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The area of the Mandelbrot set

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Summary. We obtain upper bounds for the area of the Mandelbrot set. An effective procedure is given for computing the coefficients of the conformal mapping from the exterior of the unit circle onto the exterior of the Mandelbrot set. The upper bound is obtained by computing finitely many of these coefficients and applying Green's Theorem. The error in such calculations is estimated by deriving explicit formulas for infinitely many of the coefficients and comparing.

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1 Introduction

The Mandelbrot set M arises in studying the dynamics of the complex quadratic polynomials $q_w(z)=z^2+w$. For each fixed w, the (filled in) Julia set of $q_w(z)$ consists of those values z that remain bounded under iteration; that is, the process

$$z \mapsto z^2+w \mapsto (z^2+w)^2+w \mapsto \dots$$

is bounded. The *Mandelbrot set M* consists of those parameter values w for which the Julia set is connected. It is well known that M is a compact and hence measurable subset of the plane. We are interested in finding good approximations to the area of M.

The topological structure of M is studied by considering its complement. A fundamental result of Douady and Hubbard [1] shows that the complement $\tilde{M}$ of M in the Riemann sphere is simply connected with mapping radius 1; that is, there is an analytic homeomorphism

$$\psi(z)=z+\sum_{m=0}^{\infty} b_m z^{-m} \tag{1}$$

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of $\Delta=\{z: 1<|z|\leqq\infty\}$ onto $\tilde{M}$. It is a classical result of Gronwall [3] that the area of M is determined by the coefficients b_m as

(2) $$A=\pi\left[1-\sum_{m=1}^{\infty} m|b_m|^2\right].$$

Hence, for each positive integer N, one obtains the upper bound

(2′) $$A\leqq A_N:=\pi\left[1-\sum_{m=1}^{N} m|b_m|^2\right]$$

for the area of the Mandelbrot set.

There is another more naïve way to approximate the area of M. One knows that points w in M are precisely the parameter values for which 0 remains bounded under iteration of $q_w(z)$. Moreover, all points outside the disk of radius 2 are unbounded under iteration. This provides a simple test to determine (approximately) whether a point is in M. Using this, we can subdivide a suitable region containing M and test the centers of a rectangular grid. This approximation by pixels is, of course, the way one draws computer pictures of M.

In this note, we provide an effective algorithm for computing the coefficients b_m and hence for obtaining upper bounds on the area of the Mandelbrot set. Because the upper bounds are surprisingly different from the naïve estimates obtained by pixel counting, we also provide estimates of the error in our calculations. Such estimates are obtained both by using an explicit formula for certain values of b_m, which we then compare to our computed values, and by heuristic arguments. They indicate that the algorithm is remarkably stable and our calculated values for the b_m are quite accurate. We discuss the consequences of our calculations at the end of Sect. 6.

2 Procedure for determining coefficients

Define recursively the polynomials $p_0(w)=w$ and

(3) $$p_n(w)=p_{n-1}(w)^2+w$$

for $n\geqq 1$. For $n\geqq 1$, p_n is the Faber polynomial of degree 2^n of the Mandelbrot set M (see [2]). This means that $p_n(\psi(z))=z^{2^n}+o(1)$ as $z\to\infty$. In other words, we may expand in a fashion convenient for our purposes

(4) $$p_n(\psi(z))=\sum_{m=0}^{\infty} \beta_{nm} z^{2^n-m} \quad \text{for } |z|>1$$

where $\beta_{nm}=0$ for $1\leqq m\leqq 2^n$ and $n\geqq 1$. In addition, the identity $p_0(\psi(z))=\psi(z)$ implies that $\beta_{0m}=b_{m-1}$, $m\geqq 1$. In any case, we have $\beta_{n0}=1$. (The β_{nm} are related to the Grunsky coefficients of ψ.) The recursion (3) for $p_n(\psi(z))$ may be written as

$$(5)\quad \sum_{m=0}^{\infty} \beta_{nm} z^{2^n-m} = \sum_{m=0}^{\infty} \sum_{k=0}^{m} \beta_{n-1,k}\beta_{n-1,m-k} z^{2^n-m} + \sum_{m=2^n-1}^{\infty} \beta_{0,m-2^n+1} z^{2^n-m}.$$

For $1 \leqq m \leqq 2^n - 2$, this implies

$$\beta_{nm} = 2\beta_{n-1,m} + \sum_{k=1}^{m-1} \beta_{n-1,k}\beta_{n-1,m-k}.$$

Since $\beta_{nm}=0$ for $1 \leqq m \leqq 2^n$, it follows that the left side and the sum on the right are zero. Therefore $\beta_{n-1,m}$ is zero for $1 \leqq m \leqq 2^n - 2$ and $n \geqq 2$. In other words, we have

$$(6)\qquad \beta_{nm}=0 \quad \text{for } 1 \leqq m \leqq 2^{n+1}-2 \quad \text{and} \quad n \geqq 1.$$

Thus we have extended the range in which we know the β_{nm} are zero. In terms of the Faber polynomials, it says that $p_n(\psi(z)) = z^{2^n} + O(1/z^{2^n-1})$ as $z \to \infty$.

