モノイダル圏における左右単位律子の一致の証明

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1 モノイダル圏

以下の5つの物 (テンソル積, 単位対象, 結合律子, 左単位律子, 右単位律子) が存在し、下記の二つの等式 (三角等式, 五角等式) を満たす圏をモノイダル圏と言う。

● テンソル積⊗

関手 \otimes : $C \times C \rightarrow C$

- 単位対象 *I*
 - $I\in Ob(C)$
- 結合律子 α

自然同型 α

$$(A \otimes B) \otimes C \xrightarrow{\alpha_{A,B,C}} A \otimes (B \otimes C)$$

• 左単位律子 λ

自然同型 λ

$$I \otimes A \xrightarrow{\lambda_A} A$$

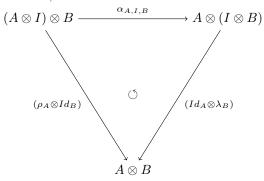
• 右単位律子 ρ

自然同型 ρ

$$A \otimes I \xrightarrow{\rho_A} A$$

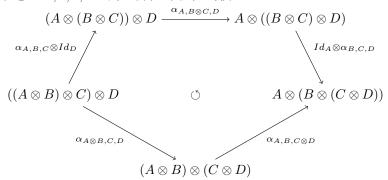
• 三角等式

任意の A,B に対し以下の図式が可換



• 五角等式

任意の A,B,C,D に対し以下の図式が可換



2 証明

Proof.

以下、 λ_I (左単位律子) に恒等射を入れて行きます。

$$\lambda_{I} = \lambda_{I}$$

$$\circ (\lambda_{I} \otimes Id_{I}) \circ (\lambda_{I}^{-1} \otimes Id_{I})$$

$$= \lambda_{I}$$

$$\circ (\lambda_{I} \otimes Id_{I})$$

$$\circ (\rho_{I}^{-1} \otimes Id_{I}) \circ (\rho_{I} \otimes Id_{I})$$

$$\circ (\lambda_{I}^{-1} \otimes Id_{I})$$

$$= \lambda_{I}$$

$$\circ (\lambda_{I} \otimes Id_{I})$$

$$\circ (\rho_{I}^{-1} \otimes Id_{I})$$

$$\circ (\rho_{I}^{-1} \otimes Id_{I})$$

$$\circ (\rho_{I} \otimes Id_{I})$$

$$\circ (\lambda_{I}^{-1} \otimes Id_{I})$$

$$\circ (\lambda_{I}^{-1} \otimes Id_{I})$$

$$\circ (\rho_{I}^{-1} \otimes Id_{I}) \circ \rho_{I}^{-1} \circ \rho_{I} \circ (\rho_{I} \otimes Id_{I})$$

$$\circ \alpha_{I,I,I}^{-1} \circ \alpha_{I,I,I}$$

$$\circ (\lambda_{I} \otimes Id_{I})$$

$$= \lambda_{I}$$

$$\circ (\lambda_{I} \otimes Id_{I})$$

$$\circ (\rho_{I}^{-1} \otimes Id_{I}) \circ \rho_{I}^{-1} \circ \rho_{I} \circ (\rho_{I} \otimes Id_{I})$$

$$\circ \alpha_{I,I,I}^{-1}$$

$$\circ (Id_{I} \otimes \lambda_{I}^{-1}) \circ (Id_{I} \otimes \lambda_{I})$$

$$\circ \alpha_{I,I,I}$$

$$\circ (\lambda_{I}^{-1} \otimes Id_{I})$$

$$= \lambda_{I}$$

$$\circ (\lambda_{I} \otimes Id_{I})$$

$$\circ (\rho_{I}^{-1} \otimes Id_{I}) \circ \rho_{I}^{-1} \circ \rho_{I} \circ (\rho_{I} \otimes Id_{I})$$

$$\circ \alpha_{I,I,I}^{-1} \circ (Id_{I} \otimes \lambda_{I}^{-1})$$

$$\circ (\lambda_{I} \otimes Id_{I}) \circ (\lambda_{I}^{-1} \otimes Id_{I}) \circ (Id_{I} \otimes \lambda_{I}) \circ \alpha_{I,I,I}$$

$$\circ (\lambda_{I}^{-1} \otimes Id_{I})$$

$$= \lambda_{I}$$

$$\circ (\lambda_{I} \otimes Id_{I})$$

$$\circ (\rho_{I}^{-1} \otimes Id_{I}) \circ \rho_{I}^{-1} \circ \rho_{I} \circ (\rho_{I} \otimes Id_{I})$$

$$\circ \alpha_{I,I,I}^{-1} \circ (Id_{I} \otimes \lambda_{I}^{-1}) \circ (\lambda_{I} \otimes Id_{I})$$

$$\circ \alpha_{I,I,I}^{-1} \circ \alpha_{I,I,I}$$

$$\circ (\lambda_{I}^{-1} \otimes Id_{I}) \circ (Id_{I} \otimes \lambda_{I}) \circ \alpha_{I,I,I}$$

