Programming with Data Structures

CMPSCI 187
Spring 2016

- **Please find a seat**
  - **Try to sit close to the center (the room will be pretty full!)**
- **Turn off or silence your mobile phone**
- **Turn off your other internet-enabled devices**

Last Lecture

- Postfix Expression, such as `2 14 + 23 *`
- It does not require any parentheses, and is easy to evaluate using a stack:
    1. Scan the expression
    2. When we see an operand, push it onto the stack.
    3. When we see an operator, pop two values off the stack, apply the operator, push the result back onto the stack.

Last Lecture

- So far we’ve assumed **binary** operators. What about **unary** operators, such as the **negate** operator: `!`
- A unary operator requires just one operand. So when you see a unary operator, pop **one** operand off the stack, apply the operator, and push it back onto the stack.
- Example: `2 ! 5 ! - 1  4 ! - *`
- You will implement the negate operator for project 4.

This Lecture

- Thinking **Recursively.**
- What is Recursion, some basic examples.
- Three questions about a recursive algorithm.
- Stack Frames and StackOverflowException
- Tower of Hanoi.

What is Recursion?

- A method that calls itself:

```
void recursiveMethod() {
   … …
   recursiveMethod();
}
```

- Like self-referential structures (e.g. LLNode), the compiler is totally cool with this.

- A conceptually simple way to solve many problems (although might not be computationally efficient).

What is Recursion?

- Recursion can also take the form of two or more methods that call each other and form a loop.

```
void first() {
  … …
  second();
}
void second() {
  … …
  first();
}
```

Basic Recursion Examples

- Say method (or function) **`intsum(n)`** computes the sum of integers from 1 to n.
- We already know two ways to write this method.
  1. Write a loop to compute the sum from 1 to n.
  2. Or mathematically **`intsum(n) = n(n+1)/2`**
- But we can also think about it recursively, that is, **`intsum(n)`** is equal to the sum of integers from 1 to (n-1) plus n. In other words:

```
intsum(n) = intsum(n-1) + n
```

Basic Recursion Examples

- To write down this idea in code:

```
int intsum(int n) {
  return intsum(n-1) + n;
}
```

- Think about “passing the buck” (i.e. the responsibility).
  - But the buck has to stop at some point!
  - When n=1, we know the answer is 1, so we can return without making further recursion.
  - This is called the ‘**base case**’.

Basic Recursion Examples

- To write down this idea in code:

```
int intsum(int n) {
  if(n==1) return 1;
  else return intsum(n-1) + n;
}
```

- Even with this base case, you still need to be careful, for example, what happens if I call **`intsum(0)`** or **`intsum(-1)`**? We will come back to this question in a little bit.

Basic Recursion Examples

- Let’s take a look at another example:
- Mathematically, what does **`n!`** mean?

Computing Factorials

- Given a non-negative integer n, its **factorial**, written as **`n!`** is the **product** of all integers from 1 to n.

  `n! = 1 x 2 x 3 x 4 …. x (n-1) x n`

- For example: `5! = 1 x 2 x 3 x 4 x 5 = 120`
- This is similar to `intsum(n)`, except using multiplication.
- Do you know what is `0!` equal to?
- In terms of grow speed, factorial **grows extremely fast**, exceeding the exponential family. A well-known example is writing down all possible permutations of n elements.

Computing Factorials

- Similar to before, you can write down a loop to calculate the factorial. But you can also think about the problem recursively:

```
factorial(n) = factorial(n-1) * n;
factorial(0) = 1  // base case
```

- This translates directly to code:

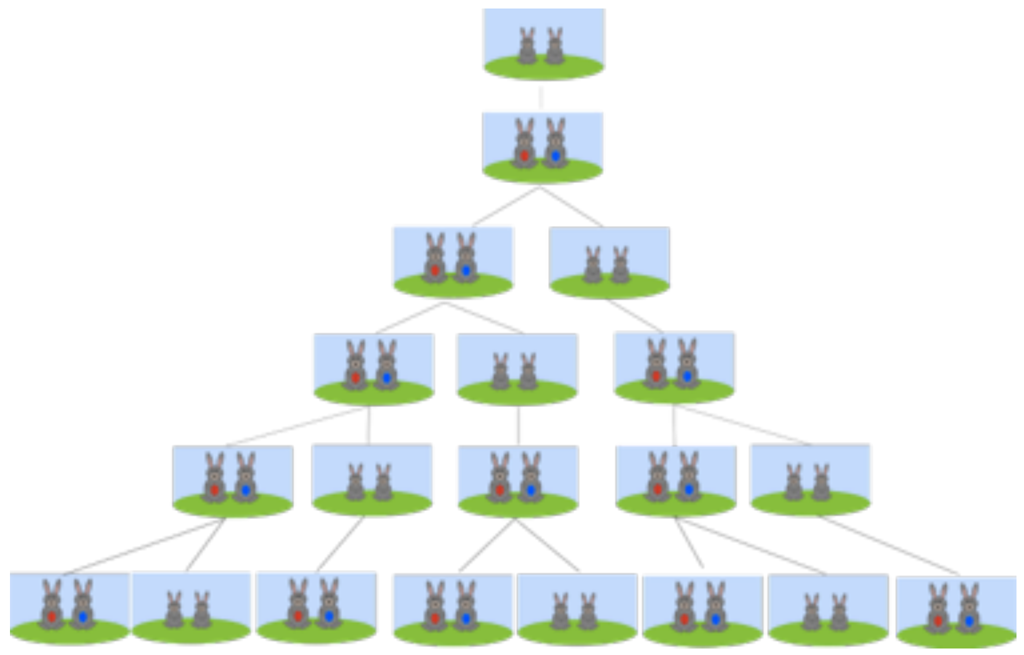
```
int factorial(int n) {
  if(n==0) return 1;
  else return (factorial(n-1)*n);
}
```

