The Black-Scholes model

1 Model theory

To date, the Black-Scholes model is perhaps the most popular tool for financial derivative valuation. It is a continuous-time model for the evolution of asset prices. Here we will discuss the main assumptions to the model, treat simulation techniques and derive the famous Black-Scholes formula for European options.

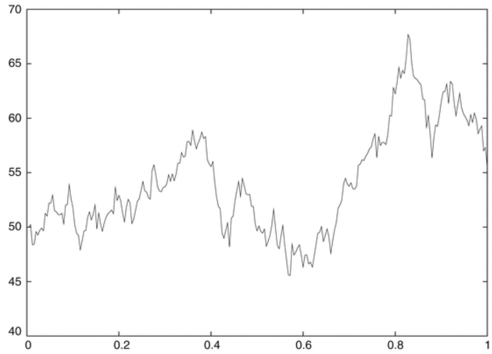

Figure 1: Stock price movements simulated by a geometric Brownian motion. Source: [1].

Although it is more realistic than the binomial tree, the Black-Scholes model is still a simplified representation of the market. Let again S denote the value of a stock. The main concept here is that the risk-neutral dynamics of the stock price are captured by a *geometric Brownian motion*. This is a stochastic process that is described by the following SDE.

$$dS_t = rS_t dt + \sigma S_t dz_t \tag{1}$$

The drift of the process is constant and equal to the risk-free rate r. The diffusion coëfficient is also constant and referred to as the volatility σ. The process z_t is a *standard Brownian motion* or *Wiener process*. Below we summarize the main assumptions of this model:

1. Cash invested in the money-market yields a continuously compounded interest at a constant rate r.
2. The stock price S_t follows a geometric Brownian motion, with constant drift and volatility.
3. The economy is free of arbitrage.
4. There are no transaction costs for trading shares of stock.

1.1 Derivation Black-Scholes formula

In this section we will derive the analytical formula to price a European call option written on an asset in the Black-Scholes model. This formula is traditionally known as the Black-Scholes formula. Recall that a call option is a contract which gives the holder the right, but not the obligation, to buy an underlying asset S at a specified strike price K at a specified future date T. The pay-off at maturity T of this security is therefore equal to

$$V(T) = (S_T - K)^+ := \max\{S_T - K, 0\}$$

Let $0 \leq t < T$ be any time before maturity. In accordance with the risk-neutral valuation principle, the value of the contract at time t is equal to the expected value of the pay-off, discounted with the risk-free interest rate. This is given by (compare to the binomial tree)

$$V(t) = \mathbb{E}\left[e^{-r(T-t)}V(T)\right] = e^{-r(T-t)}\mathbb{E}\left[(S_T - K)^+\right]$$

We will analytically compute $V(t)$ by first deriving the probability density of S_T and secondly evaluating the expectation above.

1.1.1 The distribution of S_T

The main assumption is that the underlying asset is modelled as a geometric Brownian motion. This means that the risk-neutral dynamics of S_t are given by the following stochastic process:

$$dS_t = rS_t dt + \sigma S_t dz_t$$

where r is the constant risk-free rate, σ the constant volatility and z_t a Brownian motion. We can find an exact expression for S_t through an application of Itô's lemma. Itô's lemma says that if a variable x follows a stochastic process of the form

$$dx = a(x,t)dt + b(x,t)dz_t$$

then any smooth function $G(x,t)$ follows the process

$$dG = \left(\frac{\partial G}{\partial x}a(x,t) + \frac{\partial G}{\partial t} + \frac{1}{2}\frac{\partial^2 G}{\partial x^2}b^2(x,t)\right)dt + \frac{\partial G}{\partial x}b(x,t)dz_t$$

Now let $G(S,t) = \log(S)$. Then we have

$$\frac{\partial G}{\partial S} = \frac{1}{S}, \qquad \frac{\partial^2 G}{\partial S^2} = -\frac{1}{S^2}, \qquad \frac{\partial G}{\partial t} = 0$$

and according to Itô's lemma it follows

$$d\log(S_t) = \left(r - \frac{1}{2}\sigma^2\right)dt + \sigma dz_t$$

Note that on the right hand side, the unknown, stochastic S_t has dropped out of the expression. Now we can simply integrate both sides, which would not have been possible if S_t appeared in the integrand.

$$\begin{aligned}\int_t^T d\log(S_u) &= \int_t^T \left(r - \frac{1}{2}\sigma^2\right)du + \int_t^T \sigma dz_u \\ \log(S_T) - \log(S_t) &= \left(r - \frac{1}{2}\sigma^2\right)(T-t) + \sigma(z_T - z_t)\end{aligned}$$

A property of Brownian motion is that any increment is normally distributed as $z_T - z_t \sim \mathcal{N}(0, T-t)$. Hence, if we let Z denote a standard normal random variable, we can write $z_T - z_t \simeq \sqrt{T-t}Z$. We finalise by taking the exponential on both sides of the equation above.

