3

Short-rate models

We need to choose a quantity to serve as a state variable that determines the interest rate term structure and its evolution in time. The first generation of stochastic interest rate models use the instantaneous short rate as the state variable. The two key advantages of short-rate models are their general simplicity and the fact that they often lead to analytic formulae for bonds and associated vanilla options. The tractability of short-rate models means that the price of a given derivative can often be computed quickly, important in situations where a large number of securities need to be valued. Indeed, throughout this chapter, we focus on short-rate models that allow discount bonds to be priced in closed form.

One-factor models assume that the entire interest rate term structure is driven by a one-dimensional Wiener process. Such models are usually suitable when pricing securities that depend on a single rate only, but for more complex products which depend on two or more different rates we may need to move to a multi-factor model driven by multi-dimensional Brownian motion. In the final section we present one of the most popular multi-factor short-rate models, the two-factor Hull–White model.

A weakness of the short-rate approach is that the instantaneous short rate is a mathematical idealisation rather than something that can be observed directly in the market. In the past decade, short-rate models have, to some extent, been superseded by the LIBOR market model (covered in Chapter 5), in which the stochastic state variable is a set of benchmark forward LIBOR rates. Nonetheless, short-rate models are particularly useful and remain popular due to their analytic tractability.

We continue to work under Assumption 2.1, which stipulates the existence of a probability measure Q (the risk-neutral measure) equivalent to P such that the value of any security discounted by the money market account is a martingale under Q.

3.1 General properties

In a **one-factor short-rate model** we assume that the short rate $r(t)$ satisfies an SDE of the form

$$dr(t) = \mu(t, r(t))dt + \sigma(t, r(t))dW(t), \tag{3.1}$$

where $W(t)$ is a Brownian motion under the risk-neutral measure Q. We also assume that, for any $T \geq 0$, the price of the T-bond depends on the instantaneous short rate,

$$B(t, T) = F(t, r(t); T),$$

where $F(t, r; T)$ is a sufficiently smooth function to allow all transformations that follow.

Using Itô's formula to compute $dB(t, T) = dF(t, r(t), T)$, we can write

$$dB(t, T) = \alpha(t, T)B(t, T)dt + \Sigma(t, T)B(t, T)dW(t),$$

where

$$\alpha(t, T) = \frac{\frac{\partial F}{\partial t}(t, r(t); T) + \mu(t, r(t))\frac{\partial F}{\partial r}(t, r(t); T) + \frac{1}{2}\sigma(t, r(t))^2\frac{\partial^2 F}{\partial r^2}(t, r(t); T)}{F(t, r(t); T)},$$

$$\Sigma(t, T) = \frac{\sigma(t, r(t))\frac{\partial F}{\partial r}(t, r(t); T)}{F(t, r(t); T)}.$$

Since $W(t)$ is a Brownian motion under Q, from Exercise 2.3 we know that $\alpha(t, T) = r(t)$. This shows that $F(t, r; T)$ must satisfy the partial differential equation

$$\frac{\partial F}{\partial t} + \mu\frac{\partial F}{\partial r} + \frac{1}{2}\sigma^2\frac{\partial^2 F}{\partial r^2} = rF \tag{3.2}$$

called the **term structure equation**. Because $B(T, T) = 1$, this equation

Table 3.1 *A selection of short-rate models.*

Model	$\mu(t, r(t))$	$\sigma(t, r(t))$
Merton	α	σ
Vasiček	$\theta - \alpha r(t)$	σ
Cox–Ingersoll–Ross	$\alpha(\beta - r(t))$	$\sigma \sqrt{r(t)}$
Dothan	$\alpha r(t)$	$\sigma r(t)$
Black–Derman–Toy	$\theta(t) r(t)$	$\sigma(t) r(t)$
Ho–Lee	$\theta(t)$	σ
Hull–White (extended Vasiček)	$\theta(t) - \alpha r(t)$	$\sigma(t)$
Black–Karasiński	$r(t)(\theta(t) - \alpha \ln r(t))$	$\sigma r(t)$

can be solved subject to the final condition $F(T, r; T) = 1$ to find a formula for $F(t, r; T)$ and hence for the bond price $B(t, T)$ for any $t < T$.

Nonetheless, we are going to adopt an alternative approach, utilising the fact that $\frac{B(t,T)}{B(t)}$ is a martingale under the risk-neutral measure Q and $B(t) = \exp\left(\int_0^t r(s)ds\right)$, so the bond price can be expressed as

$$B(t, T) = B(t)\mathbb{E}_Q\left(\left.\frac{B(T, T)}{B(T)}\right| \mathcal{F}_t\right) = \mathbb{E}_Q\left(\exp\left(-\int_t^T r(s)ds\right)\middle| \mathcal{F}_t\right). \tag{3.3}$$

3.2 Popular short-rate models

A number of models have been proposed for the dynamics of the short rate under the risk-neutral measure Q. These models specify a particular form for $\mu(t, r(t))$ and $\sigma(t, r(t))$. A list is shown in Table 3.1. It is not exhaustive, and various other possible functional forms for the risk-neutral drift and volatility can be proposed.

For the majority of models in Table 3.1 the short rate is either normally distributed (Merton, Vasiček, Ho–Lee and Hull–White) or log-normally distributed (Dothan, Black–Derman–Toy and Black–Krasiński). The Cox–Ingersoll–Ross model does not fit into either of these two categories as it features the non-central chi-squared distribution. Models where the short rate is normally distributed (Gaussian) are the most analytically tractable.

The model parameters are determined by calibrating to the current term structure of interest rates and to the implied volatilities of actively traded vanilla options (caps, floors and swaptions). However, on doing this calibration we quickly see that some of the models provide only an approximate fit to the present term structure of interest rates. This issue is discussed in more detail when we present the Vasiček model.

3.3 Merton model

This is arguably the simplest short-rate model. The SDE for the short rate is

$$dr(t) = \alpha dt + \sigma dW(t),$$

where α and σ are constants, and where $W(t)$ is a Brownian motion under the risk-neutral measure Q. This gives

$$r(s) = r(t) + \alpha(s - t) + \sigma(W(s) - W(t)).$$

We can see that the short rate is normally distributed.

The Merton model gives rise to a simple analytic formula for the bond price, which we derive below.

Bond pricing formula

Since the short rate is normally distributed, computing the expectation in (3.3) reduces to calculating the expected value of a log-normal random variable. Integrating between t and T, we have

$$\int_t^T r(s)ds = r(t)(T - t) + \frac{1}{2}\alpha(T - t)^2 + \sigma \int_t^T W(s)ds - \sigma W(t)(T - t).$$

Noting that

$$d\left((T - s)W(s)\right) = -W(s)ds + (T - s)dW(s),$$

we can replace the last two terms to get

$$\int_t^T r(s)ds = r(t)(T - t) + \frac{1}{2}\alpha(T - t)^2 + \sigma \int_t^T (T - s)dW(s).$$

It follows that

$$X = \frac{1}{2}\alpha(T - t)^2 + \sigma \int_t^T (T - s)dW(s)$$

is independent of $\mathcal{F}_t$ and normally distributed with mean $m = \frac{1}{2}\alpha(T - t)^2$ and variance $s^2 = \sigma^2 \int_t^T (T - s)^2 ds = \frac{1}{3}\sigma^2(T - t)^3$ under the risk-neutral measure Q. Because the expectation of e^{-X} is $e^{-m+\frac{1}{2}s^2}$, this proves the following result.

