6.1.2 Logarithmic Return Model

The binomial lattice model is predicated on the assumption that the stock price at the next time step is either upward (u) or downward (d) from the current stock price. Consequently, the random variable X representing the stock price at the next time level becomes discrete, with its range consisting of two points. Now, our interest is to consider a scenario where the price can vary across a continuum. Despite this, we will continue to use discrete time intervals, and the transition towards a continuous framework will be taken up in the subsequent sections.

Let S_0 represent the current value of an investment. According to the model of continuous compounding with a fixed annual interest rate r, the value at any future time t is given by

$$S(t) = S_0 e^{rt}.$$

Initially, let us view S_0 as the current price of a stock. Consequently, the model suggests that the stock price at any time t may be taken as $S(t)$ given by the above formula. Nevertheless, it is imperative to acknowledge that the stock price at a later time is subject to uncertainty due to random market movements. To incorporate this randomness into the model, we propose the form

$$S(t) = S_0 e^{R_t},$$

where R_t is a random variable. We first develop the model for one year holding period of a stock and then generalize it to any holding time period $[0, T]$.

Let the one year time period $[0, 1]$ be divided into n equal parts, and hence the partition is

$$\left\{ t_0 = 0, t_1 = \frac{1}{n}, t_2 = \frac{2}{n}, \ldots, t_{n-1} = \frac{n-1}{n}, t_n = 1 \right\}$$

We propose the model for the stock price at time $t = t_{k+1}$ as

$$S(t_{k+1}) = S(t_k) e^{R_{k+1}}, \tag{6.2}$$

where R_{k+1} is a random variable, for $k = 0, 1, \ldots, n-1$.

The stock price at $t_n = 1$ can be written as

$$S(t_n) = \frac{S(t_n)}{S(t_{n-1})} \times \frac{S(t_{n-1})}{S(t_{n-2})} \times \ldots \times \frac{S(t_1)}{S_0} S_0. \tag{6.3}$$

Using (6.2) in the above expression, we get

$$S(t_n) = S_0 \exp\left(\sum_{k=1}^{n} R_k \right).$$

Taking logarithm on both sides, we get

$$\ln\big(S(t_n)\big) = \ln(S_0) + \sum_{k=1}^{n} R_k,$$

which can be written as

$$\ln\left(\frac{S(t_n)}{S_0}\right) = \sum_{k=1}^{n} R_k. \tag{6.4}$$

The efficient market hypothesis (EMH) suggests that successive price changes are independent, implying that we can assume R_k's are mutually independent. Additionally, we assume that R_k's are identically distributed. By combining these assumptions, we treat $\{R_k \mid k = 1, \ldots, n\}$ as a collection of independent and identically distributed (iid) random variables. In this context, the sequence of random variables $\left\{\ln\left(\frac{S(t_k)}{S_0}\right) \mid k = 0, 1, 2, \ldots, \right\}$ is called a *random walk*.

Let R_k's have a finite mean $\tilde{\mu}$ and variance $\tilde{\sigma}^2$. Further, let

$$\left.\begin{aligned} E\left(\ln\left(\tfrac{S(t_n)}{S_0}\right)\right) &= \mu, \\ \mathrm{Var}\left(\ln\left(\tfrac{S(t_n)}{S_0}\right)\right) &= \sigma^2. \end{aligned}\right\} \tag{6.5}$$

On the other hand, taking expectation and variance in (6.4), and noting that R_k's are mutually independent, we have

$$\left.\begin{aligned} E\left(\ln\left(\tfrac{S(t_n)}{S_0}\right)\right) &= n\tilde{\mu}, \\ \mathrm{Var}\left(\ln\left(\tfrac{S(t_n)}{S_0}\right)\right) &= n\tilde{\sigma}^2. \end{aligned}\right\} \tag{6.6}$$