Fibonacci Numbers

- Now let's look at a slightly more complex problem: the **Fibonacci** numbers. The story came from a simple mathematical model of reproducing rabbits.

- The Fibonacci numbers are defined recursively:

```
F(n) = F(n-1) + F(n-2)
F(0) = 1
F(1) = 1
```



- In sequence:

```
1, 1, 2, 3, 5, 8, 13, 21
```

Fibonacci Numbers

- This directly translates to the following code:

```
int Fibonacci(int n) {
   if(n==0 || n==1)
     return 1;
   else
     return Fibonacci(n-1) + Fibonacci(n-2);
}
```

Clicker Question #1

```
public int foo (int n) {
    if (n <= 1) return 0;
    return 1 + foo(n-2);
}
```

- What's the return value of this method if it is called with a positive integer **n** as parameter?

  (a) n

  (b) n/2+1

  (c) n/2

  (d) 2*n

3 questions about Recursion

- For a recursive algorithm to work correctly on a given input value, we need to verify the following three questions:

    1. Does the algorithm have a base case?
    2. Does every recursive call make progress toward the base case?
    3. Does the call to the algorithm get the right answer if we assume that all subsequent recursive calls get the right answer (i.e. induction)?

Three Questions

- A base case is where there is no further recursion. Our factorial algorithm has a base case of n = 0.
- We need to guarantee progress towards the base case. For example, in the factorial example, parameter n gets smaller at each recursion.
- We can justify correctness. For example, assuming `factorial(n-1)` returns the correct result, we know that `factorial(n-1)*n` gives the correct result of `factorial(n)`.

Clicker Question #2

```
public void clear(Stack<Integer> s) {
   if (!s.isEmpty()) {
      s.pop();
      clear(s);
   }
}
```

- What is the base case of this recursive method?
  - (a) when `s.isEmpty()` returns true.
  - (b) when a `StackUnderflowException` is thrown.
  - (c) when `clear(s)` throws an exception.
  - (d) this method has no base case.

Another One

Clicker Question #3

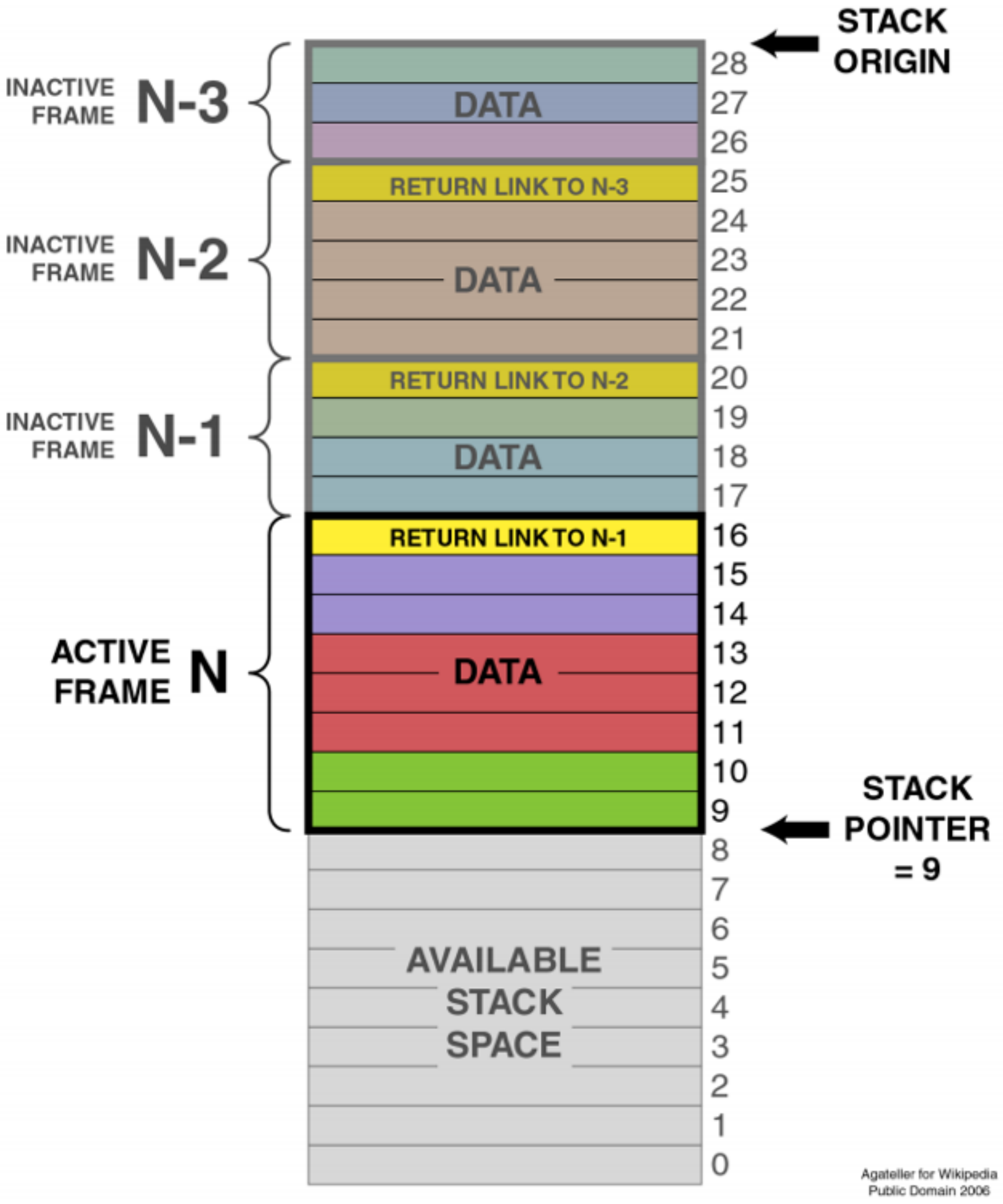
```
public int sum2(int n) {
   if(n==0)
     return 0;
   else
     return n+sum2(n-2);
}
```

