8.2 Sensitivity Analysis: Greeks

A hedge is a position that offsets the price risk of another position. The main purpose of an option is to hedge a position in an asset. A risk in holding an asset is due to the fluctuations in the asset's price, which causes volatility. Thus, the underlying asset price S and the volatility parameter σ mainly influences an option's price which is also apparent from the Black-Scholes model. It is important to understand the dependency of the option price on S and σ (also on r and t) more closely in order to use options as a powerful hedging tool. To this end, we introduce certain terms that can be used in dependency analysis, called the *sensitivity analysis*, of the option price on the variables. Some of the sensitivity terms are commonly denoted by Greek alphabets and hence the name *Greeks* is used in the literature. In this section, we introduce some important Greeks used in option price sensitivity analysis and discuss their usage in hedging strategies.

The Black-Scholes formulae for European put and call options are derived in Corollary 7.2.7 and Corollary 7.2.8, respectively. We can see that these prices depend on t, S, σ, r, and K. For a fixed strike K, we write the Black-Scholes formulae as

$$\begin{aligned} C(t,S,r,\sigma) &= S\Phi(d_+) - Ke^{-r(T-t)}\Phi(d_-), \\ P(t,S,r,\sigma) &= Ke^{-r(T-t)}\Phi(-d_-) - S\Phi(-d_+), \end{aligned}$$

where

$$d_\pm(t,S,r,\sigma) = \frac{\log\left(\frac{S}{K}\right) + \left(r \pm \frac{\sigma^2}{2}\right)(T-t)}{\sigma\sqrt{T-t}}.$$

In this section, we analyze the sensitivity of these prices with respect to the parameters t, S, σ, and r. More precisely, we study the fundamental properties of the partial derivatives of C and P (we use the notation V, the value of an option or a portfolio, in a general discussion not specific to call or put) with respect to the parameters occurring in the Black-Scholes formulae. The collection of all the partial derivatives is called the *Greeks*. The purpose of studying Greeks is to build hedging portfolios to reduce the sensitivity of the value of a portfolio (or an option) to changes in the underlying by means of diversifying the position.

8.2.1 Delta

Let us start our sensitivity analysis with the dependency of V with respect to the price of the underlying asset S.

The *delta* of an option or a portfolio of options is the sensitivity of the option or portfolio to the underlying and is given by

$$\Delta_V = \frac{\partial V}{\partial S}.$$

We shall now see how the knowledge of delta can be used in hedging a portfolio.

Delta Hedge

Consider the portfolio $\Pi = (0, s, h)$, where s and h are the number of units in the underlying and the option, respectively. Then, the value of the portfolio is given by

$$V = sS + hH,$$

where S is the price of the underlying and H is the price of the option.

Perfect hedging is to maintain the value V of the portfolio unchanged while the underlying price fluctuates. This is achieved by selecting s for a given h (or choosing h for a given s) such that

$$\frac{\partial V}{\partial S} = 0.$$

Let $\Delta_H = \partial H/\partial S$ be the delta of the option. Then, we have

$$s = -h\Delta_H \text{ or } h = -\frac{s}{\Delta_H}.$$

Problem 8.9.

Show that

1. the delta of a European call option is given by

$$\Delta_C := \frac{\partial C}{\partial S} = \Phi(d_+).$$

2. the delta of a European put option is given by

$$\Delta_P := \frac{\partial P}{\partial S} = -\Phi(-d_+).$$

From Problem 8.9, we can see that $\Delta_C > 0$ and $\Delta_P < 0$ (the delta for call and put options, respectively). Thus, a short position in call leads to a long spot position and so on.

A portfolio is called *delta neutral* at some time $t \in [0, T]$ if $\Delta_V = 0$ at t.

Building a delta neutral portfolio by having an opposite spot market position in the underlying of an option position is referred to as *delta-hedging* of a portfolio.

Example 8.2.1.

Consider a K-strike call (European) option whose delta is given by $\Delta_C = 0.6$. and the lot size is 100. Assume that a trader sold (wrote) 20 call options. Thus, the portfolio of the trader is $\Pi = (0, s, -2000)$ and the value of the portfolio is given by

$$V = sS - 2000 \times C.$$

To keep the portfolio delta neutral, we have to choose the number of units of the stock s such that the delta of the portfolio is zero. But the delta of the portfolio is

$$\Delta_V = s - 2000 \times 0.6.$$

Thus, in order to keep the portfolio delta neutral (or in order to delta-hedge the portfolio Π), the trader has to buy $s = 1200$ shares in the underlying.

