Assignment #2 (**100 points**): Due 11:59 pm, Sunday, March 27, 2022

1. In the false data injection (FDI) attack, assume that the normal measurement is $\mathbf{z} = \mathbf{Hx} + \mathbf{n}$ and the attacked measurement is $\mathbf{z_a} = \mathbf{z} + \mathbf{a}$. Assume that the weighted least square (WLS) solution is given by $\hat{\mathbf{x}} = \left(\mathbf{H^T R^{-1} H}\right)^{-1}\mathbf{H^T R^{-1} z}$ for the normal measurement $\mathbf{z}$, and let $\hat{\mathbf{x}}_\mathbf{a}$ be the WLS solution obtained from the attacked measurement $\mathbf{z_a}$. Use your linear algebra knowledge to prove that the pre-attack residual $\mathbf{r} = \mathbf{z} - \mathbf{H}\hat{\mathbf{x}}$ is the same as the post-attack residual $\mathbf{r_a} = \mathbf{z_a} - \mathbf{H}\hat{\mathbf{x}}_\mathbf{a}$ if $\mathbf{a} = \mathbf{Hc}$. (**20 points**)

Solution:

Given,

Pre-attack residual $\mathbf{r} = \mathbf{z} - \mathbf{H}\hat{\mathbf{x}}$

Attacked measurement is $\mathbf{z_a} = \mathbf{z} + \mathbf{a}$

Normal measurement is $\mathbf{z} = \mathbf{Hx} + \mathbf{n}$

and the Post-attack residual $\mathbf{r_a} = \mathbf{z_a} - \mathbf{H}\hat{\mathbf{x}}_\mathbf{a}$.

$$=> \mathbf{z} + \mathbf{a} - \mathbf{H}\,[(\mathbf{H^T R^{-1} H})^{-1}\mathbf{H^T R^{-1} z_a}]$$

$$=> \mathbf{z} + \mathbf{a} - \mathbf{H}\,[(\mathbf{H^T R^{-1} H})^{-1}\mathbf{H^T R^{-1}}(z + a)]$$

$$=> \mathbf{z} + \mathbf{a} - \mathbf{H}\,[(\mathbf{H^T R^{-1} H})^{-1}\mathbf{H^T R^{-1} z} + (\mathbf{H^T R^{-1} H})^{-1}\mathbf{H^T R^{-1}}\,\mathbf{a}]$$

$$=> \mathbf{z} + \mathbf{a} - \mathbf{H}\,[\,\mathbf{x}\text{^} + (\mathbf{H^T R^{-1} H})^{-1}\mathbf{H^T R^{-1}}\,\mathbf{a}]$$

$$=> \mathbf{z} + \mathbf{a} - \mathbf{H}\,(\mathbf{x}\text{^} + \mathbf{H^{-1} a})$$

$$=> z + a - H\left(\hat{x} + \frac{a}{H}\right)$$

$$=> \mathbf{z} + \mathbf{a} - \mathbf{H}\,(\mathbf{x}\text{^} + \mathbf{c})$$

$$=> \mathbf{z} - \mathbf{Hx}\text{^} + \mathbf{a} - \mathbf{Hc}$$

Since $\mathbf{a} = \mathbf{Hc}$ $=> \mathbf{z} - \mathbf{Hx}\text{^} + \mathbf{a} - \mathbf{a}$

$$=> \mathbf{z} - \mathbf{Hx}\text{^}$$

Which implies $\mathbf{r} = \mathbf{r_a}$

Hence it is proved that the pre-attack residual $\mathbf{r} = \mathbf{z} - \mathbf{Hx}\text{^}$ is the same as the post-attack residual $\mathbf{r_a} = \mathbf{z_a} - \mathbf{Hx}\text{^}_\mathbf{a}$, if $\mathbf{a} = \mathbf{Hc}$

2. A one-line diagram of a 3-bus power system is shown below. Assume that we have one meter installed on each line to measure the power flows. For bus i and j connected by a line, the power flow P_{ij}, the reactance X_{ij}, and voltage angles θ_i, θ_j are considered to approximately follow the relation below when no noise is considered:

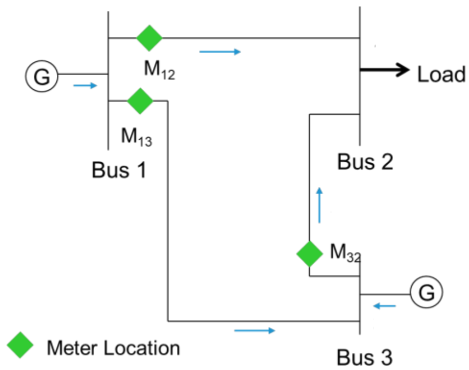
$$P_{ij} = (\theta_i - \theta_j)/X_{ij}$$

