モノイダル圏における単位対象の左右単位律子の一致の証明

西坂 毅

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1 モノイダル圏

以下の5つの物 (テンソル積, 単位対象, 結合律子, 左単位律子, 右単位律子) が存在し、下記の二つの等式 (三角等式, 五角等式) を満たす圏をモノイダル圏と言う。

● テンソル積⊗

関手 \otimes : $C \times C \rightarrow C$

- 単位対象 *I*
 - $I \in Ob(C)$
- 結合律子 α

自然同型 α

$$(A \otimes B) \otimes C \xrightarrow{\alpha_{A,B,C}} A \otimes (B \otimes C)$$

• 左単位律子 λ

自然同型 λ

$$I \otimes A \xrightarrow{\lambda_A} A$$

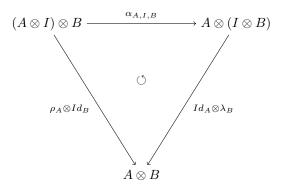
• 右単位律子 ρ

自然同型 ρ

$$A \otimes I \xrightarrow{\rho_A} A$$

• 三角等式

任意の A,B に対し以下の図式が可換

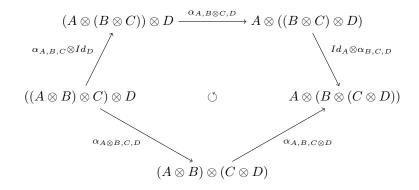


つまり以下の等式が成り立つ

$$\rho_A \otimes Id_B = (Id_A \otimes \lambda_B) \circ (\alpha_{A,I,B})$$

● 五角等式

任意の A,B,C,D に対し以下の図式が可換



つまり以下の等式が成り立つ

$$(Id_A \otimes \alpha_{B,C,D}) \circ \alpha_{A,B \otimes C,D} \circ (\alpha_{A,B,C} \otimes Id_D) = \alpha_{A,B,C \otimes D} \circ \alpha_{A \otimes B,C,D}$$

2 証明

Proof.

以下、 λ_I (左単位律子) に恒等射を入れて行きます。

$$\lambda_{I} = \lambda_{I}$$

$$\circ (\lambda_{I} \otimes Id_{I}) \circ (\lambda_{I}^{-1} \otimes Id_{I})$$

$$= \lambda_{I}$$

$$\circ (\lambda_{I} \otimes Id_{I})$$

$$\circ (\rho_{I}^{-1} \otimes Id_{I}) \circ (\rho_{I} \otimes Id_{I})$$

$$\circ (\lambda_{I}^{-1} \otimes Id_{I})$$

$$= \lambda_{I}$$

$$\circ (\lambda_{I} \otimes Id_{I})$$

$$\circ (\rho_{I}^{-1} \otimes Id_{I})$$

$$\circ (\rho_{I}^{-1} \otimes Id_{I})$$

$$\circ (\rho_{I} \otimes Id_{I})$$

$$\circ (\lambda_{I}^{-1} \otimes Id_{I})$$

$$\circ (\lambda_{I}^{-1} \otimes Id_{I})$$

$$\circ (\rho_{I}^{-1} \otimes Id_{I}) \circ \rho_{I}^{-1} \circ \rho_{I} \circ (\rho_{I} \otimes Id_{I})$$

$$\circ \alpha_{I,I,I}^{-1} \circ \alpha_{I,I,I}$$

$$\circ (\lambda_{I} \otimes Id_{I})$$

$$= \lambda_{I}$$

$$\circ (\lambda_{I} \otimes Id_{I})$$

$$\circ (\rho_{I}^{-1} \otimes Id_{I}) \circ \rho_{I}^{-1} \circ \rho_{I} \circ (\rho_{I} \otimes Id_{I})$$

$$\circ \alpha_{I,I,I}^{-1}$$

$$\circ (Id_{I} \otimes \lambda_{I}^{-1}) \circ (Id_{I} \otimes \lambda_{I})$$

$$\circ \alpha_{I,I,I}$$

$$\circ (\lambda_{I}^{-1} \otimes Id_{I})$$

$$= \lambda_{I}$$

$$\circ (\lambda_{I} \otimes Id_{I})$$

$$\circ (\rho_{I}^{-1} \otimes Id_{I}) \circ \rho_{I}^{-1} \circ \rho_{I} \circ (\rho_{I} \otimes Id_{I})$$

$$\circ \alpha_{I,I,I}^{-1} \circ (Id_{I} \otimes \lambda_{I}^{-1})$$

$$\circ (\lambda_{I} \otimes Id_{I}) \circ (\lambda_{I}^{-1} \otimes Id_{I}) \circ (Id_{I} \otimes \lambda_{I}) \circ \alpha_{I,I,I}$$

$$\circ (\lambda_{I}^{-1} \otimes Id_{I})$$

$$= \lambda_{I}$$

$$\circ (\lambda_{I} \otimes Id_{I})$$

$$\circ (\rho_{I}^{-1} \otimes Id_{I}) \circ \rho_{I}^{-1} \circ \rho_{I} \circ (\rho_{I} \otimes Id_{I})$$

$$\circ \alpha_{I,I,I}^{-1} \circ (Id_{I} \otimes \lambda_{I}^{-1}) \circ (\lambda_{I} \otimes Id_{I})$$

