Livsförsäkringsmatematik II Projekt 2

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1 Introduction (assignment 4)

In this assignment, we have studied a life insurance savings contract, which consists of an investment in a risky-asset which, upon death, is redistributed to others owners in the portfolio. In a sense, this product is an investment account discounted towards the mortality rates, yielding benefits to the survivors. We can define the policyholder value of this contract at expiration as

$$V_i = 1_i m X(20) \frac{I_{50}}{I_{70}} \tag{1}$$

Where $m = 1$ is the initial invested amount per, 1_i the binary variable if policyholder i survives, $X(20)$ the value of the risky asset at time $t = 20$, $I_{50} = 10000$ is the known number of initial policyholders at age 50 with 1_i being the binary variable if that 1_i I_{70} an unknown random variable of the amount of surviving policyholders. We easily observe that the expected, risk-neutral value of this portfolio becomes

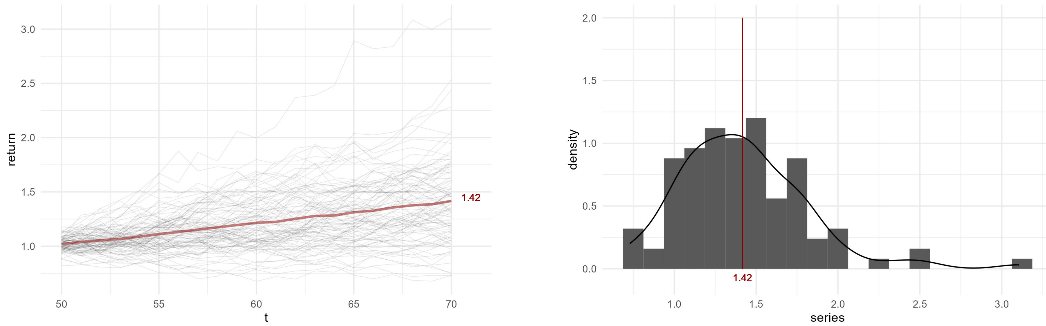
$$\begin{aligned} PV[V_i] &= e^{-20r} E[V_i] \\ &= E[1_i] e^{-20r} * m * E[X(20)] * E[\frac{I_{50}}{I_{70}}] \\ &= e^{-20r} e^{20r} * m * {}_{20}p_{50} \frac{1}{{}_{20}p_{50}} \\ &= m = 1. \end{aligned}$$

The assignment is not to calculate the expected value of this insurance contract but to simulate and observe the effect different fees has on the portfolio. This to observe the underlying variance and how we as policy issuers may want to set fees based upon such risk. In doing so, we have to simulate the distribution of the risky asset and the probability of survival for every year $t = 1, 2, .., 20$ and analysed how these compare to the risk-neutral expectation.

For the risky asset, the simulation is a standard stochastic, wiener process, and we simulate the asset value as $X_0^{sim} = 1$

$$X_t^{sim} = X_{t-1}^{sim} * exp((r - \sigma^2/2) * t) + \sigma * W_i^Q \tag{2}$$

where we draw random normal sample $W_i^Q = N(0, \sqrt{t})$ which constitutes the simulation. In figure 1, we see the distribution of these simulations and also a histogram of the final distribution of $X^{sim}(20)$ with the red line denoting the average distribution.



(a) Simulation of asset evolution. (b) Distribution of final asset value.

Figur 1: Distribution of price and portfolio simulation based on 10000 simulations.

