A Spatial Searching Method for Planning Under Time-Dependent Constraints for Eco-Driving in Signalized Traffic Intersection

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***Abstract*—Typical numerical solutions to planning problems under time-dependent constraints (like traffic signals) involve searching in time plus state space. We consider a sampled discrete spatial formulation of the vehicle dynamics. This allows us to propose an optimal planning algorithm with much reduced search-space and time complexity, for vehicles moving across signalized intersections with full knowledge of the traffic Signal, Phasing and Timing (SPaT) information. Then we extend these results to partial knowledge by casting the problem as a Markov Decision Process (MDP). The proposed algorithms are demonstrated through numerical simulations that show a five-fold improvement in runtime compared with a standard time-state formulation, while providing comparable improvements in fuel economy with no vehicle dynamic constraints or traffic rules violated.**

***Index Terms*—Intelligent transportation systems, motion and path planning, state-space searching, time-dependent constraints.**

I. Introduction

TRAFFIC signals are indispensable for traffic flow management. For individual vehicles, they are also a primary cause of energy consumption because of idling, braking and acceleration related to the traffic signals. [1] shows that 50.36% of the fuel and 68.5% of the travel time is consumed at intersections, while the intersection accounts for only 28.9% of the total testing distance. For the fuel consumed at the intersection, 78.4% is due to stopping and idling at the red light, and the subsequent acceleration. This implies a significant opportunity for energy-efficient motion planning for autonomous vehicles.

Motion planning at intersections is well-studied, which is often cast as an optimization problem and has been explored by both analytical and numerical solutions. The problem is to prescribe the desired motion of the vehicle (in terms of its velocity or acceleration) subject to constraints on its dynamics as well as obey traffic rules. The traffic signal is specified through the SPaT information.

Prior work to address the above problem has included analytical approaches e.g. applying Pontryagin's minimum principle (PMP), in [2] with both deterministic and stochastic traffic signals information, and in [3] to optimize the engine torque, the brake force, and the gearshift with known signal information in a fleet of vehicles. However, the solvability of such method is highly dependent on the convexity of the objective and constraint functions. Alternatively, numerical methods for the optimal or near-optimal solutions in discrete time and state space have been explored. Dynamic Programming (DP) is applied to search for optimal motion solution in [4] with the deterministic traffic signal. A big source of uncertainty in motion planning at traffic intersections is the dynamics of the traffic signal, which is often dealt with through the introduction of a probabilistic model for such dynamics. [5] provides a numerical solution based on the statistical average of red and green light duration with uniform distribution to define a conditional probability of the signal change. [6] leverages a DP method with final point constraints from [7] to generate optimal motion planning using a spatial trajectory formulation. A stochastic variable is introduced to capture the feasible passing time in the problem formulation. [8] inherits the eco-driving control from [6] and integrates with an adaptive cruise controller (ACC) to improve safety by avoiding front collision and traffic violation. Planning has also been approached by studying the underlying human behavior. [9] studies the human driving model under intersections in different conditions, while most human models take safety issue more than fuel economy. In psychological research, experiments capture the driving style of different ages of people [10], which observe a smoother driving style among older drivers versus young drivers in a dilemma zone.

More recently, the availability of improved connectivity has been utilized to extend the motion planning algorithms in the context of connected vehicles. Communication between connected vehicles and the traffic signal can be leveraged to eliminate the uncertainty associated with the traffic light for a group of connected vehicles. In [11], it is shown that the connectivity to a traffic signal is adequate for a single vehicle to achieve optimal motion planning in an intersection without queues. In [12], the vehicle communicates with both the traffic light and the vehicles in front and back to plan optimally while avoiding the collision with the neighboring vehicles. Instead of planning on velocity profile at the level of the vehicle, [13], [14] construct a "request-go communication system" to establish a Traffic Management System (TMS). The vehicle sends a request when approaching an intersection, and if the request is approved, this vehicle can

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proceed through the intersection regardless of the traffic. It is the responsibility of the centralized TMS to ensure safety and feasibility when responding to requests. [15] studies the optimal control problem of departing intersection of connected vehicles, considering physical factors including gear choices, road friction and road slope.

Synthesizing from the above prior work, we make two key observations on the eco-driving problem formulations: (i) Optimization methods suffer from the curse of dimensionality, so any approach to reduce the dimensionality will be desirable; (ii) the component of SPaT that has the highest uncertainty is only the phase offset (as considered in other studies such as [6]), but this uncertainty is removed as soon as the first signal change is observed, which can be detected by several types of sensors already available in the current production vehicle. These two observations motivate us to propose an optimal eco-driving solution.

In this letter, we consider a single vehicle at a single traffic intersection with known SPaT, but unknown phase offset. Similar to [6], a spatial state formulation is applied to our proposed solution. However, we refine the formulation to reduce the algorithmic computational complexity and be reactive to traffic signal sensing. Our key contributions are:

- Sampled Space Optimal Planning (SSOP): Leveraging the spatial formulation, and by imposing a simple constraint of non-negative velocity, and using piece-wise constant accelerations between samples, we can remove time as an independent state of the system during optimization, helping to reduce the system dimensionality. We detail the consequent reduction in computational complexity.
- Stochastic State Space Planning (SSSP): Use of a Markov Decision Process (MDP) to model the belief of the traffic signal, and explicit consideration of the transition of the system from stochastic to deterministic in the optimization solution.
- A data-driven motion planning, which enables table lookup for control commands given initial velocity and signal in both SSOP and SSSP algorithms to reduce the online computation.

