CARNEGIE MELLON UNIVERSITY

Department of Electrical and Computer Engineering

18-751: Applied Stochastic Processes

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Lecture 1

I. Course Overview

Applied Stochastic Processes is a fundamental class for graduate study and research in electrical engineering and computer science. The prerequisites for this class are both the knowledge of undergraduate-level probability and maturity in math (*i.e.*, familiarity with Calculus I, II, and III, Linear Algebra, *etc*.).

This class has three major components; probability ($\approx$ 3 lectures), random variables ($\approx$ 9 lectures), and stochastic (*i.e.,* random) processes ($\approx$ 12 lectures). Although the contents of this class is highly mathematical, the ultimate goal of this class is to enable students to apply the concepts of probability theory and stochastic processes in solving practical and/or real-world research problems. In order to achieve this goal, we will put a heavy emphasis on homework. More specifically, around ten problem sets will be assigned, and they will account for 25% of your final grade. Some of these problem sets will have a MATLAB® component as well. Although one of the weekly recitations by teaching assistants will cover some basics of using MATLAB®, this may not be sufficient for students who are *not* familiar with this tool at all. Therefore, students are expected and should be able to learn how to use MATLAB® by themselves. A recommended short tutorial book for MATLAB® is the book entitled "Getting Started with MATLAB 7: A Quick Introduction for Scientists and Engineers" by R. Pratap.

The required textbook is "Probability, Random Variables, and Stochastic Processes" by A. Papoulis and S. U. Pillai. This book is considered to be "the bible" in probability theory and stochastic processes. Unfortunately, its writing is quite terse and not so reader-friendly; therefore, reading this book requires some effort and patience. As an example, some proofs in this book may skip several intermediate steps. In this case, students are encouraged to figure out the intermediate steps. This is considered a good exercise for making sure that you understand the underlying concepts.

Other recommended supplementary readings are: "Probability and Random Processes for Electrical Engineering" by A. Leon-Garcia, and "Probability and Stochastic Processes for Engineers" by C. W. Helstrom. Copies of the two books will be placed on reserve in the Engineering and Science Library on the fourth floor of Wean Hall.

Please note that policies about plagiarism and late homework submission will be **strictly enforced**. Make sure you understand them and closely follow them. However, collaboration among classmates in *discussing* problem-solving ideas for the homework is encouraged. Nevertheless, you must work independently in writing your own answers for all the problem sets.

II. Probability Theory

The motivation for learning probability theory is to allow one to cope with situations where exact and precise information about all or one of the inputs, the processing, and the output is not available. For such situations, decision making must be based on *statistical averages* in order to deal with the inherent uncertainty.

Usually, a conventional approach in learning probability is to start with the axiomatic description of probability theory (*e.g.*, the probability of the whole sample space is one and the probability of the null set is zero). However, a more useful and intuitive approach is to start with the *non*-axiomatic description or the so-called "relative frequency interpretation." The non-axiomatic description consists of three components: *experiments, outcomes, and events*.

Experiments will be done repeatedly in order to allow us to assign a probability measure to an event. As an example, when we repeatedly toss a fair die, we would expect that after 100 times, we would see even numbers for 50 times.

Example: Flip a coin [Randon

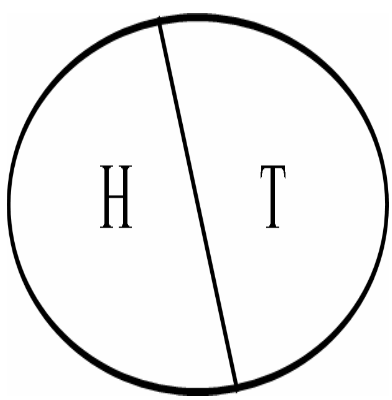

Fig. 1. Venn diagram for a coin tossing experiment with two possible outcomes: H (Head) and T (Tail).

- Two outcomes: H (Head) and T (Tail).
- In this case, random means that ***not*** all the causes are known.
- Possible outcomes can be represented in Probability Space.

