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THE AXIOM OF A CONVEX QUADRILATERAL AND A CIRCLE THAT FORMS "PASCAL POINTS"- THE PROPERTIES OF "PASCAL POINTS" ON THE SIDES OF A CONVEX QUADRILATERAL

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Abstract

Euclidean geometry is one of the oldest branches of mathematics – the properties of different shapes have been investigated for thousands of years. For this reason, many tend to believe that today it is almost impossible to discover new properties and new directions for research in Euclidean geometry.

In the present paper, we define the concepts of "Pascal points", "a circle that forms Pascal points", and "a circle coordinated with the Pascal points formed by it", and we shall prove nine theorems that describe the properties of "Pascal points" on the sides of a convex quadrilateral.

These properties concern the following subjects:

● The ratios of the distances between the Pascal points formed on a pair of opposite sides by different circles.

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- The ratios of the distances between the centers of the circles "that form Pascal points on the sides of the quadrilateral", and the ratios of the distances between the Pascal points formed using these circles.

- Special types of circles "that form Pascal points on the sides of a quadrilateral".

- The properties of Pascal points and the centers of the special circles defined.

Introduction: Definitions and Fundamental Theorem

Let us consider the convex quadrilateral $ABCD$ in which the diagonals intersect at point E, and the continuations of sides BC and AD intersect at point F. We shall assume that there exists a circle ω, which satisfies the following two requirements:

(I) It passes through points E and F.

(II) It intersects sides BC and AD at their internal points M and N, respectively (see Figure 1).

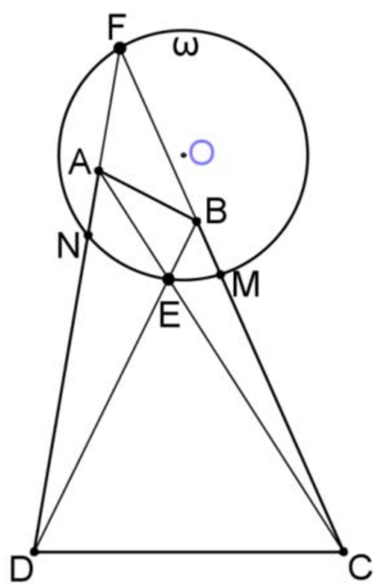

Figure 1.

Note. There exists convex quadrilaterals for which no circle satisfies both these requirements together (see Figures 2(a) and 2(b)):

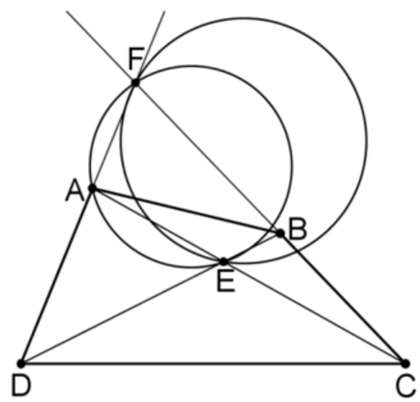

Figure 2(a).

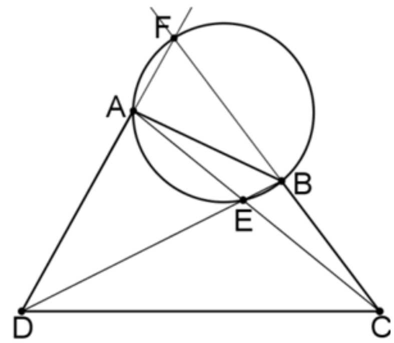

Figure 2(b).

We shall call a circle that satisfies both requirements (I) and (II) (for example, the circle in Figure 1) in the following manner:

A circle that passes through sides *BC* and *AD* and through points *E* and *F*.

We denote by K and L the points of intersection of circle ω with the continuations of diagonals BD and AC, respectively. We draw four straight lines through points K and L to points M and N (the points of intersection of circle ω with the sides of the quadrilateral, as shown in Figure 3). We denote by P the point of intersection of straight lines KN and LM, and by Q the point of intersection of lines KM and LN.

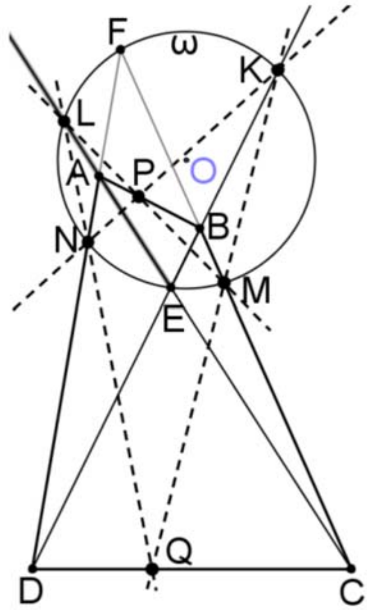

Figure 3.

It can be seen that points P and Q belong to sides AB and CD, respectively.

The property we observe is the **fundamental property** of the theory of a convex quadrilateral that is not a parallelogram, and the circle that is associated with it (see [2]).

Theorem 1 (The fundamental theorem). *Let there be*: *a convex quadrilateral*; *a circle that intersects a pair of opposite sides of the quadrilateral*, *that passes through the point of intersection of the continuations of these sides*, *and that passes through the point of intersection of the diagonals*. *In addition*, *let there be four straight lines*, *each of which passes both through the point of intersection of the circle with a side of the quadrilateral and through the point of intersection of the circle with the continuation of a diagonal*. *Then there holds*: *the straight lines intersect at two points that are located on the other pair of opposite sides in the quadrilateral*.

Or, *by notation*:

Given is: *convex quadrilateral ABCD*, *in which* $E = AC \cap BD$, $F = BC \cap AD$. *Circle* ω *that satisfies* $E, F \in \omega$; $M = \omega \cap [BC]$; $N = \omega \cap [AD]$; $K = \omega \cap BD$; $L = \omega \cap AC$.

Prove that: $KN \cap LM = P \in [AB]$; $KM \cap LN = Q \in [CD]$.

Proof. Note that we have defined six points in total on circle ω : E, M, K, F, L, and N, and that the theorem requires us to prove the collinearity of two triplet of points: the first triplet is A, P, B, and the second triplet is D, Q, C. These considerations provide a reason to use the following theorem, due to Pascal, in the proof: Opposite sides of a hexagon inscribed in a circle intersect at three points that are located on the same straight line. We note that Pascal's theorem also holds for a general hexagon, in other words a hexagon that is a closed broken line of six parts (segments) whose ends are located on a single circle.

Let us prove that points A, P, and B are located on the same straight line.

We consider the closed broken line *EKNFML* (see Figure 4(a)). Its opposite sides satisfy the following:

EL and *NF* intersect at point A, *ML* and *KN* intersect at point P, and *FM* and *EK* intersect at point B.

In addition, hexagon *EKNFML* is inscribed in a circle. Therefore, in accordance with the general Pascal's theorem, points A, P, and B are located on the same straight line. We have thus proven that there holds $P \in AB$.

