モノイダル圏における左右単位律子の一致の証明

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1　モノイダル圏

以下の５つの物 (テンソル積, 単位対象, 結合律子, 左単位律子, 右単位律子) が存在し、下記の二つの等式 (三角等式, 五角等式) を満たす圏をモノイダル圏と言う。

- テンソル積 $\otimes$
  関手 $\otimes : C \times C \to C$
- 単位対象 I
  $I \in Ob(C)$
- 結合律子 α
  自然同型 α
  $(A \otimes B) \otimes C \xrightarrow{\alpha_{A,B,C}} A \otimes (B \otimes C)$
- 左単位律子 λ
  自然同型 λ
  $I \otimes A \xrightarrow{\lambda_A} A$
- 右単位律子 ρ
  自然同型 ρ
  $A \otimes I \xrightarrow{\rho_A} A$
- 三角等式
  任意の A,B に対し以下の図式が可換

$$\begin{tikzcd}
(A \otimes I) \otimes B \arrow[rr, "\alpha_{A,I,B}"] \arrow[dr, "\rho_A \otimes Id_B"'] & & A \otimes (I \otimes B) \arrow[dl, "Id_A \otimes \lambda_B"] \\
& A \otimes B \arrow[u, phantom, "\circlearrowleft"] &
\end{tikzcd}$$

従って以下の等式が成り立つ

$$\rho_A \otimes Id_B = (Id_A \otimes \lambda_B) \circ (\alpha_{A,I,B})$$

- 五角等式
  任意の A,B,C,D に対し以下の図式が可換

$$\begin{tikzcd}
& (A \otimes (B \otimes C)) \otimes D \arrow[r, "\alpha_{A,B\otimes C,D}"] & A \otimes ((B \otimes C) \otimes D) \arrow[dr, "Id_A \otimes \alpha_{B,C,D}"] & \\
((A \otimes B) \otimes C) \otimes D \arrow[ur, "\alpha_{A,B,C} \otimes Id_D"] \arrow[drr, "\alpha_{A\otimes B,C,D}"'] & & \circlearrowleft & A \otimes (B \otimes (C \otimes D)) \\
& & (A \otimes B) \otimes (C \otimes D) \arrow[ur, "\alpha_{A,B,C\otimes D}"'] &
\end{tikzcd}$$

従って以下の等式が成り立つ

$$(Id_A \otimes \alpha_{B,C,D}) \circ \alpha_{A,B\otimes C,D} \circ (\alpha_{A,B,C} \otimes Id_D) = \alpha_{A,B,C\otimes D} \circ \alpha_{A\otimes B,C,D}$$

2　証明

Proof.

以下、λ_I(左単位律子) に恒等射を入れて行きます。

$$
\begin{aligned}
\lambda_I = {} & \lambda_I \\
& \circ (\lambda_I \otimes Id_I) \circ (\lambda_I^{-1} \otimes Id_I) \\
& = \lambda_I \\
& \circ (\lambda_I \otimes Id_I) \\
& \circ (\rho_I^{-1} \otimes Id_I) \circ (\rho_I \otimes Id_I) \\
& \circ (\lambda_I^{-1} \otimes Id_I) \\
& = \lambda_I \\
& \circ (\lambda_I \otimes Id_I) \\
& \circ (\rho_I^{-1} \otimes Id_I) \\
& \circ \rho_I^{-1} \circ \rho_I \\
& \circ (\rho_I \otimes Id_I) \\
& \circ (\lambda_I^{-1} \otimes Id_I) \\
& = \lambda_I \\
& \circ (\lambda_I \otimes Id_I) \\
& \circ (\rho_I^{-1} \otimes Id_I) \circ \rho_I^{-1} \circ \rho_I \circ (\rho_I \otimes Id_I) \\
& \circ \alpha_{I,I,I}^{-1} \circ \alpha_{I,I,I} \\
& \circ (\lambda_I^{-1} \otimes Id_I) \\
& = \lambda_I \\
& \circ (\lambda_I \otimes Id_I) \\
& \circ (\rho_I^{-1} \otimes Id_I) \circ \rho_I^{-1} \circ \rho_I \circ (\rho_I \otimes Id_I) \\
& \circ \alpha_{I,I,I}^{-1} \\
& \circ (Id_I \otimes \lambda_I^{-1}) \circ (Id_I \otimes \lambda_I) \\
& \circ \alpha_{I,I,I} \\
& \circ (\lambda_I^{-1} \otimes Id_I)
\end{aligned}
$$

$$
\begin{aligned}
&= \lambda_I \\
&\circ (\lambda_I \otimes Id_I) \\
&\circ (\rho_I^{-1} \otimes Id_I) \circ \rho_I^{-1} \circ \rho_I \circ (\rho_I \otimes Id_I) \\
&\circ \alpha_{I,I,I}^{-1} \circ (Id_I \otimes \lambda_I^{-1}) \\
&\circ (\lambda_I \otimes Id_I) \circ (\lambda_I^{-1} \otimes Id_I) \circ (Id_I \otimes \lambda_I) \circ \alpha_{I,I,I} \\
&\circ (\lambda_I^{-1} \otimes Id_I) \\
&= \lambda_I \\
&\circ (\lambda_I \otimes Id_I) \\
&\circ (\rho_I^{-1} \otimes Id_I) \circ \rho_I^{-1} \circ \rho_I \circ (\rho_I \otimes Id_I) \\
&\circ \alpha_{I,I,I}^{-1} \circ (Id_I \otimes \lambda_I^{-1}) \circ (\lambda_I \otimes Id_I) \\
&\circ \alpha_{I,I,I}^{-1} \circ \alpha_{I,I,I} \\
&\circ (\lambda_I^{-1} \otimes Id_I) \circ (Id_I \otimes \lambda_I) \circ \alpha_{I,I,I} \\
&\circ (\lambda_I^{-1} \otimes Id_I) \\
&= \lambda_I \\
&\circ (\lambda_I \otimes Id_I) \\
&\circ (\rho_I^{-1} \otimes Id_I) \circ \rho_I^{-1} \circ \rho_I \circ (\rho_I \otimes Id_I) \\
&\circ \alpha_{I,I,I}^{-1} \circ (Id_I \otimes \lambda_I^{-1}) \circ (\lambda_I \otimes Id_I) \circ \alpha_{I,I,I}^{-1} \\
&\circ \lambda_{I\otimes I}^{-1} \circ \lambda_{I\otimes I} \\
&\circ \alpha_{I,I,I} \circ (\lambda_I^{-1} \otimes Id_I) \circ (Id_I \otimes \lambda_I) \circ \alpha_{I,I,I} \\
&\circ (\lambda_I^{-1} \otimes Id_I)
\end{aligned}
$$

