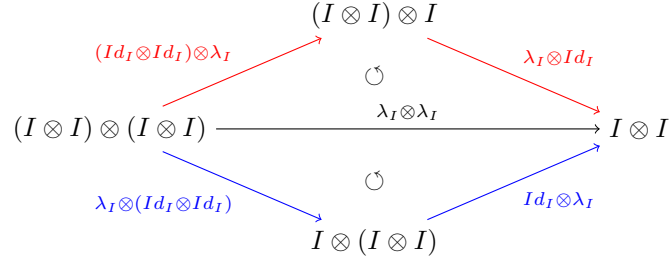
従って以下の等式が成り立つ

$$\begin{aligned}
 & \lambda_I \otimes (Id_I \otimes Id_I) \circ \alpha_{I, I, I \otimes I} = \alpha_{I, I, I} \circ (\lambda_I \otimes Id_I) \otimes Id_I \\
 & = \lambda_I \\
 & \circ (\lambda_I \otimes Id_I) \\
 & \circ (\rho_I^{-1} \otimes Id_I) \circ \rho_I^{-1} \circ \rho_I \circ (\rho_I \otimes Id_I) \\
 & \circ \alpha_{I, I, I}^{-1} \circ (Id_I \otimes \lambda_I^{-1}) \circ (\lambda_I \otimes Id_I) \circ \alpha_{I, I, I}^{-1} \circ \lambda_{I \otimes I}^{-1} \circ \lambda_I^{-1} \\
 & \circ \lambda_I \circ \lambda_{I \otimes I} \circ \alpha_{I, I, I} \circ (\lambda_I^{-1} \otimes Id_I) \circ (Id_I \otimes \lambda_I) \\
 & \circ \alpha_{I, I, I} \circ (\lambda_I \otimes Id_I) \otimes Id_I \\
 & \circ (\lambda_I \otimes (Id_I \otimes Id_I)) \circ \alpha_{I, I, I \otimes I} \\
 & \circ ((\lambda_I^{-1} \otimes Id_I) \otimes Id_I) \\
 & \circ (\lambda_I^{-1} \otimes Id_I)
 \end{aligned}$$

以下の等式が成り立つ

※交換則 $(a \otimes b) \circ (c \otimes d) = (a \circ c) \otimes (b \circ d)$ を使用

$$(\lambda_I \otimes Id_I) \circ ((Id_I \otimes Id_I) \otimes \lambda_I) = \lambda_I \otimes \lambda_I = (Id_I \otimes \lambda_I) \circ (\lambda_I \otimes (Id_I \otimes Id_I))$$



$$= \lambda_I$$

$$\circ (\lambda_I \otimes Id_I)$$

$$\circ (\rho_I^{-1} \otimes Id_I) \circ \rho_I^{-1} \circ \rho_I \circ (\rho_I \otimes Id_I)$$

$$\circ \alpha_{I,I,I}^{-1} \circ (Id_I \otimes \lambda_I^{-1}) \circ (\lambda_I \otimes Id_I) \circ \alpha_{I,I,I}^{-1} \circ \lambda_{I \otimes I}^{-1}$$

$$\circ \lambda_I^{-1} \circ \lambda_I \circ \lambda_{I \otimes I} \circ \alpha_{I,I,I} \circ (\lambda_I^{-1} \otimes Id_I)$$

$$\circ (\lambda_I \otimes Id_I) \circ ((Id_I \otimes Id_I) \otimes \lambda_I)$$

$$\circ (\lambda_I \otimes Id_I) \circ ((Id_I \otimes Id_I) \otimes \lambda_I)$$

$$\circ \alpha_{I,I,I \otimes I}$$

$$\circ ((\lambda_I^{-1} \otimes Id_I) \otimes Id_I)$$

$$\circ (\lambda_I^{-1} \otimes Id_I)$$

以下は恒等射なので削除

$$(\lambda_I^{-1} \otimes Id_I) \circ (\lambda_I \otimes Id_I)$$

$$\begin{array}{ccc} & \xrightarrow{\lambda_I \otimes Id_I} & \\ (I \otimes I) \otimes I & & I \otimes I \\ & \xleftarrow{\lambda_I^{-1} \otimes Id_I} & \end{array}$$

$$\begin{aligned} &= \lambda_I \\ &\circ (\lambda_I \otimes Id_I) \\ &\circ (\rho_I^{-1} \otimes Id_I) \circ \rho_I^{-1} \circ \rho_I \circ (\rho_I \otimes Id_I) \\ &\circ \alpha_{I,I,I}^{-1} \circ (Id_I \otimes \lambda_I^{-1}) \circ (\lambda_I \otimes Id_I) \circ \alpha_{I,I,I}^{-1} \circ \lambda_{I \otimes I}^{-1} \\ &\circ \lambda_I^{-1} \circ \lambda_I \circ \lambda_{I \otimes I} \circ \alpha_{I,I,I} \\ &\circ (\lambda_I^{-1} \otimes Id_I) \circ (\lambda_I \otimes Id_I) \\ &\circ ((Id_I \otimes Id_I) \otimes \lambda_I) \\ &\circ \alpha_{I,I,I \otimes I} \\ &\circ ((\lambda_I^{-1} \otimes Id_I) \otimes Id_I) \\ &\circ (\lambda_I^{-1} \otimes Id_I) \end{aligned}$$