$$\frac{S_T}{S_t} = \exp\left\{\left(r - \frac{1}{2}\sigma^2\right)(T-t) + \sigma\sqrt{T-t}Z\right\}$$

It can be concluded that $\frac{S_T}{S_t}$ has a *lognormal distribution.*

1.1.2 Compute the expectation

Now that we have the distribution of S_T, we can proceed by analytically computing the expectation in the expression for $V(t)$. For convenience, define $\tau := T - t$. Also, in the expression for S_T we will write $-Z$ instead of Z, which by its symmetry is equivalent, but will make the computations below easier. Substitution of S_T yields

$$\begin{aligned} V(t) &= e^{-r\tau}\mathbb{E}\left[(S_T - K)^+\right] \\ &= e^{-r\tau}\mathbb{E}\left[\left(S_t \exp\left\{\left(r - \frac{1}{2}\sigma^2\right)\tau - \sigma\sqrt{\tau}Z\right\} - K\right)^+\right] \\ &= \frac{e^{-r\tau}}{\sqrt{2\pi}}\int_{-\infty}^{\infty}\left(S_t \exp\left\{\left(r - \frac{1}{2}\sigma^2\right)\tau - \sigma\sqrt{\tau}x\right\} - K\right)^+ e^{-\frac{1}{2}x^2}dx \end{aligned}$$

The integrand of the integral above is only non-zero if $S_t \exp\left\{\left(r - \frac{1}{2}\sigma^2\right)\tau - \sigma\sqrt{\tau}x\right\} - K > 0$. This is exactly the case whenever for the integration-variable x we have

$$x < \frac{\log\left(\frac{S_t}{K}\right) + \left(r - \frac{1}{2}\sigma^2\right)\tau}{\sigma\sqrt{\tau}} := d_2$$

It follows that

$$\begin{aligned} V(t) &= \frac{e^{-r\tau}}{\sqrt{2\pi}}\int_{-\infty}^{d_2}\left(S_t \exp\left\{\left(r - \frac{1}{2}\sigma^2\right)\tau - \sigma\sqrt{\tau}x\right\} - K\right) e^{-\frac{1}{2}x^2}dx \\ &= \frac{1}{\sqrt{2\pi}}\int_{-\infty}^{d_2} e^{-r\tau}S_t \exp\left\{\left(r - \frac{1}{2}\sigma^2\right)\tau - \sigma\sqrt{\tau}x\right\} e^{-\frac{1}{2}x^2}dx - \frac{1}{\sqrt{2\pi}}\int_{-\infty}^{d_2} e^{-r\tau}Ke^{-\frac{1}{2}x^2}dx \end{aligned}$$

Let $N(x) := \frac{1}{\sqrt{2\pi}}\int_{-\infty}^{x} e^{-\frac{1}{2}u^2}du$ denote the standard normal cumulative distribution function. Then the integral on the right can easily be rewritten as

$$\frac{1}{\sqrt{2\pi}}\int_{-\infty}^{d_2} e^{-r\tau}Ke^{-\frac{1}{2}x^2}dx = e^{-r\tau}KN\left(d_2\right)$$

The integral on the left is trickier and requires a variable substitution. Define $y = x + \sigma\sqrt{\tau}$, then the integral can be rewritten as

$$\begin{aligned} &\frac{1}{\sqrt{2\pi}}\int_{-\infty}^{d_2} e^{-r\tau}S_t \exp\left\{\left(r - \frac{1}{2}\sigma^2\right)\tau - \sigma\sqrt{\tau}x\right\} e^{-\frac{1}{2}x^2}dx \\ &= \frac{S_t}{\sqrt{2\pi}}\int_{-\infty}^{d_2} \exp\left\{-\frac{1}{2}\left(x + \sigma\sqrt{\tau}\right)^2\right\}dx \\ &= \frac{S_t}{\sqrt{2\pi}}\int_{-\infty}^{d_2+\sigma\sqrt{\tau}} \exp\left\{-\frac{1}{2}y^2\right\}dy = S_tN\left(d_1\right) \end{aligned}$$

where we inherently defined $d_1 := d_2 + \sigma\sqrt{\tau}$. Our final result is what is known as the Black-Scholes formula for a call option:

$$V(t) = S_tN\left(d_1\right) - e^{-r\tau}KN\left(d_2\right) \tag{2}$$

$$d_1 = \frac{\log\left(\frac{S_t}{K}\right) + \left(r + \frac{1}{2}\sigma^2\right)\tau}{\sigma\sqrt{\tau}} \tag{3}$$

$$d_2 = d_1 - \sigma\sqrt{\tau} \tag{4}$$

1.2 Simulation techniques

In this section we treat simulation techniques for Itô processes. Itô processes form a class of stochastic processes that are commonly used to describe phenomena in physiscs, biology and finance which are subject to randomness. The dynamics of an Itô process X_t are characterized by an SDE of the form

$$dX_t = a(t, X_t)dt + b(t, X_t)dz_t \tag{5}$$

where z_t denotes a standard Brownian motion. A geometric Brownian motion as in eq. (1) is therefore an example of an Itô process, with $a(t, X_t) = rX_t$ and $b(t, X_t) = \sigma X_t$. In integral form, an Itô process can be expressed as

$$X_t = X_0 + \int_0^t a(u, X_u)du + \int_0^t b(u, X_u)dz_u \tag{6}$$

The question is now: how do you simulate trajectories of the process above. In the following sections we provide two approaches.

