CHAPTER 6

# Stochastic Models

In our study of bonds and derivatives, we have introduced pricing models and some of their important properties, primarily assuming fixed interest rates. While studying derivative prices, we also assumed the underlying asset prices to be known, providing a basis for present-time valuation. However, when aiming to predict security prices at some future time, phenomena such as market fluctuations, unpredictable events, and ever-changing economic situations introduce randomness. To address inherent randomness and gain a deeper understanding of financial markets, stochastic models emerge as indispensable tools.

Stochastic models provide a framework to analyze and predict financial phenomena by incorporating randomness into their mathematical formulations. Within these models, financial variables such as asset prices, interest rates, and market volatilities are subject to random fluctuations influenced by various factors such as investor behavior, economic indicators, and geopolitical events.

Stochastic models can be categorized into discrete-time and continuous-time frameworks. At the discrete-time level, the *binomial lattice model* stands as a foundational pricing model, assuming the *efficient market hypothesis* and constructing a random walk. In the limit as the time step tends to zero, this model converges to the *lognormal model* in the continuous-time framework to predict a risky asset price underlying the *Black-Scholes model* for options pricing.

In this chapter, we discuss some basics of stochastic models in finance. Given the role of options as hedging tools to reduce risk in risky asset portfolios, the models discussed in this chapter play a vital role in options pricing models discussed in the next chapter.

In Section 6.1, we start with discrete time models, where we discuss binomial and trinomial models. Section 6.2 focuses on continuous-time models, particularly geometric Brownian motion, and its numerical implementation through Monte Carlo simulation, with a specific emphasis on stock price modeling. Finally, in Section 6.3, we introduce Ito calculus, extending our understanding to more general stochastic models and their applications across various financial contexts.

## 6.1 Discrete Time Models

Let $t = 0$ represent the present time at which the underlying risky asset is purchased. In the discrete time setup, we consider a partition $\{t_0 = 0, t_1, \ldots, \}$ of the time interval $[0, \infty)$.

Our focus in this section is on developing the *binomial lattice model*. This model is developed under the assumption of the well-known *efficient market hypothesis*, which may be stated as

*The current price of a risky asset always reflects all available information and will change only with the arrival of new information.*

Since future information cannot be predicted in the present, the asset price at a future time remains unpredictable. Consequently, movements in the asset price from the current market value, either upward or downward, are analogous to predicting heads or tails when tossing a possibly biased coin. At each time step, the model also requires a probability of the asset price moving up and hence the probability of moving down.

In what follows, we use the word 'stock' to indicate a risky asset. However, the models can be adapted to any risky asset like currencies and commodities.

### 6.1.1 Binomial Lattice Model

The *binomial lattice model*, often referred to as the *binomial model*, assumes the stock price at the present time to be known and estimates its potential values at some future time $t = T$ under various given scenarios. It achieves this by discretizing time into intervals and simulating possible stock price movements at these discrete times until $t = T$.

We use the notation $S_k$, where $k = 0, 1, 2, \ldots$, to represent the price of a stock at time $t = t_k$. Let us first develop the model for a single time period $[t_k, t_{k+1}]$, with $k$ fixed. We then extend it to multiple time steps.

### Single-time Step

Let one share of a stock be bought at time $t = t_k$ for the price of $S_k$ and held until time $t = t_{k+1}$. Thus, $t_k$ is the present time, and the price $S_k$ is known, whereas $t_{k+1}$ is a future time, and the price of the stock $S_{k+1}$ is unknown and uncertain. Since $S_{k+1}$ is uncertain, there are at least two possible outcomes. Binomial model considers only two possibilities at every time level.

For a given pair of numbers $d_k$ and $u_k$ such that $0 < d_k < 1 < u_k$, consider two possibilities for $S_{k+1}$:

1. either $S_{k+1} = S_k u_k$ (upward movement)
2. or $S_{k+1} = S_k d_k$ (downward movement).

