CHAPTER 7

Option Pricing Models

We continue our discussion on options. The option premium depends on the dynamics of the price of the underlying asset, and is well described by continuous time stochastic models. However, discrete time models are fundamental and plays a vital role in constructing continuous models. We introduce the binomial model at the discrete time level and the celebrated Black-Scholes model at the continuous time level.

Binomial option pricing model is discussed in Section 7.1 and the Black-Scholes model is discussed in Section 7.2.

7.1 Discrete time pricing model

The ultimate aim of options theory is to develop models that govern a fair price (premium) for a given option. In this section, we study discrete-time pricing models, more precisely, the Binomial pricing model.

We partition the option period $[0, T]$ into n equally spaced subintervals, where the partition is denoted by $\mathbb{T}_n := \{t_0, t_1, \dots, t_n\}$, with $t_k = kh$, $k = 0, 1, \dots, n$, and $h = T/n$. Discrete time models assume that the transactions occur only at times $t \in \mathbb{T}_n$ and hence are developed to work on a given time partition instead of the whole interval $[0, T]$.

The stock price at each time $t = t_k$, for $k = 0, 1, \dots, n$, is denoted by $S(t_k)$ or S_k (for notational convenience).

At time $t = t_0$ $(= 0)$, we know the value of S_0. From here, we have two possibilities for S_1, namely,

1. $S_1 \geq S_0$, which we denote by U and
2. $S_1 < S_0$, which we denote by D.

Let us denote the outcome at $t = t_1$ as s_1, which is either U or D.

At time $t = t_1$, we know exactly the value of S_1 and therefore, we know the outcome s_1. Again, at t_1 we have two possibilities for S_2 as above and the outcome is denoted by s_2, which is either U or D.

In general, at time $t = t_k$, for any $k \in \{0, 1, 2, \ldots, n\}$, we know all the outcomes s_j, for $j = 1, 2, \ldots, k$ and we know exactly the value of S_k. In other words, we have the information about the stock price till time $t = t_k$. However, we do not know the value of S_{k+1} at time t_k. For S_{k+1} we again consider two possibilities U and D, and the outcome s_{k+1} is one of these two possibilities.

The process of *flow of information* continues till time $t = t_n$ $(= T)$, which are precisely the stock prices S_j, $j = 1, 2, \ldots, n$ (equivalently, s_j's). Finally, at time $t = T$, we know $S_T (= S_n)$ and all the outcomes s_k, $k = 1, 2, \ldots, n$. Let us denote the outcomes by the n-tuple $\boldsymbol{s} = (s_1, s_2, \ldots, s_n)$ and the set of all possible outcomes by S.

For any given $k \in \{1, 2, \ldots, n\}$ we use the notation

$$\mathsf{S}'_k = \mathsf{S}_1 \times \ldots \times \mathsf{S}_k \text{ and } \mathsf{S}''_k = \mathsf{S}_{k+1} \times \ldots \times \mathsf{S}_n.$$

With these notations, we can write $\mathsf{S} = \mathsf{S}'_k \times \mathsf{S}''_k$, for each $k = 1, 2, \ldots, n$, and therefore any $\boldsymbol{s} \in \mathsf{S}$ can be written $\boldsymbol{s} = (\boldsymbol{s}'_k, \boldsymbol{s}''_k)$ with $\boldsymbol{s}'_k \in \mathsf{S}'_k$ and $\boldsymbol{s}''_k \in \mathsf{S}''_k$.

For any given $\boldsymbol{s}'_k \in \mathsf{S}'_k$,

$$B_{\boldsymbol{s}'_k} = \left\{ (\boldsymbol{s}'_k, \boldsymbol{s}''_k) \in \mathsf{S} \mid \boldsymbol{s}''_k \in \mathsf{S}''_k \right\}.$$

We can see that, for any given $k \in \{1, 2, \ldots, n\}$,

$$\mathtt{P}_k = \{ B_{\boldsymbol{s}'_k} \mid \boldsymbol{s}'_k \in \mathsf{S}'_k \}$$

forms a partition of S. Also, we can see that $\mathtt{P}_{k+1}$ is finer than $\mathtt{P}_k$, for $k = 1, 2, \ldots, n-1$.

We use the convention that $\mathtt{P}_0 = \{\mathsf{S}\}$.

It is convenient to work with the generated σ-field of the partitions instead of directly working with the partitions.

We now construct the filtration from the sequence of partitions constructed above. First recall a simple exercise problem from basic probability course:

Problem 7.1.

Show that the generated σ-algebra of a partition $\mathtt{P}$ of a set S is the collection of all finite unions of the sets in $\mathtt{P}$ along with the empty set $\emptyset$.

The following lemma is a direct consequence of the above problem.

Lemma 7.1.1.

Consider a partition $\mathbb{T}_n := \{t_0, t_1, \ldots, t_n\}$ of $[0, T]$. For a given $k \in \{0, 1, 2, \ldots, n\}$, let $\mathcal{F}_k$ be the σ-algebra generated by P_k. Then the family $\{\mathcal{F}_k \mid k = 0, 1, \ldots, n\}$ is a filtration.

Remark 7.1.2 [Probability space].

Note that $\mathcal{F}_0 = \{\emptyset, \mathsf{S}\}$ and $\mathcal{F}_n$ is the power set of S, which we denote by $\mathcal{F}^*$. We consider a probability space $(\mathsf{S}, \mathcal{F}^*, \mathbb{P})$, where the discrete probability measure is suitably defined such that $\mathbb{P}(\{\boldsymbol{s}\}) > 0$, for all $\boldsymbol{s} \in \mathsf{S}$.

Note

Generally, we used the notation s for the number of units in a risky asset held in a portfolio. However, s is now reserved for a typical element of a sample space. So, we use the notation θ to denote the number of units in a risky asset held in a portfolio. Similarly, we use the notation ϕ to denote the component of a risk-free asset (or simply a savings bank account) in a portfolio.

In the derivation of the discrete time model for an option price, we always assume that there is only one risk-free asset in the portfolio. Further, we always consider the component of the derivative in the portfolio as zero. Thus, a typical portfolio that we consider is given by

$$\Pi = (\phi, \boldsymbol{\theta}, 0),$$

where ϕ is a scalar representing the number of units held in a risk-free asset and $\boldsymbol{\theta} = (\theta_1, \theta_2, \ldots, \theta_m) \in \mathbb{R}^m$ with θ_j, for $j = 1, 2, \ldots, m$, representing the number of units held in the j^{th} risky asset in the portfolio.

For the sake of convenience, we drop the derivative component in the portfolio and write

$$\Pi = (\phi, \boldsymbol{\theta}).$$

The corresponding tuple of assets $(B, \boldsymbol{S})$, where B refers to the risk-free asset and $\boldsymbol{S} = (S^1, S^2, \ldots, S^m)$ is the tuple of risky assets, is referred to as a *finance market*.

