1. Start with a positive integer, then choose a negative integer. We'll use these two numbers to generate a sequence using the following rule: create the next term in the sequence by adding the previous two. For example, if we started with 6 and -5, we would get the sequence

$$\underbrace{6, -5, 1, -4}_{\text{alternating part}}, -3, -7, -10, -17, -27, \ldots$$

which starts with 4 elements that alternate sign before the terms are all negative. If we started with 3 and -2, we would get the sequence

$$\underbrace{3, -2, 1, -1}_{\text{alternating part}}, 0, -1, -1, -2, -3, \ldots$$

which also starts with 4 elements that alternate sign before the terms are all non-positive (we don't count 0 in the alternating part).

(a) Can you find a sequence of this type that starts with 5 elements that alternate sign? With 10 elements that alternate sign? Can you find a sequence with any number of elements that alternate sign?

(b) Given a particular starting integer, what negative number should you choose to make the alternating part of the sequence as long as possible? For example, if your sequence started with 8, what negative number would give the longest alternating part? What if you started with 10? With n?

2. The tail of a giant hare is attached by a giant rubber band to a stake in the ground. A flea is sitting on top of the stake eyeing the hare (hungrily). Seeing the flea, the hare leaps into the air and lands one kilometer from the stake (with its tail still attached to the stake by the rubber band). The flea does not give up the chase but leaps into the air and lands on the stretched rubber band one centimeter from the stake. The giant hare, seeing this, again leaps into the air and lands another kilometer from the stake (i.e., a total of two kilometers from the stake). The flea is undaunted and leaps into the air again, landing on the rubber band one centimeter further along. Once again the giant hare jumps another kilometer. The flea again leaps bravely into the air and lands another centimeter along the rubber band. If this continues indefinitely, will the flea ever catch the hare? (Assume the earth is flat and continues indefinitely in all directions.)

3. The set S contains some real numbers, according to the following three rules.

(i) $\frac{1}{1}$ is in S.

(ii) If $\frac{a}{b}$ is in S, where $\frac{a}{b}$ is written in lowest terms (that is, a and b have highest common factor 1), then $\frac{b}{2a}$ is in S.

(iii) If $\frac{a}{b}$ and $\frac{c}{d}$ are in S, where they are written in lowest terms, then $\frac{a+c}{b+d}$ is in S.

These rules are exhaustive: if these rules do not imply that a number is in S, then that number is not in S. Can you describe which numbers are in S? For example, by (i), $\frac{1}{1}$ is in S. By (ii), since $\frac{1}{1}$ is in S, $\frac{1}{2\cdot 1}$ is in S. Since both $\frac{1}{1}$ and $\frac{1}{2}$ are in S, (iii) tells us $\frac{1+1}{1+2}$ is in S.

4. The sequence (x_n) of positive real numbers satisfies the relationship $x_{n-1}x_nx_{n+1} = 1$ for all $n \geqslant 2$. If $x_1 = 1$ and $x_2 = 2$, what are the values of the next few terms? What can you say about the sequence? What happens for other starting values?

The sequence (y_n) satisfies the relationship $y_{n-1}y_{n+1} + y_n = 1$ for all $n \geqslant 2$. If $y_1 = 1$ and $y_2 = 2$, what are the values of the next few terms? What can you say about the sequence? What happens for other starting values?

5. A monkey has filled in a 3×3 grid with the numbers $1, 2, \ldots, 9$. A cat writes down the three numbers obtained by multiplying the numbers in each horizontal row. A dog writes down the three numbers obtained by multiplying the numbers in each vertical column. Can the monkey fill in the grid in such a way that the cat and dog obtain the same lists of three numbers? What if the monkey writes the numbers $1, 2, \ldots, 25$ in a 5×5 grid? Or $1, 2, \ldots, 121$ in a 11×11 grid? Can you find any conditions on n that guarantee that it is possible or any conditions that guarantee that it is impossible for the monkey to write the numbers $1, 2, \ldots, n^2$ in an $n \times n$ grid so that the cat and the dog obtain the same lists of numbers?

6. Consider the polynomial $(x + y + z)^n$ for $n \geq 1$. Expand and combine like terms so that all terms of the form $x^iy^jz^k$ are together with one coefficient. Let $c_n(i, j, k)$ be this coefficient for fixed n.

(i) Can you predict when $c_n(i, j, k)$ is even or odd? For example, when $n = 1$, we have $c_1(1, 0, 0), c_1(0, 1, 0)$, and $c_1(0, 0, 1)$ are all equal to 1, hence odd, and not defined for any other values of i, j, k.

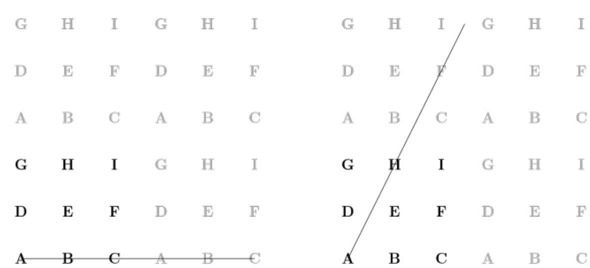
(ii) Can you say anything about the sum of the coefficients $c_n(i, j, k)$ over the values where $i, j, k > 0$? Can you find an expression for this sum?

7. Can you find an equilateral triangle in the xy-plane with all its vertices having integer coordinates? If so, give an example, and if not, explain why no such triangle exists. Can you find an equilateral triangle with two vertices that have integer coordinates while the third vertex is within $\frac{1}{10}$ distance of a point with integer coordinates? What about within $\frac{1}{100}$? Within $\frac{1}{n}$?

8. Consider the grid of letters that represent points below, which is formed by repeating the bold 3×3 grid. (The points are labeled by the letters A through I.)

G	H	I	G	H	I
D	E	F	D	E	F
A	B	C	A	B	C
G	**H**	**I**	G	H	I
D	**E**	**F**	D	E	F
A	**B**	**C**	A	B	C

We'll draw lines through the bottom left point (the bold A) and at least one other bold letter, then see what set of letters the line hits. We've drawn two example lines for the repeating 3×3 grid below. For the example on the left, the set of letters is {A, B, C}, and for the right, the set of letters is {A, F, H}.



Assuming the 3×3 grid repeats forever in every direction, do any of these lines ever pass through more than 3 different letters? Can you get the same set of letters from two different lines? Find the four different sets of letters that you can get from drawing lines in this grid.

What would happen if you had a repeated 5×5 grid of letters (and still had to draw lines through the bottom left point and at least one other bold point)? Can you predict what would happen with a repeated 7×7 grid? Does your prediction also work for 6×6? Can you justify your predictions?

End of Problem Set