$$
\begin{aligned}
&= \lambda_I \\
&\circ (\lambda_I \otimes Id_I) \\
&\circ (\rho_I^{-1} \otimes Id_I) \circ \rho_I^{-1} \circ \rho_I \circ (\rho_I \otimes Id_I) \\
&\circ \alpha_{I,I,I}^{-1} \circ (Id_I \otimes \lambda_I^{-1}) \\
&\circ (\lambda_I \otimes Id_I) \circ \alpha_{I,I,I}^{-1} \circ \lambda_{I\otimes I}^{-1} \\
&\circ \lambda_I^{-1} \circ \lambda_I \\
&\circ \lambda_{I\otimes I} \circ \alpha_{I,I,I} \circ (\lambda_I^{-1} \otimes Id_I) \circ (Id_I \otimes \lambda_I) \circ \alpha_{I,I,I} \\
&\circ (\lambda_I^{-1} \otimes Id_I) \\
&= \lambda_I \\
&\circ (\lambda_I \otimes Id_I) \\
&\circ (\rho_I^{-1} \otimes Id_I) \circ \rho_I^{-1} \circ \rho_I \circ (\rho_I \otimes Id_I) \\
&\circ \alpha_{I,I,I}^{-1} \circ (Id_I \otimes \lambda_I^{-1}) \circ (\lambda_I \otimes Id_I) \\
&\circ \alpha_{I,I,I}^{-1} \circ \lambda_{I\otimes I}^{-1} \circ \lambda_I^{-1} \circ \lambda_I \circ \lambda_{I\otimes I} \circ \alpha_{I,I,I} \circ (\lambda_I^{-1} \otimes Id_I) \circ (Id_I \otimes \lambda_I) \circ \alpha_{I,I,I} \\
&\circ ((\lambda_I \otimes Id_I) \otimes Id_I) \circ ((\lambda_I^{-1} \otimes Id_I) \otimes Id_I) \\
&\circ (\lambda_I^{-1} \otimes Id_I)
\end{aligned}
$$

α が自然変換であることから以下の図式が可換。

$$\begin{tikzcd}
((I \otimes I) \otimes I) \otimes I \arrow[r, "\alpha_{I \otimes I, I, I}"] \arrow[d, "(\lambda_I \otimes Id_I) \otimes Id_I"'] \arrow[dr, phantom, "\circlearrowleft"] & (I \otimes I) \otimes (I \otimes I) \arrow[d, "\lambda_I \otimes (Id_I \otimes Id_I)"] \\
(I \otimes I) \otimes I \arrow[r, "\alpha_{I,I,I}"'] & I \otimes (I \otimes I)
\end{tikzcd}$$

従って以下の等式が成り立つ

$$\lambda_I \otimes (Id_I \otimes Id_I) \circ \alpha_{I,I,I \otimes I} = \alpha_{I,I,I} \circ (\lambda_I \otimes Id_I) \otimes Id_I$$

$= \lambda_I$

$\circ\, (\lambda_I \otimes Id_I)$

$\circ\, (\rho_I^{-1} \otimes Id_I) \circ \rho_I^{-1} \circ \rho_I \circ (\rho_I \otimes Id_I)$

$\circ\, \alpha_{I,I,I}^{-1} \circ (Id_I \otimes \lambda_I^{-1}) \circ (\lambda_I \otimes Id_I) \circ \alpha_{I,I,I}^{-1} \circ \lambda_{I \otimes I}^{-1} \circ \lambda_I^{-1}$

$\circ\, \lambda_I \circ \lambda_{I \otimes I} \circ \alpha_{I,I,I} \circ (\lambda_I^{-1} \otimes Id_I) \circ (Id_I \otimes \lambda_I)$

$\circ\ \cancel{\alpha_{I,I,I} \circ (\lambda_I \otimes Id_I) \otimes Id_I}$

$\circ\, (\lambda_I \otimes (Id_I \otimes Id_I)) \circ \alpha_{I,I,I \otimes I}$

$\circ\, ((\lambda_I^{-1} \otimes Id_I) \otimes Id_I)$

$\circ\, (\lambda_I^{-1} \otimes Id_I)$

以下の等式が成り立つ

※交換則 $(a\otimes b)\circ(c\otimes d)=(a\circ c)\otimes(b\circ d)$ を使用

$$(\lambda_I\otimes Id_I)\circ((Id_I\otimes Id_I)\otimes\lambda_I)=\lambda_I\otimes\lambda_I=(Id_I\otimes\lambda_I)\circ(\lambda_I\otimes(Id_I\otimes Id_I))$$

$$\begin{tikzcd}
& (I\otimes I)\otimes I \arrow[dr, "\lambda_I\otimes Id_I"] \arrow[d, phantom, "\circlearrowleft"] & \\
(I\otimes I)\otimes(I\otimes I) \arrow[ur, "(Id_I\otimes Id_I)\otimes\lambda_I"] \arrow[rr, "\lambda_I\otimes\lambda_I"] \arrow[dr, "\lambda_I\otimes(Id_I\otimes Id_I)"'] & & I\otimes I \\
& I\otimes(I\otimes I) \arrow[ur, "Id_I\otimes\lambda_I"'] \arrow[u, phantom, "\circlearrowleft"] &
\end{tikzcd}$$

