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On numerical approximations of the area of the generalized Mandelbrot sets

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ABSTRACT

In the present work, the area of the generalized Mandelbrot sets is defined as the double limit of the areas of the plotted generalized Mandelbrot sets in a given square lattice, using the finite escape algorithm, while the lattice resolution and the number of iteration counts, used to plot them, tends to infinity. The asymptotic behavior of the areas of the generalized Mandelbrot sets in terms of their degree growth is investigated. Finally, numerical approximations of the area of the Mandelbrot set are proposed by using tools from regression analysis.

1. Introduction

In this work, we propose a mathematical framework for defining the area of the generalized Mandelbrot sets [1–4] based on the calculation of the areas of the numerically plotted generalized Mandelbrot sets on a square lattice using the finite escape algorithm [5].

Firstly, we consider a square lattice of N^2 points, where N denotes the lattice resolution [6]; the number of points on the side of the square lattice. Then, we define as a projected generalized Mandelbrot set in a square lattice, the plotted image of the generalized Mandelbrot set on the given lattice associated to a maximum number of iteration counts.

Then, the area of the projected generalized Mandelbrot set is calculated based on the counting pixel method as it was proposed in the cased of the area of the Mandelbrot set in [6]. Finally, we define the area of the generalized Mandelbrot set as the double limit of the area of the projected generalized Mandelbrot sets while the resolution and the maximum number of iteration counts used to plot them tend to infinity.

In the case of the Mandelbrot set, this approach was suggested by Rabenhorst in [6] who conjectured that a value of the area of the Mandelbrot set to be 1.508 (to 4 significant digits). Subsequently, Ewing and Schober [7,8], using analytical methods estimated that the area of the Mandelbrot set is 1.7274 (to 5 significant digits). Fischer and Hill [9,10] combined both analytical and numerical methods; they found an estimation of 1.5613 (to 5 significant digits). (For further details, there is the web-page maintained by Muffano [11] dedicated to the calculations of the area of the Mandelbrot set).

Then, we presented numerical plots of the generalized projected Mandelbrot sets for various degrees and we calculate the asymptotic behavior of their areas in terms of their degree growth. The numerical results obtained are in agreement with the theoretical result obtained by Boyd and Schulz [4] when they proved that the limit of the generalized Mandelbrot set as their degree goes to the infinity is the unit disk.

The current paper is divided in four parts. In the first part, we describe briefly the finite escape algorithm and how it is applied to plot a projected generalized Mandelbrot set in a square lattice with a given resolution and a given maximum

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number of iteration counts. Then, we present the application of the pixel counting method to calculate the areas of the projected generalized Mandelbrot sets as it was proposed in [6] for the case of the area of the Mandelbrot set.

In the second part we present numerical plotting of various generalized projected Mandelbrot sets and we calculate a model of the asymptotic behavior of their areas in terms of their degree growth.

In the third part we define the area of the generalized Mandelbrot sets as the double limit [12,13] of the area of the projected generalized Mandelbrot sets plotted in a square lattice as the maximum number of iteration counts and the lattice resolution, used to plot them, tend to infinity. Afterwards, using the Cauchy convergence criterion [13], we prove the existence of such a limit.

Thereafter, in the fourth part, we focus in the case of the projected Mandelbrot set and we calculate various areas of the projected Mandelbrot sets while varying the maximum number of iteration counts and the resolution of the square lattice used to plot them. Then, we construct a 3-D regression model for modelling those values. Thereafter, by using a Gaussian fit [14] we calculate an approximation value of 1.5101 (to 5 significant digits) for the area of the Mandelbrot set. Then, we consider a nested subsequence of square lattices; one contained in the subsequent one. Afterwards, while fixing the maximum number of iteration counts, we construct a subsequence of the double sequence of the areas of the projected Mandelbrot sets, by increasing the resolution of those lattices. Then, we provide a condition for the existence of the limit of that subsequence by using an assumption of asymptotically equivalent function [15]. Finally, using tools of regression analysis, we provide numerical values of that limit and hence numerical approximations of the area of the Mandelbrot set.