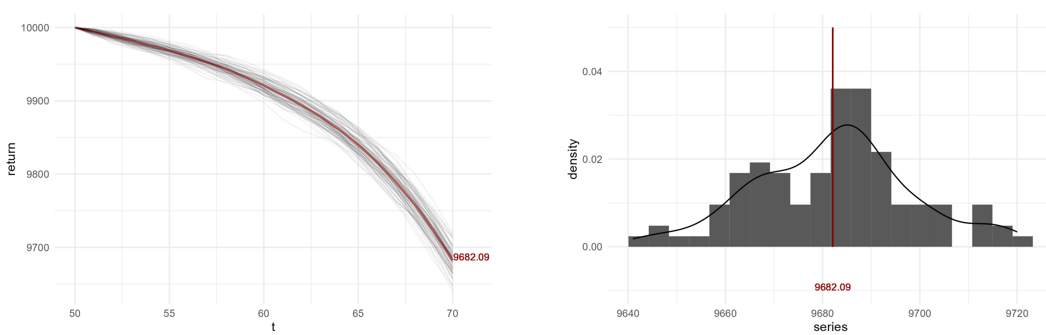
In terms of survival, we use the gombertz-makeham survival function to calculate the probability of an individual surviving a year as

$$p_x = exp\left(- \int_x^{x+1} \mu(u)du \right) = exp\left(- [ax + \frac{b}{c}e^{cx}]_x^{x+1} \right) \tag{3}$$

In evaluating this probability, we will use the mortality function given, then draw a random samples from the Bernoulli distribution $Bernoulli(p_x)$ if an individual policyholder will survive the current year x. On a portfolio level, taking repeated draws from the Bernoulli distribution, this becomes binomial, and we have from the initial known amount of policyholders $S_0^{sim} = I_{50} = 10000$

$$S_t^{sim} \sim B(S_{t-1}^{sim}, p_x) \tag{4}$$

Recursively drawing from this distribution, we can simulate the number of survivors for each timestep $t = 1, 2, .., 20$. In figure 2, 100 of these simulations are shown, and on average (in red), we find that the expected number of survivors is 9928.09 which equals $I_{50\,20}p_{50}$.



(a) Simulation of policyholder survival. (b) Distribution of survival at $t = 20$.

Figur 2: Distribution of price and portfolio simulation based on 10000 simulations.

Joining these two simulations together, we can simulate the portfolio value for a survivor by

$$V_t^{sim} = m * X_t^{sim} * \frac{I_{50}}{S_t^{sim}} \tag{5}$$

This is without any fee added, and we will, through this project, study 3 different types of portfolio fees. In conducting the assignment, we will show how we simulate each type of fee and show the evolution of the portfolio for an individual survivor.

In the concluding remarks calculate the risk-neutral value based on the expected value at $t = 0$ as

$$PV[V_t^{sim}] = e^{-20r} {}_{20}p_{50} V_t^{sim} \tag{6}$$

which will yield us the risk-neutral value based on the initial expected return of the portfolio.

1.1 Alternative approach

In this solution we simulate the portfolio on a yearly time-step. One could as well solve the problem, by continuously integrating from age 50 to 70 and solve as

$$\int_{50}^{70} \frac{e^{-r(t-t_0)} X(t)}{Bin(I_{50+t}^{sim}, \mu(t))} \, dt \tag{7}$$

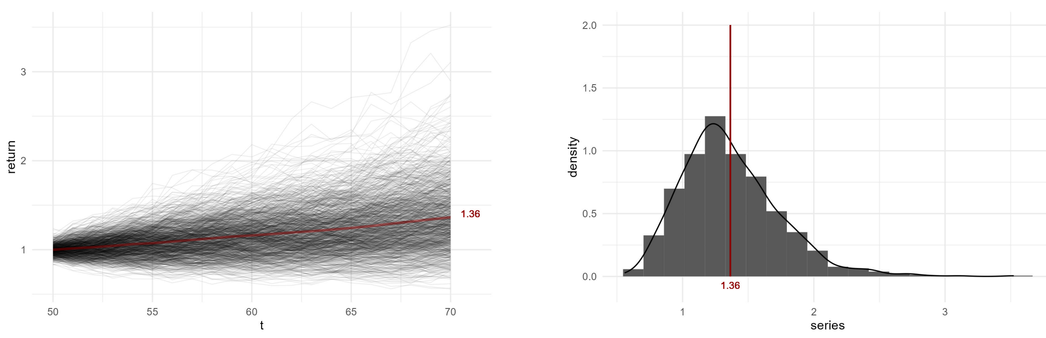
Observe that the binomial function in this case is stochastic and continuous in time for every time-step. This was an other approach we tried, but due to complexity of simulating survival continuously, and numerical instability, we went for the yearly time-step simulations instead.