$$\begin{aligned}
&=\lambda_I\\
&\circ(\lambda_I\otimes Id_I)\\
&\circ(\rho_I^{-1}\otimes Id_I)\circ\rho_I^{-1}\circ\rho_I\circ(\rho_I\otimes Id_I)\\
&\circ\alpha_{I,I,I}^{-1}\circ(Id_I\otimes\lambda_I^{-1})\circ(\lambda_I\otimes Id_I)\circ\alpha_{I,I,I}^{-1}\circ\lambda_{I\otimes I}^{-1}\\
&\circ\lambda_I^{-1}\circ\lambda_I\circ\lambda_{I\otimes I}\circ\alpha_{I,I,I}\circ(\lambda_I^{-1}\otimes Id_I)\\
&\circ\ \cancel{(Id_I\otimes\lambda_I)\circ(\lambda_I\otimes(Id_I\otimes Id_I))}\\
&\circ(\lambda_I\otimes Id_I)\circ((Id_I\otimes Id_I)\otimes\lambda_I)\\
&\circ\alpha_{I,I,I\otimes I}\\
&\circ((\lambda_I^{-1}\otimes Id_I)\otimes Id_I)\\
&\circ(\lambda_I^{-1}\otimes Id_I)
\end{aligned}$$

以下は恒等射なので削除

$$(\lambda_I^{-1} \otimes Id_I) \circ (\lambda_I \otimes Id_I)$$

$$\begin{tikzcd}
(I \otimes I) \otimes I \arrow[r, bend left, "\lambda_I \otimes Id_I"] & I \otimes I \arrow[l, bend left, "\lambda_I^{-1} \otimes Id_I"]
\end{tikzcd}$$

$= \lambda_I$

$\circ\, (\lambda_I \otimes Id_I)$

$\circ\, (\rho_I^{-1} \otimes Id_I) \circ \rho_I^{-1} \circ \rho_I \circ (\rho_I \otimes Id_I)$

$\circ\, \alpha_{I,I,I}^{-1} \circ (Id_I \otimes \lambda_I^{-1}) \circ (\lambda_I \otimes Id_I) \circ \alpha_{I,I,I}^{-1} \circ \lambda_{I \otimes I}^{-1}$

$\circ\, \lambda_I^{-1} \circ \lambda_I \circ \lambda_{I \otimes I} \circ \alpha_{I,I,I}$

$\circ\, \cancel{(\lambda_I^{-1} \otimes Id_I) \circ (\lambda_I \otimes Id_I)}$

$\circ\, ((Id_I \otimes Id_I) \otimes \lambda_I)$

$\circ\, \alpha_{I,I,I \otimes I}$

$\circ\, ((\lambda_I^{-1} \otimes Id_I) \otimes Id_I)$

$\circ\, (\lambda_I^{-1} \otimes Id_I)$

α が自然変換であることから以下の図式が可換。

$$
\begin{tikzcd}
(I \otimes I) \otimes (I \otimes I) \arrow[r, "\alpha_{I,I,I\otimes I}"] \arrow[d, "(Id_I \otimes Id_I) \otimes \lambda_I"'] \arrow[dr, phantom, "\circlearrowleft"] & I \otimes (I \otimes (I \otimes I)) \arrow[d, "Id_I \otimes (Id_I \otimes \lambda_I)"] \\
(I \otimes I) \otimes I \arrow[r, "\alpha_{I,I,I}"'] & I \otimes (I \otimes I)
\end{tikzcd}
$$

従って以下の等式が成り立つ

$$(Id_I \otimes (Id_I \otimes \lambda_I)) \circ \alpha_{I,I,I\otimes I} = \alpha_{I,I,I} \circ ((Id_I \otimes Id_I) \otimes \lambda_I)$$

$= \lambda_I$

$\circ (\lambda_I \otimes Id_I)$

$\circ (\rho_I^{-1} \otimes Id_I) \circ \rho_I^{-1} \circ \rho_I \circ (\rho_I \otimes Id_I)$

$\circ \alpha^{-1}_{I,I,I} \circ (Id_I \otimes \lambda_I^{-1}) \circ (\lambda_I \otimes Id_I) \circ \alpha^{-1}_{I,I,I} \circ \lambda^{-1}_{I\otimes I} \circ \lambda_I^{-1} \circ \lambda_I \circ \lambda_{I\otimes I}$

$\circ \cancel{\alpha_{I,I,I} \circ ((Id_I \otimes Id_I) \otimes \lambda_I)}$

$\circ (Id_I \otimes (Id_I \otimes \lambda_I)) \circ \alpha_{I,I,I\otimes I}$

$\circ \alpha_{I,I,I\otimes I}$

$\circ ((\lambda_I^{-1} \otimes Id_I) \otimes Id_I)$

$\circ (\lambda_I^{-1} \otimes Id_I)$

五角等式より以下の図式が可換と言える。

$$
\begin{tikzcd}
 & (I \otimes (I \otimes I)) \otimes I \arrow[rr, "\alpha_{I,I\otimes I,I}"] & & I \otimes ((I \otimes I) \otimes I) \arrow[dr, "Id_I \otimes \alpha_{I,I,I}"] & \\
((I \otimes I) \otimes I) \otimes I \arrow[ur, "\alpha_{I,I,I} \otimes Id_I"] \arrow[drr, "\alpha_{I\otimes I,I,I}"'] & & \circlearrowleft & & I \otimes (I \otimes (I \otimes I)) \\
 & & (I \otimes I) \otimes (I \otimes I) \arrow[urr, "\alpha_{I,I,I\otimes I}"'] & &
\end{tikzcd}
$$