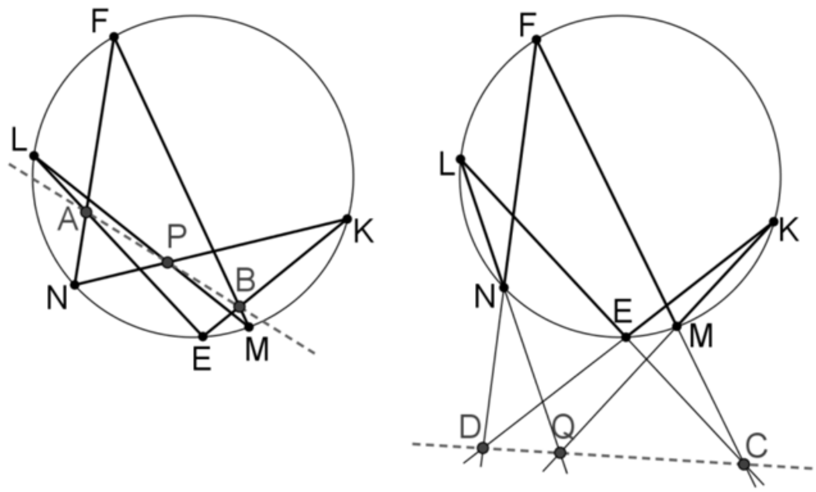

Figure 4(a). **Figure 4(b).**

Let us now prove that points C, Q, and D are located on the same straight line.

We consider the closed broken line (general hexagon) *EKMFNL* described in Figure 4(b). The continuations of the opposite sides of the hexagon satisfy: rays *LE* and *FM* intersect at point *C*, rays *KM* and *LN* – at point Q, and rays *KE* and *FN* – at point *D*. From the general Pascal theorem and from the fact that hexagon *EKMFNL* is inscribed in the circle, it follows that points C, Q, and D are located on the same straight line.

It remains to be proven that points *P* and Q belong to segments *AB* and *CD*, respectively.

We note that points *A*, *B*, and *P*, are the points of intersection of certain chords in the circle (see Figure 4(a)). In addition, point *L* and chord *KN* are located in the same part of the circle that is bounded by chord *EK* and arc $\widehat{EFK}$. On the other hand, point *K* and chord *LM* are located in the same part of the circle bounded by chord *EL* and arc $\widehat{EFL}$. Therefore *P*, as the point of intersection of chords *KN* and *LM*, belongs to the part of the circle that is the intersection of these parts and that, is also the inner part of the inscribed angle KEL. Since points A and B are located on the sides of angle KEL (on its limits), the intersection of straight line AB with the part of the circle bounded by chords KE, EL and arc $\widehat{KFL}$ is segment AB. Therefore, P is an interior point of the segments.

In the same manner, we prove that point Q is an inner point of segment CD. In fact, all the points of rays FD and MQ (aside for F and M), are located in the same half-plane with respect to straight line FM – this is the half-plane that does not contain the point K (see Figure 4(b)). All the points of rays FC and NQ (aside for F and N) are located on the same half-plane with respect to straight line FN – this is the half-plane that does not contain the point L. Point Q, being the point of intersection of the two rays MQ and NQ that belong to two different half-planes, belongs to the intersection of these half-planes, i.e., plane angle MFN. Points C and D are located on the sides of the angle (on rays FM and FN, respectively), therefore the intersection of straight line CD with plane angle MFN is segment CD, and Q is an inner point of the segment.

Q.*E.D.*

Definitions. Because we have used Pascal's theorem to prove the property of the points of intersection P and Q:

(I) We shall call our defined points "*Pascal points*" *on sides AB and CD of the quadrilateral*.

(II) We shall call a circle that passes through the points of intersection E and F, and through two opposite sides "*a circle that forms Pascal points on the sides of the quadrilateral*".

Properties that Result from the Fundamental Theorem

Let $ABCD$ be a convex quadrilateral; E – the point of intersection of the diagonals; F – the point of intersection of the continuations of the sides BC and AD; and let ω_1, ω_2, ω_3 be three circles, each of which is a circle that "passes through sides BC and AD and through points of intersection E and F" (see Figure 5).

For each of the circles, all the required constructions were performed to obtain "Pascal points" on sides *AB* and *CD*: using circle ω_1, we form "Pascal points" P_1 and Q_1; using circle ω_2, we form "Pascal points" P_2 and Q_2; using circle ω_3, we form "Pascal points" P_3 and Q_3.

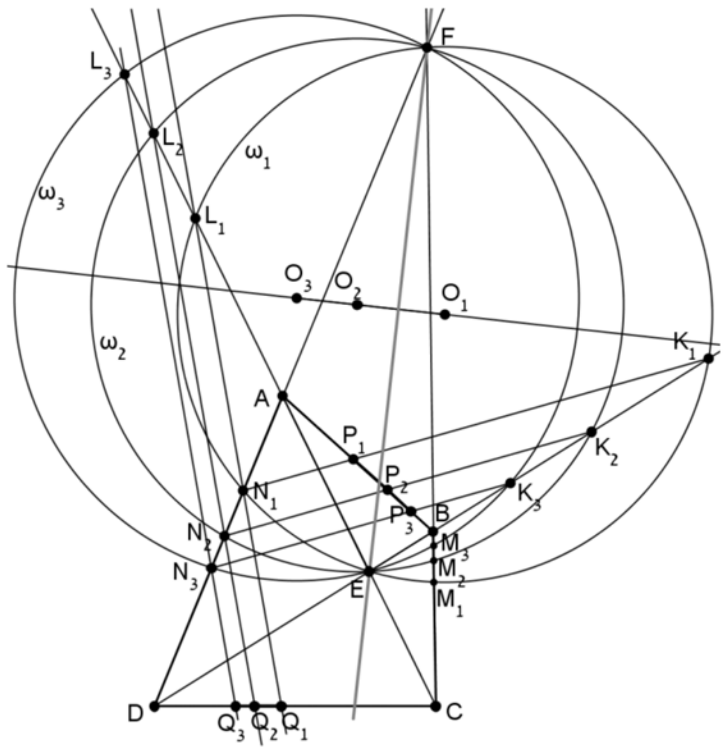

Figure 5.

The following theorem holds for these points:

Theorem 2. *Let* ω_1, ω_2, *and* ω_3 *be three circles that pass through sides BC and AD of the quadrilateral, through the point of intersection, F, of their continuation, and through the point of intersection, E, of the diagonals.*

Then, "*Pascal points*" P_1 *and* Q_1, P_2 *and* Q_2, P_3 *and* Q_3, *which are formed respectively using these circles*, *assign proportional segments on the sides AB and* CD : $\frac{P_1P_2}{P_2P_3} = \frac{Q_1Q_2}{Q_2Q_3}$.

Proof. We use the following lemma (on the intersecting lines of the circles that pass through the points of their intersection):

If circles ω_1 *and* ω_2 *intersect at points E and F; a straight line e passes through point E*, *which intersects circle* ω_1 *at point A and circle* ω_2 *at point C*; *a straight line f passes through point F*, *which intersects circle* ω_1 *at point B and circle* ω_2 *at point D. Then straight lines AB and CD are parallel.*

In proving the lemma, we distinguish between three possible cases of the reciprocal state of points *A*, *B*, *C*, and *D* with respect to straight line *EF*:

(I) Points *A* and *B*; *C* and *D* are located on different sides of straight line *EF* (see Figure 6(a)).

(II) All four points are on the same side relative to straight line *EF* (see Figure 6(b)).

(III) Points *A* and *C*; *B* and *D* are located on different sides of straight line *EF* (see Figure 6(c)).

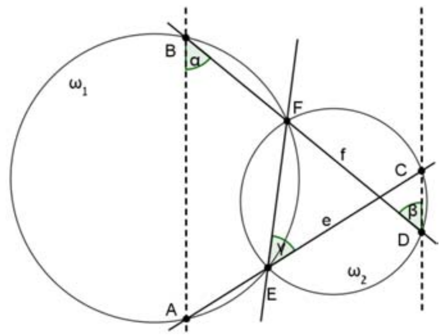

Figure 6(a).