This letter is organized in the following manner. Section II describes the mathematical formulation of the problem statement. Section III details the proposed planning under full and partial knowledge of traffic signals. Section IV presents the simulation results including the trajectory profiles of different algorithms and fuel cost statics, as well as the associated runtime costs, and the conclusion is drawn in Section V.

II. Problem Definition

A. Spatial Dynamics

We will consider the classical kinematic model for the vehicle, treating the acceleration as a controllable input to the vehicle. This treatment can be readily extended to include additional dynamics such as the powertrain dynamics by mapping applied torque to the acceleration. First, we illustrate the two possible formulations - namely spatial and temporal, in a continuous system setting:

1) Temporal and Spatial Formulation of Continuous Vehicle Dynamics: Defining x, v as the position and velocity of the vehicle at time t, we can express the vehicle dynamics through various smooth trajectories evolving under a control policy π, starting from an initial condition x_0, v_0 through the tuple $\Gamma_{x_0,v_0,\pi} = \{x, v, t\}$. These trajectories are described by a parametric equation as $x(\theta), v(\theta), t(\theta)$ with some variable θ that is smooth monotonic with the distance along the curve. The dynamics of the system is then represented by $dx(\theta)/d\theta, dv(\theta)/d\theta, dt(\theta)/d\theta$.

In the most popular temporal representation of the dynamics we set $\theta := t$, leading to the familiar dynamics given by $dx/dt = v, dv/dt = u$, where u is the acceleration. Thus there are **two** continuous states x, v representing this system.

In the alternate spatial representation of the dynamics, $\theta = x$, leading to the following dynamics given by $dv/dx = u/v, dt/dx = 1/v$. Still we have **two** continuous states v, t representing the system. The primary advantage of this representation is that it allows specification of position-based constraints, such as at a traffic intersection. This property is leveraged in the formulations in [6].

2) Sampled Discrete Temporal and Spatial Formulations of the Vehicle Dynamics: Considering a discrete sampling based representation of the dynamics by using a sampling function $\gamma : I_+ \to \mathcal{R}$ such that $\theta_k := \gamma(k) = k\Delta\theta$ where $\Delta\theta$ is a constant positive scalar, and assuming that u_k is the constant acceleration during the sampling interval, we can write the two formulations of the dynamics as below:

1) Sampled Discrete Temporal Formulation, with $\theta = t$: $t_k = k\Delta t$, the two variables x_k, v_k is updated by

$$\begin{cases} x_{k+1} = x_k + v_k\Delta t + \frac{1}{2}u_k\Delta t^2 \\ v_{k+1} = v_k + u_k\Delta t \end{cases} \tag{1}$$

In this formulation we have **two** states x_k, v_k.

2) Sampled Discrete Spatial Formulation, with $\theta = x$: $x_k = k\Delta x$, under the assumption that $\forall k, v_k \geq 0$, variables v_k is updated by

$$v_k = \sqrt{v_{k-1}^2 + 2u_k\Delta x} \tag{2}$$

Δt_k is the time duration during the interval, which is determined by

$$\Delta t_k = \begin{cases} \frac{\sqrt{v_k^2+2u_k\Delta x}-v_k}{u_k} & u_k > 0 \\ -\frac{\sqrt{v_k^2+2u_k\Delta x}-v_k}{u_k} & u_k < 0 \\ \Delta x/v_k & u_k = 0, v_k > 0 \end{cases} \tag{3}$$

It is to be noted that u_k can be obtained from (2) and therefore, in (3), Δt_k does not depend on Δt_{k-1} but depends on v_{k-1}. Effectively, the system dynamics is governed by only **one** state v_k, and Δt_k can be treated as an output function of the state v_k. This reduction in number of states will translate to reduced computational complexity in our solution approaches (as explained in Section III below).

Based on the analysis above, the state variables in our problem are defined in the discrete spatial formulation. The origin of x is upstream of the traffic signal which is located at L. Our traffic scenario will be restricted to starting at the origin, moving towards the traffic signal and going past the traffic intersection by Δx. In the discrete spatial formulation, define the following discrete variables:

$$\begin{cases} x_i \in \mathcal{X}, \mathcal{X} = \{i\Delta x : i = 1, 2, \ldots, m\}; \\ v_j \in \mathcal{V}, \mathcal{V} = \{j\Delta v : j = 0, 1, \ldots, n-1\}; \\ t_k \in \mathcal{T}, \mathcal{T} = \mathbb{R}_{\geq 0} \end{cases} \tag{4}$$

Here $m, n, \Delta x$ and Δv are constant discretization parameters. m and Δx are chosen such that $L = (m-1)\Delta x$ to specify constraints at the traffic intersection. By the definition of the origin, we have $x_0 = 0$. Further we define $v_{\max}$ as the maximum velocity that we will consider, and choose n and Δv such that $v_{\max} = (n-1)\Delta v$. For the time variable t_k, it is defined as the accumulated time from x_0 at position $k\Delta x$, $t_0 = 0$ and we have

$$t_k = \sum_{i=1}^{k} \Delta t_i \tag{5}$$

where Δt_i is from (3).