従って以下の等式が成り立つ

$$(Id_I \otimes \alpha_{I,I,I}) \circ \alpha_{I,I\otimes I,I} \circ (\alpha_{I,I,I} \otimes Id_I) = \alpha_{I,I,I\otimes I} \circ \alpha_{I\otimes I,I,I}$$

$= \lambda_I$

$\circ\, (\lambda_I \otimes Id_I)$

$\circ\, (\rho_I^{-1} \otimes Id_I) \circ \rho_I^{-1} \circ \rho_I \circ (\rho_I \otimes Id_I)$

$\circ\, \alpha_{I,I,I}^{-1} \circ (Id_I \otimes \lambda_I^{-1}) \circ (\lambda_I \otimes Id_I) \circ \alpha_{I,I,I}^{-1} \circ \lambda_{I\otimes I}^{-1} \circ \lambda_I^{-1} \circ \lambda_I \circ \lambda_{I\otimes I}$

$\circ\, (Id_I \otimes (Id_I \otimes \lambda_I))$

$\circ\, \cancel{\alpha_{I,I,I\otimes I} \circ \alpha_{I,I,I\otimes I}}$

$\circ\, (Id_I \otimes \alpha_{I,I,I}) \circ \alpha_{I,I\otimes I,I} \circ (\alpha_{I,I,I} \otimes Id_I)$

$\circ\, ((\lambda_I^{-1} \otimes Id_I) \otimes Id_I)$

$\circ\, (\lambda_I^{-1} \otimes Id_I)$

λ が自然変換であることから以下の図式が可換。

$$\begin{tikzcd}
I \otimes (I \otimes (I \otimes I)) \arrow[r, "\lambda_{I\otimes(I\otimes I)}"] \arrow[d, "Id_I\otimes(Id_I\otimes\lambda_I)"'] \arrow[dr, phantom, "\circlearrowleft"] & I \otimes (I \otimes I) \arrow[d, "Id_I\otimes\lambda_I"] \\
I \otimes (I \otimes I) \arrow[r, "\lambda_{I\otimes I}"'] & I \otimes I
\end{tikzcd}$$

従って以下の等式が成り立つ

$$(Id_I \otimes \lambda_I) \circ \lambda_{I\otimes(I\otimes I)} = \lambda_{I\otimes I} \circ (Id_I \otimes (Id_I \otimes \lambda_I))$$

$= \lambda_I$

$\circ (\lambda_I \otimes Id_I)$

$\circ (\rho_I^{-1} \otimes Id_I) \circ \rho_I^{-1} \circ \rho_I \circ (\rho_I \otimes Id_I)$

$\circ \alpha_{I,I,I}^{-1} \circ (Id_I \otimes \lambda_I^{-1}) \circ (\lambda_I \otimes Id_I) \circ \alpha_{I,I,I}^{-1} \circ \lambda_{I\otimes I}^{-1} \circ \lambda_I^{-1} \circ \lambda_I$

$\circ\ \cancel{\lambda_{I\otimes I} \circ (Id_I \otimes (Id_I \otimes \lambda_I))}$

$\circ (Id_I \otimes \lambda_I) \circ \lambda_{I\otimes(I\otimes I)}$

$\circ (Id_I \otimes \alpha_{I,I,I}) \circ \alpha_{I,I\otimes I,I} \circ (\alpha_{I,I,I} \otimes Id_I)$

$\circ ((\lambda_I^{-1} \otimes Id_I) \otimes Id_I)$

$\circ (\lambda_I^{-1} \otimes Id_I)$

λ が自然変換であることから以下の図式が可換。

$$
\begin{tikzcd}
I \otimes (I \otimes I) \arrow[rr, "\lambda_{I \otimes I}"] \arrow[d, "Id_I \otimes \lambda_I"'] \arrow[drr, phantom, "\circlearrowleft"] & & I \otimes I \arrow[d, "\lambda_I"] \\
I \otimes I \arrow[rr, "\lambda_I"'] & & I
\end{tikzcd}
$$

従って以下の等式が成り立つ

$$\lambda_I \circ \lambda_{I \otimes I} = \lambda_I \circ (Id_I \otimes \lambda_I)$$

$= \lambda_I$

$\circ\, (\lambda_I \otimes Id_I)$

$\circ\, (\rho_I^{-1} \otimes Id_I) \circ \rho_I^{-1} \circ \rho_I \circ (\rho_I \otimes Id_I)$

$\circ\, \alpha_{I,I,I}^{-1} \circ (Id_I \otimes \lambda_I^{-1}) \circ (\lambda_I \otimes Id_I) \circ \alpha_{I,I,I}^{-1} \circ \lambda_{I \otimes I}^{-1} \circ \lambda_I^{-1}$

$\circ\, \cancel{\lambda_I \circ (Id_I \otimes \lambda_I)}$

$\circ\, \lambda_I \circ \lambda_{I \otimes I}$

$\circ\, \lambda_{I \otimes (I \otimes I)}$

$\circ\, (Id_I \otimes \alpha_{I,I,I}) \circ \alpha_{I,I \otimes I,I} \circ (\alpha_{I,I,I} \otimes Id_I)$