α が自然変換であることから以下の図式が可換。

$$\begin{array}{ccc}
 (I \otimes I) \otimes (I \otimes I) & \xrightarrow{\alpha_{I,I,I \otimes I}} & I \otimes (I \otimes (I \otimes I)) \\
 \downarrow (Id_I \otimes Id_I) \otimes \lambda_I & \circlearrowleft & \downarrow Id_I \otimes (Id_I \otimes \lambda_I) \\
 (I \otimes I) \otimes I & \xrightarrow{\alpha_{I,I,I}} & I \otimes (I \otimes I)
 \end{array}$$

従って以下の等式が成り立つ

$$(Id_I \otimes (Id_I \otimes \lambda_I)) \circ \alpha_{I,I,I \otimes I} = \alpha_{I,I,I} \circ ((Id_I \otimes Id_I) \otimes \lambda_I)$$

$$\begin{aligned}
 &= \lambda_I \\
 &\circ (\lambda_I \otimes Id_I) \\
 &\circ (\rho_I^{-1} \otimes Id_I) \circ \rho_I^{-1} \circ \rho_I \circ (\rho_I \otimes Id_I) \\
 &\circ \alpha_{I,I,I}^{-1} \circ (Id_I \otimes \lambda_I^{-1}) \circ (\lambda_I \otimes Id_I) \circ \alpha_{I,I,I}^{-1} \circ \lambda_{I \otimes I}^{-1} \circ \lambda_I \circ \lambda_{I \otimes I} \\
 &\circ \alpha_{I,I,I} \circ ((Id_I \otimes Id_I) \otimes \lambda_I) \\
 &\circ (Id_I \otimes (Id_I \otimes \lambda_I)) \circ \alpha_{I,I,I \otimes I} \\
 &\circ \alpha_{I,I,I \otimes I} \\
 &\circ ((\lambda_I^{-1} \otimes Id_I) \otimes Id_I) \\
 &\circ (\lambda_I^{-1} \otimes Id_I)
 \end{aligned}$$

五角等式より以下の図式が可換と言える。

$$\begin{array}{ccc}
 (I \otimes (I \otimes I)) \otimes I & \xrightarrow{\alpha_{I, I \otimes I, I}} & I \otimes ((I \otimes I) \otimes I) \\
 \nearrow \alpha_{I, I, I} \otimes Id_I & & \searrow Id_I \otimes \alpha_{I, I, I} \\
 ((I \otimes I) \otimes I) \otimes I & \circlearrowleft & I \otimes (I \otimes (I \otimes I)) \\
 \searrow \alpha_{I \otimes I, I, I} & & \nearrow \alpha_{I, I, I \otimes I} \\
 (I \otimes I) \otimes (I \otimes I) & &
 \end{array}$$

従って以下の等式が成り立つ

$$\begin{aligned}
 & (Id_I \otimes \alpha_{I, I, I}) \circ \alpha_{I, I \otimes I, I} \circ (\alpha_{I, I, I} \otimes Id_I) = \alpha_{I, I, I \otimes I} \circ \alpha_{I \otimes I, I, I} \\
 & = \lambda_I \\
 & \circ (\lambda_I \otimes Id_I) \\
 & \circ (\rho_I^{-1} \otimes Id_I) \circ \rho_I^{-1} \circ \rho_I \circ (\rho_I \otimes Id_I) \\
 & \circ \alpha_{I, I, I}^{-1} \circ (Id_I \otimes \lambda_I^{-1}) \circ (\lambda_I \otimes Id_I) \circ \alpha_{I, I, I}^{-1} \circ \lambda_{I \otimes I}^{-1} \circ \lambda_I^{-1} \circ \lambda_I \circ \lambda_{I \otimes I} \\
 & \circ (Id_I \otimes (Id_I \otimes \lambda_I)) \\
 & \circ \alpha_{I, I, I \otimes I} \circ \alpha_{I, I, I \otimes I} \\
 & \circ (Id_I \otimes \alpha_{I, I, I}) \circ \alpha_{I, I \otimes I, I} \circ (\alpha_{I, I, I} \otimes Id_I) \\
 & \circ ((\lambda_I^{-1} \otimes Id_I) \otimes Id_I) \\
 & \circ (\lambda_I^{-1} \otimes Id_I)
 \end{aligned}$$