B. Assumptions and Variable Definitions

Assumption A1: We assume a single vehicle passing one traffic light on a flat and straight road.

Assumption A2: The velocity of the vehicle is positive unless at the intersection:

$$\forall k \neq m-1, v_k > 0 \tag{6}$$

Assumption A3: We assume that we can control the longitudinal acceleration of the vehicle. In other words, we will assume that we have adequate control over the powertrain of the vehicle to achieve this, which is similar to assumptions made in [4], [5], for simplification of the presentation of the concepts. The longitudinal acceleration is denoted by u, and the sampled acceleration at the k^{th} sample is u_k. Further we will assume that acceleration is bounded as below:

$$u \in [u_{\min}, u_{\max}] \tag{7}$$

C. Traffic Signal Dynamics

A common representation of the traffic signal dynamics is the fixed green, yellow and red SPaT $\{T_g, T_y, T_r\}$ with offset ϕ_0. We define the set of traffic signals as $\mathcal{L} = \{G, Y, R\}$, and the residual time of traffic period by

$$t_{res} = (t + \phi_0) mod (T_g + T_y + T_r) \tag{8}$$

then the traffic light function $l : \mathcal{T} \to \mathcal{L}$ that maps time to traffic lights is given by

$$l(t) = \begin{cases} \text{G} & 0 \leq t_{res} \leq T_g \\ \text{Y} & T_g < t_{res} \leq T_g + T_y \\ \text{R} & T_g + T_y < t_{res} < T_g + T_y + T_r \end{cases} \tag{9}$$

With the above, we can capture the constraint that the vehicle should stop at the traffic light as below:

$$(x_k = L) \wedge (l(t_k) = \text{R}) \to v_k = 0 \tag{10}$$

D. Fuel Consumption

In this letter the VT-CPFM-1 model from [16] is applied for fuel cost calculation, which is

$$FC(u(t), v(t)) = \begin{cases} \alpha_0 + \alpha_1 P(t) + \alpha_2 P(t)^2 & P \geq 0 \\ \alpha_0 & P < 0 \end{cases} \tag{11}$$

Here $FC(u(t), v(t))$ is the instantaneous fuel consumption rate (Liter/s) at time t with velocity $v(t)$ and acceleration $u(t)$, $P(t)$ is the power of the vehicle defined in VT-CPFM-1 model [16]. For a Toyota Camry 2016 [17]: $\alpha_0 = 6.289e - 04, \alpha_1 = 2.676e - 05, \alpha_2 = 1e - 06$, with fuel density 748.9 kg/m^3. The fuel cost $c(v_k, u_k)$ associated with the movement from x_k to x_{k+1}, starting from v_k under the constant acceleration u_k can be obtained by integrating $FC(u(t), v(t))$ as below:

$$c(v_k, u_k) = \int_{\Delta t=0}^{\Delta t_k} FC(u_k, v_k + u_k \Delta t) d\Delta t \tag{12}$$

E. Eco Driving at Traffic Signal - Problem Statement

The primary objective of eco-driving is to navigate the traffic signal such that the fuel consumed is minimized, while traffic rules are not violated. In our sampled discrete spatial formulation, given the variable domain (4) and assumptions in Section II-B, the problem to address can be expressed mathematically as the development of a control strategy, i.e. a sequence of constant acceleration commands $\mathbf{u} = \{u_0, \ldots, u_{m-1}\}$, such that expectation of the total fuel costs are minimized:

$$\min_{\mathbf{u}} \quad \sum_{k=0}^{m-1} E[c(v_k, u_k)] \tag{13a}$$

$$\text{s.t.} \quad v_k = \sqrt{v_{k-1}^2 + 2u_k \Delta x}; \tag{13b}$$

$$u_k \in [u_{\min}, u_{\max}]; \tag{13c}$$

$$\text{if } l(t_{m-1}) = \text{R}, v_{m-1} = 0; \text{else } v_{m-1} \in \mathcal{V} - \{0\}; \tag{13d}$$

Here $E[X]$ refers to the expectation of X (if X is known deterministically, $E[X] = X$).

III. Eco Driving With Traffic Signals - Solution Approach

In this section, we will describe our solution approach to the motion planning problem statement of Section II above. We consider the problem in two scenarios: (i) both the SPaT information $\{T_g, T_y, T_r\}$ and the offset ϕ_0 corresponding to the beginning of the motion planning are known to the controller, (ii) only the SPaT information is known to the controller, but not the offset ϕ_0. We will call the former scenario as having "full" knowledge and the latter "partial" knowledge. These two different problems are discussed separately, but the proposed solution to partial knowledge is based on the solution of the full knowledge.