Note

Note that the value of ϕ and the components of $\boldsymbol{\theta}$ can be positive or negative depending on the positions (long or short, respectively) held in the corresponding assets.

Definition 7.1.3 [Trading Strategy].

The stochastic process of portfolios

$$\{\Pi_k \mid k = 1, \dots, n\},$$

where Π_k corresponds to the period $[t_{k-1}, t_k]$, is called a ***trading strategy***, if each component of Π_k is $\mathcal{F}_{k-1}$-measurable.

For $k = 0, 1, \dots, n$, let $\boldsymbol{S}_k = (S_k^1, S_k^2, \dots, S_k^m)$ be the price vector at time $t = t_k$, where S_k^j, $j = 1, 2, \dots, m$, denotes the price of the j^{th} risky asset at time $t = t_k$. We also denote the risk-free asset price at $t = t_k$ as B_k, $k = 0, 1, \dots, n$. The stochastic process $\{V_k \mid k = 1, 2, \dots, n\}$, where

$$V_k = \phi_k B_k + \boldsymbol{\theta}_k \cdot \boldsymbol{S}_k =: V(\Pi_k)(t_k)$$

being the value of the portfolio Π_k, is called the *value process* or the *wealth process* of the trading strategy. The quantity $V_0 = V(\Pi_1)(t_0)$ is called the *initial investment*.

Remark 7.1.4 [Interpretation of Trading Strategy].

The trading strategy says that an investor has to construct the portfolio Π_k, $k = 1, 2, \dots, n$, at time $t = t_{k-1}$ based on the information $\boldsymbol{S}_{k-1}$ available at t_{k-1}, and this portfolio has to be held till time $t = t_k$. The value V_k of the portfolio Π_k is evaluated at time $t = t_k$ with respect to the information $\boldsymbol{S}_k$. Then the investor make suitable changes in the existing portfolio in order to get the new portfolio $\Pi_{k+1} = (\phi_{k+1}, \boldsymbol{\theta}_{k+1})$, which starts at time t_k and goes till t_{k+1}. The process $\{\boldsymbol{\theta}_k\}$ is called a *predictable* process.

Remark 7.1.5 [Discounted market].

Generally, it is assumed that $B_k > 0$, for $k = 0, 1, \dots, n$ and often one considers the normalized price of the risky assets given by

$$\widetilde{\boldsymbol{S}}_k = \frac{\boldsymbol{S}_k}{B_k}.$$

In the normalized form, the finance market is called the *normalized finance market* or *discounted market*, denoted by $(1, \widetilde{\boldsymbol{S}})$. The process $\{B_k \mid k = 0, 1, \dots, n\}$ is called the *numéraire*.

In a discounted market, the value of the portfolio Π_k at time $t = t_k$, $k = 1, 2, \dots, n$, is given by

$$\widetilde{V}_k = \phi_k + \boldsymbol{\theta}_k \cdot \widetilde{\boldsymbol{S}}_k$$

and is called the *discounted value*. The corresponding value process $\{\widetilde{V}_k\}$ is called the *discounted value process*.

Definition 7.1.6 [Self-Financing Strategy].

A trading strategy $\{\Pi_k \mid k = 1, \ldots, n\}$ is called a ***self-financing strategy*** if

$$V(\Pi_k)(t_k) = V(\Pi_{k+1})(t_k), \tag{7.1}$$

for every $k = 1, 2, \ldots, n-1$.

Note
In what follows, we always consider self-financing strategies.

Remark 7.1.7.

The self-finance condition (7.1) can be re-written as

$$\Delta\phi_k B_k + \Delta\boldsymbol{\theta}_k \cdot \boldsymbol{S}_k = 0,$$

where

$$\Delta\phi_k = (\phi_{k+1} - \phi_k) \text{ and } \Delta\boldsymbol{\theta}_k = (\boldsymbol{\theta}_{k+1} - \boldsymbol{\theta}_k),$$

for $k = 1, 2, \ldots, n-1$.

With respect to the discounted finance market, the self-finance condition (7.1) can be re-written as

$$\Delta\phi_k + \Delta\boldsymbol{\theta}_k \cdot \widetilde{\boldsymbol{S}}_k = 0,$$

for $k = 1, 2, \ldots, n-1$.

Problem 7.2.

Show that a trading strategy $\{(\phi_k, \boldsymbol{\theta_k}) \mid k = 1, 2, \ldots, n\}$ is self-financing if and only if

$$\phi_{k+1} = \widetilde{V}_0 - \sum_{j=0}^{k} \Delta\boldsymbol{\theta}_j \cdot \widetilde{\boldsymbol{S}}_j, \quad k = 0, 1, \ldots, n-1,$$

where $\boldsymbol{\theta}_0 = \boldsymbol{0}$.

Problem 7.3.

Show that a trading strategy $\{(\phi_k, \boldsymbol{\theta_k}) \mid k = 1, 2, \ldots, n\}$ is self-financing if and only if, for every $k = 1, 2, \ldots, n$,

$$\widetilde{V}_k = \widetilde{V}_0 + \widetilde{G}_k,$$

where

$$\widetilde{G}_k = \sum_{j=1}^{k} \boldsymbol{\theta}_j \cdot \Delta \widetilde{\boldsymbol{S}}_{j-1}$$

is called the *discounted gain* for the predictable process $\{\boldsymbol{\theta}_k\}$.

Problem 7.4.

Consider a trading strategy $(\phi_k, \boldsymbol{\theta}_k)$, $k = 1, 2, \ldots, n$. If the process $\{\boldsymbol{\theta}_k\}$ is a predictable process with a fixed initial investment V_0, then show that there exists a unique predictable process $\{\phi_k\}$ such that the trading strategy $(\phi_k, \boldsymbol{\theta}_k)$ is self-financing.

The task is to develop a model to obtain a price of an option at time $t = 0$, which is fair to both writer and holder and is not an easy task. This is because, at $t = 0$, we do not have any information about the price movement of the underlying asset. We derive the models using a concept called *replicating strategy* (also called *hedging strategy*) and the no-arbitrage assumption on the market.

Definition 7.1.8 [Replicating Strategy].

An option is said to be *attainable* (or *reachable*) if there exists a self-financing trading strategy whose value process satisfies

$$V_n = H_T, \tag{7.2}$$

where H_T is the payoff of the option, and n is the number of partitions of the option period.

A trading strategy that makes an option attainable is called a ***replicating strategy*** (or a *hedging strategy*). If every option is attainable in a market, then the market is said to be *complete*.

Remark 7.1.9.