$$\circ \alpha_{I,I,I}^{-1} \circ \alpha_{I,I,I}$$

$$\circ (\lambda_{I}^{-1} \otimes Id_{I}) \circ (Id_{I} \otimes \lambda_{I}) \circ \alpha_{I,I,I}$$

$$\circ (\lambda_{I}^{-1} \otimes Id_{I})$$

$$= \lambda_{I}$$

$$\circ (\lambda_{I} \otimes Id_{I})$$

$$\circ (\rho_{I}^{-1} \otimes Id_{I}) \circ \rho_{I}^{-1} \circ \rho_{I} \circ (\rho_{I} \otimes Id_{I})$$

$$\circ \alpha_{I,I,I}^{-1} \circ (Id_{I} \otimes \lambda_{I}^{-1}) \circ (\lambda_{I} \otimes Id_{I}) \circ \alpha_{I,I,I}^{-1}$$

$$\circ \lambda_{I \otimes I}^{-1} \circ \lambda_{I \otimes I}$$

$$\circ \alpha_{I,I,I} \circ (\lambda_{I}^{-1} \otimes Id_{I}) \circ (Id_{I} \otimes \lambda_{I}) \circ \alpha_{I,I,I}$$

$$\circ (\lambda_{I}^{-1} \otimes Id_{I})$$

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= \lambda_{I}
\circ (\lambda_{I} \otimes Id_{I})
\circ (\rho_{I}^{-1} \otimes Id_{I}) \circ \rho_{I}^{-1} \circ \rho_{I} \circ (\rho_{I} \otimes Id_{I})
\circ \alpha_{I,I,I}^{-1} \circ (Id_{I} \otimes \lambda_{I}^{-1})
\circ (\lambda_{I} \otimes Id_{I}) \circ \alpha_{I,I,I}^{-1} \circ \lambda_{I \otimes I}^{-1}
\circ \lambda_{I}^{-1} \circ \lambda_{I}
\circ \lambda_{I \otimes I} \circ \alpha_{I,I,I} \circ (\lambda_{I}^{-1} \otimes Id_{I}) \circ (Id_{I} \otimes \lambda_{I}) \circ \alpha_{I,I,I}
\circ (\lambda_{I}^{-1} \otimes Id_{I})
= \lambda_{I}
\circ (\lambda_{I} \otimes Id_{I})
\circ (\rho_{I}^{-1} \otimes Id_{I}) \circ \rho_{I}^{-1} \circ \rho_{I} \circ (\rho_{I} \otimes Id_{I})
\circ \alpha_{I,I,I}^{-1} \circ (Id_{I} \otimes \lambda_{I}^{-1}) \circ (\lambda_{I} \otimes Id_{I})
\circ \alpha_{I,I,I}^{-1} \circ \lambda_{I \otimes I}^{-1} \circ \lambda_{I}^{-1} \circ \lambda_{I} \circ \lambda_{I \otimes I} \circ \alpha_{I,I,I} \circ (\lambda_{I}^{-1} \otimes Id_{I}) \circ (Id_{I} \otimes \lambda_{I}) \circ \alpha_{I,I,I}
\circ ((\lambda_{I} \otimes Id_{I}) \otimes Id_{I}) \circ ((\lambda_{I}^{-1} \otimes Id_{I}) \otimes Id_{I})
\circ (\lambda_{I}^{-1} \otimes Id_{I})
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α が自然変換であることから以下の図式が可換。

$$((I \otimes I) \otimes I) \otimes I \xrightarrow{\alpha_{I \otimes I, I, I}} (I \otimes I) \otimes (I \otimes I)$$

$$(\lambda_{I} \otimes Id_{I}) \otimes Id_{I} \qquad (I \otimes I) \otimes I \xrightarrow{\alpha_{I, I, I}} I \otimes (I \otimes I)$$

従って以下の等式が成り立つ

$$\lambda_I \otimes (Id_I \otimes Id_I) \circ \alpha_{I,I,I \otimes I} = \alpha_{I,I,I} \circ (\lambda_I \otimes Id_I) \otimes Id_I$$

$$= \lambda_{I}$$

$$\circ (\lambda_{I} \otimes Id_{I})$$

$$\circ (\rho_{I}^{-1} \otimes Id_{I}) \circ \rho_{I}^{-1} \circ \rho_{I} \circ (\rho_{I} \otimes Id_{I})$$

$$\circ \alpha_{I,I,I}^{-1} \circ (Id_{I} \otimes \lambda_{I}^{-1}) \circ (\lambda_{I} \otimes Id_{I}) \circ \alpha_{I,I,I}^{-1} \circ \lambda_{I \otimes I}^{-1} \circ \lambda_{I}^{-1}$$

$$\circ \lambda_{I} \circ \lambda_{I \otimes I} \circ \alpha_{I,I,I} \circ (\lambda_{I}^{-1} \otimes Id_{I}) \circ (Id_{I} \otimes \lambda_{I})$$

$$\circ \alpha_{I,I,I} \circ (\lambda_{I} \otimes Id_{I}) \otimes Id_{I}$$

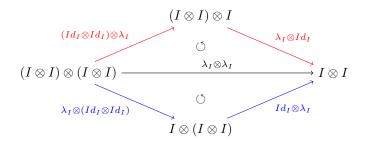
$$\circ (\lambda_{I} \otimes (Id_{I} \otimes Id_{I})) \circ \alpha_{I,I,I \otimes I}$$

$$\circ ((\lambda_{I}^{-1} \otimes Id_{I}) \otimes Id_{I})$$

$$\circ (\lambda_{I}^{-1} \otimes Id_{I})$$

以下の等式が成り立つ ※交換則 $(a\otimes b)\circ (c\otimes d)=(a\circ c)\otimes (b\circ d)$ を使用