2 Assignment 1

In assignment 1, we finance the insurance by a 0.1% yearly fee on the invested capital. As the invested capital is redistributed upon death, the yearly fee can be seen as an annual 0.1% fee on the total portfolio level. As such, at every time-step t, a value on the portfolio of

$$V_t^{sim} = m * X_t^{sim} * (1-f)^t * \frac{I_{50}}{S_t^{sim}} \tag{8}$$

where f denotes the yearly fee of the invested capital. Simulating the insurance contract with this type of fee, we get the distributions shown in figure 3. From a policyholder's perspective, these yearly fees don't affect the end result of the portfolio, as it is only payable upon the contract end date.



(a) Policy-holder portfolio value. (b) Distribution of fair market price.

Figur 3: Distribution of price and portfolio simulation based on 10000 simulations.

3 Assignment 2

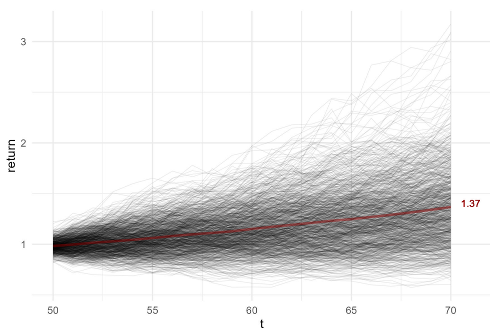
From assignment 1, we take out an annual 0.1% fee of the whole portfolio and invest it in the risky asset. In effect, this yearly fee does not affect the portfolio allocation, we still have the same amount of money invested in the risky asset. As zero assets are liquidated until expiration, the fees do nothing effectively and could be replaced with a single fee at the end of expiration. We can calculate this fee as the cumulative product of all fees as

$$f^* = 1 - (1 - f)^{20} = 1 - 0.9802 = 0.0198 \tag{9}$$

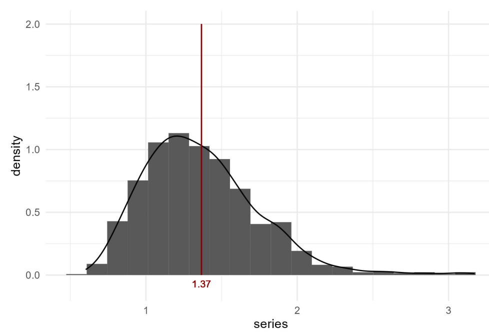
There f is the fee from assignment 1 and f^* is the initial fee. If we take this fee at expiration, the we as policyholder take on asset risk exposure, but if we take it initially, we no longer take on asset risk and have a guaranteed amount from our policy. In the end, we attain the following formula for simulating the policy value for a policy-holder

$$V_t^{sim} = m * (1 - f^*) * X_t^{sim} * \frac{I_{50}}{S_t^{sim}} \tag{10}$$

Simulating this contract with the initial fee $f^* = 0.0198$ we get a portfolio evolution shown in figure 4. As can be seen, the overall distributions look the same, and from a policyholder's perspective, the present value and expiration value of the policy are largely the same. The difference is seen in the initial part of figure 4a, where we observe that the policyholder value starts at $1 - f^* = 0.98$, which is recovered as no fee is taken through the period insured.



(a) Policy-holder portfolio value.



(b) Distribution of fair market price.

Figur 4: Distribution of a survivors portfolio evolution and final value based on 10000 simulations.

Overall, this contract is largely the same, and from a policyholder perspective, there is no difference. But from an insurer's perspective, it is no longer connected with the asset risk of the investment. This can be an attractive property as financial institutes prefer to work guaranteed returns, leaving the risk to speculators which in this case is policy holders.

