モノイダル圏における左右単位子の一致の証明

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Proof.

以下、λ_I に恒等射を入れて行きます。

$$
\begin{aligned}
\lambda_I &= \lambda_I \\
&\circ(\lambda_I \otimes Id_I) \circ (\lambda_I^{-1} \otimes Id_I) \\
&= \lambda_I \\
&\circ (\lambda_I \otimes Id_I) \\
&\circ(\rho_I^{-1} \otimes Id_I) \circ (\rho_I \otimes Id_I) \\
&\circ (\lambda_I^{-1} \otimes Id_I) \\
&= \lambda_I \\
&\circ (\lambda_I \otimes Id_I) \\
&\circ (\rho_I^{-1} \otimes Id_I) \\
&\circ\rho_I^{-1} \circ \rho_I \\
&\circ (\rho_I \otimes Id_I) \\
&\circ (\lambda_I^{-1} \otimes Id_I) \\
&= \lambda_I \\
&\circ (\lambda_I \otimes Id_I) \\
&\circ (\rho_I^{-1} \otimes Id_I) \circ \rho_I^{-1} \circ \rho_I \circ (\rho_I \otimes Id_I) \\
&\circ\alpha_{I,I,I}^{-1} \circ \alpha_{I,I,I} \\
&\circ (\lambda_I^{-1} \otimes Id_I) \\
&= \lambda_I \\
&\circ (\lambda_I \otimes Id_I) \\
&\circ (\rho_I^{-1} \otimes Id_I) \circ \rho_I^{-1} \circ \rho_I \circ (\rho_I \otimes Id_I) \\
&\circ \alpha_{I,I,I}^{-1} \circ (Id_I \otimes \lambda_I^{-1}) \circ (Id_I \otimes \lambda_I) \circ \alpha_{I,I,I} \\
&\circ (\lambda_I^{-1} \otimes Id_I)
\end{aligned}
$$

$= \lambda_I$

$\circ (\lambda_I \otimes Id_I)$

$\circ (\rho_I^{-1} \otimes Id_I) \circ \rho_I^{-1} \circ \rho_I \circ (\rho_I \otimes Id_I)$

$\circ \alpha_{I,I,I}^{-1} \circ (Id_I \otimes \lambda_I^{-1})$

$\circ (\lambda_I \otimes Id_I) \circ (\lambda_I^{-1} \otimes Id_I) \circ (Id_I \otimes \lambda_I) \circ \alpha_{I,I,I}$

$\circ (\lambda_I^{-1} \otimes Id_I)$

$= \lambda_I$

$\circ (\lambda_I \otimes Id_I)$

$\circ (\rho_I^{-1} \otimes Id_I) \circ \rho_I^{-1} \circ \rho_I \circ (\rho_I \otimes Id_I)$

$\circ \alpha_{I,I,I}^{-1} \circ (Id_I \otimes \lambda_I^{-1}) \circ (\lambda_I \otimes Id_I)$

$\circ \alpha_{I,I,I}^{-1} \circ \alpha_{I,I,I}$

$\circ (\lambda_I^{-1} \otimes Id_I) \circ (Id_I \otimes \lambda_I) \circ \alpha_{I,I,I}$

$\circ (\lambda_I^{-1} \otimes Id_I)$

$= \lambda_I$

$\circ (\lambda_I \otimes Id_I)$

$\circ (\rho_I^{-1} \otimes Id_I) \circ \rho_I^{-1} \circ \rho_I \circ (\rho_I \otimes Id_I)$

$\circ \alpha_{I,I,I}^{-1} \circ (Id_I \otimes \lambda_I^{-1}) \circ (\lambda_I \otimes Id_I) \circ \alpha_{I,I,I}^{-1}$

$\circ \lambda_{I \otimes I}^{-1} \circ \lambda_{I \otimes I}$

$\circ \alpha_{I,I,I} \circ (\lambda_I^{-1} \otimes Id_I) \circ (Id_I \otimes \lambda_I) \circ \alpha_{I,I,I}$

$\circ (\lambda_I^{-1} \otimes Id_I)$

$= \lambda_I$

$\circ (\lambda_I \otimes Id_I)$

$\circ (\rho_I^{-1} \otimes Id_I) \circ \rho_I^{-1} \circ \rho_I \circ (\rho_I \otimes Id_I)$

$\circ \alpha_{I,I,I}^{-1} \circ (Id_I \otimes \lambda_I^{-1})$

$\circ (\lambda_I \otimes Id_I) \circ \alpha_{I,I,I}^{-1} \circ \lambda_{I\otimes I}^{-1}$

$\circ \lambda_I^{-1} \circ \lambda_I$

$\circ \lambda_{I\otimes I} \circ \alpha_{I,I,I} \circ (\lambda_I^{-1} \otimes Id_I) \circ (Id_I \otimes \lambda_I) \circ \alpha_{I,I,I}$

