## Stochastic Interest Rates

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DARAGH MCINERNEY is a Director at the Valuation Modelling and Methodologies Group at UBS, and a researcher in mathematical finance at AGH University of Science and Technology in Kraków, Poland. He holds a PhD in Applied Mathematics from the University of Oxford, and has worked since 2001 as a quantitative analyst in both investment banking and fund management.

TOMASZ ZASTAWNIAK holds the Chair of Mathematical Finance at the University of York. He has authored about 50 research publications and six books. He has supervised four PhD dissertations and around 80 MSc dissertations in mathematical finance.

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# Stochastic Interest Rates

DARAGH MCINERNEY
*AGH University of Science and Technology, Kraków, Poland*

TOMASZ ZASTAWNIAK
*University of York, UK*

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To our daughters Teresa, Francesca, Karolina and Klementyna

# Contents

# Preface

In this volume of the 'Mastering Mathematical Finance' series we relax the assumption of constant interest rates adopted in the binomial or the Black–Scholes market models covered in earlier volumes, in particular [DMFM] and [BSM]. In general, interest rates are time dependent and random. Being closely linked to, and indeed determined by, fixed-income instruments traded in the market, the rates also depend on the maturity dates of the underlying instruments. This gives rise to the notion of term structure, i.e. the family of interest rates parameterised by the maturity date. We are going to study models describing the random evolution through time of the term structure, that is, of the entire family of interest rates for various maturities.

Because the rates for different maturities are related to one another and evolve simultaneously in time, their joint evolution is more intricate than that of a single quantity such as a stock price. There is not a single term structure model universally adopted as a benchmark to play a similar role as the Black–Scholes model does for stock prices. Instead, a range of alternative and to some extent complementary models are in use to capture various aspects of the evolution of the term structure. A selection of such models will be presented along with the associated interest rate derivative securities.

The prerequisites for this book are covered in some other volumes of the 'Mastering Mathematical Finance' series. These include probability theory [PF], stochastic calculus [SCF], and the Black–Scholes model [BSM]. Familiarity with Monte Carlo simulations [NMFC] will also be helpful.

We begin with various fundamental notions and properties associated with fixed-income instruments in Chapter 1 and the basic 'vanilla' interest rate derivatives in Chapter 2. Here we also cover the change of numeraire technique and introduce the notion of forward measure, a very useful alternative to the risk-neutral measure when pricing interest rate derivatives.

A number of short-rate models, in which the evolution of the entire term structure is driven by a single interest rate, namely the short rate, are covered in Chapter 3. In particular, the Merton, Vasiček and one-factor and two-factor Hull–White models are discussed in detail. In Chapter 4 we turn our attention to one-factor and multi-factor models of forward rates within what is known as the Heath–Jarrow–Morton (HJM) framework, and learn how the term structure is driven by the evolution of the family of forward rates.

Chapters 5, 6 and 7 are devoted to the LIBOR market model (LMM) and the swap market model (SMM). These models are presented and analysed in Chapter 5. In particular, Black's formula is derived for caplets and swaptions. This formula is essential for calibration to implied market volatilities, discussed in Chapter 6 along with the implementation of the LMM via Monte Carlo simulation. In Chapter 7 we introduce a range of options that can be valued within the LMM. Chapter 8 on modelling volatility skews and smiles concludes the volume.

The book contains a considerable number of examples and exercises, which are an important part of the course. The code and spreadsheets that were used to compute many of the numerical examples and plot some of the figures, along with the solutions to all exercises in this volume can be downloaded from www.cambridge.org/9781107002579.

# 1

# Fixed-income instruments



At its simplest, an interest rate is the rate that is charged or paid for the use of money. It is often expressed as an annual percentage of the notional amount. Throughout the text we will generally focus on what are known as 'interbank rates'. These are the interest rates at which banks borrow from and lend to each other in the interbank, or over-the-counter (OTC) market. The most important example of an interbank rate is the London Interbank Offered Rate, or LIBOR. The LIBOR rate is the interest rate at which banks offer to lend unsecured funds to each other in the London wholesale money market. Another related market rate is the swap rate, which is the fixed rate that a bank is willing to exchange for a series of payments based on the LIBOR rate.

In this chapter we present some basic terminology and definitions, together with an overview of fixed-income instruments such as forward rate agreements (FRAs), swaps, floating-rate notes and fixed-coupon bonds. Government-backed securities form another important class of interest rate instruments. In the USD market, securities such as Treasury bonds, Treasury notes and Treasury bills are issued by the US Treasury to finance government debt. All instruments are assumed to be default free, that is, we do not account for the possibility that the issuers may fail to honour their commitments.

We begin with the definition of the zero-coupon bond. Such bonds are not actively traded within the interbank market. The reason they are important, however, is because interbank interest rates such as LIBOR and swap rates can be defined in terms of zero-coupon bonds. The set of zero-coupon bonds for various time horizons is known as the zero-coupon curve. How the zero-coupon curve is estimated from market data such as LIBOR and swap rates is also discussed.

In this chapter, and indeed throughout this volume, time is measured in years.

## 1.1 Interest rates and bonds

A **zero-coupon bond** or **discount bond** with maturity date $T$ is a financial contract that guarantees the holder one dollar at time $T$. The bond can be thought of as the value at time $t < T$ of one dollar to be paid at time $T$. The zero-coupon bond maturing at time $T$ is often referred to as a $T$-bond, and its price at time $t$ is denoted by $B(t, T)$. Therefore, a zero-coupon bond is parameterised by two time indices, the current time $t$ and the maturity date $T$. By definition, $B(T, T) = 1$, and we have

$$0 < B(t, T) < 1$$

for $t < T$.

The dependence of $B(t, T)$ on the maturity date $T$ is known as the **term structure of discount factors** or **zero-coupon curve** at time $t$. The curve is a decreasing function of maturity.

### Spot interest rates

Having defined the zero-coupon bond $B(t, T)$, we now introduce the notion of the simply compounded interest rate. The **simply compounded spot rate** at time $t$ for maturity $T$ is defined as the annualised rate of return from holding the bond from time $t$ until maturity $T$. It is denoted by $L(t, T)$, and is defined as

$$L(t, T) = \frac{1 - B(t, T)}{(T - t)B(t, T)}. \tag{1.1}$$

The bond price $B(t, T)$ can be expressed in terms of the spot rate $L(t, T)$ as

$$B(t, T) = \frac{1}{1 + (T - t)L(t, T)}. \tag{1.2}$$

If interest rates are positive, we must have

$$B(t, S) > B(t, T)$$

for $t \leq S < T$.

An important example of a simply compounded rate is the **London Interbank Offered Rate (LIBOR)**. This is the interest rate at which banks offer to lend unsecured funds to each other in the London wholesale money market. From now on we shall identify $L(t, T)$ with the LIBOR rate.

**Remark 1.1**
LIBOR is the primary benchmark for short-term interest rates. Daily fixings of LIBOR are published by the British Bankers Association (BBA) shortly after 11 a.m. (GMT). It is a filtered average of quotes provided by a number of banks and can be thought of as representing the lowest real-world cost of unsecured funding in the London money market. It is produced for ten major currencies, Pound Sterling, US Dollar, Euro, Japanese Yen, Swiss Franc, Canadian Dollar, Australian Dollar, Swedish Krona, Danish Krona and the New Zealand Dollar. Fifteen maturities are quoted for each currency ranging from overnight to 12 months. LIBOR rates are widely used as a reference rate for a range of vanilla financial instruments such as forward rate agreements, short-term interest rate futures contracts and interest rate swaps.

The Euro Interbank Offered Rate (EURIBOR) is another example of a money-market rate and is compiled by the European Banking Federation. The EURIBOR rate is the benchmark rate for EUR-denominated instruments.

**Remark 1.2**
We defined the simply compounded spot rate at time $t$ as the rate of return over the interval $[t, T]$; see (1.1). For the case of USD LIBOR, however, the accrual period starts two London business days after time $t$ (the date on which the rate becomes fixed). For example, for the six-month USD LIBOR spot rate on 15 March 2010 the accrual period begins on 17 March 2010 and ends on 17 September 2010. These timing conventions differ from currency to currency. Interest is calculated on an Actual/360 basis (see Remark 1.6). Throughout the text we will assume that spot rates fix and start on the same date, unless explicitly stated otherwise.

Interest rates quoted in the market are almost always simply compounded. However, it can be mathematically more convenient to work with continuously compounded rates. The **continuously compounded spot rate** is the annualised logarithmic rate of return from holding the bond from time $t$ until maturity $T$. It is denoted by $R(t,T)$, and is defined as

$$R(t,T) = -\frac{\ln B(t,T)}{T-t}.$$

The zero-coupon bond price can be expressed in terms of $R(t,T)$ as

$$B(t,T) = \mathrm{e}^{-R(t,T)(T-t)}. \tag{1.3}$$

The continuously compounded spot rate can be thought of as a measure of the implied interest rate offered by the bond and is sometimes referred to as the **yield to maturity**. The graph of $R(t,T)$ versus maturity $T$ is known as the **yield curve** (see Figures 1.1 and 1.2). Yield curves are typically increasing or decreasing functions of $T$, but can often be inverted or 'hump' shaped.

**Exercise 1.1** Consider an annually compounded spot rate $L(0,T)$ maturing in one year, i.e. $T = 1$. Compute the continuously compounded spot rate $R(0,T)$ when $L(0,T) = 5\%$.

### Time value of money

The above definitions express the principle that today's value of one dollar paid at some time in the future is less than one dollar paid today. This is known as the **time value of money**.

A closely related notion is **discounted value** or **present value** (**PV**). It is the value today of a deterministic (known in advance) future payment or a series of deterministic future payments. We use the discount bond to express the present value. For example, an amount $A$ known at time $t$ to be paid at time $T > t$ has present value $B(t,T)A$ at time $t$.

The present value of a deterministic payment should not be confused with the more general concept of the discounted value of a random future payment. If we have a random payment $X$ at some future time $T > t$, its discounted value $B(t,T)X$ at time $t$ is also a random variable, whose value may be unknown at time $t$.

**Exercise 1.2** Consider a perpetual bond that pays one dollar at the end of each year forever. Assuming that $B(0,T) = (1+r)^{-T}$, where $r$ is a constant annually compounded rate of interest, show that the present value of the perpetual bond, that is, the sum of the present values of all the payments, can be written as a geometric series. Simplify the series to find the present value of the perpetual bond for $r = 5\%$.

### The bond price is a stochastic process

If the bond price were deterministic, then the following would have to be true.

**Proposition 1.3**

*Let $t < S < T$. If the zero-coupon bond price $B(S,T)$ were known at time $t$ (i.e. deterministic), then in the absence of arbitrage we would have*

$$B(t,T) = B(t,S)B(S,T). \tag{1.4}$$

*Proof* Suppose that $B(t,T) < B(t,S)B(S,T)$. Consider this strategy.

- At time $t$ we buy (go long) a $T$-bond and sell (go short) an amount $B(S,T)$ of $S$-bonds to give an income of $B(t,S)B(S,T) - B(t,T) > 0$.
- At time $S$ our short position in $S$-bonds matures and we are required to pay the amount $B(S,T)$. We raise this amount by selling one $T$-bond.
- At time $T$ our net position will be zero. The long position in $T$-bonds purchased at time $t$ cancels the short position in $T$-bonds purchased at time $S$.

Our strategy created a risk-free profit of $B(t,S)B(S,T) - B(t,T) > 0$ at time $t$, violating the no-arbitrage assumption. By adopting the opposite strategy, we can see that the reverse inequality $B(t,T) > B(t,S)B(S,T)$ would also give rise to an arbitrage opportunity. □

If we were to perform an empirical analysis of a bond price time series, it would quickly become apparent that condition (1.4) is not satisfied. The zero-coupon bond price should therefore be modelled as a stochastic process that evolves towards a known value at time $T$.

## 1.2 Forward rate agreements

Let $t < S < T$. A **forward rate agreement** (**FRA**) is a contract entered into at time $t$, when the issuer agrees to pay the holder at time $T$ the LIBOR rate $L(S,T)$ in exchange for a fixed rate $K$ applied to a notional amount $N$. The value of the payoff at time $T$ is given by

$$\tau(K - L(S,T))N, \tag{1.5}$$

where $\tau = T - S$ is the accrual period. Without loss of generality, we can assume a unit notional, $N = 1$. By definition, it costs nothing to enter into a FRA. Taking the time $t$ value of the cash flows described above and setting the resulting sum equal to zero, we can find the value of the fixed rate $K$ such that the FRA is zero. The time $t$ value of the fixed interest payment $\tau K$ is simply the discounted value $\tau K B(t, T)$. The time $t$ value of the floating payment is given by the following result.

**Proposition 1.4**
*The arbitrage-free value at time $t$ of the LIBOR-based payment $\tau L(S, T)$ at time $T$ is $B(t, S) - B(t, T)$.*

*Proof* To see this consider the following strategy.

- At time $t$ we buy (go long) an $S$-bond and sell (go short) a $T$-bond.
- At time $S$ the long position in $S$-bonds matures to yield one dollar. Use this income to buy an amount $1/B(S, T)$ of $T$-bonds.
- At time $T$ our net position will be $\dfrac{1}{B(S, T)} - 1$, which is equal to $\tau L(S, T)$.

We have replicated the payment at time $T$ using a self-financing strategy with an initial cost of $B(t, S) - B(t, T)$. In the absence of arbitrage this must be the value of the floating payment at time $t$. □

The value of the FRA at time $t$ is therefore

$$B(t, T)\tau K - B(t, S) + B(t, T).$$

Setting this equal to zero and solving for $K$, we find that the value of the fixed rate, known as the **forward LIBOR rate** or simply the **forward rate** and denoted by $F(t; S, T)$, is

$$F(t; S, T) = \frac{B(t, S) - B(t, T)}{\tau B(t, T)}. \tag{1.6}$$

The forward rate is a simply compounded rate parameterised by three time arguments: the present time $t$, the start of the spot LIBOR rate $S > t$ and the maturity date $T > S$.

**Exercise 1.3** Consider two annually compounded spot rates $L(0, S)$ and $L(0, T)$, maturing in one and two years respectively, i.e. $S = 1$ and $T = 2$. Compute $L(0, T)$ when $L(0, S) = 4\%$ and the forward rate $F(0; S, T) = 5.5\%$.

**Exercise 1.4** For two annually compounded spot rates $L(0,1) = 4\%$ and $L(0,2) = 5\%$, maturing in one and two years respectively, compute the one-year to two-year forward rate $F(0;1,2)$.

## 1.3 Forward interest rates and forward bond price

Assume we wish to enter into an agreement at time $t$ to purchase a $T$-bond at time $S$, where $t < S < T$. One of the simplest ways of determining the correct (arbitrage-free) amount $A$ known at time $t$ that we need to pay at time $S$ is to take the time $t$ value of the cash flows and set the resulting sum equal to zero,

$$-AB(t,S) + B(t,T) = 0.$$

The arbitrage-free amount we need to pay at time $S$ to purchase the $T$-bond is known as the **forward bond price** or **forward discount factor**. It is denoted by $\mathbf{FP}(t;S,T)$ and given by

$$\mathbf{FP}(t;S,T) = \frac{B(t,T)}{B(t,S)}. \tag{1.7}$$

By rearranging (1.6), it can be seen that the forward bond price can be expressed in terms of the forward rate as

$$\mathbf{FP}(t;S,T) = \frac{B(t,T)}{B(t,S)} = \frac{1}{1 + (T-S)F(t;S,T)}.$$

Therefore, we can think of the forward rate as the simply compounded rate of return over the time interval $[S,T]$ implied by the forward bond price.

We can quote forward rates as either simple rates or continuously compounded rates. The **continuously compounded forward rate** at time $t$ for expiry $S$ and maturity $T$ is denoted by $R(t;S,T)$. It is found by solving

$$\frac{B(t,T)}{B(t,S)} = \mathrm{e}^{-R(t;S,T)(T-S)}. \tag{1.8}$$

The above can be written as

$$R(t;S,T) = -\frac{\ln B(t,T) - \ln B(t,S)}{T-S}.$$

Using (1.3), we can write this in terms of the continuously compounded spot rates as

$$R(t; S, T) = \frac{R(t,T)(T-t) - R(t,S)(S-t)}{T-S}.$$

**Exercise 1.5** Given a continuously compounded spot rate $R(0,1) = 4\%$ and a continuously compounded forward rate $R(0,1,2) = 5.5\%$, compute the spot rate $R(0,2)$.

**Remark 1.5**

The forward rate $F(t; S, T)$ could be taken as a predictor of the actual spot interest rate $L(S,T)$ at time $S$. Indeed, $F(t; S, T)$ is the expectation of $L(S,T)$ under what is known as the $T$-forward measure (see Section 2.4 for more details).

### Instantaneous rates

The **instantaneous forward rate** at time $t$ for maturity $T$, which we denote by $f(t,T)$, can be thought of as the rate of return over an infinitesimally small time interval $[T, T + \delta T]$ or, more precisely,

$$f(t,T) = \lim_{\delta T \to 0} R(t, T, T + \delta T) = -\frac{\partial \ln B(t,T)}{\partial T}. \tag{1.9}$$

The dependence of $f(t,T)$ on the maturity $T$ is known as the **term structure of forward rates** (or **forward curve**) at time $t$.

A related concept is the **instantaneous short rate** or **risk-free rate** at time $t$, denoted by $r(t)$. It is the rate of return over the infinitesimal time interval $[t, t + \delta t]$, and is defined in terms of the instantaneous forward rate as

$$r(t) = f(t,t).$$

Although they are abstract concepts, the instantaneous forward and short rates play an important role in stochastic interest rate modelling. In Chapter 3 we cover models based on the short rate and then in Chapter 4 we study the seminal Heath–Jarrow–Morton model of the dynamics of the term structure of forward rates.

**Bond price formula**

Integrating (1.9) over the time interval $[t, T]$, we can see that

$$\int_t^T f(t,u)du = -\ln B(t,u)\Big|_t^T = -\ln B(t,T). \tag{1.10}$$

Hence the zero-coupon bond price can be expressed in terms of the instantaneous forward rates as

$$B(t,T) = \exp\left(-\int_t^T f(t,u)du\right). \tag{1.11}$$

**Exercise 1.6** Show that the instantaneous forward rate $f(t,T)$ and the continuously compounded spot rate $R(t,T)$ are related by

$$f(t,T) = R(t,T) + (T-t)\frac{\partial}{\partial T}R(t,T).$$

**Remark 1.6**

In the expression for the simply compounded forward rate the interest accrues over the time interval $[S, T]$. In reality, however, how interest accrues over time is determined by **day-count conventions**. There are three main cases.

- Actual/365: assume there are 365 days in a year, and calculate the actual number of days between the dates and divide by 365.
- 30/360: assume there are 30 days in a month and 360 days in a year, and calculate the number of days accordingly and divide by 360.
- Actual/360: assume there are 360 days in a year, and calculate the actual number of days between the dates and divide by 360.

For the market LIBOR rates the accrual basis is Actual/360.

**Example 1.7**

Consider a three-month USD LIBOR rate beginning on 17 March 2010 and maturing three months later on 17 June 2010. The accrual period is $92/360 = 0.25555$.

## 1.4 Money market account

The **money market account** is a risk-free security where interest accrues continuously at the instantaneous short rate $r(t)$. The short rate is typically modelled as a stochastic process with the assumption that almost all sample paths are Lebesgue integrable. The value of the money market account at time $t$ is denoted by $B(t)$, and is defined by the differential equation

$$dB(t) = r(t)B(t)dt$$

with $B(0) = 1$. Solving, we have

$$B(t) = \exp\left(\int_0^t r(u)du\right).$$

The money market account can be thought of as the amount earned by starting with a unit amount at time 0 and continually reinvesting it at the short rate $r(t)$ over the infinitesimal time interval $[t, t + \delta t]$. The money market account is often referred to as the bank-account numeraire.

## 1.5 Coupon-bearing bonds

A **fixed-coupon bond** is a financial instrument that pays the holder deterministic (known at time $t \leq T_0$) amounts $c_1, \ldots, c_n$, referred to as **coupon payments**, at times $T_1, \ldots, T_n$, where $T_0 < T_1 < \cdots < T_n$. At maturity, time $T_n$, the holder receives the notional or face value $N$ in addition to the final coupon $c_n$. Computing the price of a fixed-coupon bond is simply a matter of discounting each cash flow back to time $t$. The value of a fixed-coupon bond at time $t \leq T_0$, which we denote by $\mathbf{B}_{\text{fixed}}(t)$, is given by

$$\mathbf{B}_{\text{fixed}}(t) = \sum_{i=1}^{n} c_i B(t, T_i) + NB(t, T_n). \tag{1.12}$$

Coupons are typically quoted in terms of a fixed annualised rate of return $K$, known as the **coupon rate**. Each coupon is then defined as $c_i = \tau_i NK$ for $i = 1, \ldots, n$, where $\tau_i = T_i - T_{i-1}$.

A **floating-coupon bond** or **floating-rate note** is analogous to a fixed-coupon bond with the important difference that the coupon payment at time $T_i$ for $i = 1 \ldots, n$ is a function of the spot LIBOR rate $L(T_{i-1}, T_i)$, which is unknown (stochastic) at time $t < T_{i-1}$. For $i = 1, \ldots, n$ the coupon payment

$c_i$ at time $T_i$ is

$$c_i = \tau_i N L(T_{i-1}, T_i) = N\left(\frac{1}{B(T_{i-1}, T_i)} - 1\right),$$

where $\tau_i = T_i - T_{i-1}$.