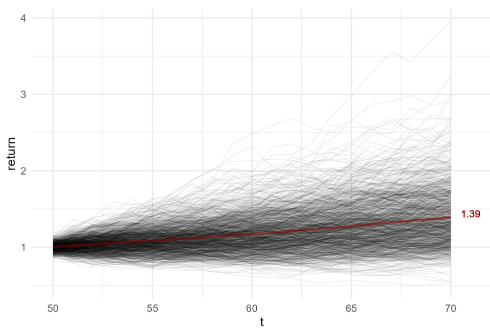
4 Assignment 3

In assignment 3 we are testing a different type of fee. A 20% inheritance fee being paid to the policy issuer upon the death of the policyholder. In effect, we are now taking on the risk of mortality instead of financial risk and simulate the portfolio level for an individual policyholder as:

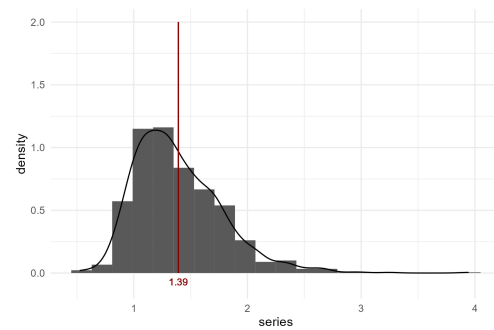
$$V_t^{sim} = m * X_t^{sim} * \frac{I_{50}}{S_t^{sim}}(1 - (1 - \frac{S_t^{sim}}{I_{50}})f). \tag{11}$$

This formula is more advanced, but easier to understand as the original contract times a fee based on the amount of mortalities. If $f = 0$ we easily observe that this is a non-fee type contract and for $f = 1$ we observe that the this just becomes a financial investment without any wealth redistribution. In this assignment we have used $f = 0.2$

In figure 5, we see the policy value evaluation for a single policyholder and the distribution of the present risk-neutral value of the policy. As can be seen, the net value is higher than previous and



(a) Policy-holder portfolio value.



(b) Distribution of fair market price.

Figur 5: Distribution of price and portfolio simulation based on 10000 simulations.

in comparison to the previous assignments, but otherwise, the distribution looks overall similar to the above.

As can be seen the final value of the contract is higher compared to previous contracts. This comes as a consequence of the net fee payment is larger compared to the inheritance fee based one. One could calculate the fee from 20% to something lower to account for this which will be done in the end of the concluding remarks. Anyhow, from a policyholder perspective this contract is less valuable, but from a policy issuer, it is a more profitable with the final distribution largely the same, as the main source of variance is the risky asset.

5 Discussion and concluding remarks

In this assignment, we have studied an insurance contract with 3 different fee structures. We observe that the different fee structures associate different types of risk to the policy issuer:

1. Risk associated with the risky asset.
2. Risk based upon the survival rate of the policy-holders.

In table 1, we compare the end results towards the risk neutral expectations based on the results from our assignment. Overall, we observe large variance in our risk-neutral value with the expected present value from a policyholder is largely the same, all lower than the expected value of 1, because of the fee introduced. As a quick conclusion we can say that there is a lot of risk associated in this investment, something which is not observed when taking the mean present value.

Going into more detail, the issuer is exposed to asset risk in the yearly fee, having collected the fee based on portfolio over the duration causing high variance in the returns of the product. In the initial fee, there is no risk exposure, the policyholder only takes a cut of the initial invested capital. and as such we have no variance from a issuer perpective. And lastly, In the inheritance-based fee, it takes on risk based on mortality. Here, the contract is more profitable for the issuer if mortality rate increases. Again we see high variance but also a lower mean value as the fees differ.

Fee type	Present Policyholder Value	Present Policy Issuer Value
Yearly Fee of 0.1%	0.98 ± 0.53	197.69 ± 106.41
Initial Fee of 1.98%	0.98 ± 0.54	198 ± 0
Inheritance Fee of 20%	0.99 ± 0.54	63.98 ± 35.99

Tabell 1: Mean ±95% standard deviation quantile of the risk-neutral final policyholder and Policy Issuer Value.

