Lecture -30

Queueing Models (weighting line model)

→ Queueing model can be of the following types

- Single Server
- Multiple Server

It can be also divided or classified as finite queue length or Infinite queue length.

→ Finite queue length → It has restricted or limited length for the queue.

→ Infinite queue length → It has no limit for the length of queue.

Another classification is

→ Finite population model : The number of members in queue is already fixed. No new addition to queue will be included.

→ Infinite population model : There is not fixed queue members.

poisson distribution

arrivals → λ/hour

Services → μ/hour (exponential)

$$\left\{ \frac{\lambda}{\mu} < 1 \right\}$$ → Infinite queue length Condition.

* Balking → If the arrival does not join the line of queue and it leaves

* Reneging → The phenomenon when the job in the queue waiting to be served moves away from the queue after waiting for a particular period of time without being Served.

* ~~Joking~~
* Jockeying : when there are multiple Servers in a System, the jobs in the queue of the Servers switch the queue between the Server lines.

⇒ Single Server Infinite queue length model

(M/M/1 : ∞/∞ model)

arrival (λ/h) — service (μ/h) — No. of server — infinite queueing length and population.

Four important System parameters:

[L_s] * length of System → No. of people in the System

[L_q] * Length of queue → No. of people waiting in queue for service

[W_s] * Weighting time in System → Total time spend in the whole System.

[W_q] * Weighting time in queue → Time Spend Spend in queue to enter in the server.

$P_0, P_1, P_2, \ldots$ → Probability of persons in the System.

Derivation

$$P_n(t+h) = \underbrace{P_{n-1}(t)}_{\text{Probability of one arrival and No service}} + \underbrace{P_{n+1}(t)}_{\text{Probability of No arrival and one service}} + \underbrace{P_n(t)}_{\text{Probability of no arrival and no service}}$$

→ This is a memoryless property.

$$\Rightarrow P_n(t+h) = P_{(n-1)}(t) * \lambda h(1-\mu h) + P_{n+1}(t) * \mu h(1-\lambda h)$$

$$+ P_n(t)(1-\lambda h)(1-\mu h)$$

$$= P_{n-1}(t)\lambda h + P_{n+1}(t)\mu h - P_n(t)(1-\lambda h-\mu h)$$

$$\frac{P_n(t+h) - P_n(t)}{h} = P_{n-1}(t)\lambda + P_{n+1}(t)\mu - P_n(t)(\lambda+\mu)$$

When we apply steady state condition,

$$\boxed{\Rightarrow \lambda P_{n-1} + \mu P_{n+1} = (\lambda+\mu)P_n \quad (1)}$$

$P_{n-1} \rightarrow$ Probability that there are $n-1$ person in s/m

$P_{n+1} \rightarrow$ Probability that there are $n+1$ persons in s/m.

$P_n \rightarrow$ Probability that there are n person's in the s/m.

Making Small changes to the initial derivation we had,

$$P_0(t+h) = P_1(t)\,(1-\lambda h)\,\mu h + P_0(t)\,(1-\lambda h)\,1$$

$$P_0(t+h) = P_1(t)\,\mu h + P_0(t)\,(1-\lambda h)$$

$$P_0 \frac{(t+h) - P_0(t)}{h} = P_1(t)\,\mu - P_0(t)\,\lambda$$

$$\boxed{\mu P_1 = \lambda P_0 \quad \text{—} \quad (2)}$$

$$\therefore \quad P_1 = \frac{\lambda}{\mu} P_0$$

$$\lambda P_0 + \mu P_2 = (\lambda + \mu) P_1$$

$$\cancel{\lambda P_0} + \mu P_2 = \lambda P_1 + \mu P_1$$

$$= \lambda P_1 + \cancel{\lambda P_0}$$

$$P_2 = \left(\frac{\lambda}{\mu}\right) P_1 = \left(\frac{\lambda}{\mu}\right)^2 P_0$$

$$\therefore \quad P_1 = \rho\, P_0$$

$$P_2 = \rho P_1 = \rho^2 P_0$$

$$P_3 = \rho P_2 = \rho^3 P_1$$

$$\therefore \boxed{P_n = \rho^n P_0}$$

$$\therefore P_0 + P_1 + P_2 + \cdots + \infty = 1$$

$$P_0 + \rho P_0 + \rho^2 P_0 + \cdots \infty = 1$$

$$P_0 (1 + \rho + \rho^2 + \cdots \infty) = 1$$

$$P_0 \left[\frac{1}{1-\rho} \right] = 1$$

$$P_0 = 1 - \rho$$

Now we will derive expression for L_s, L_q, W_s, W_q.