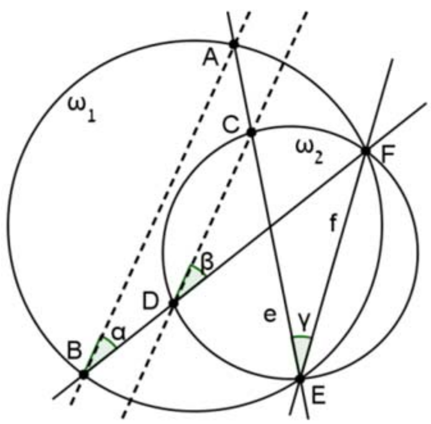

Figure 6(b).

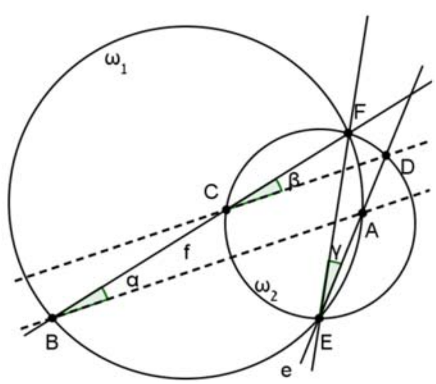

Figure 6(c).

The truthfulness of the lemma (in all three cases) is obtained from the equality of the two angles α and β (which results from the fact that each of these angles is equal to angle γ).

In proving Theorem 2, we shall use cases (II) and (III) of the lemma. First we prove that straight lines L_1N_1, L_2N_2, and L_3N_3 (see Figure 5) are parallel.

Circles ω_1 and ω_2 intersect at points E and F, and straight lines AC and AD pass through these points and intersect circle ω_1 at points L_1 and N_1, respectively, and circle ω_2 at points L_2 and N_2, respectively. Therefore, in accordance with the specified lemma (case II, in which all four points L_1, N_1, L_2, and N_2 are on the same side relative to straight line EF), there holds $L_1N_1 \| L_2N_2$.

In a similar manner, using circles ω_2 and ω_3 which intersect at points E and F, one proves that $L_2N_2 \| L_3N_3$. We thus obtain that the sides of angle ADC are intersected by three parallel straight lines. Therefore, from Thales' theorem, proportional segments are formed on the sides of the angle, satisfying: $\frac{N_1N_2}{N_2N_3} = \frac{Q_1Q_2}{Q_2Q_3}$ (*).

We shall now prove that the straight lines K_1N_1, K_2N_2, and K_3N_3 are also parallel to each other.

We return to circles ω_1 and ω_2. Straight line BD passes through point E and intersects ω_1 at point K_1, and ω_2 at point K_2. Straight line AD passes through point F and intersects ω_1 at point N_1, and ω_2 at point N_2.

From the above lemma (case III, in which points N_1 and N_2, K_1 and K_2 are on different sides relative to the straight line EF), there holds $K_1N_1 \| K_2N_2$. Similarly, we prove that $K_2N_2 \| K_3N_3$.

We obtained that the sides of angle DAB are bisected by the three parallel lines K_1N_1, K_2N_2, and K_3N_3.

Therefore proportional segments are formed on the sides of the angle, which satisfy $\frac{N_1N_2}{N_2N_3} = \frac{P_1P_2}{P_2P_3}$ $(**)$.

Following from equalities $(*)$ and $(**)$ is the required equality $\frac{P_1P_2}{P_2P_3} = \frac{Q_1Q_2}{Q_2Q_3}$.

Q.*E.D.*

Notes. (1) Since the centers of all the circles that pass through points E and F are located on the midperpendicular to segment EF, we shall call this straight line *the center-line of the circles that form "Pascal points" on sides BC and AD*, or – *the center-line determined by intersection points E and F of quadrilateral ABCD*.

(2) The equality of the ratios of the distances between the formed Pascal points (on the same pair of opposite sides in the quadrilateral) using three arbitrary circles ω_1, ω_2, and ω_3, which is the subject of Theorem 2 also holds for the distances between the centers of the circles O_1, O_2, and O_3, respectively (see Figure 5). In the other words, there holds $\frac{O_1O_2}{O_2O_3} = \frac{P_1P_2}{P_2P_3}$. (We shall discuss this equality in Theorem 5.)

We shall now consider the extreme states of a "circle that passes through sides BC and AD and through points E and F" of the quadrilateral $ABCD$.

In the first extreme state, the circle passes through points A, E, and F (we denote the circle by ω_A and its center by O_A).

In the second extreme state, the circle passes through points B, E, and F (we denote the circle by ω_B and its center by O_B).

It is clear that the center, O, of each circle ω that "passes through sides BC and AD and through points E and F" lies between points O_A and O_B (i.e., it belongs to the segment O_AO_B, see Figure 7).

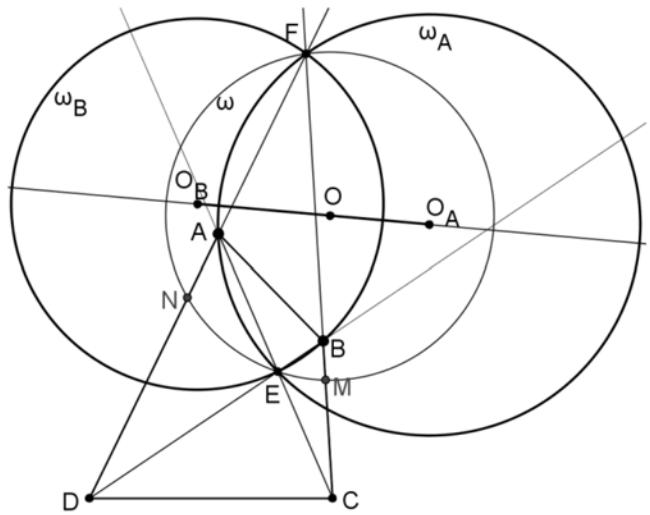

Figure 7.

In the circle ω_A, point A is both the point of intersection of the circle with side AD, and the point of intersection of the circle with diagonal AC. In the other words, $A = L_A = N_A$ (see Figure 8(a)). Therefore, for the "Pascal point" P_A, there holds:

$$P_A = K_A N_A \cap L_A M_A = K_A A \cap AM_A = A,$$

and the straight line $L_A N_A$ is actually a tangent to circle ω_A at point A.

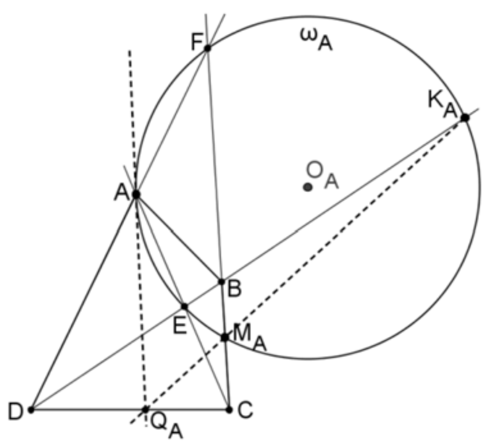

Figure 8(a).

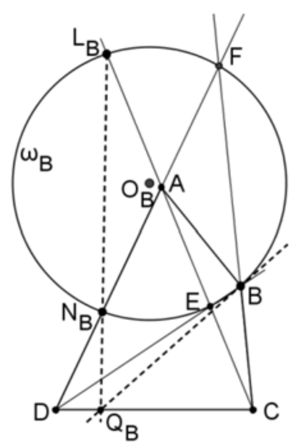

Figure 8(b).

