

8.2 Sensitivity Analysis: Greeks

A hedge is a position that offsets the price risk of another position. The main purpose of an option is to hedge a position in an asset. A risk in holding an asset is due to the fluctuations in the asset's price, which causes volatility. Thus, the underlying asset price S and the volatility parameter σ mainly influences an option's price which is also apparent from the Black-Scholes model. It is important to understand the dependency of the option price on S and σ (also on r and t) more closely in order to use options as a powerful hedging tool. To this end, we introduce certain terms that can be used in dependency analysis, called the *sensitivity analysis*, of the option price on the variables. Some of the sensitivity terms are commonly denoted by Greek alphabets and hence the name *Greeks* is used in the literature. In this section, we introduce some important Greeks used in option price sensitivity analysis and discuss their usage in hedging strategies.

The Black-Scholes formulae for European put and call options are derived in Corollary 7.2.7 and Corollary 7.2.8, respectively. We can see that these prices depend on t , S , σ , r , and K . For a fixed strike K , we write the Black-Scholes formulae as

$$\begin{aligned} C(t, S, r, \sigma) &= S\Phi(d_+) - Ke^{-r(T-t)}\Phi(d_-), \\ P(t, S, r, \sigma) &= Ke^{-r(T-t)}\Phi(-d_-) - S\Phi(-d_+), \end{aligned}$$

where

$$d_{\pm}(t, S, r, \sigma) = \frac{\log\left(\frac{S}{K}\right) + \left(r \pm \frac{\sigma^2}{2}\right)(T-t)}{\sigma\sqrt{T-t}}.$$

In this section, we analyze the sensitivity of these prices with respect to the parameters t , S , σ , and r . More precisely, we study the fundamental properties of the partial derivatives of C and P (we use the notation V , the value of an option or a portfolio, in a general discussion not specific to call or put) with respect to the parameters occurring in the Black-Scholes formulae. The collection of all the partial derivatives is called the *Greeks*. The purpose of studying Greeks is to build hedging portfolios to reduce the sensitivity of the value of a portfolio (or an option) to changes in the underlying by means of diversifying the position.

8.2.1 Delta

Let us start our sensitivity analysis with the dependency of V with respect to the price of the underlying asset S .

The *delta* of an option or a portfolio of options is the sensitivity of the option or portfolio to the underlying and is given by

$$\Delta_V = \frac{\partial V}{\partial S}.$$

We shall now see how the knowledge of delta can be used in hedging a portfolio.

Delta Hedge

Consider the portfolio $\Pi = (0, s, h)$, where s and h are the number of units in the underlying and the option, respectively. Then, the value of the portfolio is given by

$$V = sS + hH,$$

where S is the price of the underlying and H is the price of the option.

Perfect hedging is to maintain the value V of the portfolio unchanged while the underlying price fluctuates. This is achieved by selecting s for a given h (or choosing h for a given s) such that

$$\frac{\partial V}{\partial S} = 0.$$

Let $\Delta_H = \partial H / \partial S$ be the delta of the option. Then, we have

$$s = -h\Delta_H \text{ or } h = -\frac{s}{\Delta_H}.$$

Problem 8.9.

Show that

1. the delta of a European call option is given by

$$\Delta_C := \frac{\partial C}{\partial S} = \Phi(d_+).$$

2. the delta of a European put option is given by

$$\Delta_P := \frac{\partial P}{\partial S} = -\Phi(-d_+).$$

From Problem 8.9, we can see that $\Delta_C > 0$ and $\Delta_P < 0$ (the delta for call and put options, respectively). Thus, a short position in call leads to a long spot position and so on.

A portfolio is called *delta neutral* at some time $t \in [0, T]$ if $\Delta_V = 0$ at t .

Building a delta neutral portfolio by having an opposite spot market position in the underlying of an option position is referred to as *delta-hedging* of a portfolio.

Example 8.2.1.

Consider a K -strike call (European) option whose delta is given by $\Delta_C = 0.6$. and the lot size is 100. Assume that a trader sold (wrote) 20 call options. Thus, the portfolio of the trader is $\Pi = (0, s, -2000)$ and the value of the portfolio is given by

$$V = sS - 2000 \times C.$$

To keep the portfolio delta neutral, we have to choose the number of units of the stock s such that the delta of the portfolio is zero. But the delta of the portfolio is

$$\Delta_V = s - 2000 \times 0.6.$$

Thus, in order to keep the portfolio delta neutral (or in order to delta-hedge the portfolio Π), the trader has to buy $s = 1200$ shares in the underlying.