Proposition 3.1
The zero-coupon bond price in the Merton model can be expressed as

$$B(t,T) = \exp\left(-r(t)(T-t) - \frac{1}{2}\alpha(T-t)^2 + \frac{1}{6}\sigma^2(T-t)^3\right). \tag{3.4}$$

Proof By (3.3), since $r(t)$ is $\mathcal{F}_t$-measurable and X is independent of $\mathcal{F}_t$,

$$\begin{aligned} B(t,T) &= \mathbb{E}_Q\left(\exp\left(-\int_0^T r(s)ds\right)\Big|\mathcal{F}_t\right) \\ &= \mathbb{E}_Q\left(\exp\left(-r(t)(T-t)\right)\exp\left(-X\right)\big|\mathcal{F}_t\right) \\ &= \exp\left(-r(t)(T-t)\right)\mathbb{E}_Q\left(\exp\left(-X\right)\right) \\ &= \exp\left(-r(t)(T-t) - m + \frac{1}{2}s^2\right) \\ &= \exp\left(-r(t)(T-t) - \frac{1}{2}\alpha(T-t)^2 + \frac{1}{6}\sigma^2(T-t)^2\right). \end{aligned}$$

□

> **Exercise 3.1** Let $B(t,T) = F(t,r(t);T)$ be the zero-coupon bond price (3.4) in the Merton model. Show that $F(t,r;T)$ satisfies the term structure equation (3.2).

3.4 Vasiček model

A problem with the Merton model is that the short rates can be negative, but an even more pressing issue is that it fails to model the dynamics correctly. An important empirical feature is that, when interest rates are high, there is a tendency for them to fall over time and, likewise, when the rates are low, they tend to rise. This is captured by the Vasiček model

$$dr(t) = (\theta - \alpha r(t))dt + \sigma dW(t),$$

where θ, α, σ are constants and $W(t)$ is a Brownian motion under the risk-neutral measure Q.

This is considered the first realistic model of the short rate. Vasiček modelled the short rate using a mean-reverting drift. The drift is positive when $r(t)$ is below θ/α, and negative when $r(t)$ is greater than θ/α.

The SDE for $r(t)$ can be solved explicitly. Observe that

$$d\left(e^{\alpha t} r(t)\right) = \theta e^{\alpha t} dt + \sigma e^{\alpha t} dW(t).$$

Integrating from t to time $s \geq t$, and multiplying both sides of the equality by $e^{-\alpha s}$, we get

$$r(s) = r(t)e^{-\alpha(s-t)} + \theta \int_t^s e^{-\alpha(s-u)} du + \sigma \int_t^s e^{-\alpha(s-u)} dW(u).$$

From the above we can see that the short rate is normally distributed under the risk-neutral measure Q with mean

$$\mathbb{E}_{\mathbb{Q}}(r(s)) = r(0)e^{-\alpha s} + \theta \int_0^s e^{-\alpha(s-u)} du = r(0)e^{-\alpha s} + \theta \frac{1 - e^{-\alpha s}}{\alpha}$$

and variance given by the Itô isometry as

$$\mathrm{Var}(r(s)) = \sigma^2 \int_0^s e^{-2\alpha(s-u)} du = \sigma^2 \frac{1 - e^{-2\alpha s}}{2\alpha}.$$

As time s tends to infinity, the expectation of the short rate $r(s)$ tends to θ/α. The short rate is mean reverting. Moreover, because the short rate is normally distributed, it can become negative. This feature of the model might at first be considered a fatal flaw. Nonetheless, in practical applications the probability of the short rate becoming negative is often small.

Bond pricing formula

Computing the integral of $r(s)$ from t to T, we have

$$\begin{aligned}\int_t^T r(s)ds = r(t) \int_t^T & e^{-\alpha(s-t)} ds \\ & + \theta \int_t^T \left(\int_t^s e^{-\alpha(T-u)} du \right) ds + \sigma \int_t^T \left(\int_t^s e^{-\alpha(T-u)} dW(u) \right) ds.\end{aligned}$$

Let us denote the integral in the first term on the right-hand side by

$$D(t,T) = \int_t^T e^{-\alpha(s-t)} ds = \frac{1 - e^{-\alpha(T-t)}}{\alpha}. \tag{3.5}$$

To compute the second and third terms observe that

$$d\left(\int_t^s e^{-\alpha(s-u)}du\right) = ds - \alpha\left(\int_t^s e^{-\alpha(s-u)}du\right)ds,$$

$$d\left(\int_t^s e^{-\alpha(s-u)}dW(u)\right) = dW(s) - \alpha\left(\int_t^s e^{-\alpha(s-u)}dW(u)\right)ds.$$

Hence

$$\begin{aligned}\left(\int_t^s e^{-\alpha(s-u)}du\right)ds &= d\left(\int_t^s \frac{1-e^{-\alpha(s-u)}}{\alpha}du\right)\\ &= d\left(\int_t^s D(u,s)du\right),\\ \left(\int_t^s e^{-\alpha(s-u)}dW(u)\right)ds &= d\left(\int_t^s \frac{1-e^{-\alpha(s-u)}}{\alpha}dW(u)\right)\\ &= d\left(\int_t^s D(u,s)dW(u)\right).\end{aligned}$$

As a result, integrating from t to T, we find that

$$\int_t^T r(s)ds = r(t)D(t,T) + \theta\int_t^T D(u,T)du + \sigma\int_t^T D(u,T)dW(u).$$

It follows that

$$X = \theta\int_t^T D(u,T)du + \sigma\int_t^T D(u,T)dW(u)$$

is a random variable independent of $\mathcal{F}_t$, normally distributed with mean

$$m = \theta\int_t^T D(u,T)du$$

and variance given by the Itô isometry as

$$s^2 = \sigma^2\int_t^T D(u,T)^2du$$

under the risk-neutral measure Q. The expectation of e^{-X} is $e^{-m+\frac{1}{2}s^2}$. Hence, using the bond pricing formula (3.3), we arrive at the following result just like in the proof of Proposition 3.1.

Proposition 3.2
The zero-coupon bond price in the Vasiček model can be expressed as

$$B(t,T) = \exp\left(- r(t)D(t,T) - \theta \int_t^T D(u,T)du + \frac{1}{2}\sigma^2 \int_t^T D(u,T)^2 du \right), \qquad (3.6)$$

where $D(t,T)$ is given by (3.5).