$\circ (\lambda_I^{-1} \otimes Id_I)$

$= \lambda_I$

$\circ (\lambda_I \otimes Id_I)$

$\circ (\rho_I^{-1} \otimes Id_I) \circ \rho_I^{-1} \circ \rho_I \circ (\rho_I \otimes Id_I)$

$\circ \alpha_{I,I,I}^{-1} \circ (Id_I \otimes \lambda_I^{-1}) \circ (\lambda_I \otimes Id_I)$

$\circ \alpha_{I,I,I}^{-1} \circ \lambda_{I\otimes I}^{-1} \circ \lambda_I^{-1} \circ \lambda_I \circ \lambda_{I\otimes I} \circ \alpha_{I,I,I} \circ (\lambda_I^{-1} \otimes Id_I) \circ (Id_I \otimes \lambda_I) \circ \alpha_{I,I,I}$

$\circ ((\lambda_I \otimes Id_I) \otimes Id_I) \circ ((\lambda_I^{-1} \otimes Id_I) \otimes Id_I)$

$\circ (\lambda_I^{-1} \otimes Id_I)$

α が自然変換であることから以下の図式が可換。

$$\begin{tikzcd}
((I \otimes I) \otimes I) \otimes I \arrow[r, "\alpha_{I \otimes I, I, I}"] \arrow[d, "(\lambda_I \otimes Id_I) \otimes Id_I"'] \arrow[dr, phantom, "\circlearrowleft"] & (I \otimes I) \otimes (I \otimes I) \arrow[d, "\lambda_I \otimes (Id_I \otimes Id_I)"] \\
(I \otimes I) \otimes I \arrow[r, "\alpha_{I,I,I}"'] & I \otimes (I \otimes I)
\end{tikzcd}$$

従って以下の等式が成り立つ

$$\lambda_I \otimes (Id_I \otimes Id_I) \circ \alpha_{I,I,I \otimes I} = \alpha_{I,I,I} \circ (\lambda_I \otimes Id_I) \otimes Id_I$$

$$\begin{aligned}
&= \lambda_I \\
&\circ (\lambda_I \otimes Id_I) \\
&\circ (\rho_I^{-1} \otimes Id_I) \circ \rho_I^{-1} \circ \rho_I \circ (\rho_I \otimes Id_I) \\
&\circ \alpha_{I,I,I}^{-1} \circ (Id_I \otimes \lambda_I^{-1}) \circ (\lambda_I \otimes Id_I) \circ \alpha_{I,I,I}^{-1} \circ \lambda_{I \otimes I}^{-1} \circ \lambda_I^{-1} \\
&\circ \lambda_I \circ \lambda_{I \otimes I} \circ \alpha_{I,I,I} \circ (\lambda_I^{-1} \otimes Id_I) \circ (Id_I \otimes \lambda_I) \\
&\circ (\lambda_I \otimes (Id_I \otimes Id_I)) \circ \alpha_{I,I,I \otimes I} \\
&\circ ((\lambda_I^{-1} \otimes Id_I) \otimes Id_I) \\
&\circ (\lambda_I^{-1} \otimes Id_I)
\end{aligned}$$

以下の等式が成り立つ
※ λ_I を当てる順番を入れ替えただけ

$$(\lambda_I \otimes Id_I) \circ ((Id_I \otimes Id_I) \otimes \lambda_I) = \lambda_I \otimes \lambda_I = (Id_I \otimes \lambda_I) \circ (\lambda_I \otimes (Id_I \otimes Id_I))$$

$$\begin{aligned}
&= \lambda_I \\
&\circ (\lambda_I \otimes Id_I) \\
&\circ (\rho_I^{-1} \otimes Id_I) \circ \rho_I^{-1} \circ \rho_I \circ (\rho_I \otimes Id_I) \\
&\circ \alpha_{I,I,I}^{-1} \circ (Id_I \otimes \lambda_I^{-1}) \circ (\lambda_I \otimes Id_I) \circ \alpha_{I,I,I}^{-1} \circ \lambda_{I \otimes I}^{-1} \\
&\circ \lambda_I^{-1} \circ \lambda_I \circ \lambda_{I \otimes I} \circ \alpha_{I,I,I} \circ (\lambda_I^{-1} \otimes Id_I) \\
&\circ (\lambda_I \otimes Id_I) \circ ((Id_I \otimes Id_I) \otimes \lambda_I) \\
&\circ \alpha_{I,I,I \otimes I} \\
&\circ ((\lambda_I^{-1} \otimes Id_I) \otimes Id_I) \\
&\circ (\lambda_I^{-1} \otimes Id_I)
\end{aligned}$$