Let us check if the point of intersection of this straight line and straight line $K_A M_A$ (denoted by Q_A) belongs to the side CD.

One method to prove the relation $Q_A \in CD$ is based on the known fact that Pascal's theorem also holds in degenerated cases:

In cases when one pair or more of the vertices of a hexagon coincide, one can consider the pentagon inscribed in the circle as a hexagon in which a pair of adjacent vertices coincide, and therefore one side of the hexagon transforms into a tangent to the circle. Similarly, one can consider a quadrilateral inscribed in a circle as a hexagon in which two pairs of adjacent vertices coincide, and therefore two sides of the hexagon become two tangents to the circle.

Similar results are also obtained for the circle ω_B (see Figure 8(b)). In this case: points K_B and M_B coincide with point B, in the other words, $P_B = B$. The tangent to circle ω_B at point B, and its intersecting line $L_B N_B$ intersect at point Q_B, which belongs to side CD. Therefore, the following theorem holds:

Theorem 3. *Let ABCD be a convex quadrilateral.*

(1) *The "Pascal points" on sides AB* and *CD, formed using circle* ω_A *that passes through points A, E and F, with* $E = AC \cap BD$, $F = BC \cap AD$, *are the vertex A and the point* Q_A, *which is the point of intersection of the tangent to* ω_A *at the point A, with the straight line* $K_A M_A$.

(2) *The "Pascal points" on sides AB and CD, formed using circle* ω_B *that passes through points B, E and F, are the vertex B and the point* Q_B, *which is the point of intersection of the tangent to* ω_B *at the point B, with the straight line* $L_B N_B$.

Conclusions from Theorems 2 and 3

If for a quadrilateral $ABCD$ there exists a circle ω that intersects sides BC and AD, and passes through the point of intersection of their continuation F and the point of intersection of the diagonals E, then:

(a) The center O of any circle ω "that passes through sides BC and AD and through points E and F" is located on segment O_AO_B, where points O_A and O_B are the centers of the circles ω_A and ω_B.

(b) Pascal points P and Q that are formed using circle ω divide segments AB and Q_AQ_B by an equal ratio, i.e., $\dfrac{AP}{PB} = \dfrac{Q_AQ}{QQ_B}$ (see Figure 9).

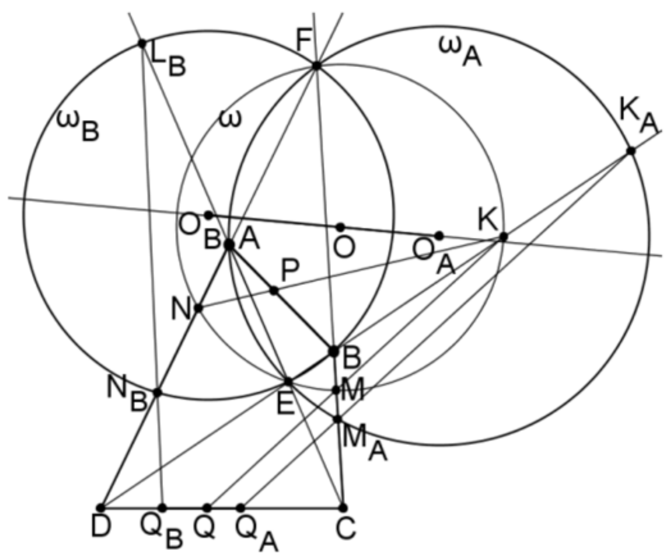

Figure 9.

Theorem 4. *Let ABCD be a quadrilateral whose diagonals intersect at point E, and the continuations of sides BC and AD intersect at the point F; and let* ω *be a circle "that passes through the sides BC and AD and through the points E and F",* ω_A *– a circle that passes through points A, E, and F;* ω_B *– a circle that passes through points B, E, and F.*

Then the center of circle ω *(point O) divides the segment that connects the centers of circles* ω_A *and* $\omega_B(O_AO_B)$ *by a ratio that is equal to the ratio by which "Pascal points"* P *and* Q *formed by circle* ω *divide segments AB and* Q_AQ_B, *respectively*.

Proof. In view of conclusion (b), above, it is sufficient to prove that there holds $\frac{AP}{PB} = \frac{O_AO}{OO_B}$.

We shall use the method of complex numbers in plane geometry (the principles of the method and the formulas we use in the proofs appear, for example, in source [4], pages 154-181).

We shall select a Cartesian system of coordinates in the following manner: the origin shall coincide with the point O and its length unit shall be equal to the radius of circle ω. In this system, circle ω is the unit circle whose equation is $z\bar{z} = 1$, where z and $\bar{z}$ are the complex coordinate and its conjugate for an arbitrary point on the circle.

All the points that appear in Figure 9 are assigned their coordinates in this system. We shall denote the complex coordinates of the points E, M, K, F, L, and N by e, m, k, f, l, and n, respectively. We shall make use of the following formulas:

Let S be the point of intersection of the straight lines that pass through chords AB and CD in the unit circle (see Figure 10).

For the complex coordinate of S and its complex conjugate, there holds:

(I) $\bar{s} = \frac{a+b-c-d}{ab-cd}$, and

(II) $s = \frac{bcd+acd-abd-abc}{cd-ab}$.

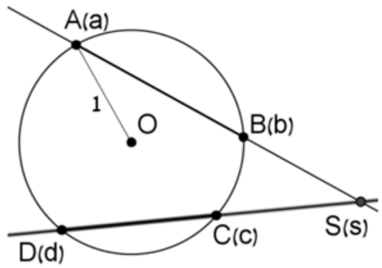

Figure 10.

From these formulas, the points of intersection of chords FN and EL, FM and EK, KN and ML, denoted by A, B, and P, respectively, shall have the following complex coordinates:

$$a = \frac{enl + efl - fnl - efn}{el - fn} \text{ and } \overline{a} = \frac{f + n - e - l}{fn - el};$$

$$b = \frac{emk + efk - fmk - efm}{ek - fm} \text{ and } \overline{b} = \frac{f + m - e - k}{fm - ek};$$

$$p = \frac{mnl + mkl - mnk - nkl}{ml - nk} \text{ and } \overline{p} = \frac{n + k - m - l}{nk - ml}.$$

We denote by λ_1 the ratio of the lengths of the segments $\frac{AP}{PB}$. This ratio can be expressed by the complex coordinates in the following manner: $\lambda_1 = \frac{p - a}{b - p}$, where $\lambda_1 = \overline{\lambda_1}$ is a real number. Hence

$$\lambda_1 = \overline{\lambda_1} = \frac{\overline{p} - \overline{a}}{\overline{b} - \overline{p}} = \frac{\dfrac{n + k - m - l}{nk - ml} - \dfrac{f + n - e - l}{fn - el}}{\dfrac{f + m - e - k}{fm - ek} - \dfrac{n + k - m - l}{nk - ml}}$$

$$= \frac{[(n + k - m - l)(fn - el) - (f + n - e - l)(nk - ml)](fm - ek)}{[(f + m - e - k)(nk - ml) - (n + k - m - l)(fm - ek)](fn - el)}$$

$$= \frac{(fm - fn + nk - ek - ml + el)(n - l)(fm - ek)}{(fm - fn + nk - ek - ml + el)(k - m)(fn - el)} = \frac{(n - l)(fm - ek)}{(k - m)(fn - el)}.$$

We obtained that $\lambda_1 = \frac{(n - l)(fm - ek)}{(k - m)(fn - el)}$.

We denote by λ_2 the ratio $\frac{O_A O}{OO_B}$ between the lengths of the segments that connect the centers of circles ω and ω_A, ω and ω_B.