For $m \geqq 2^n - 1$, formula (5) implies that

$$\beta_{nm} = \sum_{k=0}^{m} \beta_{n-1,k}\beta_{n-1,m-k} + \beta_{0,m-2^n+1},$$

and using (6) and $\beta_{n-1,0}=1$, we may rewrite it as

$$\beta_{nm} = 2\beta_{n-1,m} + \sum_{k=2^n-1}^{m-2^n+1} \beta_{n-1,k}\beta_{n-1,m-k} + \beta_{0,m-2^n+1} \quad (m \geqq 2^n-1,\ n \geqq 1).$$

This is the forward recursion, which determines β_{nm} in terms of β_{jk} with $j<n$, $k \leqq m$. There is a corresponding backward recursion formula

$$(7)\quad \beta_{n-1,m} = \tfrac{1}{2}\left[\beta_{nm} - \sum_{k=2^n-1}^{m-2^n+1} \beta_{n-1,k}\beta_{n-1,m-k} - \beta_{0,m-2^n+1}\right] (m \geqq 2^n-1,\ n \geqq 1),$$

which determines β_{nm} in terms of β_{jk} with $j>n$, $k \leqq m$.

For a fixed value of m, we know that $\beta_{nm}=0$ for all n sufficiently large according to (6). Therefore, knowing all β_{jk} for $k<m$, we can use (7) to determine β_{jm} for all j. In particular, we will have determined $\beta_{0m}=b_{m-1}$.

In order to illustrate the procedure, suppose that $m=1$. From (6) we see that $\beta_{j1}=0$ for $j \geqq 1$, and then (7) with $n=1$ gives

$$b_0 = \beta_{01} = \tfrac{1}{2}[0-\beta_{00}] = -1/2.$$

Similarly, for $m=2$ we obtain

$$b_1 = \beta_{02} = \tfrac{1}{2}[0-\beta_{01}^2 - \beta_{01}] = 1/8.$$

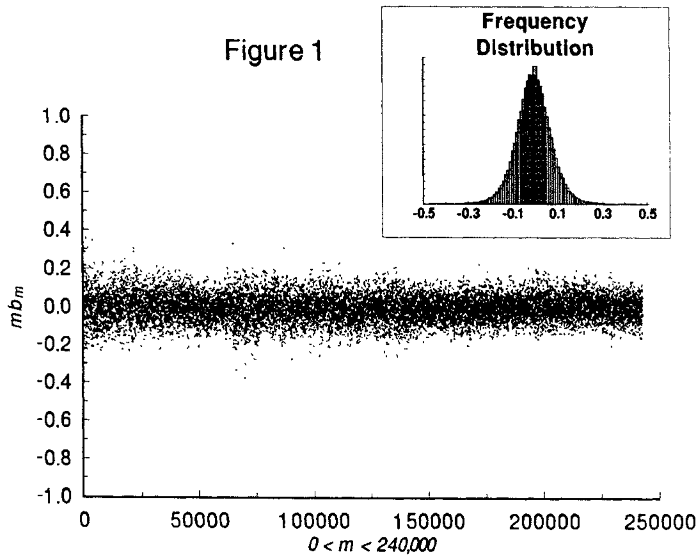

Fig. 1

Next, one can apply the procedure to find β_{nm} for $3 \leqq m \leqq 6$. Here $\beta_{jm} = 0$ for $j \geqq 2$, and so we must first determine β_{1m} before determining β_{0m}. One finds that

$$b_2 = \beta_{03} = -1/4, \quad b_3 = \beta_{04} = 15/128, \quad b_4 = \beta_{05} = 0,$$
$$\text{and} \quad b_5 = \beta_{06} = -47/1024.$$

This process can be programmed easily, and we did so using double precision arithmetic on an Alliant (parallel processor) computer. We computed the first 240000 values of b_m. (This required about 30 hours of CPU time. Since effective use of (7) for computation requires storing all previously computed values of β_{jk}, space becomes a limiting factor in implementing the algorithm.) Figure 1 shows a scatter plot and distribution of the mb_m for $0 \leqq m \leqq 240000$.

In this range it is the case that $-1 < mb_m < 1$. We conjecture that this is so for all m. One expects local variation in the values of mb_m, of course, but there are larger fluctuations in the values, which are apparent in Fig. 1 as slight 'bumps' in the scatter plot occuring near multiples of large powers of 2. There is an especially troublesome 'bump' near $m = 65536 = 2^{16}$.

We shall study the accuracy of these calculations in Sect. 5.

3 The binary form of the coefficients

The preceding algorithm can be used to easily derive a description of the coefficients in (1) and (4) as binary rationals. (This can also be deduced from [5] where Levin provides a description of the coefficients of the inverse function $\psi^{-1}(z)$ as binary rationals.)

Theorem 1. $2^{2m+3-2^{n+2}}\beta_{nm}$ *is an integer for* $n \geqq 0$ *and* $m \geqq 1$. *In particular,* $2^{2m+1} b_m$ *is an integer for* $m \geqq 0$.