General case of $t = T$

Recall that the above derivation is restricted to one-year period. Let us now extend the study to any stock holding period $[0, T]$. For simplicity, assume that $N = nT$ is an integer, where n is the number of partitions of the unit interval $[0, 1]$. Write the stock price at time $t = T$ as in (6.3) with $t_k = kT/N$, for $k = 0, 1, \ldots, N$, and proceed as in the previous case to obtain

$$\ln\left(\frac{S(T)}{S_0}\right) = \sum_{k=1}^{N} R_k.$$

Thus, we have

$$\left.\begin{aligned} E\left(\ln\left(\frac{S(T)}{S_0}\right)\right) &= N\tilde{\mu}, \\ \mathrm{Var}\left(\ln\left(\frac{S(T)}{S_0}\right)\right) &= N\tilde{\sigma}^2. \end{aligned}\right\} \tag{6.7}$$

Binomial Lattice Model: Revisited

In the BLM, we have illustrated the procedure of obtaining the probability p of success (u) using the fair game criteria. In this section, we illustrate a method which uses a matching argument with logarithmic return to obtain u, d, and p for a given annual mean μ and variance σ of the stock.

For a given S_0, μ, and σ, we have

$$E\left(\ln\left(\frac{S(\Delta t)}{S_0}\right)\right) = p\ln u + (1-p)\ln d, \quad \mathrm{Var}\left(\ln\left(\frac{S(\Delta t)}{S_0}\right)\right) = p(1-p)(\ln u - \ln d)^2.$$

Substituting these expressions into (6.7) for one period $[0, \Delta t]$, we get the matching relation

$$\left.\begin{aligned} p\ln u + (1-p)\ln d = \mu\Delta t, \\ p(1-p)(\ln u - \ln d)^2 = \sigma^2\Delta t. \end{aligned}\right\} \tag{6.8}$$

We have obtained two equation, but we have three unknowns u, d, and p. Therefore, we need one more equation to get a closed system or fix one parameter value and solve the equations for the other two. There are at least two ways that we can chose one of the three parameters:

1. **Logarithmic return binomial model with symmetric lattice:** We may simply choose $d = 1/u$. Using this symmetric lattice condition, the equations (6.8) reduce to

$$\begin{aligned} (2p-1)\ln u &= \mu\Delta t, \\ 4p(1-p)(\ln u)^2 &= \sigma^2\Delta t. \end{aligned}$$

Squaring the first equation and adding it to the second equation, we get

$$\ln u = \sqrt{(\mu\Delta t)^2 + \sigma^2\Delta t} \tag{6.9}$$

Substituting (6.9) in the first equation, we get

$$p = \frac{1}{2} + \frac{1/2}{\sqrt{\sigma^2/(\mu^2\Delta t) + 1}}.$$

Thus, we have

$$\left.\begin{aligned} u = \exp\left(\sqrt{(\mu\Delta t)^2 + \sigma^2\Delta t}\right), \\ d = \exp\left(-\sqrt{(\mu\Delta t)^2 + \sigma^2\Delta t}\right), \\ p = \tfrac{1}{2} + \tfrac{1/2}{\sqrt{\sigma^2/(\mu^2\Delta t)+1}}. \end{aligned}\right\} \tag{6.10}$$

Assuming that Δt is very small we adopt an approximation to the above three expressions by neglecting the terms with Δt and retain only the terms with $\sqrt{\Delta t}$. Thus, we get

$$\left.\begin{aligned} p &\approx \frac{1}{2} + \frac{1}{2}\left(\frac{\mu}{\sigma}\right)\sqrt{\Delta t}, \\ u &\approx e^{\sigma\sqrt{\Delta t}}, \\ d &\approx e^{-\sigma\sqrt{\Delta t}}. \end{aligned}\right\} \tag{6.11}$$

Note that we can use the formulae (6.11) only when Δt is very small. For instance, we can use these approximate formulae when $\Delta t = 1/365$ (one-day stock analysis) or $\Delta t = 1/52$ (one-week stock analysis, where it is customary to consider 52 weeks per year). For a given set of parameters μ and σ, one can use the *logarithmic return binomial model with symmetric lattice* to obtain an idea of how the stock price dynamics will be in the near future. Generally, μ and σ are chosen empirically or using a moving average formula.

2. **Risk-neutral probability:** Another commonly followed idea is to fix a value for p and solve for u and d. An alternate idea is to take p as the risk-neutral probability for a given prevailing interest rate.

Example 6.1.4.