λ が自然変換であることから以下の図式が可換。

$$\begin{array}{ccc}
 I \otimes (I \otimes (I \otimes I)) & \xrightarrow{\lambda_{I \otimes (I \otimes I)}} & I \otimes (I \otimes I) \\
 \downarrow Id_I \otimes (Id_I \otimes \lambda_I) & \circlearrowleft & \downarrow Id_I \otimes \lambda_I \\
 I \otimes (I \otimes I) & \xrightarrow{\lambda_{I \otimes I}} & I \otimes I
 \end{array}$$

従って以下の等式が成り立つ

$$(Id_I \otimes \lambda_I) \circ \lambda_{I \otimes (I \otimes I)} = \lambda_{I \otimes I} \circ (Id_I \otimes (Id_I \otimes \lambda_I))$$

$$\begin{aligned}
 &= \lambda_I \\
 &\circ (\lambda_I \otimes Id_I) \\
 &\circ (\rho_I^{-1} \otimes Id_I) \circ \rho_I^{-1} \circ \rho_I \circ (\rho_I \otimes Id_I) \\
 &\circ \alpha_{I,I,I}^{-1} \circ (Id_I \otimes \lambda_I^{-1}) \circ (\lambda_I \otimes Id_I) \circ \alpha_{I,I,I}^{-1} \circ \lambda_{I \otimes I}^{-1} \circ \lambda_I^{-1} \circ \lambda_I \\
 &\circ \lambda_{I \otimes I} \circ (Id_I \otimes (Id_I \otimes \lambda_I)) \\
 &\circ (Id_I \otimes \lambda_I) \circ \lambda_{I \otimes (I \otimes I)} \\
 &\circ (Id_I \otimes \alpha_{I,I,I}) \circ \alpha_{I,I \otimes I,I} \circ (\alpha_{I,I,I} \otimes Id_I) \\
 &\circ ((\lambda_I^{-1} \otimes Id_I) \otimes Id_I) \\
 &\circ (\lambda_I^{-1} \otimes Id_I)
 \end{aligned}$$

λ が自然変換であることから以下の図式が可換。

$$\begin{array}{ccc}
 I \otimes (I \otimes I) & \xrightarrow{\lambda_{I \otimes I}} & I \otimes I \\
 \downarrow Id_I \otimes \lambda_I & \circlearrowright & \downarrow \lambda_I \\
 I \otimes I & \xrightarrow{\lambda_I} & I
 \end{array}$$

従って以下の等式が成り立つ

$$\lambda_I \circ \lambda_{I \otimes I} = \lambda_I \circ (Id_I \otimes \lambda_I)$$

$$= \lambda_I$$

$$\circ (\lambda_I \otimes Id_I)$$

$$\circ (\rho_I^{-1} \otimes Id_I) \circ \rho_I^{-1} \circ \rho_I \circ (\rho_I \otimes Id_I)$$

$$\circ \alpha_{I,I,I}^{-1} \circ (Id_I \otimes \lambda_I^{-1}) \circ (\lambda_I \otimes Id_I) \circ \alpha_{I,I,I}^{-1} \circ \lambda_{I \otimes I}^{-1} \circ \lambda_I^{-1}$$

$$\circ \lambda_I \circ (Id_I \otimes \lambda_I)$$

$$\circ \lambda_I \circ \lambda_{I \otimes I}$$

$$\circ \lambda_{I \otimes (I \otimes I)}$$

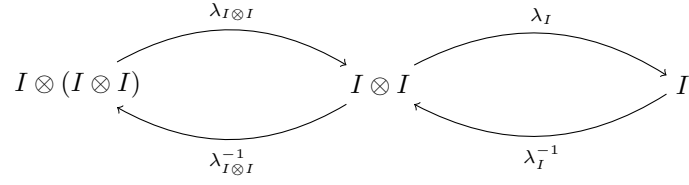
$$\circ (Id_I \otimes \alpha_{I,I,I}) \circ \alpha_{I,I \otimes I,I} \circ (\alpha_{I,I,I} \otimes Id_I)$$

$$\circ ((\lambda_I^{-1} \otimes Id_I) \otimes Id_I)$$

$$\circ (\lambda_I^{-1} \otimes Id_I)$$

以下は恒等射なので削除

$$\lambda_{I \otimes I}^{-1} \circ \lambda_I^{-1} \circ \lambda_I \circ \lambda_{I \otimes I}$$