$$\circ (\lambda_{I}^{-1} \otimes Id_{I})$$

$$= \lambda_{I}$$

$$\circ (\lambda_{I} \otimes Id_{I})$$

$$\circ (\rho_{I}^{-1} \otimes Id_{I}) \circ \rho_{I}^{-1} \circ \rho_{I} \circ (\rho_{I} \otimes Id_{I})$$

$$\circ \alpha_{I,I,I}^{-1} \circ (Id_{I} \otimes \lambda_{I}^{-1}) \circ (\lambda_{I} \otimes Id_{I}) \circ \alpha_{I,I,I}^{-1}$$

$$\circ \lambda_{I \otimes I}^{-1} \circ \lambda_{I \otimes I}$$

$$\circ \alpha_{I,I,I} \circ (\lambda_{I}^{-1} \otimes Id_{I}) \circ (Id_{I} \otimes \lambda_{I}) \circ \alpha_{I,I,I}$$

$$\circ (\lambda_{I}^{-1} \otimes Id_{I})$$

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= \lambda_{I}
\circ (\lambda_{I} \otimes Id_{I})
\circ (\rho_{I}^{-1} \otimes Id_{I}) \circ \rho_{I}^{-1} \circ \rho_{I} \circ (\rho_{I} \otimes Id_{I})
\circ \alpha_{I,I,I}^{-1} \circ (Id_{I} \otimes \lambda_{I}^{-1})
\circ (\lambda_{I} \otimes Id_{I}) \circ \alpha_{I,I,I}^{-1} \circ \lambda_{I \otimes I}^{-1}
\circ \lambda_{I}^{-1} \circ \lambda_{I}
\circ \lambda_{I \otimes I} \circ \alpha_{I,I,I} \circ (\lambda_{I}^{-1} \otimes Id_{I}) \circ (Id_{I} \otimes \lambda_{I}) \circ \alpha_{I,I,I}
\circ (\lambda_{I}^{-1} \otimes Id_{I})
= \lambda_{I}
\circ (\lambda_{I} \otimes Id_{I})
\circ (\rho_{I}^{-1} \otimes Id_{I}) \circ \rho_{I}^{-1} \circ \rho_{I} \circ (\rho_{I} \otimes Id_{I})
\circ \alpha_{I,I,I}^{-1} \circ (Id_{I} \otimes \lambda_{I}^{-1}) \circ (\lambda_{I} \otimes Id_{I})
\circ \alpha_{I,I,I}^{-1} \circ \lambda_{I \otimes I}^{-1} \circ \lambda_{I}^{-1} \circ \lambda_{I} \circ \lambda_{I \otimes I} \circ \alpha_{I,I,I} \circ (\lambda_{I}^{-1} \otimes Id_{I}) \circ (Id_{I} \otimes \lambda_{I}) \circ \alpha_{I,I,I}
\circ ((\lambda_{I} \otimes Id_{I}) \otimes Id_{I}) \circ ((\lambda_{I}^{-1} \otimes Id_{I}) \otimes Id_{I})
\circ (\lambda_{I}^{-1} \otimes Id_{I})
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α が自然変換であることから以下の図式が可換。

$$((I \otimes I) \otimes I) \otimes I \xrightarrow{\alpha_{I \otimes I, I, I}} (I \otimes I) \otimes (I \otimes I)$$

$$(\lambda_{I} \otimes Id_{I}) \otimes Id_{I} \qquad (I \otimes I) \otimes I \xrightarrow{\alpha_{I, I, I}} I \otimes (I \otimes I)$$

$$(I \otimes I) \otimes I \xrightarrow{\alpha_{I, I, I}} I \otimes (I \otimes I)$$

従って以下の等式が成り立つ

$$\lambda_I \otimes (Id_I \otimes Id_I) \circ \alpha_{I,I,I \otimes I} = \alpha_{I,I,I} \circ (\lambda_I \otimes Id_I) \otimes Id_I$$

$$= \lambda_{I}$$

$$\circ (\lambda_{I} \otimes Id_{I})$$

$$\circ (\rho_{I}^{-1} \otimes Id_{I}) \circ \rho_{I}^{-1} \circ \rho_{I} \circ (\rho_{I} \otimes Id_{I})$$

$$\circ \alpha_{I,I,I}^{-1} \circ (Id_{I} \otimes \lambda_{I}^{-1}) \circ (\lambda_{I} \otimes Id_{I}) \circ \alpha_{I,I,I}^{-1} \circ \lambda_{I \otimes I}^{-1} \circ \lambda_{I}^{-1}$$

$$\circ \lambda_{I} \circ \lambda_{I \otimes I} \circ \alpha_{I,I,I} \circ (\lambda_{I}^{-1} \otimes Id_{I}) \circ (Id_{I} \otimes \lambda_{I})$$

$$\circ \alpha_{I,I,I} \circ (\lambda_{I} \otimes Id_{I}) \otimes Id_{I}$$

$$\circ (\lambda_{I} \otimes (Id_{I} \otimes Id_{I})) \circ \alpha_{I,I,I \otimes I}$$

