5.2.2 For American Options

We now turn our attention to obtaining bounds for an American option premium.

It is important to note that American options offer more flexibility than European options. Consequently, one might expect the premium of an American option to be higher than that of a European option with the same parameters.

Problem 5.5.

Consider two options (either both call or both put):

- K-strike, call option;
- option period is $[0, T]$; and
- premium of X^e for the European option and X^a for the American option.

If the underlying stock does not pay dividend during the option period, the market does not allow arbitrage, and the prevailing interest rate is $r > 0$, then prove that

$$X^e \leq X^a.$$

Hints:
Take a long position in one contract (European or American, as appropriate) and take a short position in the other contract. Hold the positions until expiration. At expiration, we must settle both options simultaneously. It is important to note that we must justify arbitrage. For this purpose, the settlement should be viewed as follows:

- Borrow the amount S_T;
- Settle the contract where we have to buy the asset;
- Supply the asset in the other contract and receive S_T;
- Repay the borrowed money.

This way, we do not need to pay any interest for the borrowing as the borrowing and repayment occur immediately.

Note
In the above result, the absence of a specified time component indicates that the result remains consistent irrespective of the time variable.

In particular, for call options, we always have $C^e = C^a$ when the underlying asset is a non-dividend-paying stock.

Problem 5.6.

Consider two call options:

- K-strike, call option;
- option period is $[0, T]$; and
- premium of C^e for the European option and C^a for the American option.

If the underlying stock does not pay dividend during the option period, the market does not allow arbitrage, and the prevailing interest rate is $r > 0$, then prove that

$$C^e = C^a.$$

Note

The above result does not hold if the underlying stock pays dividends sometime during the option period. Also, the result does not hold for American put options.

The following theorem is a direct consequence of Theorem 5.2.1 and Problem 5.6

Theorem 5.2.4 [American Call on Non-Dividend-Paying Stock].

Consider an American call option with
- K-strike;
- period $[0, T]$; and
- premium C_0^a,

where the underlying stock with spot price S_0 pays no dividend during the option period. If the market does not allow arbitrage, then

$$\max\left(0, S_0 - Ke^{-rT}\right) \leq C_0^a \leq S_0,$$

where r is the prevailing annual interest rate continuously compounded.

As noted above, in the case of put options, we have $P^e \leq P^a$ even if the underlying stock does not pay a dividend during the option period. Therefore, it is crucial to investigate the bounds for American put options. As usual, we begin by considering non-dividend paying stocks as the underlying asset.

Problem 5.7 [American Put on Non-Dividend-Paying Stock].

Consider the American put option
- K-strike;
- period $[0, T]$;
- premium P_0^a,

where the underlying stock has a spot price of S_0 per unit and pays no dividend during the option period. If the market does not allow arbitrage, then show that

$$\max\left(0, K - S_0\right) \leq P_0^a \leq K.$$

We can obtain the bounds for American options for dividend paying stocks by combining all the results stated so far.

Problem 5.8 [American Options on Dividend-Paying Stock].

Consider the American (call or put) option

- K-strike;
- period $[0, T]$;
- premium C_0^a for call and P_0^a for put,

where the underlying stock with spot price S_0 pays a dividend D_0 at some time during the option period. If the market does not allow arbitrage, then show that

$$\max\left(0, S_0 - D_0 - Ke^{-rT}, S_0 - K\right) \leq C_0^a \leq S_0, \quad \text{for call option;}$$
$$\max\left(0, Ke^{-rT} + D_0 - S_0, K - S_0\right) \leq P_0^a \leq K, \quad \text{for put option,}$$

where r is the prevailing annual interest rate continuously compounded.

5.2.3 Put-Call Parity Estimates

So far, we have obtained the bounds for call and put options. Now, we will discuss the relationship between call and put options in both European and American types.

Let us begin by examining European options and establishing the relationship.

Theorem 5.2.5 [Put-Call Parity for European Options].