以下は恒等射なので削除

$$(\lambda_I^{-1} \otimes Id_I) \circ (\lambda_I \otimes Id_I)$$

$= \lambda_I$
$\circ (\lambda_I \otimes Id_I)$
$\circ (\rho_I^{-1} \otimes Id_I) \circ \rho_I^{-1} \circ \rho_I \circ (\rho_I \otimes Id_I)$
$\circ \alpha_{I,I,I}^{-1} \circ (Id_I \otimes \lambda_I^{-1}) \circ (\lambda_I \otimes Id_I) \circ \alpha_{I,I,I}^{-1} \circ \lambda_{I\otimes I}^{-1}$
$\circ \lambda_I^{-1} \circ \lambda_I \circ \lambda_{I\otimes I} \circ \alpha_{I,I,I}$
$\circ (\lambda_I^{-1} \otimes Id_I) \circ (\lambda_I \otimes Id_I)$
$\circ ((Id_I \otimes Id_I) \otimes \lambda_I)$
$\circ \alpha_{I,I,I\otimes I}$
$\circ ((\lambda_I^{-1} \otimes Id_I) \otimes Id_I)$
$\circ (\lambda_I^{-1} \otimes Id_I)$

α が自然変換であることから以下の図式が可換。

$$\begin{tikzcd}
(I \otimes I) \otimes (I \otimes I) \arrow[r, "\alpha_{I,I,I\otimes I}"] \arrow[d, "(Id_I\otimes Id_I)\otimes\lambda_I"'] \arrow[dr, phantom, "\circlearrowleft"] & I \otimes (I \otimes (I \otimes I)) \arrow[d, "Id_I\otimes(Id_I\otimes\lambda_I)"] \\
(I \otimes I) \otimes I \arrow[r, "\alpha_{I,I,I}"'] & I \otimes (I \otimes I)
\end{tikzcd}$$

従って以下の等式が成り立つ

$$(Id_I \otimes (Id_I \otimes \lambda_I)) \circ \alpha_{I,I,I\otimes I} = \alpha_{I,I,I} \circ ((Id_I \otimes Id_I) \otimes \lambda_I)$$

$= \lambda_I$
$\circ (\lambda_I \otimes Id_I)$
$\circ (\rho_I^{-1} \otimes Id_I) \circ \rho_I^{-1} \circ \rho_I \circ (\rho_I \otimes Id_I)$
$\circ \alpha_{I,I,I}^{-1} \circ (Id_I \otimes \lambda_I^{-1}) \circ (\lambda_I \otimes Id_I) \circ \alpha_{I,I,I}^{-1} \circ \lambda_{I\otimes I}^{-1} \circ \lambda_I^{-1} \circ \lambda_I \circ \lambda_{I\otimes I}$
$\circ (Id_I \otimes (Id_I \otimes \lambda_I)) \circ \alpha_{I,I,I\otimes I}$
$\circ \alpha_{I,I,I\otimes I}$
$\circ ((\lambda_I^{-1} \otimes Id_I) \otimes Id_I)$
$\circ (\lambda_I^{-1} \otimes Id_I)$

五角等式より

$$\begin{tikzcd}[column sep=small]
 & (I \otimes (I \otimes I)) \otimes I \arrow[rr, "\alpha_{I,I\otimes I,I}"] & & I \otimes ((I \otimes I) \otimes I) \arrow[dr, "Id_I \otimes \alpha_{I,I,I}"] & \\
((I \otimes I) \otimes I) \otimes I \arrow[ur, "\alpha_{I,I,I} \otimes I"] \arrow[drr, "\alpha_{I\otimes I,I,I}"'] & & \circlearrowleft & & I \otimes (I \otimes (I \otimes I)) \\
 & & (I \otimes I) \otimes (I \otimes I) \arrow[urr, "\alpha_{I,I,I\otimes I}"'] & &
\end{tikzcd}$$

従って以下の等式が成り立つ

$$(Id_I \otimes \alpha_{I,I,I}) \circ \alpha_{I,I\otimes I,I} \circ (\alpha_{I,I,I} \otimes Id_I) = \alpha_{I,I,I\otimes I} \circ \alpha_{I,I,I\otimes I}$$

$= \lambda_I$

$\circ (\lambda_I \otimes Id_I)$

$\circ (\rho_I^{-1} \otimes Id_I) \circ \rho_I^{-1} \circ \rho_I \circ (\rho_I \otimes Id_I)$

$\circ \alpha_{I,I,I}^{-1} \circ (Id_I \otimes \lambda_I^{-1}) \circ (\lambda_I \otimes Id_I) \circ \alpha_{I,I,I}^{-1} \circ \lambda_{I\otimes I}^{-1} \circ \lambda_I^{-1} \circ \lambda_I \circ \lambda_{I\otimes I}$