Observe that H_T is a $\mathcal{F}^*$-measurable random variable. It is often referred to as *contingent claim*.

We first get an idea of the option pricing model using the single step binomial model.

7.1.1 Single step binomial model

Our first pricing model is the binomial model with a single period. We therefore take $n = 1$. We will extend the method to multi-step in the next subsection.

The method consists of two steps:

1. *Constructing a replicating strategy*: Construct a replicating strategy using the condition (7.2) and
2. *Matching the initial investment*: The amount spent in setting up the replicating portfolio at $t = 0$ should be taken as the option premium.

Note

The second step mentioned above needs a justification, and it comes from the no-arbitrage principle, the *law of one price* (see Theorem 7.1.11).

European Option

We first develop the single step binomial pricing model for European call options. The partition is $\mathbb{T}_1 = \{t_0 = 0, t_1 = T\}$ and the underlying stock price at $t = 0$ is S_0.

Recall that in the single step binomial model, we take the stock price at time $t = T$ as

$$S_T = \begin{cases} S_0 u, & \text{if } s_1 = U \\ S_0 d, & \text{if } s_1 = D, \end{cases} \tag{7.3}$$

where

$$0 < d < (1 + r) < u. \tag{7.4}$$

Here, d is the discount factor if the stock goes downwards (*i.e.* $s_1 = D$), u is the amplification factor if the stock goes upwards (*i.e.* $s_1 = U$), and r is the per-period interest rate.

Problem 7.5.

Show that the condition (7.4) is a no-arbitrage condition.

Let us now construct the single step binomial model.

Step 1: [*Constructing a replicating portfolio*]

The portfolio $\Pi_1 = (\phi_1, \theta_1)$ needs to be constructed at $t = 0$. Assume that the corresponding prices are denoted by (B_0, S_0). We have two possibilities for S_1 at this time level. Using these possibilities in the replicating strategy condition (7.2), we get the linear system

$$\left.\begin{aligned}\phi_1 B_0(1+r) + \theta_1 S_0 d = H_T^d,\\ \phi_1 B_0(1+r) + \theta_1 S_0 u = H_T^u,\end{aligned}\right\} \tag{7.5}$$

where H_T^d and H_T^u are the values of the payoff function with respect to $S_1 = S_0 d$ and $S_1 = S_0 u$, respectively. We can obtain Π_1 by solving the linear system (7.5). Note that the no-arbitrage condition (7.4) guarantees a unique solution of (7.5) and hence, for a given d, u, and r, we have a unique replicating strategy for a given option at $t = 0$.

Solving the system (7.5), we get

$$\phi_1 = \frac{uH_T^d - dH_T^u}{B_0(1+r)(u-d)}, \quad \theta_1 = \frac{H_T^u - H_T^d}{S_0(u-d)}. \tag{7.6}$$

For a given set of parameters $\{r, u, d, B_0, S_0\}$, the portfolio Π_1 constructed using the formulae (7.6) at a given time $t = t_0$ is called a *replicating portfolio* of the option with payoff H_T.

Step 2: [*Matching the initial investment*]

The initial investment is given by

$$V_0 = \phi_1 B_0 + \theta_1 S_0.$$

Using (7.6), we get

$$V_0 = \frac{H_T^u}{(1+r)} p^* + \frac{H_T^d}{(1+r)}(1-p^*),$$

where

$$p^* = \frac{1+r-d}{(u-d)} \tag{7.7}$$

is called the *risk-neutral probability*. It is natural to use the notation

$$E^*(H_T^*) = \frac{H_T^u}{(1+r)} p^* + \frac{H_T^d}{(1+r)}(1-p^*),$$

where $H_T^* = \frac{H_T}{1+r}$ is the discounted payoff of the option.

Remark 7.1.10 [Fair game criteria].

Under the risk-neutral probability, we have

$$E^*(S_T) = S_0 u p^* + S_0 d(1-p^*) = S_0(1+r),$$

irrespective of the choice of u and d.

The following theorem provides an option pricing model under no-arbitrage assumption.

Theorem 7.1.11 [1-Step Binomial Model].

Consider a set of parameters B_0, S_0, and r, where r is the per-period interest rate with discrete compounding. For a given option, if Π_1 is a replicating portfolio in a no-arbitrage market, then the price H_0 at $t = 0$ of the option is given by

$$H_0 = E^*(H_T^*).$$

Proof.

Suppose that $H_0 > \phi_1 B_0 + \theta_1 S_0$.

In this case, we construct a portfolio Π_0 at $t = 0$ with the following trades:

- write one option with premium H_0;
- take a long position in the portfolio $\Pi_1 = (\phi_1, \theta_1)$; and
- invest remaining money in a risk-free instrument.

Then the initial investment is

$$V(\Pi_0)(0) = 1 \times (H_0 - (\phi_1 B_0 + \theta_1 S_0)) + 1 \times (\phi_1 B_0 + \theta_1 S_0) + -1 \times H_0 = 0.$$

At time $t = T$, the value of the portfolio Π_0 is

$$V(\Pi_0)(T) = \big(H_0 - (\phi_1 B_0 + \theta_1 S_0)\big)(1 + r) + \phi_1 B_0(1 + r) + \theta_1 S_T - H_T.$$

Using replicating strategy, we see that

$$V(\Pi_0)(T) = \big(H_0 - (\phi_1 B_0 + \theta_1 S_0)\big)(1 + r) > 0$$

irrespective to whether the stock goes up or down. Thus, Π_0 is an arbitrage portfolio.

Similarly, we can construct an arbitrage portfolio for the case when $H_0 < \phi_1 B_0 + \theta_1 S_0$.

Note

Note that the above theorem is not restricted to a call or a put option. It works for both cases, and hence, we have used the notation H_0 for the price. Similarly, we have used H_T for the payoff. However, when the types are specified, we will use our usual notations. For instance, if the option is a European call option, then we use the following notations:

1. $C^e = C^e(t)$ for the price (premium) as a function of $t \in [0, T]$;
2. C_0^e for the price at $t = 0$; and
3. C_T^e for the payoff.

We use similar notations for the other cases.

Whenever a result holds for both call and put, we use notations H_0 for the price at $t = 0$, $H(t)$ or H_t for price at $t \in [0, T]$, and H_T for the payoff. Note that the superscript e and a can be ignored because we mostly discuss the European and American options separately under respective subsections.

Problem 7.6.

For a fixed real numbers $d \in (0, 1)$ and $r > 0$, consider $C_0^e = C_0^e(u)$, for $u \in (1+r, \infty)$. If the option price C_0^e is governed by 1-step binomial model, then show that C_0^e is a non-decreasing function (of u).

Problem 7.7.