Consider a stock with the parameters $S_0 = 100$, $\mu = 15\%$, and $\sigma = 30\%$. We wish to construct the lattice diagram based on the above formula for the stock for a period of one month, with step length being one week.

First, let us find u, d, and p. We are given $\mu = 0.15$, $\sigma = 0.3$, $T = 1/12$ and $\Delta t = 1/52$ (assuming there are 52 weeks per year).

Using the formulae given in (6.11), we get

$$u \approx 1.0425, \quad d \approx 0.9593, \quad p \approx 0.5347.$$

The lattice diagram is given in Figure 6.3.

Problem 6.4.

Find the risk-neutral probability in the above example if the prevailing interest rate is 6% per annum. Ans: 0.5042

We refer to the procedure discussed above as *matching procedure*. The matching procedure gives different expressions for u, d, and p depending on how we choose the third condition. For instance, in the above discussion, we chose $d = 1/u$. However, one can have other choices.

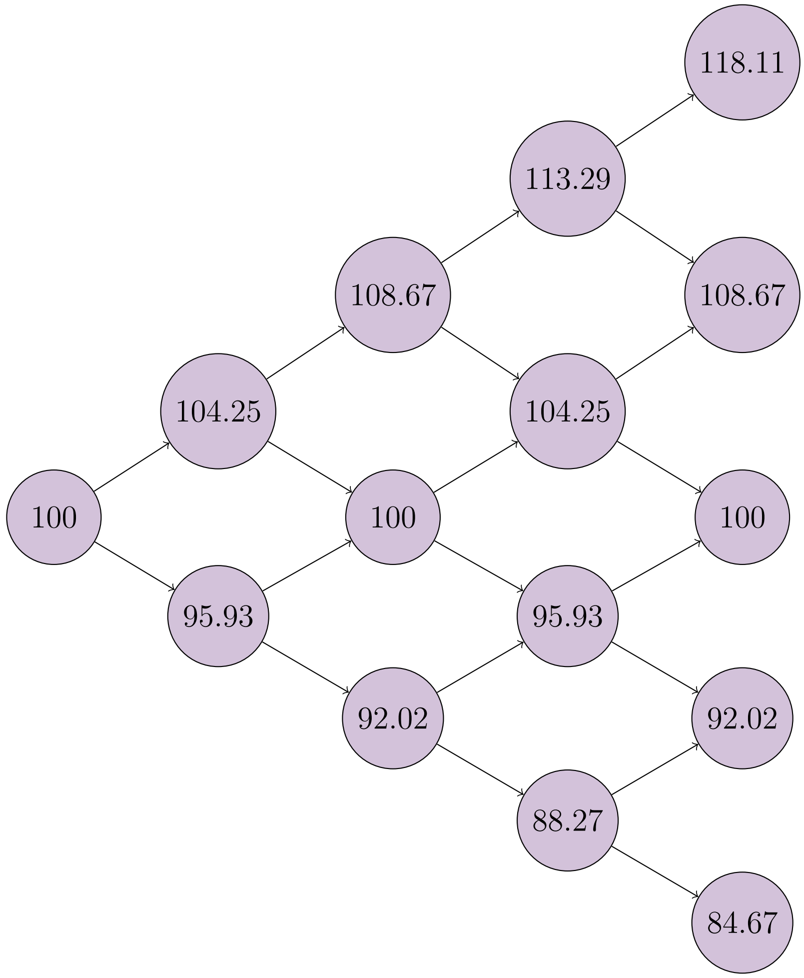

Figure 6.3: Lattice diagram for Example 6.1.4.

Problem 6.5.

In the matching procedure of the binomial lattice model with the logarithmic return binomial model, if we take $p = 1/2$ (instead of $d = 1/u$), then find the corresponding expressions for u and d.

Problem 6.6.

In the binomial lattice model's matching procedure with logarithmic return, if we take the probability of success $p = 2/3$, then find the corresponding expressions for u and d. Further, if the stock price is ₹100 per share, $\mu = 10\%$, and $\sigma = 15\%$, then construct the corresponding lattice diagram for a period of two weeks, with step length being one week. Consider 52 weeks in a year.