2. On the finite escape algorithm used to plot the projected generalized Mandelbrot sets

Let us recall the definition of the iteration process defined by a complex map $Q_{a,c}$ of degree α a real number, as:

$$Q_{\alpha,c}(z) = z^{\alpha} + c, \tag{1}$$

with c, z complex numbers.

Let us recall the definition of the generalized Mandelbrot set of the complex quadratic map $Q_{\alpha,c}$ which we denote by $M(Q_{\alpha,c})$. We fix the origin (0,0), and consider different values of the parameters $C = (c_1,c_2)$. The generalized Mandelbrot set of the map $Q_{\alpha,c}$, is the set of all the values of parameters (c_1,c_2) such that $\lim_{n\to\infty}|Q_{\alpha,c}{}^{(n)}(0,0)| < \infty$, where $Q^{(n)}_{\alpha,c}$ denoted the nth iteration of the map $Q_{\alpha,c}$.

In the following, we recall briefly the escape-time algorithm as explained in [5], for the case of the Mandelbrot set, as it is applied for the numerical calculation of the generalized Mandelbrot set. Initially, we fix an interval of the space of parameters (c_1,c_2) as follows $-2 \leqslant c_1 \leqslant 2$ and $-2 \leqslant c_2 \leqslant 2$, as those considered in [6]. Subsequently, we consider a lattice of parameters values with various numbers of points. Then, we set the maximum number of iteration counts to 500 and we calculate the value of the distance r from the origin up to the 500 iterations counts. If $r \leqslant 10$, then we keep the point (x_o,y_o) in a file, otherwise we ignore this point and, finally, we plot the resulting file, which gives us the corresponding projected generalized Mandelbrot set in the given lattice of points. We define also that this point (c_1,c_2) bares the generalized Mandelbrot Property [16].

In Fig. 1 we present the projected Mandelbrot sets, $\alpha = 2$ on a square lattice with a maximum number of iteration counts equal to 500 and with a lattice resolution (a) 500, and (b) 1000 respectively.

In Fig. 2 we present the projected generalized Mandelbrot sets, on a square lattice with a maximum number of iteration counts equal to 500 and with a lattice resolution 500 for $\alpha = 10, 30, 50, 70, 90$ and 100.

The numerical results indicate that at the limit of the degree growth to the infinity the generalized Mandelbrot set approaches the unit disk as it theoretically proved by Boyd and Schulz [4].

3. On the pixel counting method to calculate the area of a projected generalized Mandelbrot set

Let us now extend the counting pixel method used by Rabenhorst in 1987 [6] to evaluate the area of the Mandelbrot set for the case of the area of the generalized projected Mandelbrot sets based on the finite escape algorithm [5].

Firstly we consider a square lattice of points where the projected generalized Mandelbrot set is plotted, based on the calculations presented in [16]. For any four real numbers a, b, c and d such that $a < b$ and $c < d$, we consider a lattice of points (x,y) with x an element of the interval $[a,b]$ and y an element of the interval $[c,d]$. Then, for any lattice resolution N, we construct N subintervals of the x-interval $[a,b]$, defined via the formula: $a + (i-1)\left(\frac{b-a}{N-1}\right)$, with $1 \leqslant i \leqslant N$ and N subintervals of the y-interval $[c,d]$, defined via the formula: $c + (j-1)\left(\frac{d-c}{N-1}\right)$, with $1 \leqslant j \leqslant N$. Hence, we obtain N^2 points on the lattice with coordinates $P\left(a + (i-1)\left(\frac{b-a}{N-1}\right), c + (j-1)\left(\frac{d-c}{N-1}\right)\right)$, with $1 \leqslant i \leqslant N$, $1 \leqslant j \leqslant N$.