- What happens if we call `sum2(123)`?

```
(a) the return value is 123*(123+1)/2 = 7626.
(b) the return value is 123*(123+1)/4 = 3813.
(c) the return value is 122*(122+2)/4 = 3782.
(d) it throws StackOverflowException.
```

Program Stacks

- What’s really going on with recursion? How does the run-time system handle recursive calls?
- Computers use **stacks** to handle method calls and returns. Method calls are executed in **Last-In-First-Out (LIFO)** fashion (recall the ‘clean your house’ task and all the additional tasks it incurred).
- A method’s local variables and return link are kept in the stack memory. This is called a **stack frame**.
- The run-time system has a limited amount of stack memory.


STACK ORIGIN
INACTIVE FRAME N-3
DATA
INACTIVE FRAME N-2
RETURN LINK TO N-3
DATA
INACTIVE FRAME N-1
RETURN LINK TO N-2
DATA
ACTIVE FRAME N
RETURN LINK TO N-1
DATA
STACK POINTER = 9
AVAILABLE STACK SPACE
28
27
26
25
24
23
22
21
20
19
18
17
16
15
14
13
12
11
10
9
8
7
6
5
4
3
2
1
0
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Program Stacks

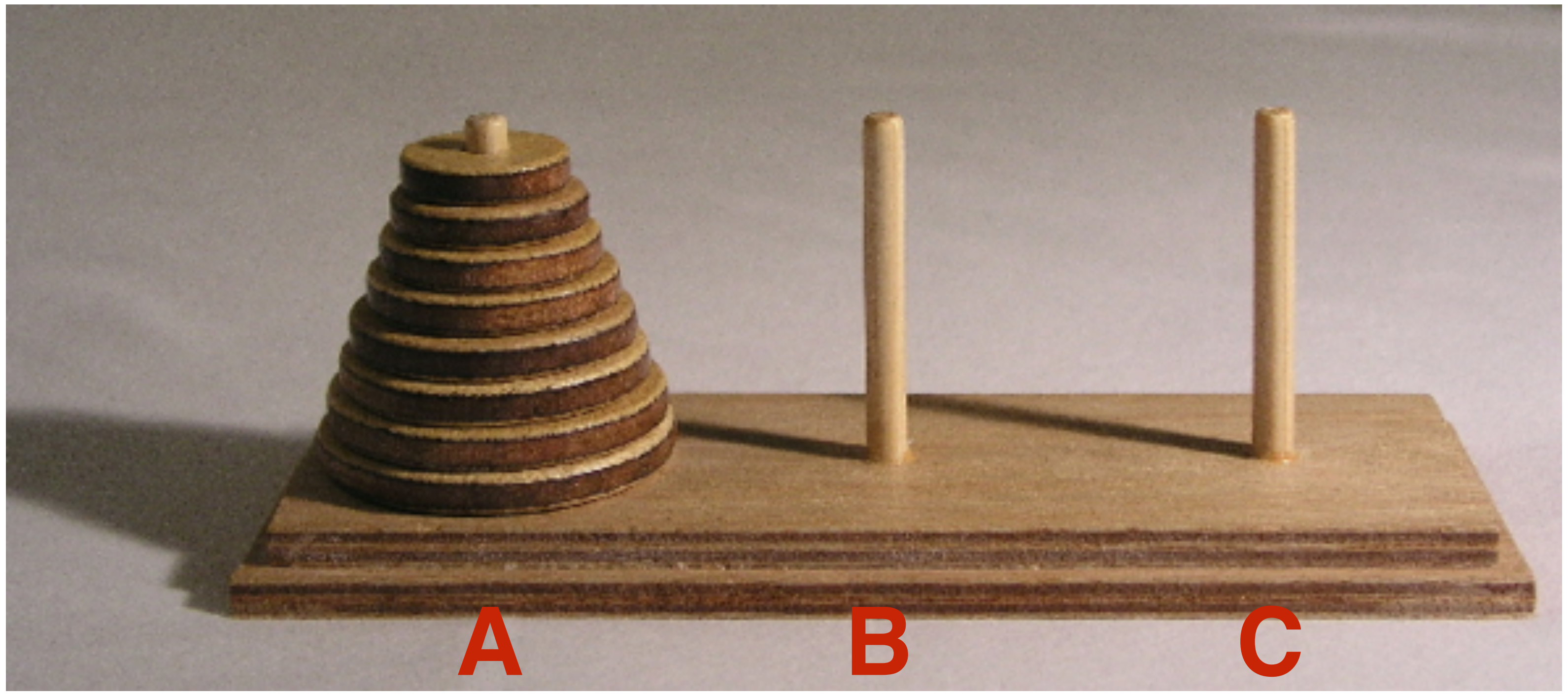
- Upon **calling** a method, the local variables and return link of the current method (caller) are preserved in the stack; then the stack pointer moves up to the new frame (callee).
- Upon **returning**, the stack pointer moves back to the caller's frame, allowing the computation to continue there.
- This calling mechanism is the same for all methods calls, regardless of whether a method calls itself or another method.

