CHAPTER 4

Derivatives: Forwards, Futures, Swaps

A *derivative* is a financial contract between two parties that obligates a trade in an asset, termed the *underlying asset*, at a future date known as the *delivery date* or *expiration date*, at a predetermined price called the *strike price*. The underlying asset can be a commodity like crude oil, gold, etc., in which case the derivative is called a *commodity derivative*, or it can be some other financial instrument like a bond or stock, in which case it is called a *financial derivative*. A derivative itself has value and is hence regarded as a financial instrument. Also, a derivative, as a financial instrument, can be traded in derivatives markets, and hence derivatives are securities.

The party who initiates (*i.e.,* sells) a derivative contract is called a *seller* or *writer* of the derivative, and the party who buys it is called a *buyer* or *holder*. Among the two parties, the one who agrees to buy the asset is said to take a *long position*, and the other is said to take a *short position*. Derivatives can be categorized into two types, namely, *contingent claims* and *noncontingent claims*. *Contingent claims* are contracts that define the holders' right but not the obligation to execute the contract on or before the expiry. Options are contingent claims. On the other hand, *noncontingent claims* are contracts where both the writer and holder have an obligation to execute the contract on or before the expiry. Forward, future, and swap are noncontingent claims.

Before proceeding further into the theory of derivatives, let us first understand the basic purpose of the derivatives. Traders in a market can be classified into two types based on their risk attitude. One type exhibits a *risk-averse* nature, endeavoring to quantify the risks associated with their trades and then striving to balance them through alternative trading or investment strategies. For instance, they may diversify their investments, allocating funds into different assets. Conversely, traders with a *risk-seeking* nature willingly embrace higher levels of risk to maximize gains. An example of this is investing all their funds into a single stock whose price exhibits significant fluctuations in a short period.

Any strategy adopted to reduce the risk of price fluctuations in an asset is called *hedging*. A trader with a risk-averse nature may be referred to as a *hedger*. Trading in a risky security with the hope of making profits in a short span of time due to market fluctuations is known as *speculation*. Traders with a risk-seeking nature may be called *speculators*.

Derivatives markets are primarily developed to serve the purpose of hedging, providing risk management tools for various financial instruments. However, due to their increasing popularity, these markets exhibit high liquidity, meaning that many traders actively participate. Consequently, securities traded in the markets experience significant price fluctuations, making them attractive to speculators. Thus, a derivative market becomes a dynamic mix of hedgers and speculators. The active participation of traders with diverse mindsets often leads to *arbitrage* opportunities, situations where guaranteed profit can be obtained without any initial investment, across markets (in combination with spot and derivative markets). As a result, *arbitrageurs* are naturally drawn to the derivatives markets.

Diverse market participants, including institutional investors, individual traders, and market makers, play pivotal roles in shaping the dynamics of derivatives markets. The positions taken in derivatives markets exert a profound influence on the corresponding dynamics of spot markets. Institutional investors, primarily acting as hedgers, utilize derivative instruments as a key hedging tool to manage and mitigate risks. Moreover, institutional investors often engage in arbitrage strategies, capitalizing on fleeting opportunities that arise and vanish rapidly. Successfully identifying and capitalizing on such opportunities demands sophisticated machine learning and artificial intelligence techniques. The development and efficient implementation of such software, powered by robust supercomputers, enable automatic identification and rapid execution in the brief window of arbitrage opportunities. The architects of these software solutions require a profound understanding of derivatives, recognizing the intricate interdependence between derivatives and spot markets. Understanding derivatives pricing models becomes paramount in this context, as it forms the foundation for developing effective strategies in both hedging and arbitrage scenarios.

In our course, we mainly concentrate on hedging strategies and derivative pricing models. In this chapter, we restrict our discuss to forwards, futures, and swaps. In Section 4.1, we introduce the notion of an arbitrage portfolio, which plays a central role in constructing pricing models for derivatives. We initiate our discussion on derivatives with forwards in Section 4.2, where we derive pricing models for forward contracts both without and with costs, and also without and with dividends. Section 4.3 is devoted to the discussion of futures, where we explain the marking to market process and hedging with futures. Finally, in Section 4.4, we discuss forward rates and end the chapter with a brief discussion on swaps.

4.1 Arbitrage Portfilio

Before proceeding further into derivatives, let's define an important concept known as an *arbitrage portfolio*, which plays a crucial role in pricing derivatives. An arbitrage portfolio refers to a carefully constructed combination of financial instruments that enables investors to exploit price discrepancies in different markets or assets. Traders and investors strategically leverage arbitrage opportunities to capitalize on market inefficiencies, contributing to the equalization of prices across different assets. Therefore, understanding the concept of an arbitrage portfolio is pivotal for comprehending the dynamics of derivative pricing.