The line reactance has been measured in advance as:

$$X_{12} = 0.50\ p.u.$$
$$X_{13} = 0.10\ p.u.$$
$$X_{32} = 0.25\ p.u.$$

The meter readings have been reported as:

$$M_{12} = 0.60\ p.u.$$
$$M_{13} = 0.05\ p.u.$$
$$M_{32} = 0.40\ p.u.$$



Solve the questions below. (**80 points**)

a) Assume that $\theta_1 = 0$ is the reference angle. Using different pairs of meter readings, find the values below:
   i. θ_2, θ_3 and P_{12}, using only the values of M_{13}, M_{32} and the reactance X.
   ii. θ_2, θ_3 and P_{13}, using only the values of M_{12}, M_{32} and the reactance X.
   iii. θ_2, θ_3 and P_{32}, using only the values of M_{12}, M_{13} and the reactance X.

Solution:

i.

Given,

Line reactance has been measured in advance are:

$X_{12} = 0.50\ p.u.$
$X_{13} = 0.10\ p.u.$
$X_{32} = 0.25\ p.u.$

Meter readings are:

$M_{12} = 0.60\ p.u.$

$M_{13} = 0.05\ p.u.$

$M_{32} = 0.40\ p.u.$

$P_{ij} = (\theta_i - \theta_j)/X_{ij}$

$M_{12} = (\theta_1 - \theta_2)/ X_{12}$

$0.60 = (0 - \theta_2) / 0.50$

$=>> 0.300 = 0 - \theta_2$

$=>> \theta_2 = -0.3$

$M_{13} = (\theta 1 - \theta 3) / X_{13}$

$0.05 = (0 - \theta_3) / 0.10$

$=>> 0.005 = 0 - \theta_3$

$=>> \theta_3 = -0.005$

Similarly,

$P32 = (\theta 3 - \theta 2) / X_{32}$

$=>> (-0.005 + 0.300) / 0.25$

P32 =>> 1.18 $p.u.$

ii.

Given,

$M_{12} = (\theta_1 - \theta_2) / X_{12}$

$0.60 = (0 - \theta_2) / 0.50$

$=> 0.3 = 0 - \theta_2$

$=> \theta_2 = -0.3$

$M_{32} = (\theta_3 - \theta_2) / X_{32}$

$0.40 = (\theta_3 + 0.300) / 0.25$

$=>> 0.10 = \theta_3 + 0.3$

$=>> \theta_3 = -0.2$

Similarly,

$P_{ij} = (\theta_i - \theta_j) / X_{ij},$

$P13 = (\theta 1 - \theta 3) / X_{13}$

$=>> (0 + 0.2) / 0.1$

P13 =>> 2$p.u.$

iii.

Given,

$M13 = (\theta 1 - \theta 3) / X13$

$0.05 = (0 - \theta 3) / 0.10$

$=>> 0.005 = 0 - \theta 3$

$=>> \theta 3 = -0.005$

$M32 = (\theta 3 - \theta 2) / X32$

$0.40 = (-0.005 - \theta 2) / 0.25$

$=>> 0.10 = -0.005 - \theta 2$

$=>> \ \theta 2 = -0.105$

Similarly,

$Pij = (\theta i - \theta j)/Xij,$
$P12 = (\theta 1 - \theta 2)/X12$
$=> (0+0.105)/0.50$
$\mathbf{P12 => 0.21}\ p.u.$

b) Let $\mathbf{z} = [M_{12}, M_{13}, M_{32}]^T$ be the measurement vector and $\mathbf{x} = [\theta_1, \theta_2, \theta_3]^T$ be the state variables, where T indicates the vector/matrix transpose. Then from the ideal case $\mathbf{z} = \mathbf{Hx}$ when there is no noise, we have:

$$\begin{bmatrix} M_{12} \\ M_{13} \\ M_{32} \end{bmatrix} = \begin{bmatrix} P_{12} \\ P_{13} \\ P_{32} \end{bmatrix} = \mathbf{H} \begin{bmatrix} \theta_1 \\ \theta_2 \\ \theta_3 \end{bmatrix}, \text{ where } \mathbf{H} = \begin{bmatrix} h_{11} & h_{12} & h_{13} \\ h_{21} & h_{22} & h_{23} \\ h_{31} & h_{32} & h_{33} \end{bmatrix}$$

Based on the relation $P_{ij} = (\theta_i - \theta_j)/X_{ij}$, find the entries h_{ij} in the 3-by-3 matrix $\mathbf{H}$.