Consider two European options, one call and one put with

- K-strike;
- period $[0, T]$; and
- the premium C_0^e for the call option and the premium P_0^e for the put option.

Let the underlying stock have a spot price of S_0 pays no dividends during the option period. If the market does not allow arbitrage, then

$$C_0^e - P_0^e = S_0 - Ke^{-rT},$$

where r is the prevailing annual interest rate continuously compounded.

Proof.

Case 1: Assume the contrary that $C_0^e - P_0^e > S_0 - Ke^{-rT}$.

Let us construct the portfolio Π_0 using the following trades:

- buy a share at the spot market for S_0 per share;
- take a long position in one put option;
- write one call option; and
- invest (or borrow if negative) the sum $C_0^e - P_0^e - S_0$ in a risk-free interest rate investment.

It can be seen that $V(\Pi_0)(0) = 0$. At time $t = T$, we have

$$V(\Pi_0)(T) = (C_0^e - P_0^e - S_0)e^{rT} + K > 0,$$

by our assumption. Complete the proof.

Case 2: Assume the contrary that $C_0^e - P_0^e < S_0 - Ke^{-rT}$.

Let us construct the portfolio Π_0 using the following trades:

- short a share at the spot market for S_0 per share;
- write one put option;
- go long in one call option; and
- invest (or borrow if negative) the sum $S_0 - C_0^e + P_0^e$ in a risk-free interest rate investment.

Complete the proof.

We now obtain the put-call parity for American options.

Theorem 5.2.6 [Put-Call Parity for American Options].

Consider two American options, one call and one put with

- K-strike;
- period $[0, T]$; and
- premium C_0^a for the call option and P_0^a for the put option,

Let the underlying stock have a spot price of S_0 pays no dividends during the option period. If the market does not allow arbitrage, then

$$S_0 - K \le C_0^a - P_0^a \le S_0 - Ke^{-rT},$$

where r is the prevailing annual interest rate continuously compounded.

Proof.

Let us first consider the upper bound.

Assume the contrary that $C_0^a - P_0^a - S_0 + Ke^{-rT} > 0$.

Construct the portfolio Π_0 with the following trades:

- write one call option and get the premium C_0^a;
- buy one put option by paying the premium P_0^a;
- buy one share by paying S_0; and
- invest (borrow if negative) the remaining $C_0^a - P_0^a - S_0$ in a risk-free interest rate investment.

The initial value of Π_0 is given by $V(\Pi_0)(0) = 0$.

Let $t = \tau \in [0, T]$ be the exercise time of the short call option. The payoff of Π_0 at $t = \tau$ is

$$V(\Pi_0)(\tau) = (C_0^a - P_0^a - S_0)e^{r\tau} + K.$$

Using our assumption, we can see that $V(\Pi_0)(\tau) > 0$ (how?). From the above inequality, we can see that, although τ and S_τ are random, finally, the required positivity does not depend on these variables. Thus, the positivity of the portfolio happens with probability 1, and therefore Π_0 is an arbitrage portfolio.

Let us now consider the lower bound.

Assume the contrary that $C_0^a - P_0^a - S_0 + K < 0$.

Construct the portfolio Π_0 with the following trades:

- write one put option and get the premium P_0^a;
- buy one call option by paying the premium C_0^a;
- short one share and get S_0; and
- invest (borrow if negative) the remaining $P_0^a - C_0^a + S_0$ in a risk-free interest rate investment.

The initial value of Π_0 is given by $V(\Pi_0)(0) = 0$.

Let $t = \tau \in [0, T]$ be the exercise time of the short put option. The payoff of Π_0 at $t = \tau$ is

$$V(\Pi_0)(\tau) = (P_0^a - C_0^a + S_0)e^{r\tau} - K.$$

From our assumption, we see that $V(\Pi_0)(\tau) > 0$ (how?). Complete the arbitrage arguement.

5.2.4 Premium Valuation

So far, we have studied some important estimates of call and put options having the same strike price. In this subsection, we study some basic properties of options prices depending on their strike prices. Let us include the strike price into the notation of option prices.