Recursion

- Therefore to the compiler and run-time system, there is no distinction between recursive vs. non-recursive calls, they are treated the same way.
- Each stack frame keeps a separate copy of the method's local variables (remember, arguments are also local variables).
- Conceptually, it's easy to understand recursion in a top-down manner (i.e. `n! = (n-1)! * n`)
- Computationally, it's really done from bottom-up.
- Show how `factorial(5)` works.

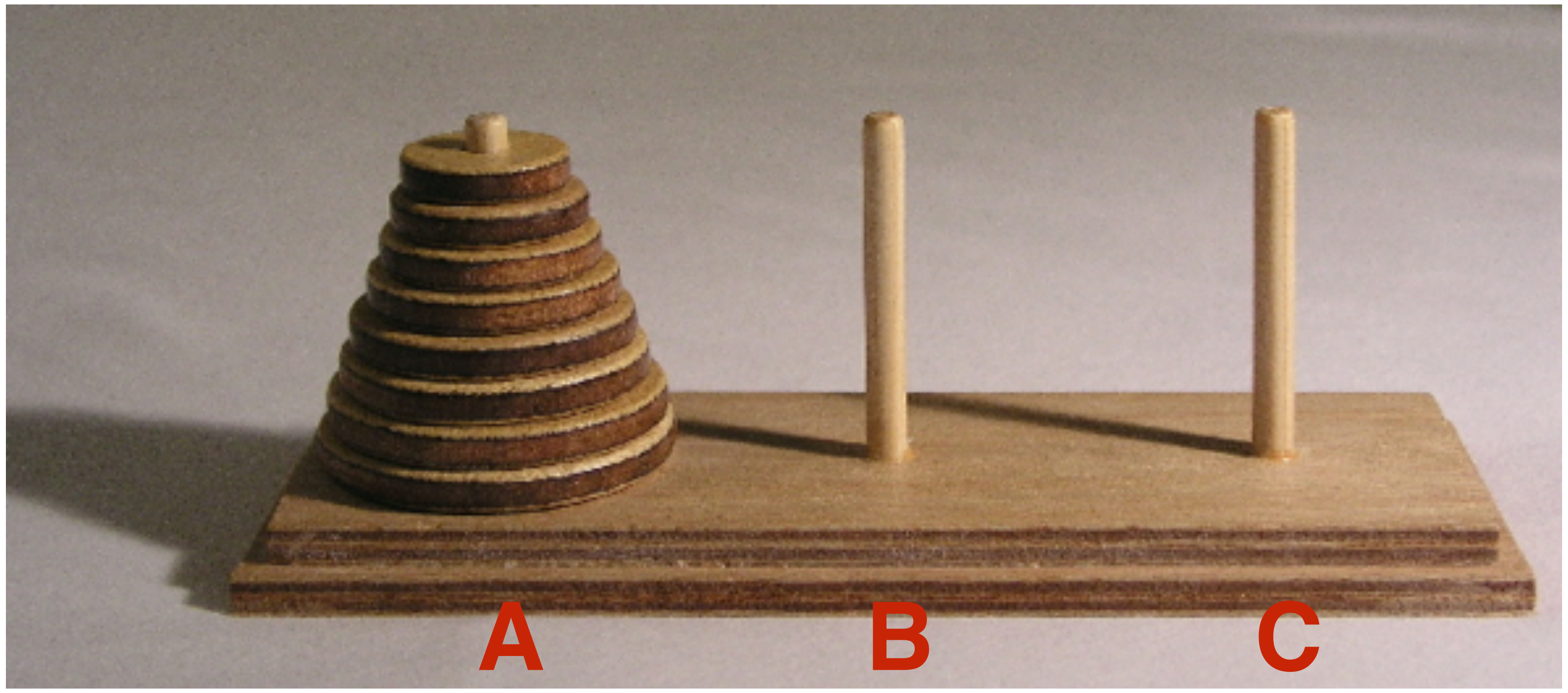
Towers of Hanoi

- A puzzle consisting of **three rods (A, B, C)**, and **n** disks, each at a different size and can slide onto any rod.