5.1 Calculating an equal inheritence fee

Do observe that an inheritance fee-based contract has a lower risk-neutra value for the issuer due to the difference in fees. This decrease in the issuer's risk-neutral value is instead migrated to the policyholders, who see an increase in their present value. To make these contracts have a equal distribution, we can change the fee based on equation 10 and 11 and solve for f:

$$m * (1 - f^*) * X(20) * \frac{I_{50}}{I_{70}} = m * X(20) * \frac{I_{50}}{I_{70}}(1 - (1 - \frac{I_{70}}{I_{50}})f) \tag{12}$$

Which leads to the solution

$$f = \frac{f^*}{1 - \frac{I_{70}}{I_{50}}} \tag{13}$$

Which if taken the expected value we have $f = (1 - {}_{20}p_{50})$

$$f = \frac{0.0198}{1 - {}_{20}p_{50}} = 0.623 \tag{14}$$

Meaning that we would require a 62% tax on inheritance to expect to break even with the fee based policies. Observe that this has other considerations outside expectations, one would also be interested to see what variance this outcome would have and what risk profile we are setting.

5.2 Final remarks

In the end, we have 3 different tools to build a risk profile on the risky asset and policyholder mortality. Likely, we can combine these fee-structures to build our unique investment opportunity based on our investment thesis and risk-appetite. Being profitable upon policyholder mortality may be a morally questionable discussion for an insurance company, but the same can be said for the policyholder investing in this product. This assignment proves that even if the contract can be largely similar from a policyholder perspective, the risk we take one, and the volatility we may observe as an issuer, can be vastly different.