 $(\lambda_I \otimes Id_I) \circ ((Id_I \otimes Id_I) \otimes \lambda_I) = \lambda_I \otimes \lambda_I = (Id_I \otimes \lambda_I) \circ (\lambda_I \otimes (Id_I \otimes Id_I))$



$$=\lambda_I$$

$$\circ (\lambda_I \otimes Id_I)$$

$$\circ (\rho_I^{-1} \otimes Id_I) \circ \rho_I^{-1} \circ \rho_I \circ (\rho_I \otimes Id_I)$$

$$\circ \alpha_{I,I,I}^{-1} \circ (Id_I \otimes \lambda_I^{-1}) \circ (\lambda_I \otimes Id_I) \circ \alpha_{I,I,I}^{-1} \circ \lambda_{I \otimes I}^{-1}$$

$$\circ \lambda_I^{-1} \circ \lambda_I \circ \lambda_{I \otimes I} \circ \alpha_{I,I,I} \circ (\lambda_I^{-1} \otimes Id_I)$$

$$\circ \ (Id_I \otimes \lambda_I) \circ (\lambda_I \otimes (Id_I \otimes Id_I))$$

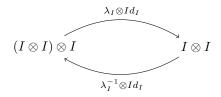
$$\circ (\lambda_I \otimes Id_I) \circ ((Id_I \otimes Id_I) \otimes \lambda_I)$$

$$\circ \alpha_{I,I,I\otimes I}$$

$$\circ ((\lambda_I^{-1} \otimes Id_I) \otimes Id_I)$$

$$\circ (\lambda_I^{-1} \otimes Id_I)$$

$$(\lambda_I^{-1} \otimes Id_I) \circ (\lambda_I \otimes Id_I)$$



- $=\lambda_I$
- $\circ (\lambda_I \otimes Id_I)$
- $\circ (\rho_I^{-1} \otimes Id_I) \circ \rho_I^{-1} \circ \rho_I \circ (\rho_I \otimes Id_I)$
- $\circ \alpha_{I,I,I}^{-1} \circ (Id_I \otimes \lambda_I^{-1}) \circ (\lambda_I \otimes Id_I) \circ \alpha_{I,I,I}^{-1} \circ \lambda_{I \otimes I}^{-1}$
- $\circ \lambda_{I}^{-1} \circ \lambda_{I} \circ \lambda_{I \otimes I} \circ \alpha_{I,I,I}$
- $\circ \ (\lambda_I^{-1} \otimes Id_I) \circ (\lambda_I \otimes Id_I)$
- $\circ \left(\left(Id_{I}\otimes Id_{I}\right) \otimes \lambda_{I}\right)$
- $\circ \alpha_{I,I,I\otimes I}$
- $\circ ((\lambda_I^{-1} \otimes Id_I) \otimes Id_I)$
- $\circ (\lambda_I^{-1} \otimes Id_I)$

α が自然変換であることから以下の図式が可換。

$$(I \otimes I) \otimes (I \otimes I) \xrightarrow{\alpha_{I,I,I \otimes I}} I \otimes (I \otimes (I \otimes I))$$

$$(Id_{I} \otimes Id_{I}) \otimes \lambda_{I}$$

$$(I \otimes I) \otimes I \xrightarrow{\alpha_{I,I,I}} I \otimes (I \otimes I)$$

従って以下の等式が成り立つ

$$(Id_I \otimes (Id_I \otimes \lambda_I)) \circ \alpha_{I,I,I \otimes I} = \alpha_{I,I,I} \circ ((Id_I \otimes Id_I) \otimes \lambda_I)$$

- $=\lambda_I$
- $\circ \left(\lambda_{I} \otimes Id_{I} \right)$
- $\circ (\rho_I^{-1} \otimes Id_I) \circ \rho_I^{-1} \circ \rho_I \circ (\rho_I \otimes Id_I)$
- $\circ \alpha_{I,I,I}^{-1} \circ (Id_I \otimes \lambda_I^{-1}) \circ (\lambda_I \otimes Id_I) \circ \alpha_{I,I,I}^{-1} \circ \lambda_{I \otimes I}^{-1} \circ \lambda_I^{-1} \circ \lambda_I \circ \lambda_{I \otimes I}$
- $\circ \alpha_{I,I,I} \circ ((Id_I \otimes Id_I) \otimes \lambda_I)$
- $\circ (Id_I \otimes (Id_I \otimes \lambda_I)) \circ \alpha_{I,I,I \otimes I}$
- $\circ \alpha_{I,I,I\otimes I}$
- $\circ ((\lambda_I^{-1} \otimes Id_I) \otimes Id_I)$
- $\circ (\lambda_I^{-1} \otimes Id_I)$

五角等式より以下の図式が可換と言える。

$$(I \otimes (I \otimes I)) \otimes I \xrightarrow{\alpha_{I,I \otimes I,I}} I \otimes ((I \otimes I) \otimes I)$$

$$\alpha_{I,I,I} \otimes Id_{I} \qquad (I \otimes I) \otimes I \qquad (I \otimes I) \otimes I$$