Proof. Since $\beta_{0m}=b_{m-1}$, it is sufficient to prove the assertion about the coefficients β_{nm}. First check the case $m=1$. Since $\beta_{01}=b_0=-1/2$ and $\beta_{n1}=0$ for $n\geqq 1$, the theorem is true for $m=1$ and all n. We shall use induction on m. That is, assume that the theorem is true for β_{nt} for all t, $1\leqq t<m$ and all n. We shall show that it is true for m and all n.

If $2^{n+1}-2\geqq m$, then (6) implies that $\beta_{nm}=0$, and so the theorem is true for these n's. Thus the theorem is true for some n with $2^n-1\leqq m$. We shall employ (reverse) induction on n. So we assume that the assertion is true for some n, $2^n-1\leqq m$, and verify it for $n-1$. We may use the backward formula (7):

$$(8)\qquad 2^{2m+3-2^{n+1}}\beta_{n-1,m} = 2^{2m+2-2^{n+1}}\left[\beta_{nm}-\sum_{k=2^n-1}^{m-2^n+1}\beta_{n-1,k}\beta_{n-1,m-k}-\beta_{0,m-2^n+1}\right].$$

To verify that this is an integer from the induction hypothesis, it is sufficient to check that $2m+2-2^{n+1}$ is at least as large as $2m+3-2^{n+2}$, $2k+3-2^{n+1}+2(m-k)+3-2^{n+1}$, and $2(m-2^n+1)+3-2^2$. These three inequalities are all satisfied for $n\geqq 1$. This completes the inductions and the proof. □

Now set $B_{nm}=2^{2m+3-2^{n+2}}\beta_{nm}$. Then Theorem 1 says that B_{nm} is an integer, and (6) implies that $B_{nm}=0$ for $1\leqq m\leqq 2^{n+1}-2$. Furthermore, the inductive formula (8) becomes

$$(9)\qquad B_{n-1,m}=2^{2^{n+1}-1}B_{nm}-2^{2^{n+1}-4}\sum_{k=2^n-1}^{m-2^n+1}B_{n-1,k}B_{n-1,m-k}-2B_{0,m-2^n+1}.$$

Using this formula, the numerical procedure of Sect. 2 can be carried out using integer arithmetic. Table 1 lists a few of the values for $B_{0,m+1}=2^{2m+1}b_m$ that have been obtained in this way.

Based on formula (9) it is possible to make a few observations concerning the integers B_{nm}. The first is that

$$B_{nm}\equiv 0 \mod 2 \qquad \text{for } n\geqq 1.$$

Since $B_{0m}\equiv \sum_{k=1}^{m-1} B_{0k}B_{0,m-k} \mod 2$, it also follows that

$$(10)\qquad B_{0m}\equiv 0 \mod 2 \qquad \text{for } m \text{ odd}$$

and

$$(11)\qquad B_{0m}\equiv B_{0,m/2}^2 \mod 2 \qquad \text{for } m \text{ even}.$$

The following corollary is a consequence of (10) and (11).

Corollary. $B_{0m}\equiv\begin{cases}1 \mod 2 & \text{for } m=2^k\\ 0 \mod 2 & \text{for } m\neq 2^k,\end{cases}\quad k\geqq 0.$

This corollary shows that the exponent $2m+1$ in the last part of Theorem 1 cannot be improved for $m=2^k-1$, $k\geqq 0$.

Table 1

m	$2^{2m-3}a_m$	$2^{2m-3}a_m$	m
2	1	6858677625918511091485784750800016	63
3	1	1827557591967968023675540391487078 4	62
4	8	418364361989948356388770791384706 4	61
5	15	104327644398783279743178673802444 8	60
6	0	17009166869343491555946167310639 2	59
7	162	−3877184394592422735859338654515 2	58
8	1024	−377933675088834328590928494372	57
9	1499	0	56
10	4096	−101841202740215917984856010148 0	55
11	33102	850744899989888213252613079 04	54
12	0	122826207823213685752601834260	53
13	−39114	8356095265176316855569088512	52
14	917504	−1141032152474257259774636420	51
15	4104516	89043359670348505869923123 2	50
16	33554432	2327797700956063736650513 4	49
17	78558483	0	48
18	117440512	39544858774458740426298040	47
19	992135190	8391437714188530312282112	46
20	0	1017232127103664499071412	45
21	−1012222110	264452523040700131966976	44
22	72477573120	−102452195858723889881 88	43
23	265656602460	27011509701057666088 96	42
24	0	13713716810890011672 10	41
25	805564272286	0	40
26	5866925326336	3517140548583868170 04	39
27	−30644822602228	1542336505386505338 88	38
28	0	233598337096249977 38	37
29	638424330487244	40352252661239644 16	36
30	1398578790531072	19576470942163294 46	35
31	11366891965625992	2162783352300503 04	34
32	72057594037927936	1188024665116372 51	33

4 Formulas for coefficients

In order to evaluate the precision of the calculations based on the procedure of Sect. 2, it will be useful to have exact values for many coefficients. Some of these values will be in terms of the (generalized) binomial coefficients $C_k(\alpha)$ generated by

$$(1+z)^\alpha = \sum_{k=0}^{\infty} C_k(\alpha) z^k.$$

In the earlier article [2] we obtained the representation

$$-mb_m = \frac{1}{2\pi i} \int_{|w|=R} p_n(w)^{m/2^n} dw \tag{12}$$

whenever $m \leqq 2^{n+1}-3$ and R is sufficiently large. Based on this representation we were able to show that many coefficients are zero [2, Theorem 2 and 3]. In addition, we determined a number of nonzero coefficients [2, Theorem 4]. These results are contained in the following proposition.