By Proposition 1.4, in the absence of arbitrage the time $t$ value of the floating coupon payment at time $T_i$ is $N(B(t, T_{i-1}) - B(t, T_i))$. The value at time $t$ of the floating-coupon bond, which we denote by $\mathbf{B}_{\text{floating}}(t)$, is therefore

$$\begin{aligned}\mathbf{B}_{\text{floating}}(t) &= N \sum_{i=1}^{n} (B(t, T_{i-1}) - B(t, T_i)) + NB(t, T_n) \\ &= NB(t, T_0). \end{aligned} \tag{1.13}$$

Note that, at time $T_0$, the value of the floating-coupon bond is equal to its notional, $\mathbf{B}_{\text{floating}}(T_0) = N$. In such cases the bond is said to be trading **at par**.

**Exercise 1.7** The table below lists some 1-month GBP LIBOR rates from the period between January and June 2010.

| $i$ | $T_{i-1}$ | $T_i$ | $L(T_{i-1}, T_i)$ |
|---|---|---|---|
| 1 | 4 Jan 2010 | 1 Feb 2010 | 0.516 25% |
| 2 | 1 Feb 2010 | 1 Mar 2010 | 0.519 38% |
| 3 | 1 Mar 2010 | 1 Apr 2010 | 0.540 00% |
| 4 | 1 Apr 2010 | 4 May 2010 | 0.547 50% |
| 5 | 4 May 2010 | 1 Jun 2010 | 0.554 69% |
| 6 | 1 Jun 2010 | 1 Jul 2010 | 0.565 94% |

What was the cash flow of a floating-coupon bond starting at time $T_0$ and maturing at $T_6$ with six coupons payable at times $T_1, \ldots, T_6$ and face value 100 GBP?

## 1.6 Interest rate swaps

An **interest rate swap** is an OTC instrument in which two counterparties exchange a set of payments at a fixed rate of interest for a set of payments at a floating rate, typically the spot LIBOR rate. If the holder is paying the floating rate and receiving the fixed rate, the swap is said be a **receiver swap**. Alternatively, if the holder is receiving the floating rate and paying the fixed rate, the swap is called a **payer swap**.

Consider a unit notional amount $N = 1$ and a set of dates $T_0 < T_1 < \cdots < T_n$ with accrual periods $\tau_i = T_i - T_{i-1}$ for $i = 1, \ldots, n$. At time $T_i$, $i = 1, \ldots, n$, the holder of a payer swap pays a fixed amount $\tau_i K$, where $K$ is a preassigned fixed rate of interest, the **swap rate**, in exchange for a floating payment of $\tau_i L(T_{i-1}, T_i)$, where $L(T_{i-1}, T_i)$ is the spot LIBOR fixing at time $T_{i-1}$ for the $i$th accrual period. The number of payments $n$ is referred to as the **length of the swap**, the payment dates $T_1, \ldots, T_n$ as the **settlement dates**, and the dates $T_0, \ldots, T_{n-1}$ as the **reset dates**. The first reset date $T_0$ is called the **start date** of the swap. If current time $t < T_0$, the swap agreement is referred to as a **forward-starting** payer or receiver swap.

Applying Proposition 1.4, we can see that the value at time $t \leq T_i$ of the time $T_i$ floating payment is $B(t, T_{i-1}) - B(t, T_i)$. The value at time $t \leq T_0$ of the forward payer swap is therefore

$$\begin{aligned}\mathbf{PS}(t) &= \sum_{i=1}^{n}(B(t, T_{i-1}) - B(t, T_i)) - K \sum_{i=1}^{n} \tau_i B(t, T_i) \\ &= (B(t, T_0) - B(t, T_n)) - K \sum_{i=1}^{n} \tau_i B(t, T_i). \end{aligned} \tag{1.14}$$

From the above expression it can be seen that the payer swap can be expressed as the difference between the floating-coupon bond (1.13) and the fixed-coupon bond (1.12),

$$\mathbf{PS}(t) = \mathbf{B}_{\text{floating}}(t) - \mathbf{B}_{\text{fixed}}(t).$$

The **forward swap rate**, denoted by $S_{0,n}(t)$, is the value of the fixed rate $K$ that makes the time $t$ value of the forward swap zero. Equivalently, it is the value of $K$ that makes the time $t$ value of the floating-coupon bond equal to that of the fixed-coupon bond. This gives

$$S_{0,n}(t) = \frac{B(t, T_0) - B(t, T_n)}{\sum_{i=1}^{n} \tau_i B(t, T_i)}. \tag{1.15}$$

The denominator is often referred to as the **swap annuity** or **level**.

The swap rate can be written in terms of the forward bond price (1.7) by dividing both the numerator and the annuity by $B(t, T_0)$,

$$S_{0,n}(t) = \frac{1 - \mathbf{FP}(t; T_0, T_n)}{\sum_{i=1}^{n} \tau_i \mathbf{FP}(t; T_0, T_i)}.$$

The above formulae for the swap rate are often expressed in terms of the forward rates $F(t, T_{i-1}, T_i)$ for $i = 1, \ldots, n$. To see this note that the time $t$ value of $\tau_i L(T_{i-1}, T_i)$, the floating-rate payment at time $T_i$, can be written as $\tau_i F(t, T_{i-1}, T_i) B(t, T_i)$. The value at time $t \leq T_0$ of the forward payer swap (1.14) can therefore be expressed in terms of the forward rates,

$$\mathbf{PS}(t) = \sum_{i=1}^{n} \tau_i F(t; T_{i-1}, T_i) B(t, T_i) - K \sum_{i=1}^{n} \tau_i B(t, T_i).$$

Setting the above formula equal to zero and rearranging, we can express $S_{0,n}(t)$ as the sum of weighted forward rates

$$S_{0,n}(t) = \sum_{i=1}^{n} w_i(t) F(t; T_{i-1}, T_i), \tag{1.16}$$

where the weights $w_i(t)$ are given by

$$w_i(t) = \frac{\tau_i B(t, T_i)}{\sum_{j=1}^{n} \tau_j B(t, T_j)} = \frac{\tau_i \mathbf{FP}(t; T_0, T_i)}{\sum_{j=1}^{n} \tau_j \mathbf{FP}(t; T_0, T_j)}.$$

**Remark 1.8**

We have expressed the swap rate in terms of zero-coupon bond prices. In reality, however, swap rates are traded benchmark securities, which we can use to determine the zero-coupon curve. In the next section we use a collection of **spot-starting** or **co-initial** swap rates, i.e. swap rates with various maturities that share the same start date, to build a zero-coupon curve by a method known as bootstrapping.

**Remark 1.9**

In the above description of the interest rate swap, the fixed and floating payments occur on the same dates. In reality, for a USD swap, fixed-payment dates or coupon dates are typically semiannual, and the floating-payment dates are quarterly, corresponding to the three-month spot LIBOR. Moreover, interest accrues on a 30/360 basis on the fixed leg and an Actual/360 basis on the floating leg.

**Exercise 1.8** Using the data in the table below, compute the swap rate starting in one year with four semiannual payments over a two-year period.

| Maturity $T_i$ | $B(0, T_i)$ |
|---|---|
| 0.5 | 0.9756 |
| 1 | 0.9518 |
| 1.5 | 0.9286 |
| 2 | 0.9060 |
| 2.5 | 0.8839 |
| 3 | 0.8623 |

**Exercise 1.9** Consider a set of dates $0 = T_0 < T_1 < \cdots < T_n$. Given a set of co-initial swap rates $S_{0,i}(0)$ for $i = 1, \ldots, n$, show how to iteratively solve for the discount factors $B(0, T_i)$.

## 1.7 Yield curve construction

In this section we examine how to calculate the zero-coupon curve, or equivalently the yield curve, from market data. We use a set of spot-starting swap rates to derive an approximation of the forward curve, that is, the term structure of forward rates $f(t, T)$ for $T > t$, where $t$ is the current date (the spot date).

The problem of computing the forward curve from a finite set of swap rates is somewhat ill posed mathematically, as we do not have enough market data to uniquely determine the forward curve. We need to employ some form of interpolation. The assumptions we make, therefore, will play a role in determining the curve.

There are a number of possible interpolation schemes. The most common is that the instantaneous forward rates are taken to be constant between the maturities of the swap contracts. Therefore, the forward curve at time $t$ will be approximated by a piecewise constant function.

The approximation for the forward curve can then be used to determine the zero-coupon curve via the bond-pricing equation (1.11) and the yield curve from (1.3). The resulting yield curve will be continuous but non-differentiable.

### Yield curve from swap rates

We are given a sequence of co-initial interest rate swaps starting at the spot date $t$ and such that $T_0, \ldots, T_{n_i-1}$ are the reset dates and $T_{n_i}$ is the maturity date for the $i$th swap, where $t \leq T_0 < T_1 < \cdots$ and where the lengths of the swaps form an increasing sequence $n_1 < n_2 < \cdots$. We denote the accrual periods by $\tau_j = T_j - T_{j-1}$ for $j = 1, 2, \ldots$.

The value on the spot date $t$ of a spot-starting swap is zero. Hence, by (1.15),

$$B(t, T_0) - B(t, T_{n_i}) = r_i \sum_{j=1}^{n_i} \tau_j B(t, T_j) \tag{1.17}$$

for each $i = 1, 2, \ldots$, where the $i$th swap rate is denoted by $r_i = S_{0,n_i}(t)$ to keep the notation simple.

We adopt the piecewise constant interpolation of the instantaneous forward rate

$$f(t, T) = \begin{cases} f_1 & \text{for } t \leq T \leq T_{n_1}, \\ f_{i+1} & \text{for } T_{n_i} < T \leq T_{n_{i+1}}, \ i = 1, 2, \ldots . \end{cases}$$

By (1.11), for $i = 1$ and $j = 0, \ldots, n_1$ we have

$$B(t, T_j) = \exp\left(-\int_t^{T_j} f(t, u)du\right) = \exp(-f_1(T_j - t)),$$

and can write (1.17) as

$$\exp(-f_1(T_0 - t)) - \exp(-f_1(T_{n_1} - t)) = r_1 \sum_{j=1}^{n_1} \tau_j \exp(-f_1(T_j - t)).$$

We then solve for $f_1$ using a root-finding algorithm such as the bisection or Newton–Raphson methods (see, for example, [NMFC]).

For the $i$th forward rate $f_i$ an iterative procedure can be set up as follows. Suppose we have calculated the forward curve out to the maturity $T_{n_i}$ of the $i$th swap. We now determine the constant forward rate $f_{i+1}$ from time $T_{n_i}$ to the next swap rate maturity $T_{n_{i+1}}$. We can write (1.17) as

$$B(t, T_{n_{i+1}}) + r_{i+1} \sum_{j=n_i+1}^{n_{i+1}} \tau_j B(t, T_j) = B(t, T_0) - r_{i+1} \sum_{j=1}^{n_i} \tau_j B(t, T_j).$$

Table 1.1 *USD swap rates on 18 May 2011.*

| Length $n_i$ | Swap rate $r_i$ | Maturity $T_{n_i}$ | Forward rate $f_i$ |
|---|---|---|---|
| 1 | 0.003 70 | 21 May 2012 | 0.003 6774 |
| 2 | 0.007 42 | 20 May 2013 | 0.011 1694 |
| 3 | 0.012 05 | 20 May 2014 | 0.021 4233 |
| 4 | 0.016 49 | 20 May 2015 | 0.030 1666 |
| 5 | 0.020 56 | 20 May 2016 | 0.037 5366 |
| 7 | 0.026 78 | 21 May 2018 | 0.043 7546 |
| 10 | 0.032 40 | 20 May 2021 | 0.047 8592 |
| 12 | 0.034 80 | 22 May 2023 | 0.049 7284 |
| 15 | 0.037 15 | 20 May 2026 | 0.049 9878 |
| 20 | 0.038 92 | 20 May 2031 | 0.046 8864 |
| 25 | 0.039 79 | 20 May 2036 | 0.045 6238 |
| 30 | 0.040 25 | 20 May 2041 | 0.044 6587 |

By (1.11), for $j = n_i + 1, \ldots, n_{i+1}$ we have

$$B(t, T_j) = \exp\left(-\int_t^{T_j} f(t,u)du\right) = B(t, T_{n_i}) \exp\left(-\int_{T_{n_i}}^{T_j} f(t,u)du\right)$$
$$= B(t, T_{n_i}) \exp(-f_{i+1}(T_j - T_{n_i})),$$

so we get

$$\exp(-f_{i+1}(T_{n_{i+1}} - T_{n_i})) + r_{i+1} \sum_{j=n_i+1}^{n_{i+1}} \tau_j \exp(-f_{i+1}(T_j - T_{n_i}))$$
$$= B(t, T_{n_i})^{-1} \left(B(t, T_0) - r_{i+1} \sum_{j=1}^{n_i} \tau_j B(t, T_j)\right).$$

The bond prices $B(t, T_j)$ are known for $j = 0, \ldots, n_i$ if the forward curve has already been computed up to maturity $T_{n_i}$, and the last equation can be solved for $f_{i+1}$ by using a root-finding algorithm once again.

**Example 1.10**

We use a set of spot-starting (co-initial) swap rates listed in Table 1.1 with maturities $1, 2, 3, 4, 5, 7, 10, 12, 15, 20, 25, 30$ years and based on USD LIBOR to build the forward curve out to 30 years.

By convention, the start date $T_0$ of such a swap is two business days after the spot date $t$. The swaps pay interest on dates that use the same modified

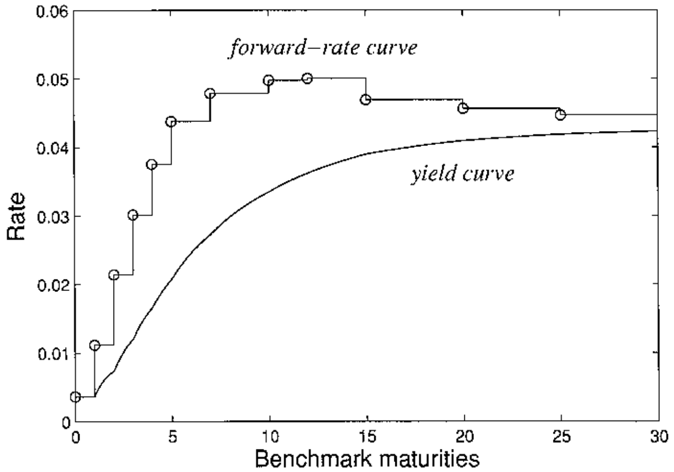


**Figure 1.1** Instantaneous forward-rate curve $f(t, T)$ and yield curve $R(t, T)$ from swap rates in Example 1.10.

following-business-day conventions as USD LIBOR rates. If the maturity date of the LIBOR rate does not fall on a business day, then the next business day will be used unless that business day extends into a next calendar month, in which case the business day that precedes the maturity date is used. The floating-payment dates are quarterly (every three months) to correspond to a three-month LIBOR, and the fixed-payment dates (coupon dates) are semiannual (every six months). Interest is computed on an Actual/360 day basis on the floating side of the swap, and on a 30/360 day basis on the fixed side.

For the data in Table 1.1 the spot date $t$ is 18 May 2011 and the start date $T_0$ is 20 May 2011 for each of the swap rates. The maturities are 21 May 2012 for the first swap, 20 May 2013 for the second swap, and so on.

In Figure 1.1 we plot the piecewise constant approximation to the instantaneous forward-rate curve derived form swap rates on 18 May 2011 as given in Table 1.1. The maturity dates for the spot swap rates are marked by circles. The (smooth) yield curve resulting from the forward-rate curve is also shown.

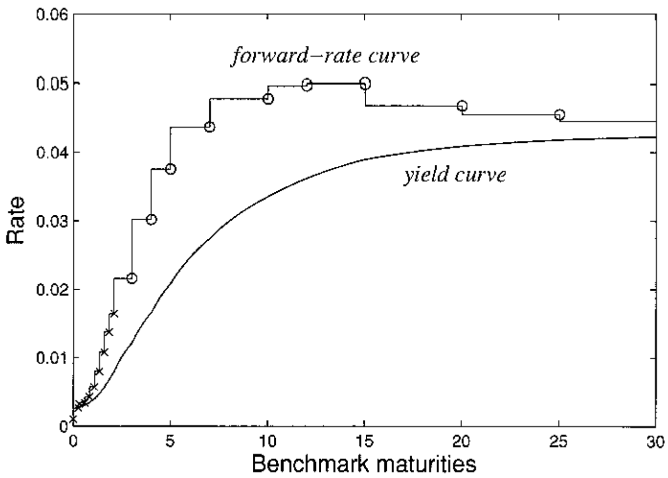


**Figure 1.2** Instantaneous forward-rate curve $f(t,T)$ and yield curve $R(t,T)$ from LIBOR rates, futures prices and swap rates on 18 May 2011; see Remark 1.11.

**Remark 1.11**

To keep the discussion simple, we have used just the spot-starting swap rates with the understanding that the short-end of the yield curve will be poorly approximated.

In practice, the data sets employed to calibrate the yield curve also include LIBOR rates and futures prices, along with swap rates. The piecewise approximation to the instantaneous forward-rate curve and the corresponding yield curve computed for such market data on 18 May 2011 are shown in Figure 1.2. LIBOR deposits are used for the first three months before moving to Eurodollar futures. The first eight quarterly futures contracts are highly liquid and are used to build the curve from between three months to two years. After two years we change from Eurodollar futures to spot-starting swaps beginning with the three-year rate. The maturity dates for LIBOR rates and Eurodollar futures are marked by crosses, while the maturity dates for the spot swap rates are indicated by circles.

The short end of the curve in Figure 1.2 will better approximate the actual interest rate term structure implied by the market as compared to Figure 1.1. Forward rates calculated at the change-over point from futures to swaps can often display large jumps or discontinuities.

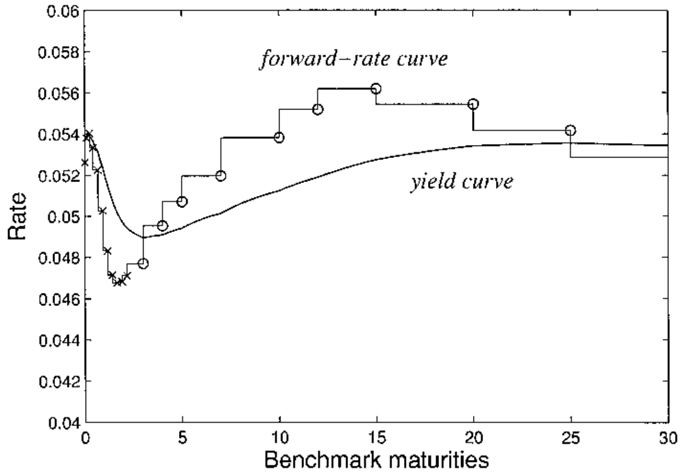


**Figure 1.3** Instantaneous forward-rate curve $f(t, T)$ and yield curve $R(t, T)$ from LIBOR rates, futures prices and swap rates on 18 Apr 2007; see Remark 1.12.

### Remark 1.12

Changes in the market due to the credit crisis have considerably complicated the construction of the yield curve. The LIBOR rate is the market rate for unsecured lending, therefore the credit worthiness of the counterparty becomes relevant. Though historically quite low, this credit-risk premium became important in the wake of the credit crisis. This makes the construction of the short end of the curve difficult. Indeed, it is now standard practice to construct a range of different yield curves rather than just one unique curve.

For comparison purposes, in Figure 1.3 we show the instantaneous forward rate computed from market data on 18 Apr 2007. This was just a few months before the start of the credit crisis. Here we can see that the yield curve is much flatter than in Figure 1.2.

## Bootstrapping the swap rates

Rather than calculating the instantaneous forward rates, where we need to employ a numerical root-finding scheme, we can estimate a set of discount factors by using a simple bootstrapping technique along with an interpolation scheme.

We begin by noting that equation (1.17) can be rearranged so that the discount bond at swap-rate maturity $T_{n_i}$ can be expressed in terms of the discount bonds on earlier dates,

$$B(t, T_{n_i}) = \frac{B(t, T_0) - \sum_{j=1}^{n_i - 1} r_i \tau_j B(t, T_j)}{1 + \tau_{n_i} r_i}. \tag{1.18}$$

This is an example of a **bootstrapping formula**.

> **Exercise 1.10** Using (1.18), compute the prices of discount bonds with maturity 1, 2, 3, 4 and 5 years from the swap rates in Table 1.1. Assume that $t = T_0$ for simplicity, so that $B(t, T_0) = 1$.

The bootstrapping formula (1.18) gives the discount bond prices $B(t, T_{n_i})$ on the swap maturity dates $T_{n_i}$. However, on the right-hand side we also need the bond prices $B(t, T_j)$ on the dates $T_j$, and some of these dates may fall between the swap maturity dates. The standard approach to this problem is to perform a cubic spline interpolation between the benchmark maturities so that we get a swap rate for each $T_j$.

**Example 1.13**

For USD interest rate swaps we are typically working with a set of spot-starting swap rates with maturities 1, 2, 3, 4, 5, 7, 10, 12, 15, 20, 25 and 30 years. But we also need the swap rates for each annual maturity between these dates. In Figure 1.4 the market swap rates from Table 1.1 are indicated by circles and the interpolated ones by crosses.

We can compare the discount factors calculated by this approach with those in Example 1.10 by plotting the discount curve implied by the forward curve shown in Figure 1.2. This is done in Figure 1.5. As we can see, the agreement, though not exact, is reasonable.

The key advantage of this approach is that it is computationally very fast. In Chapter 5 on the LIBOR market model we will use the bootstrapping formula (1.18) to calculate a set of discount factors and hence LIBOR rates between two dates; see Example 5.8.