Explanation: At time $t = 0$, let the underlying be trading at ₹100 per share and the trader sold the call options at ₹10 per share. Then the initial investment in the portfolio Π is

$$V(\Pi)(0) = 1200 \times 100 - 2000 \times 10 = 100000.$$

At time $t = \Delta t$, let the stock price goes up by ₹1.

Since $\Delta_C = 0.6$, we can say approximately that the change in the call option price

$$C(\Delta t) - C(0) = 0.6(S(\Delta t) - S(0)) = 0.6 \times 1 = 0.6.$$

Thus, the corresponding call option price is approximately $C(\Delta t) = 10.6$.

The value of the portfolio Π at time $t = \Delta t$ is given by

$$V(\Pi)(\Delta t) = 1200 \times 101 - 2000 \times 10.6 = 100000.$$

Since $V(\Pi)(0) = V(\Pi)(\Delta t)$, we see that the change in the stock price did not make any profit or loss to the trader leading to a delta neutral portfolio.

Remark 8.2.2.

Delta hedging generally works well for a short time period, *i.e.* when Δt is very small, and when the stock price variation is very small. If the time step is large or the stock price variation is very large, then one has to recompute the delta of the option (Δ_C or Δ_P depending on whether the traded option is a call or a put option, respectively) and reconstruct the delta-hedging portfolio. This process is called the *rebalancing* of the portfolio.

If a hedged position is rebalanced time to time to maintain a delta neutral portfolio, then the hedging process is called the *dynamic hedge*. On the other hand, if a hedging position is held without rebalancing until the expiry or square-off, then the hedging process is called the *static hedge*.

The information about delta of an option is (similarly all other Greeks that we discuss in the subsequent subsections are) provided by the stock exchange itself.

Example 8.2.3.

Consider two call options on the same underlying asset and expiration T:

1. $K_1 = 22,150$ with premium $C_{1,0} = 89$;
2. $K_2 = 22,250$ with premium $C_{2,0} = 48$.

Here, we used the notation $C_{1,0}$ for $C_1(t_0)$ and a similar notation for the other option.

Consider a portfolio $\Pi_1 = (s, c_1, -c_2)$ at time $t = t_0$, where s is the number of units in the underlying, and $c_i > 0$, $i = 1, 2$, is the number of units held in the K_i-strike call option.

In particular, if we take $c_1 = c_2 = c$, then the portfolio is a bull spread.

At time $t_0 < t_1 < T$, the price of K_1-strike option is $C_{1,1} = 155$ with delta $\Delta_{1,1} = 0.53$ and K_2-strike price is $C_{2,1} = 65$ with delta $\Delta_{2,1} = 0.44$.

There are two ways to achieve a delta neutral portfolio at time $t = t_1$. One is by delta hedging and another is by simply adjusting the option portfolio itself.

Delta-hedge portfolio: In this case we have to find s such that $\Delta_V = 0$. Thus, we have

$$s = -c(\Delta_{1,1} - \Delta_{2,1}) = -0.09c.$$

Option adjustment: We have $\Pi_1 = (0, 1, -1)$ created at time $t = t_0$. At time $t = t_1$, we have to make another portfolio Π_2 such that $\Delta_{V_2} = 0$. Let the portfolio Π_2 be $\Pi_2 = (0, c, c_2)$. Then, we have

$$V_2 = cC_1 - c_2 C_2.$$

Therefore, $\Delta_{V_2} = 0$ implies

$$c_2 = \frac{\Delta_{1,1}}{\Delta_{2,1}} c \approx 1.2045c.$$

We already hold c units of K_2-strike option in our portfolio. Therefore, we have to sell $c_2 - c = 0.2045c$ units of K_2-strike option additionally to make the portfolio Π_2. This results in a spread called *ratio spread*. Ratio spread is similar to bull and bear spreads but with different units in the option positions.

Note that the above adjustment is one way to achieve delta neutral portfolio. We may also adjust the K_1-strike option or we may also look for a K_3-strike option, even with a different expiration.

At time $t_0 < t_1 < t_2 < T$, let $C_{3,1} = 237$, $C_{3,2} = 129$, $\Delta_{1,2} = 0.5$ and $\Delta_{2,2} = 0.32$. Again we can adjust the K_2-strike (in particular) to achieve a delta neutral portfolio at time t_2. Hence, we made a three level dynamic delta neutral portfolio strategy $\{\Pi_1, \Pi_2, \Pi_3\}$. We can extend it to any number of time levels. Theoretically, we can even obtain a continuous time delta neutral portfolio strategy $\{\Pi_t\}$, which will obviously be a stochastic process.