Proposition 1. *If k and v are any integers satisfying $k \geqq 0$ and $2^v \geqq k+3$ and if $m=(2k+1)2^v$, then $b_m=0$. In addition, if $m=(2^{v+1}-3)2^v$ and $v \geqq 1$, then $b_m = \frac{-1}{m} C_{2^v-1}(2^v-\frac{3}{2})$.*

In all of our computations the only coefficients that have been observed to be zero are those mentioned in Proposition 1. In the following theorem we determine exact values for many more coefficients.

Theorem 2. *If $m=(2^{v+1}-1)2^v$ and $v \geqq 1$, then*

$$b_m = \frac{-1}{4m}\left[(2^{v+1}-1)C_{2^v-2}(2^v-\tfrac{5}{2}) - 2^{v+1}C_{2^v}(2^v-\tfrac{1}{2})\right]. \tag{13}$$

If $m=(2^{v+1}+1)2^v$ and $v \geqq 2$, then

$$b_m = \frac{-1}{32m}\left[(2^{v+1}+1)(2^{v+1}-3)C_{2^v-3}(2^v-\tfrac{7}{2}) - 2^{v+2}(2^{v+1}+1)C_{2^v-1}(2^v-\tfrac{3}{2}) + 2^{v+2}(2^v+2)C_{2^v+1}(2^v+\tfrac{1}{2})\right]. \tag{14}$$

If $m=(2^{v+1}+3)2^v$ and $v \geqq 2$, then

$$\begin{aligned} b_m = \frac{-1}{384m}\Big[&(2^{v+1}+3)(2^{v+1}-1)(2^{v+1}-5)C_{2^v-4}(2^v-\tfrac{9}{2}) \\ &-6(2^{v+1}+3)(2^{2v+1}-2^v-8)C_{2^v-2}(2^v-\tfrac{5}{2}) \\ &+3\cdot 2^{v+2}(2^{v+1}+3)(2^v+2)C_{2^v}(2^v-\tfrac{1}{2}) \\ &-2^{v+3}(2^{2v}+3\cdot 2^{v+1}+20)C_{2^v+2}(2^v+\tfrac{3}{2})\Big]. \end{aligned} \tag{15}$$

Proof. Represent m in the form $m=(2k+1)2^v$ where k is 2^v-1, 2^v, or 2^v+1. Fix n sufficiently large that $m \leqq 2^{n+1}-3$. As in the proof of Proposition 1, we shall use the recursion (3) to write

(16)

$$\begin{aligned} p_n(w)^{m/2^n} &= p_{n-1}(w)^{m/2^{n-1}}\left[1+\frac{w}{p_{n-1}(w)^2}\right]^{w/2^n} = \sum_{j=0}^{\infty} C_j(m/2^n)\, w^j p_{n-1}(w)^{m/2^{n-1}-2j} \\ &= \sum_{j=0}^{\infty} C_j(m/2^n)\, w^j p_{n-2}(w)^{m/2^{n-2}-2^2 j}\left[1+\frac{w}{p_{n-2}(w)^2}\right]^{m/2^{n-1}-2j} \\ &= \sum_{j_1=0}^{\infty}\sum_{j_2=0}^{\infty} C_{j_1}(m/2^n)\, C_{j_2}(m/2^{n-1}-2j_1)\, w^{j_1+j_2} p_{n-2}(w)^{m/2^{n-2}-2^2 j_1-2j_2} = \dots \\ &= \sum_{j_1=0}^{\infty}\dots\sum_{j_N=0}^{\infty} C_{j_1}(m/2^n)\dots C_{j_N}(m/2^{n-N+1}-2^{N-1}j_1-\dots-2j_{N-1}) \\ &\quad\cdot w^{j_1+\dots+j_N} p_{n-N}(w)^{m/2^{n-N}-2^N j_1-\dots-2j_N}. \end{aligned}$$

Choose $N = n - v$. Then (16) reduces to

$$p_n(w)^{m/2^n} = \sum_{j_1=0}^{\infty} \dots \sum_{j_N=0}^{\infty} C_{j_1}((2k+1)2^{-N}) \dots C_{j_N}((2k+1)2^{-1} - 2^{N-1}j_1 - \dots - 2j_{N-1})$$
$$\cdot w^{j_1+\dots+j_N} p_v(w)^u$$

where $u = 2k+1-2^N j_1 - \dots - 2j_N$. Thus the integral (12) depends on the coefficient of $1/w$ in the expansion of $w^{j_1+\dots+j_N} p_v(w)^u$ near infinity. Consequently, we may confine our attention to indices $j_1, \dots, j_N$ for which $u \leqq -1$.