To find a probability value for an event, we do the experiment repeatedly for N times. Assume that there are k events, each of which occurs: $n_{\mathrm{a}}, n_{\mathrm{b}}, \cdots, n_{\mathrm{k}}$ times. Therefore, we can assign each event a probability value based on the frequency interpretation, *e.g.*, $\mathbf{P}[a] = \frac{n_{\mathrm{a}}}{N}$. Note that an event can be a *simple event* (*e.g.*, getting a 3 when throwing a die.) or a *compound event* (*e.g.*, getting an even number when throwing a die).

III. Notation

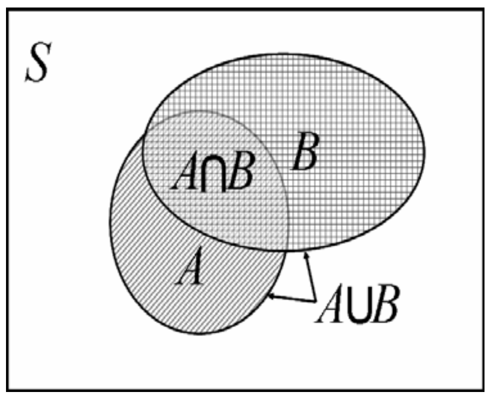

Fig. 2. A simple Venn diagram illustrating the union operator $\cup$ and the intersection operator $\cap$.

- **Sample Space** S: the space that includes all the possible outcomes; thus, $\mathbf{P}[S] = 1$.
- **Null Space** ϕ: the space that contains no possible outcome; thus, $\mathbf{P}[\phi] = 0$.
- Probability of Event A occurring is denoted as $\mathbf{P}[A]$.
- Probability of Event A not occuring is denoted as $\mathbf{P}\left[\bar{A}\right] = 1 - \mathbf{P}[A]$, where $\bar{\cdot}$ represents the complement operator.
- The probability of Event A ***or*** Event B is denoted as $\mathbf{P}[A \text{ or } B]$ or $\mathbf{P}[A + B]$ or $\mathbf{P}[A \cup B]$ where $\cup$ represents the union operator.
- The probability of Event A ***and*** Event B is denoted as $\mathbf{P}[A \text{ and } B]$ or $\mathbf{P}[A \cap B]$ or $\mathbf{P}[AB]$ where cap represents the intersection operator.
- Event A and Event B are **mutually exclusive** when $\mathbf{P}[AB] = 0$.

Note that

$$\mathbf{P}[A+B] = \mathbf{P}[A\bar{B}] + \mathbf{P}[\bar{A}B] + P[AB].$$

Example: Let Event A, Event B, and AB occur n_{a} times, n_{b} times, and n_{ab} times, respectively. Therefore, $\mathbf{P}[A] = \frac{n_{\mathrm{a}}}{N}$; $\mathbf{P}[B] = \frac{n_{\mathrm{b}}}{N}$; and $\mathbf{P}[AB] = \frac{n_{\mathrm{ab}}}{N}$

Conditional Probability: Probability of Event A **given** Event B is denoted as $\mathbf{P}[A/B]$. This is called "conditional probability." In this case, we are interested in the probability of Event A when we know that Event B has already occurred. Saying that Event B has already occurred effectively implies that one changes the probability space from the sample space S (the "whole universe") to the space of Event B.

IV. Bayes Rule

Continuing with the example just given in the previous paragraph, we can compute the conditional probabilty $\mathbf{P}[A/B]$ as:

$$\mathbf{P}[A/B] = \frac{n_{\mathrm{ab}}}{n_{\mathrm{b}}} = \frac{\frac{n_{\mathrm{ab}}}{N}}{\frac{n_{\mathrm{b}}}{N}} = \frac{\mathbf{P}[AB]}{\mathbf{P}[B]}, \text{ when } \mathbf{P}[B] \neq 0.$$

The last equality is called "Bayes Rule"−the conditional probability can be computed as the ratio of the joint probability (*e.g.*, $\mathbf{P}[AB]$) and the nonzero marginal probability (*e.g.*, $\mathbf{P}[B]$). We will see in the remainder of this course that Bayes Rule plays a fundamental role both in the theory and applications of probability, random variables, and stochastic processes.