Thereafter, we denote by $M_{S;N}(Q_{\alpha,C})$ the projected Mandelbrot set with S the maximum number of iteration counts and N the lattice resolution. Then we define the number of generalized Mandelbrot points n $(M_{S;N}(Q_{\alpha,C})$ is the counting of all the points of the square lattice that belongs to $M_{S;N}(Q_{\alpha,C})$. Then, we denote the area of the $M_{S;N}(Q_{\alpha,C})$, based on the counting pixel method, as $\mathbf{A}^{\alpha}(S,N)$ which is defined as the ratio of the number of the generalized Mandelbrot points over all the total number of the square lattice point times the area of the lattice:

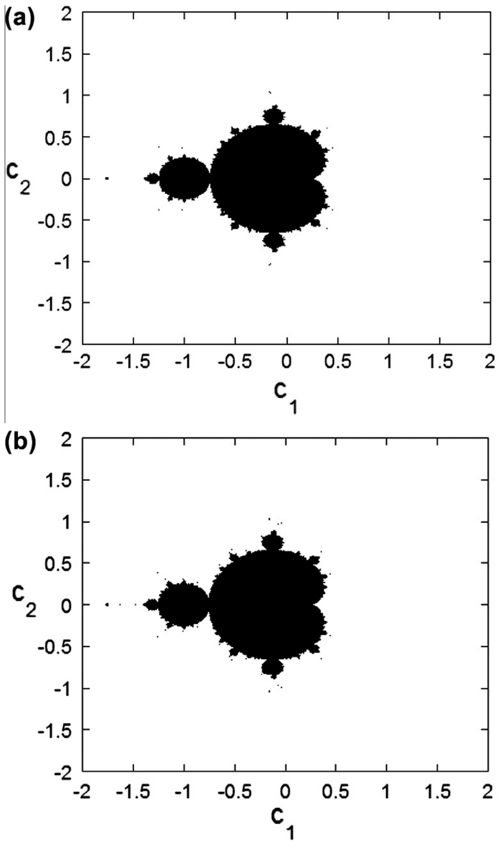

Fig. 1. The projected Mandelbrot sets plotted based on the finite escape algorithm with a maximum number of iteration counts equal to 500, in a square lattice with resolution (a) 500, and (b) 1000 respectively.

$$\mathbf{A}^{\alpha}(S,N) = \frac{n(M_{S;N}(Q_{\alpha,C}))}{N^2}(b-a)(c-d), \tag{2}$$

Then, we define the area of the generalized Mandelbrot set $M(Q_{\alpha,c})$, denoted by $\mathbf{A}^{\alpha}$ $(M(Q_{\alpha,c}))$, as the double limit of $\mathbf{A}^{\alpha}(S,N)$ when the maximum number of the resolution $N \to \infty$ and the maximum number of iteration counts $S \to \infty$:

$$\mathbf{A}^{\alpha}(M(Q_{\alpha,c})) = \lim_{N,S\to\infty} \mathbf{A}^{\alpha}(S,N). \tag{3}$$

In the following Table 1, we present the areas of the generalized projected Mandelbrot sets for various values of their degree in ascending order.

Then in the Fig. 3, we determine an exponential model can be used to model the values of the areas of the generalized projected Mandelbrot sets $\mathbf{A}^{\alpha}$ (500,500) under degree growth. The limiting value is close to 3.14 to the area of the unit disk as it was expected from the Fig. 2 and in agreement with [4].

By applying the same methods, similar model can be obtained for the values of $\mathbf{A}^{\alpha}$ (S,N) wile fixing the values of S and N. However, for a fixed degree α, the limiting value of the sequence $\mathbf{A}^{\alpha}$ (S,N) is a numerical approximation of the area of the generalized Mandelbrot set $\mathbf{A}^{\alpha}(M(Q_{\alpha,c}))$. Indeed, the generalized Mandelbrot set $M(Q_{\alpha,c})$ is defined via a continuum process, i.e. all the points of the complex plane for which the iteration process applied to the origin remains bounded. In this work we make clear that we could approximate the area of $M(Q_{\alpha,c})$ via the calculation of the areas of its plotted image $\mathbf{A}^{\alpha}(M(Q_{\alpha,c}))$ which is a discrete process based on the lattice resolution S and of the iteration maximum N.

In the next paragraph, we are going to show the existence of the area of the generalized Mandelbrot sets by showing that the sequence $\mathbf{A}^{\alpha}(S,N)$ is a Cauchy sequence and hence it is convergent [12,13].