Note that $w^{j_1+\dots+j_N} p_v(w)^u = w^{j_1+\dots+j_N+2^v u}[1+O(1/w)]$ as $w \to \infty$. Rewrite the definition of u in the form

$$j_1 + \dots + j_N = \tfrac{1}{2}[2k+1-u] - (2^{N-1}-1)j_1 - \dots - j_{N-1}$$

so that

$$(17) \quad j_1 + \dots + j_N + 2^v u = \tfrac{1}{2}[2k+1+(2^{v+1}-1)u] - (2^{N-1}-1)j_1 - \dots - j_{N-1}.$$

Observe that u is an odd integer. If $u \leqq -3$, then the exponent

$$j_1 + \dots + j_N + 2^v u \leqq \tfrac{1}{2}[2k+1-3(2^{v+1}-1)] = k+2-3\,2^v \leqq -3$$

in all cases. That is, such indices do not contribute to the integral. Thus we may assume that $u = -1$ and that the exponent (17) is

$$(18) \qquad j_1 + \dots + j_N - 2^v = k+1-2^v-(2^{N-1}-1)j_1 - \dots - j_{N-1}.$$

It is straightforward to verify by induction that

$$p_v(w) = w^{2^v} + \tfrac{1}{2}2^v w^{2^v-1} + (\tfrac{1}{8}2^{2v} - \tfrac{1}{4}2^v) w^{2^v-2}$$
$$+ (\tfrac{1}{48}2^{3v} - \tfrac{1}{8}2^{2v} + \tfrac{5}{12}2^v) w^{2^v-3} + \dots + w$$

for $v \geqq 2$, and it follows that

$$w^{j_1+\dots+j_N} p_v(w)^{-1} = w^{j_1+\dots+j_N-2^v} - \tfrac{1}{2}2^v$$
$$\cdot w^{j_1+\dots+j_N-2^v-1} + (\tfrac{1}{8}2^{2v} + \tfrac{1}{4}2^v) w^{j_1+\dots+j_N-2^v-2}$$
$$-(\tfrac{1}{48}2^{3v} + \tfrac{1}{8}2^{2v} + \tfrac{5}{12}2^v) w^{j_1+\dots+j_N-2^v-3} + \dots .$$

First, in order to verify (13) assume that $k = 2^v - 1$. In this case (18) becomes

$$(19) \qquad j_1 + \dots + j_N - 2^v = -(2^{N-1}-1)j_1 - \dots - 3j_{N-2} - j_{N-1}.$$

The right side is never positive. It can be zero only if $j_1 = \dots = j_{N-1} = 0$ and $j_N = 2^v$. These indices contribute $-\tfrac{1}{2}2^v C_{2^v}(2^v - \tfrac{1}{2})$ to the integral. The only other indices of interest are those for which (19) equals -1. This occurs only if $j_1 = \dots = j_{N-2} = 0$, $j_{N-1} = 1$, and $j_N = 2^v - 2$, and these indices contribute $C_1((2^{v+1}-1)2^{-2}) C_{2^v-2}((2^{v+1}-1)2^{-1}-2)$ to the integral. These two terms make up the formula (13).

In order to verify (14) assume that $k = 2^v$. In this case (18) becomes

$$(20) \qquad j_1 + \dots + j_N - 2^v = 1 - (2^{N-1}-1)j_1 - \dots - 3j_{N-2} - j_{N-1}.$$

The right side is at most one, and it can be one only if $j_1=\ldots=j_{N-1}=0$ and $j_N=2^\nu+1$. These indices contribute $(\frac{1}{8}2^{2\nu}+\frac{1}{4}2^\nu)C_{2^\nu+1}(2^\nu+\frac{1}{2})$ to the integral. The right side of (20) can be zero only if $j_1=\ldots=j_{N-2}=0$, $j_{N-1}=1$, and $j_N=2^\nu-1$, and these indices contribute $-\frac{1}{2}2^\nu C_1((2^{\nu+1}+1)2^{-2})C_{2^\nu-1}((2^{\nu+1}+1)2^{-1}-2)$ to the integral. Next, the right side of (20) can be -1 only if $j_1=\ldots=j_{N-2}=0$, $j_{N-1}=2$, and $j_N=2^\nu-3$. These indices contribute $C_2((2^{\nu+1}+1)2^{-2})C_{2^\nu-3}((2^{\nu+1}+1)2^{-1}-4)$ to the integral. The three terms given make up the formula (14). Third, in order to verify (15) let $k=2^\nu+1$. Then (18) becomes

