Market Simulator with Deep Propagator

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Introduction

- We are proposing a new framework for non-parametric estimation of the asset price process using option prices written on it, and also the interest rates.
- Contrary to the traditional approaches where only the dynamics under the risk-neutral measure is retrieved, our method recovers the one under the physical measure— by imposing a "Ross's recovery theorem" type assumption.
- In the numerical part, we rely on existing machine-learning architectures to estimate the transition density of the diffusion, and then the diffusion- and drift- coefficients of the asset price process.

Introduction

- In this presentation, I will survey the framework quickly,
- and give an example of its implementation where we model the price process of multiple assets,
- and finally using dummy data I will discuss the efficiency of our framework.

Outline

Backgrounds 1. Interest Rate Modeling

In the classical "modeling approach" in quantitative finance, interest rates are modeled by way of

- spot rate — Vasicek, Cox-Ingersoll-Ross, Hull-White, ...
- yield curve —Heath-Jarrow-Morton,...
- Libor rates — Brace-Gatrek-Musiela,...
- state price density (=pricing kernel) — Flesaker-Hughston, Rogers, Hunt-Kennedy,... $\Rightarrow$ often called "Markov functional model".

Backgrounds 2. Markov Functional– State Price Density Approach

- Recall that state price density is a strictly positive process $\{\gamma_t\}$ such that for the price $\{S_t\}$ of any traded asset, $\{\gamma S\}$ is a (local) martingale under the physical measure.
- Consequently, the fair price at time t of an option paying Z at the maturity T is given by $\gamma_t^{-1}\mathrm{E}[\gamma_T Z|\mathcal{F}_t]$, where $\{\mathcal{F}_t\}$ is the market filtration, and the expectation is taken by the physical measure.
- In particular, the price at time t of the zero-coupon-bond paying 1 at T is given by $\gamma_t^{-1}\mathrm{E}[\gamma_T|\mathcal{F}_t]$.
- In a Markov functional model, it is assumed that $\gamma_t = U(t, X_t)$ for a Markov process X and such a function U as allows an explicit calculation of the expectation so that the bond price is also given by a function of X_t.

Backgrounds 3.Ross's Recovery Theorem

- "Ross's Recovery (Type) Theorem" states that under the separation assumption $U(t,x) = f(t)g(x)$, $f = e^{\lambda t}$ where λ is the first eigenvalue of the infinitesimal generator of X, and $1/g$ is its eigenfunction.
- This theorem enables a "non-parametric estimation" of the "market price of risk" = instantaneous volatility of the state price density.

Lifting to an infinite dimensional separation

- Ross's recovery is, however, not fully "market-driven"; we need to model X and the function $r(x)$ that gives the spot rate r_t by $r(X_t)$.
- Motivated by Ross's recovery theorem Vu and Akahori [3] proposed a method to estimate not only the market price of risk but also the volatility and the drift of the state price, that is to say, the process.
- The key idea is to lift the separation assumption $U(t,x) = f(t)g(x)$ to an infinite dimensional one; $U(t,x) = T_t(g)(x)$, where T_t is a semigroup.
- Our recovery theorem then claims that T_t must be the transition semigroup of X.

Deep Propagator Method

- The idea dates back to the paper by Akahori, Hishida, Teichmann, Tsuchiya [1], but it was still in the context of the classical modeling paradigm.
- We have found that the approach, what they call the "heat kernel approach", might work quite well in a market-driven way.

Deep Propagator Method

- The semigroup property can be rephrased as follows; let X is a Markov process on a Polish space $\mathcal{S}$. The state price density is given by $\gamma_t = u(t, X_t)$, where u is a function with the so-called *propagation property*;

$$\begin{aligned} u(t+s, x) &= \mathbb{E}[u(t, X_s) | X_0 = x] = \int u(t, y) \mathbb{P}(X_s \in \mathrm{d}y | X_0 = x) \\ &= \int u(t, y) p(s, x, y) \mathrm{d}y \end{aligned}$$

  for any $t, s > 0$ and $x \in \mathcal{S}$.

- Then, the zero-coupon-bond price process is given by

$$P(t, T) = \frac{u(2T - t, X_t)}{u(t, X_t)}. \tag{1}$$

Deep Propagator Method

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- In general, with the Markov function assumption $P(t, T) = U_T(t, X_t)$, the function U_T is observed only once at the time t, so we can not recover the values of the function from the market.
- But the formula (1) enables us to recover the function form from the market values of the zero coupon bond prices, once we get $u(t, x)$ for some t.