- Initially all disks are stacked onto stack A in ascending order of size (smallest at the top), forming a cone shape.

Towers of Hanoi



- **Objective**: move all disks from **A** to **C**.
- **Rules**:

Youtube demo

1. Only one disk can be moved at a time (from the top of one stack to the top of another stack)
2. No disk can be placed on top of a smaller disk (i.e. the disks on each stack must be sorted at all times)

Towers of Hanoi

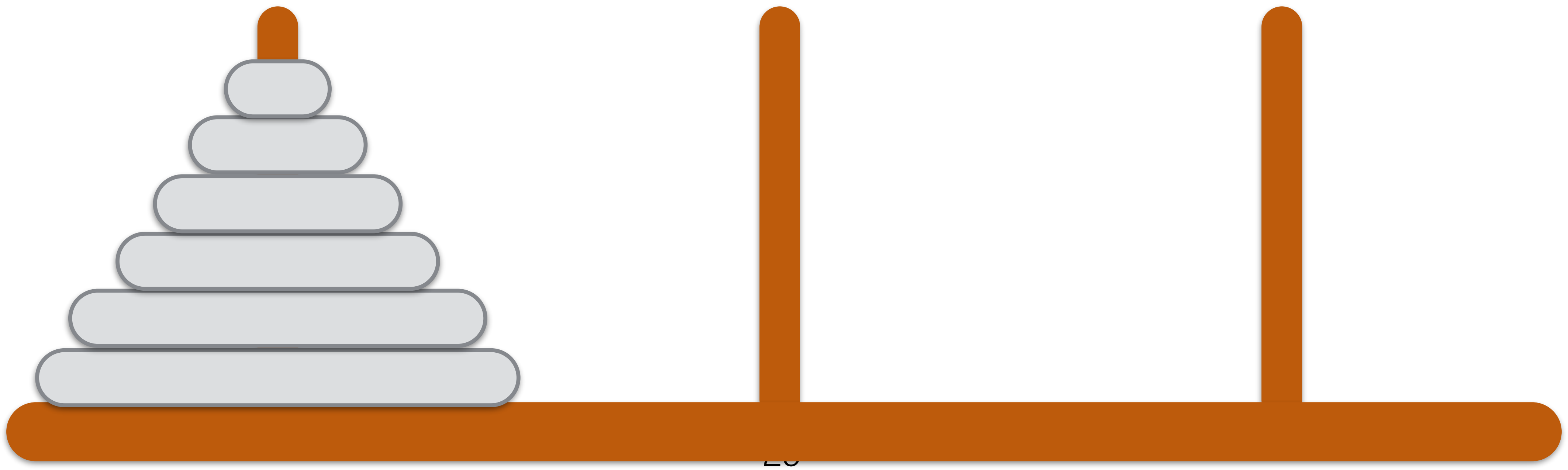
- Play the game to gain some intuitions first. Also pay attention to the number of steps involved in each case.
  - n=1
  - n=2
  - n=3
  - n=4

Towers of Hanoi

- Play the game to gain some intuitions first. Also pay attention to the number of steps involved in each case.
    - n=1: 1 step (A->C)
    - n=2: 3 steps (A->B, A->C, B->C)
    - n=3: 7 steps
    - n=4: what's your guess?

Towers of Hanoi

- Initially we have **n** disks stacked on column A. Let's call the top (**n-1**) disks a **sub-stack**.