1.2.1 Exact sampling of an SDE

An explicit solution to the integral of eq. (6) is not always known. If however a solution is available, then we can directly sample trajectories of the Itô process, because its distribution is known. In the specific case of a geometric Brownian motion (GBM), there is an exact solution to the SDE. We derived it in section 1.1.1, where we showed that

$$S_T = S_t \exp\left\{\left(r - \frac{1}{2}\sigma^2\right)(T - t) + \sigma(z_T - z_t)\right\}$$

The increments $z_T - z_t$ of a standard Brownian motion are independent, Gaussian random variables with distribution $z_T - z_t \sim \mathcal{N}(0, T - t)$. The process can hence be simulated as described in Algorithm 1.

Algorithm 1: Exact simulation GBM

Select $M \in \mathbb{N}$, set $\Delta t = \frac{T}{M}$;
Initialize S_0, the stock price today;
for $m = 1, \ldots, M$ **do**
  Sample $Z_m \sim \mathcal{N}(0, 1)$;
  $S_{m \cdot \Delta t} = S_{(m-1)\cdot\Delta t} \exp\left\{\left(r - \frac{1}{2}\sigma^2\right)\Delta t + \sigma\sqrt{\Delta t}Z_m\right\}$
end

1.2.2 Discrete Euler method for an SDE

In case an explicit solution to the integral of eq. (6) is not available, one has to settle with an approximation. Perhaps the most intuitive approach for an Itô process is the Euler method. The main idea is that the SDE is discretized. A trajectory of the process can then be approximated on a dicretized time-grid. A discretization of the Itô SDE in eq. (5) is given by

$$\Delta X_t = a(t, X_t)\Delta t + b(t, X_t)\Delta z_t$$

As a property of the Brownian motion, we know that $\Delta z_t := z_{t+\Delta t} - z_t \sim \mathcal{N}(0, \Delta t)$. The accuracy of the Euler discretization depends on the step-size Δt. A smaller Δt implies a higher accuracy, but also requires a larger number of simulation steps. The stock-price process can hence by approximation be simulated as described in Algorithm 2.

Algorithm 2: Approximate simulation GBM with Euler method

Select $M \in \mathbb{N}$, set $\Delta t = \frac{T}{M}$;
Initialize S_0, the stock price today;
for $m = 1, \ldots, M$ **do**
  Sample $Z_m \sim \mathcal{N}(0, 1)$;
  $S_{m\cdot\Delta t} = S_{(m-1)\cdot\Delta t} + rS_{(m-1)\cdot\Delta t}\Delta t + \sigma S_{(m-1)\cdot\Delta t}\sqrt{\Delta t}Z_m$
end

Figure 2 shows a trajectory of a geometric Brownian motion. One line represents the path that is simulated with the exact method. The other two are approximations based on the Euler method, using 100 and 25 steps respectively.

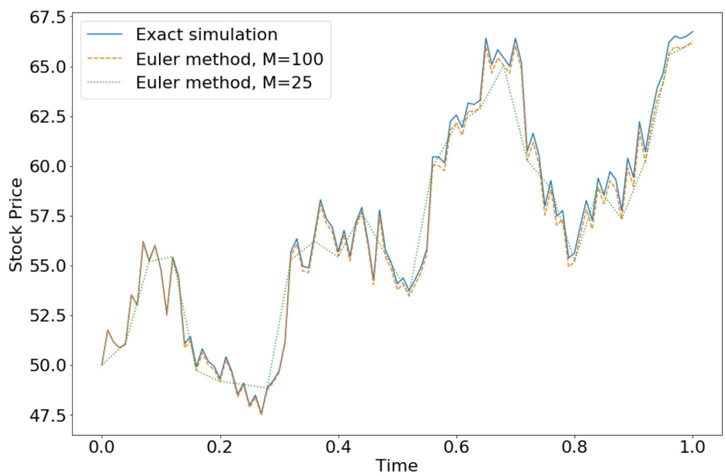

Figure 2: Simulation of a geometric Brownian motion, exact and using the Euler method.

References

[1] R. Seydel and R. Seydel, *Tools for computational finance*, vol. 3. Springer, 2006.