$$\circ ((\lambda_{I}^{-1} \otimes Id_{I}) \otimes Id_{I})$$

$$\circ (\lambda_{I}^{-1} \otimes Id_{I})$$

以下の等式が成り立つ $%\lambda_I$ を当てる順番を入れ替えただけ

$$(\lambda_I \otimes Id_I) \circ ((Id_I \otimes Id_I) \otimes \lambda_I) = \lambda_I \otimes \lambda_I = (Id_I \otimes \lambda_I) \circ (\lambda_I \otimes (Id_I \otimes Id_I))$$

$$= \lambda_{I}$$

$$\circ (\lambda_{I} \otimes Id_{I})$$

$$\circ (\rho_{I}^{-1} \otimes Id_{I}) \circ \rho_{I}^{-1} \circ \rho_{I} \circ (\rho_{I} \otimes Id_{I})$$

$$\circ \alpha_{I,I,I}^{-1} \circ (Id_{I} \otimes \lambda_{I}^{-1}) \circ (\lambda_{I} \otimes Id_{I}) \circ \alpha_{I,I,I}^{-1} \circ \lambda_{I \otimes I}^{-1}$$

$$\circ \lambda_{I}^{-1} \circ \lambda_{I} \circ \lambda_{I \otimes I} \circ \alpha_{I,I,I} \circ (\lambda_{I}^{-1} \otimes Id_{I})$$

$$\circ (Id_{I} \otimes \lambda_{I}) \circ (\lambda_{I} \otimes (Id_{I} \otimes Id_{I}))$$

$$\circ (\lambda_{I} \otimes Id_{I}) \circ ((Id_{I} \otimes Id_{I}) \otimes \lambda_{I})$$

$$\circ \alpha_{I,I,I \otimes I}$$

$$\circ ((\lambda_{I}^{-1} \otimes Id_{I}) \otimes Id_{I})$$

$$\circ (\lambda_{I}^{-1} \otimes Id_{I})$$

以下は恒等射なので削除

$$(\lambda_I^{-1} \otimes Id_I) \circ (\lambda_I \otimes Id_I)$$

$$= \lambda_{I}$$

$$\circ (\lambda_{I} \otimes Id_{I})$$

$$\circ (\rho_{I}^{-1} \otimes Id_{I}) \circ \rho_{I}^{-1} \circ \rho_{I} \circ (\rho_{I} \otimes Id_{I})$$

$$\circ \alpha_{I,I,I}^{-1} \circ (Id_{I} \otimes \lambda_{I}^{-1}) \circ (\lambda_{I} \otimes Id_{I}) \circ \alpha_{I,I,I}^{-1} \circ \lambda_{I \otimes I}^{-1}$$

$$\circ \lambda_{I}^{-1} \circ \lambda_{I} \circ \lambda_{I \otimes I} \circ \alpha_{I,I,I}$$

$$\circ (\lambda_{I}^{-1} \otimes Id_{I}) \circ (\lambda_{I} \otimes Id_{I})$$

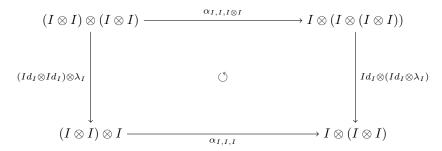
$$\circ ((Id_{I} \otimes Id_{I}) \otimes \lambda_{I})$$

$$\circ \alpha_{I,I,I \otimes I}$$

$$\circ ((\lambda_{I}^{-1} \otimes Id_{I}) \otimes Id_{I})$$

$$\circ (\lambda_{I}^{-1} \otimes Id_{I})$$

α が自然変換であることから以下の図式が可換。



従って以下の等式が成り立つ

$$(Id_I \otimes (Id_I \otimes \lambda_I)) \circ \alpha_{I,I,I \otimes I} = \alpha_{I,I,I} \circ ((Id_I \otimes Id_I) \otimes \lambda_I)$$

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= \lambda_{I}
\circ (\lambda_{I} \otimes Id_{I})
\circ (\rho_{I}^{-1} \otimes Id_{I}) \circ \rho_{I}^{-1} \circ \rho_{I} \circ (\rho_{I} \otimes Id_{I})
\circ \alpha_{I,I,I}^{-1} \circ (Id_{I} \otimes \lambda_{I}^{-1}) \circ (\lambda_{I} \otimes Id_{I}) \circ \alpha_{I,I,I}^{-1} \circ \lambda_{I}^{-1} \circ \lambda_{I} \circ
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五角等式より

$$(I \otimes (I \otimes I)) \otimes I \xrightarrow{\alpha_{I,I \otimes I,I}} I \otimes ((I \otimes I) \otimes I)$$

$$\alpha_{I,I,I} \otimes Id_{I} / Id_{I} \otimes \alpha_{I,I,I}$$

$$((I \otimes I) \otimes I) \otimes I / I \otimes (I \otimes I)$$

$$\alpha_{I,I,I \otimes I} / (I \otimes I) \otimes (I \otimes I)$$

従って以下の等式が成り立つ

$$(Id_I \otimes \alpha_{I,I,I}) \circ \alpha_{I,I \otimes I,I} \circ (\alpha_{I,I,I} \otimes Id_I) = \alpha_{I,I,I \otimes I} \circ \alpha_{I,I,I \otimes I}$$