> **Exercise 3.2** Compute the mean m and variance s^2 of X and hence express the bond price $B(t,T)$ in the Vasiček model explicitly in terms of the parameters θ, α, σ.

> **Exercise 3.3** Let $B(t,T) = F(t, r(t); T)$ be the zero-coupon bond price (3.6) in the Vasiček model. Show that $F(t, r; T)$ satisfies the term structure equation (3.2).

Remark 3.3
Note that, in both the Merton and Vasiček models, bond prices can be written as

$$B(t,T) = e^{f(t,T)-g(t,T)r(t)},$$

where $f(t,T)$ and $g(t,T)$ are deterministic functions. Models of this type are referred to as **affine term structure models**. The value of affine term structure models lies in their relative simplicity. In general, if the drift term $\mu(t, r(t))$ and the square of the volatility $\sigma(t, r(t))^2$ are affine functions of the short rate $r(t)$ in the SDE (3.1), that is, if

$$\mu(t, r(t)) = \gamma(t)r(t) + \delta(t), \qquad \sigma(t, r(t))^2 = \eta(t)r(t) + \epsilon(t),$$

where the time-dependent functions are suitably well behaved, then the model is said to possess an affine term structure. All models in Table 3.1 have this property, except for the Black–Derman–Toy, Black–Karasiński and Dothan models.

Table 3.2 *Calibration to zero-coupon bond prices implied by the USD interest rate curve on 18 May 2011 in Example 3.4.*

Maturity	$B^{\text{mkt}}(0, T)$	$B(0, T)$	% Error
1	0.9962	0.9943	0.00
2	0.9851	0.9803	0.01
3	0.9645	0.9594	0.03
4	0.9359	0.9329	0.06
5	0.9013	0.9020	0.10
6	0.8628	0.8678	0.14
7	0.8258	0.8315	0.18
8	0.7873	0.7936	0.22
9	0.7504	0.7550	0.26
10	0.7153	0.7163	0.29

Calibration

The model parameters are chosen to match the initial interest rate term structure. Suppose that, at time 0, we are given a set of N zero-coupon bond prices for $i = 1, \dots, N$ derived from a set of actively traded benchmark securities. Denoting the analytic expression found in Exercise 3.2 for the bond prices in terms of the model parameters by $B(0, T_i; \theta, \alpha, \sigma)$, we can perform the least squares optimisation

$$\min_{\theta,\alpha,\sigma} \sum_{i=1}^{N} \left(B^{\text{mkt}}(0, T_i) - B(0, T_i; \theta, \alpha, \sigma)\right)^2 \tag{3.7}$$

to compute the parameter values for the best match between the model and market data.

However, as N is typically far greater than the number of model parameters (three in the Vasiček model), we find that after performing the optimisation the zero-coupon curve implied by the model often fails to accurately match that given by the market. This is particularly true when the market curve is inverted. It is a serious flaw in models such as Vasiček where we have only a limited number of constant parameters. In general, the current price of a zero-coupon T-bond given by the model will rarely match the market price. The model's failure to match even the current zero-coupon curve means it cannot be used for more exotic interest rate derivatives.

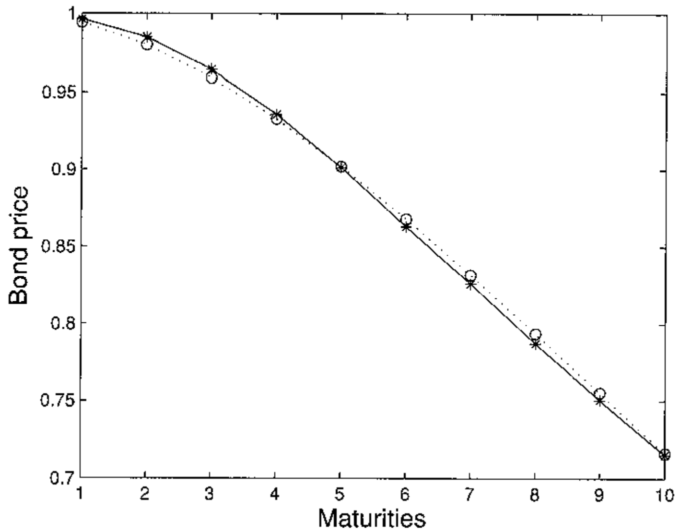

Figure 3.1 Results for least squares optimisation for market price data on 18 May 2011 in Example 3.4. The market-implied zero-coupon bond prices $B^{\mathrm{mkt}}(t,T)$ are indicated by asterisks, and the model prices $B(t,T)$ by circles.

Example 3.4
In this numerical example we calibrate to a set of bond prices in Table 3.2 derived from the interest rate curve on 18 May 2011. The parameters computed by least squares optimisation are $\theta = 0.0099$, $\alpha = 0.131$, $\sigma = 0.01$.

To check if the parameters are in any sense financially realistic we can consider the ratio θ/α, which is the expectation of the short rate as time tends to infinity. This value is 0.075 66, which is plausible. However, for the given set of parameters the model provides only an approximate match to the current discount curve, as can be seen in Figure 3.1.

3.5 Hull–White model

To reproduce exactly the initial zero-coupon curve we need a time-varying parameter. This parameter is specifically chosen to provide an exact match to the initial term structure. Arguably, the most popular model with time-dependent parameters is the Hull–White model, in which the parameters corresponding to θ and σ appearing in the Vasiček model are chosen to be deterministic functions of time,

$$dr(t) = (\theta(t) - \alpha r(t))dt + \sigma(t)dW(t), \tag{3.8}$$

where α is constant and $W(t)$ is a Brownian motion under the risk-neutral measure Q.

Integrating (3.8) from t to $s \geq t$, we have

$$r(s) = r(t)e^{-\alpha(s-t)} + \int_t^s \theta(u)e^{-\alpha(s-u)}du + \int_t^s \sigma(u)e^{-\alpha(s-u)}dW(u). \tag{3.9}$$

Adopting an approach analogous to that in the Vasiček model, we can derive an analytic expression for the zero-coupon bond price using the risk-neutral pricing formula (3.3). This yields

$$B(t,T) = \exp\Bigg(- r(t)D(t,T) - \int_t^T \theta(u)D(u,T)du + \frac{1}{2}\int_t^T \sigma(u)^2 D(u,T)^2 du\Bigg), \tag{3.10}$$

where $D(t,T)$ is given by (3.5).

Exercise 3.4 Verify (3.10) for the Hull–White model by following a similar argument to that leading to formula (3.6) for the zero-coupon bond price in the Vasiček model.