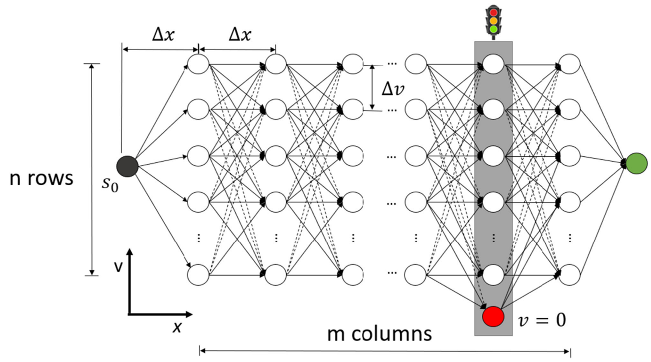


Fig. 1. Graph representation of planning with traffic signal. Black vertex is the initial state s_0, k^{th} column of vertices represents the states of the same position $x_k = k\Delta x$ but different velocities $v \in \mathcal{V}$. The second right column includes the states at intersection, and the red vertex is the only possible state of zero velocity due to traffic signal, with other states with zero velocity omitted in this graph. The green vertex is the termination of the problem which has no actual value. The solid edges are valid state transitions, and dashed edges between vertices are inapplicable state transitions.

A. SSOP: Plan Under Full Traffic Signal Knowledge

As discussed in Section II-A2, the independent state of the system can be specified to be the velocity, indexed by distance. However, for convenience of notation, we will define the state of vehicle as the tuple of position, velocity: $\mathcal{S}_f = \mathcal{X} \times \mathcal{V}$ with domain $\mathcal{X}, \mathcal{V}$ defined in (4), the initial time $t_0 = 0$, and the initial state $s_0 = (0, v_0)$, the initial velocity of vehicle is $v_0 \in [v_{\min}, v_{\max}]$. Explicitly the state transition function is

$$s_{k+1} = f(s_k, u_k) = \begin{bmatrix} x_{k+1} \\ v_{k+1} \end{bmatrix} = \begin{bmatrix} x_k + \Delta x \\ \sqrt{v_{k-1}^2 + 2u_k \Delta x} \end{bmatrix} \quad (14)$$

The discrete spatial formulation of the optimization problem allows us to the describe the entire state space as a digraph $G = (V(G), E(G))$ as Fig. 1, the reachability graph [18] by considering all sequences of discretized actions. Each vertex $\mathbf{v}$ represents a state $s_i := (x, v)$ for $s \in \mathcal{S}$, i.e. there exists a bijective mapping $h : \mathcal{S}_f \to V(G)$, such that every element of state $s_i \in \mathcal{S}_f$ maps to a unique vertex $\mathbf{v}_i \in V(G)$. Additionally, if we denote s_0 as the initial vertex $\mathbf{v}_0$ and s_T as the terminal vertex $\mathbf{v}_T$ which has no physical meaning, then we can write $V(G) = \{h(\mathcal{S}_f), \mathbf{v}_0, \mathbf{v}_T\}$.

A state transition represents a *valid* change of state from $s_k = (x_k, v_k)$ to $s_{k+1} = (x_{k+1}, v_{k+1})$ according to dynamics (14) if the constraints (13c), (13d) are not violated. Every directed edge of G represents a valid state transition, i.e. $\exists e = \{\mathbf{v}_i, \mathbf{v}_j\} \in E(G)$ if and only if $h(s_i) = \mathbf{v}_i, h(s_j) = \mathbf{v}_j$ s.t. there is a valid transition from s_i to s_j.

Define a *path* p in the directed graph G as a sequence of edges $p = \{e_1, e_2, \ldots e_k\}$ such that a list of vertices and edges $\mathbf{v}_0, e_1, \mathbf{v}_1, \ldots e_k, \mathbf{v}_k$ can be obtained from path p, and for $1 \leq i \leq k, e_i = \{\mathbf{v}_{i-1}, \mathbf{v}_i\}$ with all vertices in this list are unique; the number of edges in the path is the length of the path. A path p with two endpoints $\mathbf{v}_0$ and $\mathbf{v}_T$ defines a unique motion planning solution from the initial state s_0 to pass the intersection. The set of all such paths $\mathcal{P}$ defines all possible unique solutions to the system dynamics, and our problem is to identify the specific valid path with *optimal* properties.

For each edge $e = \{h(s_k), h(s_{k+1})\} \in E(G)$, considering the associated state transition $s_{k+1} = f(s_k, u_k)$, define the *weight* of the edge as $w(e) = w(\{h(s_k), h(f(s_k, u_k))\}) = c(s_k, u_k)$ by (12). Treating the fuel cost as the weight associated with the edge e, we can restate the optimization problem 13 as finding the unique path p^* with minimum total weight under same constraints in (13)b-d.

$$p^* = \arg\min_{p \in \mathcal{P}} \sum_{e \in p} w(e) \quad (15)$$

Thus the problem is converted to a graph based shortest path finding problem. However, the traffic signal constraint (10) makes this problem unique to the common shortest path finding: not all paths could satisfy the time-dependent constraint (13d). Specifically, a motion planning solution needs to be validated for all constraints aforementioned. The velocity dynamics (13b) and control constraint (13c) has been validated when constructing edge set $E(G)$. For the traffic light constraint (13d), which is time-dependent, it is required to be validated based on the specific solution, or path p. Define the time taken for in a state transition represented by an edge $e = \{h(s_k), h(s_{k+1})\}$ as $\delta t(e) = \Delta t_k$ defined in (3). Given path p with 2 endpoints $\mathbf{v}_0$ and $\mathbf{v}_T$, we can calculate the time taken from initial state s_0 to intersection state s_{m-1} as

$$t_{m-1} = \sum_{i=0}^{m-1} \delta t(e_i) \quad (16)$$

then use (13d) to verify the feasibility of such solution. While classical shortest path searching algorithms such as Dijkstra's algorithm and DP return the optimal solution when there are no terminal constraints, they are not directly applicable to the problem described herein, because of the time-dependent constraint (13d).