$$\begin{aligned}
&= \lambda_I \\
&\circ (\lambda_I \otimes Id_I) \\
&\circ (\rho_I^{-1} \otimes Id_I) \circ \rho_I^{-1} \circ \rho_I \circ (\rho_I \otimes Id_I) \\
&\circ \alpha_{I,I,I}^{-1} \circ (Id_I \otimes \lambda_I^{-1}) \circ (\lambda_I \otimes Id_I) \circ \alpha_{I,I,I}^{-1} \\
&\circ \lambda_{I \otimes I}^{-1} \circ \lambda_I^{-1} \circ \lambda_I \circ \lambda_{I \otimes I} \\
&\circ \lambda_{I \otimes (I \otimes I)} \\
&\circ (Id_I \otimes \alpha_{I,I,I}) \circ \alpha_{I,I \otimes I,I} \circ (\alpha_{I,I,I} \otimes Id_I) \\
&\circ ((\lambda_I^{-1} \otimes Id_I) \otimes Id_I) \\
&\circ (\lambda_I^{-1} \otimes Id_I)
\end{aligned}$$

λ が自然変換であることから以下の図式が可換。

$$\begin{array}{ccc}
 I \otimes ((I \otimes (I \otimes I)) & \xrightarrow{\lambda_{(I \otimes I) \otimes I}} & (I \otimes I) \otimes I \\
 \downarrow Id_I \otimes \alpha_{I,I,I} & \circlearrowleft & \downarrow \alpha_{I,I,I} \\
 I \otimes (I \otimes (I \otimes I)) & \xrightarrow{\lambda_{I \otimes (I \otimes I)}} & I \otimes (I \otimes I)
 \end{array}$$

従って以下の等式が成り立つ

$$\begin{aligned}
 \alpha_{I,I,I} \circ \lambda_{(I \otimes I) \otimes I} &= \lambda_{I \otimes (I \otimes I)} \circ (Id_I \otimes \alpha_{I,I,I}) \\
 &= \lambda_I \\
 &\circ (\lambda_I \otimes Id_I) \\
 &\circ (\rho_I^{-1} \otimes Id_I) \circ \rho_I^{-1} \circ \rho_I \circ (\rho_I \otimes Id_I) \\
 &\circ \alpha_{I,I,I}^{-1} \circ (Id_I \otimes \lambda_I^{-1}) \circ (\lambda_I \otimes Id_I) \circ \alpha_{I,I,I}^{-1} \\
 &\circ \lambda_{I \otimes (I \otimes I)} \circ (Id_I \otimes \alpha_{I,I,I}) \\
 &\circ \alpha_{I,I,I} \circ \lambda_{(I \otimes I) \otimes I} \\
 &\circ \alpha_{I,I \otimes I,I} \circ (\alpha_{I,I,I} \otimes Id_I) \\
 &\circ ((\lambda_I^{-1} \otimes Id_I) \otimes Id_I) \\
 &\circ (\lambda_I^{-1} \otimes Id_I)
 \end{aligned}$$

以下の恒等射を挿入

$$(\lambda_I^{-1} \otimes Id_I) \circ \lambda_I^{-1} \circ \lambda_I \circ (\lambda_I \otimes Id_I)$$

$$\begin{array}{ccccc}
 & \xrightarrow{\lambda_I \otimes Id_I} & & \xrightarrow{\lambda_I} & \\
 (I \otimes I) \otimes I & & I \otimes I & & I \\
 & \xleftarrow{\lambda_I^{-1} \otimes Id_I} & & \xleftarrow{\lambda_I^{-1}} &
 \end{array}$$

$$\begin{aligned}
 &= \lambda_I \\
 &\circ (\lambda_I \otimes Id_I) \\
 &\circ (\rho_I^{-1} \otimes Id_I) \circ \rho_I^{-1} \circ \rho_I \circ (\rho_I \otimes Id_I) \\
 &\circ \alpha_{I,I,I}^{-1} \circ (Id_I \otimes \lambda_I^{-1}) \circ (\lambda_I \otimes Id_I) \circ \alpha_{I,I,I}^{-1} \\
 &\circ \alpha_{I,I,I} \\
 &\circ (\lambda_I^{-1} \otimes Id_I) \circ \lambda_I^{-1} \circ \lambda_I \circ (\lambda_I \otimes Id_I) \\
 &\circ \lambda_{(I \otimes I) \otimes I} \\
 &\circ \alpha_{I,I \otimes I,I} \circ (\alpha_{I,I,I} \otimes Id_I) \\
 &\circ ((\lambda_I^{-1} \otimes Id_I) \otimes Id_I) \\
 &\circ (\lambda_I^{-1} \otimes Id_I)
 \end{aligned}$$