In this section, we introduce the notion of a discrete market model, where we define portfolios and arbitrage portfolios restricted to a single trading period $[0, T]$. These concepts can be easily extended to any given partition of this interval.

4.1.1 Portfolio

We start with a rigorous definition of portfolio. We denote a collection of risk-free assets by

$$\boldsymbol{B} = (B_1, B_2, \ldots, B_{n_b}),$$

where each B_i, $i = 1, 2, \ldots, n_b$, is a risk-free asset like a bond. Let us denote a collection of risky assets by

$$\boldsymbol{S} = (S_1, S_2, \ldots, S_{n_s}),$$

where each S_i, $i = 1, 2, \ldots, n_s$, is a risky asset like a stock of a company. Finally, we denote a collection of derivatives by

$$\boldsymbol{D} = (D_1, D_2, \ldots, D_{n_d}),$$

where each D_i, $i = 1, 2, \ldots, n_d$, is one of the derivative instruments.

A *market* is a collection of assets $(\boldsymbol{B}, \boldsymbol{S}, \boldsymbol{D})$.

Note

From a real-world perspective, $\boldsymbol{B}$ can be considered as the bond market, $\boldsymbol{S}$ as the equity market, and $\boldsymbol{D}$ as the derivatives market. However, at an individual level, whether at an institutional level or even at the level of an individual portfolio manager, a market can be formulated as per their requirements.

Example 4.1.1.

Mrs. Sahana has ₹1,00,000 surplus money, which she kept in her savings account at a bank with the intention of investing the money in the stock market. She selected her 5 favorite stocks $\boldsymbol{S} = (S_1, S_2, \ldots, S_5)$. For instance, S_1 represents TCS, S_2 is SBI bank, S_3 denotes Reliance, S_4 is Infosys, and S_5 corresponds to Bajaj Auto. Then, she considered her own market $(B, \mathbf{S})$ where B represents her savings account, which pays interest at the rate of 4% per annum paid quarterly.

Definition 4.1.2 [Portfolio].

A ***portfolio*** of a given market $(\boldsymbol{B}, \boldsymbol{S}, \boldsymbol{D})$ is an ordered $(n_b + n_s + n_d)$-tuple of real numbers

$$\Pi := (\boldsymbol{b}, \boldsymbol{s}, \boldsymbol{d}),$$

where $\boldsymbol{b} = (b_1, b_2, \ldots, b_{n_b})$ with b_i being the number of units of the asset B_i, $i = 1, 2, \ldots, n_b$, $\boldsymbol{s} = (s_1, s_2, \ldots, s_{n_s})$ with s_i being the number of units of the asset S_i, $i = 1, 2, \ldots, n_s$, and $\boldsymbol{d} = (d_1, d_2, \ldots, d_{n_d})$ with d_i being the number of units of the derivative D_i, $i = 1, 2, \ldots, n_d$.

Note

By assuming each component of a portfolio to be a real number, we assume that trading fractional quantities is allowed in the asset. Often, this is not allowed in a market; for instance, stocks are always traded in integer quantities. This assumption is therefore only for theoretical purposes with caution.

If a component of a portfolio is negative, it means that the portfolio has a *short position* in the corresponding asset. The portfolio is said to have a *long position* in an asset if the corresponding component has a positive value.

Given a market, building a portfolio consists of three steps:

1. Select the assets to be included in the portfolio and determine their quantities.
2. Plan the order in which they are to be traded (sometimes this ordering may not matter).
3. Trade (buy and sell) the assets in the market as planned.

The first time the portfolio is completely built is the starting time of the portfolio, and the time at which any change is made in the portfolio is the end time of the current portfolio. The modification time is marked as the starting time for a new portfolio with present adjustments.

Once the portfolio is built, its value can be determined at every time instance.

Definition 4.1.3 [Portfolio Value].

Assume that a portfolio is built at time $t = 0$ and is held until time $t = T$. The ***portfolio value*** at any time $t \in [0, T]$ is defined as

$$V(\Pi)(t) = \sum_{i=1}^{n_b} b_i B_i(t) + \sum_{i=1}^{n_s} s_i S_i(t) + \sum_{i=1}^{n_d} d_i D_i(t),$$

where $S_i(t)$ denotes the current market price for the asset S_i at time t and similarly for B_i's and D_i's. The above expression can also be written in the vector notation as

$$V(\Pi)(t) = \boldsymbol{b} \cdot \boldsymbol{B}(t) + \boldsymbol{s} \cdot \boldsymbol{S}(t) + \boldsymbol{d} \cdot \boldsymbol{D}(t).$$

Example 4.1.4.