Define a probability space $(\mathsf{S}_k, \mathscr{S}_*, \mathbb{P}_k)$, where $\mathsf{S}_k = \{d_k, u_k\}$ along with the power set $\mathscr{S}_*$ of $\mathsf{S}_k$ forms a sample space $(\mathsf{S}_k, \mathscr{S}_*)$ and the discrete probability $\mathbb{P}_k$ is given by

$$\mathbb{P}_k(\{s\}) = \begin{cases} p_k, & \text{if } s = u_k, \\ 1 - p_k, & \text{if } s = d_k, \end{cases}$$

for some $0 < p_k < 1$. Consider the random variable

$$X_{k+1}(s) = S_k s, \quad s \in \mathsf{S}_k.$$

The broad idea of the *single-step binomial lattice model* is the following:

Let $t = t_k$ be the present time and let $S_k$ be given. Define $S_{k+1} = X_{k+1}$ with appropriate values of $d_k$, $u_k$, and $p_k$. In other words, the model assumes

$$S_{k+1} := \begin{cases} S_k u_k, & \text{with probability } p_k, \\ S_k d_k, & \text{with probability } 1 - p_k. \end{cases}$$

**Remark 6.1.1.**

It is appropriate to choose $d_k$ and $u_k$ such that

$$S_k d_k \le B_{k+1} \le S_k u_k,$$

where $B$ is the amount obtained from a risk-free investment for $S_k$. More precisely, if $r_k$ is the per-period interest rate offered in the *money market* (a market where risk-free assets, like bonds, are issued/traded) for the period $[t_k, t_{k+1}]$, then $d_k$ and $u_k$ are to be chosen such that

$$0 < d_k \le I(r_k) \le u_k, \tag{6.1}$$

where

$$I(r_k) = \begin{cases} e^{r_k}, & \text{for continuously compounded interest} \\ (1 + r_k), & \text{otherwise} \end{cases}$$

The condition (6.1) is called the *no-arbitrage condition*.

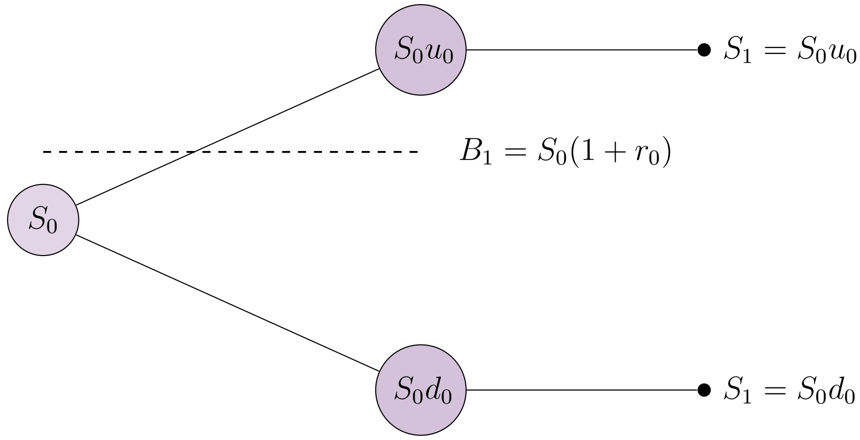


Figure 6.1: An illustration of 1-step binomial tree for $k = 0$ and $d_0 < I(r_0) < u_0$ with a discrete interest scheme.

**Problem 6.1.**

Show that any violation of the inequality (6.1) leads to an arbitrage opportunity for any given $d_k$ and $u_k$ such that $0 < d_k < u_k$.

Figure 6.1 illustrates one-time step of the binomial lattice model (with $k = 0$).

In the following example, we adopt the *fair game* criteria to obtain the probability $p$ for a given parameters $r$, $u$, and $d$ (we drop the suffix $k$ for the sake of convenience).

**Example 6.1.2.**

Let the prevailing annual interest rate be 6% and the stock at time $t = 0$ be $S_0 =$₹750. If the stock price in one year can be one of the two values ₹1260 or ₹675, then the aim is to find the probability that it takes the value 1260.