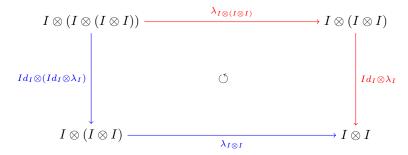
$$\alpha_{I,I,I} \otimes Id_{I} \qquad (I \otimes I) \otimes I \qquad (I \otimes I)$$

$$\alpha_{I,I,I \otimes I} \qquad (I \otimes I) \otimes I \otimes I \qquad (I \otimes I)$$

従って以下の等式が成り立つ

$$(Id_I \otimes \alpha_{I,I,I}) \circ \alpha_{I,I \otimes I,I} \circ (\alpha_{I,I,I} \otimes Id_I) = \alpha_{I,I,I \otimes I} \circ \alpha_{I \otimes I,I,I}$$

- $=\lambda_I$
- $\circ \left(\lambda_{I} \otimes Id_{I} \right)$
- $\circ (\rho_I^{-1} \otimes Id_I) \circ \rho_I^{-1} \circ \rho_I \circ (\rho_I \otimes Id_I)$
- $\circ \alpha_{I,I,I}^{-1} \circ (Id_I \otimes \lambda_I^{-1}) \circ (\lambda_I \otimes Id_I) \circ \alpha_{I,I,I}^{-1} \circ \lambda_{I \otimes I}^{-1} \circ \lambda_I^{-1} \circ \lambda_I \circ \lambda_{I \otimes I}$
- $\circ \left(Id_I \otimes \left(Id_I \otimes \lambda_I\right)\right)$
- $\circ \alpha_{I,I,I\otimes I} \circ \alpha_{I,I,I\otimes I}$
- $\circ \left(Id_{I} \otimes \alpha_{I,I,I}\right) \circ \alpha_{I,I \otimes I,I} \circ \left(\alpha_{I,I,I} \otimes Id_{I}\right)$
- $\circ ((\lambda_I^{-1} \otimes Id_I) \otimes Id_I)$
- $\circ (\lambda_I^{-1} \otimes Id_I)$



従って以下の等式が成り立つ

$$(Id_I \otimes \lambda_I) \circ \lambda_{I \otimes (I \otimes I)} = \lambda_{I \otimes I} \circ (Id_I \otimes (Id_I \otimes \lambda_I))$$

$$= \lambda_{I}$$

$$\circ (\lambda_{I} \otimes Id_{I})$$

$$\circ (\rho_{I}^{-1} \otimes Id_{I}) \circ \rho_{I}^{-1} \circ \rho_{I} \circ (\rho_{I} \otimes Id_{I})$$

$$\circ \alpha_{I,I,I}^{-1} \circ (Id_{I} \otimes \lambda_{I}^{-1}) \circ (\lambda_{I} \otimes Id_{I}) \circ \alpha_{I,I,I}^{-1} \circ \lambda_{I}^{-1} \circ \lambda_{I}^{-1} \circ \lambda_{I}$$

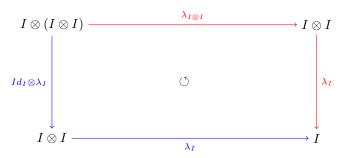
$$\circ \lambda_{I \otimes I} \circ (Id_{I} \otimes (Id_{I} \otimes \lambda_{I}))$$

$$\circ (Id_{I} \otimes \lambda_{I}) \circ \lambda_{I \otimes (I \otimes I)}$$

$$\circ (Id_{I} \otimes \alpha_{I,I,I}) \circ \alpha_{I,I \otimes I,I} \circ (\alpha_{I,I,I} \otimes Id_{I})$$

$$\circ ((\lambda_{I}^{-1} \otimes Id_{I}) \otimes Id_{I})$$

$$\circ (\lambda_{I}^{-1} \otimes Id_{I})$$



従って以下の等式が成り立つ

$$\lambda_I \circ \lambda_{I \otimes I} = \lambda_I \circ (Id_I \otimes \lambda_I)$$

$$= \lambda_{I}$$

$$\circ (\lambda_{I} \otimes Id_{I})$$

$$\circ (\rho_{I}^{-1} \otimes Id_{I}) \circ \rho_{I}^{-1} \circ \rho_{I} \circ (\rho_{I} \otimes Id_{I})$$

$$\circ \alpha_{I,I,I}^{-1} \circ (Id_{I} \otimes \lambda_{I}^{-1}) \circ (\lambda_{I} \otimes Id_{I}) \circ \alpha_{I,I,I}^{-1} \circ \lambda_{I \otimes I}^{-1} \circ \lambda_{I}^{-1}$$

$$\circ \lambda_{I} \circ (Id_{I} \otimes \lambda_{I})$$

$$\circ \lambda_{I} \circ \lambda_{I \otimes I}$$

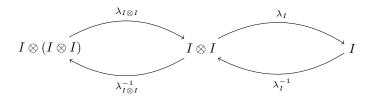
$$\circ \lambda_{I \otimes (I \otimes I)}$$

$$\circ (Id_{I} \otimes \alpha_{I,I,I}) \circ \alpha_{I,I \otimes I,I} \circ (\alpha_{I,I,I} \otimes Id_{I})$$