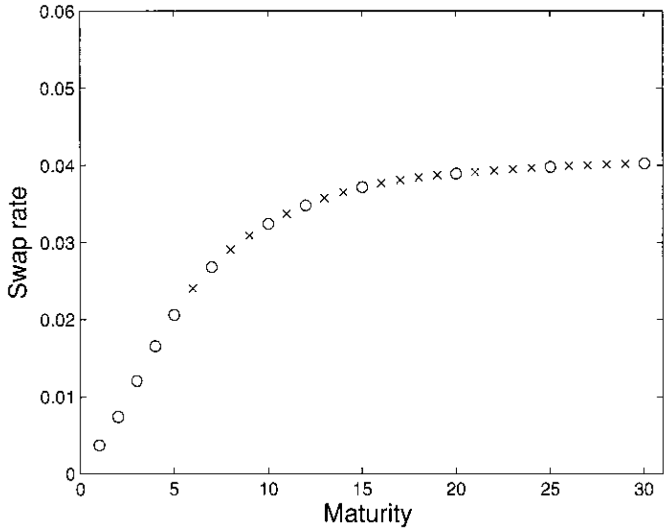


**Figure 1.4** Market swap rates (circles) and interpolated swap rates (crosses) for data in Table 1.1; see Example 1.13.

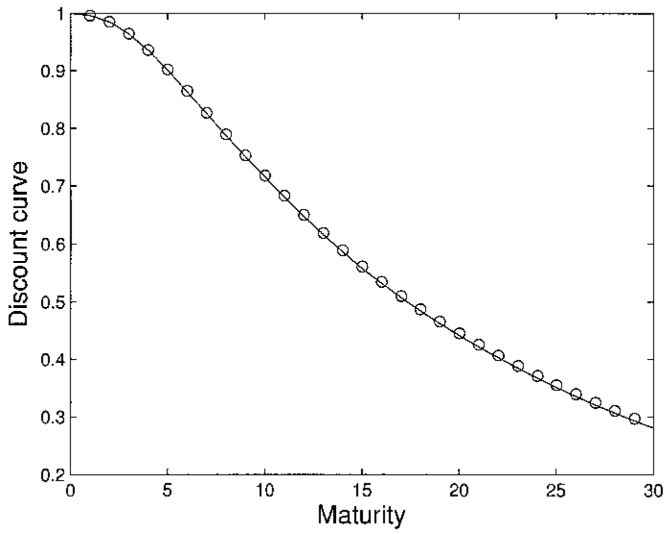


**Figure 1.5** Discount curve $B(t,T)$ implied by the yield curve in Figure 1.1 (solid line) and discount bond prices computed by bootstrapping with interpolation (circles); see Example 1.13.

# 2

# Vanilla interest rate options and forward measure



We begin this chapter by assuming the existence of a risk-neutral (or martingale) measure, before going on to present the change of numeraire technique. Then we define the forward measure and discuss vanilla interest rate options.

We work on a probability space $(\Omega, \mathcal{F}, P)$, with $P$ being the empirical (market) probability measure, equipped with a filtration $\mathcal{F}_t$, which captures the information available up to and including time $t \geq 0$. Suppose that the market consists of a set of securities and the money market account such that the price $V(t)$ of each security is adapted to the filtration, as is the price $B(t)$ of a unit of the money market account.

**Assumption 2.1**

There exists a probability measure $Q$ equivalent to $P$ such that the price $V(t)$ of any security discounted by $B(t)$ is a martingale under $Q$, that is, for any $0 \leq t < T$

$$\mathbb{E}_Q\left(\frac{V(T)}{B(T)}\middle|\mathcal{F}_t\right) = \frac{V(t)}{B(t)}. \tag{2.1}$$

We say that $Q$ is a **risk-neutral measure** or **martingale measure**. This assumption is closely linked to the absence of arbitrage.

## 2.1 Change of numeraire

In the context of stochastic interest rates the risk-neutral measure $Q$ is generally difficult to work with. In contrast to the Black–Scholes framework, the money market account is now stochastic. For example, consider a payoff at time $T$ that depends on a LIBOR rate $L(S,T)$, where $t < S \leq T$, such as the FRA covered in Chapter 1. The FRA payoff at time $T$ is $\tau(K - L(S,T))$, where $\tau = T - S$ is the accrual period of the LIBOR rate. Under the risk-neutral measure $Q$, the value of the FRA at time $t < S$ is

$$V(t) = B(t)\mathbb{E}_Q\left(B(T)\tau(K - L(S,T))|\mathcal{F}_t\right).$$

To calculate this expectation we need to model the joint probability distribution of the money market account $B(T)$ and the LIBOR rate $L(S,T)$. This can be avoided and replaced by the distribution of a single random variable under a suitable choice of measure.

When changing from one measure to another, a key concept is that of a **numeraire**, defined as any traded asset that pays no dividends and whose price $A(t)$ is positive at any time $t \geq 0$. The money market account $B(t)$ is an example of a numeraire.

**Proposition 2.2**

*For any choice of numeraire $A(t)$ and for any $T > 0$ there is a probability measure $P_A$ equivalent to $Q$ such that the price $V(t)$ of any security discounted by $A(t)$ becomes a martingale under $P_A$ in the time interval $[0,T]$, that is,*

$$\mathbb{E}_{P_A}\left(\frac{V(T)}{A(T)}\middle|\mathcal{F}_t\right) = \frac{V(t)}{A(t)} \quad \text{for} \quad 0 \leq t \leq T.$$

*Proof* According to Assumption 2.1, since the numeraire is a tradeable security, its price $\frac{A(t)}{B(t)}$ discounted by the money market account is a martingale under the risk-neutral measure $Q$. It follows that

$$\mathbb{E}_Q\left(\frac{A(T)}{B(T)}\right) = \frac{A(0)}{B(0)}.$$

As a result,

$$Z = \frac{B(0)}{A(0)}\frac{A(T)}{B(T)}$$

is a positive random variable such that $\mathbb{E}_Q(Z) = 1$. This enables us to define a probability measure $P_A$ by taking $Z$ as its density (Radon–Nikodym derivative) with respect to $Q$,

$$\frac{dP_A}{dQ} = Z.$$

Since $Z > 0$, it follows that $P_A$ is equivalent to $Q$ and $\frac{dQ}{dP_A} = \frac{1}{Z}$ is also an integrable random variable under $Q$.

We know that for any security its discounted price $\frac{V(t)}{B(t)}$ is a martingale under $Q$. This means, in particular, that $\frac{V(T)}{B(T)}$ is integrable under $Q$, which in turn implies that the random variable

$$\frac{V(T)}{A(T)} = \frac{B(0)}{A(0)}\frac{V(T)}{B(T)}\frac{1}{Z}$$

is integrable under $P_A$, and the Bayes formula for conditional expectation

$$\mathbb{E}_{P_A}\left(\frac{V(T)}{A(T)}\middle|\mathcal{F}_t\right) = \frac{\mathbb{E}_Q\left(\frac{dP_A}{dQ}\frac{V(T)}{A(T)}\middle|\mathcal{F}_t\right)}{\mathbb{E}_Q\left(\frac{dP_A}{dQ}\middle|\mathcal{F}_t\right)} \tag{2.2}$$

holds for any $t$ such that $0 \leq t \leq T$; see [PF]. Observe that

$$\mathbb{E}_Q\left(\frac{dP_A}{dQ}\middle|\mathcal{F}_t\right) = \frac{B(0)}{A(0)}\mathbb{E}_Q\left(\frac{A(T)}{B(T)}\middle|\mathcal{F}_t\right) = \frac{B(0)}{A(0)}\frac{A(t)}{B(t)}$$

because $\frac{A(t)}{B(t)}$ is a martingale under $Q$. Futhermore,

$$\mathbb{E}_Q\left(\frac{dP_A}{dQ}\frac{V(T)}{A(T)}\middle|\mathcal{F}_t\right) = \frac{B(0)}{A(0)}\mathbb{E}_Q\left(\frac{V(T)}{B(T)}\middle|\mathcal{F}_t\right) = \frac{B(0)}{A(0)}\frac{V(t)}{B(t)}$$

since $\frac{V(t)}{B(t)}$ is a martingale under $Q$. It follows that

$$\mathbb{E}_{P_A}\left(\frac{V(T)}{A(T)}\middle|\mathcal{F}_t\right) = \frac{V(t)}{A(t)},$$

completing the proof. □

**Exercise 2.1** Show that the time 0 price of a European derivative security with payoff $X$ and exercise time $T > 0$ can be written as

$$V(0) = A(0)\mathbb{E}_{P_A}\left(\frac{X}{A(T)}\right),$$

irrespective of the choice of numeraire $A$.

We have seen in the proof of Proposition 2.2 that the **Radon–Nikodym derivative** of $P_A$ with respect to $Q$ is given by

$$\frac{dP_A}{dQ} = \frac{B(0)}{A(0)}\frac{A(T)}{B(T)}. \tag{2.3}$$

The **Radon–Nikodym density process** is defined as

$$\left.\frac{dP_A}{dQ}\right|_t = \mathbb{E}_Q\left(\left.\frac{dP_A}{dQ}\right|\mathcal{F}_t\right)$$

for $0 \le t \le T$. It is clearly a martingale under $Q$, and

$$\left.\frac{dP_A}{dQ}\right|_t = \mathbb{E}_Q\left(\left.\frac{B(0)}{A(0)}\frac{A(T)}{B(T)}\right|\mathcal{F}_t\right) = \frac{B(0)}{A(0)}\frac{A(t)}{B(t)}.$$

We frequently switch between different numeraires when working with stochastic interest rates, which distinguishes this case from the Black–Scholes setting, where typically the money market account plays the role of numeraire. A popular choice of numeraire is the zero-coupon bond. The associated measure is known as the forward measure.

## 2.2 Forward measure

Consider the price at time $t$ of a European derivative security with time $T$ payoff $X$. It is given by the usual valuation formula

$$V(t) = B(t)\mathbb{E}_Q\left(\left.\frac{X}{B(T)}\right|\mathcal{F}_t\right). \tag{2.4}$$

To calculate the expectation we need to evaluate the joint distribution of the stochastic money market account $B(T)$ and the random payoff $X$. A simpler and more natural way of computing the above expectation is to use the zero-coupon $T$-bond $B(t,T)$ as numeraire.

The measure associated with the choice of $B(t,T)$ as numeraire is called the **forward measure** for settlement date $T$, and will be denoted by $P_T$.

It is a probability measure equivalent to $Q$ defined on the $\sigma$-field $\mathcal{F}_T$ with Radon–Nikodym derivative

$$\frac{dP_T}{dQ} = \frac{B(0)}{B(0,T)} \frac{B(T,T)}{B(T)} = \frac{1}{B(T)B(0,T)}. \tag{2.5}$$

The Radon–Nikodym density process is

$$\left.\frac{dP_T}{dQ}\right|_t = \mathbb{E}_Q\left(\frac{1}{B(T)B(0,T)}\middle|\mathcal{F}_t\right) = \frac{B(t,T)}{B(t)B(0,T)} \tag{2.6}$$

for $0 \le t \le T$, where clearly $\left.\frac{dP_T}{dQ}\right|_0 = 1$.

**Exercise 2.2** Show that

$$\frac{dP_S}{dP_T} = \frac{B(0,T)}{B(0,S)} \frac{1}{B(S,T)} \quad \text{when } S \le T,$$

and

$$\frac{dP_S}{dP_T} = \frac{B(0,T)}{B(0,S)} B(T,S) \quad \text{when } S \ge T.$$

The price at time $t$ of the time $T$ payoff $X$ can be written as

$$V(t) = B(t,T)\mathbb{E}_{P_T}\left(\frac{X}{B(T,T)}\middle|\mathcal{F}_t\right) = B(t,T)\mathbb{E}_{P_T}(X|\mathcal{F}_t). \tag{2.7}$$

This is consistent with Exercise 2.1 for $t = 0$. What have we gained? Instead of the joint distribution of $X$ and $B(T)$ under $Q$, all we need to know is the distribution of $X$ under $P_T$.

To derive an explicit expression for the change of measure we make the assumption that $B(t,T)$ obeys the SDE

$$dB(t,T) = B(t,T)r(t)dt + B(t,T)\Sigma(t,T)dW(t), \tag{2.8}$$

where $W(t)$ is standard Brownian motion under the risk-neutral measure $Q$, and $r(t)$ is the instantaneous short rate. By definition, $B(T,T) = 1$.

**Exercise 2.3** Suppose that

$$dB(t,T) = B(t,T)\mu(t,T)dt + B(t,T)\Sigma(t,T)dW(t),$$

where $W(t)$ is standard Brownian motion under the risk-neutral measure $Q$. Show that $\mu(t,T) = r(t)$ for each $t \in [0,T]$.

**Exercise 2.4** Solve the SDE (2.8) for $B(t,T)$ with the final condition $B(T,T) = 1$.

We can apply the Itô formula to show that the Radon–Nikodym density process $\xi(t) = \left.\frac{dP_T}{dQ}\right|_t$ given by (2.6) satisfies the SDE

$$d\xi(t) = \xi(t)\Sigma(t,T)dW(t). \tag{2.9}$$

**Exercise 2.5** Show that the Radon–Nikodym density process does indeed satisfy (2.9).

Solving SDE (2.9) for $\xi(t)$ with the initial condition $\xi(0) = 1$, we obtain

$$\xi(t) = \left.\frac{dP_T}{dQ}\right|_t = \exp\left(\int_0^t \Sigma(u,T)dW(u) - \frac{1}{2}\int_0^t \Sigma(u,T)^2 du\right).$$

For $t = T$ this gives the Radon–Nikodym derivative of $P_T$ with respect to $Q$.

**Proposition 2.3**

*If the zero-coupon bond price $B(t,T)$ obeys the SDE* (2.8), *then*

$$\frac{dP_T}{dQ} = \exp\left(\int_0^T \Sigma(u,T)dW(u) - \frac{1}{2}\int_0^T \Sigma(u,T)^2 du\right).$$

**Exercise 2.6** Suppose that $B(t,T)$ satisfies the SDE

$$dB(t,T) = B(t,T)r(t)dt + B(t,T)\sum_{i=1}^{n}\Sigma_i(t,T)dW_i(t),$$

where the $W_i(t)$ are independent standard Brownian motions under the risk-neutral measure $Q$ for $i = 1,\ldots,n$. Derive a formula for $\frac{dP_T}{dQ}$ in terms of $\Sigma_i(t,T)$ and $W_i(t)$.

## 2.3 Forward contract

Consider a forward contract where we agree to buy an asset on some future delivery date $T$ for an amount $F(t,T)$ determined at time $t < T$. The forward price $F(t,T)$ is such that the value of the forward contract, denoted by $V(t)$, is zero at time $t$. The cash flow at time $T$ is $X(T) - F(t,T)$, where $X(T)$ is the asset price. The value of the forward contract at time $t$ is

$$V(t) = B(t,T)\mathbb{E}_{P_T}(X(T) - F(t,T)|\mathcal{F}_t) = 0.$$

Since $F(t,T)$ is an $\mathcal{F}_t$-measurable random variable, the forward price is

$$F(t,T) = \mathbb{E}_{P_T}(X(T)|\mathcal{F}_t) \tag{2.10}$$

for any $t \in [0,T]$. Therefore the forward price is a martingale under $P_T$. In other words, the forward price can be regarded as an unbiased estimator of $X(T)$, where the expectation is taken under the forward measure.

## 2.4 Martingales under the forward measure

Recall the definition of the forward bond price $\mathbf{FP}(t;S,T)$ in Chapter 1. It is the price set at time $t$ to be paid at time $S$ for a zero-coupon $T$-bond. By (2.10), the forward price can be expressed as

$$\mathbf{FP}(t;S,T) = \mathbb{E}_{P_S}(B(S,T)|\mathcal{F}_t) = \mathbb{E}_{P_S}\left(\frac{B(S,T)}{B(S,S)}\middle|\mathcal{F}_t\right) = \frac{B(t,T)}{B(t,S)}.$$

The above is a specific instance of the general principle that the price of any non-dividend-paying tradeable asset divided by a numeraire is a martingale under the measure associated with the numeraire. This is a consequence of Proposition 2.2. In the case under consideration the numeraire is the $S$-bond.

Another example of a martingale under the forward measure is the forward rate. By (1.6),

$$F(t;S,T) = \frac{1}{\tau}\left(\frac{B(t,S)}{B(t,T)} - 1\right),$$

where $\tau = T - S$. The simply compounded forward rate $F(t;S,T)$ is a martingale under the forward measure $P_T$.

For $t = S$ the forward rate $F(S;S,T)$ is the spot LIBOR rate $L(S,T)$. Via the martingale property we have

$$F(t;S,T) = \mathbb{E}_{P_T}(L(S,T)|\mathcal{F}_t) \tag{2.11}$$

for each $t \in [0,S]$. The forward rate at time $t$ can be regarded as an unbiased estimator of the future spot LIBOR rate under the forward measure $P_T$.

## 2.5 FRAs and interest rate swaps: the forward measure

In Chapter 1 we derived the value of an FRA by using a portfolio replication argument. Here we derive the same expression by valuing the cash flow at maturity under the appropriate forward measure. A payment at time $T$ based on the spot LIBOR rate $L(S,T)$ is exchanged for a payment based on a fixed rate $K$. The cash flow at time $T$ is $\tau(K-L(S,T))$, where $\tau = T-S$ is the accrual period of the spot LIBOR rate. The fixed rate $K$ is the rate chosen so that the value $V(t)$ of the FRA is zero at time $t$. Under the forward measure $P_T$ the value of the FRA is

$$V(t) = B(t,T)\mathbb{E}_{P_T}(\tau(K - L(S,T))|\mathcal{F}_t) = 0.$$

Using the martingale property (2.11) and the fact that $K$ is $\mathcal{F}_t$-measurable, we can simplify the above expression to

$$V(t) = B(t,T)\tau K - B(t,T)\tau F(t;S,T) = 0.$$

Therefore the fixed rate is the forward rate, $K = F(t;S,T)$.

Formula (1.14) for the forward payer swap can also be derived by using forward measures. Consider a unit notional amount and a set of dates $T_0 < T_1 < \cdots < T_n$ with accrual periods $\tau_i = T_i - T_{i-1}$ for $i = 1,\ldots,n$. At time $T_i$ the holder of the payer swap pays a fixed amount $\tau_i K$ in exchange for a floating payment of $\tau_i L(T_{i-1},T_i)$. The value at time $t$ of the cash flow can be evaluated as

$$\begin{aligned}
\mathbf{PS}(t) &= \sum_{i=1}^{n} B(t,T_i)\mathbb{E}_{P_{T_i}}(\tau_i(L(T_{i-1},T_i) - K)|\mathcal{F}_t) \\
&= \sum_{i=1}^{n} B(t,T_i)\tau_i(F(t;T_{i-1},T_i) - K) \\
&= B(t,T_0) - B(t,T_n) - K\sum_{i=1}^{n} \tau_i B(t,T_i),
\end{aligned}$$

where we use the fact that the forward rate $F(t,T_{i-1},T_i)$ is a martingale under the forward measure $P_{T_i}$.

## 2.6 Option pricing in the forward measure

We now apply the techniques developed in this chapter to price a European option written on a zero-coupon bond. If we assume that, for each $T > 0$, the zero-coupon bond $B(t,T)$ follows the SDE (2.8) with a deterministic

$\Sigma(t,T)$, which means that $B(t,T)$ is log-normally distributed, then we can derive a Black–Scholes-type formula for the option price.

Let us consider a call option with expiry time $S$ and strike $K$ written on a zero-coupon bond $B(t,T)$, where $0 < S < T$. At any time $t < S$ the option can be priced as

$$\begin{aligned} V(t) &= B(t)\mathbb{E}_Q\left( \frac{(B(S,T)-K)^+}{B(S)} \middle| \mathcal{F}_t \right) \\ &= B(t)\mathbb{E}_Q\left( \frac{B(S,T)}{B(S)} \mathbf{1}_{\{B(S,T)\geq K\}} \middle| \mathcal{F}_t \right) - KB(t)\mathbb{E}_Q\left( \frac{1}{B(S)} \mathbf{1}_{\{B(S,T)\geq K\}} \middle| \mathcal{F}_t \right), \end{aligned}$$

where $\{B(S,T) \geq K\}$ is the exercise set. Using Proposition 2.2, we can change the numeraire to $B(t,T)$ in the first expectation in the last line, and to $B(t,S)$ in the second expectation. This gives

$$\begin{aligned} V(t) &= B(t,T)\mathbb{E}_{P_T}\left( \{\mathbf{1}_{\{B(S,T)\geq K\}} \middle| \mathcal{F}_t \right) - KB(t,S)\mathbb{E}_{P_S}\left( \mathbf{1}_{\{B(S,T)\geq K\}} \middle| \mathcal{F}_t \right) \\ &= B(t,T)P_T(B(S,T) \geq K | \mathcal{F}_t) - KB(t,S)P_S(B(S,T) \geq K | \mathcal{F}_t). \end{aligned} \tag{2.12}$$

To calculate these probabilities we need to know the distribution of $B(S,T)$ under $P_T$ and also under $P_S$.