The time-dependent parameter $\theta(t)$ can be chosen to match the current term structure. In fact, from (3.10) it can be seen that what is really needed is an expression for the integral $\int_t^T \theta(u)D(u,T)du$ rather than $\theta(t)$ itself. To this end we take

$$\begin{aligned}\ln\frac{B(0,T)}{B(0,t)} &= -r(0)(D(0,T) - D(0,t)) - \int_t^T \theta(u)D(u,T)du \\ &\quad - \int_0^t \theta(u)\left(D(u,T) - D(u,t)\right)du + \frac{1}{2}\int_t^T \sigma(u)^2 D(u,T)^2 du \\ &\quad + \frac{1}{2}\int_0^t \sigma(u)^2\left(D(u,T)^2 - D(u,t)^2\right)du.\end{aligned}$$

The integrals from 0 to t can be rewritten by using the relation

$$D(u,T) = D(u,t) + D_t(u,t)D(t,T), \tag{3.11}$$

where $D_t(u,t)$ is the partial derivative of $D(u,t)$ with respect to t, to yield

$$\ln\frac{B(0,T)}{B(0,t)} = -r(0)D_t(0,t)D(t,T) - \int_t^T \theta(u)D(u,T)du$$
$$- D(t,T)\int_0^t \theta(u)D_t(u,t)du + \frac{1}{2}\int_t^T \sigma(u)^2 D(u,T)^2 du$$
$$+ D(t,T)\int_0^t \sigma(u)^2 D(u,t)D_t(u,t)du + \frac{1}{2}D(t,T)^2\int_0^t \sigma(u)^2 D_t(u,t)^2 du.$$

From (3.10), using formula (1.9) for the instantaneous forward rate, we obtain

$$f(0,t) = r(0)D_t(0,t) + \int_0^t \theta(u)D_t(u,t)du$$
$$- \int_0^t \sigma^2(u)D(u,t)D_t(u,t)du. \tag{3.12}$$

It follows that

$$\ln\frac{B(0,T)}{B(0,t)} = -f(0,t)D(t,T) - \int_t^T \theta(u)D(u,T)du$$
$$+ \frac{1}{2}\int_t^T \sigma(u)^2 D(u,T)^2 du + \frac{1}{2}D(t,T)^2\int_0^t \sigma(u)^2 D_t(u,t)^2 du,$$

which gives the desired expression for $\int_t^T \theta(u)D(u,T)du$ in terms of $B(0,t)$, $B(0,T)$ and $f(0,t)$, that is, in terms of the current term structure. Substituting this expression into (3.10), we get the following result.

Proposition 3.5

In the Hull–White model the zero-coupon bond price at time $t \geq 0$ that gives an exact fit to the term structure of interest rates at time 0 is

$$B(t,T) = \frac{B(0,T)}{B(0,t)}\exp\Bigg(- (r(t) - f(0,t))D(t,T)$$
$$- \frac{1}{2}D(t,T)^2\int_0^t \sigma(u)^2 D_t(u,t)^2 du\Bigg), \tag{3.13}$$

where $D(t,T)$ is given by (3.5).

In addition to the bond price, it is also convenient to have the Hull–White short-rate process that gives an exact fit to the term structure at time 0.

From (3.12) we can see that

$$f(0,s) - e^{-\alpha(s-t)} f(0,t) = \int_t^s \theta(u) e^{-\alpha(s-u)} du - \int_0^s \sigma(u)^2 D(u,s) e^{-\alpha(s-u)} du + \int_0^t \sigma(u)^2 D(u,t) e^{-\alpha(s-u)} du.$$

Therefore (3.9) becomes

$$r(s) = (r(t) - f(0,t)) e^{-\alpha(s-t)} + f(0,s) + \int_0^s \sigma(u)^2 D(u,s) e^{-\alpha(s-u)} du - \int_0^t \sigma(u)^2 D(u,t) e^{-\alpha(s-u)} du + \int_t^s \sigma(u) e^{-\alpha(s-u)} dW(u). \quad (3.14)$$

Using the above expression in the pricing formula (3.3) for zero-coupon bonds provides another way of deriving (3.13).

Exercise 3.5 Derive the zero-coupon bond price (3.13) in the Hull–White model by using the Hull–White short-rate process (3.14).

Hint: To simplify the calculations use the formulae

$$\int_t^T e^{-\alpha(s-t)} ds = D(t,T), \quad \int_t^T D(t,s) e^{-\alpha(s-t)} ds = \frac{1}{2} D(t,T)^2, \quad (3.15)$$

where $D(t,T)$ is given by (3.5).

Bond option formula

Consider a call and a put option with strike K and expiry S written on a zero-coupon bond with maturity $T > S$. From Theorem 2.4 and Exercise 2.9 we know that if the zero-coupon bond obeys (2.8) with deterministic log-volatility, then the prices at time 0 of the call and the put are

$$\mathbf{BC}(0; S, T, K) = B(0,T) N(d_+) - K B(0,S) N(d_-), \quad (3.16)$$

$$\mathbf{BP}(0; S, T, K) = K B(0,S) N(-d_-) - B(0,T) N(-d_+), \quad (3.17)$$

where

$$d_+ = \frac{\ln \frac{B(0,T)}{B(0,S)K} + \frac{1}{2} v(0,S)}{\sqrt{v(0,S)}}, \qquad d_- = d_+ - v(0,S), \quad (3.18)$$

and $v(0,S)$ is the variance of $\ln B(S,T)$.

In the Hull–White model the zero-coupon bond price $B(S,T)$ is given by (3.10) with S substituted for t. The variance of $\ln B(S,T)$ is therefore equal to the variance of $-r(S)D(S,T)$, where $r(S)$ is given by (3.9) with S substituted for s and 0 for t. As a result,

$$\begin{aligned} v(0,S) &= \mathrm{Var}\left(D(S,T)\int_0^S \sigma(u)e^{-\alpha(S-u)}dW(u)\right) \\ &= D(S,T)^2\int_0^S \sigma(u)^2 e^{-2\alpha(S-u)}du. \end{aligned} \tag{3.19}$$

This gives an analytic formula for the bond option price. When the time-dependent volatility term $\sigma(t)$ is chosen to be constant, (3.19) becomes

$$v(0,S) = \frac{\sigma^2}{2\alpha^3}\left(1 - e^{-\alpha(T-S)}\right)^2\left(1 - e^{-2\alpha S}\right). \tag{3.20}$$

Exercise 3.6 Derive formulae for calls and puts on a zero-coupon bond in the Merton model.

Exercise 3.7 Derive formulae for calls and puts on a zero-coupon bond in the Vasiček model. Compare these formulae with those for the Hull–White model with $\sigma(t)$ chosen to be constant. They should be identical. Why?