On 1st March 2021, Infosys was trading at ₹1267 per share. Mr. Megh predicted that the stock would either trade at ₹1340 or ₹1160 per share on 25th March 2021. Mr. Megh also has an opportunity to invest in a risk-free instrument offering $r = 0.0039$ with discrete compounding during the period of 1st to 25th March 2021. If Mr. Megh wants to buy a 1260-strike European call option with expiration 25th March 2021, then find the fair price of the option with respect to the predicted spot market prices.
Ans: 49.58

Note

It is important to observe that the model discussed in this subsection (and also in the next subsection) considered the discrete interest scheme and the interest rate given is the per-period interest rate. Generally, we take the interest rate to be per-annum in which case, an obvious modification needs to be considered. Also, we can write the model for continuous compounding interest scheme with a suitable modification.

7.1.2 Multi-step model

We extend the binomial model to the multi-step case. The idea is to divide the entire binomial lattice into single subtrees, start from $t = t_n$, and go backward by applying the single-step binomial model for each subtree. We illustration the idea in the 2-step binomial case.

Example 7.1.12 [2-step binomial model].

Consider the partition $\mathbb{T}_2 = \{t_0 = 0, t_1, t_2 = T\}$. In this case, the sample set is given by

$$\mathsf{S} = \{(U, U), (U, D), (D, U), (D, D)\}$$

and the partitions of S at $t = t_1$ and $t = t_2$ are given by, respectively,

$$\begin{aligned}\mathtt{P}_1 &= \{B_U, B_D\};\ B_U = \{(U,U),(U,D)\}, B_D = \{(D,U),(D,D)\},\\ \mathtt{P}_2 &= \{B_{UU}, B_{UD}, B_{DU}, B_{DD}\};\ B_{UU} = \{(U,U)\},\ \text{and so on.}\end{aligned}$$

We use the following notations for the payoff H_T (since $n = 2$, we can also use the notation H_2 instead of H_T):

$$H_2 = \begin{cases} H_T^{uu}, & \text{on } B_{UU},\\ H_T^{ud}, & \text{on } B_{UD},\\ H_T^{du}, & \text{on } B_{DU},\\ H_T^{dd}, & \text{on } B_{DD}.\end{cases}$$

Using 1-step binomial model, the intrinsic value of the option at $t = t_1$ is given by

$$H_1 = \begin{cases} E^*(H_2^{u*}), & \text{on } B_U,\\ E^*(H_2^{d*}), & \text{on } B_D,\end{cases} \quad \text{where} \quad H_2^{u*} = \frac{1}{1+r}\begin{cases} H_T^{uu}, & \text{on } B_{UU},\\ H_T^{ud}, & \text{on } B_{UD},\end{cases}$$

$$H_2^{d*} = \frac{1}{1+r}\begin{cases} H_T^{du}, & \text{on } B_{DU},\\ H_T^{dd}, & \text{on } B_{DD},\end{cases}$$

where the expectation E^* is defined with respect to the risk-neutral probability p^* restricted to the corresponding subtree. Note that H_1 is a random variable and the expectation used in defining H_1 is conditioned on the σ-field $\mathcal{F}_1$. Hence, H_1 can be written in terms of the conditional expectation as $H_1 = E^*(H_2^* \mid \mathcal{F}_1)$ (as we will see more precisely in the following discussions).

We now consider H_1 as the payoff of the option at time level $t = t_1$ and apply the 1-step binomial model once again (now at $t = t_0$) to obtain H_0. At this stage, the formula for H_0 is given by

$$H_0 = E^*(H_1^*); \quad H_1^* = \frac{H_1}{1+r}. \tag{7.8}$$

which we finally take as the price of the option at time $t = 0$.

As in the case of the 1-step method, we can prove that H_0 given by (7.8) is the unique no-arbitrage price for a European option.

Problem 7.8 [2-Step Binomial Model].

For a given option, if the process $\{\Pi_1, \Pi_2\}$ is obtained by replicating strategy in a no-arbitrage market, then show that the price H_0 at $t = 0$ of the option is uniquely given by (7.8).

Problem 7.9.

Obtain the fair price for the option given in Problem 7.7 using the 2-step model.

Let us now discuss the detailed construction of the multi-step model using *backward induction* technique illustrated in the above example. Note that in the above example, we have constructed the 2-step model by combining three 1-step models. Now, we will explain the entire process from the basic assumptions.

We consider a European option with payoff H_T. Consider the partition $\mathbb{T}_n = \{t_0 = 0, t_1, t_2, \ldots, t_{n-1}, t_n = T\}$ of the option period $[0, T]$. It is convenient to denote the payoff H_T by H_n.

Given the $\mathcal{F}_{n-1}$-measurable random variable S_{n-1}, define

$$S_n = \begin{cases} S_{n-1}u, & \text{if } s_n = U, \\ S_{n-1}d, & \text{if } s_n = D, \end{cases} \tag{7.9}$$

where d and n are chosen to satisfy the no-arbitrage condition (7.4) for a given per period interest rate r. Then the payoff is given by

$$H_n = \begin{cases} H_n^u, & \text{if } s_n = U, \\ H_n^d, & \text{if } s_n = D, \end{cases}$$

where $H_n^d = \max(0, S_{n-1}d - K)$ for a European call option and similarly for the other cases.

Step 1: [*Constructing a replicating portfolio*]

Next, let us impose the condition (7.2) for a replicating strategy. Since

$$V_n = \phi_n B_n + \theta_n S_n,$$

the condition (7.2) can be written as

$$\left.\begin{array}{r} \phi_n B_{n-1}(1+r) + \theta_n S_{n-1}d = H_n^d, \\ \phi_n B_{n-1}(1+r) + \theta_n S_{n-1}u = H_n^u. \end{array}\right\}$$

Under the no-arbitrage assumption (7.4), we can obtain a unique solution of the above linear system and is given by

$$\phi_n = \frac{uH_n^d - dH_n^u}{B_{n-1}(1+r)(u-d)}, \quad \theta_n = \frac{H_n^u - H_n^d}{S_{n-1}(u-d)}. \tag{7.10}$$

Note that $\Pi_n = (\phi_n, \theta_n)$ is the final term in the trading strategy $\{\Pi_k \mid k = 1, 2, \ldots, n\}$ that makes the entire strategy a replicating strategy. Hence, it is called a *replicating portfolio*.

Remark 7.1.13.

Often, θ_n is called *Delta* as it is the ratio of the variation in payoff and the variation in the underlying.