$$= \lambda_{I}$$

$$\circ (\lambda_{I} \otimes Id_{I})$$

$$\circ (\rho_{I}^{-1} \otimes Id_{I}) \circ \rho_{I}^{-1} \circ \rho_{I} \circ (\rho_{I} \otimes Id_{I})$$

$$\circ \alpha_{I,I,I}^{-1} \circ (Id_{I} \otimes \lambda_{I}^{-1}) \circ (\lambda_{I} \otimes Id_{I}) \circ \alpha_{I,I,I}^{-1} \circ \lambda_{I}^{-1} \circ \lambda_{I}^{-1} \circ \lambda_{I} \circ \lambda_{I \otimes I}$$

$$\circ (Id_{I} \otimes (Id_{I} \otimes \lambda_{I}))$$

$$\circ \alpha_{I,I,I \otimes I} \circ \alpha_{I,I,I \otimes I}$$

$$\circ (Id_{I} \otimes \alpha_{I,I,I}) \circ \alpha_{I,I \otimes I,I} \circ (\alpha_{I,I,I} \otimes Id_{I})$$

$$\circ ((\lambda_{I}^{-1} \otimes Id_{I}) \otimes Id_{I})$$

 $\circ \left(\lambda_I^{-1} \otimes Id_I \right)$

λ が自然変換であることから以下の図式が可換。

$$I \otimes (I \otimes (I \otimes I)) \xrightarrow{\lambda_{I \otimes (I \otimes I)}} I \otimes (I \otimes I)$$

$$Id_{I} \otimes (Id_{I} \otimes \lambda_{I}) \downarrow \qquad \qquad \downarrow Id_{I} \otimes \lambda_{I}$$

$$I \otimes (I \otimes I) \xrightarrow{\lambda_{I \otimes I}} I \otimes I$$

従って以下の等式が成り立つ

$$(Id_I \otimes \lambda_I) \circ \lambda_{I \otimes (I \otimes I)} = \lambda_{I \otimes I} \circ (Id_I \otimes (Id_I \otimes \lambda_I))$$

$$= \lambda_{I}$$

$$\circ (\lambda_{I} \otimes Id_{I})$$

$$\circ (\rho_{I}^{-1} \otimes Id_{I}) \circ \rho_{I}^{-1} \circ \rho_{I} \circ (\rho_{I} \otimes Id_{I})$$

$$\circ \alpha_{I,I,I}^{-1} \circ (Id_{I} \otimes \lambda_{I}^{-1}) \circ (\lambda_{I} \otimes Id_{I}) \circ \alpha_{I,I,I}^{-1} \circ \lambda_{I}^{-1} \circ \lambda_{I}^{-1} \circ \lambda_{I}$$

$$\circ \lambda_{I \otimes I} \circ (Id_{I} \otimes (Id_{I} \otimes \lambda_{I}))$$

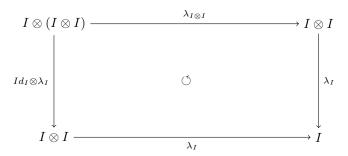
$$\circ (Id_{I} \otimes \lambda_{I}) \circ \lambda_{I \otimes (I \otimes I)}$$

$$\circ (Id_{I} \otimes \alpha_{I,I,I}) \circ \alpha_{I,I \otimes I,I} \circ (\alpha_{I,I,I} \otimes Id_{I})$$

$$\circ ((\lambda_{I}^{-1} \otimes Id_{I}) \otimes Id_{I})$$

$$\circ (\lambda_{I}^{-1} \otimes Id_{I})$$

λが自然変換であることから以下の図式が可換。



従って以下の等式が成り立つ

$$\lambda_I \circ \lambda_{I \otimes I} = \lambda_I \circ (Id_I \otimes \lambda_I)$$

$$= \lambda_{I}$$

$$\circ (\lambda_{I} \otimes Id_{I})$$

$$\circ (\rho_{I}^{-1} \otimes Id_{I}) \circ \rho_{I}^{-1} \circ \rho_{I} \circ (\rho_{I} \otimes Id_{I})$$

$$\circ \alpha_{I,I,I}^{-1} \circ (Id_{I} \otimes \lambda_{I}^{-1}) \circ (\lambda_{I} \otimes Id_{I}) \circ \alpha_{I,I,I}^{-1} \circ \lambda_{I \otimes I}^{-1} \circ \lambda_{I}^{-1}$$

$$\circ \lambda_{I} \circ (Id_{I} \otimes \lambda_{I})$$

$$\circ \lambda_{I} \circ \lambda_{I \otimes I}$$

$$\circ \lambda_{I \otimes (I \otimes I)}$$

$$\circ (Id_{I} \otimes \alpha_{I,I,I}) \circ \alpha_{I,I \otimes I,I} \circ (\alpha_{I,I,I} \otimes Id_{I})$$