In this letter, we propose an exhaustive searching algorithm which uses Depth First Search (DFS) to collect all solutions, exclude solutions not satisfying the traffic signal constraint and pick the one with minimum cost from the rest. In relevant work on optimal control [6], all solutions including waiting at traffic intersection are ignored by arrival time constraint for concern of the traveling time, but our method picks the best solution purely based on fuel cost.

The procedure is described in Algorithm 1. The idea for such an optimization approach is mentioned in [19] in the context of motion planning to avoid obstacles. This approach is typically called "sample-based planner" or "layered state lattice". The time complexity of SSOP is computed in Table I, which is $O(mn^m)$. Typical DP methods applied in literature (e.g. [4], [5] search in the time state discrete space, i.e. the space of $\mathcal{S}_{ts} = \mathcal{X} \times \mathcal{V} \times T$, here T is discrete time space with some fixed interval Δt, we term it as Time State Optimal Planning (TSOP). Suppose $T = \{i\Delta t : i = 0, 1, \ldots, N-1\}$, N needs to be chosen sufficiently large to provide full coverage of the traffic lights: $N\Delta t \geq T_g + T_r + T_y$. In each state transition, a state goes from state at t_k to state at t_{k+1} in fixed time Δt by

TABLE I
TIME COMPLEXITY OF ALGORITHM 1

Step	Complexity
1. Graph initialization	$O(\lvert G(V)\rvert) = O(mn)$
2. Build directed graph	$O(\lvert G(E)\rvert + \lvert G(V)\rvert) = O(mn^2)$
3. Find all possible solutions	$O(mn^m)$
4. Find the optimal solution	$O(n^m)$
Total complexity	$O(mn^m)$

Algorithm 1: SSOP.

Require: A reachability graph $G = (V, E)$, initial state $s_0 = (0, v_0)$ and known traffic signal time offset ϕ_0.
1. Graph vertex initialization
Initialize 2-d `graph` with size (m+2, n), (`graph`[i][j] = $(i\Delta x, j\Delta v)$, `graph`[0] = $\mathbf{v}_0$, `graph`[m+1] = $\mathbf{v}_T$).
2. Build directed edges
for k=0; k < m+1; k++ **do**
  for vertex $\mathbf{v}_1 = (x_k, v_i)$ in `graph`[k] **do**
    for vertex $\mathbf{v}_2 = (x_{k+1}, v_j)$ in `graph`[k+1] **do**
      $u = g(v_i, v_j)$, if $u \in [u_{\min}, u_{\max}]$, add edge $e = (\mathbf{v}_1, \mathbf{v}_2)$ to `graph` with weight $w(e)$ and time duration $\delta t(e)$;
    end for
  end for
end for
3. Find all applicable solutions
`solutions`, `cost` ← DFS(s_0, `graph`);
Apply DFS algorithm to find all possible paths from $\mathbf{v}_0$ to $\mathbf{v}_T$ in `graph`, save solutions satisfying traffic signal constraints to list `solutions` and the accumulated fuel cost to list `cost`;
4. Find solution with minimum fuel cost
i = min(`cost`).index();
u = `solutions`[i];
return u;

control u_k. All constraints are avoided by making conflicting states unavailable. To build the time-state space, the complexity is $O(mnN)$, and the number of all possible transitions are $O((\frac{m^2n}{2})^N)$. Finally to find optimal planning takes $O(N)$ steps. In total, the time complexity of TSOP is $O((m^2n)^N)$.

In order to compare the our proposed SSOP with TSOP, parameters Δx and Δv, m and n should be set to the same value. Then the complexity will depend on the relation between N (time dimension of TSOP) and m (state dimension of both the methods). However, the choice of N is arrived at as a trade-off between approximation and computation complexity: the larger N, the larger computation complexity; the smaller N, the larger is Δt and correspondingly the potential for larger error in constraint violation detection. On the other hand, using a sampled-space approach, we can always ensure that the traffic rules violation is precisely captured. Thus we believe, Algorithm SSOP is a better alternative in planning under full knowledge of traffic signal.

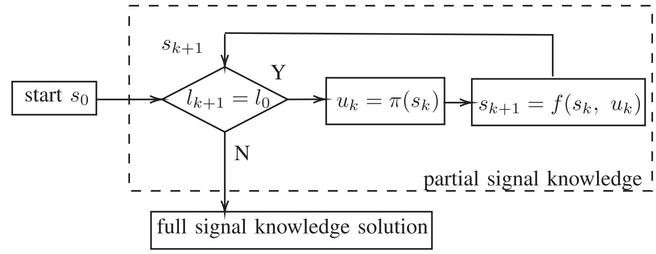


Fig. 2. The temporal logic diagram for planning under partial traffic light knowledge.

B. SSSP: Plan Under Partial Traffic Signal Knowledge

The methodology of the previous section requires knowledge of the timing offset ϕ_0, which is equivalent to knowing apriori when the traffic signal will change. Using localization from GPS [20] it is possible to obtain the signal information $\{T_g, T_y, T_r\}$. However, the current state of smart traffic infrastructure and V2I communications is not adequate yet for the vehicle to predict the timing information apriori. Therefore, we have to treat the timing information as a random variable. On the other hand, advances in computer vision technology have facilitated realtime recognition of traffic signals [21], [22]. Thus, the timing information becomes a deterministic variable as soon as we detect a change in the traffic signal.