Remark 5.2.7.

For a given expiration T, we use the following notations:

For any $t \in [0, T]$ and for a positive real number K,

1. $C^e(t, K)$ or $C^e_t(K)$ denotes the European call option price (or premium) at time t whose strike price is K. For an American call option, we use the notation $C^a(t, K)$ or $C^a_t(K)$.
2. $P^e(t, K)$ or $C^e_t(K)$ denotes the European put option price (or premium) at time t whose strike price is K. For an American put option, we use the notation $P^a(t, K)$ or $P^a_t(K)$.

Theorem 5.2.8 [Dependency on strike price: European].

If the underlying stock does not pay a dividend during the option period and the market does not allow arbitrage, then

$$0 \le C^e_0(K_1) - C^e_0(K_2) \le e^{-rT}\big(K_2 - K_1\big),$$
$$0 \le P^e_0(K_2) - P^e_0(K_1) \le e^{-rT}\big(K_2 - K_1\big),$$

for any $0 \le K_1 \le K_2$, where r is the prevailing interest rate continuously compounded and all the options have the same period $[0, T]$.

Proof.

Let us first consider the lower bound.

1. **Lower bound for call options:**
   Assume the contrary that $C^e_0(K_1) < C^e_0(K_2)$.
   Construct a portfolio Π_0 using the following trades:
   - write the K_2-strike call option;
   - buy the K_1-strike call option; and
   - invest $C^e_0(K_2) - C^e_0(K_1)$ in a risk-free interest rate investment.

   The initial value of Π_0 is $V(\Pi_0)(0) = 0$.

At time $t = T$, the value of Π_0 is

$$V(\Pi_0)(T) = (C_0^e(K_2) - C_0^e(K_1))e^{rT} + x_0,$$

where

$$x_0 = \begin{cases} K_2 - K_1, & \text{if } K_2 < S_T \\ S_T - K_1, & \text{if } K_1 < S_T \leq K_2 \\ 0, & \text{if } S_T \leq K_1. \end{cases}$$

Since $K_1 \leq K_2$ and by our assumption, we see that $V(\Pi_0)(T) > 0$, leading to a contradiction to the no-arbitrage principle.

2. **Lower bound for put option:**
   Left as an exercise.
3. **Upper bound for both:** By put-call parity result from Theorem 5.2.5, we have

$$C_0^e(K_1) - P_0^e(K_1) = S_0 - K_1 e^{-rT},$$
$$C_0^e(K_2) - P_0^e(K_2) = S_0 - K_2 e^{-rT}.$$

Subtracting, we get

$$\big(C_0^e(K_1) - C_0^e(K_2)\big) + \big(P_0^e(K_2) - P_0^e(K_1)\big) = (K_2 - K_1)e^{-rT}.$$

Since both the terms in the brackets on the left-hand side are nonnegative as per the lower bounds, we see that each one is less than or equal to the right-hand side.

Problem 5.9.

Prove the upper bounds in the above theorem using arbitrage arguments.

Remark 5.2.9.

From the lower bound, we see that the call option price is a non-increasing function of K and the put option price is a non-decreasing function of K. Also, we see from the upper bound that both the call option and the put option prices are Lipschitz functions.

In the next theorem, we show that C^e and P^e are convex functions of K, a characteristic often observed in the market. However, the mathematical proof requires the assumption that the market allows trading fractional units.

Theorem 5.2.10 [Convexity property: European].

Assume that the market does not allow arbitrage, permits fractional unit trades, and allows short selling.

For any $\alpha \in [0,1]$ and for any positive real numbers K_1 and K_2, we have

$$C^e\big(\alpha K_1 + (1-\alpha)K_2\big) \leq \alpha C^e(K_1) + (1-\alpha)C^e(K_2),$$
$$P^e\big(\alpha K_1 + (1-\alpha)K_2\big) \leq \alpha P^e(K_1) + (1-\alpha)P^e(K_2),$$

where all the options have the same expiration.

Proof.