We begin with $P_S$. Recall that the forward bond price $\mathbf{FP}(t;S,T) = \frac{B(t,T)}{B(t,S)}$ is a martingale under $P_S$. Given that the bond prices follow (2.8), we can verify by means of the Itô formula that

$$\begin{aligned} d\mathbf{FP}(t;S,T) &= \mathbf{FP}(t;S,T)\,(\Sigma(t,T) - \Sigma(t,S))\,dW(t) \\ &\quad + \mathbf{FP}(t;S,T)\Sigma(t,S)\,(\Sigma(t,S) - \Sigma(t,T))\,dt. \end{aligned} \tag{2.13}$$

**Exercise 2.7** Show that the forward bond price process $\mathbf{FP}(t;S,T)$ satisfies the SDE (2.13).

From Proposition 2.3 we know that

$$\frac{dP_S}{dQ} = \exp\left( \int_0^S \Sigma(t,S)dW(u) - \frac{1}{2}\int_0^S \Sigma(u,S)^2 du \right).$$

Hence, by Girsanov's theorem (see [BSM]), the process

$$W^S(t) = W(t) - \int_0^t \Sigma(u,S)du \quad \text{for} \quad t \in [0,S] \tag{2.14}$$

is a Brownian motion under $P_S$. Because $\mathbf{FP}(t;S,T)$ is a martingale under $P_S$, there should be no $dt$ term if we write $d\mathbf{FP}(t;S,T)$ in terms of

$dW^S(t)$ rather than $dW(t)$. Indeed, substituting $dW(t) = dW^S(t) + \Sigma(t,S)dt$ in (2.13), we obtain

$$d\mathbf{FP}(t;S,T) = \mathbf{FP}(t;S,T)\,(\Sigma(t,T) - \Sigma(t,S))\,dW^S(t). \tag{2.15}$$

Solving this linear SDE with $\mathbf{FP}(S;S,T) = B(S,T)$ gives

$$B(S,T) = \mathbf{FP}(t;S,T)\exp\left(\int_t^S (\Sigma(u,T) - \Sigma(u,S))\,dW^S(u) - \frac{1}{2}\int_t^S (\Sigma(u,T) - \Sigma(u,S))^2\,du\right). \tag{2.16}$$

Since $\Sigma(t,S)$ and $\Sigma(t,T)$ are deterministic and $W^S(t)$ is a Brownian motion under $P_S$, the expression in the exponent, which is equal to $\ln\frac{B(S,T)}{\mathbf{FP}(t;S,T)}$, is independent of $\mathcal{F}_t$ and normally distributed with variance

$$v(t,S) = \int_t^S (\Sigma(u,T) - \Sigma(u,S))^2\,du$$

and mean $-\frac{1}{2}v(t,S)$ under $P_S$. Moreover, $\mathbf{FP}(t;S,T)$ is $\mathcal{F}_t$-measurable. It follows that

$$P_S\left(B(S,T) \geq K|\mathcal{F}_t\right) = N(d_-), \tag{2.17}$$

where

$$N(x) = \frac{1}{\sqrt{2\pi}}\int_{-\infty}^x e^{-\frac{1}{2}y^2}dy$$

is the standard normal distribution function and

$$d_- = \frac{\ln\frac{\mathbf{FP}(t;S,T)}{K} - \frac{1}{2}v(t,S)}{\sqrt{v(t,S)}} = \frac{\ln\frac{B(t,T)}{KB(t,S)} - \frac{1}{2}v(t,S)}{\sqrt{v(t,S)}}. \tag{2.18}$$

To compute the first probability in (2.12) we take

$$W^T(t) = W(t) - \int_0^t \Sigma(u,T)du \quad \text{for} \quad t \in [0,S],$$

which, by Proposition 2.3 and Girsanov's theorem, is a Brownian motion under $P_T$. Since

$$W^S(t) = W(t) - \int_0^t \Sigma(u,S)du = W^T(t) + \int_0^t (\Sigma(u,T) - \Sigma(u,S))\,du,$$

substituting this into (2.16), we get

$$B(S,T) = \mathbf{FP}(t;S,T)\exp\left(\int_t^S (\Sigma(u,T) - \Sigma(u,S))\,dW^T(u) + \frac{1}{2}\int_t^S (\Sigma(u,T) - \Sigma(u,S))^2\,du\right).$$

The expression in the exponent, which is equal to $\ln \frac{B(S,T)}{\mathbf{FP}(t;S,T)}$, is independent of $\mathcal{F}_t$ and normally distributed with mean $\frac{1}{2}v(t,S)$ and variance $v(t,S)$ under $P_T$. As a result,

$$P_T\left(B(S,T) \geq K|\mathcal{F}_t\right) = N(d_+), \tag{2.19}$$

where

$$d_+ = \frac{\ln \frac{\mathbf{FP}(t;S,T)}{K} + \frac{1}{2}v(t,S)}{\sqrt{v(t,S)}} = \frac{\ln \frac{B(t,T)}{KB(t,S)} + \frac{1}{2}v(t,S)}{\sqrt{v(t,S)}}. \tag{2.20}$$

**Exercise 2.8** Verify (2.17) and (2.19) in detail using the fact that $\ln \frac{B(S,T)}{\mathbf{FP}(t;S,T)}$ is independent of $\mathcal{F}_t$ and normally distributed with mean $-\frac{1}{2}v(t,S)$ and variance $v(t,S)^2$ under $P_S$ and with mean $\frac{1}{2}v(t,S)$ and variance $v(t,S)^2$ under $P_T$.

We have established the following result, similar to the classical Black–Scholes formula; see [BSM].

**Theorem 2.4**

*If zero-coupon bonds follow the SDE* (2.8) *with deterministic log-volatility* $\Sigma(t,T)$, *then the time $t$ price* $\mathbf{BC}(t;S,T,K)$ *of a call option with strike $K$ and expiry $S$ written on a zero-coupon bond with maturity $T$, where* $0 \leq t < S < T$, *is*

$$\mathbf{BC}(t;S,T,K) = B(t,T)N(d_+) - KB(t,S)N(d_-) \tag{2.21}$$

*with $d_+, d_-$ given by* (2.18) *and* (2.20), *respectively.*

We shall return to this formula again in Chapters 3 and 4, where we study some term structure models which yield an analytic formula for the zero-coupon bond price.

### Put-call parity for bond options

We now use a replication argument to outline a put-call parity relationship for bond options. At time $t$ consider a portfolio that consists of a long position in one call option, $\mathbf{BC}(t; S, T, K)$, and a short position in one put option, $\mathbf{BP}(t; S, T, K)$. Both options have strike $K$ and expiry $S > t$, and are written on a zero-coupon bond with maturity $T > S$. The value at time $S$ of this portfolio is

$$(B(S,T) - K)^+ - (K - B(S,T))^+ = B(S,T) - K.$$

It can be replicated by a portfolio consisting of a long position in one zero-coupon bond maturing at time $T$ and a short position in $K$ zero-coupon bonds maturing at time $S$. In the absence of arbitrage both portfolios must have the same value at time $t$, that is,

$$\mathbf{BC}(t; S, T, K) - \mathbf{BP}(t; S, T, K) = B(t,T) - KB(t,S).$$

This relation is known as **put-call parity** for bond options.

**Exercise 2.9** Given formula (2.21) for the price of a call option, use put-call parity to derive the formula

$$\mathbf{BP}(t; S, T, K) = KB(t,S)N(-d_-) - B(t,T)N(-d_+) \qquad (2.22)$$

for the corresponding put option.

## 2.7 Caps and floors

An interest rate **cap** or **floor** can be thought of as an option on an interest rate (typically a spot LIBOR rate), or equivalently on a zero-coupon bond. Consider a unit notional amount and a set of dates $T_0 < T_1 < \cdots < T_n$ with accrual periods $\tau_i = T_i - T_{i-1}$ for $i = 1, \ldots, n$. At time $T_i$, where $i = 1, \ldots, n$, the holder of an interest rate cap or floor receives $\tau_i(L(T_{i-1}, T_i) - K)^+$ or $\tau_i(K - L(T_{i-1}, T_i))^+$, respectively, where $K$ is a preassigned fixed rate, known as the **cap rate**.

The payment at time $T_i$ for any $i = 1, \ldots, n$ is called a **caplet** or **floorlet**. The $i$th caplet is a European option with expiry $T_i$ written on the spot LIBOR rate $L(T_{i-1}, T_i)$ with payoff

$$\tau_i(L(T_{i-1}, T_i) - K)^+. \qquad (2.23)$$

The payoff of the caplet becomes known at time $T_{i-1}$ (when the LIBOR rate fixes), and it pays at time $T_i$.

In the following we show that an interest rate caplet is equivalent to a put option on a zero-coupon bond. We begin by noting that the caplet payoff (2.23) at time $T_i$ is an $\mathcal{F}_{T_{i-1}}$-measurable random variable, and is therefore equivalent to the payoff

$$\tau_i B(T_{i-1}, T_i)(L(T_{i-1}, T_i) - K)^+$$

at time $T_{i-1}$. Using the definition (1.1) of the LIBOR rate $L(T_{i-1}, T_i)$, we can write this time $T_{i-1}$ payoff as

$$(1 + \tau_i K)\left(\frac{1}{1 + \tau_i K} - B(T_{i-1}, T_i)\right)^+ .$$

The caplet can be viewed as a portfolio of $1 + \tau_i K$ put options with strike $(1+\tau_i K)^{-1}$ and expiry $T_{i-1}$, written on a zero-coupon bond with maturity $T_i$.

Summing the $n$ caplets, we get the price at time $t$ of the interest rate cap,

$$\mathbf{Cap}(t) = \sum_{i=1}^{n}(1 + \tau_i K)\mathbf{BP}\left(t; T_{i-1}, T_i, \frac{1}{1 + \tau_i K}\right),$$

where the formula for the zero-coupon bond put option is given by (2.22).

Similarly, we can show that the equivalent floorlet is a portfolio of $1+\tau_i K$ call options with strike $(1+\tau_i K)^{-1}$ and expiry $T_{i-1}$, written on a zero-coupon bond with maturity $T_i$. Again, summing the $n$ floorlets, we find that the price at time $t$ of the interest rate floor is

$$\mathbf{Flr}(t) = \sum_{i=1}^{n}(1 + \tau_i K)\mathbf{BC}\left(t; T_{i-1}, T_i, \frac{1}{1 + \tau_i K}\right),$$

with the zero-coupon bond call option price given by (2.21).

**Example 2.5**

Like interest rate swaps, caps and floors can either be spot or forward starting. For example, a 1-by-5 forward-starting cap consists of 16 caplets. The first caplet payoff becomes known one year from today and the last in four years and nine months. The contract matures in five years' time, when the last caplet is paid. Meanwhile, a seven-year spot-starting cap consists of 27 caplets and matures in seven years' time, when the last caplet is paid, with the first caplet payment becoming known in three months from today and the last in six years and nine months.

### Put-call parity for caps and floors

At time $t$ consider a portfolio that is long one cap, $\mathbf{Cap}(t)$, and short one floor, $\mathbf{Flr}(t)$, where both options have the same contractual features. At time $T_i$ the cash flow from this portfolio is $\tau_i B(T_{i-1}, T_i)(L(T_{i-1}, T_i) - K)$. This is none other than the cash flow at time $T_i$ for a payer swap. Therefore, at time $t$ we can replicate the cash flows generated by the portfolio by entering into a forward-starting payer swap $\mathbf{PS}(t)$ with settlement dates $T_1, \dots, T_n$ and swap rate $K$. This implies that

$$\mathbf{Cap}(t) - \mathbf{Flr}(t) = \mathbf{PS}(t).$$

## 2.8 Swaptions

Swaptions are European calls and puts on interest rate swaps. The expiry time of a swaption is typically the first reset date $T_0$ of the underlying interest rate swap. The holder of a **payer** (or **receiver**) **swaption** has the right to enter a payer (or receiver) swap at time $T_0$.

**Example 2.6**

A 2-into-5 year payer swaption with strike 3.5% gives the holder the right to enter a five-year payer swap starting in two years. That is, in two years' time the holder has the right (but not the obligation) to enter a five-year payer interest rate swap to receive a spot LIBOR floating rate in return for a fixed rate of 3.5%.

The length of the underlying interest rate swap (measured in years) is referred to as the **tenor**.

Consider a payer interest rate swap $\mathbf{PS}(t)$ with settlement dates $T_1, \dots, T_n$ and reset dates $T_0, \dots, T_{n-1}$. The payer swaption payoff is

$$\mathbf{PSwpt}_{0,n}(T_0) = (\mathbf{PS}(T_0))^+,$$

and its value at time $t$ is given by

$$\mathbf{PSwpt}_{0,n}(t) = B(t)\mathbb{E}_Q\left( \frac{(\mathbf{PS}(T_0))^+}{B(T_0)} \,\middle|\, \mathcal{F}_t \right).$$

The underlying payer swap can be expressed in a number of different ways.

Using the results in Section 1.6, we have the price of the payer swap at the first reset date $T_0$ (the swaption expiry) given by

$$\begin{aligned}\mathbf{PS}(T_0) &= \sum_{i=1}^{n} B(T_0, T_i)\tau_i(F(T_0; T_{i-1}, T_i) - K) \\ &= 1 - B(T_0, T_n) - K\sum_{i=1}^{n} \tau_i B(T_0, T_i).\end{aligned}$$

Hence the swaption can be thought of as a put option with strike 1 written on a coupon-bearing bond with coupon rate $K$. Furthermore, using formula (1.15) for the swap rate, we can write the payer swap as

$$\mathbf{PS}(T_0) = (S_{0,n}(T_0) - K)\sum_{i=1}^{n} \tau_i B(T_0, T_i).$$

It is clear that the payer swaption will be exercised if and only if the swap rate at expiry is greater than the strike.

### Put-call parity for swaptions

There is a simple put-call parity relationship for swaptions. At time $t$, consider a portfolio that consists of a long position in one payer swaption and a short position in one receiver swaption. Both options have strike $K$ and expiry $T_0$. The payoff at time $T_0$ for this portfolio is

$$(\mathbf{PS}(T_0))^+ - (-\mathbf{PS}(T_0))^+ = \mathbf{PS}(T_0).$$

Therefore, in the absence of arbitrage we must have, at time $t \le T_0$,

$$\mathbf{PSwpt}_{0,n}(t) - \mathbf{RSwpt}_{0,n}(t) = \mathbf{PS}(t).$$

The difference between a payer swaption and a receiver swaption is equal to a forward-starting swap.

A swaption is said to be at-the-money when the strike rate $K$ is equal to the forward swap rate $S_{0,n}(t)$. In this case the value of the forward-starting swap is zero, and by put-call parity the at-the-money payer swaption is equal to the at-the-money receiver swaption.

## 2.9 Implied Black volatility

Interest rate caps and floors can be expressed in terms of a series of caplets or floorlets. The market convention is to quote their prices as implied volatilities.

If the market quotes prices as implied volatilities, then how do we obtain the actual price? We need to use what is known as **Black's formula**. For the caplet payoff (2.23) at time $T_i$ this formula reads

$$\mathbf{Cpl}_i^{\text{Black}}(t;\sigma) = \tau_i B(t,T_i)(F(t;T_{i-1},T_i)N(d_+) - KN(d_-)), \tag{2.24}$$

where

$$d_+ = \frac{\ln\frac{F(t;T_{i-1},T_i)}{K} + \frac{1}{2}\sigma^2(T_{i-1}-t)}{\sigma\sqrt{T_{i-1}-t}}, \quad d_- = d_+ - \sigma\sqrt{T_{i-1}-t},$$

and where $F(t;T_{i-1},T_i)$ is the simply compounded forward LIBOR rate. The **implied Black volatility** or **spot volatility** $\hat{\sigma}_i^{\text{caplet}}$ of the $i$th caplet is defined as the unique solution to the equation

$$\mathbf{Cpl}_i^{\text{mkt}}(t) = \mathbf{Cpl}_i^{\text{Black}}(t;\hat{\sigma}_i^{\text{caplet}}), \tag{2.25}$$

where $\mathbf{Cpl}_i^{\text{mkt}}(t)$ is the caplet price derived from the quoted cap prices; see Remark 2.7.

For a cap with payment dates $T_1,\ldots,T_n$ the typical market convention is to assume that the same implied volatility is used for each caplet that constitutes the $n$-period cap. This volatility is denoted by $\hat{\sigma}_n^{\text{cap}}$ and referred to as the **flat volatility**. By definition, $\hat{\sigma}_n^{\text{cap}}$ solves the equation

$$\mathbf{Cap}^{\text{mkt}}(t) = \sum_{i=1}^{n} \mathbf{Cpl}_i^{\text{Black}}(t;\hat{\sigma}_n^{\text{cap}})$$

for the time $t$ market price of the cap. The flat volatility is therefore the unique volatility that must be inserted into Black's formula for each caplet to obtain the market price of the cap.

For example, the flat volatility for various cap maturities on 18 May 2011 (post credit crisis) is presented in Figure 2.1. The set of flat volatilities for each cap maturity is referred to as the **volatility term structure**. Before the credit crisis the graph of the volatility term structure was typically hump shaped. The implied volatility would be upward sloping for maturities up to two or three years and would fall gradually for caps with a longer time to maturity. However, since the credit crisis the implied volatilities for short maturities have become very high relative to other maturities.

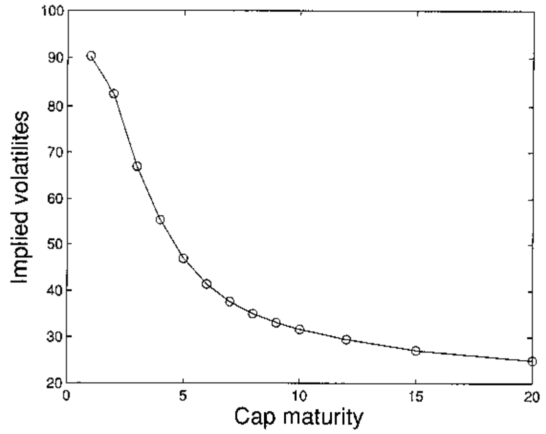


**Figure 2.1** USD at-the-money flat volatility for different cap maturities on 18 May 2011.

### Remark 2.7

The market quotes cap (floor) prices in terms of the flat volatility. The set of spot volatilities can be derived by applying a bootstrapping procedure to the set of flat volatilities quoted in the market.

### Black's formula for swaptions

As is the case with caps (and floors), the swaption price is quoted in the market as an implied Black volatility. Black's formula for a $T_0$ into $T_n - T_0$ payer swaption written on a swap with settlement dates $T_1, \ldots, T_n$ and reset dates $T_0, \ldots, T_{n-1}$ as described in Section 2.8 reads

$$\mathbf{PSwpt}_{0,n}^{\mathrm{Black}}(t;\sigma) = \sum_{i=1}^{n} \tau_i B(t,T_i)(S_{0,n}(t)N(d_+) - KN(d_-)), \qquad (2.26)$$

where

$$d_+ = \frac{\ln \dfrac{S_{0,n}(t)}{K} + \dfrac{1}{2}\sigma^2(T_0 - t)}{\sigma\sqrt{T_0 - t}}, \quad d_- = d_+ - \sigma\sqrt{T_0 - t},$$

with $S_{0,n}(t)$ being the swap rate. By definition, the **implied Black swaption volatility** $\hat{\sigma}_{0,n}^{\mathrm{swpt}}$ solves the equation

$$\mathbf{PSwpt}_{0,n}^{\mathrm{mkt}}(t) = \mathbf{PSwpt}_{0,n}^{\mathrm{Black}}(t;\hat{\sigma}_{0,n}^{\mathrm{swpt}})$$

Table 2.1 *At-the-money implied swaption volatilities for the USD market on 18 May 2011. The rows labelled 1y, ...,30y denote the time to expiry of the swaption. The tenor of the underlying swap is given by the columns labelled 1y, ...,20y.*

| | 1y | 2y | 3y | 5y | 7y | 10y | 15y | 20y |
|---|---|---|---|---|---|---|---|---|
| 1y | 74.70 | 59.70 | 49.40 | 37.90 | 32.40 | 28.60 | 24.60 | 23.30 |
| 2y | 51.90 | 43.50 | 38.10 | 32.20 | 29.10 | 26.60 | 23.70 | 22.60 |
| 3y | 38.90 | 34.30 | 31.50 | 28.40 | 26.50 | 24.70 | 22.50 | 21.70 |
| 5y | 27.50 | 26.40 | 25.60 | 24.50 | 23.50 | 22.40 | 20.70 | 20.30 |
| 10y | 20.90 | 20.70 | 20.40 | 20.10 | 19.70 | 19.40 | 18.00 | 17.50 |
| 15y | 18.10 | 18.10 | 18.10 | 18.00 | 17.90 | 17.50 | 16.00 | 15.80 |
| 20y | 17.30 | 17.10 | 16.90 | 16.40 | 16.30 | 16.10 | 15.20 | 15.00 |
| 30y | 17.40 | 17.40 | 17.10 | 16.60 | 16.70 | 17.00 | 16.70 | 17.00 |

for the market price of the swaption.

**Example 2.8**
In the market, at-the-money swaption prices are quoted as a grid of implied volatilities where one axis is the time to expiry and the other is the tenor of the underlying swap. See Table 2.1 for the at-the-money implied swaption volatilities for the USD market on 18 May 2011. We can see, for example, that on this date the implied volatility for the at-the-money 2-into-5 year payer swaption was 32.20%.

The market provides swaption volatilities only for certain standard maturities and tenors. If the volatility for 6-into-10 year swaptions is needed, then this will have to be inferred from the market quotes. To this end, market practitioners will take a set of quoted volatilities such as those in Table 2.1 and use interpolation to create a volatility surface such as that shown in Figure 2.2.