On 2nd of January 2023, Mrs. Sahana had ₹1,00,000 in her saving account with a bank. On the same day, she made the following investments:

- 10 shares of TCS at ₹3285 per share
- 5 shares of State Bank of India at ₹610 per share
- 7 shares of Reliance at ₹2333 each
- 8 shares of Bajaj Auto at ₹3588

The remaining amount was kept in the savings account with an interest rate of 4% per annum, paid quarterly.

The portfolio out of the market considered in Example 4.1.1 is

$$\Pi = (b_1, \boldsymbol{s}),$$

where we take $b_1 = 1$ and $\boldsymbol{s} = (10, 5, 7, 0, 8)$.

We always consider the first component of the vector $\boldsymbol{B}$ as the savings bank account with its unit set as $b_1 = \pm 1$. Note that $b_1 = -1$ indicates that the investor had borrowed money from the bank to invest in assets. In such cases, it is understood that the investor has to pay interest to the bank as per the specified interest scheme.

We now compute the value of the portfolio at $t = 0$, representing the *initial investment* of the portfolio. We always neglect the cost associated with brokerage, taxes, and any other charges involved in maintaining the portfolio. Such a valuation is referred to as the *frictionless valuation*.

From the buy price of the stocks, let us take $\boldsymbol{S}(0) = (3285, 610, 2333, 0, 3588)$. Therefore the total investment in stocks is given by

$$\boldsymbol{s} \cdot \boldsymbol{S}(0) = 80935.$$

For the risk-free investment we take $B_1(0) = 19065$ as the initial value. Therefore, the total investment in the risk-free assets is

$$\boldsymbol{b} \cdot \boldsymbol{B}(0) = b_1 \times B_1(0) = 19065.$$

The initial value of the portfolio is $V(\Pi)(0) = b_1 \times B(0) + \boldsymbol{s} \cdot \boldsymbol{S}(0) = 100000$.

After the end of one year (denoted as t_4 due to the quarterly interest payments by B_1), it is found from the market that

- the market prices of TCS is $S_1(t_4) = 3675$
- the market prices of SBI is $S_2(t_4) = 637$
- the market prices of Reliance is $S_3(t_4) = 2599$
- the market prices of Bajaj Auto is $S_5(t_4) = 6961$

Therefore,

$$\boldsymbol{s} \cdot \boldsymbol{S}(t_4) = 113816 \quad \text{and} \quad \boldsymbol{b} \cdot \boldsymbol{B}(t_4) \approx 19839.12.$$

The frictionless value of the portfolio (neglecting the dividend payments of the stocks) at $t = t_4$ is given by

$$V(\Pi)(t_4) \approx 133655.12.$$

Definition 4.1.5 [Gain].

Let Π be a portfolio created at time $t = 0$ and held until time $t = T$. The ***gain*** of a portfolio between two times $t \in [0, T]$ is defined as

$$G(\Pi)(t) = \boldsymbol{b} \cdot \Delta \boldsymbol{B}(t) + \boldsymbol{s} \cdot \Delta \boldsymbol{S}(t) + \boldsymbol{d} \cdot \Delta \boldsymbol{D}(t),$$

where $\Delta \boldsymbol{S}(t) = \boldsymbol{S}(t) - \boldsymbol{S}(0)$ and similarly for other components.

Problem 4.1.

Mr. Megh purchased 500 shares of Infosys and 200 shares of SBI on the 2^{nd} of January 2023, financed by borrowing money from a bank with an annual interest rate of 9%, compounded quarterly. Write down the portfolio created by Mr. Megh with respect to the market given in Example 4.1.1. Calculate the gain of his portfolio as of the 2^{nd} of January 2024. **Answer:** $G(\Pi)(t_4) \approx -81393.28$

Hints:
Obtain the corresponding share prices from the website

`https://www.nseindia.com/report-detail/eq_security`

Take the day's high price for each stock on the 2nd of January 2023 and then take the day's low price on the 2nd of January 2024 in your calculations.

4.1.2 Arbitrage

We finally define the notion of *arbitrage portfolio*.

Definition 4.1.6 [Arbitrage Portfolio].

A portfolio Π of a market is said to be an ***arbitrage portfolio*** or simply *arbitrage* in a time period $[0, T]$, say, if it satisfies the following conditions:

1. $V(\Pi)(0) = 0$,
2. $V(\Pi)(T) \geq 0$, a.s. and
3. $\mathbb{P}\big(V(\Pi)(T) > 0\big) > 0$.

Remark 4.1.7 [Arbitrage-free market].

If at least one arbitrage portfolio exists in a market, then we say that the market has *arbitrage opportunity*. A market is said to be an *arbitrage-free market* or a *viable market* if it provides no arbitrage opportunities.

Example 4.1.8.

Consider a hypothetical situation with two bank offers, Bank-L and Bank-D. Bank-L offers a loan at the continuously compounded rate of 8% per annum, while Bank-D offers a deposit at the rate of 9% per annum, continuously compounded.