Step 2: [*Self-financing condition*]

The total amount needed to make the portfolio Π_n is

$$V(\Pi_n)(t_{n-1}) = \phi_n B_{n-1} + \theta_n S_{n-1}.$$

Since we assume that the trading strategy is self-financing, by (7.1), we get

$$V_{n-1} = \phi_n B_{n-1} + \theta_n S_{n-1}.$$

Substituting (7.10) in the above expression, we get

$$V_{n-1} = \frac{1}{1+r}\Big(H_n^u p^* + H_n^d(1-p^*)\Big), \tag{7.11}$$

where p^* is the risk-neutral probability given by (7.7).

Remark 7.1.14 [Conditional Expectation].

The payoff H_n can be taken as a random variable defined on the probability space $(\mathsf{S}, \mathcal{F}^*, \mathbb{P}^*)$, where the probability measure $\mathbb{P}^*$ is defined in such a way that

$$\mathbb{P}^*\big(\{S_k = S_{k-1}u\}\big) = p^*, \quad k = 1, 2, \ldots, n.$$

The expression $(H_n^u p^* + H_n^d(1-p^*))$ on the right hand side of (7.11) is the expectation of H_n under the probability measure $\mathbb{P}^*$. Note that both H_n^u and H_n^d depends on S_{n-1}. Since the stock price process $\{S_k \mid k = 1, 2, \ldots, n\}$ is defined on the filtered probability space $(\mathsf{S}, \mathcal{F}, \mathbb{P}^*, \{\mathcal{F}_k\})$, we see that S_{n-1} is a $\mathcal{F}_{n-1}$-measurable random variable. Hence, both H_n^u and H_n^d are $\mathcal{F}_{n-1}$-measurable random variable. This shows that the expectation $(H_n^u p^* + H_n^d(1-p^*))$ is a $\mathcal{F}_{n-1}$-measurable random variable. Further, it can be shown that this is precisely the *conditional expectation* of H_n given $\mathcal{F}_{n-1}$ ($\subseteq \mathcal{F}^*$). In notation, we write it as $E(H_n \mid \mathcal{F}_{n-1})$. Since, we have been denoting the expectation with respect to the risk-neutral probability as $E^*(\cdot)$, and the discounted payoff as H_n^*, we use the notation

$$E^*(H_n^* \mid \mathcal{F}_{n-1}) := \frac{1}{1+r}\Big(H_n^u p^* + H_n^d(1-p^*)\Big) \tag{7.12}$$

The rigorous definition is given in Definition ??.

Using the conditional expectation notation, we can write

$$V_{n-1} = E^*(H_n^* \mid \mathcal{F}_{n-1}).$$

Let H_{n-1} denotes the value of the option at time $t = t_{n-1}$, then by following the idea of the proof of Theorem 7.1.11, we can prove the following theorem.

Theorem 7.1.15.

For a given option, if a replicating strategy exists in a no-arbitrage market, then the price H_k of the option at $t = t_k$ is given by

$$H_k = V_k,\ \ k = 0, 1, \ldots, n-1.$$

By the above theorem, we can write

$$H_{n-1} = E^*(H_n^* \mid \mathcal{F}_{n-1}).$$

Step 3: [*Backward Induction*]

For every $k = n, n-2, \ldots, 1$, assume that we have the $\mathcal{F}_{k-1}$-measurable random variable H_k.

1. Using H_k construct the replicating portfolio Π_k at time level $t = t_{k-1}$.
2. Impose the self-financing condition and obtain V_{k-1}.
3. Take
$$H_{k-1} = E^*(H_k^* \mid \mathcal{F}_{k-1}).$$

Repeat the above steps until $k = 1$ and at $k = 1$, we have the required price of the option as

$$H_0 = E^*(H_1^*). \tag{7.13}$$

Note that the conditioning on $\mathcal{F}_0$ is not required because, it is a trivial σ-field.

The stock price S_k at every time level $t = t_k$, $k = 1, 2, \ldots, n$, takes one of the values

$$S_{k,j} := S_0 u^j d^{k-j};\ \ j = 0, 1, 2, \ldots, k.$$

Correspondingly, we use the notation $H_k(S_{k,j}) = H_{k,j}$ for the value of the option at each time level $t = t_k$.

With this notation, we can write (7.13) for 3-step binomial model as

$$H_0 \ \ = \ \ \frac{1}{(1+r)^2}\Big(H_{2,2}p^{*2} + 2H_{2,1}(1-p^*)p^* + H_{2,0}(1-p^*)^2\Big)$$

Since,

$$\begin{aligned} H_{2,2} &= E^*(H_3^* \mid S_{2,2}) = H_{3,3}^* p^* + H_{3,2}^*(1-p^*), \\ H_{2,1} &= E^*(H_3^* \mid S_{2,1}) = H_{3,2}^* p^* + H_{3,1}^*(1-p^*), \\ H_{2,0} &= E^*(H_3^* \mid S_{2,0}) = H_{3,1}^* p^* + H_{3,0}^*(1-p^*), \end{aligned}$$

the above expression can be written as

$$H_0 = \frac{1}{(1+r)^2}\Big(\Big\{H^*_{3,3}p^* + H^*_{3,2}(1-p^*)\Big\}p^{*2} + 2\Big\{H^*_{3,2}p^* + H^*_{3,1}(1-p^*)\Big\}(1-p^*)p^* + \Big\{H^*_{3,1}p^* + H^*_{3,0}(1-p^*)\Big\}(1-p^*)^2\Big)$$

Rearranging the terms, we get

$$H_0 = \frac{1}{(1+r)^3}\left(H_{3,3}p^{*3} + 3H_{3,2}(1-p^*)p^{*2} + 3H_{3,1}(1-p^*)^2p^* + H_{3,0}(1-p^*)^3\right).$$

This is the option price given by the *3-step binomial model*.

Problem 7.10.

Obtain the fair price for the option given in Problem 7.7 using the 3-step model.

Continuing in this way, we get the n-step binomial model, which we state in the form of a theorem.

Theorem 7.1.16 [Multi-step binomial model].

Let the option period $[0,T]$ be partitioned into n subinterval. The price of an attainable European option in a no-arbitrage market is uniquely given by

$$H_0 = \frac{1}{(1+r)^n}\sum_{j=0}^{n}\binom{n}{k}p^{*j}(1-p^*)^{n-j}H_{n,j}.$$

Note

The binomial model is also called *CRR model* (Cox-Ross-Rubinstein).

Problem 7.11.

Write the CRR model for European K-strike call and put options.

In the following problem, we illustrate a particular case of *trinomial model*.

Problem 7.12 [Trinomial model].