**Remark 2.9**
The Black formulae, which have been used since the 1970s, were motivated by Black's model for options on commodity futures, the key assumption being that the underlying variable follows a driftless log-normal process under some probability measure. It was not until the development of the LIBOR market model (LMM) in the late 1990s, however, that the Black

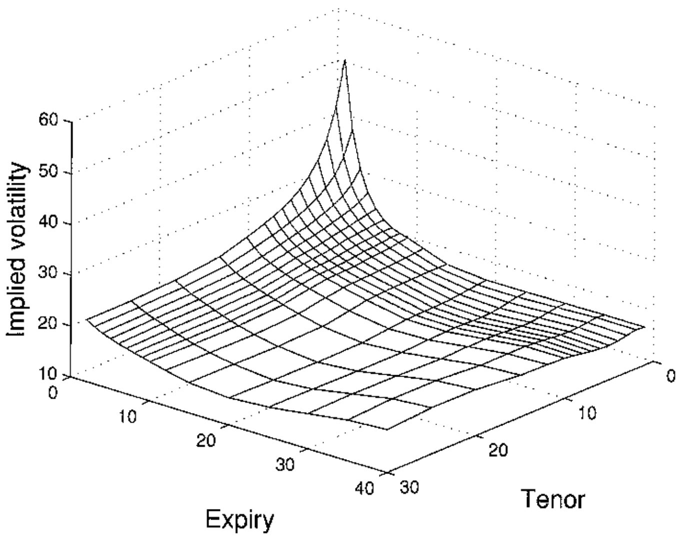


**Figure 2.2** USD at-the-money implied swaption surface for 18 May 2011.

formula for caps was given a logically consistent theoretical basis. Likewise, Black's formula for swaptions can be derived within the framework of the swap market model. We defer the derivation of Black's formulae until Chapter 5.

**Remark 2.10**
An implicit assumption in Black's formula is that for a given underlying the volatility is constant. In reality the volatility we need to insert into Black's formula to match the market price varies with the strike price. For a given swaption the graph of the implied volatility as a function of the strike $K$ is typically smile shaped. The strike price adds a 'third dimension' (in addition to the expiry and tenor) to the market quotes. Practitioners represent this data as a three-dimensional matrix, called the **volatility cube**.

In the following chapters we cover a number of stochastic interest rate models. To calibrate the parameters used in these models we need to use prices in the vanilla market. Therefore the set of implied Black volatilities quoted in the market (along with the interest rate term structure) play a key role in specifying the model parameters.

# 3

# Short-rate models



We need to choose a quantity to serve as a state variable that determines the interest rate term structure and its evolution in time. The first generation of stochastic interest rate models use the instantaneous short rate as the state variable. The two key advantages of short-rate models are their general simplicity and the fact that they often lead to analytic formulae for bonds and associated vanilla options. The tractability of short-rate models means that the price of a given derivative can often be computed quickly, important in situations where a large number of securities need to be valued. Indeed, throughout this chapter, we focus on short-rate models that allow discount bonds to be priced in closed form.

One-factor models assume that the entire interest rate term structure is driven by a one-dimensional Wiener process. Such models are usually suitable when pricing securities that depend on a single rate only, but for more complex products which depend on two or more different rates we may need to move to a multi-factor model driven by multi-dimensional Brownian motion. In the final section we present one of the most popular multi-factor short-rate models, the two-factor Hull–White model.

A weakness of the short-rate approach is that the instantaneous short rate is a mathematical idealisation rather than something that can be observed directly in the market. In the past decade, short-rate models have, to some extent, been superseded by the LIBOR market model (covered in Chapter 5), in which the stochastic state variable is a set of benchmark forward LIBOR rates. Nonetheless, short-rate models are particularly useful and remain popular due to their analytic tractability.

We continue to work under Assumption 2.1, which stipulates the existence of a probability measure $Q$ (the risk-neutral measure) equivalent to $P$ such that the value of any security discounted by the money market account is a martingale under $Q$.

## 3.1 General properties

In a **one-factor short-rate model** we assume that the short rate $r(t)$ satisfies an SDE of the form

$$dr(t) = \mu(t, r(t))dt + \sigma(t, r(t))dW(t), \tag{3.1}$$

where $W(t)$ is a Brownian motion under the risk-neutral measure $Q$. We also assume that, for any $T \geq 0$, the price of the $T$-bond depends on the instantaneous short rate,

$$B(t, T) = F(t, r(t); T),$$

where $F(t, r; T)$ is a sufficiently smooth function to allow all transformations that follow.

Using Itô's formula to compute $dB(t, T) = dF(t, r(t), T)$, we can write

$$dB(t, T) = \alpha(t, T)B(t, T)dt + \Sigma(t, T)B(t, T)dW(t),$$

where

$$\alpha(t, T) = \frac{\frac{\partial F}{\partial t}(t, r(t); T) + \mu(t, r(t))\frac{\partial F}{\partial r}(t, r(t); T) + \frac{1}{2}\sigma(t, r(t))^2\frac{\partial^2 F}{\partial r^2}(t, r(t); T)}{F(t, r(t); T)},$$

$$\Sigma(t, T) = \frac{\sigma(t, r(t))\frac{\partial F}{\partial r}(t, r(t); T)}{F(t, r(t); T)}.$$

Since $W(t)$ is a Brownian motion under $Q$, from Exercise 2.3 we know that $\alpha(t, T) = r(t)$. This shows that $F(t, r; T)$ must satisfy the partial differential equation

$$\frac{\partial F}{\partial t} + \mu\frac{\partial F}{\partial r} + \frac{1}{2}\sigma^2\frac{\partial^2 F}{\partial r^2} = rF \tag{3.2}$$

called the **term structure equation**. Because $B(T, T) = 1$, this equation

Table 3.1 *A selection of short-rate models.*

| Model | $\mu(t, r(t))$ | $\sigma(t, r(t))$ |
|---|---|---|
| Merton | $\alpha$ | $\sigma$ |
| Vasiček | $\theta - \alpha r(t)$ | $\sigma$ |
| Cox–Ingersoll–Ross | $\alpha(\beta - r(t))$ | $\sigma\sqrt{r(t)}$ |
| Dothan | $\alpha r(t)$ | $\sigma r(t)$ |
| Black–Derman–Toy | $\theta(t)r(t)$ | $\sigma(t)r(t)$ |
| Ho–Lee | $\theta(t)$ | $\sigma$ |
| Hull–White (extended Vasiček) | $\theta(t) - \alpha r(t)$ | $\sigma(t)$ |
| Black–Karasiński | $r(t)(\theta(t) - \alpha \ln r(t))$ | $\sigma r(t)$ |

can be solved subject to the final condition $F(T, r; T) = 1$ to find a formula for $F(t, r; T)$ and hence for the bond price $B(t, T)$ for any $t < T$.

Nonetheless, we are going to adopt an alternative approach, utilising the fact that $\frac{B(t,T)}{B(t)}$ is a martingale under the risk-neutral measure $Q$ and $B(t) = \exp\left(\int_0^t r(s)ds\right)$, so the bond price can be expressed as

$$B(t, T) = B(t)\mathbb{E}_Q\left(\frac{B(T,T)}{B(T)}\bigg|\mathcal{F}_t\right) = \mathbb{E}_Q\left(\exp\left(-\int_t^T r(s)ds\right)\bigg|\mathcal{F}_t\right). \tag{3.3}$$

## 3.2 Popular short-rate models

A number of models have been proposed for the dynamics of the short rate under the risk-neutral measure $Q$. These models specify a particular form for $\mu(t, r(t))$ and $\sigma(t, r(t))$. A list is shown in Table 3.1. It is not exhaustive, and various other possible functional forms for the risk-neutral drift and volatility can be proposed.

For the majority of models in Table 3.1 the short rate is either normally distributed (Merton, Vasiček, Ho–Lee and Hull–White) or log-normally distributed (Dothan, Black–Derman–Toy and Black–Krasiński). The Cox–Ingersoll–Ross model does not fit into either of these two categories as it features the non-central chi-squared distribution. Models where the short rate is normally distributed (Gaussian) are the most analytically tractable.

The model parameters are determined by calibrating to the current term structure of interest rates and to the implied volatilities of actively traded vanilla options (caps, floors and swaptions). However, on doing this calibration we quickly see that some of the models provide only an approximate fit to the present term structure of interest rates. This issue is discussed in more detail when we present the Vasiček model.

## 3.3 Merton model

This is arguably the simplest short-rate model. The SDE for the short rate is

$$dr(t) = \alpha dt + \sigma dW(t),$$

where $\alpha$ and $\sigma$ are constants, and where $W(t)$ is a Brownian motion under the risk-neutral measure $Q$. This gives

$$r(s) = r(t) + \alpha(s - t) + \sigma(W(s) - W(t)).$$

We can see that the short rate is normally distributed.

The Merton model gives rise to a simple analytic formula for the bond price, which we derive below.

### Bond pricing formula

Since the short rate is normally distributed, computing the expectation in (3.3) reduces to calculating the expected value of a log-normal random variable. Integrating between $t$ and $T$, we have

$$\int_t^T r(s)ds = r(t)(T - t) + \frac{1}{2}\alpha(T - t)^2 + \sigma \int_t^T W(s)ds - \sigma W(t)(T - t).$$

Noting that

$$d\left((T - s)W(s)\right) = -W(s)ds + (T - s)dW(s),$$

we can replace the last two terms to get

$$\int_t^T r(s)ds = r(t)(T - t) + \frac{1}{2}\alpha(T - t)^2 + \sigma \int_t^T (T - s)dW(s).$$

It follows that

$$X = \frac{1}{2}\alpha(T - t)^2 + \sigma \int_t^T (T - s)dW(s)$$

is independent of $\mathcal{F}_t$ and normally distributed with mean $m = \frac{1}{2}\alpha(T - t)^2$ and variance $s^2 = \sigma^2 \int_t^T (T - s)^2 ds = \frac{1}{3}\sigma^2(T - t)^3$ under the risk-neutral measure $Q$. Because the expectation of $\mathrm{e}^{-X}$ is $\mathrm{e}^{-m+\frac{1}{2}s^2}$, this proves the following result.

**Proposition 3.1**
*The zero-coupon bond price in the Merton model can be expressed as*

$$B(t,T) = \exp\left(-r(t)(T-t) - \frac{1}{2}\alpha(T-t)^2 + \frac{1}{6}\sigma^2(T-t)^3\right). \tag{3.4}$$

*Proof* By (3.3), since $r(t)$ is $\mathcal{F}_t$-measurable and $X$ is independent of $\mathcal{F}_t$,

$$\begin{aligned} B(t,T) &= \mathbb{E}_Q\left(\exp\left(-\int_t^T r(s)ds\right)\middle|\mathcal{F}_t\right) \\ &= \mathbb{E}_Q\left(\exp\left(-r(t)(T-t)\right)\exp\left(-X\right)\middle|\mathcal{F}_t\right) \\ &= \exp\left(-r(t)(T-t)\right)\mathbb{E}_Q\left(\exp\left(-X\right)\right) \\ &= \exp\left(-r(t)(T-t) - m + \frac{1}{2}s^2\right) \\ &= \exp\left(-r(t)(T-t) - \frac{1}{2}\alpha(T-t)^2 + \frac{1}{6}\sigma^2(T-t)^2\right). \end{aligned}$$

□

**Exercise 3.1** Let $B(t,T) = F(t,r(t);T)$ be the zero-coupon bond price (3.4) in the Merton model. Show that $F(t,r;T)$ satisfies the term structure equation (3.2).

## 3.4 Vasiček model

A problem with the Merton model is that the short rates can be negative, but an even more pressing issue is that it fails to model the dynamics correctly. An important empirical feature is that, when interest rates are high, there is a tendency for them to fall over time and, likewise, when the rates are low, they tend to rise. This is captured by the Vasiček model

$$dr(t) = (\theta - \alpha r(t))dt + \sigma dW(t),$$

where $\theta, \alpha, \sigma$ are constants and $W(t)$ is a Brownian motion under the risk-neutral measure $Q$.

This is considered the first realistic model of the short rate. Vasiček modelled the short rate using a mean-reverting drift. The drift is positive when $r(t)$ is below $\theta/\alpha$, and negative when $r(t)$ is greater than $\theta/\alpha$.

The SDE for $r(t)$ can be solved explicitly. Observe that

$$d\left(e^{\alpha t}r(t)\right) = \theta e^{\alpha t}dt + \sigma e^{\alpha t}dW(t).$$

Integrating from $t$ to time $s \geq t$, and multiplying both sides of the equality by $e^{-\alpha s}$, we get

$$r(s) = r(t)e^{-\alpha(s-t)} + \theta\int_t^s e^{-\alpha(s-u)}du + \sigma\int_t^s e^{-\alpha(s-u)}dW(u).$$

From the above we can see that the short rate is normally distributed under the risk-neutral measure $Q$ with mean

$$\mathbb{E}_Q(r(s)) = r(0)e^{-\alpha s} + \theta\int_0^s e^{-\alpha(s-u)}du = r(0)e^{-\alpha s} + \theta\frac{1-e^{-\alpha s}}{\alpha}$$

and variance given by the Itô isometry as

$$\mathrm{Var}(r(s)) = \sigma^2\int_0^s e^{-2\alpha(s-u)}du = \sigma^2\frac{1-e^{-2\alpha s}}{2\alpha}.$$

As time $s$ tends to infinity, the expectation of the short rate $r(s)$ tends to $\theta/\alpha$. The short rate is mean reverting. Moreover, because the short rate is normally distributed, it can become negative. This feature of the model might at first be considered a fatal flaw. Nonetheless, in practical applications the probability of the short rate becoming negative is often small.

### Bond pricing formula

Computing the integral of $r(s)$ from $t$ to $T$, we have

$$\begin{aligned}\int_t^T r(s)ds = r(t)&\int_t^T e^{-\alpha(s-t)}ds \\ &+ \theta\int_t^T\left(\int_t^s e^{-\alpha(T-u)}du\right)ds + \sigma\int_t^T\left(\int_t^s e^{-\alpha(T-u)}dW(u)\right)ds.\end{aligned}$$

Let us denote the integral in the first term on the right-hand side by

$$D(t,T) = \int_t^T e^{-\alpha(s-t)}ds = \frac{1-e^{-\alpha(T-t)}}{\alpha}. \qquad (3.5)$$

To compute the second and third terms observe that

$$d\left(\int_t^s e^{-\alpha(s-u)}du\right) = ds - \alpha\left(\int_t^s e^{-\alpha(s-u)}du\right)ds,$$

$$d\left(\int_t^s e^{-\alpha(s-u)}dW(u)\right) = dW(s) - \alpha\left(\int_t^s e^{-\alpha(s-u)}dW(u)\right)ds.$$

Hence

$$\begin{aligned}\left(\int_t^s e^{-\alpha(s-u)}du\right)ds &= d\left(\int_t^s \frac{1-e^{-\alpha(s-u)}}{\alpha}du\right)\\ &= d\left(\int_t^s D(u,s)du\right),\\ \left(\int_t^s e^{-\alpha(s-u)}dW(u)\right)ds &= d\left(\int_t^s \frac{1-e^{-\alpha(s-u)}}{\alpha}dW(u)\right)\\ &= d\left(\int_t^s D(u,s)dW(u)\right).\end{aligned}$$

As a result, integrating from $t$ to $T$, we find that

$$\int_t^T r(s)ds = r(t)D(t,T) + \theta\int_t^T D(u,T)du + \sigma\int_t^T D(u,T)dW(u).$$

It follows that

$$X = \theta\int_t^T D(u,T)du + \sigma\int_t^T D(u,T)dW(u)$$

is a random variable independent of $\mathcal{F}_t$, normally distributed with mean

$$m = \theta\int_t^T D(u,T)du$$

and variance given by the Itô isometry as

$$s^2 = \sigma^2\int_t^T D(u,T)^2du$$

under the risk-neutral measure $Q$. The expectation of $e^{-X}$ is $e^{-m+\frac{1}{2}s^2}$. Hence, using the bond pricing formula (3.3), we arrive at the following result just like in the proof of Proposition 3.1.

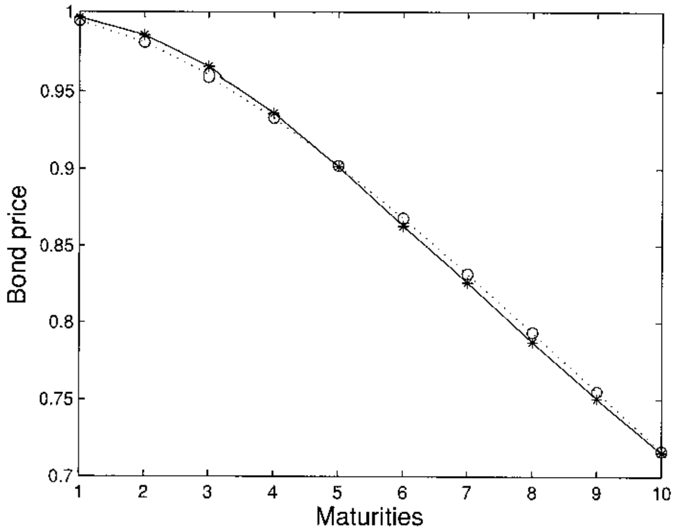


**Figure 3.1** Results for least squares optimisation for market price data on 18 May 2011 in Example 3.4. The market-implied zero-coupon bond prices $B^{\mathrm{mkt}}(t,T)$ are indicated by asterisks, and the model prices $B(t,T)$ by circles.

**Example 3.4**

In this numerical example we calibrate to a set of bond prices in Table 3.2 derived from the interest rate curve on 18 May 2011. The parameters computed by least squares optimisation are $\theta = 0.0099, \alpha = 0.131, \sigma = 0.01$.

To check if the parameters are in any sense financially realistic we can consider the ratio $\theta/\alpha$, which is the expectation of the short rate as time tends to infinity. This value is 0.075 66, which is plausible. However, for the given set of parameters the model provides only an approximate match to the current discount curve, as can be seen in Figure 3.1.

## 3.5 Hull–White model

To reproduce exactly the initial zero-coupon curve we need a time-varying parameter. This parameter is specifically chosen to provide an exact match to the initial term structure. Arguably, the most popular model with time-dependent parameters is the Hull–White model, in which the parameters corresponding to $\theta$ and $\sigma$ appearing in the Vasiček model are chosen to be deterministic functions of time,

$$dr(t) = (\theta(t) - \alpha r(t))dt + \sigma(t)dW(t), \tag{3.8}$$

where $\alpha$ is constant and $W(t)$ is a Brownian motion under the risk-neutral measure $Q$.

Integrating (3.8) from $t$ to $s \geq t$, we have

$$r(s) = r(t)\mathrm{e}^{-\alpha(s-t)} + \int_t^s \theta(u)\mathrm{e}^{-\alpha(s-u)}du + \int_t^s \sigma(u)\mathrm{e}^{-\alpha(s-u)}dW(u). \tag{3.9}$$

Adopting an approach analogous to that in the Vasiček model, we can derive an analytic expression for the zero-coupon bond price using the risk-neutral pricing formula (3.3). This yields

$$B(t,T) = \exp\Bigg( - r(t)D(t,T) - \int_t^T \theta(u)D(u,T)du + \frac{1}{2}\int_t^T \sigma(u)^2 D(u,T)^2 du\Bigg), \tag{3.10}$$

where $D(t,T)$ is given by (3.5).

**Exercise 3.4** Verify (3.10) for the Hull–White model by following a similar argument to that leading to formula (3.6) for the zero-coupon bond price in the Vasiček model.

The time-dependent parameter $\theta(t)$ can be chosen to match the current term structure. In fact, from (3.10) it can be seen that what is really needed is an expression for the integral $\int_t^T \theta(u)D(u,T)du$ rather than $\theta(t)$ itself. To this end we take

$$\begin{aligned}\ln\frac{B(0,T)}{B(0,t)} &= -r(0)(D(0,T) - D(0,t)) - \int_t^T \theta(u)D(u,T)du \\ &\quad - \int_0^t \theta(u)\left(D(u,T) - D(u,t)\right)du + \frac{1}{2}\int_t^T \sigma(u)^2 D(u,T)^2 du \\ &\quad + \frac{1}{2}\int_0^t \sigma(u)^2\left(D(u,T)^2 - D(u,t)^2\right)du.\end{aligned}$$

The integrals from 0 to $t$ can be rewritten by using the relation

$$D(u,T) = D(u,t) + D_t(u,t)D(t,T), \tag{3.11}$$

where $D_t(u,t)$ is the partial derivative of $D(u,t)$ with respect to $t$, to yield

$$\begin{aligned}\ln\frac{B(0,T)}{B(0,t)} &= -r(0)D_t(0,t)D(t,T) - \int_t^T \theta(u)D(u,T)du \\ &- D(t,T)\int_0^t \theta(u)D_t(u,t)du + \frac{1}{2}\int_t^T \sigma(u)^2 D(u,T)^2 du \\ &+ D(t,T)\int_0^t \sigma(u)^2 D(u,t)D_t(u,t)du + \frac{1}{2}D(t,T)^2\int_0^t \sigma(u)^2 D_t(u,t)^2 du.\end{aligned}$$

From (3.10), using formula (1.9) for the instantaneous forward rate, we obtain

$$f(0,t) = r(0)D_t(0,t) + \int_0^t \theta(u)D_t(u,t)du - \int_0^t \sigma^2(u)D(u,t)D_t(u,t)du. \qquad (3.12)$$

It follows that

$$\begin{aligned}\ln\frac{B(0,T)}{B(0,t)} &= -f(0,t)D(t,T) - \int_t^T \theta(u)D(u,T)du \\ &+ \frac{1}{2}\int_t^T \sigma(u)^2 D(u,T)^2 du + \frac{1}{2}D(t,T)^2\int_0^t \sigma(u)^2 D_t(u,t)^2 du,\end{aligned}$$

which gives the desired expression for $\int_t^T \theta(u)D(u,T)du$ in terms of $B(0,t)$, $B(0,T)$ and $f(0,t)$, that is, in terms of the current term structure. Substituting this expression into (3.10), we get the following result.