$$\circ ((\lambda_{I}^{-1} \otimes Id_{I}) \otimes Id_{I})$$

$$\circ (\lambda_{I}^{-1} \otimes Id_{I})$$

$$\lambda_{I\otimes I}^{-1}\circ\lambda_I^{-1}\circ\lambda_I\circ\lambda_{I\otimes I}$$

$$= \lambda_{I}$$

$$\circ (\lambda_{I} \otimes Id_{I})$$

$$\circ (\rho_{I}^{-1} \otimes Id_{I}) \circ \rho_{I}^{-1} \circ \rho_{I} \circ (\rho_{I} \otimes Id_{I})$$

$$\circ \alpha_{I,I,I}^{-1} \circ (Id_{I} \otimes \lambda_{I}^{-1}) \circ (\lambda_{I} \otimes Id_{I}) \circ \alpha_{I,I,I}^{-1}$$

$$\circ \lambda_{I \otimes I}^{-1} \circ \lambda_{I}^{-1} \circ \lambda_{I} \circ \lambda_{I \otimes I}$$

$$\circ \lambda_{I \otimes (I \otimes I)}$$

$$\circ (Id_{I} \otimes \alpha_{I,I,I}) \circ \alpha_{I,I \otimes I,I} \circ (\alpha_{I,I,I} \otimes Id_{I})$$

$$\circ ((\lambda_{I}^{-1} \otimes Id_{I}) \otimes Id_{I})$$

$$\circ (\lambda_{I}^{-1} \otimes Id_{I})$$

λが自然変換であることから以下の図式が可換。

$$I \otimes ((I \otimes (I \otimes I) \xrightarrow{\lambda_{(I \otimes I) \otimes I}} (I \otimes I) \otimes I$$

$$Id_{I} \otimes \alpha_{I,I,I} \qquad \circlearrowleft \qquad \qquad \downarrow \alpha_{I,I,I}$$

$$I \otimes (I \otimes (I \otimes I)) \xrightarrow{\lambda_{I \otimes (I \otimes I)}} I \otimes (I \otimes I)$$

従って以下の等式が成り立つ

$$\alpha_{I,I,I} \circ \lambda_{(I \otimes I) \otimes I} = \lambda_{I \otimes (I \otimes I)} \circ (Id_I \otimes \alpha_{I,I,I})$$

$$= \lambda_{I}$$

$$\circ (\lambda_{I} \otimes Id_{I})$$

$$\circ (\rho_{I}^{-1} \otimes Id_{I}) \circ \rho_{I}^{-1} \circ \rho_{I} \circ (\rho_{I} \otimes Id_{I})$$

$$\circ \alpha_{I,I,I}^{-1} \circ (Id_{I} \otimes \lambda_{I}^{-1}) \circ (\lambda_{I} \otimes Id_{I}) \circ \alpha_{I,I,I}^{-1}$$

$$\circ \lambda_{I \otimes (I \otimes I)} \circ (Id_{I} \otimes \alpha_{I,I,I})$$

$$\circ \alpha_{I,I,I} \circ \lambda_{(I \otimes I) \otimes I}$$

$$\circ \alpha_{I,I \otimes I,I} \circ (\alpha_{I,I,I} \otimes Id_{I})$$

$$\circ ((\lambda_{I}^{-1} \otimes Id_{I}) \otimes Id_{I})$$

$$\circ (\lambda_{I}^{-1} \otimes Id_{I})$$

以下の恒等射を挿入

$$(\lambda_I^{-1} \otimes I) \circ \lambda_I^{-1} \circ \lambda_I \circ (\lambda_I \otimes I)$$

$$= \lambda_{I}$$

$$\circ (\lambda_{I} \otimes Id_{I})$$

$$\circ (\rho_{I}^{-1} \otimes Id_{I}) \circ \rho_{I}^{-1} \circ \rho_{I} \circ (\rho_{I} \otimes Id_{I})$$

$$\circ \alpha_{I,I,I}^{-1} \circ (Id_{I} \otimes \lambda_{I}^{-1}) \circ (\lambda_{I} \otimes Id_{I}) \circ \alpha_{I,I,I}^{-1}$$

$$\circ \alpha_{I,I,I}$$

$$\circ (\lambda_{I}^{-1} \otimes I) \circ \lambda_{I}^{-1} \circ \lambda_{I} \circ (\lambda_{I} \otimes I)$$

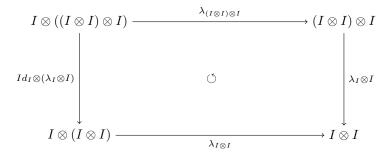
$$\circ \lambda_{(I \otimes I) \otimes I}$$

$$\circ \alpha_{I,I \otimes I,I} \circ (\alpha_{I,I,I} \otimes Id_{I})$$

$$\circ ((\lambda_{I}^{-1} \otimes Id_{I}) \otimes Id_{I})$$

$$\circ (\lambda_{I}^{-1} \otimes Id_{I})$$

λ が自然変換であることから以下の図式が可換。



従って以下の等式が成り立つ

$$\lambda_{I\otimes I}\circ (Id_I\otimes (\lambda_I\otimes I))=(\lambda_I\otimes I)\circ \lambda_{(I\otimes I)\otimes I}$$