$$\circ ((\lambda_{I}^{-1} \otimes Id_{I}) \otimes Id_{I})$$

$$\circ (\lambda_{I}^{-1} \otimes Id_{I})$$

$$\lambda_{I\otimes I}^{-1}\circ\lambda_{I}^{-1}\circ\lambda_{I}\circ\lambda_{I\otimes I}$$



$$= \lambda_I$$

$$\circ (\lambda_I \otimes Id_I)$$

$$\circ (\rho_I^{-1} \otimes Id_I) \circ \rho_I^{-1} \circ \rho_I \circ (\rho_I \otimes Id_I)$$

$$\circ \, \alpha_{I,I,I}^{-1} \circ (Id_I \otimes \lambda_I^{-1}) \circ (\lambda_I \otimes Id_I) \circ \alpha_{I,I,I}^{-1}$$

$$\circ \ \lambda_{I \otimes I}^{-1} \circ \lambda_{I}^{-1} \circ \lambda_{I} \circ \lambda_{I \otimes I}$$

$$\circ \lambda_{I\otimes (I\otimes I)}$$

$$\circ \left(Id_{I} \otimes \alpha_{I,I,I}\right) \circ \alpha_{I,I \otimes I,I} \circ \left(\alpha_{I,I,I} \otimes Id_{I}\right)$$

$$\circ ((\lambda_I^{-1} \otimes Id_I) \otimes Id_I)$$

$$\circ (\lambda_I^{-1} \otimes Id_I)$$

$$I \otimes ((I \otimes (I \otimes I) \xrightarrow{\lambda_{(I \otimes I) \otimes I}} (I \otimes I) \otimes I$$

$$Id_{I} \otimes \alpha_{I,I,I} \qquad \circlearrowleft \qquad \qquad \downarrow \alpha_{I,I,I}$$

$$I \otimes (I \otimes (I \otimes I)) \xrightarrow{\lambda_{I \otimes (I \otimes I)}} I \otimes (I \otimes I)$$

従って以下の等式が成り立つ

$$\alpha_{I,I,I} \circ \lambda_{(I \otimes I) \otimes I} = \lambda_{I \otimes (I \otimes I)} \circ (Id_I \otimes \alpha_{I,I,I})$$

$$= \lambda_{I}$$

$$\circ (\lambda_{I} \otimes Id_{I})$$

$$\circ (\rho_{I}^{-1} \otimes Id_{I}) \circ \rho_{I}^{-1} \circ \rho_{I} \circ (\rho_{I} \otimes Id_{I})$$

$$\circ \alpha_{I,I,I}^{-1} \circ (Id_{I} \otimes \lambda_{I}^{-1}) \circ (\lambda_{I} \otimes Id_{I}) \circ \alpha_{I,I,I}^{-1}$$

$$\circ \lambda_{I \otimes (I \otimes I)} \circ (Id_{I} \otimes \alpha_{I,I,I})$$

$$\circ \alpha_{I,I,I} \circ \lambda_{(I \otimes I) \otimes I}$$

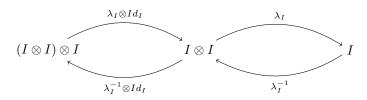
$$\circ \alpha_{I,I,SI,I} \circ (\alpha_{I,I,I} \otimes Id_{I})$$

$$\circ ((\lambda_{I}^{-1} \otimes Id_{I}) \otimes Id_{I})$$

$$\circ (\lambda_{I}^{-1} \otimes Id_{I})$$

以下の恒等射を挿入

$$(\lambda_I^{-1} \otimes Id_I) \circ \lambda_I^{-1} \circ \lambda_I \circ (\lambda_I \otimes Id_I)$$



$$=\lambda_I$$

$$\circ (\lambda_I \otimes Id_I)$$

$$\circ (\rho_I^{-1} \otimes Id_I) \circ \rho_I^{-1} \circ \rho_I \circ (\rho_I \otimes Id_I)$$

$$\circ \, \alpha_{I,I,I}^{-1} \circ (Id_I \otimes \lambda_I^{-1}) \circ (\lambda_I \otimes Id_I) \circ \alpha_{I,I,I}^{-1}$$

$$\circ \alpha_{I,I,I}$$

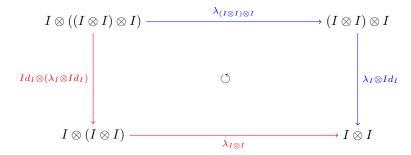
$$\circ \left(\lambda_I^{-1} \otimes Id_I\right) \circ \lambda_I^{-1} \circ \lambda_I \circ \left(\lambda_I \otimes Id_I\right)$$

$$\circ \, \lambda_{(I \otimes I) \otimes I}$$

$$\circ \alpha_{I,I\otimes I,I} \circ (\alpha_{I,I,I}\otimes Id_I)$$

$$\circ ((\lambda_I^{-1} \otimes Id_I) \otimes Id_I)$$

$$\circ (\lambda_I^{-1} \otimes Id_I)$$



従って以下の等式が成り立つ

$$\lambda_{I\otimes I}\circ (Id_I\otimes (\lambda_I\otimes Id_I))=(\lambda_I\otimes Id_I)\circ \lambda_{(I\otimes I)\otimes I}$$

$$= \lambda_{I}$$

$$\circ (\lambda_{I} \otimes Id_{I})$$

$$\circ (\rho_{I}^{-1} \otimes Id_{I}) \circ \rho_{I}^{-1} \circ \rho_{I} \circ (\rho_{I} \otimes Id_{I})$$

$$\circ \alpha_{I,I,I}^{-1} \circ (Id_{I} \otimes \lambda_{I}^{-1}) \circ (\lambda_{I} \otimes Id_{I}) \circ \alpha_{I,I,I}^{-1}$$

$$\circ \alpha_{I,I,I}$$

$$\circ (\lambda_{I}^{-1} \otimes Id_{I}) \circ \lambda_{I}^{-1} \circ \lambda_{I}$$