Consider the financial market $(B, S^{(1)}, S^{(2)})$, where $\{B_k\}$, $\{S^{(1)}_k\}$, and $\{S^{(2)}\}$ are the corresponding prices process at time levels $\mathbb{T}_n = \{t_0, t_1, \ldots, t_n\}$ with $B_0 \neq 0$, $S^{(j)}_0 \neq 0$,

$j = 1, 2$. Let the sample space be $(\mathsf{S}, \mathcal{F}^*)$, where

$$\mathsf{S} = \{D, M, U\}$$

with U being the upward movement of a risky asset, M being the sideway movement, and D being the downward movement. As usual, $\mathcal{F}^*$ is the power set of S and the discrete probability be given by

$$\mathbb{P}(\{U\}) = p,\ \mathbb{P}(\{M\}) = q, \mathbb{P}(\{D\}) = 1 - p - q,$$

for some $p, q \in (0, 1)$ with $0 < 1 - p - q < 1$.

Let d_j, m_j, u_j, $j = 1, 2$, and $r \geq 0$ be such that $0 < d_j < m_j < u_j$ and $0 < d_j < 1 + r < u_j$ (no-arbitrage condition) and $m_j \neq 1 + r$, $j = 1, 2$. Define

$$S_k^{(j)}(\boldsymbol{s}) = \begin{cases} S_{k-1}^{(j)} u_j, & \text{if } s_k = U \\ S_{k-1}^{(j)} m_j, & \text{if } s_k = M \\ S_{k-1}^{(j)} d_j, & \text{if } s_k = U \end{cases} ; \quad j = 1, 2; k = 1, 2, \ldots, n,$$

and r is the per period interest rate.

1. Show that there exists a unique replicating strategy if and only if

$$\begin{vmatrix} 1 & u_1 & u_2 \\ 1 & m_1 & m_2 \\ 1 & d_1 & d_2 \end{vmatrix} \neq 0.$$

2. Show that the system

$$\begin{aligned} (u_1 - d_1)p^* + (m_1 - d_1)q^* &= 1 + r - d_1 \\ (u_2 - d_2)p^* + (m_2 - d_2)q^* &= 1 + r - d_2 \end{aligned}$$

has a unique solution if and only if a unique replicating strategy exists.

7.2 Continuous time pricing model

In the discrete-time model, we have partitioned the option period into n sub-periods and developed a model to obtain option price at time $t = 0$. The price obviously depends on n (see for instance Theorem 7.1.16). The question now is 'how well the pricing model behaves as we keep on increasing n and what if we take $n \to \infty$?'

Let us assign to each n, the corresponding price as $H_0^{(n)}$. Then we have a sequence of option prices $\{H_0^{(n)}\}$, where the n^{th} term of the sequence corresponds to the price of the option at time $t = 0$ computed using the partition $\mathbb{T}_n$ of $[0, T]$, for $n = 1, 2, \ldots$. With this notation, we can precisely pose the above question as 'does the price sequence converge as $n \to \infty$?'. This question is equivalent to asking, 'is the pricing model *stable*?'. It can be shown that the binomial model is stable, *i.e.* the price sequence converges as $n \to \infty$, and in the limiting case, we obtain the well-known *Black-Sholes model*.

Our aim in this section is to introduce the Black-Sholes model.

For a given number of partitions $n = 1, 2, \ldots$, let us denote the interest rate, up and down movements of the stock as r_n, u_n, and d_n, respectively. Also, define $\Delta t_n = T/n$ and the equivalent martingale measure as $\mathbb{P}_n^*(\{u_n\}) = p_n^*$.

Recall, by the matching procedure of the binomial lattice model with the lognormal model, we obtained u_n, d_n, and p_n in two ways, namely,

1. by taking $d_n = 1/u_n$, where the expressions are given by

$$\begin{aligned} u_n &= \exp\left(\sqrt{(\mu\Delta t_n)^2 + \sigma^2\Delta t_n}\right) \\ d_n &= \exp\left(-\sqrt{(\mu\Delta t_n)^2 + \sigma^2\Delta t_n}\right) \\ p_n &= \frac{1}{2} + \frac{1/2}{\sqrt{\sigma^2/(\mu^2\Delta t_n) + 1}}; \end{aligned}$$

   and

2. by taking $p_n = 1/2$. and the expressions are given by

$$\begin{aligned} u_n &= \exp\left(\mu\Delta t_n + \sigma\sqrt{\Delta t_n}\right) \\ d_n &= \exp\left(\mu\Delta t_n - \sigma\sqrt{\Delta t_n}\right) \\ p_n &= \frac{1}{2}. \end{aligned}$$

To have a unified study, we assume that u_n and d_n take the form:

$$\left.\begin{aligned} u_n &= \exp\left(\sigma\sqrt{\Delta t_n} + \alpha\Delta t_n\right) \\ d_n &= \exp\left(-\sigma\sqrt{\Delta t_n} + \beta\Delta t_n\right), \end{aligned}\right\} \tag{7.14}$$

where α and β are assumed to be known. For instance, $\alpha = \beta = \mu$ gives the second case listed above.

We further consider the continuous compounding scheme so that the annual interest rate r is chosen such that

$$1 + r_n = e^{r\Delta t_n}. \tag{7.15}$$

In what follows, we consider u_n and d_n as given by (7.14) and r given by (7.15) as the risk-free interest rate. Since we always consider no-arbitrage market, by first fundamental theorem of asset pricing, we can take the EMM as the risk-neutral probability

$$p_n^* = \frac{e^{r\Delta t_n} - d_n}{u_n - d_n},$$

where u_n and d_n are given by (7.14) and r is given by (7.15).

First, let us observe that the probability p^* is independent of α and β as $n \to \infty$.

Lemma 7.2.1.

If u_n and d_n are given by (7.14) and r is given by (7.15), then

$$\lim_{n\to\infty} p_n^* = \frac{1}{2}.$$

Proof.

By definition, we have

$$p_n^* = \frac{e^{r\Delta t_n} - e^{\left(-\sigma\sqrt{\Delta t_n}+\beta\Delta t_n\right)}}{e^{\left(\sigma\sqrt{\Delta t_n}+\alpha\Delta t_n\right)} - e^{\left(-\sigma\sqrt{\Delta t_n}+\beta\Delta t_n\right)}}.$$

Therefore, we have

$$2p_n^* - 1 = \frac{2e^{r\Delta t_n} - e^{\left(\sigma\sqrt{\Delta t_n}+\alpha\Delta t_n\right)} - e^{\left(-\sigma\sqrt{\Delta t_n}+\beta\Delta t_n\right)}}{e^{\left(\sigma\sqrt{\Delta t_n}+\alpha\Delta t_n\right)} - e^{\left(-\sigma\sqrt{\Delta t_n}+\beta\Delta t_n\right)}}.$$

Using Taylor expansion for each exponential, we get

$$2p_n^* - 1 = \frac{\left(r - \frac{\sigma^2}{2} - \frac{\alpha+\beta}{2}\right)\Delta t_n + o(\Delta t_n)}{\sigma\sqrt{\Delta t_n}(1 + o(1))}, \text{ as } n \to \infty. \tag{7.16}$$

We can see that the RHS tends to zero as $n \to \infty$. Hence the result.