8.2.2 Gamma

In the last subsection we have seen that using delta hedge we can remove the risk fully. Since delta varies with respect to time (for a sufficiently large time, for instance, on daily basis), one needs to frequently rebalance the portfolio to maintain the delta neutral portfolio. In fact, a continuous rebalancing of delta hedge may be required, which is not practically feasible. One way to improve the hedge position accuracy is to include the gamma hedge in the dynamic delta hedge strategy. Let us first introduce the formula for gamma.

The *gamma* of an option or a portfolio of options is defined as

$$\Gamma = \frac{\partial^2 V}{\partial S^2}.$$

The gamma of an European call and put options are defined similarly and are denoted by Γ_C and Γ_P, respectively.

Problem 8.10.

Using Black-Scholes formula for European call options show that

$$\Gamma_C = \frac{e^{-\frac{d_+^2}{2}}}{\sqrt{2\pi(T-t)}S\sigma}.$$

Similarly, one can obtain a formula for Γ_P. Portfolio hedging can be made more accurate by considering a *delta-gamma-hedging*. This is achieved by setting gamma equals zero along with delta-hedging. We illustrate this with an example.

Example 8.2.4.

Consider a European call options with two strikes, K_1 and K_2, with prices denoted by C_1 and C_2, respectively. The portfolio is given by $\Pi = (0, s, \boldsymbol{c})$, where $\boldsymbol{c} = (c_1, c_2)$ is the vector with components being the number of units in respectively options. The value of the portfolio is given by

$$V = sS + c_1 C_1 + c_2 C_2.$$

Differentiating with respect to S twice, we get

$$\frac{\partial V}{\partial S} = s + c_1 \Delta_{C_1} + c_2 \Delta_{C_2}$$

and

$$\frac{\partial^2 V}{\partial S^2} = c_1 \Gamma_{C_1} + c_2 \Gamma_{C_2}.$$

Equating both the above expressions to zero, we get the linear system

$$\begin{aligned} s + c_1 \Delta_{C_1} + c_2 \Delta_{C_2} &= 0 \\ c_1 \Gamma_{C_1} + c_2 \Gamma_{C_2} &= 0. \end{aligned}$$

Given the values of Δ_{C_j} and Γ_{C_j}, $j = 1, 2$, and one of s and c_j, $j = 1, 2$, we can solve the linear system (if possible) to obtain the delta-gamma-hedging portfolio.

Note

Since Γ_{C_j}, $j = 1, 2$, are positive, we need one of c_1 and c_2 to be negative in the above system to have a nonzero solution for the second equation. Hence one needs to take opposite positions in the two call options in order to obtain a delta-gamma-hedging porfolio.

8.2.3 Theta

The *theta* of an option or a portfolio of options is defined as

$$\Theta = \frac{\partial V}{\partial t}$$

The theta of a European call and put options are defined similarly and are denoted by Θ_C and Θ_P, respectively.

Problem 8.11.

Using Black-Scholes formula for European K-strike call option show that

$$\Theta_C = -\frac{S\sigma}{2\sqrt{2\pi(T-t)}}e^{-\frac{d_+^2}{2}} - rKe^{-r(T-t)}\Phi(d_-).$$

8.2.4 Vega

The *vega* of an option or a portfolio of options is defined as

$$\mathcal{V} = \frac{\partial V}{\partial \sigma}$$

The vega of a European call and put options are defined similarly and are denoted by $\mathcal{V}_C$ and $\mathcal{V}_P$, respectively.

Problem 8.12.

Using Black-Scholes formula for European call option show that

$$\mathcal{V}_C = \frac{S\sqrt{T-t}}{\sqrt{2\pi}}e^{-\frac{d_+^2}{2}}.$$

To hedge against the underlying volatility, one can make a *vega-hedging* by equating $\mathcal{V} = 0$.

8.2.5 Rho

The *rho* of an option or a portfolio of options is defined as

$$\rho = \frac{\partial V}{\partial r}$$

The rho of a European call and put options are defined similarly and are denoted by ρ_C and ρ_P, respectively.

Problem 8.13.

Using Black-Scholes formula for European K-strike call option show that

$$\rho_C = (T-t)Ke^{-r(T-t)}\Phi(d_-).$$

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