In this section, we will expand the approach of the previous section to the stochastic case where the timing information for the traffic signal is partially unknown until the first detection of a light change. Specifically, we will assume that the SPaT information $\{T_g, T_y, T_r\}$ is known, the effective time of $t + \phi_0$ is not known apriori in (8) and (9). Consequently, we do not know the traffic signal in future $l(t)$, therefore we cannot verify constraint (13d) deterministically. We will capture this uncertainty by treating l_k as a random variable. As the vehicle moves by one spatial step, there are two possible outcomes for the traffic light: it either remains the same, or changes to the next signal (for example, from green to yellow).

Until a change of light is observed, there are two possible outcomes: $l_{k+1} = l_k$ or $l_{k+1} \neq l_k$. However, once a change of light has been observed, the problem becomes deterministic because of our prior knowledge of the SPaT information. This allows us to consider the following temporal logic to address the optimization problem in two stages: before the vehicle observes a traffic signal change, the problem is treated as a stochastic problem (and described in the rest of this section). Once the vehicle observes the traffic signal change, the offset ϕ_0 is captured, this problem is converted to the deterministic problem described in Section III-A. This is demonstrated schematically in Fig. 2, here $\pi(s_k)$ is the control policy before any traffic signal change is captured.

For the stochastic formulation, we will add the traffic light state l_k as an additional variable to represent the state of the system. Thus, the tuple of (v_k, l_k) represents the state of the system $\mathcal{S}_p = \mathcal{V} \times \mathcal{L}$, indexed by the position x_k. The initial value of the state $s_0 = (v_0, l_0)$ with same control input $u \in \mathcal{U}$

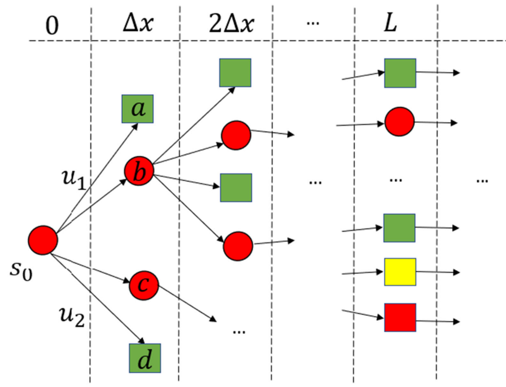


Fig. 3. A tree representation of MDP. States are sorted in the columns by position s. Color of a state represents the signal. States has 2 shapes: states with round shape are in S_1, states with square shape are in S_2, intersection locates at $x = L$.

in SSOP, which is a constant longitudinal acceleration in fixed spatial interval Δx.

Given a state $s_k = (v_k, l_k) \in \mathcal{S}_p$ and control u_k of acceleration in the corresponding space interval, v_{k+1} is deterministically calculated from (14). Further, t_k, the total time traveled up to step k can be calculated progressively as $t_{k+1} = t_k + \Delta t_k$, with Δt_k coming from (14). For the stochastic case where a light change has not been observed, since the only part of the state s_k that is random is l_k, state transition probability $\mathcal{P}(s_{k+1}|s_k, u_k)$ will degenerate to a probability model for only the light change $\mathcal{P}(l_{k+1}|s_k, u_k)$.

We now cast the fuel optimization problem as a Markov Decision Process problem. The one-step reward $\mathcal{R}(u_k, s_k)$ associated with state s_k and corresponding action u_k essentially is the fuel cost associated with such an action.

$$\mathcal{R}(u_k, s_k) = -c(u_k, s_k) \tag{17}$$

The aforementioned problem is the MDP $\mathcal{M} = \{\mathcal{S}_p, \mathcal{U}, \mathcal{P}, \mathcal{R}\}$, where $\mathcal{M}$, $V : \mathcal{S}_p \to \mathbb{R}$ denotes the optimal utility function at the state, $\pi : \mathcal{S}_p \to \mathcal{U}$ denotes the policy of the state. The optimal solution satisfies

$$V(s_k) = \max_{u_k \in \mathcal{U}(s_k)} \mathcal{R}(s_k, u_k) + \sum_{s_{k+1}} \mathcal{P}(l_{k+1}|s_k, u_k) V(s_{k+1}) \tag{18}$$

$$\pi(s_k) = \arg\max_{u_k \in \mathcal{U}(s_k)} \mathcal{R}(s_k, u_k) + \sum_{s_{k+1}} \mathcal{P}(l_{k+1}|s_k, u_k) V(s_{k+1}) \tag{19}$$

Fig. 3 is a simple visualization of the MDP described above. From initial state s_0 with red signal, state a, b are two possible states by action u_1: $a \in S_2$ has signal changed to green, $b \in S_1$ has same signal as s_0. Since all states in S_2 have deterministic optimal solution, in this MDP only the descendant states of S_1 need to be calculated by Bellman's principle. At the position of $x = L$, all states have deterministic optimal solution and utility function as well. Thus to calculate all utility value and policy in this MDP (red circles), one backward value update with (18) and (19) is sufficient. As the primary stochastic component is the traffic light change, corresponding state-transition probability can be written as $\mathcal{P}(l_{k+1}|s_k, u_k)$. Recognizing further that the only dependency on the light change is the phase offset ϕ_0, which is in turn captured by t_k, state transition probability can be written as $\mathcal{P}(l_{k+1}|l_k, t_k, \Delta t_k(v_k))$.