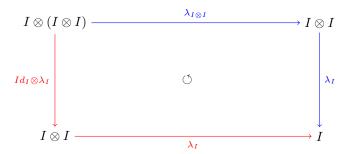
$$\circ (\lambda_{I} \otimes Id_{I}) \circ \lambda_{(I \otimes I) \otimes I}$$

$$\circ \lambda_{I \otimes I} \circ (Id_{I} \otimes (\lambda_{I} \otimes Id_{I}))$$

$$\circ \alpha_{I,I \otimes I,I} \circ (\alpha_{I,I,I} \otimes Id_{I})$$

$$\circ ((\lambda_{I}^{-1} \otimes Id_{I}) \otimes Id_{I})$$

$$\circ (\lambda_{I}^{-1} \otimes Id_{I})$$



従って以下の等式が成り立つ

$$\lambda_I \circ (Id_I \otimes \lambda_I) = \lambda_I \circ \lambda_{I \otimes I}$$

$$= \lambda_{I}$$

$$\circ (\lambda_{I} \otimes Id_{I})$$

$$\circ (\rho_{I}^{-1} \otimes Id_{I}) \circ \rho_{I}^{-1} \circ \rho_{I} \circ (\rho_{I} \otimes Id_{I})$$

$$\circ \alpha_{I,I,I}^{-1} \circ (Id_{I} \otimes \lambda_{I}^{-1}) \circ (\lambda_{I} \otimes Id_{I}) \circ \alpha_{I,I,I}^{-1}$$

$$\circ (\lambda_{I}^{-1} \otimes I) \circ \lambda_{I}^{-1}$$

$$\circ \lambda_{I} \circ \lambda_{I \otimes I}$$

$$\circ \lambda_{I} \circ (Id_{I} \otimes \lambda_{I})$$

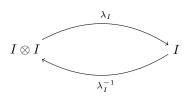
$$\circ (Id_{I} \otimes (\lambda_{I} \otimes I))$$

$$\circ \alpha_{I,I \otimes I,I} \circ (\alpha_{I,I,I} \otimes Id_{I})$$

$$\circ ((\lambda_{I}^{-1} \otimes Id_{I}) \otimes Id_{I})$$

$$\circ (\lambda_{I}^{-1} \otimes Id_{I})$$

$$\lambda_I^{-1} \circ \lambda_I$$



- $=\lambda_I$
- $\circ (\lambda_I \otimes Id_I)$
- $\circ (\rho_I^{-1} \otimes Id_I) \circ \rho_I^{-1} \circ \rho_I \circ (\rho_I \otimes Id_I)$
- $\circ \alpha_{I,I,I}^{-1} \circ (Id_I \otimes \lambda_I^{-1}) \circ (\lambda_I \otimes Id_I) \circ \alpha_{I,I,I}^{-1}$
- $\circ \alpha_{I,I,I}$
- $\circ \left(\lambda_{I}^{-1}\otimes I\right)$
- $\circ \ \ {\textstyle \frac{\lambda^{-1}}{\lambda_I}} \circ {\textstyle \frac{\lambda}{I}}$
- $\circ (Id_I \otimes \lambda_I)$
- $\circ \left(Id_{I} \otimes \left(\lambda_{I} \otimes I\right)\right)$
- $\circ \alpha_{I,I\otimes I,I} \circ (\alpha_{I,I,I} \otimes Id_I)$
- $\circ ((\lambda_I^{-1} \otimes Id_I) \otimes Id_I)$
- $\circ \left(\lambda_{I}^{-1}\otimes Id_{I}\right)$

 $(Id_I \otimes \lambda_I^{-1}) \circ (\lambda_I \otimes Id_I) \circ \alpha_{I,I,I}^{-1} \circ \alpha_{I,I,I} \circ (\lambda_I^{-1} \otimes Id_I) \circ (Id_I \otimes \lambda_I)$

$$I \otimes (I \otimes I) \qquad I \otimes I \qquad \qquad \lambda_I^{-1} \otimes Id_I \qquad \qquad \alpha_{I,I,I} \qquad \qquad I \otimes (I \otimes I) \qquad \qquad I \otimes (I \otimes I)$$

$$I \otimes (I \otimes I) \qquad \qquad \lambda_I \otimes Id_I \qquad \qquad \alpha_{I,I,I} \qquad \qquad \alpha_{I,I,I,I} \qquad \qquad \alpha_{I,I,I} \qquad \qquad \alpha_{I,I,I,I} \qquad \qquad \alpha_{I,I,I} \qquad \qquad \alpha_{I,I,I,I} \qquad \qquad \alpha_{I,I,I,I}$$