4. On the existence of the area of the generalized Mandelbrot sets

Firstly we prove the following lemma:

Lemma 1. *For any fixed square lattice resolution N the sequence* $\mathbf{A}^{\alpha}_N(S) = \mathbf{A}^{\alpha}(S,N)$ *is decreasing.*

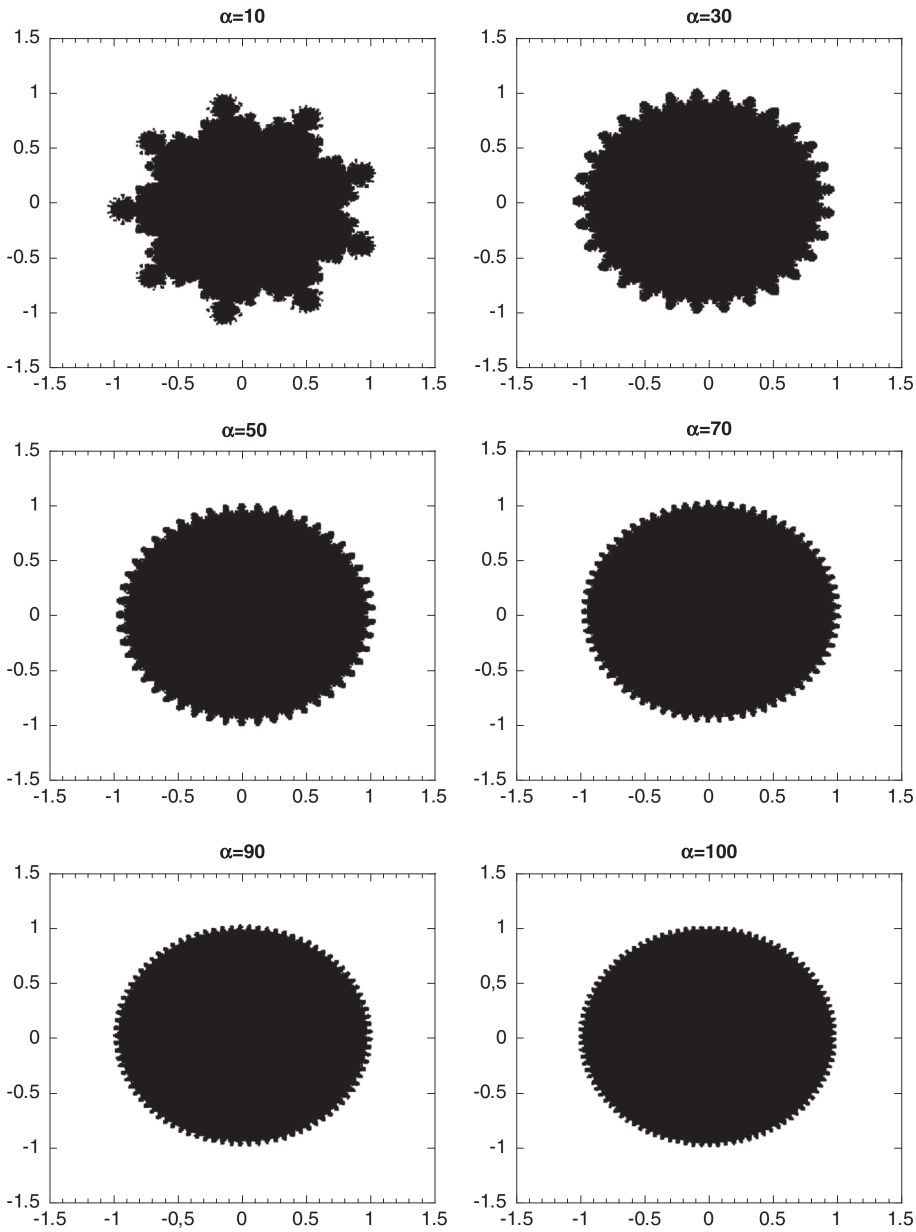

Fig. 2. The projected generalized Mandelbrot sets plotted based on the finite escape algorithm with a maximum number of iteration counts equal to 500, in a square lattice with resolution 500 and with values of $\alpha = 10, 30, 50, 70, 90$ and 100.