Formula for caps and floors

Consider a caplet with strike K, unit notional $N = 1$ and expiry T_i written on the spot LIBOR rate $L(T_{i-1}, T_i)$. In Section 2.7 we saw how the caplet payoff can be expressed as $1 + \tau_i K$ put options with strike $(1 + \tau_i K)^{-1}$ and expiry T_{i-1} written on a zero-coupon bond with maturity T_i. The price at time 0 of the caplet in the Hull–White model is

$$\mathbf{Cpl}_i(0) = (1 + \tau_i K)\mathbf{BP}\left(0; T_{i-1}, T_i, \frac{1}{1 + \tau_i K}\right), \tag{3.21}$$

where **BP** is the price of a put option on a zero-coupon bond given by (3.17) and (3.18), with the variance of $\ln B(S,T)$ given by (3.19). Similarly, the price of the corresponding floorlet is

$$\mathbf{Flr}_i(0) = (1 + \tau_i K)\mathbf{BC}\left(0; T_{i-1}, T_i, \frac{1}{1 + \tau_i K}\right), \tag{3.22}$$

where **BC** is given by (3.16) and (3.18), with the variance of $\ln B(S, T)$ given by (3.19). The price of a cap (or floor) is simply the sum of the prices of the constituent caplets (or floorlets).

In general, the prices of caplets and floorlets are derived (via bootstrapping) from the market prices of caps and floors. Formulae (3.21) and (3.22) are used in calibrating the Hull–White model, where we use the known market prices to help us derive the model parameters.

Formula for swaptions

Consider an option with strike K and expiry T_0 on a payer interest rate swap with unit notional, settlement dates $T_1, \ldots, T_n$ and reset dates $T_0, \ldots, T_{n-1}$. In Section 2.8 we saw how the swaption payoff can be written as a put option with strike 1 on a coupon-bearing bond with coupon rate K, that is,

$$\left(1 - K\sum_{i=1}^{n} \tau_i B(T_0, T_i) - B(T_0, T_n)\right)^+. \tag{3.23}$$

A method referred to as **Jamshidian's trick** can be applied to write this option on a coupon-bearing bond as a linear combination of put options on zero-coupon bonds. According to (3.10), the zero-coupon bond price $B(t, T) = F(t, r(t); T)$ in the Hull–White model is a decreasing function of the short rate $r(t)$. It follows that the coupon-bearing bond price $K\sum_{i=1}^{n} \tau_i F(t, r(t); T_i) + F(t, r(t); T_n)$ is also a decreasing function of $r(t)$, and there exists a critical value $\tilde{r}$ of the short rate such that

$$K\sum_{i=1}^{n} \tau_i F(T_0, \tilde{r}; T_i) + F(T_0, \tilde{r}; T_n) = 1. \tag{3.24}$$

Letting $K_i = F(T_0, \tilde{r}; T_i)$ for $i = 1, \ldots, n$, so $K\sum_{i=1}^{n} \tau_i K_i + K_n = 1$, and observing that $B(T_0, T_i) < K_i$ if and only if $r(T_0) > \tilde{r}$, we can write the payoff (3.23) as

$$K\sum_{i=1}^{n} \tau_i (K_i - B(T_0, T_i))^+ + (K_n - B(T_0, T_n))^+.$$

Therefore the option on a coupon-bearing bond is a linear combination of options on the underlying zero-coupon bonds.

As a result, the price at time 0 of the swaption in the Hull–White model is given by

$$\mathbf{PSwpt}_{0,n}(0) = \sum_{i=1}^{n} K(T_i - T_{i-1})\mathbf{BP}(0; T_0, T_i, K_i) + \mathbf{BP}(0; T_0, T_n, K_n), \quad (3.25)$$

where **BP** is given by (3.17) and (3.18), with the variance of $\ln B(S,T)$ given by (3.19).

> **Exercise 3.8** Suppose we have specified the value of α and the functional form of the volatility $\sigma(t)$. Explain how to estimate K_i for $i = 1, \ldots, n$ in the Hull–White swaption formula (3.25).

Calibrating the Hull–White model

The prices of the most liquid caps, floors and swaptions are given by the market. Therefore the left-hand sides of formulae (3.21), (3.22) and (3.25) are known. We can use these formulae to estimate the Hull–White model parameters given the market prices. Once we know the model parameters, we can then price complex or non-vanilla instruments. A common example of a non-vanilla instrument is the Bermudan swaption, which we will discuss in Section 3.6.

In Section 2.9 we introduced the notion of the implied Black caplet volatility (or spot volatility), which is the volatility parameter that must be inserted into the Black formula to obtain the market price of the caplet. Recall that, for the caplet maturing at time T_{i-1} and paying at time T_i, we denote the implied caplet volatility by $\hat{\sigma}_i^{\text{caplet}}$.

Formula (3.21) can be used to compute the volatility σ_i^{caplet} of the ith caplet implied by the Hull–White model. To this end we solve the equation

$$\mathbf{Cpl}_i(0) = \mathbf{Cpl}_i^{\text{Black}}(0; \sigma_i^{\text{caplet}}) \quad (3.26)$$

with $\mathbf{Cpl}_i(0)$ given by (3.21) and $\mathbf{Cpl}_i^{\text{Black}}(0; \sigma_i^{\text{caplet}})$ by Black's formula (2.24).

The volatility term structure for the Hull–White model is given by solving equation (3.26) for each caplet. The model parameters are chosen so that the implied volatility σ_i^{caplet} of the ith caplet in the Hull–White model computed via (3.26) matches as best as possible the caplet spot volatility $\hat{\sigma}_i^{\text{caplet}}$ given by the market.

Similarly, we can calibrate to vanilla swaptions by choosing the model parameters so that the volatility implied by the swaption formula in the Hull–White model is as close as possible to the Black swaption volatility $\hat{\sigma}_{0,n}^{\text{swpt}}$ given by the market.

As a first attempt at calibration to the caplet or swaption market we could choose $\sigma(t)$ to be constant. However, in such cases we will generally not have enough degrees of freedom for an accurate fit. Therefore the typical assumption is that $\sigma(t)$ is piecewise constant. We partition the time axis by a sequence of dates $T_0 < T_1 < \cdots < T_n < T$ so that

$$\sigma(t) = \sigma_i \quad \text{for } T_{i-1} < t \le T_i, \quad i = 1, \dots, n, \tag{3.27}$$

and then choose a set of vanilla options to calibrate to. According to Proposition 3.5, the T-bond price at time T_i that gives an exact fit to the market term structure of interest rates at time 0 is

$$B(T_i, T) = \frac{B^{\text{mkt}}(0,T)}{B^{\text{mkt}}(0,T_i)} \exp\Bigg(- (r(T_i) - f^{\text{mkt}}(0,T_i))D(T_i,T) \\ - \frac{1}{2}D^2(T_i,T) \sum_{k=1}^{i} \frac{\sigma_k}{2\alpha} \left(\mathrm{e}^{-2\alpha(T_i - T_k)} - \mathrm{e}^{-2\alpha(T_i - T_{k-1})} \right) \Bigg),$$

where $D(T_i, T)$ is given by (3.5). The bond price depends only on the short-rate volatilities $\sigma_1, \dots, \sigma_i$ and the mean-reversion term α.