$$= \lambda_I$$

$$\circ (\lambda_I \otimes Id_I)$$

$$\circ (\rho_I^{-1} \otimes Id_I) \circ \rho_I^{-1} \circ \rho_I \circ (\rho_I \otimes Id_I)$$

$$\circ \alpha_{I,I,I}^{-1}$$

- $\circ (Id_{I} \otimes \lambda_{I}^{-1}) \circ (\lambda_{I} \otimes Id_{I}) \circ \alpha_{I,I,I}^{-1} \circ \alpha_{I,I,I} \circ (\lambda_{I}^{-1} \otimes Id_{I}) \circ (Id_{I} \otimes \lambda_{I})$
- $\circ \left(Id_I \otimes (\lambda_I \otimes Id_I)\right)$
- $\circ \alpha_{I,I\otimes I,I} \circ (\alpha_{I,I,I} \otimes Id_I)$
- $\circ ((\lambda_I^{-1} \otimes Id_I) \otimes Id_I)$
- $\circ \left(\lambda_{I}^{-1}\otimes Id_{I}\right)$

α が自然変換であることから以下の図式が可換。

$$(I \otimes (I \otimes I)) \otimes I \xrightarrow{\alpha_{I,I \otimes I,I}} I \otimes ((I \otimes I) \otimes I)$$

$$(Id_{I} \otimes \lambda_{I}) \otimes Id_{I} \qquad (I \otimes I) \otimes I \xrightarrow{\alpha_{I,I,I}} I \otimes ((I \otimes I) \otimes I)$$

従って以下の等式が成り立つ

$$\alpha_{I,I,I} \circ ((Id_I \otimes \lambda_I) \otimes Id_I) = (Id_I \otimes (\lambda_I \otimes Id_I)) \circ \alpha_{I,I \otimes I,I}$$

$$=\lambda_I$$

$$\circ (\lambda_I \otimes Id_I)$$

$$\circ (\rho_I^{-1} \otimes Id_I) \circ \rho_I^{-1} \circ \rho_I \circ (\rho_I \otimes Id_I)$$

$$\circ \alpha_{I,I,I}^{-1}$$

$$\circ \ (Id_I \otimes (\lambda_I \otimes Id_I)) \circ \alpha_{I,I \otimes I,I}$$

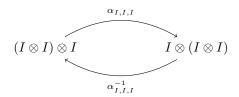
$$\circ \alpha_{I,I,I} \circ ((Id_I \otimes \lambda_I) \otimes Id_I)$$

$$\circ \left(\alpha_{I,I,I} \otimes Id_{I}\right)$$

$$\circ ((\lambda_I^{-1} \otimes Id_I) \otimes Id_I)$$

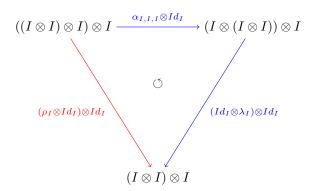
$$\circ (\lambda_I^{-1} \otimes Id_I)$$

$$\alpha_{I,I,I}^{-1} \circ \alpha_{I,I,I}$$



- $=\lambda_I$
- $\circ \left(\lambda_{I}\otimes Id_{I}\right)$
- $\circ (\rho_I^{-1} \otimes Id_I) \circ \rho_I^{-1} \circ \rho_I \circ (\rho_I \otimes Id_I)$
- $\circ \ \alpha_{I,I,I}^{-1} \circ \alpha_{I,I,I}$
- $\circ \left(\left(Id_{I}\otimes \lambda_{I}\right) \otimes Id_{I}\right)$
- $\circ \left(\alpha_{I,I,I} \otimes Id_{I}\right)$
- $\circ ((\lambda_I^{-1} \otimes Id_I) \otimes Id_I)$
- $\circ (\lambda_I^{-1} \otimes Id_I)$

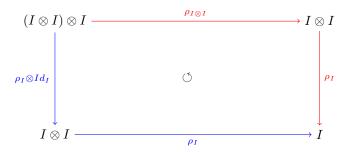
三角等式より以下の図式が可換と言える。



従って以下の等式が成り立つ

$$(\rho_I \otimes Id_I) \otimes Id_I = ((Id_I \otimes \lambda_I) \otimes Id_I) \circ (\alpha_{I,I,I} \otimes Id_I)$$

- $=\lambda_I$
- $\circ (\lambda_I \otimes Id_I)$
- $\circ (\rho_I^{-1} \otimes Id_I) \circ \rho_I^{-1} \circ \rho_I \circ (\rho_I \otimes Id_I)$
- $\circ \ ((Id_{I} \otimes \lambda_{I}) \otimes Id_{I}) \circ (\alpha_{I,I,I} \otimes Id_{I})$
- $\circ \left(\left(\rho_{I} \otimes Id_{I} \right) \otimes Id_{I} \right)$
- $\circ ((\lambda_I^{-1} \otimes Id_I) \otimes Id_I)$
- $\circ \left(\lambda_I^{-1} \otimes Id_I \right)$