Proof. Let us consider a square lattice with resolution N. Then it is clear that for two different numbers of iteration counts S and T with $S \leqslant T$, if we assume that the number of generalized Mandelbrot points with maximum number of iterations counts T equal to q, then the number of generalized Mandelbrot points with maximum number of iterations counts S, will be $q + e(t)$, where $e(t)$ is a positive function of the integer t such that $t \geqslant T$ with $e(T) = 0$. The function is positive as every point that satisfy the generalized Mandelbrot property with maximum number of iteration counts T, it also satisfies it with S as maximum number of iteration counts, given that $S \leqslant T$.

It results from Eq. (2),

$$\mathbf{A}^{\alpha}(T,N) = \frac{q}{N^2}(b-a)(c-d) \text{ and } \mathbf{A}^{\alpha}(S,N) = \frac{q+e(S)}{N^2}(b-a)(c-d).$$

Thus $\mathbf{A}^{\alpha}(S,N) = \left(1 + \frac{e(S)}{q}\right)\mathbf{A}^{\alpha}(T,N)$ and consequently $\mathbf{A}^{\alpha}(\mathrm{S,N}) \geqslant \mathbf{A}^{\alpha}(\mathrm{T,N})$.

Table 1
The areas of the generalized projected Mandelbrot sets $\mathbf{A}^{\alpha}(500,500)$.

a	$\mathbf{A}^{\alpha}(500,500)$
10	2.455168
20	2.708480
30	2.813312
40	2.870144
50	2.911104
60	2.941568
70	2.962304
80	2.975616
90	2.991616
100	3.002112
150	3.036282

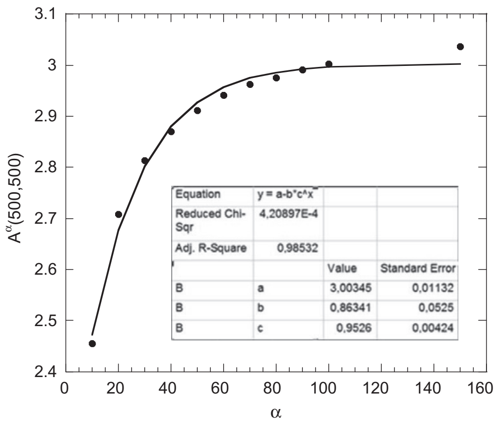

Fig. 3. An exponential model of $\mathbf{A}^{\alpha}(500,500)$ in terms of a.

Below, we show the existence of $\mathbf{A}^{\alpha}(M(Q_{\alpha,c}))$, by proving that the double sequence $\mathbf{A}^{\alpha}$ (S,N) is a Cauchy double sequence. □

Theorem 1. *The sequence $\mathbf{A}^{\alpha}$ (S,N) is a Cauchy double sequence.*

Proof. Let us consider a positive real $\varepsilon > 0$. Then for any positive integer values T, S, N, M such that: $T \geqslant S \geqslant d$ and $N \geqslant M \geqslant d$, where d will be depending on ε, we have, based on the previous Lemma,

$$\begin{aligned} |\mathbf{A}^{\alpha}(T,N) - \mathbf{A}^{\alpha}(S,M)| &\leqslant |\mathbf{A}^{\alpha}(T,N) - \mathbf{A}^{\alpha}(T,M)| + |\mathbf{A}^{\alpha}(T,M) - \mathbf{A}^{\alpha}(S,M)| \leqslant |\mathbf{A}^{\alpha}(T,N) - \mathbf{A}^{\alpha}(T,M)| + 2|\mathbf{A}^{\alpha}(S,M)| \\ &\leqslant |\mathbf{A}^{\alpha}(T,N)| + 3|\mathbf{A}^{\alpha}(S,M)|, \end{aligned}$$

Thus, via Eq. (2) we obtain

$$|\mathbf{A}^{\alpha}(T,N)| \leqslant \frac{n(M_{T,N}(Q_{\alpha,C}))}{N^2}(b-a)(c-d), \text{ and } |\mathbf{A}^{\alpha}(S,M)| \leqslant \frac{n(M_{S;M}(Q_{\alpha,C}))}{M^2}(b-a)(c-d).$$