Solution:

Given,

line reactance has been measured are:

$X_{12} = 0.50\ p.u.$
$X_{13} = 0.10\ p.u.$
$X_{32} = 0.25\ p.u.$

$$\begin{bmatrix} P_{12} \\ P_{13} \\ P_{32} \end{bmatrix} = \mathbf{H} \begin{bmatrix} \theta_1 \\ \theta_2 \\ \theta_3 \end{bmatrix},$$

And we have, $\mathbf{H} = \begin{bmatrix} h_{11} & h_{12} & h_{13} \\ h_{21} & h_{22} & h_{23} \\ h_{31} & h_{32} & h_{33} \end{bmatrix}$

This implies, $\begin{bmatrix} P_{12} \\ P_{13} \\ P_{32} \end{bmatrix} = \begin{bmatrix} h_{11} & h_{12} & h_{13} \\ h_{21} & h_{22} & h_{23} \\ h_{31} & h_{32} & h_{33} \end{bmatrix} \begin{bmatrix} \theta_1 \\ \theta_2 \\ \theta_3 \end{bmatrix}$

$$=> \begin{bmatrix} P_{12} \\ P_{13} \\ P_{32} \end{bmatrix} = \begin{bmatrix} h_{11}\theta_1 & h_{12}\theta_2 & h_{13}\theta_3 \\ h_{21}\theta_1 & h_{22}\theta_2 & h_{23}\theta_3 \\ h_{31}\theta_1 & h_{32}\theta_2 & h_{33}\theta_3 \end{bmatrix}$$

$=> \ P_{12} = h_{11}\theta_1 + h_{12}\theta_2 + h_{13}\theta_3$

$=> \ P_{13} = h_{21}\theta_1 + h_{22}\theta_2 + h_{23}\theta_3$

$=> \ P_{32} = h_{31}\theta_1 + h_{32}\theta_2 + h_{33}\theta_3$

We know that,

$$P_{ij} = (\theta_i - \theta_j)/X_{ij}$$

$$P_{12} = (\theta_1 - \theta_2)/\ X_{12}$$

$$= (\theta_1 - \theta_2)/\ 0.50$$

$$= 2\ \theta_1 - 2\ \theta_2$$

$$P_{13} = (\theta_1 - \theta_3)/\ X_{13}$$

$$= (\theta_1 - \theta_3)/\ 0.10$$

$$= 10\ \theta_1 - 10\ \theta_3$$

$$P_{32} = (\theta_3 - \theta_2)/\ X_{32}$$

$$= (\theta_3 - \theta_2)/\ 0.25$$

$$= 4\ \theta_3 - 4\ \theta_2$$

$$\begin{bmatrix} P_{12} \\ P_{13} \\ P_{32} \end{bmatrix} = \begin{bmatrix} 2 & -2 & 0 \\ 10 & 0 & -10 \\ 0 & -4 & 4 \end{bmatrix} \begin{bmatrix} \theta_1 \\ \theta_2 \\ \theta_3 \end{bmatrix} \quad \Rightarrow\Rightarrow \quad \mathbf{H} = \begin{bmatrix} 2 & -2 & 0 \\ 10 & 0 & -10 \\ 0 & -4 & 4 \end{bmatrix}$$

c) As we use $\theta_1 = 0$ as the reference angle, we do not actually need to estimate θ_1, and the state variables to be estimated are only $\mathbf{x}' = [\theta_2, \theta_3]^T$. For $\mathbf{z} = \mathbf{Hx}$, this suggests the column in $\mathbf{H}$ corresponding to the coefficients of θ_1 should be removed. Find the new 3-by-2 matrix $\mathbf{H}'$ with the column in $\mathbf{H}$ corresponding to θ_1 is removed.