従って以下の等式が成り立つ

$$\rho_I \circ \rho_{I \otimes I} = \rho_I \circ (\rho_I \otimes Id_I)$$

$$=\lambda_I$$

$$\circ (\lambda_I \otimes Id_I)$$

$$\circ \left(\rho_I^{-1} \otimes Id_I \right) \circ \rho_I^{-1}$$

$$\circ \ \rho_{I} \circ (\rho_{I} \otimes Id_{I})$$

$$\circ \rho_I \circ \rho_{I \otimes I}$$

$$\circ \left(\left(\rho_{I} \otimes Id_{I} \right) \otimes Id_{I} \right)$$

$$\circ ((\lambda_I^{-1} \otimes Id_I) \otimes Id_I)$$

$$\circ (\lambda_I^{-1} \otimes Id_I)$$

$$((I \otimes I) \otimes I) \otimes I \xrightarrow{\rho_{(I \otimes I) \otimes I}} (I \otimes I) \otimes I$$

$$(\rho_{I} \otimes Id_{I}) \otimes Id_{I} \qquad (I \otimes I) \otimes I \xrightarrow{\rho_{I \otimes I}} I \otimes I$$

従って以下の等式が成り立つ

$$(\rho_{I} \otimes Id_{I}) \circ \rho_{(I \otimes I) \otimes I} = \rho_{I \otimes I} \circ ((\rho_{I} \otimes Id_{I}) \otimes Id_{I})$$

$$= \lambda_{I}$$

$$\circ (\lambda_{I} \otimes Id_{I})$$

$$\circ (\rho_{I}^{-1} \otimes Id_{I})$$

$$\circ \rho_{I}^{-1}$$

$$\circ \rho_{I}$$

$$\circ \rho_{I \otimes Id_{I}} \circ ((\rho_{I} \otimes Id_{I}) \otimes Id_{I})$$

$$\circ (\rho_{I} \otimes Id_{I}) \circ \rho_{(I \otimes I) \otimes I}$$

$$\circ ((\lambda_{I}^{-1} \otimes Id_{I}) \otimes Id_{I})$$

$$\circ (\lambda_{I}^{-1} \otimes Id_{I})$$

以下は恒等射なので削除 $(\rho_I^{-1}\otimes Id_I)\circ\rho_I^{-1}\circ\rho_I\circ(\rho_I\otimes Id_I)$

$$(I \otimes I) \otimes I \qquad I \otimes I \qquad \rho_I$$

$$= \lambda_I$$

$$\circ (\lambda_I \otimes Id_I)
\circ (\rho_I^{-1} \otimes Id_I) \circ \rho_I^{-1} \circ \rho_I \circ (\rho_I \otimes Id_I)
\circ \rho_{(I \otimes I) \otimes I}
\circ ((\lambda_I^{-1} \otimes Id_I) \otimes Id_I)
\circ (\lambda_I^{-1} \otimes Id_I)$$

$$((I \otimes I) \otimes I) \otimes I \xrightarrow{\rho_{(I \otimes I) \otimes I}} (I \otimes I) \otimes I$$

$$(\lambda_{I} \otimes Id_{I}) \otimes Id_{I} \qquad (I \otimes I) \otimes I \xrightarrow{\rho_{I \otimes I}} I \otimes I$$

従って以下の等式が成り立つ

$$\rho_{I\otimes I} \circ ((\lambda_I \otimes Id_I) \otimes Id_I) = (\lambda_I \otimes Id_I) \circ \rho_{(I\otimes I)\otimes I}$$

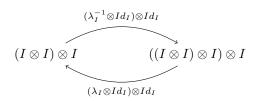
$$= \lambda_I$$

$$\circ \frac{(\lambda_I \otimes Id_I) \circ \rho_{(I\otimes I)\otimes I}}{\circ \rho_{I\otimes I}} \circ ((\lambda_I \otimes Id_I) \otimes Id_I)$$

$$\circ ((\lambda_I^{-1} \otimes Id_I) \otimes Id_I)$$

$$\circ (\lambda_I^{-1} \otimes Id_I)$$

$$((\lambda_I \otimes Id_I) \otimes Id_I) \circ ((\lambda_I^{-1} \otimes Id_I) \otimes Id_I)$$

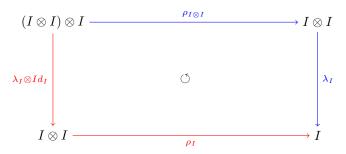


$$= \lambda_I$$

$$\circ \rho_{I \otimes I}$$

$$\circ ((\lambda_I \otimes Id_I) \otimes Id_I) \circ ((\lambda_I^{-1} \otimes Id_I) \otimes Id_I)$$

$$\circ (\lambda_I^{-1} \otimes Id_I)$$



従って以下の等式が成り立つ

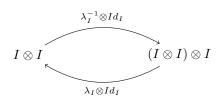
$$\rho_{I} \circ (\lambda_{I} \otimes Id_{I}) = \lambda_{I} \circ \rho_{I \otimes I}$$

$$= \frac{\lambda_{I} \circ \rho_{I \otimes I}}{\rho_{I} \circ (\lambda_{I} \otimes Id_{I})}$$

$$\circ (\lambda_{I}^{-1} \otimes Id_{I})$$

以下は恒等射なので削除

$$(\lambda_I \otimes Id_I) \circ (\lambda_I^{-1} \otimes Id_I)$$



 $= \rho_I \circ \frac{(\lambda_I \otimes Id_I) \circ (\lambda_I^{-1} \otimes Id_I)}{\rho_I(右単位律子)}$