Consider the continuously compounded interest scheme. The present value of the two possible values are given by

$$PV(1260) = 1260e^{-0.06} \approx 1186.62 \quad \text{and} \quad PV(675) = 675e^{-0.06} \approx 635.69.$$

We use the *fair game* criteria, which can be stated as

*the expected present value should be equal to the current stock price.*

That is,

$$1186.62p + 635.69(1-p) = 750.$$

This gives $p \approx 0.2075$.

**Remark 6.1.3 [Risk-neutral Probability].**

The probability obtained in the above example is called the *risk-neutral probability*. Let us now derive the formula for this probability (also see Example 4.1.9).

Let the annual interest rate be $r$ with continuously compounding scheme, and let $u$ and $d$ be such that $0 < d < e^{rT} < u$ (no-arbitrage condition for continuous compounding). Let $S_1$ be the price of the stock at time $t = T$, which is given as per the 1-step binomial lattice model. Then, the expectation of $S_1$ is given by

$$E(S_1) = S_0 up + S_0 d(1 - p).$$

The probability $p$ which makes the initial stock price $S_0$ equal to the present value of $E(S_1)$ (*fair game criteria* or *expected return*) under the continuous compounding scheme with annual interest rate $r$ is called the ***risk-neutral probability*** with respect to the given interest rate and is given by the

$$p = \frac{e^{rT} - d}{u - d}.$$

Note that the no-arbitrage condition makes $p$ to lie between 0 and 1.

**Problem 6.2.**

If the annual interest rate $r$ is taken with a quarterly compounding scheme, then show that the risk-neutral probability is given by

$$p = \frac{\left(1 + \frac{r}{4}\right)^{4T} - d}{u - d}.$$

### Multi-time Step

One can understand the multi-step binomial model by considering the analogy of repeatedly tossing several coins or tossing a single coin multiple times.

Starting from $t_0 = 0$, one can apply the one-step binomial lattice model sequentially to progress to any time $t_{k+1}$, where $k = 0, 1, 2, \ldots$. This process yields the *multi-step binomial lattice model*, more precisely termed the $(k+1)$-step binomial lattice model.

Assume that a stock is purchased at the present time $t = 0$ for an amount of $S_0$ per share. Also, assume that the stockholder has to sell the stock at some time $t = T > 0$.

To construct an $n$-step binomial lattice model, first consider the uniform partition

$$\left\{ t_0 = 0, t_1 = \frac{T}{n}, t_2 = \frac{2T}{n}, \ldots, t_{n-1} = \frac{(n-1)T}{n}, t_n = T \right\}$$

of the holding period $[0, T]$.