**Proposition 3.5**
*In the Hull–White model the zero-coupon bond price at time $t \geq 0$ that gives an exact fit to the term structure of interest rates at time 0 is*

$$B(t,T) = \frac{B(0,T)}{B(0,t)}\exp\left( - (r(t) - f(0,t))D(t,T) - \frac{1}{2}D(t,T)^2\int_0^t \sigma(u)^2 D_t(u,t)^2 du\right), \qquad (3.13)$$

*where $D(t,T)$ is given by* (3.5).

In addition to the bond price, it is also convenient to have the Hull–White short-rate process that gives an exact fit to the term structure at time 0.

From (3.12) we can see that

$$f(0,s) - e^{-\alpha(s-t)} f(0,t) = \int_t^s \theta(u) e^{-\alpha(s-u)} du - \int_0^s \sigma(u)^2 D(u,s) e^{-\alpha(s-u)} du$$
$$+ \int_0^t \sigma(u)^2 D(u,t) e^{-\alpha(s-u)} du.$$

Therefore (3.9) becomes

$$r(s) = (r(t) - f(0,t)) e^{-\alpha(s-t)} + f(0,s) + \int_0^s \sigma(u)^2 D(u,s) e^{-\alpha(s-u)} du$$
$$- \int_0^t \sigma(u)^2 D(u,t) e^{-\alpha(s-u)} du + \int_t^s \sigma(u) e^{-\alpha(s-u)} dW(u). \quad (3.14)$$

Using the above expression in the pricing formula (3.3) for zero-coupon bonds provides another way of deriving (3.13).

**Exercise 3.5** Derive the zero-coupon bond price (3.13) in the Hull–White model by using the Hull–White short-rate process (3.14).
*Hint:* To simplify the calculations use the formulae

$$\int_t^T e^{-\alpha(s-t)} ds = D(t,T), \quad \int_t^T D(t,s) e^{-\alpha(s-t)} ds = \frac{1}{2} D(t,T)^2, \quad (3.15)$$

where $D(t,T)$ is given by (3.5).

### Bond option formula

Consider a call and a put option with strike $K$ and expiry $S$ written on a zero-coupon bond with maturity $T > S$. From Theorem 2.4 and Exercise 2.9 we know that if the zero-coupon bond obeys (2.8) with deterministic log-volatility, then the prices at time 0 of the call and the put are

$$\mathbf{BC}(0; S, T, K) = B(0,T) N(d_+) - K B(0,S) N(d_-), \quad (3.16)$$
$$\mathbf{BP}(0; S, T, K) = K B(0,S) N(-d_-) - B(0,T) N(-d_+), \quad (3.17)$$

where

$$d_+ = \frac{\ln \frac{B(0,T)}{B(0,S)K} + \frac{1}{2} v(0,S)}{\sqrt{v(0,S)}}, \qquad d_- = d_+ - v(0,S), \quad (3.18)$$

and $v(0,S)$ is the variance of $\ln B(S,T)$.

In the Hull–White model the zero-coupon bond price $B(S,T)$ is given by (3.10) with $S$ substituted for $t$. The variance of $\ln B(S,T)$ is therefore equal to the variance of $-r(S)D(S,T)$, where $r(S)$ is given by (3.9) with $S$ substituted for $s$ and 0 for $t$. As a result,

$$\begin{aligned} v(0,S) &= \mathrm{Var}\left(D(S,T)\int_0^S \sigma(u)\mathrm{e}^{-\alpha(S-u)}dW(u)\right) \\ &= D(S,T)^2\int_0^S \sigma(u)^2\mathrm{e}^{-2\alpha(S-u)}du. \end{aligned} \tag{3.19}$$

This gives an analytic formula for the bond option price. When the time-dependent volatility term $\sigma(t)$ is chosen to be constant, (3.19) becomes

$$v(0,S) = \frac{\sigma^2}{2\alpha^3}\left(1-\mathrm{e}^{-\alpha(T-S)}\right)^2\left(1-\mathrm{e}^{-2\alpha S}\right). \tag{3.20}$$

**Exercise 3.6** Derive formulae for calls and puts on a zero-coupon bond in the Merton model.

**Exercise 3.7** Derive formulae for calls and puts on a zero-coupon bond in the Vasiček model. Compare these formulae with those for the Hull–White model with $\sigma(t)$ chosen to be constant. They should be identical. Why?

### Formula for caps and floors

Consider a caplet with strike $K$, unit notional $N = 1$ and expiry $T_i$ written on the spot LIBOR rate $L(T_{i-1}, T_i)$. In Section 2.7 we saw how the caplet payoff can be expressed as $1 + \tau_i K$ put options with strike $(1 + \tau_i K)^{-1}$ and expiry $T_{i-1}$ written on a zero-coupon bond with maturity $T_i$. The price at time 0 of the caplet in the Hull–White model is

$$\mathbf{Cpl}_i(0) = (1+\tau_i K)\mathbf{BP}\left(0; T_{i-1}, T_i, \frac{1}{1+\tau_i K}\right), \tag{3.21}$$

where **BP** is the price of a put option on a zero-coupon bond given by (3.17) and (3.18), with the variance of $\ln B(S,T)$ given by (3.19). Similarly, the price of the corresponding floorlet is

$$\mathbf{Flr}_i(0) = (1 + \tau_i K)\mathbf{BC}\left(0; T_{i-1}, T_i, \frac{1}{1 + \tau_i K}\right), \tag{3.22}$$

where **BC** is given by (3.16) and (3.18), with the variance of $\ln B(S, T)$ given by (3.19). The price of a cap (or floor) is simply the sum of the prices of the constituent caplets (or floorlets).

In general, the prices of caplets and floorlets are derived (via bootstrapping) from the market prices of caps and floors. Formulae (3.21) and (3.22) are used in calibrating the Hull–White model, where we use the known market prices to help us derive the model parameters.

### Formula for swaptions

Consider an option with strike $K$ and expiry $T_0$ on a payer interest rate swap with unit notional, settlement dates $T_1, \ldots, T_n$ and reset dates $T_0, \ldots, T_{n-1}$. In Section 2.8 we saw how the swaption payoff can be written as a put option with strike 1 on a coupon-bearing bond with coupon rate $K$, that is,

$$\left(1 - K\sum_{i=1}^{n} \tau_i B(T_0, T_i) - B(T_0, T_n)\right)^+ . \tag{3.23}$$

A method referred to as **Jamshidian's trick** can be applied to write this option on a coupon-bearing bond as a linear combination of put options on zero-coupon bonds. According to (3.10), the zero-coupon bond price $B(t, T) = F(t, r(t); T)$ in the Hull–White model is a decreasing function of the short rate $r(t)$. It follows that the coupon-bearing bond price $K\sum_{i=1}^{n} \tau_i F(t, r(t); T_i) + F(t, r(t); T_n)$ is also a decreasing function of $r(t)$, and there exists a critical value $\tilde{r}$ of the short rate such that

$$K\sum_{i=1}^{n} \tau_i F(T_0, \tilde{r}; T_i) + F(T_0, \tilde{r}; T_n) = 1. \tag{3.24}$$

Letting $K_i = F(T_0, \tilde{r}; T_i)$ for $i = 1, \ldots, n$, so $K\sum_{i=1}^{n} \tau_i K_i + K_n = 1$, and observing that $B(T_0, T_i) < K_i$ if and only if $r(T_0) > \tilde{r}$, we can write the payoff (3.23) as

$$K\sum_{i=1}^{n} \tau_i (K_i - B(T_0, T_i))^+ + (K_n - B(T_0, T_n))^+ .$$

Therefore the option on a coupon-bearing bond is a linear combination of options on the underlying zero-coupon bonds.

As a result, the price at time 0 of the swaption in the Hull–White model is given by

$$\mathbf{PSwpt}_{0,n}(0) = \sum_{i=1}^{n} K(T_i - T_{i-1})\mathbf{BP}(0; T_0, T_i, K_i) + \mathbf{BP}(0; T_0, T_n, K_n), \quad (3.25)$$

where **BP** is given by (3.17) and (3.18), with the variance of $\ln B(S,T)$ given by (3.19).

**Exercise 3.8** Suppose we have specified the value of $\alpha$ and the functional form of the volatility $\sigma(t)$. Explain how to estimate $K_i$ for $i = 1, \ldots, n$ in the Hull–White swaption formula (3.25).

### Calibrating the Hull–White model

The prices of the most liquid caps, floors and swaptions are given by the market. Therefore the left-hand sides of formulae (3.21), (3.22) and (3.25) are known. We can use these formulae to estimate the Hull–White model parameters given the market prices. Once we know the model parameters, we can then price complex or non-vanilla instruments. A common example of a non-vanilla instrument is the Bermudan swaption, which we will discuss in Section 3.6.

In Section 2.9 we introduced the notion of the implied Black caplet volatility (or spot volatility), which is the volatility parameter that must be inserted into the Black formula to obtain the market price of the caplet. Recall that, for the caplet maturing at time $T_{i-1}$ and paying at time $T_i$, we denote the implied caplet volatility by $\hat{\sigma}_i^{\text{caplet}}$.

Formula (3.21) can be used to compute the volatility $\sigma_i^{\text{caplet}}$ of the $i$th caplet implied by the Hull–White model. To this end we solve the equation

$$\mathbf{Cpl}_i(0) = \mathbf{Cpl}_i^{\text{Black}}(0; \sigma_i^{\text{caplet}}) \quad (3.26)$$

with $\mathbf{Cpl}_i(0)$ given by (3.21) and $\mathbf{Cpl}_i^{\text{Black}}(0; \sigma_i^{\text{caplet}})$ by Black's formula (2.24).

The volatility term structure for the Hull–White model is given by solving equation (3.26) for each caplet. The model parameters are chosen so that the implied volatility $\sigma_i^{\text{caplet}}$ of the $i$th caplet in the Hull–White model computed via (3.26) matches as best as possible the caplet spot volatility $\hat{\sigma}_i^{\text{caplet}}$ given by the market.

Similarly, we can calibrate to vanilla swaptions by choosing the model parameters so that the volatility implied by the swaption formula in the Hull–White model is as close as possible to the Black swaption volatility $\hat{\sigma}^{\text{swpt}}_{0,n}$ given by the market.

As a first attempt at calibration to the caplet or swaption market we could choose $\sigma(t)$ to be constant. However, in such cases we will generally not have enough degrees of freedom for an accurate fit. Therefore the typical assumption is that $\sigma(t)$ is piecewise constant. We partition the time axis by a sequence of dates $T_0 < T_1 < \cdots < T_n < T$ so that

$$\sigma(t) = \sigma_i \quad \text{for} \;\; T_{i-1} < t \le T_i, \;\; i = 1, \dots, n, \tag{3.27}$$

and then choose a set of vanilla options to calibrate to. According to Proposition 3.5, the $T$-bond price at time $T_i$ that gives an exact fit to the market term structure of interest rates at time 0 is

$$\begin{aligned} B(T_i, T) = \frac{B^{\text{mkt}}(0,T)}{B^{\text{mkt}}(0,T_i)} \exp\Bigg( & -(r(T_i) - f^{\text{mkt}}(0,T_i))D(T_i,T) \\ & - \frac{1}{2}D^2(T_i,T)\sum_{k=1}^{i} \frac{\sigma_k}{2\alpha}\left(\mathrm{e}^{-2\alpha(T_i-T_k)} - \mathrm{e}^{-2\alpha(T_i-T_{k-1})}\right)\Bigg), \end{aligned}$$

where $D(T_i, T)$ is given by (3.5). The bond price depends only on the short-rate volatilities $\sigma_1, \dots, \sigma_i$ and the mean-reversion term $\alpha$.

**Remark 3.6**

Given the range of caps, floors and swaptions that are actively traded in the market, a natural question to ask is what subset of these do we actually use to calibrate our model? The standard market practice is to calibrate to what traders refer to as the 'natural hedging instruments'. These are the vanilla options used to hedge risk in the exotic instrument we are attempting to model. For the case of a Bermudan swaption, which we will discuss in Section 3.6, the risk is hedged by the underlying co-terminal swaptions.

**Remark 3.7**

For a given expiry $T_i$ the Black implied volatility for strike $K$ is read from a 'volatility cube' as described in Remark 2.10. The Hull–White short-rate model in itself does not account for the fact that the implied volatility for a given maturity varies with strike. However, correctly calibrated it will reproduce the correct market price of the underlying swaptions.

## 3.6 Bermudan swaptions in the Hull–White model

In this section we will use the formulae derived for the Hull–White model to price a non-vanilla instrument. The example we choose is a Bermudan swaption as defined in what follows.

The Hull–White short-rate process that gives an exact fit to the term-structure at time 0 is given by (3.14) under the risk-neutral measure $Q$. However, when we use the Hull–White model to price an exotic instrument or structured product, it can be convenient to know the drift of the short-rate process under the forward measure to calculate the option price. Therefore, we derive the dynamics of the short-rate process under the forward measure as a first step.

### Hull–White under the forward measure

In Section 2.6 we saw in the case when the zero-coupon bond numeraire $B(t,T)$ obeys (2.8) that

$$W^T(t) = W(t) - \int_0^t \Sigma(u,T)du \quad \text{for all } t \in [0,T],$$

where $W(t)$ and $W^T(t)$ are Brownian motions under the risk-neutral measure $Q$ and the forward measure $P_T$, respectively.

Applying the Itô formula to the bond price (3.10) and then using the above relationship between $W^T(t)$ and $W(t)$, we can show that the Hull–White SDE (3.8) for the short rate becomes

$$dr(t) = (\theta(t) - \alpha r(t) - \sigma(t)^2 D(t,T))dt + \sigma(t)dW^T(t) \tag{3.28}$$

under the forward measure.

**Exercise 3.9** Show that (3.28) holds true.

Furthermore, using (3.14) we can write an expression for the Hull–White short rate at time $T > t$ that gives an exact fit to the term structure at time 0. Applying the same change of measure and then simplifying, we can see that $r(T)$ is given by

$$\begin{aligned} r(T) &= (r(t) - f(0,t))\mathrm{e}^{-\alpha(T-t)} + f(0,T) \\ &\quad + \int_0^t \sigma(u)^2 \frac{\mathrm{e}^{-\alpha(T+t-2u)} - \mathrm{e}^{-2\alpha(T-u)}}{\alpha} du + \int_t^T \sigma(u)\mathrm{e}^{-\alpha(T-u)} dW^T(u) \end{aligned} \tag{3.29}$$

under the forward measure $P_T$. This expression forms the basis of the numerical example we outline below. Note how the expectation of $r(T)$ given $r(t)$ depends only on $\sigma(u)$ for $u \in [0, t]$. Indeed, if we compare this to (3.14) (where we replace time $s$ by $T$), we can see that working under the forward measure yields a simpler formula.

### Bermudan swaption

Consider a unit notional amount and a set of dates $0 < T_0 < T_1 < \cdots < T_n$. The holder of a payer (or receiver) **Bermudan swaption** with strike $K$ has the right to enter a payer (or receiver) interest rate swap at any time $T_k$ for $k = 0, \ldots, l$ with $l < n$. The swap has reset dates $T_0, \ldots, T_n - 1$, settlement dates $T_1, \ldots, T_n$ and swap rate $K$. We assume that $l = n - 1$ and consider a payer Bermudan swaption, whose value at time 0 we denote by $\mathbf{Berm}(0)$. Note how this differs form the vanilla swaption, where the exercise date is fixed.

If the Bermudan swaption has not yet been exercised at time $T_i$ for some $i < l$, then the holder has to decide whether to continue holding the option or to exercise immediately. The exercise value is

$$\mathrm{E}(T_i) = (\mathbf{PS}(T_i))^+ .$$

Whether or not it is optimal to exercise at time $T_i$ will depend on the value of holding the option until time $T_{i+1}$. We denote this value by $\mathrm{C}(T_i)$ and refer to it as the continuation value. The value of the swaption at time $T_i$ is

$$\mathbf{Berm}(T_i) = \max(\mathrm{E}(T_i), \mathrm{C}(T_i)). \tag{3.30}$$

The continuation value at time $T_i$ can be computed recursively. We have

$$\mathrm{C}(T_i) = B(T_i, T_{i+1}) \mathbb{E}_{P_{T_{i+1}}} (\mathbf{Berm}(T_{i+1}) | \mathcal{F}_{T_i}) ,$$

where the expectation is taken under the forward measure $P_{T_{i+1}}$.

### Calibrating to co-terminal swaptions

As explained in Remark 3.6, we calibrate to the set of vanilla options used to hedge the exotic instrument. For the Bermudan swaption this is the underlying set of co-terminal swaptions. We minimise the difference between the Hull–White model price $\mathbf{PSwpt}_{i,n}(0)$ and the market price (as expressed by Black's swaption formula) $\mathbf{PSwpt}^{\mathrm{mkt}}_{i,n}(t) = \mathbf{PSwpt}^{\mathrm{Black}}_{i,n}(t; \hat{\sigma}^{\mathrm{swpt}}_{i,n})$ of the underlying set of swaptions for $i = 0, \ldots, n - 1$. More precisely, we solve the non-linear least squares optimisation problem

$$\min_{\sigma_1,\ldots,\sigma_n} \sum_{i=0}^{n-1} \left(\mathbf{PSwpt}_{i,n}(0) - \mathbf{PSwpt}_{i,n}^{\mathrm{mkt}}(0)\right)^2,$$

where $\mathbf{PSwpt}_{i,n}(0)$ depends on the short-rate volatility values $\sigma_1, \ldots, \sigma_n$. To ensure that the parameters are positive we generally perform a constrained optimisation, where we specify an upper and lower bound.

This calibrates the short-rate volatility values. For the mean-reversion parameter $\alpha$ we use the autocorrelation between the sort rates.

### Mean-reversion parameter and autocorrelation

The mean-reversion parameter $\alpha$ controls the the correlation between the short rates at different points in time, i.e. the autocorrelation. For $s > t$ the autocorrelation is

$$\begin{aligned}\mathrm{Corr}(r(t), r(s)) &= \frac{\mathbb{E}(r(t)r(s))}{\sqrt{\mathrm{Var}(r(t))\mathrm{Var}(r(s))}} \\ &= \frac{\int_0^t \sigma^2(u)\mathrm{e}^{-\alpha(t+s-2u)}du}{\sqrt{\int_0^t \sigma^2(u)\mathrm{e}^{-2\alpha(t-u)}du \int_0^s \sigma^2(u)\mathrm{e}^{-2\alpha(s-u)}du}}.\end{aligned}$$

If we assume that the volatility is constant, this can be simplified to

$$\mathrm{Corr}[r(t), r(s)] = \sqrt{\frac{\mathrm{e}^{2\alpha t} - 1}{\mathrm{e}^{2\alpha s} - 1}}.$$

Increasing the mean-reversion parameter $\alpha$ lowers the autocorrelation. This may have a significant impact on the valuation of exotic derivatives, and can be used to calibrate the value of $\alpha$.

### Numerical method

To apply the above recursive relation we begin by discretising the short rate domain, creating a grid of $N + 1$ values $r_0 < r_1 < \cdots < r_N$, where $r(0)$ is at a midpoint of the grid. Suppose that the time $T_{i+1}$ Bermudan price $\mathbf{Berm}(T_{i+1}; r(T_{i+1}))$ is known for each grid point $r(T_{i+1}) = r_j$, and we want to compute $\mathbf{Berm}(T_i; r(T_i))$ at time $T_i$ for each grid point $r(T_{i+1}) = r_j$. At time $T_i$ the continuation value is

$$\mathrm{C}(T_i; r(T_i)) = B(T_i, T_{i+1})\mathbb{E}_{P_{T_{i+1}}}\left(\mathbf{Berm}(T_{i+1}; r(T_{i+1}))|\mathcal{F}_{T_i}\right).$$

The zero-coupon bond price $B(T_i, T_{i+1})$ calibrated to the term structure at time 0 is given by (3.13), while the short rate calibrated to the term structure at time 0 is given by (3.29), namely

$$
\begin{aligned}
r(T_{i+1}) = {} & (r(T_i) - f(0,T_i))\,\mathrm{e}^{-\alpha(T_{i+1}-T_i)} + f(0,T_{i+1}) \\
& + \int_0^{T_i} \sigma(u)^2 \frac{\mathrm{e}^{-\alpha(T_{i+1}+T_i-2u)} - \mathrm{e}^{-2\alpha(T_{i+1}-u)}}{\alpha} du \\
& + \int_{T_i}^{T_{i+1}} \sigma(u) \mathrm{e}^{-\alpha(T_{i+1}-u)} dW^{T_{i+1}}(u).
\end{aligned}
$$

The stochastic integral in the last expression is independent of $\mathcal{F}_{T_i}$ and normally distributed with mean 0 and variance

$$
s_i^2 = \int_{T_i}^{T_{i+1}} \sigma(u)^2 \mathrm{e}^{-2\alpha(T_{i+1}-u)} du
$$

under the forward measure $P_{T_{i+1}}$. Putting

$$
\begin{aligned}
m_i = {} & (r(T_i) - f(0,T_i))\,\mathrm{e}^{-\alpha(T_{i+1}-T_i)} + f(0,T_{i+1}) \\
& + \int_0^{T_i} \sigma(u)^2 \frac{\mathrm{e}^{-\alpha(T_{i+1}+T_i-2u)} - \mathrm{e}^{-2\alpha(T_{i+1}-u)}}{\alpha} du,
\end{aligned}
$$

we therefore have

$$
\begin{aligned}
& \mathbb{E}_{P_{T_{i+1}}}(\mathbf{Berm}(T_{i+1}; r(T_{i+1})) | \mathcal{F}_{T_i}) \\
& \quad = \frac{1}{\sqrt{2\pi s_i^2}} \int_{-\infty}^{\infty} \mathbf{Berm}(T_{i+1}; x) \exp\left( -\frac{(x-m_i)^2}{2s_i^2} \right) dx.
\end{aligned}
$$

The last integral can be evaluated numerically by using the known values of $\mathbf{Berm}(T_{i+1}; r(T_{i+1}))$ at the grid points $r(T_{i+1}) = r_j$. This makes it possible to compute the continuation value $\mathrm{C}(T_i; r(T_i))$ and hence the Bermudan price $\mathbf{Berm}(T_i; r(T_i))$ from the recursive relation (3.30) at each grid point $r(T_i) = r_j$.