Thereafter, we have:

$$|\mathbf{A}^{\alpha}(T,N) - \mathbf{A}^{\alpha}(S,M)| \leqslant \frac{n(M_{T,N}(Q_{\alpha,C}))}{N^2}(b-a)(c-d) + 3\frac{n(M_{S;M}(Q_{\alpha,C}))}{M^2}(b-a)(c-d).$$

As by hypothesis, $N \geqslant M \geqslant d$, it results from the inequality above:

$$|\mathbf{A}^{\alpha}(T,N) - \mathbf{A}^{\alpha}(S,M)| \leqslant \frac{(n(M_{T,N}(Q_{\alpha;C})) + 3n(M_{S;M}(Q_{\alpha,C}))(a-b)(c-d)}{M^2} \leqslant \frac{(n(M_{T;N}(Q_{\alpha;C})) + 3n(M_{S;M}(Q_{\alpha,C}))(a-b)(c-d)}{d^2}.$$

Hence if we choose for any $\varepsilon > 0,\ d > \sqrt{\frac{(n(M_{T;N}(Q_{\alpha;C}))+3n(M_{S;M}(Q_{\alpha,C}))(a-b)(c-d)}{\varepsilon}}$, the double sequence will satisfy the Cauchy condition $|\mathbf{A}^{\alpha}$ (T,N)-$\mathbf{A}^{\alpha}$ $(S,M)| \leqslant \varepsilon$.

In the next paragraph we focus our work in the case of the Area of the Mandelbrot set. □

Table 2
Various values of $\mathbf{A}^2$ (S,N).

Lattice resolution (N)	Maximum iteration counts (S)					
	200	500	1000	2000	4000	10000
500	1.5205	1.5075	1.5034	1.5023	1.5017	1.5011
1000	1.5232	1.5109	1.5068	1.5045	1.5035	1.5029
1500	1.5239	1.5121	1.5082	1.5063	1.5054	1.5048
2000	1.5249	1.5124	1.5088	1.5068	1.5059	1.5054
2500	1.5248	1.5132	1.5092	1.5073	1.5063	1.5058
3000	1.5251	1.5133	1.5094	1.5075	1.5066	1.5060
3500	1.5253	1.5132	1.5094	1.5076	1.5066	1.5061
4000	1.5254	1.5136	1.5096	1.5078	1.5069	1.5063

5. On a numerical approximation of the area of the Mandelbrot set

For the case of the Mandelbrot set $\alpha = 2$, Rapenhorst considered in [6] the square $[-2,2] \times [-2,2]$ and by using a linear interpolation method he conjectured that $\mathbf{A}^2\left(M\left(Q_c^2\right)\right)$ should be equal to the numerical value of $\mathbf{A}^2$ (8192,8193) which is equal to 1.508 (to 4 significant digits) [6].

Thereafter, various authors, see for example [11], have considered various increasing values of the lattice resolution in order to provide an accurate numerical approximation of the above $\mathbf{A}^2\left(M\left(Q_c^2\right)\right)$.

In Table 2, we present values of $\mathbf{A}^2(S,N)$, given up to 5 significant digits, for various increasing values of the maximum number of iteration counts S and of the lattice resolution N.

These results are in agreement with the results obtained in [6].

Now, we construct a numerical 3-D model using a Gaussian surface fit to approximate the area of the Mandelbrot set. More precisely firstly, in Fig. 4 we present a 3-D graph of the results presented in the Table 1, using the Origin Pro8 software [14], where the x-axis represent the square lattice resolution N, the y-axis represent the maximum number of iteration counts and the z-axis represents the areas of the projected Mandelbrot sets $\mathbf{A}^2(S,N)$.

Then, by using a Gaussian surface fit [14] we obtain a model $\mathbf{A}_{\text{fit}}{}^2(S,N)$ of the area given by the formula below:

$$\mathbf{A}^2_{fit}(S,N) = 1.51013 + 0.01219 \exp\left\{-0.5\left[\left(\frac{S-320.45}{3.14367}\right)^2 + \left(\frac{266.28+N}{1291.64}\right)^2\right]\right\}. \tag{4}$$

Thus, by applying the values of $S = 10000$ and $N = 10000$ into the model provided in Eq. (4) the area of the $\mathbf{A}^2\left(M\left(Q_c^2\right)\right)$ is approximated by the value $\mathbf{A}_{\text{fit}}{}^2(10000, 10000) = 1.5101$ (up to 5 significant digits).