A Code

Code can also be found on our github repo our github repo

```
## Assignment 2 Livsf rs rking

# Load packages
library(tidyverse)
library(dplyr)
library(ggplot2)
library(reshape2)


#### Parameters

# Set Seed
set.seed(1234)

# Makeham function
a <- 3.5*10^(-4)
b <- 7*10^(-8)
c <- 0.157
makeham <- function(x){
  a + b * exp(c * x)
}

# Parameters for simulation
t_0 = 50
tau = 70
t_step <- 1
n <- 10000
n_simulations <- 1000
time_steps <- seq(0, tau-t_0, t_step)
z <- qnorm(0.975)

# Wiener process brownian motion
r <- 0.015
sigma <- 0.06

# Helper function to plot simulated brownian motions
plot_simulations <- function(simulations){
    Xs <- data.frame(simulations)
    Xs$t <- time_steps+t_0
    X_melt <- melt(Xs, id.vars="t")
    colnames(X_melt) <- c("t", "run", "return")
    mean_res <- mean(simulations[dim(simulations)[1],])

    p <- ggplot(data=X_melt,aes(x=t, y=return, group=run)) +
        theme_minimal() +
        geom_line(size=0.2, alpha=0.1,color='black')+
        stat_summary(fun.y=mean,geom="line",lwd=1,color='darkred',alpha=0.5,aes(group=1))+
        geom_text(aes(label=round(mean_res,2),y=mean_res,x=tau+1),
            vjust=0,col='darkred',size=3)

    return(p)
}

plot_histogram <- function(series){
    p  <- ggplot(data.frame(series), aes(x=series,y=..density..)) +
        geom_histogram(bins=20) +
        geom_density(alpha=.2)+
        theme_minimal()+
        geom_segment(aes(x = mean(series),xend=mean(series), y = 0, yend = 2),col='darkred',size=0.5)+
        geom_text(aes(label=round(mean(series),2),y=-0.1,x=mean(series)),
            vjust=0,col='darkred',size=3)
    return(p)
}


###### Simulate price development of the risky asset X(t)

sim_X <- function(){
    W <- rnorm(length(time_steps), mean = 0, sd = 1)
    W_cum <- cumsum(W)
    X <- exp((r - sigma^2/2)*time_steps+ sigma * W_cum)
    return(X)
}


####### Simulate survival percentage
```

```
p_t <- function(t_0,t){
  exp( - (a*(t-t_0)+b/c*(exp(c*t)-exp(c*t_0))))
}

sim_S <- function(){
    S <- rep(n,length(time_steps))
    for (i in 2:length(time_steps)){
        p <- p_t(t_0+time_steps[i-1],t_0+time_steps[i])
        S[i] <- round(rbinom(1,S[i-1],p))
    }
    return(S/n)
}

############ A.1
annual_fee <- 0.001
X_sim <-  replicate(n_simulations,sim_X())
S_sim <- replicate(n_simulations,sim_S())
portfolio_value <- X_sim*(1 -annual_fee)^(time_steps)/S_sim

# Plot simulations
p <- plot_simulations(portfolio_value)
ggsave("plots/a1_sim.jpg", p, width = 15, height = 10, units = "cm")

# Plot distribution of risk neutral value
p <- plot_histogram(portfolio_value[length(time_steps),])
ggsave("plots/a1_price.jpg", p, width = 15, height = 10, units = "cm")

# Calculate Risk neutral Value at t=20
risk_neutral_value <- portfolio_value[length(time_steps),]*exp(-r*(length(time_steps)-1))*p_t(50,70)
mean(risk_neutral_value)
z*sd(risk_neutral_value)

acc_fee = 1-(1 -annual_fee)^(tau-t_0)
policy_issuer_value = n*acc_fee*X_sim[length(time_steps),]*exp(-r*(length(time_steps)-1))
mean(policy_issuer_value)
z*sd(policy_issuer_value)

################ A.2
initial_fee <- 1-(1 -annual_fee)^(tau-t_0)
X_sim <-  replicate(n_simulations,sim_X())
S_sim <- replicate(n_simulations,sim_S())
portfolio_value <- X_sim*(1-initial_fee)/S_sim

p <- plot_simulations(portfolio_value)
ggsave("plots/a2_sim.jpg", p, width = 15, height = 10, units = "cm")

p <- plot_histogram(portfolio_value[length(time_steps),])
ggsave("plots/a2_price.jpg", p, width = 15, height = 10, units = "cm")

# Calculate Risk neutral Value at t=20
risk_neutral_value <- portfolio_value[length(time_steps),]*exp(-r*(length(time_steps)-1))*p_t(50,70)
mean(risk_neutral_value)
z*sd(risk_neutral_value)

############## A.3
inheritence_tax <- 0.20
X_sim <-  replicate(n_simulations,sim_X())
S_sim <- replicate(n_simulations,sim_S())
portfolio_value <- X_sim*(1 - (1-S_sim)*inheritence_tax)/S_sim

p <- plot_simulations(portfolio_value)
ggsave("plots/a3_sim.jpg", p, width = 15, height = 10, units = "cm")

p <- plot_histogram(portfolio_value[length(time_steps),])
ggsave("plots/a3_price.jpg", p, width = 15, height = 10, units = "cm")

# Calculate Risk neutral Value at t=20
risk_neutral_value <- portfolio_value[length(time_steps),]*exp(-r*(length(time_steps)-1))*p_t(50,70)
mean(risk_neutral_value)
z*sd(risk_neutral_value)

X_sim_20 <- X_sim[length(time_steps),]
S_sim_20 <- S_sim[length(time_steps),]
policy_issuer_value = n*X_sim_20*(1-S_sim_20)*inheritence_tax*exp(-r*(length(time_steps)-1))
mean(policy_issuer_value)
z*sd(policy_issuer_value)

############## A.4
# 1 Simulate 100 times
S <- replicate(100, sim_S())*n
```

```
p <- plot_simulations(S)
# Save plot
ggsave("plots/a4_survival_simulation.jpg", p, width = 15, height = 10, units = "cm")
p <- plot_histogram(S[21,])
ggsave("plots/a4_survival_distribution.jpg", p, width = 15, height = 10, units = "cm")


#2 Simulate 100 times ant plot
Xs <- replicate(100, sim_X())
p <- plot_simulations(Xs)
# Save plot
ggsave("plots/a4_ass_simulation.jpg", p, width = 15, height = 10, units = "cm")
p <- plot_histogram(Xs[21,])
ggsave("plots/a4_ass_distribution.jpg", p, width = 15, height = 10, units = "cm")

# Calculate New Inheritence fee to match other fee structures
f_star = 0.0198/(1-p_t(50,70))
f_star
```