To evaluate the option we start at the last exercise date $T_l$ and work backwards in time. At $T_l$ the continuation value is zero as there is no benefit in holding the option beyond the last exercise date. Then we proceed backwards from $T_i$ to $T_{i-1}$ for $i = l, \dots, 0$ using the above recursive relationships to arrive at the Bermudan prices $\mathbf{Berm}(0; r(0))$ at time 0 for all grid points $r(0) = r_j$.

## 3.7 Two-factor Hull–White model

Many vanilla options are only weakly affected by the correlation between rates of different maturities and can be adequately priced using a one-factor model. Even more complex options such as Bermudans can be handled using a one-factor model. However, many exotic options are particularly sensitive to how rates of different maturities are correlated and must be modelled in a multi-factor framework. These include options which depend in a non-linear way on the difference between two rates or options whose payoff depends on two different interest rate curves.

The model we are going to examine is a simple extension of the one-factor Hull–White model with a stochastic mean-reversion level added to the drift term. To assist in the derivation of analytic formulae this model is then reformulated as a Gaussian two-factor model. The calculations are slightly more involved than in the one-factor case, but the overall approach is similar.

The SDE (3.8) for the short rate in the Hull–White model can be modified by adding a stochastic term $u(t)$ to the drift,

$$dr(t) = (\theta(t) + u(t) - \alpha r(t))dt + \delta dW(t), \tag{3.31}$$

where $\alpha$, $\delta$ are constants, $\theta(t)$ is a deterministic function of time and $W(t)$ is a Brownian motion under the risk-neutral measure $Q$. Moreover, $u(t)$ satisfies the SDE

$$du(t) = -\beta u(t)dt + \varepsilon dZ(t) \tag{3.32}$$

with initial value $u(0) = 0$, where $\beta$ and $\varepsilon$ are constants and $Z(t)$ is another Brownian motion under the risk-neutral measure $Q$ such that

$$dW(t)dZ(t) = \varrho dt \tag{3.33}$$

for a constant $\varrho$, the correlation between the two Brownian motions. Additionally, we assume that $\alpha \neq \beta$.

It is possible to obtain a bond pricing formula in terms of $r(t)$, $u(t)$ and the model parameters by following a similar argument to the derivation of formula (3.6) for the zero-coupon bond price in the Vasiček model and (3.10) in the one-factor Hull–White model. Though not difficult, this is rather involved. Instead, we represent the short rate in the two-factor Hull–White model as

$$r(t) = \phi(t) + x(t) + y(t), \tag{3.34}$$

where $\phi(t)$ is a deterministic function and

$$dx(t) = -\alpha x(t)dt + \sigma dU(t), \tag{3.35}$$

$$dy(t) = -\beta y(t)dt + \eta dV(t), \tag{3.36}$$

with initial conditions $x(0) = 0$, $y(0) = 0$, and where $\alpha, \beta, \sigma$ and $\eta$ are constants (in fact $\alpha, \beta$ are the same constants as in (3.31) and (3.32)), and $U(t)$ and $V(t)$ are Brownian motions under the risk-neutral measure $Q$ such that

$$dU(t)dV(t) = \rho dt. \tag{3.37}$$

Note that the correlation $\rho$ between the Brownian motions $U(t)$ and $V(t)$ is not the same as the correlation $\varrho$ between $W(t)$ and $Z(t)$.

**Exercise 3.10** Show that in the two-factor Hull–White model defined by (3.31), (3.32) and (3.33) we can satisfy (3.34), (3.35), (3.36) and (3.37) if we set

$$\phi(t) = r(0)e^{-\alpha t} + \int_0^t \theta(s)e^{-\alpha(t-s)}ds,$$

$$y(t) = \frac{u(t)}{\alpha - \beta},$$

$$x(t) = r(t) - \phi(t) - y(t),$$

with constants $\sigma, \eta$ and $\rho$, and Brownian motions $U(t)$ and $V(t)$ suitably defined in terms of $\alpha, \beta, \delta, \varepsilon, \varrho$ and $W(t), Z(t)$.

### Gaussian two-factor approach

We have represented the two-factor Hull–White model (3.31), (3.32) as a **Gaussian two-factor model** (3.34), (3.35), (3.36), which helps us to derive a formula for zero-coupon bond prices.

Integrating (3.35) and (3.36) from $t$ to $s$, we have

$$r(s) = \phi(s) + x(t)e^{-\alpha(s-t)} + y(t)e^{-\beta(s-t)} + \sigma\int_t^s e^{-\alpha(s-u)}dU(u) + \eta\int_t^s e^{-\beta(s-u)}dV(u). \tag{3.38}$$

As was the case with the Vasiček and one-factor Hull–White models, we can derive an analytic expression for zero-coupon bonds using the pricing formula (3.3). Integrating the short rate and taking expectation yields

$$B(t,T) = \exp\Bigg( - x(t)\frac{1-\mathrm{e}^{-\alpha(T-t)}}{\alpha} - y(t)\frac{1-\mathrm{e}^{-\beta(T-t)}}{\beta}, \\ - \int_t^T \phi(s)ds + \frac{1}{2}V(t,T)\Bigg), \tag{3.39}$$

where

$$V(t,T) = \frac{\sigma^2}{\alpha^2}\int_t^T \left(1-\mathrm{e}^{-\alpha(T-u)}\right)^2 du + \frac{\eta^2}{\beta^2}\int_t^T \left(1-\mathrm{e}^{-\beta(T-u)}\right)^2 du \\ + 2\rho\frac{\sigma\eta}{\alpha\beta}\int_t^T \left(1-\mathrm{e}^{-\alpha(T-u)}\right)(1-\mathrm{e}^{-\beta(T-u)})du. \tag{3.40}$$

Integrating, we get

$$V(t,T) = \frac{\sigma^2}{2\alpha^3}\left(-3-\mathrm{e}^{-2\alpha(T-t)}+4\mathrm{e}^{-\alpha(T-t)}+2\alpha(T-t)\right) \\ + \frac{\eta^2}{2\beta^3}\left(-3-\mathrm{e}^{-2\beta(T-t)}+4\mathrm{e}^{-\beta(T-t)}+2\beta(T-t)\right) \\ + 2\rho\frac{\sigma\eta}{\alpha\beta}\left(T-t-\frac{1-\mathrm{e}^{-\alpha(T-t)}}{\alpha}-\frac{1-\mathrm{e}^{-\beta(T-t)}}{\beta}+\frac{1-\mathrm{e}^{-(\alpha+\beta)(T-t)}}{\alpha+\beta}\right).$$

**Exercise 3.11** Derive formula (3.39) for the bond price $B(t,T)$.

### Fitting the current term structure

The time-dependent parameter $\phi(t)$ is chosen to fit the current interest rate term structure. If the model price fits the interest rate term structure at time 0 for any given maturity $T$, then we must have

$$B(0,T) = \exp\left(-\int_0^T \phi(s)ds + \frac{1}{2}V(0,T)\right).$$

Using (1.9), we find that the model fits the term structure at time 0 if

$$f(0,T) = \phi(T) - \frac{1}{2}\frac{\partial V(0,T)}{\partial T}, \tag{3.41}$$

where

$$\frac{\partial V(0,T)}{\partial T} = \frac{\sigma^2}{\alpha^2}\left(1-\mathrm{e}^{-\alpha T}\right)^2 + \frac{\eta^2}{\beta^2}\left(1-\mathrm{e}^{-\beta T}\right)^2 + 2\rho\frac{\sigma\eta}{\alpha\beta}\left(1-\mathrm{e}^{-\alpha T}\right)\left(1-\mathrm{e}^{-\beta T}\right).$$

Note that (3.41) gives us an expression for $\phi$ as a function of the instantaneous forward curve. However, we just need an expression for the integral of $\phi$ from time $t$ to $T$ rather than an explicit formula for $\phi$. It is given by

$$\int_t^T \phi(s)ds = \ln \frac{B(0,t)}{B(0,T)} + \frac{1}{2}(V(0,T) - V(0,t)).$$

On substituting this into (3.39), we have following proposition.

**Proposition 3.8**

*In the two-factor Hull–White model the zero-coupon bond price that gives an exact fit to the term structure of interest rates at time* 0 *is*

$$B(t,T) = \frac{B(0,T)}{B(0,t)} \exp\Bigg( - x(t)\frac{1 - \mathrm{e}^{-\alpha(T-t)}}{\alpha} - y(t)\frac{1 - \mathrm{e}^{-\beta(T-t)}}{\beta} \\ + \frac{1}{2}(V(0,t) - V(0,T) + V(t,T))\Bigg), \qquad (3.42)$$

*where* $V$ *is given by* (3.40) *and* $x(t), y(t)$ *solve the SDEs* (3.35), (3.36) *with* $x(0) = 0$ *and* $y(0) = 0$.

## Bond option

In the two-dimensional Hull–White model the volatility of the logarithm of the bond price (3.42) is deterministic (see Exercise 3.12) and therefore the model yields an analytic formula for the bond option. The price at time 0 of a call option with strike $K$ and expiry $S$ written on a zero-coupon bond with maturity $T > S$ is given by

$$\mathbf{BC}(0; S, T, K) = B(0,T)N(d_+) - KB(0,S)N(d_-),$$

where

$$d_+ = \frac{\ln \frac{B(0,S)K}{B(0,T)} + \frac{1}{2}v(0,S)}{\sqrt{v(0,S)}}, \qquad d_- = d_+ - v(0,S)$$

with

$$v(0,S) = \frac{\sigma^2}{2\alpha^3}\left(1-\mathrm{e}^{-\alpha(T-S)}\right)^2\left(1-\mathrm{e}^{-2\alpha S}\right) + \frac{\eta^2}{2\beta^3}\left(1-\mathrm{e}^{-\beta(T-S)}\right)^2\left(1-\mathrm{e}^{-2\beta S}\right) \\ + \frac{2\rho\sigma\eta}{\alpha\beta(\alpha+\beta)}\left(1 - \mathrm{e}^{-\alpha(T-S)}\right)\left(1 - \mathrm{e}^{-\beta(T-S)}\right)\left(1 - \mathrm{e}^{-(\alpha+\beta)S}\right). \qquad (3.43)$$

**Exercise 3.12** Show that the variance of $\ln B(S, T)$ is given by (3.43).

### Caps and floors

Since both caps and floors can be expressed in terms of bond options, it is possible to derive an analytic expression for these instruments within the two-factor Hull–White model. The approach is identical to that in the one-factor case, so is not reproduced here. The key ingredient for these formulae is the log-variance of the bond price given by the formula above.

# 4

# Models of the forward rate



Heath–Jarrow–Morton (HJM) models are driven by the evolution in time $t$ of the instantaneous forward-rate curve $f(t, T)$ parameterised by the maturity date $T$. The entire curve serves as the state variable. This is in contrast to short-rate models, which are driven by the evolution of a single point on the curve, the short rate $r(t)$.

Just like in the case of short-rate models, we adopt Assumption 2.1, i.e. we assume the existence of a risk-neutral measure $Q$, which transforms all security prices discounted by the money market account into martingales.

The key result in this framework is that the drift of the forward rate $f(t, T)$ under the risk-neutral measure $Q$ is determined by the volatility. This is different to short-rate models, where we are free to specify the drift for the short rate. Compare this to the classical Black–Scholes model, where the drift of the underlying stock price process under the risk-neutral measure is equal to the spot interest rate; see [BSM]. In the HJM framework, just like in the Black–Scholes model, the drift of the underlying process (the instantaneous forward rate and the stock price, respectively) is fixed.

The main benefit of HJM models is that they allow for a perfect fit to the initial interest rate term structure and offer more flexibility than short-rate models. However, they can be difficult to apply in practice.

## 4.1 One-factor HJM models

Heath, Jarrow and Morton proposed a framework for modelling stochastic interest rates based on the dynamics of the instantaneous forward rate $f(t,T)$. We begin with the general assumption that the forward rate is an Itô process such that, for each maturity $T > 0$,

$$df(t,T) = \alpha(t,T)dt + \sigma(t,T)dW(t) \tag{4.1}$$

for $t \in [0,T]$, where $W(t)$ is a Brownian motion under the risk-neutral measure $Q$. The stochastic differential $df(t,T)$ is applied to the $t$ variable, whereas $T$ is treated as a parameter. We have infinitely many processes $f(t,T)$ parameterised by $T$, each driven by the same Brownian motion $W(t)$.

Next we present a heuristic argument to establish a relationship between the drift $\alpha(t,T)$ and volatility $\sigma(t,T)$ of the forward-rate process and obtain an SDE for the zero-coupon bond price $B(t,T)$. By (1.11), the bond price can be expressed in terms of forward rates as

$$B(t,T) = \exp\left(-\int_t^T f(t,u)du\right).$$

To apply the Itô formula to the exponent on the right-hand side we need to compute the stochastic differential of $\int_t^T f(t,u)du$ with respect to the $t$ variable, which appears in two places, in the lower integration limit and as an argument of $f(t,u)$. An informal calculation gives

$$\begin{aligned}
&d\left(\int_t^T f(t,u)du\right)\\
&= -f(t,t)dt + \int_t^T \Big(\alpha(t,u)dt + \sigma(t,u)dW(t)\Big)du\\
&= -r(t)dt + \left(\int_t^T \alpha(t,u)du\right)dt + \left(\int_t^T \sigma(t,u)du\right)dW(t).
\end{aligned} \tag{4.2}$$

The term $f(t,t)dt = r(t)dt$ comes from differentiating with respect to the lower integration limit $t$, while the remaining terms come from using (4.1) to write the stochastic differential $df(t,u)$ with respect to $t$ as $\alpha(t,u)dt + \sigma(t,u)dW(t)$ and moving $dt$ and $dW(t)$ outside the integral. We have no guarantee that these transformations are legitimate, though they appear natural. A precise argument requires more work and some technical assumptions. This will be done below.

Now that we have the above expression for $d\left(\int_t^T f(t,u)du\right)$, it is a matter of using the Itô formula to obtain

$$dB(t,T) = \left( r(t) - \int_t^T \alpha(t,u)du + \frac{1}{2}\left(\int_t^T \sigma(t,u)du\right)^2 \right) B(t,T)dt$$
$$- \left(\int_t^T \sigma(t,u)du\right) B(t,T)dW(t).$$

From Exercise 2.3 we know that

$$r(t) - \int_t^T \alpha(t,u)du + \frac{1}{2}\left(\int_t^T \sigma(t,u)du\right)^2 = r(t),$$

which means that

$$\int_t^T \alpha(t,u)du = \frac{1}{2}\left(\int_t^T \sigma(t,u)du\right)^2.$$

Differentiating both sides with respect to $T$ gives

$$\alpha(t,T) = \sigma(t,T)\int_t^T \sigma(t,u)du.$$

We are ready to summarise these results as a theorem. The technical assumptions concerning $\alpha$ and $\sigma$ in this theorem allow us to swap the order of integration for ordinary and stochastic iterated integrals, which lends legitimacy to the above informal argument where we move $dt$ and $dW(t)$ outside the integral with respect to $du$.

**Theorem 4.1**
*Suppose that the forward rates $f(t,T)$ satisfy* (4.1), *that is,*

$$df(t,T) = \alpha(t,T)dt + \sigma(t,T)dW(t),$$

*where $W(t)$ is a Brownian motion under the risk-neutral measure $Q$, and where we assume that $\sigma(t,T)$ is adapted to the filtration $\mathcal{F}_t$ for each $T > 0$, and*

$$\int_0^T \int_0^T |\alpha(s,u)|\, ds du < \infty, \qquad \int_0^T \left(\int_0^T |\sigma(s,u)|^2 ds\right)^{1/2} du < \infty$$

*almost surely. Then*

$$\alpha(t,T) = \sigma(t,T)\int_t^T \sigma(t,u)du. \tag{4.3}$$

*Therefore the dynamics of $f(t,T)$ under the risk-neutral measure $Q$ is*

$$f(t,T) = f(0,T) + \int_0^t \sigma(s,T)\left(\int_s^T \sigma(s,u)du\right)ds + \int_0^t \sigma(s,T)dW(s).$$

*Furthermore the zero-coupon bond prices satisfy the SDE*

$$dB(t,T) = r(t)B(t,T)dt + \Sigma(t,T)B(t,T)dW(t) \tag{4.4}$$

*with log-volatility*

$$\Sigma(t,T) = -\int_t^T \sigma(t,u)du.$$

*Proof* A rigorous proof of (4.2) is all that remains to be done. We write (4.1) in integral form as

$$f(t,T) = f(0,T) + \int_0^t \alpha(s,T)ds + \int_0^t \sigma(s,T)dW(s). \tag{4.5}$$

It follows that

$$r(t) = f(t,t) = f(0,t) + \int_0^t \alpha(s,t)ds + \int_0^t \sigma(s,t)dW(s).$$

Integrating, we get

$$\begin{aligned}&\int_t^T f(t,u)du\\ &= \int_t^T f(0,u)du + \int_t^T \left(\int_0^t \alpha(s,u)ds\right)du + \int_t^T \left(\int_0^t \sigma(s,u)dW(s)\right)du\end{aligned}$$

and

$$\begin{aligned}&\int_0^t r(u)du\\ &= \int_0^t f(0,u)du + \int_0^t \left(\int_0^u \alpha(s,u)ds\right)du + \int_0^t \left(\int_0^u \sigma(s,u)dW(s)\right)du.\end{aligned}$$

The assumptions on $\alpha$ and $\sigma$ allow us to apply the Fubini theorem (see [PF]) to swap the iterated integrals with respect to $ds$ and $du$, and the stochastic Fubini theorem[1] to swap the integrals with respect to $dW(s)$ and $du$,

$$\begin{aligned}&\int_t^T f(t,u)du\\ &= \int_t^T f(0,u)du + \int_0^t \left(\int_t^T \alpha(s,u)du\right)ds + \int_0^t \left(\int_t^T \sigma(s,u)du\right)dW(s)\end{aligned}$$

[1] See M. Veraar, The stochastic Fubini theorem revisited, *Stochastics 84*, (2012), 543–551.

and

$$\begin{aligned}
&\int_0^t r(u)du \\
&= \int_0^t f(0,u)du + \int_0^t \left( \int_s^t \alpha(s,u)du \right) ds + \int_0^t \left( \int_s^t \sigma(s,u)du \right) dW(s).
\end{aligned}$$

Hence

$$\begin{aligned}
&\int_t^T f(t,u)du \\
&= \int_0^T f(0,u)du + \int_0^t \left( \int_s^T \alpha(s,u)du \right) ds + \int_0^t \left( \int_s^T \sigma(s,u)du \right) dW(s) \\
&\quad - \int_0^t f(0,u)du - \int_0^t \left( \int_s^t \alpha(s,u)du \right) ds - \int_0^t \left( \int_s^t \sigma(s,u)du \right) dW(s) \\
&= \int_0^T f(0,u)du + \int_0^t \left( \int_s^T \alpha(s,u)du \right) ds + \int_0^t \left( \int_s^T \sigma(s,u)du \right) dW(s) \\
&\quad - \int_0^t r(u)du.
\end{aligned}$$

This proves (4.2). □

## 4.2 Gaussian models

Models with deterministic forward-rate volatility $\sigma(t,T)$ are referred to as **Gaussian models**. Their advantage is that it is possible to derive analytic formulae for calls and puts on a zero-coupon bond. When $\sigma(t,T)$ is deterministic, then both the forward rate and the short rate are normally distributed and the zero-coupon bond price is log-normally distributed under the risk-neutral measure $Q$. This leads to the familiar Black–Scholes type formulae for bond options.