Explanation: At time $t = 0$, let the underlying be trading at ₹100 per share and the trader sold the call options at ₹10 per share. Then the initial investment in the portfolio Π is

$$V(\Pi)(0) = 1200 \times 100 - 2000 \times 10 = 100000.$$

At time $t = \Delta t$, let the stock price goes up by ₹1.

Since $\Delta_C = 0.6$, we can say approximately that the change in the call option price

$$C(\Delta t) - C(0) = 0.6(S(\Delta t) - S(0)) = 0.6 \times 1 = 0.6.$$

Thus, the corresponding call option price is approximately $C(\Delta t) = 10.6$.

The value of the portfolio Π at time $t = \Delta t$ is given by

$$V(\Pi)(\Delta t) = 1200 \times 101 - 2000 \times 10.6 = 100000.$$

Since $V(\Pi)(0) = V(\Pi)(\Delta t)$, we see that the change in the stock price did not make any profit or loss to the trader leading to a delta neutral portfolio.

Remark 8.2.2.

Delta hedging generally works well for a short time period, *i.e.* when Δt is very small, and when the stock price variation is very small. If the time step is large or the stock price variation is very large, then one has to recompute the delta of the option (Δ_C or Δ_P depending on whether the traded option is a call or a put option, respectively) and reconstruct the delta-hedging portfolio. This process is called the *rebalancing* of the portfolio.

If a hedged position is rebalanced time to time to maintain a delta neutral portfolio, then the hedging process is called the *dynamic hedge*. On the other hand, if a hedging position is held without rebalancing until the expiry or square-off, then the hedging process is called the *static hedge*.

The information about delta of an option is (similarly all other Greeks that we discuss in the subsequent subsections are) provided by the stock exchange itself.

Example 8.2.3.

Consider two call options on the same underlying asset and expiration T :

1. $K_1 = 22, 150$ with premium $C_{1,0} = 89$;
2. $K_2 = 22, 250$ with premium $C_{2,0} = 48$.

Here, we used the notation $C_{1,0}$ for $C_1(t_0)$ and a similar notation for the other option.

Consider a portfolio $\Pi_1 = (s, c_1, -c_2)$ at time $t = t_0$, where s is the number of units in the underlying, and $c_i > 0$, $i = 1, 2$, is the number of units held in the K_i -strike call option.

In particular, if we take $c_1 = c_2 = c$, then the portfolio is a bull spread.

At time $t_0 < t_1 < T$, the price of K_1 -strike option is $C_{1,1} = 155$ with delta $\Delta_{1,1} = 0.53$ and K_2 -strike price is $C_{2,1} = 65$ with delta $\Delta_{2,1} = 0.44$.

There are two ways to achieve a delta neutral portfolio at time $t = t_1$. One is by delta hedging and another is by simply adjusting the option portfolio itself.

Delta-hedge portfolio: In this case we have to find s such that $\Delta_V = 0$. Thus, we have

$$s = -c(\Delta_{1,1} - \Delta_{2,1}) = -0.09c.$$

Option adjustment: We have $\Pi_1 = (0, 1, -1)$ created at time $t = t_0$. At time $t = t_1$, we have to make another portfolio Π_2 such that $\Delta_{V_2} = 0$. Let the portfolio Π_2 be $\Pi_2 = (0, c, c_2)$. Then, we have

$$V_2 = cC_1 - c_2C_2.$$

Therefore, $\Delta_{V_2} = 0$ implies

$$c_2 = \frac{\Delta_{1,1}}{\Delta_{2,1}}c \approx 1.2045c.$$

We already hold c units of K_2 -strike option in our portfolio. Therefore, we have to sell $c_2 - c = 0.2045c$ units of K_2 -strike option additionally to make the portfolio Π_2 . This results in a spread called *ratio spread*. Ratio spread is similar to bull and bear spreads but with different units in the option positions.

Note that the above adjustment is one way to achieve delta neutral portfolio. We may also adjust the K_1 -strike option or we may also look for a K_3 -strike option, even with a different expiration.