λ が自然変換であることから以下の図式が可換。

$$\begin{array}{ccc}
 I \otimes ((I \otimes I) \otimes I) & \xrightarrow{\lambda_{(I \otimes I) \otimes I}} & (I \otimes I) \otimes I \\
 \downarrow \textcolor{red}{Id_I \otimes (\lambda_I \otimes Id_I)} & \circlearrowleft & \downarrow \textcolor{blue}{\lambda_I \otimes Id_I} \\
 I \otimes (I \otimes I) & \xrightarrow{\textcolor{red}{\lambda_{I \otimes I}}} & I \otimes I
 \end{array}$$

従って以下の等式が成り立つ

$$\begin{aligned}
 \textcolor{red}{\lambda_{I \otimes I} \circ (Id_I \otimes (\lambda_I \otimes Id_I))} &= (\textcolor{blue}{\lambda_I \otimes Id_I}) \circ \lambda_{(I \otimes I) \otimes I} \\
 &= \lambda_I \\
 &\circ (\lambda_I \otimes Id_I) \\
 &\circ (\rho_I^{-1} \otimes Id_I) \circ \rho_I^{-1} \circ \rho_I \circ (\rho_I \otimes Id_I) \\
 &\circ \alpha_{I,I,I}^{-1} \circ (Id_I \otimes \lambda_I^{-1}) \circ (\lambda_I \otimes Id_I) \circ \alpha_{I,I,I}^{-1} \\
 &\circ \alpha_{I,I,I} \\
 &\circ (\lambda_I^{-1} \otimes Id_I) \circ \lambda_I^{-1} \circ \lambda_I \\
 &\circ \textcolor{blue}{(\lambda_I \otimes Id_I) \circ \lambda_{(I \otimes I) \otimes I}} \\
 &\circ \textcolor{red}{\lambda_{I \otimes I} \circ (Id_I \otimes (\lambda_I \otimes Id_I))} \\
 &\circ \alpha_{I,I \otimes I,I} \circ (\alpha_{I,I,I} \otimes Id_I) \\
 &\circ ((\lambda_I^{-1} \otimes Id_I) \otimes Id_I) \\
 &\circ (\lambda_I^{-1} \otimes Id_I)
 \end{aligned}$$

λ が自然変換であることから以下の図式が可換。

$$\begin{array}{ccc}
 I \otimes (I \otimes I) & \xrightarrow{\lambda_{I \otimes I}} & I \otimes I \\
 \textcolor{red}{Id_I \otimes \lambda_I} \downarrow & \circlearrowright & \downarrow \lambda_I \\
 I \otimes I & \xrightarrow{\lambda_I} & I
 \end{array}$$

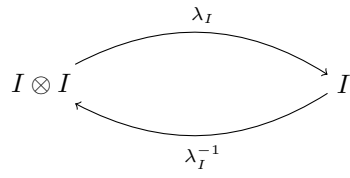
従って以下の等式が成り立つ

$$\textcolor{red}{\lambda_I \circ (Id_I \otimes \lambda_I)} = \textcolor{blue}{\lambda_I \circ \lambda_{I \otimes I}}$$

$$\begin{aligned}
 &= \lambda_I \\
 &\circ (\lambda_I \otimes Id_I) \\
 &\circ (\rho_I^{-1} \otimes Id_I) \circ \rho_I^{-1} \circ \rho_I \circ (\rho_I \otimes Id_I) \\
 &\circ \alpha_{I,I,I}^{-1} \circ (Id_I \otimes \lambda_I^{-1}) \circ (\lambda_I \otimes Id_I) \circ \alpha_{I,I,I}^{-1} \\
 &\circ \alpha_{I,I,I} \\
 &\circ (\lambda_I^{-1} \otimes I) \circ \lambda_I^{-1} \\
 &\circ \textcolor{blue}{\lambda_I \circ \lambda_{I \otimes I}} \\
 &\circ \textcolor{red}{\lambda_I \circ (Id_I \otimes \lambda_I)} \\
 &\circ (Id_I \otimes (\lambda_I \otimes I)) \\
 &\circ \alpha_{I,I \otimes I,I} \circ (\alpha_{I,I,I} \otimes Id_I) \\
 &\circ ((\lambda_I^{-1} \otimes Id_I) \otimes Id_I) \\
 &\circ (\lambda_I^{-1} \otimes Id_I)
 \end{aligned}$$