**Example 4.2**

As a simple example, we take constant $\sigma(t,T)$, that is, $\sigma(t,T) = \sigma$ for all $t \leq T$. By Theorem 4.1, we must have

$$\alpha(t,T) = \sigma \int_t^T \sigma du = \sigma^2(T-t)$$

and

$$f(t,T) = f(0,T) + \int_0^t \alpha(s,T)ds + \int_0^t \sigma(s,T)dW(s)$$

$$= f(0,T) + \frac{1}{2}\sigma^2 t(2T - t) + \sigma W(t).$$

It follows that the short rate

$$r(t) = f(t,t) = f(0,t) + \frac{1}{2}\sigma^2 t^2 + \sigma W(t)$$

satisfies the SDE of the Ho–Lee model (see Table 3.1)

$$dr(t) = \theta(t)dt + \sigma dW(t)$$

with

$$\theta(t) = \frac{\partial f(0,t)}{\partial t} + \sigma^2 t.$$

**Exercise 4.1** In the Ho–Lee model, i.e. when $\sigma(t,T) = \sigma$ is constant, show that

$$B(t,T) = \exp\left(-(T-t)(r(t) - f(0,t)) - \int_t^T f(0,s)ds - \frac{1}{2}\sigma^2 t(T-t)^2\right).$$

**Example 4.3**

Take $\sigma(t,T) = \sigma(t)e^{-\alpha(T-t)}$, where $\sigma(t)$ is a deterministic function and $\alpha$ is a constant. By Theorem 4.1, we have

$$\alpha(t,T) = \sigma(t,T)\int_t^T \sigma(t,u)du$$

$$= \sigma(t)^2 e^{-\alpha(T-t)} \int_t^T e^{-\alpha(T-u)}du = \sigma(t)^2 e^{-\alpha(T-t)} D(t,T),$$

where $D(t,T)$ is given by (3.5). As a result,

$$f(t,T) = f(0,T) + \int_0^t \alpha(s,T)ds + \int_0^t \sigma(s,T)dW(s)$$

$$= f(0,T) + \int_0^t \sigma(s)^2 e^{-\alpha(T-s)} D(s,T)ds + \int_0^t \sigma(s)e^{-\alpha(T-s)}dW(s)$$

and

$$r(t) = f(t,t) = f(0,t) + \int_0^t \sigma(s)^2 e^{-\alpha(t-s)} D(s,t) ds + \int_0^t \sigma(s) e^{-\alpha(t-s)} dW(s).$$

The short rate satisfies the SDE of the Hull–White model

$$dr(t) = (\theta(t) - \alpha r(t))dt + \sigma(t)dW(t)$$

with

$$\theta(t) = \frac{\partial f(0,t)}{\partial t} + \alpha f(0,t) + \int_0^t \sigma(s)^2 e^{-2\alpha(t-s)} ds.$$

### Gaussian HJM with separable volatility

For a general $\sigma(t,T)$ the drift term of the forward rate (and the short rate) is rather complicated. To simplify $\sigma(t,T)$ is often expressed as

$$\sigma(t,T) = \xi(t)\eta(T), \tag{4.6}$$

where $\xi(t)$ and $\eta(T)$ are strictly positive deterministic functions. With this choice of so-called **separable volatility**, the forward rate $f(t,T)$ is given by

$$f(t,T) = f(0,T) + \int_0^t \xi(s)\eta(T)\left(\int_s^T \xi(s)\eta(u)du\right)ds + \int_0^t \xi(s)\eta(T)dW(s),$$

and the short rate $r(t)$ is

$$r(t) = f(0,t) + \int_0^t \xi(s)\eta(t)\left(\int_s^t \xi(s)\eta(u)du\right)ds + \int_0^t \xi(s)\eta(t)dW(s).$$

Using the above, we can replace the stochastic integral in the expression for $f(t,T)$ to get

$$\begin{aligned} f(t,T) =& f(0,T) + \frac{\eta(T)}{\eta(t)}(r(t) - f(0,t)) \\ &+ \int_0^t \xi(s)\eta(T)\left(\int_t^T \xi(s)\eta(u)du\right)ds. \end{aligned} \tag{4.7}$$

We can see that the forward rate can be expressed in terms of the short rate. Moreover, by (1.11), we find that

$$B(t,T) = \exp\left( - \int_t^T f(0,u)du - \frac{r(t) - f(0,t)}{\eta(t)} \int_t^T \eta(u)du \right.$$
$$\left. - \int_t^T \int_0^t \xi(s)\eta(v)\left( \int_t^v \xi(s)\eta(u)du \right) ds dv \right).$$

We can simplify the above by changing the order of integration in the triple integral and calculating

$$\begin{aligned}
&\int_t^T \int_0^t \xi(s)\eta(v)\left( \int_t^v \xi(s)\eta(u)du \right) ds dv \\
&\qquad = \int_0^t \int_t^T \xi(s)\eta(v)\left( \int_t^v \xi(s)\eta(u)du \right) dv ds \\
&\qquad = \int_0^t \xi(s)^2 \left[ \int_t^T \eta(v)\left( \int_t^v \eta(u)du \right) dv \right] ds.
\end{aligned}$$

The integral inside the square brackets can be simplified by noting that

$$\int_t^T \eta(v)\left( \int_t^v \eta(u)du \right) dv = \frac{1}{2}\left( \int_t^T \eta(u)du \right)^2.$$

Setting

$$I(t,T) = \frac{1}{\eta(t)} \int_t^T \eta(u)du,$$

we can write the zero-coupon bond price in a Gaussian HJM model with separable volatility as

$$\begin{aligned}
B(t,T) = \exp\bigg( &- \int_t^T f(0,u)du - (r(t) - f(0,t))I(t,T) \\
&- \frac{1}{2} I(t,T)^2 \int_0^t \xi(s)^2 \eta(t)^2 ds \bigg). \qquad (4.8)
\end{aligned}$$

**Exercise 4.2** In Example 4.3 we chose $\sigma(t,T) = \sigma(t)e^{-\alpha(T-t)}$, where $\sigma(t)$ is a deterministic function and $\alpha$ is a constant, and saw that the short-rate process satisfies the SDE of the Hull–White model. Using (4.8), show that the zero-coupon bond pricing formula is given by (3.13).

**Example 4.4**
An example of a Gaussian HJM model with separable volatility is the **Ritchken–Sankarasubramanian model**. Here the functions $\xi(t)$ and $\eta(t)$ take the form

$$\xi(t) = \sigma(t)\exp\left(\int_0^t \alpha(u)du\right), \quad \eta(t) = \exp\left(-\int_0^t \alpha(u)du\right),$$

where $\sigma$ and $\alpha$ are deterministic functions.

For this choice of volatility the HJM model is equivalent to the general Hull–White model (i.e. the version where the mean-reversion parameter $\alpha$ is time dependent). This is often how the Hull–White model is implemented in practice.

**Exercise 4.3** Show that in the Ritchken–Sankarasubramanian model the short rate satisfies the SDE

$$dr(t) = \left(\frac{\partial f(0,t)}{\partial t} + \phi(t) + \alpha(t)(f(0,t) - r(t))\right)dt + \sigma(t)dW(t),$$

where

$$\phi(t) = \int_0^t \sigma(s)^2 \exp\left(-2\int_s^t \alpha(u)du\right)ds.$$

### Bond option price in Gaussian HJM model

In the HJM model the bond price satisfies the SDE (4.4),

$$dB(t,T) = r(t)B(t,T)dt + \Sigma(t,T)B(t,T)dW(t)$$

with log-volatility

$$\Sigma(t,T) = -\int_t^T \sigma(t,u)du.$$

For deterministic $\sigma(t,T)$ the log-volatility $\Sigma(t,T)$ is also deterministic. From Theorem 2.4 we know that the price at time $t$ of a call option with strike $K$ and expiry $S$ written on a zero-coupon bond with maturity $T > S$ is

$$\mathbf{BC}(t;S,T,K) = B(t,T)N(d_+) - KB(t,S)N(d_-),$$

where

$$d_+ = \frac{\ln \frac{B(t,T)}{B(t,S)K} + \frac{1}{2}v(t,S)}{\sqrt{v(t,S)}}, \qquad d_- = d_+ - \sqrt{v(t,S)},$$

with

$$v(t,S) = \int_t^S (\Sigma(u,T) - \Sigma(u,S))^2 \, du = \int_t^S \left( \int_S^T \sigma(u,v) dv \right)^2 du.$$

### Mercurio–Moreleda model

**Example 4.5**

A financially plausible choice of volatility in a Gaussian HJM model is one that depends on the time to maturity. For example, the volatility of the forward rate can be taken as

$$\sigma(t,T) = \sigma(\gamma(T-t)+1)e^{-\frac{\lambda}{2}(T-t)}, \tag{4.9}$$

where $\sigma, \gamma$ and $\lambda$ are positive constants such that $2\gamma > \lambda$. This is known as the **Mercurio–Moraleda model.**

**Exercise 4.4** Show that $\sigma(t,T)$ in the Mercurio–Moraleda model is a hump-shaped function of $T$. Specifically, show that $\sigma(t,T)$ is increasing in $T$ for $t \le T \le t + \frac{2\gamma-\lambda}{\gamma\lambda}$, has a maximum at $T = t + \frac{2\gamma-\lambda}{\gamma\lambda}$, and is decreasing in $T$ for $t + \frac{2\gamma-\lambda}{\gamma\lambda} \le T$.

In the next exercise we compute the prices of calls and puts written on a zero-coupon bond in the Mercurio–Moraleda model in terms of the parameters $\sigma, \gamma, \lambda$.

**Exercise 4.5** Show that the prices at time $t$ of European calls and puts with maturity $S$ and strike $K$ written on a $T$-bond, where $t < S < T$, in the Mercurio–Moraleda model (4.9) are given by (2.21), (2.22), (2.18) and (2.20) with

$$\begin{aligned}
v(t,S) &= \frac{4\sigma^2}{\lambda^4}\int_t^S (A-Bu)^2\,\mathrm{e}^{\lambda u}du \\
&= \frac{4\sigma^2}{\lambda^5}B^2\left(S^2\mathrm{e}^{\lambda S}-t^2\mathrm{e}^{\lambda t}\right)-\frac{8\sigma^2}{\lambda^6}B(B+A\lambda)\left(S\,\mathrm{e}^{\lambda S}-t\mathrm{e}^{\lambda t}\right) \\
&\quad+\frac{4\sigma^2}{\lambda^7}\left(2B^2+2AB\lambda+A^2\lambda^2\right)\left(\mathrm{e}^{\lambda S}-\mathrm{e}^{\lambda t}\right),
\end{aligned}$$

where

$$\begin{aligned}
A &= (\gamma\lambda T+\lambda+2\gamma)\,\mathrm{e}^{-\frac{\lambda}{2}T}-(\gamma\lambda S+\lambda+2\gamma)\,\mathrm{e}^{-\frac{\lambda}{2}S}, \\
B &= \gamma\lambda\left(\mathrm{e}^{-\frac{\lambda}{2}T}-\mathrm{e}^{-\frac{\lambda}{2}S}\right).
\end{aligned}$$

## 4.3 Calibration

To obtain a working model within the HJM framework we can proceed as follows.

First we specify the functional form of the volatility $\sigma(t,T)$ of the forward rate, depending on some constant parameters. Functional forms where the volatility is a decreasing function of time to maturity are popular, such as the Hull–White forward-rate volatility in Example 4.3.

Once we have specified the functional form for $\sigma(t,T)$, the drift $\alpha(t,T)$ of the forward rate can be computed from (4.3). The initial term structure $f^{\mathrm{mkt}}(0,T)$ of forward rates given by the market is then substituted for $f(0,T)$ in (4.5), so we get

$$f(t,T) = f^{\mathrm{mkt}}(0,T)+\int_0^t \alpha(t,u)du+\int_0^t \sigma(t,u)dW(u).$$

Finally, the parameters in the functional form adopted for $\sigma(t,T)$ are chosen so that we match the market prices (or equivalently the implied volatilities) of a set of vanilla instruments such as caps or swaptions.

## 4.4 Multi-factor HJM models

In a one-factor model such as (4.1) the forward rates are driven by a one-dimensional Brownian motion. By contrast, in a multi-factor HJM model the forward-rate process is driven by $n$ independent Brownian motions. For each maturity $T > 0$ the forward rate evolves as

$$df(t,T) = \alpha(t,T)dt + \sum_{i=1}^{n} \sigma_i(t,T)dW_i(t) \tag{4.10}$$

for $t \in [0,T]$, where the volatility $\sigma(t,T) = (\sigma_1(t,T),\ldots,\sigma_n(t,T))$ is an $n$-dimensional vector-valued process, and $W(t) = (W_1(t),\ldots,W_n(t))$ is an $n$-dimensional Brownian motion under the risk-neutral measure $Q$.

The analysis in Section 4.1 can be applied to (4.10) to prove the following result.

**Theorem 4.6**

*Suppose that the forward rates $f(t,T)$ satisfy* (4.10) *and assume that the volatilities $\sigma_1(t,T),\ldots,\sigma_n(t,T)$ are adapted to the filtration $\mathcal{F}_t$ for each $T > 0$, and*

$$\int_0^T \int_0^T |\alpha(s,u)|\, ds du < \infty, \qquad \int_0^T \left( \int_0^T \sum_{i=1}^{n} |\sigma_i(s,u)|^2\, ds \right)^{1/2} du < \infty$$

*almost surely. Then*

$$\alpha(t,T) = \sum_{i=1}^{n} \sigma_i(t,T) \int_t^T \sigma_i(t,u)du. \tag{4.11}$$

*Therefore the dynamics of $f(t,T)$ under the risk-neutral measure $Q$ is*

$$\begin{aligned} f(t,T) = f(0,T) &+ \int_0^t \sum_{i=1}^{n} \sigma_i(s,T) \left( \int_s^T \sigma_i(s,u)du \right) ds \\ &+ \int_0^t \sum_{i=1}^{n} \sigma_i(s,T)dW_i(s). \end{aligned}$$

*Furthermore, the zero-coupon bond prices satisfy the SDE*

$$dB(t,T) = r(t)B(t,T)dt + \sum_{i=1}^{n} \Sigma_i(t,T)B(t,T)dW_i(t), \tag{4.12}$$

*with*

$$\Sigma_i(t,T) = -\int_t^T \sigma_i(t,u)du$$

*for $i = 1,\ldots,n$.*

**Two-factor Gaussian model**

In the HJM two-factor framework the forward rate is given by

$$df(t,T) = \sum_{i=1}^{2}\left(\sigma_i(t,T)\int_t^T \sigma_i(t,u)du\right)dt + \sum_{i=1}^{2}\sigma_i(t,T)dW_i(t),$$

where $W_1(t)$, $W_2(t)$ are independent Brownian motions under the risk-neutral measure $Q$.

A popular choice for the volatilities is

$$\sigma_1(t,T) = \sigma_{11}e^{-\alpha_1(T-t)}, \quad \sigma_2(t,T) = \sigma_{21}e^{-\alpha_1(T-t)} + \sigma_{22}e^{-\alpha_2(T-t)}, \tag{4.13}$$

where $\sigma_{11}, \sigma_{21}, \sigma_{22}$ and $\alpha_1, \alpha_2$ are constants. For a suitable choice of constants this is the HJM equivalent of the Gaussian two-factor model (3.34), (3.35), (3.36), (3.37), or equivalently the two-factor Hull–White model in Section 3.7.

To see this we first express the correlated Brownian motions $U(t)$ and $V(t)$ in (3.35) and (3.36) as

$$\begin{aligned} U(t) &= \sqrt{1-\rho^2}W_1(t) + \rho W_2(t), \\ V(t) &= W_2(t). \end{aligned}$$

Therefore (3.35) and (3.36) become

$$\begin{aligned} dx(t) &= -\alpha x(t)dt + \sigma(\sqrt{1-\rho^2}dW_1(t) + \rho dW_2(t)), \\ dy(t) &= -\beta y(t)dt + \eta dW_2(t). \end{aligned}$$

By (3.42), for the two-factor Hull–White short-rate process the zero-coupon bond price at time $t \geq 0$ is

$$\begin{aligned} B(t,T) = \frac{B(0,T)}{B(0,t)}\exp\bigg(&-x(t)\frac{1-e^{-\alpha(T-t)}}{\alpha} - y(t)\frac{1-e^{-\beta(T-t)}}{\beta} \\ &+ \frac{1}{2}(V(0,t) - V(0,T) + V(t,T))\bigg), \end{aligned}$$

where $V$ is given by (3.40). Hence we have

$$dB(t,T) = B(t,T)r(t)dt + B(t,T)\left(\Sigma_1(t,T)dW_1(t) + \Sigma_2(t,T)W_2(t)\right)$$

with

$$\begin{aligned} \Sigma_1(t,T) &= -\int_t^T \sigma_1(t,u)du = -\sigma\sqrt{1-\rho^2}\frac{1-e^{-\alpha(T-t)}}{\alpha}, \\ \Sigma_2(t,T) &= -\int_t^T \sigma_2(t,u)du = -\sigma\rho\frac{1-e^{-\alpha(T-t)}}{\alpha} - \eta\frac{1-e^{-\beta(T-t)}}{\beta}. \end{aligned}$$

Therefore

$$\sigma_1(t,T) = \sigma\sqrt{1-\rho^2}\mathrm{e}^{-\alpha(T-t)},$$
$$\sigma_2(t,T) = \sigma\rho\mathrm{e}^{-\alpha(T-t)} + \eta\mathrm{e}^{-\beta(T-t)}.$$

Comparing this with (4.13), we can see that $\sigma_{11} = \sigma\sqrt{1-\rho^2}$, $\sigma_{21} = \sigma\rho$, $\sigma_{22} = \eta$ and $\alpha_1 = \alpha$, $\alpha_2 = \beta$.

## 4.5 Forward rate under the forward measure

The forward-rate dynamics under the risk-neutral measure $Q$ is given by (4.10). We now derive the dynamics of $f(t,T)$ under the forward measure $P_U$ for a settlement date $U \geq T$. Recall that this is the measure associated with $B(t,U)$ as numeraire.

From Section 2.2 we know that $P_U$ is a probability measure equivalent to $Q$ with Radon–Nikodym derivative

$$\frac{dP_U}{dQ} = \frac{1}{B(U)B(0,U)}.$$

Since $\frac{B(t,U)}{B(t)}$ is a martingale under $Q$, we have the density process

$$\xi(t) := \left.\frac{dP_U}{dQ}\right|_t = \mathbb{E}_Q\left(\frac{1}{B(U)B(0,U)}\middle|\mathcal{F}_t\right) = \frac{B(t,U)}{B(t)B(0,U)}$$

with $\xi(0) = 1$. The zero-coupon bond prices $B(t,U)$ satisfy SDE (4.12), hence $\xi(t)$ satisfies the SDE

$$\frac{d\xi(t)}{\xi(t)} = \sum_{i=1}^{n} \Sigma_i(t,U)dW_i(t),$$

with solution

$$\xi(t) = \exp\left(-\frac{1}{2}\int_0^t \sum_{i=1}^{n} \Sigma_i(s,U)^2 ds + \int_0^t \sum_{i=1}^{n} \Sigma_i(s,U)dW_i(s)\right),$$

where

$$\Sigma_i(t,U) = -\int_t^U \sigma_i(t,u)du \quad \text{for } i = 1,\ldots,n.$$

By the Girsanov theorem (see [BSM]), the process $(W_1^U(t), \ldots, W_n^U(t))$ with

$$\begin{aligned} W_i^U(t) &= W_i(t) - \int_0^t \Sigma_i(s, U)ds \\ &= W_i(t) + \int_0^t \left( \int_s^U \sigma_i(s, u)du \right) ds \quad \text{for } i = 1, \ldots, n \end{aligned}$$

is an $n$-dimensional Brownian motion under $P_U$. Substitution of the above into (4.10) with $\alpha(t, T)$ given by (4.11) yields

$$\begin{aligned} df(t, T) &= \sum_{i=1}^{n} \sigma_i(t, T) \left( \int_t^T \sigma_i(t, u)du \right) dt + \sum_{i=1}^{n} \sigma_i(t, T) dW_i(t) \\ &= \sum_{i=1}^{n} \sigma_i(t, T) \left( \int_t^T \sigma_i(t, u)du \right) dt \\ &\quad + \sum_{i=1}^{n} \sigma_i(t, T) dW_i^U(t) - \sum_{i=1}^{n} \sigma_i(t, T) \left( \int_t^U \sigma_i(t, u)du \right) dt \\ &= - \sum_{i=1}^{n} \sigma_i(t, T) \left( \int_T^U \sigma_i(t, u)du \right) dt + \sum_{i=1}^{n} \sigma_i(t, T) dW_i^U(t). \end{aligned} \quad (4.14)$$

**Exercise 4.6** Suppose that $\mathbb{E}_{P_T} \left( \int_0^T \sum_{i=1}^n \sigma_i(t, T)^2 dt \right) < \infty$. Show that the forward rate $f(t, T)$ is a martingale under the forward measure $P_T$ and, in particular,

$$f(t, T) = \mathbb{E}_{P_T} \left( r(T) | \mathcal{F}_t \right).$$

**Exercise 4.7** From (4.14) we have that the short rate $r(T)$ is given by

$$r(T) = f(t, T) + \int_t^T \sum_{i=1}^{n} \sigma_i(s, T) dW_i^T(s). \quad (4.15)$$

For $\sigma(t, T) = \sigma(t)e^{-\alpha(T-t)}$, where $\sigma(t)$ is a deterministic function and $\alpha$ is a constant, derive (3.29), the Hull–White short-rate process under the forward measure $P_T$ at time $T > t$ that gives an exact fit to the term structure at time 0.

# 5

# LIBOR and swap market models



In the previous chapters we presented models based on the instantaneous short rate and the instantaneous forward rate. These models suffer from a number of drawbacks. Firstly, calibration to the prices of commonly traded vanilla instruments such as caps, floors or swaptions can be quite involved. Exotic derivatives depending on the volatilities of many different rates may need to be calibrated to a large set of market instruments, which is difficult when using a short-rate model. Secondly, although instantaneous rates are mathematically convenient, they are not directly observable in the market, nor are they related in a straightforward manner to the prices of any traded instruments. It can be difficult to relate the model parameters, such as mean-reversion in the Hull–White model, to a market-observable quantity.

In the LIBOR market model (LMM) we are going to use market rates, namely the forward LIBOR rates, as state variables modelled by a set of stochastic differential equations. For a suitable choice of numeraire we will