Given current observation of a light signal, the SPaT information, and the time traveled up to now, we can calculate the maximum time that the light signal can continue to remain the same. Inside this time interval, we assume that the probability of a light change happening is uniformly distributed. For example $l_0 = \text{G}$, at state $s_k = (v_k, l_k)$ with next state-transition duration Δt_k,

$$P(l_{k+1} = \text{G}|l_k = \text{G}, t_k, \Delta t_k) = \begin{cases} \frac{T_g - t_k - \Delta t_k}{T_g - t_k} & \Delta t_k < T_g - t_k \\ 0 & \text{else} \end{cases} \tag{20}$$

And $P(l_{k+1} = \text{Y}|l_k = \text{G}, t_k, \Delta t_k) = 1 - P(l_{k+1} = \text{R}|l_k = \text{G}, t_k, \Delta t_k)$.

C. Summary

In this section, two planning algorithms are proposed for passing a single traffic signal: SSOP takes the input of initial state (x_0, v_0) with full knowledge of traffic signal, outputs the optimal plan of a sequence of accelerations that the vehicle has the constant acceleration as control and updates in each position interval Δx; SSSP takes the input of initial state (x_0, v_0, l_0) with only knowledge of current traffic signal, outputs the same format of plan as SSOP with condition of traffic signal change, once the signal change is observed, it switched back to the SSOP deterministic solution.

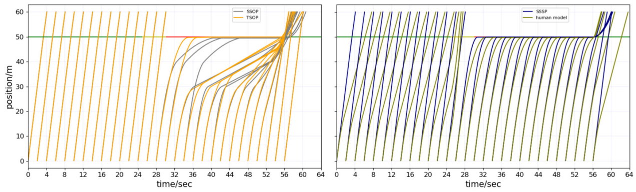
IV. Simulation Result

A series of simulations are presented to validate the proposed algorithms in Section III: 1). optimal planning of the vehicle trajectory profiles for both full and partial knowledge of the signal offset information, with specific values for signal timing and offset information; 2). Monte Carlo simulations over a range of traffic signal offset timing to evaluate the fuel cost benefits of the proposed algorithms.

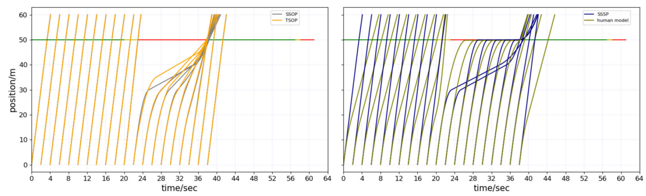
To demonstrate the effectiveness of our proposed algorithm, a TSOP algorithm similar to [4], [5] is implemented in simulation for comparison. As a reference, a human driving model is also added in the simulations based on [9]. The human driver nominally tries to maintain a preferred speed $v^* \in \mathcal{V}$. If a yellow light is observed, an analysis is performed to see if the vehicle could go through the yellow light without violating traffic rules or vehicle dynamic constraints, and if that is not possible, the vehicle will plan a safe stop at the intersection. For simplicity, term "human" is used to represent the aforementioned simplistic human baseline behavior.

A. Simulation Parameter

The simulation scenario consists of one vehicle passing one intersection with traffic signal signal located ar $L = 50$ m, and with dynamics in (9). Two different SPaTs $\{T_g, T_y, T_r\}$ are



(a) signal 1, $v_0 = 15$m/s



(b) signal 2, $v_0 = 15$m/s

Fig. 4. Trajectory profiles of four approaches in two scenarios. Traffic intersection is located at position = 50 m; the vehicle starts with different time offset in the signal period, and ends at the position = 60 m.

tested: signal 1 = $\{25, 5, 26\}$ s and signal 2 = $\{20, 3, 15\}$ s. Initial position of the vehicle is $x = 0$, and initial velocities considered are $v_0 = \{5, 10, 15, 20\}$ m/s. In SSOP and SSSP, other parameters include $\Delta x = 10$ m, $\Delta v = 1$ m/s, integer $m = 6, n = 23$, $v_{\min} = 0$ m/s, and $v_{\max} = 22$ m/s ≈ 50 mph; $u_{\min} = -5$ m/s^2 and $u_{\max} = 8$ m/s^2.

For TSOP, we need to consider discretization in both time and space domain. To keep it comparable to the SSOP simulation, we choose $\Delta t = 0.5$ s so that the corresponding change in position does not exceed the grid size of 10 m used in SSOP even at the maximum speed of 22 m/s. Δx is set to 0.25 m so that it corresponds to the minimum change in position, corresponding to $\Delta v = 1$ m/s in 0.5 s. The preferred velocity for the human model is $v^* = 7$ m/s, which is the optimal nominal speed among the velocity domain defined above. For all simulations, the dynamic state updating rate is 100 Hz.