At time $t_0 < t_1 < t_2 < T$, let $C_{3,1} = 237$, $C_{3,2} = 129$, $\Delta_{1,2} = 0.5$ and $\Delta_{2,2} = 0.32$. Again we can adjust the K_2 -strike (in particular) to achieve a delta neutral portfolio at time t_2 . Hence, we made a three level dynamic delta neutral portfolio strategy $\{\Pi_1, \Pi_2, \Pi_3\}$. We can extend it to any number of time levels. Theoretically, we can even obtain a continuous time delta neutral portfolio strategy $\{\Pi_t\}$, which will obviously be a stochastic process.

8.2.2 Gamma

In the last subsection we have seen that using delta hedge we can remove the risk fully. Since delta varies with respect to time (for a sufficiently large time, for instance, on daily basis), one needs to frequently rebalance the portfolio to maintain the delta neutral portfolio. In fact, a continuous rebalancing of delta hedge may be required, which is not practically feasible. One way to improve the hedge position accuracy is to include the gamma hedge in the dynamic delta hedge strategy. Let us first introduce the formula for gamma.

The *gamma* of an option or a portfolio of options is defined as

$$\Gamma = \frac{\partial^2 V}{\partial S^2}.$$

The gamma of an European call and put options are defined similarly and are denoted by Γ_C and Γ_P , respectively.

Problem 8.10.

Using Black-Scholes formula for European call options show that

$$\Gamma_C = \frac{e^{-\frac{d_+^2}{2}}}{\sqrt{2\pi(T-t)}S\sigma}.$$

Similarly, one can obtain a formula for Γ_P . Portfolio hedging can be made more accurate by considering a *delta-gamma-hedging*. This is achieved by setting gamma equals zero along with delta-hedging. We illustrate this with an example.

Example 8.2.4.

Consider a European call options with two strikes, K_1 and K_2 , with prices denoted by C_1 and C_2 , respectively. The portfolio is given by $\Pi = (0, s, \mathbf{c})$, where $\mathbf{c} = (c_1, c_2)$ is the vector with components being the number of units in respectively options. The value of the portfolio is given by

$$V = sS + c_1C_1 + c_2C_2.$$

Differentiating with respect to S twice, we get

$$\frac{\partial V}{\partial S} = s + c_1\Delta_{C_1} + c_2\Delta_{C_2}$$

and

$$\frac{\partial^2 V}{\partial S^2} = c_1\Gamma_{C_1} + c_2\Gamma_{C_2}.$$

Equating both the above expressions to zero, we get the linear system

$$\begin{aligned} s + c_1\Delta_{C_1} + c_2\Delta_{C_2} &= 0 \\ c_1\Gamma_{C_1} + c_2\Gamma_{C_2} &= 0. \end{aligned}$$

Given the values of Δ_{C_j} and Γ_{C_j} , $j = 1, 2$, and one of s and c_j , $j = 1, 2$, we can solve the linear system (if possible) to obtain the delta-gamma-hedging portfolio.

Note

Since Γ_{C_j} , $j = 1, 2$, are positive, we need one of c_1 and c_2 to be negative in the above system to have a nonzero solution for the second equation. Hence one needs to take opposite positions in the two call options in order to obtain a delta-gamma-hedging portfolio.

8.2.3 Theta

The *theta* of an option or a portfolio of options is defined as

$$\Theta = \frac{\partial V}{\partial t}$$

The theta of a European call and put options are defined similarly and are denoted by Θ_C and Θ_P , respectively.

Problem 8.11.

Using Black-Scholes formula for European K -strike call option show that

$$\Theta_C = -\frac{S\sigma}{2\sqrt{2\pi}(T-t)}e^{-\frac{d_+^2}{2}} - rKe^{-r(T-t)}\Phi(d_-).$$

8.2.4 Vega

The *vega* of an option or a portfolio of options is defined as

$$\mathcal{V} = \frac{\partial V}{\partial \sigma}$$

The vega of a European call and put options are defined similarly and are denoted by \mathcal{V}_C and \mathcal{V}_P , respectively.

Problem 8.12.

Using Black-Scholes formula for European call option show that

$$\mathcal{V}_C = \frac{S\sqrt{T-t}}{\sqrt{2\pi}}e^{-\frac{d_+^2}{2}}.$$

To hedge against the underlying volatility, one can make a *vega-hedging* by equating $\mathcal{V} = 0$.

8.2.5 Rho

The *rho* of an option or a portfolio of options is defined as

$$\rho = \frac{\partial V}{\partial r}$$

The rho of a European call and put options are defined similarly and are denoted by ρ_C and ρ_P , respectively.

Problem 8.13.