Remark 3.6

Given the range of caps, floors and swaptions that are actively traded in the market, a natural question to ask is what subset of these do we actually use to calibrate our model? The standard market practice is to calibrate to what traders refer to as the 'natural hedging instruments'. These are the vanilla options used to hedge risk in the exotic instrument we are attempting to model. For the case of a Bermudan swaption, which we will discuss in Section 3.6, the risk is hedged by the underlying co-terminal swaptions.

Remark 3.7

For a given expiry T_i the Black implied volatility for strike K is read from a 'volatility cube' as described in Remark 2.10. The Hull–White short-rate model in itself does not account for the fact that the implied volatility for a given maturity varies with strike. However, correctly calibrated it will reproduce the correct market price of the underlying swaptions.

3.6 Bermudan swaptions in the Hull–White model

In this section we will use the formulae derived for the Hull–White model to price a non-vanilla instrument. The example we choose is a Bermudan swaption as defined in what follows.

The Hull–White short-rate process that gives an exact fit to the term-structure at time 0 is given by (3.14) under the risk-neutral measure Q. However, when we use the Hull–White model to price an exotic instrument or structured product, it can be convenient to know the drift of the short-rate process under the forward measure to calculate the option price. Therefore, we derive the dynamics of the short-rate process under the forward measure as a first step.

Hull–White under the forward measure

In Section 2.6 we saw in the case when the zero-coupon bond numeraire $B(t,T)$ obeys (2.8) that

$$W^T(t) = W(t) - \int_0^t \Sigma(u,T)du \quad \text{for all } t \in [0,T],$$

where $W(t)$ and $W^T(t)$ are Brownian motions under the risk-neutral measure Q and the forward measure P_T, respectively.

Applying the Itô formula to the bond price (3.10) and then using the above relationship between $W^T(t)$ and $W(t)$, we can show that the Hull–White SDE (3.8) for the short rate becomes

$$dr(t) = (\theta(t) - \alpha r(t) - \sigma(t)^2 D(t,T))dt + \sigma(t)dW^T(t) \tag{3.28}$$

under the forward measure.

Exercise 3.9 Show that (3.28) holds true.

Furthermore, using (3.14) we can write an expression for the Hull–White short rate at time $T > t$ that gives an exact fit to the term structure at time 0. Applying the same change of measure and then simplifying, we can see that $r(T)$ is given by

$$\begin{aligned} r(T) &= (r(t) - f(0,t))e^{-\alpha(T-t)} + f(0,T) \\ &+ \int_0^t \sigma(u)^2 \frac{e^{-\alpha(T+t-2u)} - e^{-2\alpha(T-u)}}{\alpha} du + \int_t^T \sigma(u)e^{-\alpha(T-u)}dW^T(u) \end{aligned} \tag{3.29}$$

under the forward measure P_T. This expression forms the basis of the numerical example we outline below. Note how the expectation of $r(T)$ given $r(t)$ depends only on $\sigma(u)$ for $u \in [0, t]$. Indeed, if we compare this to (3.14) (where we replace time s by T), we can see that working under the forward measure yields a simpler formula.

Bermudan swaption

Consider a unit notional amount and a set of dates $0 < T_0 < T_1 < \cdots < T_n$. The holder of a payer (or receiver) **Bermudan swaption** with strike K has the right to enter a payer (or receiver) interest rate swap at any time T_k for $k = 0, \ldots, l$ with $l < n$. The swap has reset dates $T_0, \ldots, T_n - 1$, settlement dates $T_1, \ldots, T_n$ and swap rate K. We assume that $l = n - 1$ and consider a payer Bermudan swaption, whose value at time 0 we denote by $\mathbf{Berm}(0)$. Note how this differs form the vanilla swaption, where the exercise date is fixed.

If the Bermudan swaption has not yet been exercised at time T_i for some $i < l$, then the holder has to decide whether to continue holding the option or to exercise immediately. The exercise value is

$$\mathrm{E}(T_i) = (\mathbf{PS}(T_i))^+ .$$

Whether or not it is optimal to exercise at time T_i will depend on the value of holding the option until time T_{i+1}. We denote this value by $\mathrm{C}(T_i)$ and refer to it as the continuation value. The value of the swaption at time T_i is

$$\mathbf{Berm}(T_i) = \max(\mathrm{E}(T_i), \mathrm{C}(T_i)). \tag{3.30}$$

The continuation value at time T_i can be computed recursively. We have

$$\mathrm{C}(T_i) = B(T_i, T_{i+1})\mathbb{E}_{P_{T_{i+1}}}(\mathbf{Berm}(T_{i+1})|\mathcal{F}_{T_i}),$$

where the expectation is taken under the forward measure $P_{T_{i+1}}$.

Calibrating to co-terminal swaptions

As explained in Remark 3.6, we calibrate to the set of vanilla options used to hedge the exotic instrument. For the Bermudan swaption this is the underlying set of co-terminal swaptions. We minimise the difference between the Hull–White model price $\mathbf{PSwpt}_{i,n}(0)$ and the market price (as expressed by Black's swaption formula) $\mathbf{PSwpt}^{\mathrm{mkt}}_{i,n}(t) = \mathbf{PSwpt}^{\mathrm{Black}}_{i,n}(t; \hat{\sigma}^{\mathrm{swpt}}_{i,n})$ of the underlying set of swaptions for $i = 0, \ldots, n-1$. More precisely, we solve the non-linear least squares optimisation problem

$$\min_{\sigma_1,\dots,\sigma_n} \sum_{i=0}^{n-1} \left(\mathbf{PSwpt}_{i,n}(0) - \mathbf{PSwpt}^{\text{mkt}}_{i,n}(0)\right)^2,$$

where $\mathbf{PSwpt}_{i,n}(0)$ depends on the short-rate volatility values $\sigma_1, \dots, \sigma_n$. To ensure that the parameters are positive we generally perform a constrained optimisation, where we specify an upper and lower bound.

This calibrates the short-rate volatility values. For the mean-reversion parameter α we use the autocorrelation between the sort rates.

Mean-reversion parameter and autocorrelation

The mean-reversion parameter α controls the the correlation between the short rates at different points in time, i.e. the autocorrelation. For $s > t$ the autocorrelation is

$$\begin{aligned}
\operatorname{Corr}(r(t), r(s)) &= \frac{\mathbb{E}(r(t)r(s))}{\sqrt{\operatorname{Var}(r(t))\operatorname{Var}(r(s))}} \\
&= \frac{\int_0^t \sigma^2(u)\mathrm{e}^{-\alpha(t+s-2u)}du}{\sqrt{\int_0^t \sigma^2(u)\mathrm{e}^{-2\alpha(t-u)}du \int_0^s \sigma^2(u)\mathrm{e}^{-2\alpha(s-u)}du}}.
\end{aligned}$$

If we assume that the volatility is constant, this can be simplified to

$$\operatorname{Corr}[r(t), r(s)] = \sqrt{\frac{\mathrm{e}^{2\alpha t} - 1}{\mathrm{e}^{2\alpha s} - 1}}.$$

Increasing the mean-reversion parameter α lowers the autocorrelation. This may have a significant impact on the valuation of exotic derivatives, and can be used to calibrate the value of α.