Towers of Hanoi

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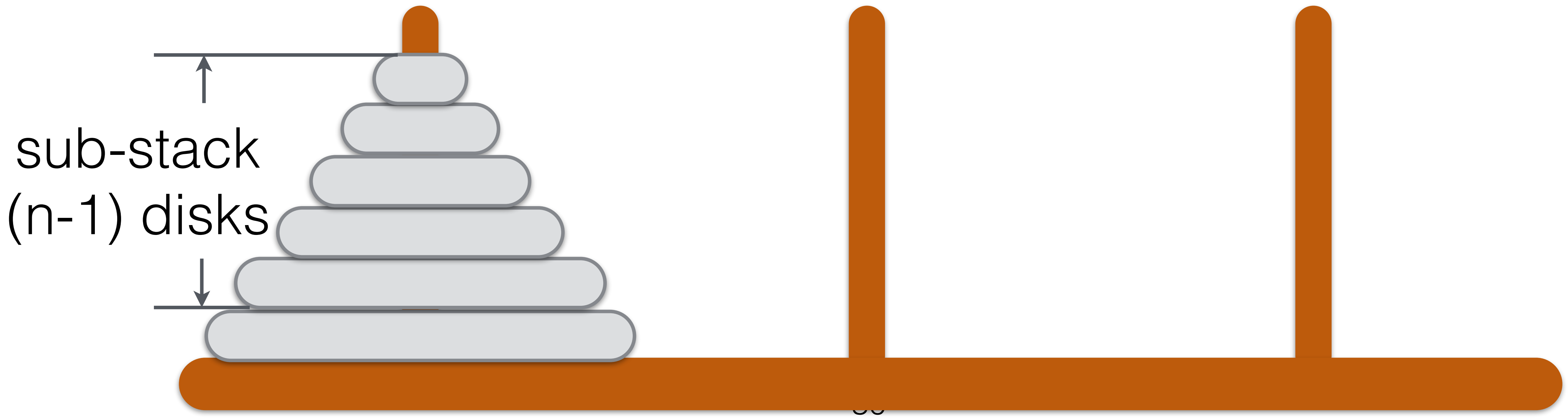
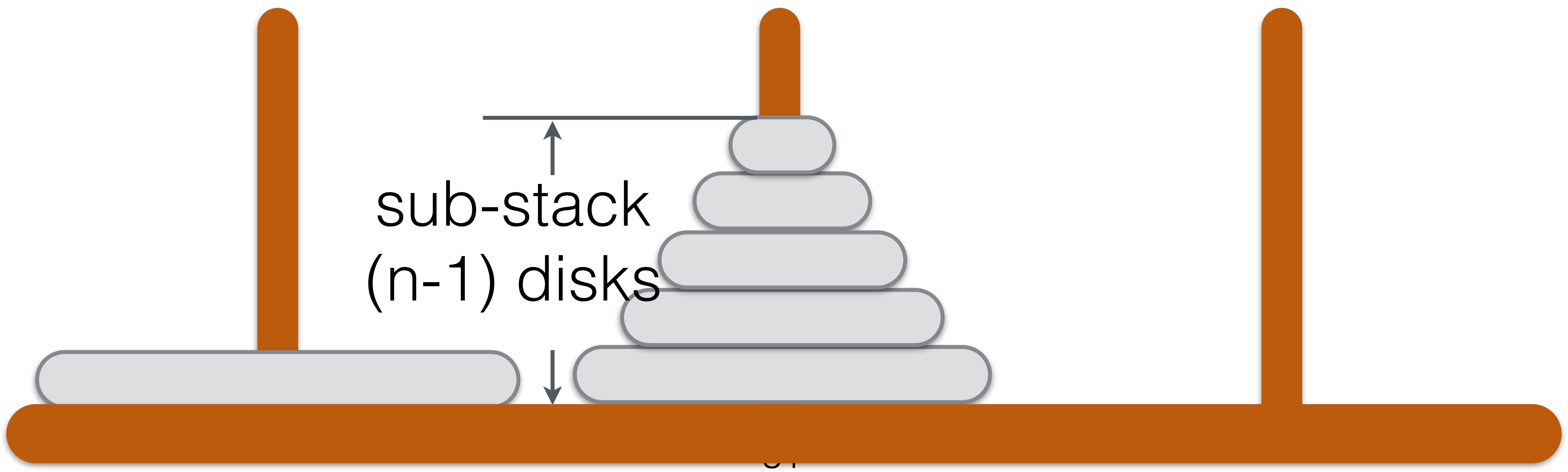


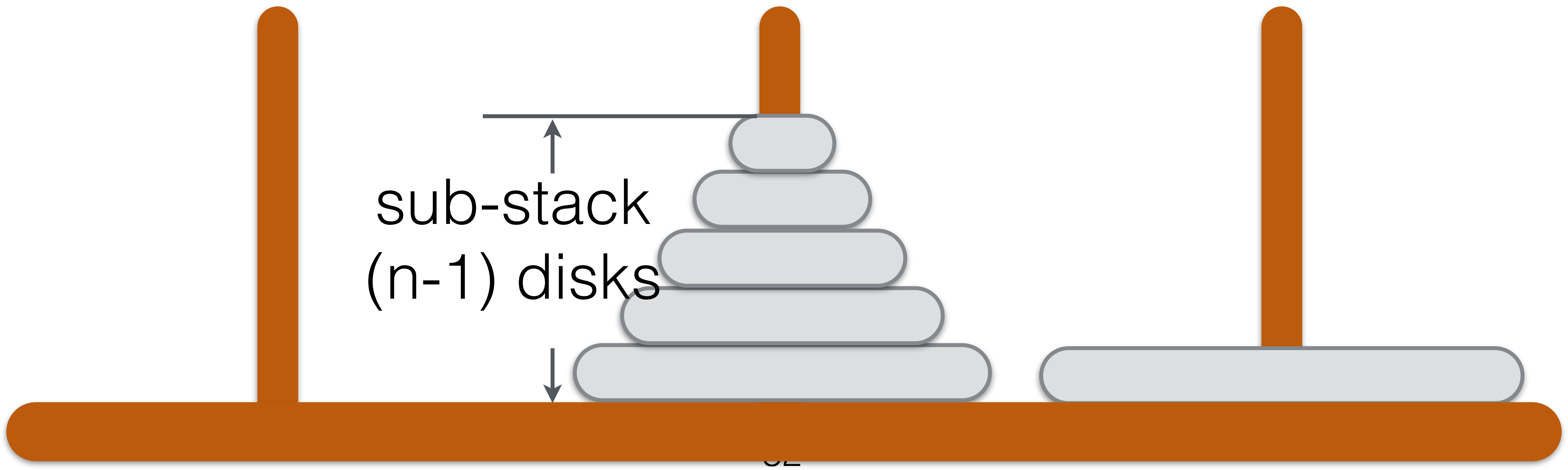
Towers of Hanoi

- Initially we have **n** disks stacked on column A. Let's call the top (**n-1**) disks a **sub-stack**.
- Assume you have some 'magical' way to move the sub-stack **from A to B via column C**.