$\circ\, ((\lambda_I^{-1} \otimes Id_I) \otimes Id_I)$

$\circ\, (\lambda_I^{-1} \otimes Id_I)$

以下は恒等射なので削除

$$\lambda^{-1}_{I\otimes I} \circ \lambda_I^{-1} \circ \lambda_I \circ \lambda_{I\otimes I}$$

$$\begin{tikzcd}
I \otimes (I \otimes I) \arrow[r, bend left, "\lambda_{I\otimes I}"] & I \otimes I \arrow[l, bend left, "\lambda^{-1}_{I\otimes I}"] \arrow[r, bend left, "\lambda_I"] & I \arrow[l, bend left, "\lambda_I^{-1}"]
\end{tikzcd}$$

$$\begin{aligned}
&= \lambda_I \\
&\circ (\lambda_I \otimes Id_I) \\
&\circ (\rho_I^{-1} \otimes Id_I) \circ \rho_I^{-1} \circ \rho_I \circ (\rho_I \otimes Id_I) \\
&\circ \alpha^{-1}_{I,I,I} \circ (Id_I \otimes \lambda_I^{-1}) \circ (\lambda_I \otimes Id_I) \circ \alpha^{-1}_{I,I,I} \\
&\circ \cancel{\lambda^{-1}_{I\otimes I} \circ \lambda_I^{-1} \circ \lambda_I \circ \lambda_{I\otimes I}} \\
&\circ \lambda_{I\otimes(I\otimes I)} \\
&\circ (Id_I \otimes \alpha_{I,I,I}) \circ \alpha_{I,I\otimes I,I} \circ (\alpha_{I,I,I} \otimes Id_I) \\
&\circ ((\lambda_I^{-1} \otimes Id_I) \otimes Id_I) \\
&\circ (\lambda_I^{-1} \otimes Id_I)
\end{aligned}$$

λ が自然変換であることから以下の図式が可換。

$$\begin{tikzcd}
I \otimes ((I \otimes (I \otimes I) \arrow[rr, "\lambda_{(I\otimes I)\otimes I}"] \arrow[d, "Id_I \otimes \alpha_{I,I,I}"'] \arrow[drr, phantom, "\circlearrowleft"] & & (I \otimes I) \otimes I \arrow[d, "\alpha_{I,I,I}"] \\
I \otimes (I \otimes (I \otimes I)) \arrow[rr, "\lambda_{I\otimes(I\otimes I)}"'] & & I \otimes (I \otimes I)
\end{tikzcd}$$

従って以下の等式が成り立つ

$$\alpha_{I,I,I} \circ \lambda_{(I\otimes I)\otimes I} = \lambda_{I\otimes(I\otimes I)} \circ (Id_I \otimes \alpha_{I,I,I})$$

$$\begin{aligned}
&= \lambda_I \\
&\circ (\lambda_I \otimes Id_I) \\
&\circ (\rho_I^{-1} \otimes Id_I) \circ \rho_I^{-1} \circ \rho_I \circ (\rho_I \otimes Id_I) \\
&\circ \alpha_{I,I,I}^{-1} \circ (Id_I \otimes \lambda_I^{-1}) \circ (\lambda_I \otimes Id_I) \circ \alpha_{I,I,I}^{-1} \\
&\circ \ \cancel{\lambda_{I\otimes(I\otimes I)} \circ (Id_I \otimes \alpha_{I,I,I})} \\
&\circ \alpha_{I,I,I} \circ \lambda_{(I\otimes I)\otimes I} \\
&\circ \alpha_{I,I\otimes I,I} \circ (\alpha_{I,I,I} \otimes Id_I) \\
&\circ ((\lambda_I^{-1} \otimes Id_I) \otimes Id_I) \\
&\circ (\lambda_I^{-1} \otimes Id_I)
\end{aligned}$$

以下の恒等射を挿入

$$(\lambda_I^{-1} \otimes Id_I) \circ \lambda_I^{-1} \circ \lambda_I \circ (\lambda_I \otimes Id_I)$$

$$\begin{tikzcd}
(I \otimes I) \otimes I \arrow[r, bend left, "\lambda_I \otimes Id_I"] & I \otimes I \arrow[l, bend left, "\lambda_I^{-1} \otimes Id_I"] \arrow[r, bend left, "\lambda_I"] & I \arrow[l, bend left, "\lambda_I^{-1}"]
\end{tikzcd}$$

$= \lambda_I$

$\circ\, (\lambda_I \otimes Id_I)$

$\circ\, (\rho_I^{-1} \otimes Id_I) \circ \rho_I^{-1} \circ \rho_I \circ (\rho_I \otimes Id_I)$

$\circ\, \alpha_{I,I,I}^{-1} \circ (Id_I \otimes \lambda_I^{-1}) \circ (\lambda_I \otimes Id_I) \circ \alpha_{I,I,I}^{-1}$

$\circ\, \alpha_{I,I,I}$

$\circ\, (\lambda_I^{-1} \otimes Id_I) \circ \lambda_I^{-1} \circ \lambda_I \circ (\lambda_I \otimes Id_I)$

$\circ\, \lambda_{(I \otimes I) \otimes I}$

$\circ\, \alpha_{I,I \otimes I,I} \circ (\alpha_{I,I,I} \otimes Id_I)$

$\circ\, ((\lambda_I^{-1} \otimes Id_I) \otimes Id_I)$

$\circ\, (\lambda_I^{-1} \otimes Id_I)$

λ が自然変換であることから以下の図式が可換。

$$\begin{tikzcd}
I \otimes ((I \otimes I) \otimes I) \arrow[r, "\lambda_{(I\otimes I)\otimes I}"] \arrow[d, "Id_I \otimes (\lambda_I \otimes Id_I)"'] \arrow[dr, phantom, "\circlearrowleft"] & (I \otimes I) \otimes I \arrow[d, "\lambda_I \otimes Id_I"] \\
I \otimes (I \otimes I) \arrow[r, "\lambda_{I\otimes I}"'] & I \otimes I
\end{tikzcd}$$