$\circ (Id_I \otimes (Id_I \otimes \lambda_I))$

$\circ (Id_I \otimes \alpha_{I,I,I}) \circ \alpha_{I,I\otimes I,I} \circ (\alpha_{I,I,I} \otimes Id_I)$

$\circ ((\lambda_I^{-1} \otimes Id_I) \otimes Id_I)$

$\circ (\lambda_I^{-1} \otimes Id_I)$

λ が自然変換であることから以下の図式が可換。

$$
\begin{tikzcd}
I \otimes (I \otimes (I \otimes I)) \arrow[r, "\lambda_{I\otimes(I\otimes I)}"] \arrow[d, "Id_I\otimes(Id_I\otimes\lambda_I)"'] \arrow[dr, phantom, "\circlearrowleft"] & I \otimes (I \otimes I) \arrow[d, "Id_I\otimes\lambda_I"] \\
I \otimes (I \otimes I) \arrow[r, "\lambda_{I\otimes I}"'] & I \otimes I
\end{tikzcd}
$$

従って以下の等式が成り立つ

$$(Id_I \otimes \lambda_I) \circ \lambda_{I\otimes(I\otimes I)} = \lambda_{I\otimes I} \circ (Id_I \otimes (Id_I \otimes \lambda_I))$$

$= \lambda_I$

$\circ (\lambda_I \otimes Id_I)$

$\circ (\rho_I^{-1} \otimes Id_I) \circ \rho_I^{-1} \circ \rho_I \circ (\rho_I \otimes Id_I)$

$\circ \alpha_{I,I,I}^{-1} \circ (Id_I \otimes \lambda_I^{-1}) \circ (\lambda_I \otimes Id_I) \circ \alpha_{I,I,I}^{-1} \circ \lambda_{I\otimes I}^{-1} \circ \lambda_I^{-1} \circ \lambda_I$

$\circ (Id_I \otimes \lambda_I) \circ \lambda_{I\otimes(I\otimes I)}$

$\circ (Id_I \otimes \alpha_{I,I,I}) \circ \alpha_{I,I\otimes I,I} \circ (\alpha_{I,I,I} \otimes Id_I)$

$\circ ((\lambda_I^{-1} \otimes Id_I) \otimes Id_I)$

$\circ (\lambda_I^{-1} \otimes Id_I)$

λ が自然変換であることから以下の図式が可換。

$$
\begin{tikzcd}
I \otimes (I \otimes I) \arrow[r, "\lambda_{I\otimes I}"] \arrow[d, "Id_I \otimes \lambda_I"'] \arrow[dr, phantom, "\circlearrowleft"] & I \otimes I \arrow[d, "\lambda_I"] \\
I \otimes I \arrow[r, "\lambda_I"'] & I
\end{tikzcd}
$$

従って以下の等式が成り立つ

$$\lambda_I \circ \lambda_{I\otimes I} = \lambda_I \circ (Id_I \otimes \lambda_I)$$

$= \lambda_I$
$\circ (\lambda_I \otimes Id_I)$
$\circ (\rho_I^{-1} \otimes Id_I) \circ \rho_I^{-1} \circ \rho_I \circ (\rho_I \otimes Id_I)$
$\circ \alpha_{I,I,I}^{-1} \circ (Id_I \otimes \lambda_I^{-1}) \circ (\lambda_I \otimes Id_I) \circ \alpha_{I,I,I}^{-1} \circ \lambda_{I\otimes I}^{-1} \circ \lambda_I^{-1}$
$\circ \lambda_I \circ \lambda_{I\otimes I}$
$\circ \lambda_{I\otimes(I\otimes I)}$
$\circ (Id_I \otimes \alpha_{I,I,I}) \circ \alpha_{I,I\otimes I,I} \circ (\alpha_{I,I,I} \otimes Id_I)$
$\circ ((\lambda_I^{-1} \otimes Id_I) \otimes Id_I)$
$\circ (\lambda_I^{-1} \otimes Id_I)$

以下は恒等射なので削除

$$\lambda_{I\otimes I}^{-1} \circ \lambda_I^{-1} \circ \lambda_I \circ \lambda_{I\otimes I}$$

$= \lambda_I$
$\circ (\lambda_I \otimes Id_I)$
$\circ (\rho_I^{-1} \otimes Id_I) \circ \rho_I^{-1} \circ \rho_I \circ (\rho_I \otimes Id_I)$
$\circ \alpha_{I,I,I}^{-1} \circ (Id_I \otimes \lambda_I^{-1}) \circ (\lambda_I \otimes Id_I) \circ \alpha_{I,I,I}^{-1}$
$\circ \lambda_{I\otimes I}^{-1} \circ \lambda_I^{-1} \circ \lambda_I \circ \lambda_{I\otimes I}$
$\circ \lambda_{I\otimes(I\otimes I)}$
$\circ (Id_I \otimes \alpha_{I,I,I}) \circ \alpha_{I,I\otimes I,I} \circ (\alpha_{I,I,I} \otimes Id_I)$
$\circ ((\lambda_I^{-1} \otimes Id_I) \otimes Id_I)$
$\circ (\lambda_I^{-1} \otimes Id_I)$