B. Simulation Results

1) Experiment 1: Given SPaT and initial state, the trajectory profiles of the four planning algorithms with different time offset are plotted in Fig. 4. A notable observation is that the trajectory profiles of SSOP, TSOP and SSSP algorithm with green initial signal are the same, and human model drives slower due to its nominal speed v^*. When the initial signal is not green, SSOP and TSOP pick the optimal solutions that avoid stopping at the intersection, while there is a trade-off between driving at low speed and restarting the vehicle from idling, consistently they stop at the intersection when remaining red signal's time is too long (grey and orange trajectories in Fig. 4(a) with $\phi_0 = 28, 30$ s); SSSP algorithm's action depends on the expectation of traffic signal, it takes the conservative way to stop at the intersection but has the idling-avoiding behavior (Fig. 4(b) with $\phi_0 = 20, 22$ s) since it switches to the deterministic optimal solution after observing light change; human model has a similar behavior as SSSP due to the lack of knowledge in the time offset of signal phase, its departure motion costs more fuel than SSSP with larger acceleration rate.

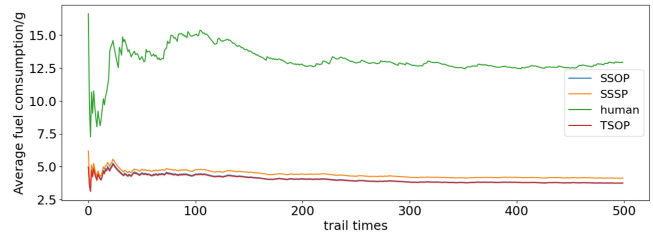


Fig. 5. Fuel cost statics of Monte Carlo simulation with signal 2, $v_0 = 15$ m/s.

TABLE II
MONTE CARLO AVERAGE FUEL COST, UNIT IN GRAM

signal	v_0 (m/s)	SSOP	SSSP	TSOP	human
1	5	7.320	8.588	7.347	20.739
	10	6.013	6.792	5.994	17.204
	15	5.595	6.491	5.440	14.031
	20	5.085	5.796	4.889	13.524
2	5	5.833	7.022	5.894	19.323
	10	4.544	4.942	4.549	14.825
	15	3.766	4.130	3.744	12.940
	20	3.607	3.833	3.458	11.684

TABLE III
TWO DETERMINISTIC ALGORITHMS AVERAGE RUNTIME, UNIT IN SECOND

	signal 1				signal 2			
v_0(m/s)	5	10	15	20	5	10	15	20
SSOP	9.0	9.3	2.3	0.7	8.6	8.9	2.2	0.7
TSOP	51.8	36.9	19.7	10.0	49.1	35.0	17.6	7.9

2) Experiment 2: Given SPaT and initial state, the average fuel cost of the four approaches to pass a single intersection is investigated. For each condition, 500 Monte Carlo trials with stochastic time offset ϕ_0 are conducted as Fig. 5, average fuel cost of all trials are shown in Table II and the average runtime of the two deterministic algorithms in Table III. From both tables and figures, the proposed SSOP and SSSP algorithms have significant improvement in fuel saving than human model, which reduces more than 50% of fuel cost. TSOP approach has the same level of fuel performance as our proposed SSOP and SSSP approaches, while it searches in the space of a finer sampling in displacement and extra time dimension, which shows the advantage of our algorithm in computational efficiency by the sufficient runtime difference in Table III. Runtime measurement is conducted in a machine with 16 cores Intel i9-9880H CPU @ 2.30 GHz in Python 3.6.9 environment. The runtime difference demonstrates the reduction of dimension in SSOP algorithm ($O(mn^m)$ of SSOP and $O((m^2n)^N)$ of TSOP). Between SSOP and SSSP algorithm, the fuel cost difference is typically around 1 g, which shows that SSSP efficiently takes the advantage of the experience-based MDP model to minimize the impact from the uncertainty of traffic signal, along with the expected additional runtime cost.

V. Conclusion

A single vehicle crossing a single intersection is considered in this letter, with full knowledge of the traffic signal timing information. We present planning algorithms for eco-driving, both when the traffic signal initial offset is known (SSOP) and unknown (SSSP). A key component of our approach is the use of a spatially indexed representation of the system dynamics, compared with the typically used time-indexed representation. This formulation allows us to develop a numerical optimization algorithm under full knowledge that is significantly less computationally complex compared with similar dynamic programming based approaches (e.g. [4], [5]). Not only in this specific problem, but this algorithm can also be adapted to other planning problems with the feature of one incremental dimension in space.

Even in the case of partial information, the spatial formulation also allows us to represent the optimization as a compact MDP that can be readily solved with the addition of only one stochastic state. Further, the stochastic optimization problem transitions back to the deterministic problem as soon as a change in the traffic light signal is observed. The proposed SSOP and SSSP algorithm is expected to be implemented in the data-driven methods with table lookup for solutions given input, which releases the online computational burden. Except in the fixed phase signal, the SSOP is also adaptable to actuated phase signal as long as the signal phase information is passed to the vehicle so that optimal solution can be approached by offline calculation.

We demonstrate the validity of the proposed algorithms through numerical simulations, that show the algorithms (i) ensure that the traffic rules are not violated and (ii) result in significant fuel savings compared with a baseline human-driven model, and (iii) consume 5 times less run-time than the classic planning in time-state space. Future work will extend the results of this work to traffic scenarios with multiple vehicles, to discover a cooperative control strategy which is closer to real-life situation.

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