Deep Propagator Method: Market Generator

Under the Deep Propagator assumption, the propagator (=transition density of X) $p(t, x, y)$ and the function $u(t_0, \cdot)$ can determine $u(t_0, \cdot)$ for $t > t_0$ by

$$u(t, x) \equiv u^{u(t_0, \cdot), p}(t, x) = \int p(t - t_0, x, y) u(t_0, y) dy,$$

and then all the zero coupon bond prices by the equation (1), and also the price at $t \geq t_0$ of an European option paying $F(X_T)$ at the maturity $T > t_0$ by

$$\pi^{T,F}(t, x) = \frac{1}{u(t, x)} \int u(T, y) p(T - t, x, y) F(y) dy. \qquad (2)$$

Deep Propagator Method: Estimator

Suppose that we are given the data of $P_o(t, T)$ and X_t and also $\pi_o^{T,F}(t)$ for $t, T \in \mathbb{T} \subset [0, \infty)$ and $F \in \mathbb{F}$, a class of functions. Our target is to find the optimal market generator $(u^*(t_0, \cdot), p^*(t, x, y))$ that minimize the total error Err, which is (roughly) defined as follows. For a given market generator (u, p) and the market observations $\{P_o, \pi_o^{T,F}\}$ define

$$\mathrm{Err}(u(t_0, \cdot), p) = \sum_{t,T} \left(P_o(t, T) - \frac{u(2T - t, X_t)}{u(t, X_t)} \right)^2 + \sum_{F,T,t} \left(\pi_o^{T,F}(t) - \pi^{T,F}(t, X_t) \right)^2$$

Deep Propagator Method: An Algorithm

We choose a "prior generator" $u_0(t_0, x)$ and a "prior propagator" $p_0(t, x, y)$. Then, we calculate

$$u_0(t, x) = \int p_0(t, x, y) u_0(t_0, y) dy,$$

for t, $T \in \mathbb{T}$ and then calculate

$$P_0(t, T) = \frac{u_0(2T - t, X_t)}{u_0(t, X_t)}$$

and

$$\pi_0^{T,F}(t, x) = \frac{1}{u_0(t, x)} \int u_0(T, y) p_0(T - t, x, y) F(y) dy$$

for randomly chosen t, T and F. According to the stochastic gradient descent algorithm solving the above optimization problem, update the prior *market generator* to a posterior. We repeat this procedure.

An Numerical Experiment Using Dummy Data

The first step in testing this theory is to verify whether $u_0(t_0, x)$ can be guessed when $p_0(t, x, y)$ is known. Firstly, we generate dummy data as follows.

- Observation dates: $\mathbb{T} := \{\frac{k}{n} : k = 0, 1, \cdots, T_x \cdot n\}$.
- Assuming that X_t is a geometric Brownian motion, dummy data is generated by substituting the parameters $(n, T_x, sigma, mu, X_0)$.
- Let $u(t, x) = 1 + sin^2(x)$ be the correct state price density function and compute $P(t, T)$ for all t in $\mathbb{T}$ and $T > t$ to create the price matrix.

An Numerical Experiment Using Dummy Data

Next, the "prior generator" $u_0(t_0, x)$ is updated by machine learning.

- $u_0(t_0, x)$ is configured as follows.
  Optimizer is Adam.
  Activation function is Soft plus.
  Layers are ◯.
  Nodes are ◯.
- Extract 200 arbitrary (t, T) and define the loss function as

$$\frac{1}{200}\sum\left(P(t,T) - \frac{u_0(2T-t, X_t)}{u_0(t, X_t)}\right)^2$$

  The parameters are updated once by back propagation for this loss.The number of epochs is assumed to be ◯.

1. Backgrounds
2. Deep Propagator Method
3. An Numerical Experiment Using Dummy Data
4. References

References

[1] J. Akahori, Y. Hishida, J. Teichmann, T. Tsuchiya, "A heat kernel approach to interest rate models" JJIAM, 31, 419 – 439 (2014)

[2] S. Ross, "The recovery theorem", The Journal of Finance 70(2), (2015) 615–648.

[3] H. Vu, and J. Akahori, "Back to the future: A deep propagator approach to estimate the market expectation" , preprint (2022)

Thank You