λが自然変換であることから以下の図式が可換。

$$\begin{tikzcd}
I \otimes ((I \otimes (I \otimes I) \arrow[r, "\lambda_{(I\otimes I)\otimes I}"] \arrow[d, "Id_I \otimes \alpha_{I,I,I}"'] \arrow[dr, phantom, "\circlearrowleft"] & (I \otimes I) \otimes I \arrow[d, "\alpha_{I,I,I}"] \\
I \otimes (I \otimes (I \otimes I)) \arrow[r, "\lambda_{I\otimes(I\otimes I)}"'] & I \otimes (I \otimes I)
\end{tikzcd}$$

従って以下の等式が成り立つ

$$\alpha_{I,I,I} \circ \lambda_{(I\otimes I)\otimes I} = \lambda_{I\otimes(I\otimes I)} \circ (Id_I \otimes \alpha_{I,I,I})$$

$= \lambda_I$
$\circ (\lambda_I \otimes Id_I)$
$\circ (\rho_I^{-1} \otimes Id_I) \circ \rho_I^{-1} \circ \rho_I \circ (\rho_I \otimes Id_I)$
$\circ \alpha_{I,I,I}^{-1} \circ (Id_I \otimes \lambda_I^{-1}) \circ (\lambda_I \otimes Id_I) \circ \alpha_{I,I,I}^{-1}$
$\circ \alpha_{I,I,I} \circ \lambda_{(I\otimes I)\otimes I}$
$\circ \alpha_{I,I\otimes I,I} \circ (\alpha_{I,I,I} \otimes Id_I)$
$\circ ((\lambda_I^{-1} \otimes Id_I) \otimes Id_I)$
$\circ (\lambda_I^{-1} \otimes Id_I)$

以下の恒等射を挿入

$$(\lambda_I^{-1} \otimes I) \circ \lambda_I^{-1} \circ \lambda_I \circ (\lambda_I \otimes I)$$

$= \lambda_I$
$\circ (\lambda_I \otimes Id_I)$
$\circ (\rho_I^{-1} \otimes Id_I) \circ \rho_I^{-1} \circ \rho_I \circ (\rho_I \otimes Id_I)$
$\circ \alpha_{I,I,I}^{-1} \circ (Id_I \otimes \lambda_I^{-1}) \circ (\lambda_I \otimes Id_I) \circ \alpha_{I,I,I}^{-1}$
$\circ \alpha_{I,I,I}$
$\circ (\lambda_I^{-1} \otimes I) \circ \lambda_I^{-1} \circ \lambda_I \circ (\lambda_I \otimes I)$
$\circ \lambda_{(I\otimes I)\otimes I}$
$\circ \alpha_{I,I\otimes I,I} \circ (\alpha_{I,I,I} \otimes Id_I)$
$\circ ((\lambda_I^{-1} \otimes Id_I) \otimes Id_I)$
$\circ (\lambda_I^{-1} \otimes Id_I)$

λ が自然変換であることから以下の図式が可換。

$$
\begin{tikzcd}
I \otimes ((I \otimes I) \otimes I) \arrow[r, "\lambda_{(I\otimes I)\otimes I}"] \arrow[d, "Id_I \otimes (\lambda_I \otimes I)"'] \arrow[dr, phantom, "\circlearrowleft"] & (I \otimes I) \otimes I \arrow[d, "\lambda_I \otimes I"] \\
I \otimes (I \otimes I) \arrow[r, "\lambda_{I\otimes I}"'] & I \otimes I
\end{tikzcd}
$$

従って以下の等式が成り立つ

$$\lambda_{I\otimes I} \circ (Id_I \otimes (\lambda_I \otimes I)) = (\lambda_I \otimes I) \circ \lambda_{(I\otimes I)\otimes I}$$

$$= \lambda_I$$
$$\circ (\lambda_I \otimes Id_I)$$
$$\circ (\rho_I^{-1} \otimes Id_I) \circ \rho_I^{-1} \circ \rho_I \circ (\rho_I \otimes Id_I)$$
$$\circ \alpha_{I,I,I}^{-1} \circ (Id_I \otimes \lambda_I^{-1}) \circ (\lambda_I \otimes Id_I) \circ \alpha_{I,I,I}^{-1}$$
$$\circ \alpha_{I,I,I}$$
$$\circ (\lambda_I^{-1} \otimes I) \circ \lambda_I^{-1} \circ \lambda_I$$
$$\circ \lambda_{I\otimes I} \circ (Id_I \otimes (\lambda_I \otimes I))$$
$$\circ \alpha_{I,I\otimes I,I} \circ (\alpha_{I,I,I} \otimes Id_I)$$
$$\circ ((\lambda_I^{-1} \otimes Id_I) \otimes Id_I)$$
$$\circ (\lambda_I^{-1} \otimes Id_I)$$