従って以下の等式が成り立つ

$$\lambda_{I\otimes I} \circ (Id_I \otimes (\lambda_I \otimes Id_I)) = (\lambda_I \otimes Id_I) \circ \lambda_{(I\otimes I)\otimes I}$$

$$\begin{aligned}
&= \lambda_I \\
&\circ (\lambda_I \otimes Id_I) \\
&\circ (\rho_I^{-1} \otimes Id_I) \circ \rho_I^{-1} \circ \rho_I \circ (\rho_I \otimes Id_I) \\
&\circ \alpha_{I,I,I}^{-1} \circ (Id_I \otimes \lambda_I^{-1}) \circ (\lambda_I \otimes Id_I) \circ \alpha_{I,I,I}^{-1} \\
&\circ \alpha_{I,I,I} \\
&\circ (\lambda_I^{-1} \otimes Id_I) \circ \lambda_I^{-1} \circ \lambda_I \\
&\circ \cancel{(\lambda_I \otimes Id_I) \circ \lambda_{(I\otimes I)\otimes I}} \\
&\circ \lambda_{I\otimes I} \circ (Id_I \otimes (\lambda_I \otimes Id_I)) \\
&\circ \alpha_{I,I\otimes I,I} \circ (\alpha_{I,I,I} \otimes Id_I) \\
&\circ ((\lambda_I^{-1} \otimes Id_I) \otimes Id_I) \\
&\circ (\lambda_I^{-1} \otimes Id_I)
\end{aligned}$$

λ が自然変換であることから以下の図式が可換。

$$
\begin{tikzcd}
I \otimes (I \otimes I) \arrow[r, "\lambda_{I \otimes I}"] \arrow[d, "Id_I \otimes \lambda_I"'] \arrow[dr, phantom, "\circlearrowleft"] & I \otimes I \arrow[d, "\lambda_I"] \\
I \otimes I \arrow[r, "\lambda_I"'] & I
\end{tikzcd}
$$

従って以下の等式が成り立つ

$$\lambda_I \circ (Id_I \otimes \lambda_I) = \lambda_I \circ \lambda_{I \otimes I}$$

$= \lambda_I$

$\circ (\lambda_I \otimes Id_I)$

$\circ (\rho_I^{-1} \otimes Id_I) \circ \rho_I^{-1} \circ \rho_I \circ (\rho_I \otimes Id_I)$

$\circ \alpha_{I,I,I}^{-1} \circ (Id_I \otimes \lambda_I^{-1}) \circ (\lambda_I \otimes Id_I) \circ \alpha_{I,I,I}^{-1}$

$\circ \alpha_{I,I,I}$

$\circ (\lambda_I^{-1} \otimes I) \circ \lambda_I^{-1}$

$\circ \cancel{\lambda_I \circ \lambda_{I \otimes I}}$

$\circ \lambda_I \circ (Id_I \otimes \lambda_I)$

$\circ (Id_I \otimes (\lambda_I \otimes I))$

$\circ \alpha_{I,I \otimes I,I} \circ (\alpha_{I,I,I} \otimes Id_I)$

$\circ ((\lambda_I^{-1} \otimes Id_I) \otimes Id_I)$

$\circ (\lambda_I^{-1} \otimes Id_I)$

以下は恒等射なので削除

$$\lambda_I^{-1} \circ \lambda_I$$

$$\begin{tikzcd} I \otimes I \arrow[r, bend left, "\lambda_I"] & I \arrow[l, bend left, "\lambda_I^{-1}"] \end{tikzcd}$$

$= \lambda_I$
$\circ (\lambda_I \otimes Id_I)$
$\circ (\rho_I^{-1} \otimes Id_I) \circ \rho_I^{-1} \circ \rho_I \circ (\rho_I \otimes Id_I)$
$\circ \alpha_{I,I,I}^{-1} \circ (Id_I \otimes \lambda_I^{-1}) \circ (\lambda_I \otimes Id_I) \circ \alpha_{I,I,I}^{-1}$
$\circ \alpha_{I,I,I}$
$\circ (\lambda_I^{-1} \otimes I)$
$\circ \cancel{\lambda_I^{-1} \circ \lambda_I}$
$\circ (Id_I \otimes \lambda_I)$
$\circ (Id_I \otimes (\lambda_I \otimes I))$
$\circ \alpha_{I,I\otimes I,I} \circ (\alpha_{I,I,I} \otimes Id_I)$
$\circ ((\lambda_I^{-1} \otimes Id_I) \otimes Id_I)$
$\circ (\lambda_I^{-1} \otimes Id_I)$

以下は恒等射なので削除

$$(Id_I \otimes \lambda_I^{-1}) \circ (\lambda_I \otimes Id_I) \circ \alpha_{I,I,I}^{-1} \circ \alpha_{I,I,I} \circ (\lambda_I^{-1} \otimes Id_I) \circ (Id_I \otimes \lambda_I)$$

$$\begin{tikzcd}
I \otimes (I \otimes I) \arrow[r, bend left, "Id_I \otimes \lambda_I"] & I \otimes I \arrow[l, bend left, "Id_I \otimes \lambda_I^{-1}"] \arrow[r, bend left, "\lambda_I^{-1} \otimes Id_I"] & (I \otimes I) \otimes I \arrow[l, bend left, "\lambda_I \otimes Id_I"] \arrow[r, bend left, "\alpha_{I,I,I}"] & I \otimes (I \otimes I) \arrow[l, bend left, "\alpha_{I,I,I}^{-1}"]
\end{tikzcd}$$

$= \lambda_I$

$\circ\, (\lambda_I \otimes Id_I)$

$\circ\, (\rho_I^{-1} \otimes Id_I) \circ \rho_I^{-1} \circ \rho_I \circ (\rho_I \otimes Id_I)$

$\circ\, \alpha_{I,I,I}^{-1}$

$\circ\, \cancel{(Id_I \otimes \lambda_I^{-1}) \circ (\lambda_I \otimes Id_I) \circ \alpha_{I,I,I}^{-1} \circ \alpha_{I,I,I} \circ (\lambda_I^{-1} \otimes Id_I) \circ (Id_I \otimes \lambda_I)}$