Through complex coordinates, this ratio is expressed as

$$\lambda_2 = \frac{0 - o_A}{o_B - 0} = -\frac{o_A}{o_B},$$

where $\lambda_2 = \overline{\lambda_2}$. We express the complex coordinates of the centers O_A and O_B by the coordinates of the points that are located on the unit circle ω.

Point O_A is the intersection of the mid-perpendicular to segment EA and the mid-perpendicular to segment EF (the midline).

We use the equation of the mid-perpendicular to the segment (given in complex numbers): $(b-a)(\overline{z}-\overline{c}) + (\overline{b}-\overline{a})(z-c) = 0$, where TZ is the mid-perpendicular to segment AB (see Figure 11): T is the middle of segment AB, and Z is an arbitrary point on the perpendicular.

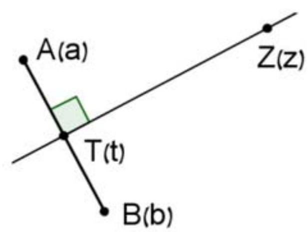

Figure 11.

This equation can be transformed into the following form:

(III) $$\overline{z} = \frac{\overline{a}-\overline{b}}{b-a} z + \frac{(b-a)\overline{t} + (\overline{b}-\overline{a})t}{b-a}.$$

Therefore for segment EF whose middle is at point O, the equation of the mid-perpendicular is $\overline{z} = \frac{\overline{f}-\overline{e}}{e-f} z + \frac{(e-f)\overline{o} + (\overline{e}-\overline{f})o}{e-f}$. In our case, point

O is the origin and the center of the unit circle, EF is the diameter of the unit circle. Therefore, for the complex coordinates of points O, E, and F, there holds: $o = \bar{o} = 0$, $\bar{e} = \frac{1}{e}$, and $\bar{f} = \frac{1}{f}$, and the equation of the mid-perpendicular to the segment EF shall become

$$\bar{z} = \frac{1}{ef} z \ (*).$$

For the segment EA, whose middle is at point T, the coordinate of T is expressed using the coordinates of the ends of segment EA as follows: $t = \frac{1}{2}(a + e)$.

We substitute the expression for t in equation (III), and we substitute e instead of b, and obtain

$$\bar{z} = \frac{\bar{a} - \bar{e}}{e - a} z + \frac{(e - a)\frac{1}{2}(\bar{a} + \bar{e}) + (\bar{e} - \bar{a})\frac{1}{2}(a + e)}{e - a},$$

which can be presented in the following manner:

$$\bar{z} = -\frac{\bar{a} - \bar{e}}{a - e} z + \frac{1}{2}\left[(\bar{a} + \bar{e}) + (a + e) \cdot \frac{\bar{a} - \bar{e}}{a - e}\right].$$

We substitute (one by one) expressions for the letters $\bar{e}$, a and $\bar{a}$ in this equation, and obtain

$$\bar{a} + \bar{e} = \frac{f + n - e - l}{fn - el} + \frac{1}{e} = \frac{ef + en - e^2 - 2el + fn}{e(fn - el)};$$

$$a + e = \frac{enl + efl - fnl - efn}{el - fn} + e = \frac{enl + efl - fnl - 2efn + e^2 l}{el - fn};$$

$$\frac{\bar{a} - \bar{e}}{a - e} = \frac{\frac{f + n - e - l}{fn - el} - \frac{1}{e}}{\frac{enl + efl - fnl - efn}{el - fn} - e} = \frac{\frac{ef + en - e^2 - fn}{e(fn - el)}}{\frac{enl + efl - fnl - e^2 l}{el - fn}} = \frac{-(ef + en - e^2 - fn)}{el\left(en + ef - fn - e^2\right)} = -\frac{1}{el}.$$

Therefore, the equation of the mid-perpendicular to the segment EA shall become

$$\bar{z} = \frac{1}{el}z + \frac{1}{2}\left(\frac{ef + en - e^2 - 2el + fn}{e(fn - el)} - \frac{enl + efl - fnl - 2efn + e^2l}{el - fn} \cdot \frac{1}{el}\right),$$

or, after simplification

$$\bar{z} = \frac{1}{el}z + \frac{(l - n)(f - l)}{l(fn - el)} \ (**).$$

By solving the system of equations $(*)$ and $(**)$, we will obtain the complex coordinates of $O_A : \frac{1}{ef}z = \frac{1}{el}z + \frac{(l - n)(f - l)}{l(fn - el)}$, and hence

$$z = o_A = \frac{\frac{(l - n)(f - l)}{l(fn - el)}}{\frac{1}{ef} - \frac{1}{el}} = \frac{(n - l)ef}{fn - el}.$$

In a similar manner, for the point O_B (the point of intersection of the midline and the mid-perpendicular to the segment *EB*), we obtain $o_B = \frac{(m - k)ef}{fm - ek}$.

Now we calculate the ratio λ_2:

$$\lambda_2 = -\frac{o_A}{o_B} = -\frac{\frac{(n - l)ef}{fn - el}}{\frac{(m - k)ef}{fm - ek}} = \frac{(n - l)(fm - ek)}{(k - m)(fn - el)}.$$

Therefore, we obtain that $\lambda_1 = \lambda_2$, and thus we have proven that $\frac{AP}{PB} = \frac{O_AO}{OO_B}$.

Q.*E.D.*

Conclusion from Theorems 2-4.

Let ω be a circle that forms "Pascal points" P and Q on sides *BC* and *AD*, respectively; and let points O_A and O_B be the centers of circles ω_A and ω_B which pass through the points *A*, *E*, *F* and *B*, *E*, *F*, respectively; and let the center, *O*, of circle ω be the middle of the center-line O_AO_B.

Then, points P and Q are the middles of segments AB and Q_AQ_B, respectively (where the points Q_A and Q_B are the Pascal points on side CD, formed using circles ω_A and ω_B, respectively).

Note. This conclusion applies to a quadrilateral with a very general shape, for example, for a quadrilateral that is not circumscribable. For such a quadrilateral, if "Pascal point" Q is the middle of segment Q_AQ_B, then it is not the middle of side CD. Therefore, for a general quadrilateral, there is no existing circle "that passes through sides BC and $AD\ldots$", which forms "Pascal points" that are the middles of sides AB and CD simultaneously.

Theorem 5. *Let ABCD be a quadrilateral whose diagonals intersect at point E, and whose continuations of sides AD and BC intersect at point F*; ω_1, ω_2, ω_3 *are three circles that "pass through sides AD and BC and through points E and F".*

Points O_1, O_2, *and* O_3 *are the centers of these circles, respectively*; P_1 *and* Q_1, P_2 *and* Q_2, P_3 *and* Q_3 *are Pascal points that are formed using these circles. Then the following proportion holds*:

$$\frac{P_1P_2}{P_2P_3} = \frac{Q_1Q_2}{Q_2Q_3} = \frac{O_1O_2}{O_2O_3}.$$

Proof. In view of Theorem 2, it is sufficient to prove that there holds: $\frac{P_1P_2}{P_2P_3} = \frac{O_1O_2}{O_2O_3}$.

On the midline (line O_1O_2), we add points O_A and O_B, which are the centers of the circles that pass through the points A, E, F and B, E, F, respectively (see Figure 12).