7.2.1 Black-Scholes Model for European Option

We consider a European option with strike K and period $[0, T]$. From Theorem 7.1.16 we can see that for a given $n = 1, 2, \ldots$, the price of the option is given by

$$H_0^{(n)} = E_n^*(e^{-rT} H_T^{(n)}),$$

where $H_T^{(n)}$ is the payoff of the option.

Let us first consider a European put option, where the payoff is given by

$$P_T^{(n)} = \max(0, K - S_T^{(n)}) \tag{7.17}$$

where $S_T^{(n)} = S_0 e^{X_n}$, with X_n being the random variable on the sample space $(\mathsf{S}, \mathcal{F}^*)$ defined by

$$X_n(\boldsymbol{s}) = \log \prod_{k=1}^{n} \overline{s}_k = \sum_{k=1}^{n} Y_k^{(n)}(\boldsymbol{s}), \tag{7.18}$$

for each $\boldsymbol{s} = (s_1, s_2, \ldots, s_n) \in \mathsf{S}$,

$$Y_k^{(n)}(\boldsymbol{s}) = \log \overline{s}_k \quad \text{and} \quad \overline{s}_k = \begin{cases} u_n, & \text{if } s_k = U, \\ d_n, & \text{if } s_k = D, \end{cases} \quad k = 1, 2, \ldots, n.$$

With the above notation, the initial option price can be written as

$$P_0^{(n)} = E_n^*\Big(e^{-rT} \max\big(0, K - S_0 e^{X_n}\big)\Big). \tag{7.19}$$

We assume that the stock movement during the time period $(t_k, t_{k+1}]$ is independent of its movement in $(t_{k-1}, t_k]$. With this assumption, we can see that Y_k's are iid random variables and we have (for each $k = 1, 2, \ldots, n$)

$$\left.\begin{array}{rcl} \mathbb{P}_n^*\Big(\{Y_k^{(n)} = \sigma\sqrt{\Delta t_n} + \alpha \Delta t_n\}\Big) & = & p_n^*, \\ \mathbb{P}_n^*\Big(\{Y_k^{(n)} = -\sigma\sqrt{\Delta t_n} + \beta \Delta t_n\}\Big) & = & 1 - p_n^*. \end{array}\right\} \tag{7.20}$$

We now state two important properties of X_n.

Lemma 7.2.2.

We have:

$$\lim_{n\to\infty} E_n^*(X_n) = \left(r - \frac{\sigma^2}{2}\right) T, \tag{7.21}$$

$$\lim_{n\to\infty} \mathrm{Var}_n^*(X_n) = \sigma^2 T. \tag{7.22}$$

Proof.

First let us prove (7.21). From (7.20) we see that, for each $k = 1, 2, \ldots, n$,

$$E_n^*(Y_k^{(n)}) = (2p_n^* - 1)\sigma\sqrt{\Delta t_n} + \Delta t_n\big(\alpha p_n^* + \beta(1 - p_n^*)\big)$$

Using Lemma 7.2.1 and (7.16), we get

$$E_n^*(Y_k^{(n)}) = \frac{\left(r - \frac{\sigma^2}{2} - \frac{\alpha+\beta}{2}\right)\Delta t_n + o(\Delta t_n)}{(1 + o(1))} + \Delta t_n \left(\frac{\alpha + \beta}{2} + o(1)\right), \text{ as } n \to \infty.$$

This can be written as

$$E_n^*(Y_k^{(n)}) = \left(r - \frac{\sigma^2}{2}\right)\Delta t_n + o(\Delta t_n), \text{ as } n \to \infty.$$

Observe that the right hand side is independent of k and therefore, we have

$$E_n^*(X_n) = nE_n^*(Y_1^{(n)}) = n\left\{\left(r - \frac{\sigma^2}{2}\right)\Delta t_n + o(\Delta t_n)\right\}, \text{ as } n \to \infty.$$

Since $\Delta t_n = T/n$, we get

$$E_n^*(X_n) = \left(r - \frac{\sigma^2}{2}\right)T + o(1), \text{ as } n \to \infty,$$

which proves (7.21).

Let us now prove (7.22). Since $Y_k^{(n)}$'s are iid, we have

$$\mathrm{Var}_n^*(X_n) = n\mathrm{Var}_n^*(Y_1^{(n)}) = n\left(E_n^*({Y_1^{(n)}}^2) - E_n^*(Y_1^{(n)})^2\right).$$

But, we have

$$\begin{aligned} E_n^*({Y_1^{(n)}}^2) &= (\log u_n + \log d_n)E_n^*(Y_1^{(n)}) - \log u_n \log d_n \\ &= \sigma^2 \Delta t_n + o(\Delta t_n), \text{ as } n \to \infty. \end{aligned}$$

Substituting this, we get

$$\begin{aligned} \mathrm{Var}_n^*(X_n) &= n\left(\sigma^2\Delta t_n + o(\Delta t_n) - \left(\left(r - \frac{\sigma^2}{2}\right)\Delta t_n + o(\Delta t_n)\right)^2\right) \\ &= \sigma^2 T + o(1), \text{ as } n \to \infty. \end{aligned}$$

This proves (7.22).

The next property is a variation of central limit theorem. So, we state the property and omit the proof.

Lemma 7.2.3.

The sequence of random variables $\{X_n\}$ defined by (7.18) converges in distribution to a random variable

$$X \sim \mathcal{N}\left(\left(r - \frac{\sigma^2}{2}\right)T, \sigma^2 T\right).$$

The above two lemmas can be combined to get the following theorem defining the *Black-Scholes price* formula for a European put option.

Theorem 7.2.4 [Black-Scholes pricing model for Put].

Let $H_0^{(n)}$ be the price of a European K-strike put option (denoted by $P_0^{(n)}$) with period $[0, T]$ partitioned into n sub-periods. Let the parameters u_n, d_n, and r be given by (7.14)-(7.15). Then the limit

$$\lim_{n\to\infty} P_0^{(n)} = P_0$$

exists and we have

$$P_0 = E(e^{-rT} P_T), \tag{7.23}$$

where $P_T = \max(0, K - S_0 e^X)$ with $X \sim \mathcal{N}\left(\left(r - \frac{\sigma^2}{2}\right) T, \sigma^2 T\right)$.

The proof of the above theorem is advanced and we omit it for our course. However, we make the following remark about the proof of the theorem.

Remark 7.2.5.