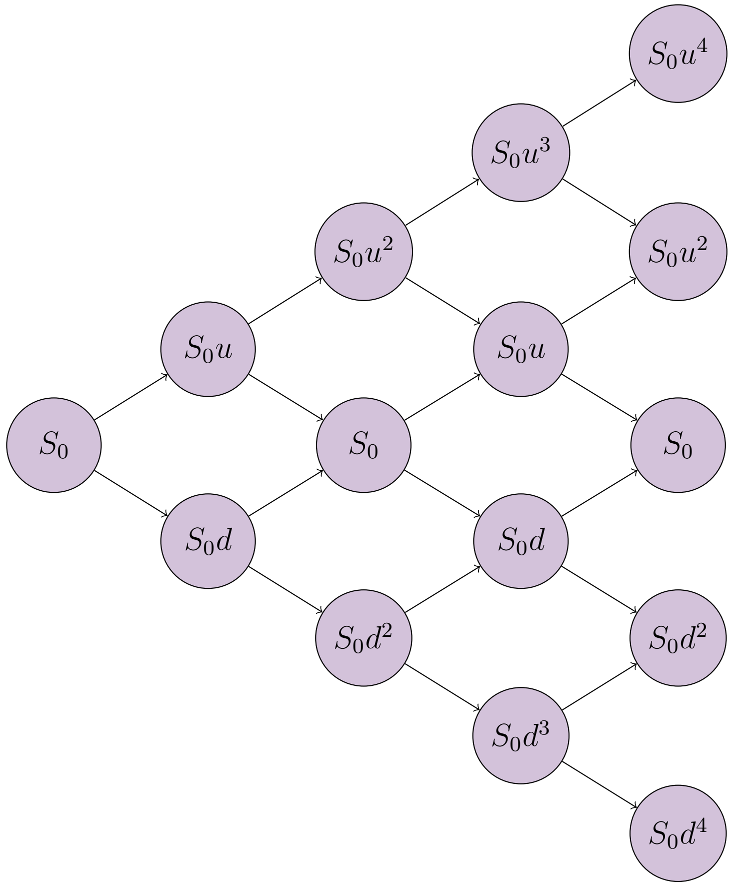


Figure 6.2: Lattice diagram for a partition $\{t_0, t_1, t_2, t_3, t_4\}$ and $d = 1/u$.

By virtue of the efficient market hypothesis, we assume that the outcome at any time step $t = t_k$ is independent of the outcomes at all the previous time steps. Consequently, we can consider the product space as the probability space $(\mathsf{S}, \mathscr{S}_*, \mathbb{P})$ given by

$$\mathsf{S} = \mathsf{S}_0 \times \mathsf{S}_1 \times \ldots \times \mathsf{S}_{n-1}, \quad \mathbb{P}(\{\boldsymbol{s}\}) = \prod_{k=0}^{n-1} \mathbb{P}_k(\{s_k\}),$$

for $\boldsymbol{s} = (s_0, s_1, \ldots, s_{n-1}) \in \mathsf{S}$.

The ($n$-step) *binomial lattice model* defines the stock price at $t = T$ as $S_n = X$, where $X$ is the random variable defined as

$$X(\boldsymbol{s}) = S_0 \prod_{j=0}^{n-1} s_j, \quad \boldsymbol{s} = (s_0, s_1, \ldots, s_{n-1}) \in \mathsf{S}.$$

Note that each $s_k$ is either $u_k$ or $d_k$, $k = 0, 1, \ldots, n-1$.

In particular, if we assume $u_0 = u_1 = \ldots = u_{n-1} = u$ and $d_0 = d_1 = \ldots = d_{n-1} = d$, then for each $\boldsymbol{s} \in \mathsf{S}$, there exists an integer $0 \leq j \leq n$ such that

$$S_n(\boldsymbol{s}) = S_0 u^j d^{n-j}.$$

Furthermore, assuming an equal probability $p$ of obtaining $u$ at every time step, we get

$$\mathbb{P}(\{S_n = S_0 u^j d^{n-j}\}) = \binom{n}{j} p^j (1-p)^{n-j}.$$

Under the fair game criteria, we can show (left as an exercise) that $p$ is given by

$$p = \frac{e^{\frac{rT}{n}} - d}{u - d},$$

leading to the risk-neutral probability.

The above discussion shows that in the binomial lattice model, with a specific choice of parameters $u_i = u$ and $d_i = d$ for $i = 0, 1, 2, \ldots, n-1$, the stock price $S_n$ at a future time $t = T$ follows a binomial distribution.

Each element $\boldsymbol{s} \in \mathsf{S}$ can be interpreted geometrically as a path, referred to as a *sample price path*, leading to a possible stock price $S_n(\boldsymbol{s})$. The lattice diagram of a 4-step binomial lattice model is illustrated in Figure 6.2 in a specific case where $d = 1/u$.

**Problem 6.3.**

Consider a stock with a 6-month holding period, currently priced at ₹100. Using a 30-step binomial lattice model (also referred simply as binomial model) with parameters

$$u = 1.12, \quad d = 0.975, \quad p = 0.5156,$$

determine the following:

1. Find the stock price at the end of 6 months if the sample price path involves 7 upward movements.
2. Find $\mathbb{P}(\{S_n \leq S_0 u^7 d^{23}\})$.

**Answer:** (i) 123.49 (ii) 0.0015294354

Different ideas can be adapted to arrive at a feasible values for $d_k$, $u_k$, and $p_k$. One approach is to utilize an approximation to the lognormal model discussed in the following subsection. Note that the risk-neutral probability is one particular choice for $p_k$ within a given pair of values for $d_k$ and $u_k$.