Numerical method

To apply the above recursive relation we begin by discretising the short rate domain, creating a grid of $N + 1$ values $r_0 < r_1 < \cdots < r_N$, where $r(0)$ is at a midpoint of the grid. Suppose that the time T_{i+1} Bermudan price $\mathbf{Berm}(T_{i+1}; r(T_{i+1}))$ is known for each grid point $r(T_{i+1}) = r_j$, and we want to compute $\mathbf{Berm}(T_i; r(T_i))$ at time T_i for each grid point $r(T_{i+1}) = r_j$. At time T_i the continuation value is

$$\mathrm{C}(T_i; r(T_i)) = B(T_i, T_{i+1})\mathbb{E}_{P_{T_{i+1}}}\left(\mathbf{Berm}(T_{i+1}; r(T_{i+1}))|\mathcal{F}_{T_i}\right).$$

The zero-coupon bond price $B(T_i, T_{i+1})$ calibrated to the term structure at time 0 is given by (3.13), while the short rate calibrated to the term structure at time 0 is given by (3.29), namely

$$\begin{aligned} r(T_{i+1}) = {} & (r(T_i) - f(0,T_i))\,\mathrm{e}^{-\alpha(T_{i+1}-T_i)} + f(0,T_{i+1}) \\ & + \int_0^{T_i} \sigma(u)^2 \frac{\mathrm{e}^{-\alpha(T_{i+1}+T_i-2u)} - \mathrm{e}^{-2\alpha(T_{i+1}-u)}}{\alpha} du \\ & + \int_{T_i}^{T_{i+1}} \sigma(u)\mathrm{e}^{-\alpha(T_{i+1}-u)} dW^{T_{i+1}}(u). \end{aligned}$$

The stochastic integral in the last expression is independent of $\mathcal{F}_{T_i}$ and normally distributed with mean 0 and variance

$$s_i^2 = \int_{T_i}^{T_{i+1}} \sigma(u)^2 \mathrm{e}^{-2\alpha(T_{i+1}-u)} du$$

under the forward measure $P_{T_{i+1}}$. Putting

$$\begin{aligned} m_i = {} & (r(T_i) - f(0,T_i))\,\mathrm{e}^{-\alpha(T_{i+1}-T_i)} + f(0,T_{i+1}) \\ & + \int_0^{T_i} \sigma(u)^2 \frac{\mathrm{e}^{-\alpha(T_{i+1}+T_i-2u)} - \mathrm{e}^{-2\alpha(T_{i+1}-u)}}{\alpha} du, \end{aligned}$$

we therefore have

$$\begin{aligned} & \mathbb{E}_{P_{T_{i+1}}}(\mathbf{Berm}(T_{i+1}; r(T_{i+1}))|\mathcal{F}_{T_i}) \\ & \quad = \frac{1}{\sqrt{2\pi s_i^2}} \int_{-\infty}^{\infty} \mathbf{Berm}(T_{i+1}; x) \exp\left(-\frac{(x-m_i)^2}{2s_i^2}\right) dx. \end{aligned}$$

The last integral can be evaluated numerically by using the known values of $\mathbf{Berm}(T_{i+1}; r(T_{i+1}))$ at the grid points $r(T_{i+1}) = r_j$. This makes it possible to compute the continuation value $\mathrm{C}(T_i; r(T_i))$ and hence the Bermudan price $\mathbf{Berm}(T_i; r(T_i))$ from the recursive relation (3.30) at each grid point $r(T_i) = r_j$.

To evaluate the option we start at the last exercise date T_l and work backwards in time. At T_l the continuation value is zero as there is no benefit in holding the option beyond the last exercise date. Then we proceed backwards from T_i to T_{i-1} for $i = l, \ldots, 0$ using the above recursive relationships to arrive at the Bermudan prices $\mathbf{Berm}(0; r(0))$ at time 0 for all grid points $r(0) = r_j$.

3.7 Two-factor Hull–White model

Many vanilla options are only weakly affected by the correlation between rates of different maturities and can be adequately priced using a one-factor model. Even more complex options such as Bermudans can be handled using a one-factor model. However, many exotic options are particularly sensitive to how rates of different maturities are correlated and must be modelled in a multi-factor framework. These include options which depend in a non-linear way on the difference between two rates or options whose payoff depends on two different interest rate curves.

The model we are going to examine is a simple extension of the one-factor Hull–White model with a stochastic mean-reversion level added to the drift term. To assist in the derivation of analytic formulae this model is then reformulated as a Gaussian two-factor model. The calculations are slightly more involved than in the one-factor case, but the overall approach is similar.

The SDE (3.8) for the short rate in the Hull–White model can be modified by adding a stochastic term $u(t)$ to the drift,

$$dr(t) = (\theta(t) + u(t) - \alpha r(t))dt + \delta dW(t), \tag{3.31}$$

where α, δ are constants, $\theta(t)$ is a deterministic function of time and $W(t)$ is a Brownian motion under the risk-neutral measure Q. Moreover, $u(t)$ satisfies the SDE

$$du(t) = -\beta u(t)dt + \varepsilon dZ(t) \tag{3.32}$$

with initial value $u(0) = 0$, where β and ε are constants and $Z(t)$ is another Brownian motion under the risk-neutral measure Q such that

$$dW(t)dZ(t) = \varrho dt \tag{3.33}$$

for a constant ϱ, the correlation between the two Brownian motions. Additionally, we assume that $\alpha \neq \beta$.

It is possible to obtain a bond pricing formula in terms of $r(t)$, $u(t)$ and the model parameters by following a similar argument to the derivation of formula (3.6) for the zero-coupon bond price in the Vasiček model and (3.10) in the one-factor Hull–White model. Though not difficult, this is rather involved. Instead, we represent the short rate in the two-factor Hull–White model as

$$r(t) = \phi(t) + x(t) + y(t), \tag{3.34}$$

where $\phi(t)$ is a deterministic function and

$$dx(t) = -\alpha x(t)dt + \sigma dU(t), \tag{3.35}$$
$$dy(t) = -\beta y(t)dt + \eta dV(t), \tag{3.36}$$

with initial conditions $x(0) = 0$, $y(0) = 0$, and where α, β, σ and η are constants (in fact α, β are the same constants as in (3.31) and (3.32)), and $U(t)$ and $V(t)$ are Brownian motions under the risk-neutral measure Q such that

$$dU(t)dV(t) = \rho dt. \tag{3.37}$$

Note that the correlation ρ between the Brownian motions $U(t)$ and $V(t)$ is not the same as the correlation ϱ between $W(t)$ and $Z(t)$.