$$= \lambda_{I}$$

$$\circ (\lambda_{I} \otimes Id_{I})$$

$$\circ (\rho_{I}^{-1} \otimes Id_{I}) \circ \rho_{I}^{-1} \circ \rho_{I} \circ (\rho_{I} \otimes Id_{I})$$

$$\circ \alpha_{I,I,I}^{-1} \circ (Id_{I} \otimes \lambda_{I}^{-1}) \circ (\lambda_{I} \otimes Id_{I}) \circ \alpha_{I,I,I}^{-1}$$

$$\circ \alpha_{I,I,I}$$

$$\circ (\lambda_{I}^{-1} \otimes I) \circ \lambda_{I}^{-1} \circ \lambda_{I}$$

$$\circ (\lambda_{I} \otimes I) \circ \lambda_{(I \otimes I) \otimes I}$$

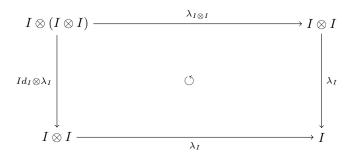
$$\circ \lambda_{I \otimes I} \circ (Id_{I} \otimes (\lambda_{I} \otimes I))$$

$$\circ \alpha_{I,I \otimes I,I} \circ (\alpha_{I,I,I} \otimes Id_{I})$$

$$\circ ((\lambda_{I}^{-1} \otimes Id_{I}) \otimes Id_{I})$$

$$\circ (\lambda_{I}^{-1} \otimes Id_{I})$$

λ が自然変換であることから以下の図式が可換。



従って以下の等式が成り立つ

$$\lambda_I \circ (Id_I \otimes \lambda_I) = \lambda_I \circ \lambda_{I \otimes I}$$

$$= \lambda_{I}$$

$$\circ (\lambda_{I} \otimes Id_{I})$$

$$\circ (\rho_{I}^{-1} \otimes Id_{I}) \circ \rho_{I}^{-1} \circ \rho_{I} \circ (\rho_{I} \otimes Id_{I})$$

$$\circ \alpha_{I,I,I}^{-1} \circ (Id_{I} \otimes \lambda_{I}^{-1}) \circ (\lambda_{I} \otimes Id_{I}) \circ \alpha_{I,I,I}^{-1}$$

$$\circ (\lambda_{I}^{-1} \otimes I) \circ \lambda_{I}^{-1}$$

$$\circ \lambda_{I} \circ \lambda_{I} \otimes I$$

$$\circ \lambda_{I} \circ (Id_{I} \otimes \lambda_{I})$$

$$\circ (Id_{I} \otimes (\lambda_{I} \otimes I))$$

$$\circ \alpha_{I,I \otimes I,I} \circ (\alpha_{I,I,I} \otimes Id_{I})$$

$$\circ (\lambda_{I}^{-1} \otimes Id_{I}) \otimes Id_{I})$$

$$\circ (\lambda_{I}^{-1} \otimes Id_{I})$$

以下は恒等射なので削除

$$\lambda_I^{-1} \circ \lambda_I$$

$$= \lambda_{I}$$

$$\circ (\lambda_{I} \otimes Id_{I})$$

$$\circ (\rho_{I}^{-1} \otimes Id_{I}) \circ \rho_{I}^{-1} \circ \rho_{I} \circ (\rho_{I} \otimes Id_{I})$$

$$\circ \alpha_{I,I,I}^{-1} \circ (Id_{I} \otimes \lambda_{I}^{-1}) \circ (\lambda_{I} \otimes Id_{I}) \circ \alpha_{I,I,I}^{-1}$$

$$\circ (\lambda_{I}^{-1} \otimes I)$$

$$\circ \frac{\lambda_{I}^{-1} \circ \lambda_{I}}{\lambda_{I}^{-1} \circ \lambda_{I}^{-1}}$$

$$\circ (Id_{I} \otimes (\lambda_{I} \otimes I))$$

$$\circ \alpha_{I,I \otimes I,I} \circ (\alpha_{I,I,I} \otimes Id_{I})$$

$$\circ ((\lambda_{I}^{-1} \otimes Id_{I}) \otimes Id_{I})$$

$$\circ (\lambda_{I}^{-1} \otimes Id_{I})$$

$$(Id_I \otimes \lambda_I^{-1}) \circ (\lambda_I \otimes Id_I) \circ \alpha_{I,I,I}^{-1} \circ \alpha_{I,I,I} \circ (\lambda_I^{-1} \otimes Id_I) \circ (Id_I \otimes \lambda_I)$$

$$= \lambda_{I}$$

$$\circ (\lambda_{I} \otimes Id_{I})$$

$$\circ (\rho_{I}^{-1} \otimes Id_{I}) \circ \rho_{I}^{-1} \circ \rho_{I} \circ (\rho_{I} \otimes Id_{I})$$

$$\circ \alpha_{I,I,I}^{-1}$$

$$\circ (Id_{I} \otimes \lambda_{I}^{-1}) \circ (\lambda_{I} \otimes Id_{I}) \circ \alpha_{I,I,I}^{-1} \circ \alpha_{I,I,I} \circ (\lambda_{I}^{-1} \otimes Id_{I}) \circ (Id_{I} \otimes \lambda_{I})$$