6. On a numerical approximation of the areas of the Mandelbrot set by using of nested square lattices

Firstly, we fix the maximum number of iteration counts equal to $S = S_0$ and we fix a square Lattice resolution $N = N_0$. Then we consider a nested subsequence of square lattices L_{nN_0} one contained in the subsequent one; $L_{nN_0} \subseteq L_{(n+1)N_0}$ and we define a subsequence u_{S_0,N_0} of $\mathbf{A}^2$ (S,N), via the formula:

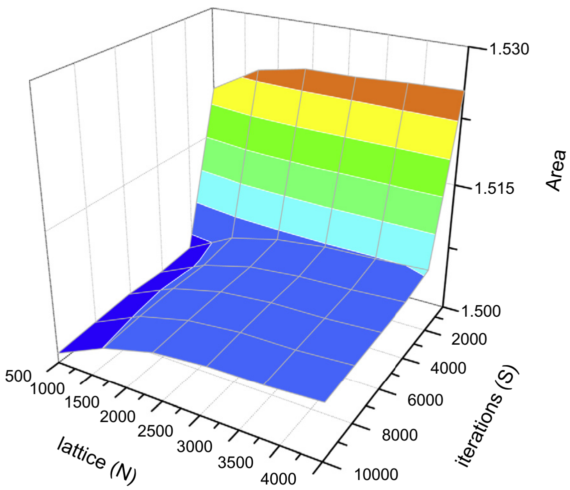

Fig. 4. A 3-D plot of $\mathbf{A}^2(S,N)$ versus the square lattice resolution S and the maximum number of iteration counts N.

Table 3
A power regression model for $\frac{d(n)}{m}$.

(S_0,N_0)	$\frac{d(n)}{m} \approx n^k$
(2000, 500)	1.0003 $n^{2.0018}$
(4000, 500)	1.0002 $n^{2.0017}$
(10000, 500)	1.0002 $n^{2.0017}$

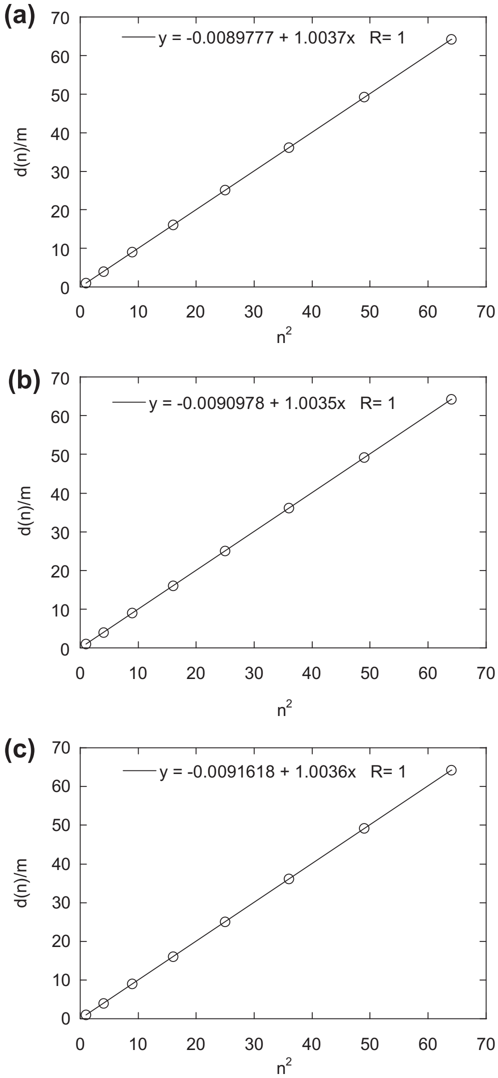

Fig. 5. The linear regression models of the $\frac{d(n)}{m}$ against n^2 with (S_0,N_0) equal to (a) (2000, 500) (b) (4000, 500) (c) (10000, 500) respectively. (circles correspond to numerical values obtained and lines correspond to linear regression model).

$$u_{S_0,N_0}(n) = \mathbf{A}^2(S_0, nN_0), \tag{5}$$

for any positive integer n.