以下は恒等射なので削除

$$\lambda_I^{-1} \circ \lambda_I$$



$$\begin{aligned}
 &= \lambda_I \\
 &\circ (\lambda_I \otimes Id_I) \\
 &\circ (\rho_I^{-1} \otimes Id_I) \circ \rho_I^{-1} \circ \rho_I \circ (\rho_I \otimes Id_I) \\
 &\circ \alpha_{I,I,I}^{-1} \circ (Id_I \otimes \lambda_I^{-1}) \circ (\lambda_I \otimes Id_I) \circ \alpha_{I,I,I}^{-1} \\
 &\circ \alpha_{I,I,I} \\
 &\circ (\lambda_I^{-1} \otimes I) \\
 &\circ \lambda_I^{-1} \circ \lambda_I \\
 &\circ (Id_I \otimes \lambda_I) \\
 &\circ (Id_I \otimes (\lambda_I \otimes I)) \\
 &\circ \alpha_{I,I \otimes I,I} \circ (\alpha_{I,I,I} \otimes Id_I) \\
 &\circ ((\lambda_I^{-1} \otimes Id_I) \otimes Id_I) \\
 &\circ (\lambda_I^{-1} \otimes Id_I)
 \end{aligned}$$

以下は恒等射なので削除

$$(Id_I \otimes \lambda_I^{-1}) \circ (\lambda_I \otimes Id_I) \circ \alpha_{I,I,I}^{-1} \circ \alpha_{I,I,I} \circ (\lambda_I^{-1} \otimes Id_I) \circ (Id_I \otimes \lambda_I)$$

$$\begin{array}{ccccc}
 I \otimes (I \otimes I) & \xrightarrow{Id_I \otimes \lambda_I} & I \otimes I & \xrightarrow{\lambda_I^{-1} \otimes Id_I} & (I \otimes I) \otimes I & \xrightarrow{\alpha_{I,I,I}} & I \otimes (I \otimes I) \\
 & \xleftarrow{Id_I \otimes \lambda_I^{-1}} & & \xleftarrow{\lambda_I \otimes Id_I} & & \xleftarrow{\alpha_{I,I,I}^{-1}} &
 \end{array}$$

$$\begin{aligned}
 &= \lambda_I \\
 &\circ (\lambda_I \otimes Id_I) \\
 &\circ (\rho_I^{-1} \otimes Id_I) \circ \rho_I^{-1} \circ \rho_I \circ (\rho_I \otimes Id_I) \\
 &\circ \alpha_{I,I,I}^{-1} \\
 &\circ (Id_I \otimes \lambda_I^{-1}) \circ (\lambda_I \otimes Id_I) \circ \alpha_{I,I,I}^{-1} \circ \alpha_{I,I,I} \circ (\lambda_I^{-1} \otimes Id_I) \circ (Id_I \otimes \lambda_I) \\
 &\circ (Id_I \otimes (\lambda_I \otimes Id_I)) \\
 &\circ \alpha_{I,I \otimes I,I} \circ (\alpha_{I,I,I} \otimes Id_I) \\
 &\circ ((\lambda_I^{-1} \otimes Id_I) \otimes Id_I) \\
 &\circ (\lambda_I^{-1} \otimes Id_I)
 \end{aligned}$$

α が自然変換であることから以下の図式が可換。

$$\begin{array}{ccc}
 (I \otimes (I \otimes I)) \otimes I & \xrightarrow{\alpha_{I, I \otimes I, I}} & I \otimes ((I \otimes I) \otimes I) \\
 \downarrow (Id_I \otimes \lambda_I) \otimes Id_I & \circlearrowleft & \downarrow Id_I \otimes (\lambda_I \otimes Id_I) \\
 (I \otimes I) \otimes I & \xrightarrow{\alpha_{I, I, I}} & I \otimes (I \otimes I)
 \end{array}$$

従って以下の等式が成り立つ

$$\begin{aligned}
 \alpha_{I, I, I} \circ ((Id_I \otimes \lambda_I) \otimes Id_I) &= (Id_I \otimes (\lambda_I \otimes Id_I)) \circ \alpha_{I, I \otimes I, I} \\
 &= \lambda_I \\
 &\circ (\lambda_I \otimes Id_I) \\
 &\circ (\rho_I^{-1} \otimes Id_I) \circ \rho_I^{-1} \circ \rho_I \circ (\rho_I \otimes Id_I) \\
 &\circ \alpha_{I, I, I}^{-1} \\
 &\circ (Id_I \otimes (\lambda_I \otimes Id_I)) \circ \alpha_{I, I \otimes I, I} \\
 &\circ \alpha_{I, I, I} \circ ((Id_I \otimes \lambda_I) \otimes Id_I) \\
 &\circ (\alpha_{I, I, I} \otimes Id_I) \\
 &\circ ((\lambda_I^{-1} \otimes Id_I) \otimes Id_I) \\
 &\circ (\lambda_I^{-1} \otimes Id_I)
 \end{aligned}$$