Build a portfolio by making the following investments:

- Borrow ₹1,00,000 from Bank-L for one year.
- Deposit ₹1,00,000 in Bank-D.

In this case, we take $\boldsymbol{B} =$ (Bank-L, Bank-D) as our market and the portfolio as

$$\Pi = (-1, 1).$$

Then we can see that $V(\Pi)(0) = 0$. After one year, the deposit in Bank-D gives

$$100000 \times e^{0.09} \approx ₹1,09,417.43,$$

while the liability with Bank-L is

$$100000 \times e^{0.08} \approx ₹1,08,328.71.$$

Therefore,

$$V(\Pi)(T) = -108328.71 + 109417.43 \approx 1088.72 > 0.$$

Assuming no credit risk, this positive gain with probability 1 implies an arbitrage opportunity in the market.

Note

In the above example, the bank offers are often conceptualized as bonds. Therefore, the portfolio is considered to take a short position in the bond offered by Bank-L and a long position in the bond offered by Bank-D.

In the above definition, $V(\Pi)(T)$ is a random variable defined on a probability space $(\mathsf{S}, \mathscr{S}, \mathbb{P})$. Thus, the arbitrage condition of a market depends on the probability measure that we use. In the above example, we have not explicitly defined the probability measure to check the arbitrage condition because the situation is intuitively clear. However, in other scenarios, we may need to consider a suitable probability measure to verify arbitrage conditions.

Problem 4.2.

Construct a suitable probability space $(\mathsf{S}, \mathscr{S}, \mathbb{P})$ with respect to which the portfolio constructed in Example 4.1.8 becomes an arbitrage portfolio.

The following example serves to illustrate two fundamental concepts in financial mathematics: risk-neutral price and risk-neutral probability.

Example 4.1.9 [Risk-neutral price vs. Risk-neutral probability].

Consider a game where a six-faced die is rolled once. If the outcome belongs to $E_1 = \{3, 4, 5, 6\}$, the player earns $₹S_u = 100$. Conversely, if the outcome is in $E_2 = \{1, 2\}$, the player receives only $₹S_d = 20$. An entry fee of $₹S_0$ is imposed to play the game. The fundamental question is: What is the fair value of S_0?

To determine the fair price, it is necessary to decide upon the probability of winning the game. In this context, the choice of probability is clear: $\mathbb{P}(E_1) = 2/3$. The estimated average value of the bet is therefore $S_T \approx 73.33$.

Assuming that the entry fee must be paid today, and the game will be played after one year. Then, $T = 1$ and the value S_T correspond to this future time. To determine

S_0, it is essential to discount S_T to the present time using a particular interest rate scheme. Let us assume continuous compounding scheme with an annual rate of $r = 0.07$. Consequently, we find $S_0 \approx 68.37$. This price is referred to as the *risk-neutral price* for this game.

Now, let us pose a reverse question: If the entry fee is fixed at ₹S_0 (not necessarily the risk-neutral price obtained above), what is the implied probability consistent with the given entry fee S_0 regarding a continuously compounding prevailing interest rate of r per annum?

Let $q \in (0,1)$ be the required probability, then we have

$$S_0 = e^{-rT}\big(qS_u + (1-q)S_d\big),$$

which leads to

$$q = \frac{e^{rT}S_0 - S_d}{S_u - S_d}.$$

Here, q is called the *risk-neutral probability*. This is the unique probability to be assigned to the given game for which the given entry fee S_0 is a risk-neutral price.

In particular, if we take $S_0 = 68.37$, then $q \approx 0.6666$ as expected.

Problem 4.3.

For a given continuously compounding interest rate r per annum, show that the game defined in Example 4.1.9 gives an arbitrage opportunity if and only if the entry fee S_0 is such that

$$e^{rT}S_0 < S_d.$$

Problem 4.4.

Let the prevailing continuously compounding interest rate be 7% per annum. On the 2$^{\text{nd}}$ of January 2023, Mrs. Sahana conducted her own analysis of the TCS stock price and predicted that the per-share price will either touch $S_u = 3650$ on the 2$^{\text{nd}}$ of January 2024 or it will touch $S_d = 3175$ on the same day. Answer the following:

1. If Mrs. Sahana is confident about her prediction that the share price will touch S_u with her observed probability 0.7, then find the risk-neutral price of the stock on the 2$^{\text{nd}}$ of January 2023. **Answer:** ≈ 3270.37
2. If the traded price of the stock on the 2$^{\text{nd}}$ of January 2023 is ₹3285 per share, then find the risk-neutral probability that the stock will touch S_u on the 2$^{\text{nd}}$ of January 2024. **Answer:** ≈ 0.73303