$\circ\, (Id_I \otimes (\lambda_I \otimes Id_I))$

$\circ\, \alpha_{I,I\otimes I,I} \circ (\alpha_{I,I,I} \otimes Id_I)$

$\circ\, ((\lambda_I^{-1} \otimes Id_I) \otimes Id_I)$

$\circ\, (\lambda_I^{-1} \otimes Id_I)$

α が自然変換であることから以下の図式が可換。

$$
\begin{tikzcd}
(I \otimes (I \otimes I)) \otimes I \arrow[r, "\alpha_{I,I\otimes I,I}"] \arrow[d, "(Id_I \otimes \lambda_I) \otimes Id_I"'] \arrow[dr, phantom, "\circlearrowleft"] & I \otimes ((I \otimes I) \otimes I) \arrow[d, "Id_I \otimes (\lambda_I \otimes Id_I)"] \\
(I \otimes I) \otimes I \arrow[r, "\alpha_{I,I,I}"'] & I \otimes (I \otimes I)
\end{tikzcd}
$$

従って以下の等式が成り立つ

$$\alpha_{I,I,I} \circ ((Id_I \otimes \lambda_I) \otimes Id_I) = (Id_I \otimes (\lambda_I \otimes Id_I)) \circ \alpha_{I,I\otimes I,I}$$

$$
\begin{aligned}
&= \lambda_I \\
&\circ (\lambda_I \otimes Id_I) \\
&\circ (\rho_I^{-1} \otimes Id_I) \circ \rho_I^{-1} \circ \rho_I \circ (\rho_I \otimes Id_I) \\
&\circ \alpha_{I,I,I}^{-1} \\
&\circ \cancel{(Id_I \otimes (\lambda_I \otimes Id_I)) \circ \alpha_{I,I\otimes I,I}} \\
&\circ \alpha_{I,I,I} \circ ((Id_I \otimes \lambda_I) \otimes Id_I) \\
&\circ (\alpha_{I,I,I} \otimes Id_I) \\
&\circ ((\lambda_I^{-1} \otimes Id_I) \otimes Id_I) \\
&\circ (\lambda_I^{-1} \otimes Id_I)
\end{aligned}
$$

以下は恒等射なので削除

$\alpha^{-1}_{I,I,I} \circ \alpha_{I,I,I}$

$$\begin{tikzcd}
(I \otimes I) \otimes I \arrow[rr, bend left, "\alpha_{I,I,I}"] & & I \otimes (I \otimes I) \arrow[ll, bend left, "\alpha^{-1}_{I,I,I}"]
\end{tikzcd}$$

$= \lambda_I$

$\circ\, (\lambda_I \otimes Id_I)$

$\circ\, (\rho_I^{-1} \otimes Id_I) \circ \rho_I^{-1} \circ \rho_I \circ (\rho_I \otimes Id_I)$

$\circ\ \cancel{\alpha^{-1}_{I,I,I} \circ \alpha_{I,I,I}}$

$\circ\, ((Id_I \otimes \lambda_I) \otimes Id_I)$

$\circ\, (\alpha_{I,I,I} \otimes Id_I)$

$\circ\, ((\lambda_I^{-1} \otimes Id_I) \otimes Id_I)$

$\circ\, (\lambda_I^{-1} \otimes Id_I)$

三角等式より以下の図式が可換と言える。

$$\begin{tikzcd}
((I \otimes I) \otimes I) \otimes I \arrow[rr, "\alpha_{I,I,I} \otimes Id_I"] \arrow[dr, "(\rho_I \otimes Id_I) \otimes Id_I"'] & \circlearrowleft & (I \otimes (I \otimes I)) \otimes I \arrow[dl, "(Id_I \otimes \lambda_I) \otimes Id_I"] \\
& (I \otimes I) \otimes I &
\end{tikzcd}$$

従って以下の等式が成り立つ

$$(\rho_I \otimes Id_I) \otimes Id_I = ((Id_I \otimes \lambda_I) \otimes Id_I) \circ (\alpha_{I,I,I} \otimes Id_I)$$

$$= \lambda_I$$
$$\circ (\lambda_I \otimes Id_I)$$
$$\circ (\rho_I^{-1} \otimes Id_I) \circ \rho_I^{-1} \circ \rho_I \circ (\rho_I \otimes Id_I)$$
$$\circ \cancel{((Id_I \otimes \lambda_I) \otimes Id_I) \circ (\alpha_{I,I,I} \otimes Id_I)}$$
$$\circ ((\rho_I \otimes Id_I) \otimes Id_I)$$
$$\circ ((\lambda_I^{-1} \otimes Id_I) \otimes Id_I)$$
$$\circ (\lambda_I^{-1} \otimes Id_I)$$

ρ が自然変換であることから以下の図式が可換。

$$\begin{tikzcd}
(I \otimes I) \otimes I \arrow[r, "\rho_{I \otimes I}"] \arrow[d, "\rho_I \otimes Id_I"'] \arrow[dr, phantom, "\circlearrowleft"] & I \otimes I \arrow[d, "\rho_I"] \\
I \otimes I \arrow[r, "\rho_I"'] & I
\end{tikzcd}$$

従って以下の等式が成り立つ

$$\rho_I \circ \rho_{I \otimes I} = \rho_I \circ (\rho_I \otimes Id_I)$$

$$\begin{aligned}
&= \lambda_I \\
&\circ (\lambda_I \otimes Id_I) \\
&\circ (\rho_I^{-1} \otimes Id_I) \circ \rho_I^{-1} \\
&\circ \cancel{\rho_I \circ (\rho_I \otimes Id_I)} \\
&\circ \rho_I \circ \rho_{I \otimes I} \\
&\circ ((\rho_I \otimes Id_I) \otimes Id_I) \\
&\circ ((\lambda_I^{-1} \otimes Id_I) \otimes Id_I) \\
&\circ (\lambda_I^{-1} \otimes Id_I)
\end{aligned}$$

ρ が自然変換であることから以下の図式が可換。

$$\begin{tikzcd}
((I \otimes I) \otimes I) \otimes I \arrow[rr, "\rho_{(I\otimes I)\otimes I}"] \arrow[d, "(\rho_I \otimes Id_I)\otimes Id_I"'] \arrow[drr, phantom, "\circlearrowleft"] & & (I \otimes I) \otimes I \arrow[d, "\rho_I \otimes Id_I"] \\
(I \otimes I) \otimes I \arrow[rr, "\rho_{I\otimes I}"'] & & I \otimes I
\end{tikzcd}$$