以下は恒等射なので削除

$$\alpha_{I,I,I}^{-1} \circ \alpha_{I,I,I}$$

$$\begin{array}{ccc} & \alpha_{I,I,I} & \\ \curvearrowright & & \curvearrowleft \\ (I \otimes I) \otimes I & & I \otimes (I \otimes I) \\ \curvearrowleft & & \curvearrowright \\ & \alpha_{I,I,I}^{-1} & \end{array}$$

$$\begin{aligned} &= \lambda_I \\ &\circ (\lambda_I \otimes Id_I) \\ &\circ (\rho_I^{-1} \otimes Id_I) \circ \rho_I^{-1} \circ \rho_I \circ (\rho_I \otimes Id_I) \\ &\circ \alpha_{I,I,I}^{-1} \circ \alpha_{I,I,I} \\ &\circ ((Id_I \otimes \lambda_I) \otimes Id_I) \\ &\circ (\alpha_{I,I,I} \otimes Id_I) \\ &\circ ((\lambda_I^{-1} \otimes Id_I) \otimes Id_I) \\ &\circ (\lambda_I^{-1} \otimes Id_I) \end{aligned}$$

三角等式より以下の図式が可換と言える。

$$\begin{array}{ccc}
 ((I \otimes I) \otimes I) \otimes I & \xrightarrow{\alpha_{I,I,I} \otimes Id_I} & (I \otimes (I \otimes I)) \otimes I \\
 \searrow (\rho_I \otimes Id_I) \otimes Id_I & \circlearrowleft & \swarrow (Id_I \otimes \lambda_I) \otimes Id_I \\
 & (I \otimes I) \otimes I &
 \end{array}$$

従って以下の等式が成り立つ

$$\begin{aligned}
 (\rho_I \otimes Id_I) \otimes Id_I &= ((Id_I \otimes \lambda_I) \otimes Id_I) \circ (\alpha_{I,I,I} \otimes Id_I) \\
 &= \lambda_I \\
 &\circ (\lambda_I \otimes Id_I) \\
 &\circ (\rho_I^{-1} \otimes Id_I) \circ \rho_I^{-1} \circ \rho_I \circ (\rho_I \otimes Id_I) \\
 &\circ \cancel{((Id_I \otimes \lambda_I) \otimes Id_I)} \circ \cancel{(\alpha_{I,I,I} \otimes Id_I)} \\
 &\circ ((\rho_I \otimes Id_I) \otimes Id_I) \\
 &\circ ((\lambda_I^{-1} \otimes Id_I) \otimes Id_I) \\
 &\circ (\lambda_I^{-1} \otimes Id_I)
 \end{aligned}$$

ρ が自然変換であることから以下の図式が可換。

$$\begin{array}{ccc}
 (I \otimes I) \otimes I & \xrightarrow{\rho_{I \otimes I}} & I \otimes I \\
 \downarrow \rho_I \otimes Id_I & \circlearrowright & \downarrow \rho_I \\
 I \otimes I & \xrightarrow{\rho_I} & I
 \end{array}$$

従って以下の等式が成り立つ

$$\rho_I \circ \rho_{I \otimes I} = \rho_I \circ (\rho_I \otimes Id_I)$$

$$= \lambda_I$$

$$\circ (\lambda_I \otimes Id_I)$$

$$\circ (\rho_I^{-1} \otimes Id_I) \circ \rho_I^{-1}$$

$$\circ \rho_I \circ (\rho_I \otimes Id_I)$$

$$\circ \rho_I \circ \rho_{I \otimes I}$$

$$\circ ((\rho_I \otimes Id_I) \otimes Id_I)$$

$$\circ ((\lambda_I^{-1} \otimes Id_I) \otimes Id_I)$$

$$\circ (\lambda_I^{-1} \otimes Id_I)$$

ρ が自然変換であることから以下の図式が可換。

$$\begin{array}{ccc}
 ((I \otimes I) \otimes I) \otimes I & \xrightarrow{\rho_{(I \otimes I) \otimes I}} & (I \otimes I) \otimes I \\
 \downarrow (\rho_I \otimes Id_I) \otimes Id_I & \circlearrowleft & \downarrow \rho_I \otimes Id_I \\
 (I \otimes I) \otimes I & \xrightarrow{\rho_{I \otimes I}} & I \otimes I
 \end{array}$$

従って以下の等式が成り立つ

$$\begin{aligned}
 (\rho_I \otimes Id_I) \circ \rho_{(I \otimes I) \otimes I} &= \rho_{I \otimes I} \circ ((\rho_I \otimes Id_I) \otimes Id_I) \\
 &= \lambda_I \\
 &\circ (\lambda_I \otimes Id_I) \\
 &\circ (\rho_I^{-1} \otimes Id_I) \\
 &\circ \rho_I^{-1} \\
 &\circ \rho_I \\
 &\circ \rho_{I \otimes I} \circ ((\rho_I \otimes Id_I) \otimes Id_I) \\
 &\circ (\rho_I \otimes Id_I) \circ \rho_{(I \otimes I) \otimes I} \\
 &\circ ((\lambda_I^{-1} \otimes Id_I) \otimes Id_I) \\
 &\circ (\lambda_I^{-1} \otimes Id_I)
 \end{aligned}$$