λ が自然変換であることから以下の図式が可換。

$$\begin{tikzcd}
I \otimes (I \otimes I) \arrow[r, "\lambda_{I\otimes I}"] \arrow[d, "Id_I \otimes \lambda_I"'] \arrow[dr, phantom, "\circlearrowleft"] & I \otimes I \arrow[d, "\lambda_I"] \\
I \otimes I \arrow[r, "\lambda_I"'] & I
\end{tikzcd}$$

従って以下の等式が成り立つ

$$\lambda_I \circ (Id_I \otimes \lambda_I) = \lambda_I \circ \lambda_{I\otimes I}$$

$= \lambda_I$

$\circ\, (\lambda_I \otimes Id_I)$

$\circ\, (\rho_I^{-1} \otimes Id_I) \circ \rho_I^{-1} \circ \rho_I \circ (\rho_I \otimes Id_I)$

$\circ\, \alpha_{I,I,I}^{-1} \circ (Id_I \otimes \lambda_I^{-1}) \circ (\lambda_I \otimes Id_I) \circ \alpha_{I,I,I}^{-1}$

$\circ\, \alpha_{I,I,I}$

$\circ\, (\lambda_I^{-1} \otimes I) \circ \lambda_I^{-1}$

$\circ \lambda_I \circ (Id_I \otimes \lambda_I)$

$\circ\, (Id_I \otimes (\lambda_I \otimes I))$

$\circ\, \alpha_{I,I\otimes I,I} \circ (\alpha_{I,I,I} \otimes Id_I)$

$\circ\, ((\lambda_I^{-1} \otimes Id_I) \otimes Id_I)$

$\circ\, (\lambda_I^{-1} \otimes Id_I)$

以下は恒等射なので削除

$$\lambda_I^{-1} \circ \lambda_I$$

$$
\begin{aligned}
&= \lambda_I \\
&\circ (\lambda_I \otimes Id_I) \\
&\circ (\rho_I^{-1} \otimes Id_I) \circ \rho_I^{-1} \circ \rho_I \circ (\rho_I \otimes Id_I) \\
&\circ \alpha_{I,I,I}^{-1} \circ (Id_I \otimes \lambda_I^{-1}) \circ (\lambda_I \otimes Id_I) \circ \alpha_{I,I,I}^{-1} \\
&\circ \alpha_{I,I,I} \\
&\circ (\lambda_I^{-1} \otimes I) \\
&\circ \lambda_I^{-1} \circ \lambda_I \\
&\circ (Id_I \otimes \lambda_I) \\
&\circ (Id_I \otimes (\lambda_I \otimes I)) \\
&\circ \alpha_{I,I \otimes I,I} \circ (\alpha_{I,I,I} \otimes Id_I) \\
&\circ ((\lambda_I^{-1} \otimes Id_I) \otimes Id_I) \\
&\circ (\lambda_I^{-1} \otimes Id_I)
\end{aligned}
$$

以下は恒等射なので削除

$$(Id_I \otimes \lambda_I^{-1}) \circ (\lambda_I \otimes Id_I) \circ \alpha_{I,I,I}^{-1} \circ \alpha_{I,I,I} \circ (\lambda_I^{-1} \otimes Id_I) \circ (Id_I \otimes \lambda_I)$$

$$
\begin{aligned}
&= \lambda_I \\
&\circ (\lambda_I \otimes Id_I) \\
&\circ (\rho_I^{-1} \otimes Id_I) \circ \rho_I^{-1} \circ \rho_I \circ (\rho_I \otimes Id_I) \\
&\circ \alpha_{I,I,I}^{-1} \\
&\circ (Id_I \otimes \lambda_I^{-1}) \circ (\lambda_I \otimes Id_I) \circ \alpha_{I,I,I}^{-1} \circ \alpha_{I,I,I} \circ (\lambda_I^{-1} \otimes Id_I) \circ (Id_I \otimes \lambda_I) \\
&\circ (Id_I \otimes (\lambda_I \otimes Id_I)) \\
&\circ \alpha_{I,I \otimes I,I} \circ (\alpha_{I,I,I} \otimes Id_I) \\
&\circ ((\lambda_I^{-1} \otimes Id_I) \otimes Id_I) \\
&\circ (\lambda_I^{-1} \otimes Id_I)
\end{aligned}
$$

