

Sixteenth International Olympiad, 1974

1974/1 Algebra

1. Three players A, B and C play the following game. On each of three cards an integer is written. These three numbers p, q, r satisfy $0 < p < q < r$. The three cards are shuffled and one is dealt to each player. Each then receives the number of counters indicated by the card he holds. Then the cards are shuffled again, the counters remain with the players. This process (*shuffling, dealing, giving out counters*) takes place for at least two rounds. After the last round, A has 20 counters in all, B has 10 and C has 9. At the last round B received r counters. Who received q counters on the first round?

1974/2 Geometry

2. In the triangle ABC , prove that there is a point D on side AB such that CD is the geometric mean of AD and DB if and only if

$$\sin A \sin B \leq \sin 2\frac{C}{2}. \quad (1)$$

1974/3 Number System

3. Prove that the number

$$\sum_{k=0}^n \binom{2n+1}{2k+1} 2^{3k} \quad (2)$$

is not divisible by 5 for any integer $n \geq 0$.

1974/4 Geometry

4. Consider decompositions of an 8×8 chessboard into p non-overlapping rectangles subject to the following conditions.
(i) Each rectangle has as many white squares as black squares.
(ii) If a_i is the number of white squares in the i -th rectangle, then $a_1 < a_2 < \dots < a_p$. Find the maximum value of p for which such a decomposition is possible. For this value of p , determine all possible sequences a_1, a_2, \dots, a_p .

1974/5 Number System

5. Determine all possible values of

$$S = \frac{a}{a+b+d} + \frac{b}{a+b+c} + \frac{c}{b+c+d} + \frac{d}{a+c+d} \quad (3)$$

Where a, b, c, d are arbitrary numbers.

1974/6 Algebra

6. Let P be a non-constant polynomial with integer coefficients. If $n(P)$ is the number of distinct integers k such that

$$(P(k))^2 = 1 \quad (4)$$

, prove that

$$n(P) - \deg(P) \geq 2 \quad (5)$$

, where $\deg(P)$ denotes the degree of the polynomial P .

Seventeenth International Olympiad, 1975

1975/1 Algebra

7. Let x_i, y_i ($i = 1, 2, \dots, n$) be real numbers such that $x_1 \geq x_2 \geq \dots \geq x_n$ and $y_1 \geq y_2 \geq \dots \geq y_n$. Prove that, if z_1, z_2, \dots, z_n is any permutation of y_1, y_2, \dots, y_n , then

$$\sum_{i=1}^n (x_i - y_i)^2 \leq \sum_{i=1}^n (x_i - z_i)^2 \quad (6)$$

1975/2 Number System

8. Let $a_1, a_2, a_3, \dots, a_n$ infinite increasing sequence of positive integers. Prove that for every $p \geq 1$ there are infinitely many a_m which can be written in the form

$$a_m = x_{ap} + y_{aq} \quad (7)$$

with x, y positive integers and $q > p$.

1975/3 Geometry

9. On the sides of an arbitrary triangle ABC , triangles ABR, BCP, CAQ are constructed externally with

$$\angle CBP = \angle CAQ = 45^\circ, \angle BCP = \angle ACQ = 30^\circ, \angle ABR = \angle BAR = 15^\circ \quad (8)$$

. Prove that

$$\angle QRP = 90^\circ \text{ and } QR = RP \quad (9)$$

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1975/4 Number System

10. When 4444^{4444} is written in decimal notation, the sum of its digits is A . Let B be the sum of the digits of A . Find the sum of the digits of B . (A and B are written in decimal notation).

1975/5 Geometry

11. Determine, with proof, whether or not one can find 1975 points on the circumference of a circle with unit radius such that the distance between any two of them is a rational number.

1975/6 Algebra

12. Find all polynomials P , in two variables, with the following properties.

(i) for a positive integer n and all real t, x, y

$$P(ta, ty) = t^n P(x, y) \quad (10)$$

(that is, P is homogeneous of degree n),

(ii) for all real a, b, c

$$P(b+c, a) + P(c+a, b) + P(a+b, c) = 0 \quad (11)$$

,
(iii) $P(1, 0) = 1$.

Eighteenth International Olympiad, 1976

1976/1 Geometry

13. In a plane convex quadrilateral of area 32, the sum of the lengths of two opposite sides and one diagonal is 16. Determine all possible lengths of the other diagonal.

1976/2 Algebra

14. Let

$$P_1(x) = x^2 - 2 \text{ and } P_j(x) = P_1(P_{j-1}(x)) \text{ for } j = 2, 3, \dots \quad (12)$$

. Show that, for any positive integer n , the roots of the equation $P_n(x) = x$ are real and distinct.

1976/3 Geometry

15. A rectangular box can be filled completely with unit cubes. If one places as many cubes as possible, each with volume 2, in the box, so that their edges are parallel to the edges of the box, one can fill exactly 40% of the box. Determine the possible dimensions of all such boxes.

1976/4 Number systems

16. Determine, with proof, the largest number which is the product of positive integers whose sum is 1976.

1976/5 Algebra

17. Consider the system of p equations in $q = 2p$ unknowns x_1, x_2, \dots, x_q .

$$a_{11}x_1 + a_{12}x_2 + \dots + a_{1q}x_q = 0 \quad a_{21}x_1 + a_{22}x_2 + \dots + a_{2q}x_q = 0 \dots \quad (13)$$

$$a_{p1}x_1 + a_{p2}x_2 + \dots + a_{pq}x_q = 0 \quad (14)$$

with every coefficient a_{ij} member of the set $\{-1, 0, 1\}$. Prove that the system has a solution (x_1, x_2, \dots, x_q) such that.

(a) $a_{ij}x_j$ ($j = 1, 2, \dots, q$) are integers,

(b) there is at least one value of j for which $x_j \neq 0$,

$$(c) |x_j| \leq q \quad (j = 1, 2, \dots, q) \quad (15)$$

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1976/6 Number Systems

18. A sequence $\{u_n\}$ is defined by

$$u_0 = 2, u_1 = \frac{5}{2}, u_{n+1} = u_n (u_2^{n-1} - 2) - u_1 \text{ for } n = 1, 2, \dots \quad (16)$$

Prove that for positive integers n ,

$$(u_n) = 2^{\frac{(2n - (-1)^n)}{3}} \quad (17)$$

where (x) denotes the greatest integer $\leq x$.