Now, we assume that the number of Mandelbrot points in the Lattice L_{N_0} equal to m; $n\left(M_{S_0,N_0}\left(Q_C^2\right)\right) = m$. Then the number of Mandelbrot points in the lattice L_{nN_0} will be equal $n\left(M_{S_0,nN_0}\left(Q_C^2\right)\right) = d(n)$ with $d(n)$ is a positive an increasing function of the integer n, and $d(1) = m > 1$. In addition $d(n+1) = d(n) + m(n)$, with $m(n)$ is an increasing sequence of positive integers.

Using Eq. (2), we obtain:

$$u_{S_0,N_0}(n) = \frac{d(n)}{n^2 N_0^2}\ (b-a)(c-d) = \frac{\frac{d(n)}{m}}{n^2}\mathbf{A}^2(S_0, N_0), \tag{6}$$

Table 4
Numerical approximations of the area of the Mandelbrot set.

(S_0,N_0)	$\mathbf{A}(S_0,N_0)$.	$\lim_{n\to\infty}\left(\frac{\frac{d(n)}{m}}{n^2}\right)$	$\mathbf{A}$
(2000, 500)	1.5023	1.0037	1.5079
(4000, 500)	1.5017	1.0035	1.5070
(10000, 500)	1.5011	1.0036	1.5065

It is clear that u_{S_0,N_0} is a bounded as $0 \leqslant u_{S_0,N_0} \leqslant 16$.

It results for Eq. (5) that if $\frac{d(n)}{m}$ is asymptotically equivalent to a function n^k with $k > 2$ [15], then the subsequence u_{S_0,N_0} is increasing and as it is bounded by above is convergent. Thus we show the validity of the following proposition.

Proposition 1. *For any positive integer values of a maximum number of iteration counts S_0,N_0 , such that $\frac{d(n)}{m}$ is asymptotically equivalent to a function n^k with $k > 2$ the subsequence u_{S_0,N_0} converges.*

In the following Table 3, by using power regression models, while fixing the value of the square lattice resolution $N_0 = 500$ and varying the maximum number of iterations counts as (a) $S_0 = 2000$ (b) $S_0 = 4000$ (c) $S_0 = 10000$, we show that $\frac{d(n)}{m}$ satisfies the condition of the Proposition 1.

We denote by $A = \lim_{n\to\infty} u_{S_0,N_0}(n)$ Then, it results from Proposition 1, that the $A = \mathbf{A}^2\left(M\left(Q_c^2\right)\right)$, due to the uniqueness of the limit of a double sequence [13]. Thereafter, we present in Fig. 5*a b*, *c*, the linear regression models of the $\frac{d(n)}{m}$ against n^2 while fixing the square lattice resolution $N = N_0 = 500$ and varying the maximum number of iterations counts as (a) $S_0 = 2000$ (b) $S_0 = 4000$ (c) $S_0 = 10000$.

In Table 4, we, provide three numerical approximation of the Area of the Mandelbrot set using the linear regression obtained in Fig. 5*a*, *b* and *c*

The numerical value of $\mathbf{A} = 1.5065$ (to 5 significant digits) is close to those obtained using the counting pixel method [6,10], or using tools from statistical analysis [11] for approximating the Area of the Mandelbrot set.

7. Conclusions

In this work, we proposed a mathematical definition for the Area of the generalized Mandelbrot sets by bringing together tools from numerical mathematics, the finite escape algorithm, and analysis, the limit of a double sequence. We proved the existence of such a limit. Then, by constructing a nested subsequence of square lattice one included into the other and using tools from regression analysis, we provided numerical approximations of the area of the Mandelbrot set.

The methods presented in this work are independent of the numerical methods used for plotting the projected generalized Mandelbrot sets, as those presented in [17,18], for example the distance estimation method, as long as a square lattice is used.

In the future we will be interested to apply our numerical calculations to the areas of noise perturbed generalized Mandelbrot sets as presented in the works by Wang et al. [19–21].

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