Recall that the convergence in distribution of random variables is defined only in terms of convergence of the respective distributions. Consequently, it is not necessary that the random variables are defined on the same probability space. If we denote by $(\mathsf{S}_n, \mathcal{F}_n, \mathbb{P}_n)$ the probability space on which X_n is defined and $(\mathsf{S}, \mathcal{F}, \mathbb{P})$ the probability space on which X is defined, then $\{X_n\}$ converges in distribution to X if and only if

$$\lim_{n\to\infty} E_n(\phi(X_n)) = E(\phi(X)),$$

for all $\phi \in C_b(\mathbb{R}^m)$ (set of all bounded continuous functions), where E_n and E are the expectations taken with respect to the probability $\mathbb{P}_n$ and $\mathbb{P}$, respectively.

We used the above property to conclude the convergence of the sequence $\{P_0^{(n)}\}$ given by (7.19) to P_0. Note that to use the above property, we need $P_T^{(n)}$ given by (7.17) to be a bounded continuous function and this is the reason why we restricted to put options. We can extend the pricing formula to call options using put-call parity relation.

Definition 7.2.6 [Black-Scholes Price for Put].

The number P_0 defined by (7.23) is called the *Black-Scholes price* of a European put option with strike K and maturity T.

Note that (7.23) does not give an explicit formula for the price. However, it is possible to derive an explicit formula for P_0, which is one of the advantages of the Black-Scholes' model.

Corollary 7.2.7 [Black-Scholes Formula for European Put].

The Black-Scholes price P_0 can be written as

$$P_0 = Ke^{-rT}\Phi(-d_-) - S_0\Phi(-d_+),$$

where Φ is the standard normal distribution function

$$\Phi(x) = \frac{1}{\sqrt{2\pi}}\int_{-\infty}^{x} e^{-\frac{s^2}{2}}ds, \;\; x \in \mathbb{R},$$

and

$$d_{\pm} = \frac{\log\left(\frac{S_0}{K}\right) + \left(r \pm \frac{\sigma^2}{2}\right)T}{\sigma\sqrt{T}}, \tag{7.24}$$

r is the prevailing interest rate.

Proof.

From (7.23), we see that

$$P_0 \;\;=\;\; e^{-rT}E\Big(\max\left(0, K - S_0e^X\right)\Big).$$

Since $X \sim \mathcal{N}\left(\left(r - \frac{\sigma^2}{2}\right)T, \sigma^2 T\right)$, we see that

$$X = \left(r - \frac{\sigma^2}{2}\right)T + \sigma\sqrt{T}Z,$$

where $Z \sim \mathcal{N}(0,1)$ is the standard normal variable. Therefore,

$$S_0e^X < K \quad \Longleftrightarrow \quad Z < -d_-.$$

We have

$$E\Big(\max\left(0, K - S_0e^X\right)\Big) = KE(\mathbb{1}_{\{S_T<K\}}) - E(S_T\mathbb{1}_{\{S_T<K\}}).$$

Let us compute the two quantities on the right hand side.

For the first term, we have

$$\begin{aligned} E(\mathbb{1}_{\{S_T<K\}}) &= E(\mathbb{1}_{\{Z<-d_-\}}) \\ &= \Phi(-d_-). \end{aligned}$$

Second term gives

$$\begin{aligned} E(S_T\mathbb{1}_{\{S_T<K\}}) &= E(S_0e^X\mathbb{1}_{\{S_T<K\}}) \\ &= S_0E\left(\exp\left(\left(r-\frac{\sigma^2}{2}\right)T+\sigma\sqrt{T}Z\right)\mathbb{1}_{\{Z<-d_-\}}\right) \\ &= e^{rT}S_0\int_{-\infty}^{-d_-}\frac{1}{\sqrt{2\pi}}\exp\left(-\frac{\sigma^2}{2}T+\sigma\sqrt{T}\xi-\frac{\xi^2}{2}\right)d\xi \end{aligned}$$

Using the change of variable $s=\xi-\sigma\sqrt{T}$, we get

$$E(S_T\mathbb{1}_{\{S_T<K\}}) = e^{rT}S_0\int_{-\infty}^{-d_--\sigma\sqrt{T}}\frac{1}{\sqrt{2\pi}}e^{-\frac{s^2}{2}}ds.$$

Since $d_+=d_-+\sigma\sqrt{T}$, we obtained the required formula.

We can now obtain the price for a European call option.

Corollary 7.2.8 [Black-Scholes Formula for Call].

The Black-Scholes price C_0 for a European call option can be written as

$$C_0 = S_0\Phi(d_+) - Ke^{-rT}\Phi(d_-),$$

where Φ is the standard normal distribution function and $d_\pm$ are given by (7.24).

Proof is left as an exercise.

Problem 7.13.

Use Black-Scholes formula to find the price of the European call option with the following details:

- 1 contract consists of 100 units;

- 25-strike;
- the underlying is trading at 20 per share;
- 3 months expiration;
- the stock's volatility is 24%; and
- the risk-less interest rate is 5% continuously compounded.

Ans: 4.24

7.2.2 Black-Scholes Differential Equation

We can derive a parabolic partial differential equation (PDE) as a limiting case of the binomial model and the PDE is the well known *Black-Scholes equation* given by

$$\frac{\partial H}{\partial t} + \frac{\sigma^2 S^2}{2}\frac{\partial^2 H}{\partial S^2} + rS\frac{\partial H}{\partial S} - rH = 0. \tag{7.25}$$

The starting level of the backward induction at time $t = t_n$ $(= T)$ can be taken as

$$H(T, S) = H_T,$$

where H_T is the payoff of the European option.

The above two equations form a *terminal value problem*. Using the change of variable

$$H(t, S) = u(\tau, x),$$

where $\tau = T - t$ and $x = \log S$, we obtain the *initial value problem* (or the *Cauchy problem*)

$$\begin{aligned} -\frac{\partial u}{\partial \tau} + \left(r - \frac{\sigma^2}{2}\right)\frac{\partial u}{\partial x} + \frac{\sigma^2}{2}\frac{\partial^2 u}{\partial x^2} - ru = 0, &\quad (\tau, x) \in (0, T] \times \mathbb{R}, \\ u(0, x) = H_T(e^x), &\quad x \in \mathbb{R}. \end{aligned}$$

Deriving the above IVP is a simple calculus exercise.

Problem 7.14.

Let $H(t, S)$ be the European option premium that satisfies the Black-Scholes partial differential equation for all $(t, S) \in [0, T] \times (0, \infty)$. Consider the portfolio $\Pi = (0, s, h)$, where s and h are the number of units held in the underlying stock and the option, respectively. Show that the value of the portfolio V considered as a function of (t, S) (*i.e.* $V(t, S)$) also satisfies the Black-Scholes partial differential equation for all $(t, S) \in [0, T] \times (0, \infty)$.