Exercise 3.10 Show that in the two-factor Hull–White model defined by (3.31), (3.32) and (3.33) we can satisfy (3.34), (3.35), (3.36) and (3.37) if we set

$$\phi(t) = r(0)e^{-\alpha t} + \int_0^t \theta(s)e^{-\alpha(t-s)}ds,$$
$$y(t) = \frac{u(t)}{\alpha - \beta},$$
$$x(t) = r(t) - \phi(t) - y(t),$$

with constants σ, η and ρ, and Brownian motions $U(t)$ and $V(t)$ suitably defined in terms of $\alpha, \beta, \delta, \varepsilon, \varrho$ and $W(t), Z(t)$.

Gaussian two-factor approach

We have represented the two-factor Hull–White model (3.31), (3.32) as a **Gaussian two-factor model** (3.34), (3.35), (3.36), which helps us to derive a formula for zero-coupon bond prices.

Integrating (3.35) and (3.36) from t to s, we have

$$r(s) = \phi(s) + x(t)e^{-\alpha(s-t)} + y(t)e^{-\beta(s-t)} + \sigma \int_t^s e^{-\alpha(s-u)}dU(u) + \eta \int_t^s e^{-\beta(s-u)}dV(u). \tag{3.38}$$

As was the case with the Vasiček and one-factor Hull–White models, we can derive an analytic expression for zero-coupon bonds using the pricing formula (3.3). Integrating the short rate and taking expectation yields

$$B(t,T) = \exp\Bigg(- x(t)\frac{1 - e^{-\alpha(T-t)}}{\alpha} - y(t)\frac{1 - e^{-\beta(T-t)}}{\beta}, \\ - \int_t^T \phi(s)ds + \frac{1}{2}V(t,T)\Bigg), \tag{3.39}$$

where

$$V(t,T) = \frac{\sigma^2}{\alpha^2}\int_t^T \left(1 - e^{-\alpha(T-u)}\right)^2 du + \frac{\eta^2}{\beta^2}\int_t^T \left(1 - e^{-\beta(T-u)}\right)^2 du \\ + 2\rho\frac{\sigma\eta}{\alpha\beta}\int_t^T \left(1 - e^{-\alpha(T-u)}\right)(1 - e^{-\beta(T-u)})du. \tag{3.40}$$

Integrating, we get

$$V(t,T) = \frac{\sigma^2}{2\alpha^3}\left(-3 - e^{-2\alpha(T-t)} + 4e^{-\alpha(T-t)} + 2\alpha(T-t)\right) \\ + \frac{\eta^2}{2\beta^3}\left(-3 - e^{-2\beta(T-t)} + 4e^{-\beta(T-t)} + 2\beta(T-t)\right) \\ + 2\rho\frac{\sigma\eta}{\alpha\beta}\left(T - t - \frac{1 - e^{-\alpha(T-t)}}{\alpha} - \frac{1 - e^{-\beta(T-t)}}{\beta} + \frac{1 - e^{-(\alpha+\beta)(T-t)}}{\alpha+\beta}\right).$$

Exercise 3.11 Derive formula (3.39) for the bond price $B(t,T)$.

Fitting the current term structure

The time-dependent parameter $\phi(t)$ is chosen to fit the current interest rate term structure. If the model price fits the interest rate term structure at time 0 for any given maturity T, then we must have

$$B(0,T) = \exp\left(-\int_0^T \phi(s)ds + \frac{1}{2}V(0,T)\right).$$

Using (1.9), we find that the model fits the term structure at time 0 if

$$f(0,T) = \phi(T) - \frac{1}{2}\frac{\partial V(0,T)}{\partial T}, \tag{3.41}$$

where

$$\frac{\partial V(0,T)}{\partial T} = \frac{\sigma^2}{\alpha^2}\left(1 - e^{-\alpha T}\right)^2 + \frac{\eta^2}{\beta^2}\left(1 - e^{-\beta T}\right)^2 + 2\rho\frac{\sigma\eta}{\alpha\beta}\left(1 - e^{-\alpha T}\right)\left(1 - e^{-\beta T}\right).$$

Note that (3.41) gives us an expression for ϕ as a function of the instantaneous forward curve. However, we just need an expression for the integral of ϕ from time t to T rather than an explicit formula for ϕ. It is given by

$$\int_t^T \phi(s)ds = \ln \frac{B(0,t)}{B(0,T)} + \frac{1}{2}(V(0,T) - V(0,t)).$$

On substituting this into (3.39), we have following proposition.

Proposition 3.8
In the two-factor Hull–White model the zero-coupon bond price that gives an exact fit to the term structure of interest rates at time 0 *is*

$$B(t,T) = \frac{B(0,T)}{B(0,t)} \exp\Bigg(- x(t)\frac{1-e^{-\alpha(T-t)}}{\alpha} - y(t)\frac{1-e^{-\beta(T-t)}}{\beta} + \frac{1}{2}(V(0,t) - V(0,T) + V(t,T))\Bigg), \tag{3.42}$$

where V *is given by* (3.40) *and* $x(t), y(t)$ *solve the SDEs* (3.35), (3.36) *with* $x(0) = 0$ *and* $y(0) = 0$.

Bond option

In the two-dimensional Hull–White model the volatility of the logarithm of the bond price (3.42) is deterministic (see Exercise 3.12) and therefore the model yields an analytic formula for the bond option. The price at time 0 of a call option with strike K and expiry S written on a zero-coupon bond with maturity $T > S$ is given by

$$\mathbf{BC}(0; S, T, K) = B(0,T)N(d_+) - KB(0,S)N(d_-),$$

where

$$d_+ = \frac{\ln \frac{B(0,S)K}{B(0,T)} + \frac{1}{2}v(0,S)}{\sqrt{v(0,S)}}, \qquad d_- = d_+ - v(0,S)$$

with

$$v(0,S) = \frac{\sigma^2}{2\alpha^3}\left(1-e^{-\alpha(T-S)}\right)^2\left(1-e^{-2\alpha S}\right) + \frac{\eta^2}{2\beta^3}\left(1-e^{-\beta(T-S)}\right)^2\left(1-e^{-2\beta S}\right) + \frac{2\rho\sigma\eta}{\alpha\beta(\alpha+\beta)}\left(1-e^{-\alpha(T-S)}\right)\left(1-e^{-\beta(T-S)}\right)\left(1-e^{-(\alpha+\beta)S}\right). \tag{3.43}$$

Exercise 3.12 Show that the variance of $\ln B(S, T)$ is given by (3.43).

Caps and floors

Since both caps and floors can be expressed in terms of bond options, it is possible to derive an analytic expression for these instruments within the two-factor Hull–White model. The approach is identical to that in the one-factor case, so is not reproduced here. The key ingredient for these formulae is the log-variance of the bond price given by the formula above.