$$\circ (Id_{I} \otimes (\lambda_{I} \otimes Id_{I}))$$

$$\circ \alpha_{I,I \otimes I,I} \circ (\alpha_{I,I,I} \otimes Id_{I})$$

$$\circ ((\lambda_{I}^{-1} \otimes Id_{I}) \otimes Id_{I})$$

$$\circ (\lambda_{I}^{-1} \otimes Id_{I})$$

α が自然変換であることから以下の図式が可換。

$$(I \otimes (I \otimes I)) \otimes I \xrightarrow{\alpha_{I,I \otimes I,I}} I \otimes ((I \otimes I) \otimes I)$$

$$(Id_{I} \otimes \lambda_{I}) \otimes Id_{I} \qquad \circlearrowleft \qquad \downarrow Id_{I} \otimes (\lambda_{I} \otimes Id_{I})$$

$$(I \otimes I) \otimes I \xrightarrow{\alpha_{I,I,I}} I \otimes (I \otimes I)$$

従って以下の等式が成り立つ

$$\alpha_{I,I,I} \circ ((Id_I \otimes \lambda_I) \otimes Id_I) = (Id_I \otimes (\lambda_I \otimes Id_I)) \circ \alpha_{I,I \otimes I,I}$$

$$= \lambda_I$$

$$\circ (\lambda_I \otimes Id_I)$$

$$\circ (\rho_I^{-1} \otimes Id_I) \circ \rho_I^{-1} \circ \rho_I \circ (\rho_I \otimes Id_I)$$

$$\circ \alpha_{I,I,I}^{-1}$$

$$\circ (Id_I \otimes (\lambda_I \otimes Id_I)) \circ \alpha_{I,I \otimes I,I}$$

$$\circ \alpha_{I,I,I} \circ ((Id_I \otimes \lambda_I) \otimes Id_I)$$

$$\circ (\alpha_{I,I,I} \otimes Id_I)$$

$$\circ ((\lambda_I^{-1} \otimes Id_I) \otimes Id_I)$$

以下は恒等射なので削除

 $\circ (\lambda_I^{-1} \otimes Id_I)$

$$\alpha_{I,I,I}^{-1} \circ \alpha_{I,I,I}$$

$$= \lambda_{I}$$

$$\circ (\lambda_{I} \otimes Id_{I})$$

$$\circ (\rho_{I}^{-1} \otimes Id_{I}) \circ \rho_{I}^{-1} \circ \rho_{I} \circ (\rho_{I} \otimes Id_{I})$$

$$\circ \frac{\alpha_{I,I,I}^{-1} \circ \alpha_{I,I,I}}{(Id_{I} \otimes \lambda_{I}) \otimes Id_{I})}$$

$$\circ (\alpha_{I,I,I} \otimes Id_{I})$$

$$\circ ((\lambda_{I}^{-1} \otimes Id_{I}) \otimes Id_{I})$$

$$\circ (\lambda_{I}^{-1} \otimes Id_{I})$$

三角等式より $((I\otimes I)\otimes I)\otimes I \xrightarrow{\alpha_{I,I,I}\otimes Id_I} (I\otimes (I\otimes I))\otimes I$ $(\rho_I\otimes Id_I)\otimes Id_I \qquad \circlearrowleft \qquad (Id_I\otimes \lambda_I)\otimes Id_I$

従って以下の等式が成り立つ

 $(I \otimes I) \otimes I$

$$(\rho_I \otimes Id_I) \otimes Id_I = ((Id_I \otimes \lambda_I) \otimes Id_I) \circ (\alpha_{I,I,I} \otimes Id_I)$$

$$=\lambda_I$$

$$\circ (\lambda_I \otimes Id_I)$$

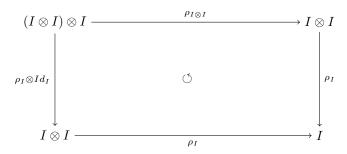
$$\circ (\rho_I^{-1} \otimes Id_I) \circ \rho_I^{-1} \circ \rho_I \circ (\rho_I \otimes Id_I)$$

$$\circ \ ((Id_{I} \otimes \lambda_{I}) \otimes Id_{I}) \circ (\alpha_{I,I,I} \otimes Id_{I})$$

$$\circ ((\rho_I \otimes Id_I) \otimes Id_I)$$

$$\circ ((\lambda_I^{-1} \otimes Id_I) \otimes Id_I)$$

$$\circ \left(\lambda_I^{-1} \otimes Id_I \right)$$



従って以下の等式が成り立つ

$$\rho_I \circ \rho_{I \otimes I} = \rho_I \circ (\rho_I \otimes Id_I)$$

$$=\lambda_I$$

$$\circ (\lambda_I \otimes Id_I)$$

$$\circ \left(\rho_I^{-1} \otimes Id_I \right) \circ \rho_I^{-1}$$

$$\circ \ \rho_{I} \circ (\rho_{I} \otimes Id_{I})$$

$$\circ\,\rho_I\circ\rho_{I\otimes I}$$

$$\circ \left(\left(\rho_I \otimes Id_I \right) \otimes Id_I \right)$$

$$\circ ((\lambda_I^{-1} \otimes Id_I) \otimes Id_I)$$

$$\circ (\lambda_I^{-1} \otimes Id_I)$$

$$((I \otimes I) \otimes I) \otimes I \xrightarrow{\rho_{(I \otimes I) \otimes I}} (I \otimes I) \otimes I$$