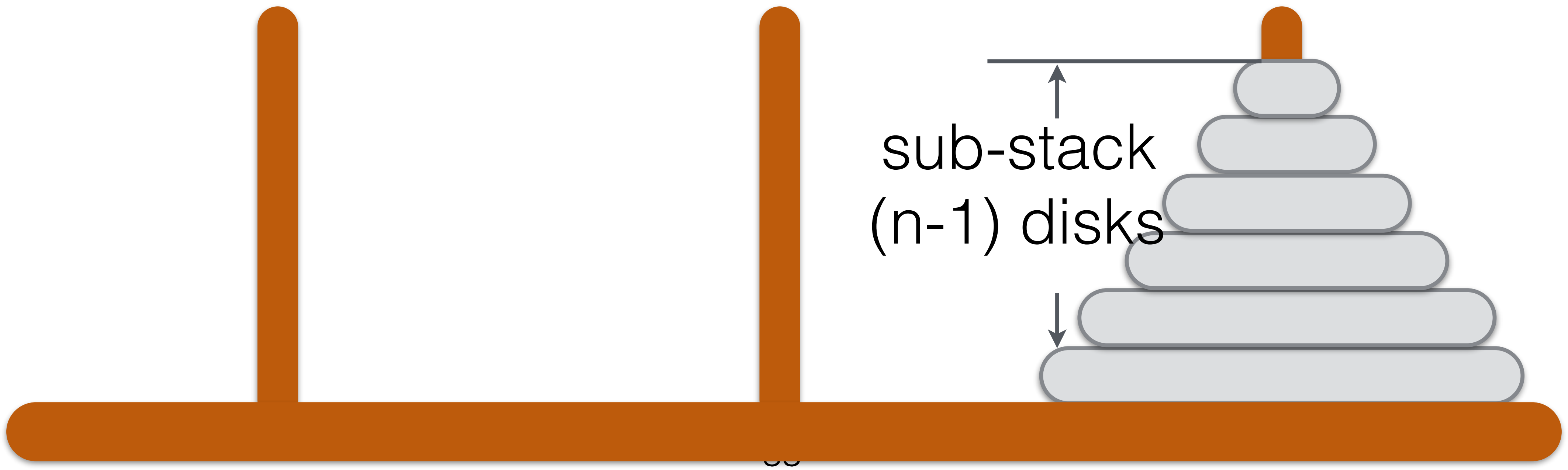
Towers of Hanoi

- Initially we have **n** disks stacked on column A. Let's call the top (**n-1**) disks a **sub-stack**.
- Assume you have some 'magical' way to move the sub-stack **from A to B via column C**.
- Move the remaining one disk from A to C.



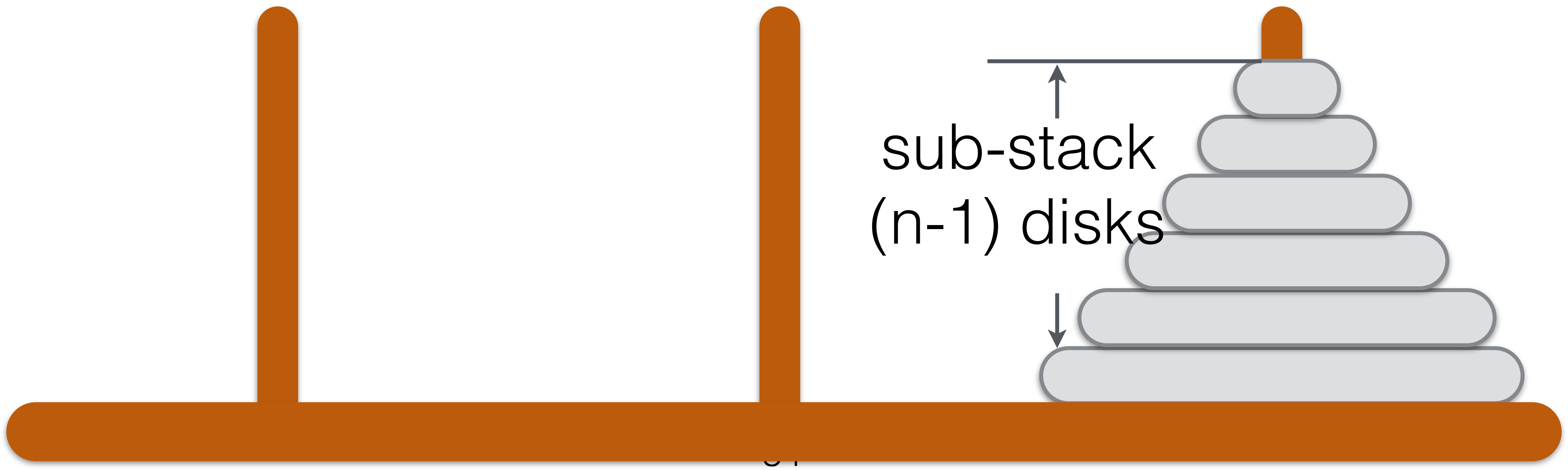
Towers of Hanoi

- Initially we have **n** disks stacked on column A. Let's call the top (**n-1**) disks a **sub-stack**.
- Assume you have some 'magical' way to move the sub-stack **from A to B via column C**.
- Move the remaining one disk from A to C.
- 'Magically' move the sub-stack **from B to C via A**.