従って以下の等式が成り立つ

$$(\rho_I \otimes Id_I) \circ \rho_{(I\otimes I)\otimes I} = \rho_{I\otimes I} \circ ((\rho_I \otimes Id_I) \otimes Id_I)$$

$= \lambda_I$
$\circ\, (\lambda_I \otimes Id_I)$
$\circ\, (\rho_I^{-1} \otimes Id_I)$
$\circ\, \rho_I^{-1}$
$\circ\, \rho_I$
$\circ\, \cancel{\rho_{I\otimes I} \circ ((\rho_I \otimes Id_I) \otimes Id_I)}$
$\circ\, (\rho_I \otimes Id_I) \circ \rho_{(I\otimes I)\otimes I}$
$\circ\, ((\lambda_I^{-1} \otimes Id_I) \otimes Id_I)$
$\circ\, (\lambda_I^{-1} \otimes Id_I)$

以下は恒等射なので削除

$$(\rho_I^{-1} \otimes Id_I) \circ \rho_I^{-1} \circ \rho_I \circ (\rho_I \otimes Id_I)$$

$$\begin{tikzcd}
(I \otimes I) \otimes I \arrow[r, bend left, "\rho_I \otimes Id_I"] & I \otimes I \arrow[l, bend left, "\rho_I^{-1} \otimes Id_I"] \arrow[r, bend left, "\rho_I"] & I \arrow[l, bend left, "\rho_I^{-1}"]
\end{tikzcd}$$

$= \lambda_I$
$\circ\, (\lambda_I \otimes Id_I)$
$\circ\, \cancel{(\rho_I^{-1} \otimes Id_I) \circ \rho_I^{-1} \circ \rho_I \circ (\rho_I \otimes Id_I)}$
$\circ\, \rho_{(I\otimes I)\otimes I}$
$\circ\, ((\lambda_I^{-1} \otimes Id_I) \otimes Id_I)$
$\circ\, (\lambda_I^{-1} \otimes Id_I)$

ρ が自然変換であることから以下の図式が可換。

$$\begin{tikzcd}
((I \otimes I) \otimes I) \otimes I \arrow[rr, "\rho_{(I\otimes I)\otimes I}"] \arrow[d, "(\lambda_I \otimes Id_I)\otimes Id_I"'] \arrow[drr, phantom, "\circlearrowleft"] & & (I \otimes I) \otimes I \arrow[d, "\lambda_I \otimes Id_I"] \\
(I \otimes I) \otimes I \arrow[rr, "\rho_{I\otimes I}"'] & & I \otimes I
\end{tikzcd}$$

従って以下の等式が成り立つ

$$\rho_{I\otimes I} \circ ((\lambda_I \otimes Id_I) \otimes Id_I) = (\lambda_I \otimes Id_I) \circ \rho_{(I\otimes I)\otimes I}$$

$$\begin{aligned}
&= \lambda_I \\
&\circ \cancel{(\lambda_I \otimes Id_I) \circ \rho_{(I\otimes I)\otimes I}} \\
&\circ \rho_{I\otimes I} \circ ((\lambda_I \otimes Id_I) \otimes Id_I) \\
&\circ ((\lambda_I^{-1} \otimes Id_I) \otimes Id_I) \\
&\circ (\lambda_I^{-1} \otimes Id_I)
\end{aligned}$$

以下は恒等射なので削除

$$((\lambda_I \otimes Id_I) \otimes Id_I) \circ ((\lambda_I^{-1} \otimes Id_I) \otimes Id_I)$$

$$\begin{tikzcd}
(I \otimes I) \otimes I \arrow[rr, bend left, "(\lambda_I^{-1}\otimes Id_I)\otimes Id_I"] & & ((I \otimes I) \otimes I) \otimes I \arrow[ll, bend left, "(\lambda_I\otimes Id_I)\otimes Id_I"]
\end{tikzcd}$$

$$\begin{aligned}
&= \lambda_I \\
&\circ \rho_{I\otimes I} \\
&\circ \cancel{((\lambda_I \otimes Id_I) \otimes Id_I) \circ ((\lambda_I^{-1} \otimes Id_I) \otimes Id_I)} \\
&\circ (\lambda_I^{-1} \otimes Id_I)
\end{aligned}$$

ρ が自然変換であることから以下の図式が可換。

$$\begin{tikzcd}
(I \otimes I) \otimes I \arrow[r, "\rho_{I \otimes I}"] \arrow[d, "\lambda_I \otimes Id_I"'] \arrow[dr, phantom, "\circlearrowleft"] & I \otimes I \arrow[d, "\lambda_I"] \\
I \otimes I \arrow[r, "\rho_I"'] & I
\end{tikzcd}$$

従って以下の等式が成り立つ

$$\rho_I \circ (\lambda_I \otimes Id_I) = \lambda_I \circ \rho_{I \otimes I}$$

$$= \cancel{\lambda_I \circ \rho_{I \otimes I}}$$
$$\rho_I \circ (\lambda_I \otimes Id_I)$$
$$\circ (\lambda_I^{-1} \otimes Id_I)$$

以下は恒等射なので削除ううううううううう

$$(\lambda_I \otimes Id_I) \circ (\lambda_I^{-1} \otimes Id_I)$$

$$\begin{tikzcd}
I \otimes I \arrow[r, bend left, "\lambda_I^{-1} \otimes Id_I"] & (I \otimes I) \otimes I \arrow[l, bend left, "\lambda_I \otimes Id_I"]
\end{tikzcd}$$

$$= \rho_I \circ \cancel{(\lambda_I \otimes Id_I) \circ (\lambda_I^{-1} \otimes Id_I)}$$
$$= \rho_I (\text{右単位律子})$$

$\square$