$$(21)\qquad j_1+\ldots+j_N-2^\nu=2-(2^{N-1}-1)j_1-\ldots-3j_{N-2}-j_{N-1}.$$

The right side is at most two, and it can be two only if $j_1=\ldots=j_{N-1}=0$ and $j_N=2^\nu+2$. These indices contribute $-(\frac{1}{48}2^{3\nu}+\frac{1}{8}2^{2\nu}+\frac{5}{12}2^\nu)C_{2^\nu+2}(2^\nu+\frac{3}{2})$ to the integral. The right side of (21) can be one only if $j_1=\ldots=j_{N-2}=0$, $j_{N-1}=1$, and $j_N=2^\nu$, and these indices contribute $(\frac{1}{8}2^{2\nu}+\frac{1}{4}2^\nu)C_1((2^{\nu+1}+3)2^{-2})C_{2^\nu}((2^{\nu+1}+3)2^{-1}-2)$ to the integral. Next, the right side of (21) can be zero only if $j_1=\ldots=j_{N-2}=0$, $j_{N-1}=2$, and $j_N=2^\nu-2$. These indices contribute $\frac{-1}{2}2^\nu C_2((2^{\nu+1}+3)2^{-2})C_{2^\nu-2}((2^{\nu+1}+3)2^{-1}-4)$ to the integral. Finally, the right side of (21) can be -1 only in two different ways. One way is if $j_1=\ldots=j_{N-2}=0$, $j_{N-1}=3$, and $j_N=2^\nu-4$, and it contributes $C_3((2^{\nu+1}+3)2^{-2})C_{2^\nu-4}((2^{\nu+1}+3)2^{-1}-6)$ to the integral. The other way is if $j_1=\ldots=j_{N-3}=0$, $j_{N-2}=1$, $j_{N-1}=0$, and $j_N=2^\nu-2$, and it contributes $C_1((2^{\nu+1}+3)2^{-3})C_{2^\nu-2}((2^{\nu+1}+3)2^{-1}-4)$ to the integral. Combination of these five terms adds up to formula (15). □

Since $p_0(w)=w$, the choice $N=n$ in formula (16) leads to the following corollary.

Corollary. *If* $m\leqq 2^{n+1}-3$, *then*

$$-mb_m=\sum C_{j_1}(m/2^n)\ldots C_{j_n}(m/2-2^{n-1}j_1-\ldots-2j_{n-1})$$

where the sum is over all indices $j_1,\ldots,j_n$ *such that*

$$(2^n-1)j_1+\ldots+j_n=m+1.$$

Remark. By using Stirling's formula one can show that $b_m=O\left(\frac{1}{m^{5/4}}\right)$ as $m\to\infty$ through any of the sequences in Proposition 1 and Theorem 2.

5 Accuracy of the algorithm

The algorithm for calculating the coefficients b_m is an application of the backwards recursion given by Eq. (7). There is, of course, a great deal of computation involved for each new coefficient. For example, in order to compute b_{240000} $=(\beta_{0,240001})$ one starts by noting from (6) that $\beta_{n,240001}=0$ for $n\geqq 17$. Then applying (7) one sees that $\beta_{16,240001}=-\frac{1}{2}\beta_{0,108930}$, and the $\beta_{n,240001}$ for $n<16$ are computed recursively. There are approximately 1788964 multiplications in this whole process and a similar number of additions. The final value of b_{240000} depends on the 4080017 previously computed values of β_{nm}. Fewer than 300000 of these are known exactly – the rest are computed by the algorithm itself.

As usual, the error in the computation of each new β_{nm} arises from both machine error (due to roundoff and loss of significance) in a single application of Eq. (7) together with the propogated error due to errors in the previously computed coefficients. Machine error is easily seen to be relatively small in such a computation and tends to be dominated by roundoff (using a 53 bit mantissa). We would expect such error to be approximately $m \cdot 10^{-16}$ when computing b_m.

The propogated error is much more difficult to explicitly bound, but heuristic arguments suggest that it should be of the same order as the machine error; that is, one expects the algorithm to be stable in the sense that small errors in the previously computed coefficients produce small errors in subsequent computations. To see this, suppose we let $\hat{\beta}_{nm} = \beta_{nm} + \varepsilon_{nm}$, where β_{nm} is the true value and $\hat{\beta}_{nm}$ is our computed value. Substituting into Eq. (7), we see that the propogated error in calculating $\beta_{n-1,m}$ is given by:

$$\beta_{n-1,m} - \tfrac{1}{2}\left[\hat{\beta}_{nm} - \sum_{k=2^n-1}^{m-2^n+1} \hat{\beta}_{n-1,k}\hat{\beta}_{n-1,m-k} - \hat{\beta}_{0,m-2^n+1}\right]$$

$$\approx \tfrac{1}{2}\left[-\varepsilon_{nm} + \varepsilon_{0,m-2^n+1}\right] + \sum_{k=2^n-1}^{m-2^n+1} \beta_{n-1,k}\varepsilon_{n-1,m-k},$$

where we have ignored products of ε's. First, we note that computation shows the sums

$$\sum_{k=2^n-1}^{m-2^n+1} |\beta_{n-1,k}| < 2$$

for all n and m in the range we are considering here (and in most cases, the bound is much smaller). The sum in this expression for the propogated error is therefore always bounded by twice the maximum previous error. Plainly, the best one can obtain from these observations is that the propogated error is always less than 3 times the maximum previous error. Without further analysis, this might lead one to expect that the propogated error grows exponentially.