$$L_s = \sum_{j=0}^{\infty} j P_j = \sum j \rho^j P_0$$

$$= P_0 \rho \sum j \rho^{j-1}$$

$$= P_0 \rho \sum \frac{d}{d\rho} \rho^j$$

$$= P_0 \rho \frac{d}{d\rho} \sum \rho^j$$

$$= P_0 \rho \frac{d}{d\rho} [1 + \rho + \rho^2 + \cdots \infty]$$

$$= P_0 \rho \frac{d}{d\rho} \frac{1}{1-\rho}$$

$$= P_0 \rho \frac{1}{(1-\rho)^2}$$

$$= \frac{(1-\rho)\rho}{(1-\rho)^2} = \frac{\rho}{(1-\rho)}$$

$$\therefore \boxed{L_s = \frac{\rho}{1-\rho}}$$

$L_s = L_q +$ expected served.

$$L_s = L_q + \frac{\lambda}{\mu}$$

$$\therefore \left.\begin{aligned} L_s &= \lambda W_s \\ L_q &= \lambda W_q \end{aligned}\right\} \text{Little's equation.}$$

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Next model is $[M/M/1 : N/\infty]$

Here we don't have infinite no. of persons.

$$\therefore P_0 + P_1 + P_2 + \cdots + P_N = 1$$

$$P_0 + \rho P_0 + \rho^2 P_0 + \cdots \rho^N P_0 = 1$$

$$P_0 \left[1 + \rho + \rho^2 + \cdots + \rho^N \right] = 1$$

$$P_0 \left[\frac{1 - \rho^{N+1}}{1 - \rho} \right] = 1$$

$$P_0 = \frac{1 - \rho}{1 - \rho^{N+1}}$$

$$\therefore P_n = \rho^n P_0 \quad , n = 1, 2, \ldots$$

$$L_s = \sum_{n=0}^{N} n P_n$$

$$= \sum_{n=0}^{N} n \rho^n P_0$$

$$= P_0 \rho \sum_{n=0}^{N} n \rho^{n-1}$$

$$= P_0 \rho \frac{d}{d\rho} \sum \rho^n$$

$$= P_0 \rho \frac{d}{d\rho}\left[\frac{1-\rho^{N+1}}{1-\rho}\right]$$

$$= P_0 \rho \left[\frac{(1-\rho)(N+1)\rho^N - (1-\rho^{N+1})}{(1-\rho)^2}\right]$$

$$= \frac{P_0 \rho}{(1-\rho)^2}\left[1-\rho^{N+1} + (N+1)\rho^N (1-\rho)\right]$$

$$= \frac{P_0 \rho}{(1-\rho)^2}\left[1-\rho^{N+1} - (N+1)\rho^{N+1} + (N+1)\rho^N\right]$$

$$= \frac{P_0 \rho}{(1-\rho)^2}\left[1+N\rho^{N+1} - (N+1)\rho^N\right]$$

$$= \frac{1-\rho}{(1-\rho^{N+1})(1-\rho)^2}$$

$$= \frac{\rho}{(1-\rho)(1-\rho^{N+1})}\left[1+N\rho^{N+1} - (N+1)\rho^N\right]$$

$\therefore \quad t_s = L_q \times \frac{\lambda}{\mu}$ This happens due to Belking.

Next model (multiple server model)

$$[M/M/C : \infty/\infty]$$

Here $\frac{\lambda}{C\mu} < 1$ (C = No. of server).

$\lambda_n = \lambda$

$\mu_n = n\mu$

$= C\mu$

General expression, $P_n = \rho^n P_0$

(we consider 4 server sm)

$$= \frac{\lambda^n}{\mu_n} P_0$$

$$P_n = \frac{\lambda^n}{\mu, 2\mu, \ldots n\mu} P_0$$

$$= \left(\frac{\lambda}{\mu}\right)^n \frac{1}{n!} P_0 \quad ; \; n < C$$

$$= \frac{\lambda^n}{\mu \, 2\mu \, 3\mu \, 4\mu \, 4\mu \, 4\mu}$$

$$P_n = \frac{\lambda^n}{C! \, \mu^n C^{n-C}} P_0$$

$$P_0 + P_1 + P_2 \cdots P_\infty = 1$$

$$\sum_{n=0}^{c-1} \frac{\rho^n}{n!} P_0 + \sum_{n=c}^{\infty} \frac{\rho^n}{c!\, c^{n-c}} P_0 = 1$$

$$= P_0 \left[\sum_{0}^{c-1} \frac{\rho^n}{n!} + \sum_{n=c}^{\infty} \frac{\rho^c}{c!} \frac{\rho^{n-1}}{c^{n-c}} \right] = 1$$

$$= P_0 \left[\sum_{0}^{c-1} \frac{\rho^n}{n!} + \frac{\rho^c}{c!} \left\{ 1 + \rho/c + (\rho/c)^2 + \cdots \infty \right\} \right]$$

$$\left[\frac{\rho}{c} = \frac{\lambda}{c\mu} < 1 \right]$$

$$\left[P_0 = \frac{1}{\sum_{0}^{c-1} \frac{\rho^n}{n!} + \frac{\rho^c}{c!} \frac{1}{(1-\rho/c)}} \right]$$

So with this probability equation we can calculate L_s, L_q, W_s, W_q