Let us use the notation

$$K = \alpha K_1 + (1-\alpha)K_2.$$

Call Option:
Assume the contrary that

$$C^e\big(K\big) > \alpha C^e(K_1) + (1-\alpha)C^e(K_2).$$

Construct a portfolio Π_0 using the following strategies:

- write the K-strike call option;
- buy α units of K_1-strike call options;
- buy $1-\alpha$ units of K_2-strike call options; and
- invest $C^e\big(K\big) - \big(\alpha C^e(K_1) + (1-\alpha)C^e(K_2)\big)$ in a risk-free interest rate instrument.

We have $V(\Pi_0)(0) = 0$, and at time $t = T$, we have

$$V(\Pi_0)(T) \;=\; \Big(C^e\big(K\big) - \big(\alpha C^e(K_1) + (1-\alpha)C^e(K_2)\big)\Big)e^{rT} + x_0,$$

where

$$x_0 = \begin{cases} \begin{cases} \alpha(S_T - K_1), & \text{if } K_1 < S_T \text{ and } K_2 \geq S_T \\ (1-\alpha)(S_T - K_2), & \text{if } K_1 \geq S_T \text{ and } K_2 < S_T \\ 0, & \text{if } K_1 \geq S_T \text{ and } K_2 \geq S_T \end{cases}, & \text{if } K \geq S_T, \\ K - \begin{cases} \alpha K_1 + (1-\alpha)K_2, & \text{if } K_1 < S_T \text{ and } K_2 < S_T \\ \alpha K_1 + (1-\alpha)S_T, & \text{if } K_1 < S_T \text{ and } K_2 \geq S_T \\ \alpha S_T + (1-\alpha)K_2, & \text{if } K_1 \geq S_T \text{ and } K_2 < S_T \end{cases}, & \text{if } K < S_T. \end{cases}$$

It can be proved that $x_0 \geq 0$.

Put Option:

Convexity of the put option price can be proved in two ways,

1. by constructing an arbitrage portfolio; and
2. using convexity of call option price and the put-call parity result in Theorem 5.2.5.

We leave the first method as an exercise and prove the result using the second method.

We prove the result for $t = 0$.

From Theorem 5.2.5, we have

$$C_0^e(K) - P_0^e(K) \;=\; S_0 - Ke^{-rT}$$

This implies

$$\begin{aligned} P_0^e(K) &\leq \alpha C_0^e(K_1) + (1-\alpha)C_0^e(K_2) + Ke^{-rT} - S_0 \\ &= \alpha\Big(C_0^e(K_1) + K_1e^{-rT} - S_0\Big) + (1-\alpha)\Big(C_0^e(K_2) + K_2e^{-rT} - S_0\Big) \end{aligned}$$

Again using Theorem 5.2.5, we get the desired result.

We now state the equivalent theorems in the case of American options. The proofs are left as exercises.

Problem 5.10 [Dependency on strike price: American].

If the underlying stock does not pay a dividend during the option period and the market does not allow arbitrage, then show that

$$\begin{aligned} 0 \leq C_0^a(K_1) - C_0^a(K_2) \leq e^{-rT}\big(K_2 - K_1\big), \\ 0 \leq P_0^a(K_2) - P_0^a(K_1) \leq K_2 - K_1, \end{aligned}$$

for any $0 \leq K_1 \leq K_2$, where r is the prevailing interest rate continuously compounded and all the options have the same period $[0, T]$.

Problem 5.11 [Convexity property: American].

For any $\alpha \in [0, 1]$ and for any positive real numbers K_1 and K_2, show that

$$\begin{aligned} C^a\big(\alpha K_1 + (1-\alpha)K_2\big) \leq \alpha C^a(K_1) + (1-\alpha)C^a(K_2), \\ P^a\big(\alpha K_1 + (1-\alpha)K_2\big) \leq \alpha P^a(K_1) + (1-\alpha)P^a(K_2), \end{aligned}$$

where all the options have the same period $[0, T]$, and the market does not allow arbitrage.

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