If we assume that the errors are roughly normally distributed, however, with small probability of exceeding ε, then we can make a heuristic argument which makes the situation look quite different. In this case, the sum is essentially a weighted mean of (independent) random variables. The law of large numbers implies that as the size of the sum increases, the sum approaches zero with probability one. We therefore expect the sum to contribute little to the propogated error. The propogated error in computing $\beta_{n-1,m}$ should be closely approximated by $\frac{1}{2}[-\varepsilon_{nm} + \varepsilon_{0,m-2^n+1}]$ when m is large. Clearly, if previous errors are bounded by ε, then the propogated error is bounded by ε as well.

This is merely a heuristic argument. However, because we have an independent method to compute a number of exact values of the b_m (using Proposition 1 and Theorem 2), we can test the validity of the argument by comparing the numerical values derived from our algorithm with the true values.

On the basis of Proposition 1, the coefficients b_m with $m = (2k+1)2^\nu$ and $2^\nu \geqq k+3$ should be zero. There are 963 such values for $m \leqq 240000$ and the corresponding coefficients were computed from our algorithm. The largest value for $|b_m|$ for these values of m was less than $1.1 \cdot 10^{-18}$, and the values decrease as m becomes larger. The larger values for small m are due to loss of significance; one expects $m|b_m|$ to be (roughly) independent of m. In fact, the largest value computed for $m|b_m|$ was less than $5.6 \cdot 10^{-15}$, which is of the order of magnitude of machine error. One might argue, of course, that the zero coefficients are special in some way, and so as additional evidence we explicitly computed a number of nonzero coefficients using the results of both Proposition 1 and Theorem 2. For this purpose, it is convenient to rewite (13)–(15) in the form

$$b_m = \frac{-1}{2^{\nu+3}(2^\nu - 1)} C_{2^\nu - 2}(2^\nu - \tfrac{5}{2}) \tag{13'}$$

for $m = (2^{\nu+1} - 1)2^\nu$ and $\nu \geqq 1$,

$$b_m = \frac{3(2^\nu - 6)}{2^{\nu+5}(2^\nu + 1)(2^{\nu+1} - 5)} C_{2^\nu - 1}(2^\nu - \tfrac{3}{2}) \tag{14'}$$

for $m = (2^{\nu+1} + 1)2^\nu$ and $\nu \geqq 2$, and

$$b_m = \frac{-(214 \cdot 2^{3\nu} - 767 \cdot 2^{2\nu} + 146 \cdot 2^\nu + 452)}{2^{\nu+8}(2^{\nu+1} - 7)(2^{2\nu} - 1)(2^\nu + 2)} C_{2^\nu - 2}(2^\nu - \tfrac{5}{2}) \tag{15'}$$

for $m = (2^{\nu+1} + 3)2^\nu$ and $\nu \geqq 2$. (There is machine error in computing the above expressions, of course, but our aim is to show that the propogated error is of the same order of magnitude as the machine error and so we can ignore this.) For these values of m, the greatest magnitude of the difference of b_m with the given values for $m \leqq 240000$ turned out to be less than $2.2 \cdot 10^{-18}$. In fact, the magnitude of the largest difference for mb_m turned out to be less than $6.0 \cdot 10^{-16}$.

The preceding analysis indicates that the computations based on the procedure in Sect. 2 are remarkably stable, and that machine error dominates the propogated error for large values of m. The computations suggest the maximum error in the calculated value of b_m for $m \leqq 240000$ is approximately 10^{-17}.

6 The area of M

We can express the area of M in terms of the coefficients b_m of ψ using a well known application of Green's Theorem (used first by Gronwall [3] to obtain his inequality). Specifically, suppose

$$\psi(z) = \sum_{m=-1}^{\infty} b_m z^{-m}$$

is a conformal diffeomorphism of the exterior of the unit disk to the exterior of M. For any circle C_r with $r > 1$, the image $\psi(C_r)$ is a simple closed curve

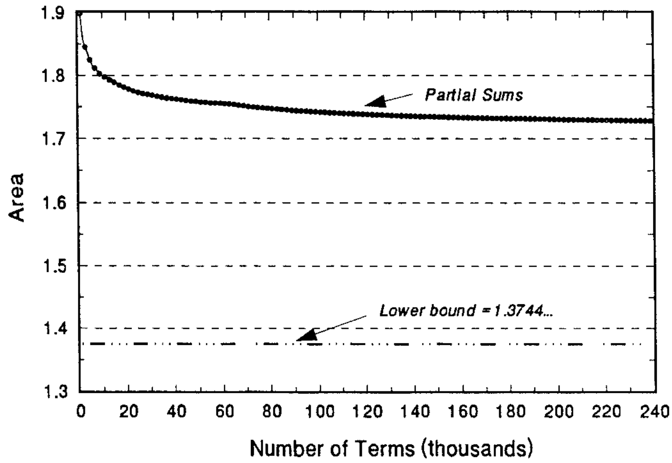

Fig. 2

enclosing a region $M(r)$. Applying Green's Theorem to $\psi(z)=w=u+iv$, we see that

$$M(r)=\iint du\,dv=\oint u\,dv=-\oint v\,du=\frac{1}{2i}\oint \bar{w}\,dw.$$

Now substituting $w=\psi(re^{i\theta})$, we obtain

$$\begin{aligned}M(r)&=\tfrac{1}{2}\sum_{m,n\geqq -1} m\bar{b}_n b_m r^{m+n}\oint e^{(m-n)i\theta}d\theta\\&=\pi\sum_{m\geqq -1} m|b_m|^2 r^{2m}.\end{aligned}$$