α が自然変換であることから以下の図式が可換。

$$\begin{tikzcd}
(I \otimes (I \otimes I)) \otimes I \arrow[r, "\alpha_{I,I\otimes I,I}"] \arrow[d, "(Id_I \otimes \lambda_I) \otimes Id_I"'] \arrow[dr, phantom, "\circlearrowleft"] & I \otimes ((I \otimes I) \otimes I) \arrow[d, "Id_I \otimes (\lambda_I \otimes Id_I)"] \\
(I \otimes I) \otimes I \arrow[r, "\alpha_{I,I,I}"'] & I \otimes (I \otimes I)
\end{tikzcd}$$

従って以下の等式が成り立つ

$$\alpha_{I,I,I} \circ ((Id_I \otimes \lambda_I) \otimes Id_I) = (Id_I \otimes (\lambda_I \otimes Id_I)) \circ \alpha_{I,I\otimes I,I}$$

$$\begin{aligned}
&= \lambda_I \\
&\circ (\lambda_I \otimes Id_I) \\
&\circ (\rho_I^{-1} \otimes Id_I) \circ \rho_I^{-1} \circ \rho_I \circ (\rho_I \otimes Id_I) \\
&\circ \alpha_{I,I,I}^{-1} \\
&\circ \alpha_{I,I,I} \circ ((Id_I \otimes \lambda_I) \otimes Id_I) \\
&\circ (\alpha_{I,I,I} \otimes Id_I) \\
&\circ ((\lambda_I^{-1} \otimes Id_I) \otimes Id_I) \\
&\circ (\lambda_I^{-1} \otimes Id_I)
\end{aligned}$$

以下は恒等射なので削除

$$\alpha_{I,I,I}^{-1} \circ \alpha_{I,I,I}$$

$$\begin{aligned}
&= \lambda_I \\
&\circ (\lambda_I \otimes Id_I) \\
&\circ (\rho_I^{-1} \otimes Id_I) \circ \rho_I^{-1} \circ \rho_I \circ (\rho_I \otimes Id_I) \\
&\circ \alpha_{I,I,I}^{-1} \circ \alpha_{I,I,I} \\
&\circ ((Id_I \otimes \lambda_I) \otimes Id_I) \\
&\circ (\alpha_{I,I,I} \otimes Id_I) \\
&\circ ((\lambda_I^{-1} \otimes Id_I) \otimes Id_I) \\
&\circ (\lambda_I^{-1} \otimes Id_I)
\end{aligned}$$

三角等式より

$$\begin{tikzcd}
((I \otimes I) \otimes I) \otimes I \arrow[rr, "\alpha_{I,I,I} \otimes Id_I"] \arrow[dr, "(\rho_I \otimes Id_I) \otimes Id_I"'] & \circlearrowleft & (I \otimes (I \otimes I)) \otimes I \arrow[dl, "(Id_I \otimes \lambda_I) \otimes Id_I"] \\
& (I \otimes I) \otimes I &
\end{tikzcd}$$

従って以下の等式が成り立つ

$$(\rho_I \otimes Id_I) \otimes Id_I = ((Id_I \otimes \lambda_I) \otimes Id_I) \circ (\alpha_{I,I,I} \otimes Id_I)$$

$$= \lambda_I$$
$$\circ\, (\lambda_I \otimes Id_I)$$
$$\circ\, (\rho_I^{-1} \otimes Id_I) \circ \rho_I^{-1} \circ \rho_I \circ (\rho_I \otimes Id_I)$$
$$\circ((\rho_I \otimes Id_I) \otimes Id_I)$$
$$\circ\, ((\lambda_I^{-1} \otimes Id_I) \otimes Id_I)$$
$$\circ\, (\lambda_I^{-1} \otimes Id_I)$$

ρ が自然変換であることから以下の図式が可換。

$$
\begin{tikzcd}
(I \otimes I) \otimes I \arrow[r, "\rho_{I\otimes I}"] \arrow[d, "\rho_I \otimes Id_I"'] \arrow[dr, phantom, "\circlearrowleft"] & I \otimes I \arrow[d, "\rho_I"] \\
I \otimes I \arrow[r, "\rho_I"'] & I
\end{tikzcd}
$$

従って以下の等式が成り立つ

$$\rho_I \circ \rho_{I\otimes I} = \rho_I \circ (\rho_I \otimes Id_I)$$

$$
\begin{aligned}
&= \lambda_I \\
&\circ (\lambda_I \otimes Id_I) \\
&\circ (\rho_I^{-1} \otimes Id_I) \circ \rho_I^{-1} \\
&\circ \rho_I \circ \rho_{I\otimes I} \\
&\circ ((\rho_I \otimes Id_I) \otimes Id_I) \\
&\circ ((\lambda_I^{-1} \otimes Id_I) \otimes Id_I) \\
&\circ (\lambda_I^{-1} \otimes Id_I)
\end{aligned}
$$