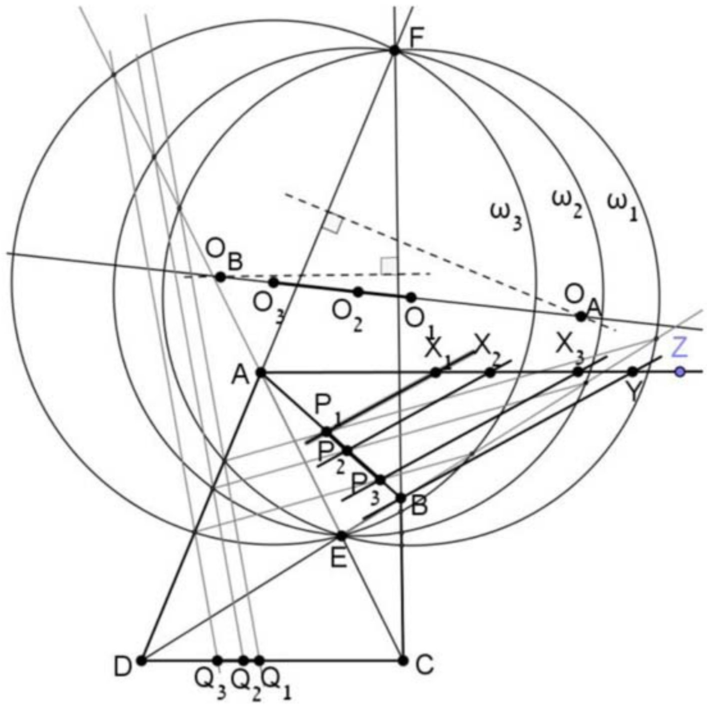

Figure 12.

From Theorem 4, the following proportions hold:

$$\frac{AP_1}{P_1B} = \frac{O_AO_1}{O_1O_B}, \quad \frac{AP_2}{P_2B} = \frac{O_AO_2}{O_2O_B}, \text{ and } \frac{AP_3}{P_3B} = \frac{O_AO_3}{O_3O_B}.$$

We add to Figure 12 the following auxiliary constructions:

We draw an arbitrary ray AZ, on which we mark four points, X_1, X_2, X_3, and Y, so that there holds: $AX_1 = O_AO_1$, $AX_2 = O_AO_2$, $AX_3 = O_AO_3$, and $AY = O_AO_B$.

Therefore, the following proportions will also hold: $\frac{AP_1}{P_1B} = \frac{AX_1}{X_1Y}$, $\frac{AP_2}{P_2B} = \frac{AX_2}{X_2Y}$, and $\frac{AP_3}{P_3B} = \frac{AX_3}{X_3Y}$. We draw four straight lines through points P_1 and X_1, P_2 and X_2, P_3 and X_3, B and Y.

From the inverse of Thales' theorem, it follows that the last proportion suggests that each of the straight lines P_1X_1, P_2X_2 and P_3X_3 is parallel to the line BY. Therefore, there holds: $P_1X_1 \| P_2X_2 \| P_3X_3$.

Therefore, from Thales' theorem, it follows that $\frac{P_1P_2}{P_2P_3} = \frac{X_1X_2}{X_2X_3}$.

From the last proportion and the fact that $X_1X_2 = O_1O_2$ and $X_2X_3 = O_2O_3$, we obtain that $\frac{P_1P_2}{P_2P_3} = \frac{O_1O_2}{O_2O_3}$.

Q.*E.D.*

Now let us consider the special circle ω that is a "circle that passes through two opposite sides of the quadrilaterial…", and that also satisfies the following additional property: *a straight line that passes through the "Pascal points" P and Q formed using circle* ω, *also passes through the center of circle O.*

Definition. A circle, whose center is collinear with the "Pascal points" formed by it shall be called:

The circle coordinated with the Pascal points formed by it.

For example, in Figure 13 shown are the "Pascal points" P and Q that were formed by circle ω, and the straight line PQ that passes through the center of the circle O. Therefore, circle ω is the circle coordinated with the "Pascal points" formed by it on sides AB and CD.

Theorem 6. *Let ABCD be a convex quadrilateral, and let* ω *be a circle coordinated with the "Pascal points" P and Q formed by it, where* ω *intersects a pair of opposite sides of the quadrilateral at points M and N, and also intersects the continuations of the diagonals at points K and L* (see Figure 13).

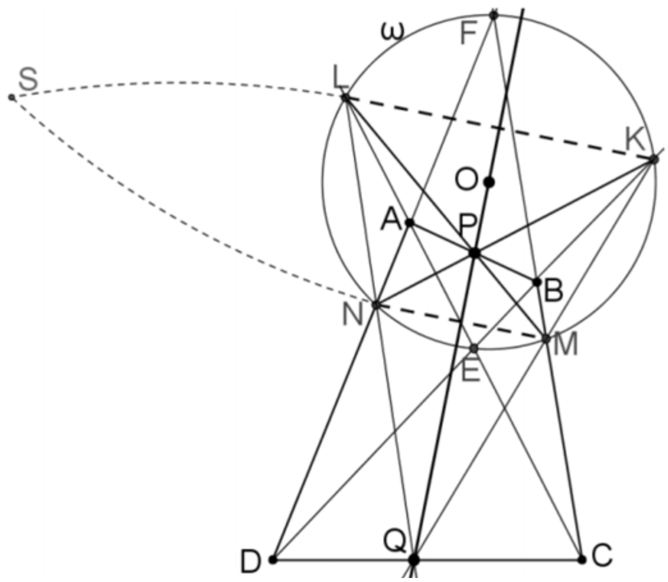

Figure 13.

Then there holds:

(a) $KL \| MN$;

(b) *in a system in which circle* ω *is the unit circle, the complex coordinates of the points* K, L, M, *and* N *satisfy the equality* $mn = kl$.

Proof. (a) Let us consider quadrilateral $KLNM$ that is inscribed in circle ω. The diagonals of this quadrilateral intersect at point P, and the continuations of the sides KM and LN intersect at point Q.

Let us assume that the continuations of sides KL and MN also intersect at point S (see Figure 13).

Then, straight line PQ is a polar of point S relative to circle ω (see [3], Section 211).

In addition, line PQ passes through point O, which is the center of circle ω. Therefore, pole S of straight line PQ is an infinity point, and therefore lines KL and MN are parallel.

(b) From (a), it follows that vectors $\overrightarrow{KL}$ and $\overrightarrow{MN}$ are parallel, and that the complex coordinates of points K, L, M and N satisfy the following equality:

$$(l-k)(\overline{n}-\overline{m}) = (\overline{l}-\overline{k})(n-m) \; (*).$$

We choose a system of coordinates such that circle ω is the unit circle (O is the origin and the radius is $OE = 1$). Since points K, L, M, and N belong to the unit circle (whose equation is $z\overline{z} = 1$), there holds $\overline{k} = \frac{1}{k}$, $\overline{l} = \frac{1}{l}$, $\overline{m} = \frac{1}{m}$, and $\overline{n} = \frac{1}{n}$. We substitute these expressions into formula $(*)$ and obtain $(l-k)\left(\frac{1}{n}-\frac{1}{m}\right) = \left(\frac{1}{l}-\frac{1}{k}\right)(n-m)$. After simplification, we obtain: $mn = kl$.

Q.*E.D.*

Theorem 7. *Let ABCD be a convex quadrilateral, and let* ω *be circle coordinated with "Pascal points"* P *and* Q *formed by it. Then, points* P *and* Q *transform one into the other by inversion relative to circle* ω. *In other words there holds the equality* $OP \cdot OQ = r^2$, *where r and O are the radius and the center of* ω, *respectively.*

Proof. We select a system of coordinates so that circle ω is the unit circle (O is the origin and the radius is $r = OE = 1$). Therefore, one must prove that $OP \cdot OQ = 1 \, (*)$.