以下は恒等射なので削除

$$(\rho_I^{-1} \otimes Id_I) \circ \rho_I^{-1} \circ \rho_I \circ (\rho_I \otimes Id_I)$$

$$\begin{array}{ccccc}
 & \rho_I \otimes Id_I & & \rho_I & \\
 (I \otimes I) \otimes I & \xrightarrow{\quad} & I \otimes I & \xrightarrow{\quad} & I \\
 & \rho_I^{-1} \otimes Id_I & & \rho_I^{-1} &
 \end{array}$$

$$\begin{aligned}
 &= \lambda_I \\
 &\circ (\lambda_I \otimes Id_I) \\
 &\circ (\rho_I^{-1} \otimes Id_I) \circ \rho_I^{-1} \circ \rho_I \circ (\rho_I \otimes Id_I) \\
 &\circ \rho_{(I \otimes I) \otimes I} \\
 &\circ ((\lambda_I^{-1} \otimes Id_I) \otimes Id_I) \\
 &\circ (\lambda_I^{-1} \otimes Id_I)
 \end{aligned}$$

ρ が自然変換であることから以下の図式が可換。

$$\begin{array}{ccc}
 ((I \otimes I) \otimes I) \otimes I & \xrightarrow{\rho_{(I \otimes I) \otimes I}} & (I \otimes I) \otimes I \\
 \downarrow (\lambda_I \otimes Id_I) \otimes Id_I & \circlearrowright & \downarrow \lambda_I \otimes Id_I \\
 (I \otimes I) \otimes I & \xrightarrow{\rho_{I \otimes I}} & I \otimes I
 \end{array}$$

従って以下の等式が成り立つ

$$\begin{aligned}
 \rho_{I \otimes I} \circ ((\lambda_I \otimes Id_I) \otimes Id_I) &= (\lambda_I \otimes Id_I) \circ \rho_{(I \otimes I) \otimes I} \\
 &= \lambda_I \\
 &\circ (\lambda_I \otimes Id_I) \circ \rho_{(I \otimes I) \otimes I} \\
 &\circ \rho_{I \otimes I} \circ ((\lambda_I \otimes Id_I) \otimes Id_I) \\
 &\circ ((\lambda_I^{-1} \otimes Id_I) \otimes Id_I) \\
 &\circ (\lambda_I^{-1} \otimes Id_I)
 \end{aligned}$$

以下は恒等射なので削除

$$\begin{array}{ccc}
 & \xrightarrow{(\lambda_I^{-1} \otimes Id_I) \otimes Id_I} & \\
 (I \otimes I) \otimes I & & ((I \otimes I) \otimes I) \otimes I \\
 & \xleftarrow{(\lambda_I \otimes Id_I) \otimes Id_I} &
 \end{array}$$

$$\begin{aligned}
 &= \lambda_I \\
 &\circ \rho_{I \otimes I} \\
 &\circ ((\lambda_I \otimes Id_I) \otimes Id_I) \circ ((\lambda_I^{-1} \otimes Id_I) \otimes Id_I) \\
 &\circ (\lambda_I^{-1} \otimes Id_I)
 \end{aligned}$$

ρ が自然変換であることから以下の図式が可換。

$$\begin{array}{ccc}
 (I \otimes I) \otimes I & \xrightarrow{\rho_{I \otimes I}} & I \otimes I \\
 \lambda_I \otimes Id_I \downarrow & \circlearrowright & \downarrow \lambda_I \\
 I \otimes I & \xrightarrow{\rho_I} & I
 \end{array}$$

従って以下の等式が成り立つ

$$\begin{aligned}
 \rho_I \circ (\lambda_I \otimes Id_I) &= \lambda_I \circ \rho_{I \otimes I} \\
 &= \lambda_I \circ \rho_{I \otimes I} \\
 \rho_I \circ (\lambda_I \otimes Id_I) & \\
 \circ (\lambda_I^{-1} \otimes Id_I) &
 \end{aligned}$$

以下は恒等射なので削除

$$(\lambda_I \otimes Id_I) \circ (\lambda_I^{-1} \otimes Id_I)$$

$$\begin{array}{ccc}
 & \lambda_I^{-1} \otimes Id_I & \\
 I \otimes I & \xrightarrow{\quad} & (I \otimes I) \otimes I \\
 & \lambda_I \otimes Id_I &
 \end{array}$$

$$\begin{aligned}
 &= \rho_I \circ (\lambda_I \otimes Id_I) \circ (\lambda_I^{-1} \otimes Id_I) \\
 &= \rho_I(\text{右単位律子})
 \end{aligned}$$

□