Using Black-Scholes formula for European K -strike call option show that

$$\rho_C = (T-t)Ke^{-r(T-t)}\Phi(d_-).$$

Index

- 3-step binomial model, 157
- accrued interest, 36
- accumulation factor, 13, 17
- adapted, 130
- American option, 95
- annuity, 25
- annuity-immediate, 25
- arbitrage, 50, 55
 - strategy, 64
- arbitrage opportunity, 55
- arbitrage portfolio, 51, 55
- arbitrage strategy, 64
- arbitrage-free market, 55
- arbitrageurs, 50
- asset, 11
- asset-liability, 11
- at-the-money, 96
- attainable, 148

- backward induction, 154
- basis, 79
- bear spread, 173
- bearish, 174
- binomial lattice model, 115, 116, 120
- binomial model, 116
- Black-Scholes equation, 166
- Black-Scholes model, 115
- Black-Scholes price, 162, 163
- Black-Sholes model, 159
- Bombay stock exchange, 8
- bond, 31
 - market, 7
 - primary, 32
 - secondary, 32
- par, 35
- Brownian motion, 20
- bull spread, 171
- bullish, 171
- butterfly spread, 175
- buy-side, 2
- buyer, 49, 94

- calendar spread, 170
- call option, 94
- call options, 94
- capital, 11
- capital market, 9
- carry costs, 62
- cash flow, 11, 14
- cash flow stream, 14, 21
- Cauchy problem, 166
- clean price, 36
- color noise, 134
- commodity derivative, 49
- commodity market, 8
- complete, 148
- compound interest, 12, 15
- conditional expectation, 155
- contingent claim, 148
- Contingent claims, 49
- contingent claims, 49
- continuous compound interest, 16
- Contract size, 95
- coupon, 31
- coupon bonds, 32

- coupon payment, 32
- coupon rate, 32
- coupon value, 32
- covered put, 170
- credit risk, 45
- creditor, 11
- CRR model, 157
- currency markets, 8

- debt
 - security, 6
- debtor, 11
- delivery date, 49
- delivery price, 58, 69
- Delta, 155
- delta, 180
- delta neutral, 181
- delta-gamma-hedging, 184
- delta-hedging, 181
- derivative, 49
 - security, 7
- derivatives
 - market, 8
 - security, 6
- deterministic model, 132
- diagonal spread, 170
- dirty price, 36
- discount factor, 21
- discount value, 21
- discounted gain, 148
- discounted market, 146
- discounted value, 147
- discounted value process, 147
- discounting, 21
- discrete compound interest, 17
- dividend, 6
- drift, 20, 132, 136
- duration, 41
- dynamic hedge, 183

- effective interest rate, 16
- efficient market hypothesis, 115, 116
- equity
 - investment, 5
 - market, 7
 - security, 6
- European option, 95
- exchange market, 7
 - Bombay stock exchange, 8
 - National stock exchange, 8
- exchange-traded markets, 9
- expected return, 119
- expiration date, 49

- face value, 32
- fair game, 118
- fair game criteria, 119
- fair swap rate, 91
- fair value, 61
- filtered probability space, 129
- filtration, 129
- Fin-Tech firms, 3
- finance market, 145
- financial asset, 6
- financial derivative, 49
- financial investment, 5
- financial market, 7
- financial system, 5
- fixed leg, 89
- Fixed-income securities, 31
- fixed-income securities, 45
- floating leg, 89
- flow of information, 144
- forward contract, 58
- forward price, 58
- forwards, 7
- frequency, 12
- frictionless valuation, 54
- future, 69

- future price, 69
- future value, 20, 23
- future value of money, 11
- futures, 7

- gain, 54
- gamma, 184
- Gaussian white noise, 133
- generalized Wiener process, 136
- generated filtration, 130
- geometric Brownian motion, 129, 132, 136
- Greeks, 180
- gross price, 36
- growth factor
 - compound interest, 15
 - simple interest, 13

- hedge, 169
- hedger, 50
- hedging, 50
- hedging strategy, 148
- holder, 49, 94
- horizontal spread, 170

- in-the-money, 96
- Independent increment property, 130
- inflation risk, 45, 46
- initial investment, 54, 146
- initial value problem, 166
- interest, 6, 11
 - compound, 15
 - rate, 12
 - simple, 13
- interest rate, 16
- interest rate risk, 47
- interest rate swap, 89
- interest rates, 12
- internal rate of return, 28
- intrinsic value, 97
- IRR equation, 28