For the complex coordinate of point P (and its conjugate), there holds (see the proof of Theorem 4):

$$\overline{p} = \frac{n+k-m-l}{nk-ml} \text{ and } p = \frac{mkl+mnl-nkl-mnk}{ml-nk}.$$

Or (since $mn = kl$) we can obtain a simpler expression for p:

$$p = \frac{mn(m+l-n-k)}{ml-nk}.$$

Similarly, for the complex coordinates of point Q (and its conjugate), we obtain

$$\overline{q} = \frac{m + k - n - l}{mk - nl} \text{ and } q = \frac{mn(n + l - m - k)}{nl - mk}.$$

We use the formula for the distances between two points $A(a)$ and $B(b)$

$$|AB|^2 = (b - a)\left(\overline{b} - \overline{a}\right).$$

For distance OP, we obtain

$$OP = \sqrt{(p - o)(\overline{p} - \overline{o})} = \sqrt{\left(\frac{mn(m + l - n - k)}{ml - nk} - 0\right)\left(\frac{n + k - m - l}{nk - ml} - 0\right)}$$

$$= \sqrt{\frac{mn(m + l - n - k)^2}{(ml - nk)^2}},$$

and similarly, for distance OQ, we obtain $OQ = \sqrt{(q - o)(\overline{q} - \overline{o})} = \sqrt{\frac{mn(n + l - m - k)^2}{(nl - mk)^2}}$. Therefore,

$$OP \cdot OQ = \sqrt{\frac{mn(m + l - n - k)^2}{(ml - nk)^2}} \cdot \sqrt{\frac{mn(n + l - m - k)^2}{(nl - mk)^2}}$$

$$= \frac{mn(m + l - n - k)(n + l - m - k)}{(ml - nk)(nl - mk)}$$

$$= \frac{mn[(l - k) + (m - n)] \cdot [(l - k) - (m - n)]}{mnl^2 - m^2kl - n^2kl + mnk^2}$$

$$= \frac{mn\left(l^2 - 2lk + k^2 - m^2 + 2mn - n^2\right)}{mn\left(l^2 - m^2 - n^2 + k^2\right)}$$

$$= \frac{mn\left(l^2 + k^2 - m^2 - n^2\right)}{mn\left(l^2 - m^2 - n^2 + k^2\right)} = 1.$$

We have obtained that points P and Q satisfy equality $(*)$ relative to circle $\omega = (O, r)$. Therefore, points P and Q transform one into the other by inversion relative to circle ω.

Q.*E.D.*

Theorem 8. *Let ABCD be a convex quadrilateral in which E is the point of intersection of the diagonals, and F is the point of intersection of the continuations of sides BC and AD; and let* ω *be a circle that intersect sides BC and AD at points M and N, respectively, and also intersects the continuations of diagonals BD and AC at points K and L, respectively. In addition,* ω *is coordinated with "Pascal points" P and* Q *formed by it; and let PQ be a straight line that intersects* ω *at points I and J* (see Figure 14(a)).

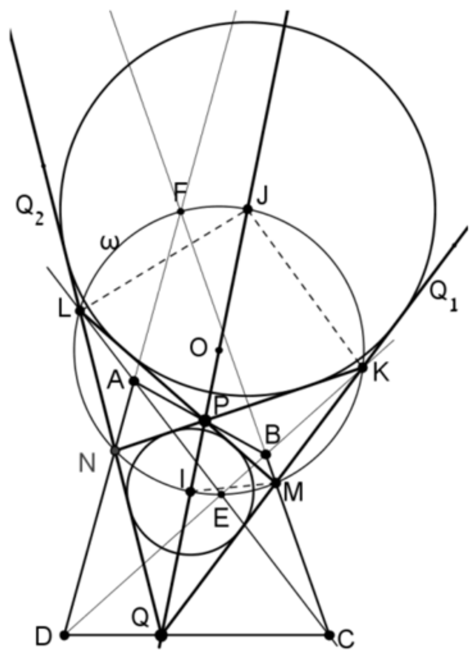

Figure 14(a).

Then there holds:

(a) *quadrilateral* $PMQN$ *is a kite*;

(b) *point* I *is the center of the circle inscribed in quadrilateral* $PMQN$, *and point* J *is the center of the circle that is tangent to the continuations of the sides of the quadrilateral* $PMQN$.

Proof. (a) From section (a) of Theorem 6 $(KL \parallel MN)$, quadrilateral $KMNL$ is a trapezoid (see Figure 14(b)). Since $KMNL$ is inscribed in a circle, it follows that it is an isosceles trapezoid in which $KM = LN$ (equal sides), and $KN = LM$ (equal diagonals). Now, it is easy to see that $PM = PN$ and $QM = QN$, in the other words, quadrilateral $PMQN$ is a kite.

(b) The main diagonal of the kite (segment PQ) bisects the two angles MPN and MQN. Given is the fact that ω is a circle whose center, O, is collinear with "Pascal points" P and Q formed by it. Therefore, in accordance with Theorem 7, points P and Q transform one into the other by inversion relative to circle ω. Hence, these points together with the points of intersection I and J form a **harmonic quadruplet** (see [3], Section 204). Therefore, points I and J divide segment PQ by a harmonic division: I – by internal division, J – by external division, and therefore circle ω is a circle of Appolonius for the segment PQ.

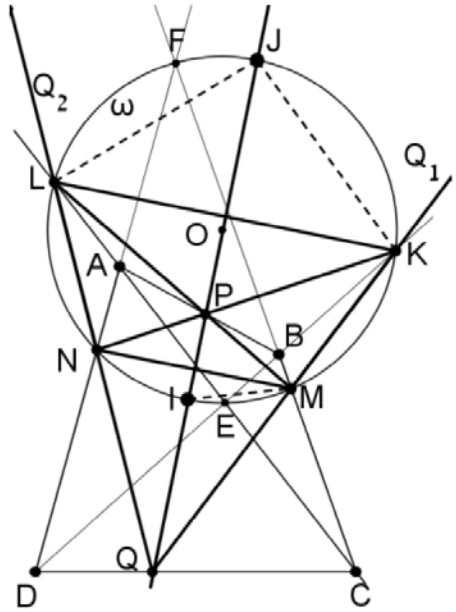

Figure 14(b).

Regarding point M, as a point that belongs to the circle of Appolonius ω, there holds that segment MI bisects angle PMQ in triangle PMQ.

Therefore point I, as the point of intersection of three angle bisectors in the quadrilateral, lies at equal distances from all four sides of the quadrilateral $PMQN$. It follows that point I is the center of the circle inscribed in a quadrilateral $PMQN$.

We now consider segment KJ. Since point K belongs to circle of Appolonius ω (whose diameter is IJ), it follows that segment KJ bisects the exterior angle of triangle PKQ (angle PKQ_1). Similarly, we prove that LJ bisects angle PLQ_2. In addition, ray PJ bisects angle KPL (since PQ bisects angle MPN which is vertically opposite KPL). It follows that point J is located at equal distances from four rays: PK, PL, KQ, and LQ_2, all of which are continuations of the sides of kite $PMQN$.

Therefore, J is the center of the circle that is tangent to the continuations of the sides of quadrilateral $PMQN$ (see Figure 14(a)).

Q.*E.D.*

Theorem 9. *Let $ABCD$ be a convex quadrilateral in which E is the point of intersection of the diagonals, and F is the point of intersection of the continuations of sides BC and AD; ω is a circle coordinated with the "Pascal points" P and Q formed by it; O_A and O_B are the centers of the circles ω_A and ω_B that pass through the points A, E, F and B, E, F, respectively. Then:*

(a) *The polar of point O_A relative to circle ω and the polar of point O relative to circle ω_B coincide* (see Figure 15).