ρ が自然変換であることから以下の図式が可換。

$$\begin{tikzcd}
((I \otimes I) \otimes I) \otimes I \arrow[rr, "\rho_{(I\otimes I)\otimes I}"] \arrow[d, "(\rho_I \otimes Id_I)\otimes Id_I"'] \arrow[drr, phantom, "\circlearrowleft"] & & (I \otimes I) \otimes I \arrow[d, "\rho_I \otimes Id_I"] \\
(I \otimes I) \otimes I \arrow[rr, "\rho_{I\otimes I}"'] & & I \otimes I
\end{tikzcd}$$

従って以下の等式が成り立つ

$$(\rho_I \otimes Id_I) \circ \rho_{(I\otimes I)\otimes I} = \rho_{I\otimes I} \circ ((\rho_I \otimes Id_I) \otimes Id_I)$$

$$\begin{aligned}
&= \lambda_I \\
&\circ (\lambda_I \otimes Id_I) \\
&\circ (\rho_I^{-1} \otimes Id_I) \\
&\circ \rho_I^{-1} \\
&\circ \rho_I \\
&\circ (\rho_I \otimes Id_I) \circ \rho_{(I\otimes I)\otimes I} \\
&\circ ((\lambda_I^{-1} \otimes Id_I) \otimes Id_I) \\
&\circ (\lambda_I^{-1} \otimes Id_I)
\end{aligned}$$

以下は恒等射なので削除

$$(\rho_I^{-1} \otimes Id_I) \circ \rho_I^{-1} \circ \rho_I \circ (\rho_I \otimes Id_I)$$

$$\begin{aligned}
&= \lambda_I \\
&\circ (\lambda_I \otimes Id_I) \\
&\circ (\rho_I^{-1} \otimes Id_I) \circ \rho_I^{-1} \circ \rho_I \circ (\rho_I \otimes Id_I) \\
&\circ \rho_{(I\otimes I)\otimes I} \\
&\circ ((\lambda_I^{-1} \otimes Id_I) \otimes Id_I) \\
&\circ (\lambda_I^{-1} \otimes Id_I)
\end{aligned}$$

ρ が自然変換であることから以下の図式が可換。

$$\begin{tikzcd}
((I \otimes I) \otimes I) \otimes I \arrow[r, "\rho_{(I \otimes I) \otimes I}"] \arrow[d, "(\lambda_I \otimes Id_I) \otimes Id_I"'] \arrow[dr, phantom, "\circlearrowleft"] & (I \otimes I) \otimes I \arrow[d, "\lambda_I \otimes Id_I"] \\
(I \otimes I) \otimes I \arrow[r, "\rho_{I \otimes I}"'] & I \otimes I
\end{tikzcd}$$

従って以下の等式が成り立つ

$$\rho_{I \otimes I} \circ ((\lambda_I \otimes Id_I) \otimes Id_I) = (\lambda_I \otimes Id_I) \circ \rho_{(I \otimes I) \otimes I}$$

$$\begin{aligned}
&= \lambda_I \\
&\circ \rho_{I \otimes I} \circ ((\lambda_I \otimes Id_I) \otimes Id_I) \\
&\circ ((\lambda_I^{-1} \otimes Id_I) \otimes Id_I) \\
&\circ (\lambda_I^{-1} \otimes Id_I)
\end{aligned}$$

以下は恒等射なので削除

$$((\lambda_I \otimes Id_I) \otimes Id_I) \circ ((\lambda_I^{-1} \otimes Id_I) \otimes Id_I)$$

$$\begin{aligned}
&= \lambda_I \\
&\circ \rho_{I \otimes I} \\
&\circ ((\lambda_I \otimes Id_I) \otimes Id_I) \circ ((\lambda_I^{-1} \otimes Id_I) \otimes Id_I) \\
&\circ (\lambda_I^{-1} \otimes Id_I)
\end{aligned}$$

ρ が自然変換であることから以下の図式が可換。

$$\begin{tikzcd}
(I \otimes I) \otimes I \arrow[r, "\rho_{I\otimes I}"] \arrow[d, "\lambda_I \otimes Id_I"'] \arrow[dr, phantom, "\circlearrowleft"] & I \otimes I \arrow[d, "\lambda_I"] \\
I \otimes I \arrow[r, "\rho_I"'] & I
\end{tikzcd}$$

従って以下の等式が成り立つ

$$\rho_I \circ (\lambda_I \otimes Id_I) = \lambda_I \circ \rho_{I\otimes I}$$

$$= \rho_I \circ (\lambda_I \otimes Id_I)$$
$$\circ (\lambda_I^{-1} \otimes Id_I)$$

以下は恒等射なので削除

$$(\lambda_I \otimes Id_I) \circ (\lambda_I^{-1} \otimes Id_I)$$

$$= \rho_I \circ (\lambda_I \otimes Id_I) \circ (\lambda_I^{-1} \otimes Id_I)$$
$$= \rho_I$$

□