Taking the limit as $r\to 1$ and noting that in our case $b_{-1}=1$, we see that the area of the Mandelbrot set is given by

$$M(1)=\pi\left[1-\sum_{m>0} m|b_m|^2\right].$$

It follows that for every N, the partial sum

$$A_N=\pi\left[1-\sum_{m=1}^{N} m|b_m|^2\right]$$

is an upper bound for the area of M.

We can use the values of b_m calculated from our algorithm to find these upper bounds. Figure 2 gives a graph of the partial sums A_N as a function of N. Clearly the A_N form a sequence which (on a large scale) decreases asymptotically to a limiting value. The smallest upper bound we obtain in this way, using all of the coefficients indicated in Fig. 1, is $A_{240000}=1.7274\ldots$.

How quickly are the partial sums converging? It is difficult to extrapolate with assurance from Fig. 2. Nonetheless, elementary numerical techniques

(Richardson extrapolation) suggest that A_N approaches a limit in the range of 1.66–1.71.

How accurate are these upper bounds? If we write $\hat{b}_m = b_m + \varepsilon_m$, where ε_m is the error in computing b_m, then we see that the error in computing A_N is approximately

$$2\pi \left| \sum_{m=1}^{N} m b_m \varepsilon_m \right| \leqq 2\pi \sum_{m=1}^{N} |\varepsilon_m|$$

since $|m b_m| \leqq 1$, at least in the range in which we are working. From the remarks of Sect. 5, we therefore expect the error to be bounded by $2\pi \cdot 240000 \cdot 10^{-17} \approx 1.5 \cdot 10^{-11}$.

Can we obtain lower bounds for the area of M (and hence measure the rate of convergence of the A_N)? The interior of M contains the *hyperbolic* components, which consists of w for which q_w has an attracting periodic point. (It is conjectured that the union of the hyperbolic components *is* the interior of M.) We can obtain a precise number by computing the area of the two largest hyperbolic components of M. Of course, this is a rather crude lower bound.

The main cardioid of the Mandelbrot set consists of those values w for which $q_w(z)$ has an attracting fixed point z_0. In other words, w satisfies $q_w(z_0) = z_0$ and $|q'_w(z_0)| < 1$ for some z_0. Elimination of z_0 in these relations leads to the inequality $|1 \pm \sqrt{1+4w}| < 1$. The w's satisfying this inequality fill in a cardioid that is symmetric with respect to the real axis and intersects the real axis in the interval $(-3/4, 1/4)$. The area of this cardioid is $3\pi/8$.

The Mandelbrot set also contains a disk immediately to the left of the main cardioid. It consists of those values w for which $q_w(z)$ has a attracting periodic point of period two. That is, w satisfies $q_w(q_w(z_0)) = z_0$, $q_w(z_0) \neq z_0$, and $|q_w(q_w(z_0))'| < 1$ for some z_0. Elimination of z_0 leads to the disk $|w+1| < 1/4$, which has area $\pi/16$. Thus a crude lower bound for the area of M is $3\pi/8 + \pi/16 = 7\pi/16 = 1.3744\ldots$, which is indicated in Fig. 2.

It is possible, of course, to find reasonable approximations to the area of other hyperbolic components, giving better approximations to a lower bound of the area. It is also possible to obtain larger estimates of the area by pixel counting as described in the introduction. By covering M with a fine grid, we can iterate q_w for each w at the center of a grid point. If the origin remains within the disk of radius 2 after a fixed number of iterations T, we conclude that w is in M; otherwise, we know (approximately) that it is not in M. The resulting estimate depends on T, of course, but for a fixed mesh one can repeat the procedure for increasing values of T until the estimate is stable. The entire procedure can be refined by estimating the distance of a pixel to M using the results of [6].

In all cases, the estimates that we have obtained by refined pixel counting, appear to be no larger than approximately $1.52(\pm 0.1)$. This surprising discrepancy leads one to conclude that either the partial sums A_N converge remarkably slowly or pixel counting leads to an incorrect value of the area (and hence an incorrect 'picture' of the Mandelbrot set near the boundary).

Of course, our lower bounds and the estimates from pixel counting provide an approximation for the area of the *interior* of M. The upper bounds we obtain are upper bounds on the area of M itself. Soon after people began to study the Mandelbrot set, Mandelbrot conjectured that the boundary of M has Hausdorff dimension 2. The present calculations of area seem to suggest the possibility that the boundary might be even more pathological than first imagined, and may have positive Lebesgue measure.

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