(b) *The polar of point O_B relative to circle ω and the polar of point O relative to circle ω_A coincide.*

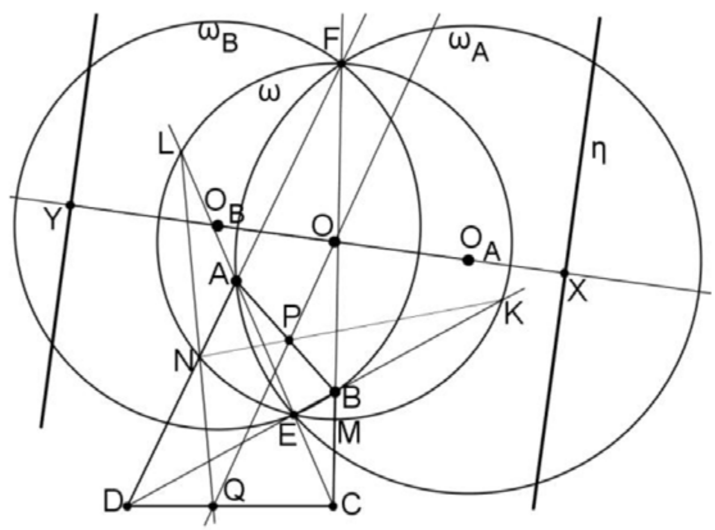

Figure 15.

Proof. We choose a system of coordinates, so that circle ω is the unit circle (O is the origin and the radius is $OE = 1$). We use the formulas that relate the complex coordinates of centers O_A and O_B with the complex coordinates of the six points on the circle ω (see the proof of Theorem 4):

$$o_A = \frac{(n-l)ef}{fn-el} \text{ and } o_B = \frac{(m-k)ef}{fm-ek}.$$

The formulas of the conjugates shall be $\overline{o_A} = \dfrac{n-l}{fn-el}$ and $\overline{o_B} = \dfrac{m-k}{fm-ek}$.

We denote by η the polar of point O_A relative to circle ω, and by X the point of intersection of η with center-line O_AO_B. From the definition of a pole and its polar relative to a given circle (see [1], Chapter 6, Paragraph 1), points O_A and X transform into one another by inversion relative to circle ω.

Therefore the following relation holds between the complex coordinates of points O_A and X (see [5], Paragraph 13): $x = \dfrac{1}{\overline{o_A}}$ and $\overline{x} = \dfrac{1}{o_A}$.

We substitute the expressions for o_A and $\overline{o_A}$ into these equations, and obtain $x = \dfrac{fn-el}{n-l}$ and also $\overline{x} = \dfrac{fn-el}{(n-l)ef}$.

By equating expressions for x and $\overline{x}$, o_A and $\overline{o_A}$, o_B and $\overline{o_B}$, it is easy to see that $\overline{x} = \dfrac{x}{ef}$, $\overline{o_A} = \dfrac{o_A}{ef}$ and $\overline{o_B} = \dfrac{o_B}{ef}$.

Therefore, $x\overline{x} = \dfrac{x^2}{ef}$, $o_A\overline{o_A} = \dfrac{{o_A}^2}{ef}$, and $o_B\overline{o_B} = \dfrac{{o_B}^2}{ef}$ $(*)$ (we use these relations later in the proof).

We now prove that points O and X transform one into the other by inversion relative to circle ω_B. To this end, we verify that the equality $OO_B \cdot O_BX = r^2_{\omega_B}$ holds (see [3], Section 204). We select $r_{\omega_B} = O_BF$, and check whether the relation $OO_B \cdot O_BX = (O_BF)^2$ $(**)$ holds. In complex coordinates, equation $(**)$ is written as follows:

$$\sqrt{\left((o_B - 0)(\overline{o_B} - \overline{0})\right)\left((o_B - x)(\overline{o_B} - \overline{x})\right)} = (o_B - f)\left(\overline{o_B} - \bar{f}\right),$$

and after opening the parentheses, we have

$$\sqrt{o_B\overline{o_B}(o_B\overline{o_B} - o_B\overline{x} - \overline{o_B}x + x\overline{x})} = o_B\overline{o_B} - o_B\bar{f} - \overline{o_B}f + 1.$$

We use formulas $(*)$, and obtain

$$\sqrt{\frac{o_B{}^2}{ef}\left(\frac{o_B{}^2}{ef} - o_B\frac{x}{ef} - \frac{o_B}{ef}x + \frac{x^2}{ef}\right)} = \frac{o_B{}^2}{ef} - o_B\frac{1}{f} - \frac{o_B}{ef}f + 1,$$

and finally $\sqrt{\left(\frac{o_B}{\sqrt{ef}}\right)^2 \cdot \left(\frac{o_B}{\sqrt{ef}} - \frac{x}{\sqrt{ef}}\right)^2} = \frac{o_B{}^2}{ef} - \frac{o_B}{ef}(e + f) + 1$ $(***)$.

Equation $(***)$ is satisfied if at least one of the following equalities is satisfied:

(a) $$\frac{o_B}{\sqrt{ef}} \cdot \left(\frac{o_B}{\sqrt{ef}} - \frac{x}{\sqrt{ef}}\right) = \frac{o_B{}^2}{ef} - \frac{o_B}{ef}(e + f) + 1,$$

or

(b) $$\frac{o_B}{\sqrt{ef}} \cdot \left(\frac{x}{\sqrt{ef}} - \frac{o_B}{\sqrt{ef}}\right) = \frac{o_B{}^2}{ef} - \frac{o_B}{ef}(e + f) + 1.$$

Let us check the correctness of equality (a):

We simplify equality (a) to the following form:

$$2o_B - x - (e + f) + \frac{ef}{o_B} = 0.$$

We substitute the expressions for o_B and x, and obtain

$$2\frac{(m - k)ef}{fm - ek} - \frac{fn - el}{n - l} - (e + f) + \frac{ef(fm - ek)}{(m - k)ef} = 0.$$

Hence, after algebraic simplification and division of the two sides by $mn = kl$ (which follows from the data that circle ω is coordinated with the "Pascal points" formed by it), we obtain

$$4fe(m-k) - 2f^2(m-k) - 2e^2(m-k) = 0,$$

and hence, since $m \neq k$, we obtain the following impossible condition: $(f-e)^2 = 0.$

Let us check the correctness of equality (b):

The equality can be transformed into $x - e - f + \frac{ef}{o_B} = 0.$

We substitute the expressions for o_B and x, and obtain

$$\frac{fn - el}{n - l} - e - f + \frac{fm - ek}{m - k} = 0.$$

After adding fractions and collecting similar terms, we obtain $\frac{ekl - fkl - emn + fmn}{(n-l)(m-k)} = 0,$ and finally $(e-f)(kl - mn) = 0.$

Since the factor $kl - mn$ in the last equality equal to zero (see Section b, Theorem 6), we conclude that this equality is a true statement, and therefore equality $(***)$ is satisfied, which suggests that equality $(**)$ is also true.

Hence it follows that points O and X are transformed into one another relative to circle ω_B. From the definition of a pole and its polar, straight line η (a line that passes through X and is perpendicular to the center-line) is a polar of pole O relative to circle ω_B. We have obtained that straight line η is both a polar of the point O_A relative to circle ω, and a polar of point O relative to circle ω_B.

The second part (part b) of Theorem 9 is proved in a similar manner.

Q.*E.D.*

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