$$\downarrow^{(\rho_{I} \otimes Id_{I}) \otimes Id_{I}} \qquad \downarrow^{\rho_{I} \otimes Id_{I}}$$

$$(I \otimes I) \otimes I \xrightarrow{\rho_{I \otimes I}} I \otimes I$$

従って以下の等式が成り立つ

$$(\rho_{I} \otimes Id_{I}) \circ \rho_{(I \otimes I) \otimes I} = \rho_{I \otimes I} \circ ((\rho_{I} \otimes Id_{I}) \otimes Id_{I})$$

$$= \lambda_{I}$$

$$\circ (\lambda_{I} \otimes Id_{I})$$

$$\circ (\rho_{I}^{-1} \otimes Id_{I})$$

$$\circ \rho_{I}^{-1}$$

$$\circ \rho_{I}$$

$$\circ \rho_{I} \otimes Id_{I}) \circ \rho_{(I \otimes Id_{I}) \otimes Id_{I}}$$

$$\circ (\rho_{I} \otimes Id_{I}) \circ \rho_{(I \otimes I) \otimes I}$$

$$\circ ((\lambda_{I}^{-1} \otimes Id_{I}) \otimes Id_{I})$$

$$\circ (\lambda_{I}^{-1} \otimes Id_{I})$$

$$(\rho_{I}^{-1} \otimes Id_{I}) \circ \rho_{I}^{-1} \circ \rho_{I} \circ (\rho_{I} \otimes Id_{I})$$

$$= \lambda_{I}$$

$$\circ (\lambda_{I} \otimes Id_{I})$$

$$\circ (\rho_{I}^{-1} \otimes Id_{I}) \circ \rho_{I}^{-1} \circ \rho_{I} \circ (\rho_{I} \otimes Id_{I})$$

$$\circ \rho_{(I \otimes I) \otimes I}$$

$$\circ ((\lambda_{I}^{-1} \otimes Id_{I}) \otimes Id_{I})$$

$$\circ (\lambda_{I}^{-1} \otimes Id_{I})$$

$$((I \otimes I) \otimes I) \otimes I \xrightarrow{\rho_{(I \otimes I) \otimes I}} (I \otimes I) \otimes I$$

$$(\lambda_{I} \otimes Id_{I}) \otimes Id_{I} \qquad (I \otimes I) \otimes I \xrightarrow{\rho_{I \otimes I}} I \otimes I$$

従って以下の等式が成り立つ

$$\rho_{I\otimes I} \circ ((\lambda_I \otimes Id_I) \otimes Id_I) = (\lambda_I \otimes Id_I) \circ \rho_{(I\otimes I)\otimes I}$$

$$= \lambda_I$$

$$\circ \frac{(\lambda_I \otimes Id_I) \circ \rho_{(I\otimes I)\otimes I}}{\circ \rho_{I\otimes I} \circ ((\lambda_I \otimes Id_I) \otimes Id_I)}$$

$$\circ ((\lambda_I^{-1} \otimes Id_I) \otimes Id_I)$$

$$\circ (\lambda_I^{-1} \otimes Id_I)$$

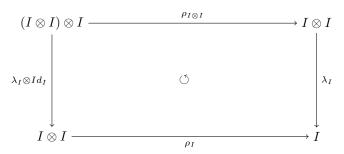
$$((\lambda_I \otimes Id_I) \otimes Id_I) \circ ((\lambda_I^{-1} \otimes Id_I) \otimes Id_I)$$

$$= \lambda_I$$

$$\circ \rho_{I \otimes I}$$

$$\circ \ ((\lambda_I \otimes Id_I) \otimes Id_I) \circ ((\lambda_I^{-1} \otimes Id_I) \otimes Id_I)$$

$$\circ (\lambda_I^{-1} \otimes Id_I)$$



従って以下の等式が成り立つ

$$\rho_{I} \circ (\lambda_{I} \otimes Id_{I}) = \lambda_{I} \circ \rho_{I \otimes I}$$

$$= \frac{\lambda_{I} \circ \rho_{I \otimes I}}{\rho_{I} \circ (\lambda_{I} \otimes Id_{I})}$$

$$\circ (\lambda_{I}^{-1} \otimes Id_{I})$$

以下は恒等射なので削除

$$(\lambda_I \otimes Id_I) \circ (\lambda_I^{-1} \otimes Id_I)$$

$$= \rho_I \circ (\lambda_I \otimes Id_I) \circ (\lambda_I^{-1} \otimes Id_I)$$
$$= \rho_I (右単位律子)$$