- issue price, 32
- Itô formula, 139
- Itô integral, 139
- Itô process, 139

- junk bonds, 45

- Lévy process, 20
- law of one price, 73, 149
- leverage, 167, 168
- liability, 11
- limited loss, 169
- linear price sensitivity model, 43
- liquidating, 7
- liquidity risk, 45, 46
- logarithmic return
 - binomial model
 - symmetric lattice, 125
- lognormal model, 115, 127
- long hedge, 81
- long position, 49, 52, 94
- long strangle, 179

- Macaulay duration, 42
- maintenance margin, 70
- margin, 70
- market, 7, 51
 - arbitrage-free, 55
 - bond, 7
 - primary, 32
 - capital, 9
 - commodity, 8
 - currency, 8
 - derivatives, 8
 - equity, 8
 - exchange, 7
 - financial, 7
 - money, 8
 - over-the-counter, 9
 - primary, 8

- bond, 32
- secondary, 8
- viable, 55
- marking to market, 69, 70
- marking to market table, 71
- matching procedure, 126
- maturity time, 12, 32
- mean-reverting process, 20
- minimum-variance, 86
- minimum-variance hedge, 85
- modified duration, 43
- modified Macaulay duration, 43
- money market, 8, 117
- Monte Carlo, 136
- multi-step binomial lattice model, 119
- National stock exchange, 8
- net gain, 169
- net gain function, 171
- net present value, 26
- no-arbitrage condition, 117
- nominal interest rate, 16
- nominal value, 13, 15, 17
- noncontingent claim, 58
- noncontingent claims, 49
- normalized finance market, 146
- notional principal, 89
- numéraire, 146
- offset, 72
- options, 7, 93
 - premium, 93
- ordinary annuity, 25
- Ornstein-Uhlenbeck process, 20
- out-of-the-money, 96
- over the counter, 58
- Over-the-Counter, 9
- par bond, 35, 40
- par value, 32
- payoff, 59, 96
- per period, 12, 23
- Perfect hedging, 181
- perfect hedging, 83
- plain vanilla swap, 89
- portfolio, 52
- portfolio value, 53
- predictable, 146
- premium, 93, 94
- present value, 20, 24
- present value of money, 11
- price, 94
- price spread, 170
- price-yield, 39
- primary market, 8
 - bond, 32
- principal, 11
- proxy hedging, 88
- pullback factor, 20
- pure discount bond, 32
- put option, 94
- put options, 94
- Quantitative finance, 1
- quants, 2
- random walk, 123, 131, 134
- rate of return, 128
- ratio spread, 184
- reachable, 148
- rebalancing, 182
- replicating portfolio, 150, 154
- replicating strategy, 148
- replication, 74
- return, 58
- rho, 186
- risk-averse, 49
- risk-free investment, 5
- risk-neutral price, 57
- risk-neutral probability, 57, 119, 150

risk-seeking, 49
risky investment, 5

sample price path, 121
secondary bond market, 32
secondary market, 8, 32
security, 6
 debt, 6
 derivative, 7
 equity, 6
securityderivatives, 6
self-finance strategy, 64
self-financing strategy, 147
sell-side, 2
seller, 49, 94
sensitivity analysis, 180
settle price, 69
short hedge, 78
short position, 49, 52, 94
short straddle, 178
short strangle, 179
simple interest, 13
single-step binomial lattice model, 117
speculation, 50
speculator, 168
speculators, 50
spot price, 59, 69
spread, 170
square off, 72
stable, 159
standard Brownian motion, 130
static hedge, 183
stochastic differential equation, 136
stochastic integral, 133
stochastic integral equation, 139
stochastic model, 136
stochastic process, 128
stochastically integrable, 134
straddle, 178

strangle, 178, 179
strike price, 49, 93
swap, 89
swaps, 7

terminal value problem, 166
theta, 185
time value, 99
time value of money, 11
total return, 128
trade, 8
trading strategy, 64, 146
trinomial model, 157

underlying asset, 49, 58
underlying price, 59

value process, 146
variation margin, 70
Vasicek model, 20
vega, 186
vega-hedging, 186
vertical spread, 170
viable market, 55
volatility, 132, 136

wealth process, 146
Wiener process, 20, 130
writer, 49, 94

yield, 33
yield curve, 40
yield to maturity, 33

zero-coupon bond, 32