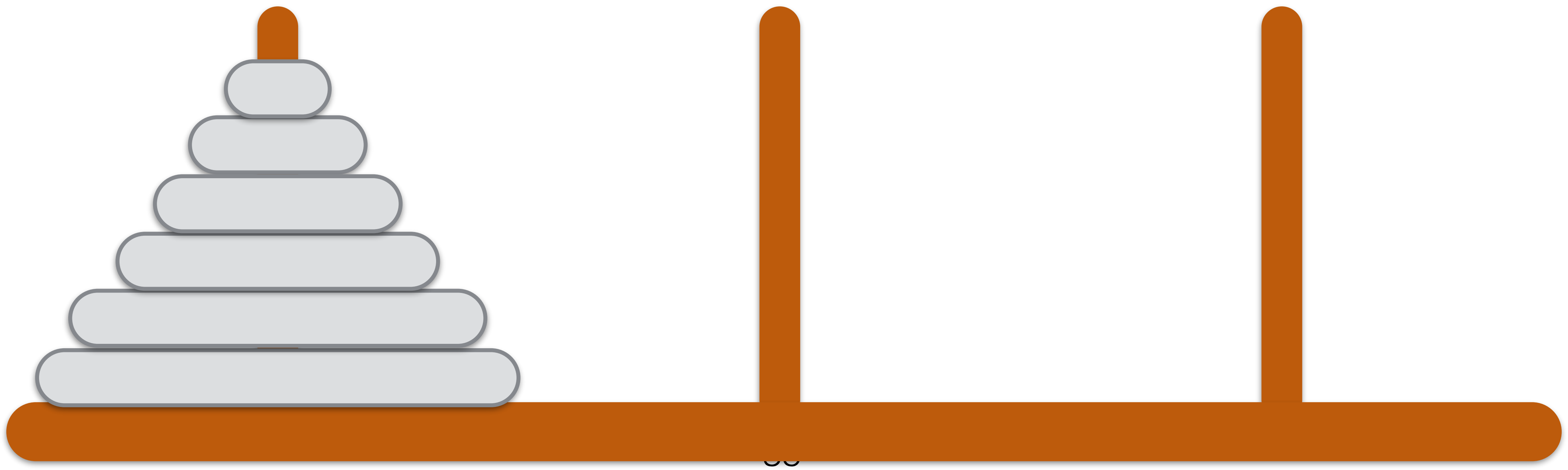
Towers of Hanoi

- This is similar to the n=2 case, except the top disk there is now a sub-stack.
- So how do we 'magically' move the sub-stack? This is where recursion comes handy.
- Moving (n-1) disks from A to B via C is similar to the original problem, but with 1 less disk. We can recursively break it down to a (n-2) problem and so on.



Towers of Hanoi

- Say we want to move n disks from a **From column F** to a **To column T**, using a **Via column V**.
- We use different terms than A, B, C, because during recursion, the associations of A, B, C to F, T, V will change.
- Initially:
  - **F**rom is A, **V**ia is B, **T**o is C



Towers of Hanoi

To move the entire stack of **all disks** from **F** (via V) **to T**

1. Move the sub-stack (consisting of the top n-1 disks) from **F** (via T) **to V.**
2. Move the remaining (1) disk from **F to T**.
3. Move the sub-stack (consisting of the same n-1 disks as in step 1) from **V** (via F) **to T**.

Steps 1 and 3 (moving sub-stacks) involve recursion.

The base case is when the sub-stack has only 1 disk, in which case if can be trivially moved.

Recursive doTowers()

```
public void doTowers(int n, char from, char via, char to)
{
  if (n > 0) {
    // move n-1 disks from --> via
    doTowers(n-1, from, to, via);
    // move the nth disk from --> to
    System.out.println("Move disk from "+from+" to "+to);
    // move n-1 disks via --> to
    doTowers(n-1, via, from, to);
  }
}

public static void main(String[] args) {
  doTowers(10, ‘A’, ‘B’, ‘C’);
}
```

Clicker Question #4

```
public void doTowers(int n, int from, int via, int to) {
  if (n > 0) {
    doTowers(n-1, from, to, via);
    System.out.println("Move disk from "+from+" to "+to);
    doTowers(n-1, via, from, to);
  }
}
```

- How do we know that this method will make progress toward its base case?

  (a) we move at least one disk with each call

  (b) the base case has n equal to 1

  (c) the parameter n decreases with each call

  (d) each call has only two recursive calls