Solution:

Given,

Z = Hx

When we use,

$\theta_1 = 0$ as the reference angle, the state variables to be estimated are only $\mathbf{x}' = [\theta_2, \theta_3]^T$

So, by removing the column in H corresponding to the coefficients of θ_1

$\mathbf{z}' = \mathbf{H}'\ \mathbf{x}'$

$$\mathbf{H} = \begin{bmatrix} 2 & -2 & 0 \\ 10 & 0 & -10 \\ 0 & -4 & 4 \end{bmatrix}$$

$$\mathbf{z}' = \begin{bmatrix} -2 & 0 \\ 0 & -10 \\ -4 & 4 \end{bmatrix} \begin{bmatrix} \theta_2 \\ \theta_3 \end{bmatrix}$$

$$\mathbf{H}' = \begin{bmatrix} -2 & 0 \\ 0 & -10 \\ -4 & 4 \end{bmatrix}$$

d) Let $\mathbf{R} = [\sigma_{ij}^2]$ be the 3-by-3 covariance matrix of the additive white Gaussian noise $\mathbf{n}$, where σ_{ii}^2 is the variance of noise of meter i and σ_{ij}^2 $(i \neq j)$ is the covariance between noises on two different meters i and j. Normally we assume the noises on different meters are independent, so the covariance $\sigma_{ij}^2 = 0$ for $i \neq j$; we also know the variances of noise on meters is: $\sigma_{11}^2 = 0.05$ (for M_{12}), $\sigma_{22}^2 = 0.045$ (for M_{13}) and $\sigma_{33}^2 = 0.05$ (for M_{32}) Find the covariance matrix $\mathbf{R}$ by specifying all its entries σ_{ij}^2 in the matrix form.

Solution:

Given,

$$\sigma_{11}^2 = 0.05$$

$$\sigma_{22}^2 = 0.045$$

$$\sigma_{33}^2 = 0.05$$

According to the above, covariance $\sigma^2 = 0$ for $i \neq j$, So

$$\sigma_{12}^2 = \sigma_{13}^2 = \sigma_{21}^2 = \sigma_{23}^2 = \sigma_{31}^2 = \sigma_{32}^2 = 0$$

Covariance matrix,

$$\mathrm{R} = \begin{bmatrix} \sigma_{11}^2 & \sigma_{12}^2 & \sigma_{13}^2 \\ \sigma_{21}^2 & \sigma_{22}^2 & \sigma_{23}^2 \\ \sigma_{31}^2 & \sigma_{32}^2 & \sigma_{33}^2 \end{bmatrix}$$

So,

$$\mathbf{R} = \begin{bmatrix} 0.05 & 0 & 0 \\ 0 & 0.045 & 0 \\ 0 & 0 & 0.05 \end{bmatrix}$$

e) With the answers to $\mathbf{H}'$, $\mathbf{R}$, and $\mathbf{z}$ from Questions 2-4, the weighted least square (WLS) solution is given by $\hat{\mathbf{x}} = (\mathbf{H}'^{\mathrm{T}}\mathbf{R}^{-1}\mathbf{H}')^{-1}\mathbf{H}'^{\mathrm{T}}\mathbf{R}^{-1}\mathbf{z}$. Use MATLAB, Excel, or any other program that can solve the matrix inverse to find $\mathbf{R}^{-1}$, $(\mathbf{H}^{\mathrm{T}}\mathbf{R}^{-1}\mathbf{H})^{-1}$, and finally, the state variable estimate $\hat{\mathbf{x}}$.

We know that

$$\mathbf{R} = \begin{bmatrix} 0.05 & 0 & 0 \\ 0 & 0.045 & 0 \\ 0 & 0 & 0.05 \end{bmatrix}$$

So, $$\mathbf{R}^{-1} = \begin{bmatrix} 0.05 & 0 & 0 \\ 0 & 0.045 & 0 \\ 0 & 0 & 0.05 \end{bmatrix}^{-1} = \begin{bmatrix} 20 & 0 & 0 \\ 0 & 22.22 & 0 \\ 0 & 0 & 20 \end{bmatrix}$$

Now, by removing the coefficients of θ_1, $\mathbf{H} = \begin{bmatrix} 2 & -2 & 0 \\ 10 & 0 & -10 \\ 0 & -4 & 4 \end{bmatrix}$ is replaced by $\mathbf{H}' = \begin{bmatrix} -2 & 0 \\ 0 & -10 \\ -4 & 4 \end{bmatrix}$

Now,

$$(\mathbf{H}^{\mathrm{T}}\mathbf{R}^{-1}\mathbf{H})^{-1} = \left(\begin{bmatrix} -2 & 0 \\ 0 & -10 \\ -4 & 4 \end{bmatrix}^{\mathrm{T}} \begin{bmatrix} 0.05 & 0 & 0 \\ 0 & 0.045 & 0 \\ 0 & 0 & 0.05 \end{bmatrix}^{-1} \begin{bmatrix} -2 & 0 \\ 0 & -10 \\ -4 & 4 \end{bmatrix} \right)^{-1}$$

$$=> \left(\begin{bmatrix} -2 & 0 & -4 \\ 0 & -10 & 4 \end{bmatrix} \begin{bmatrix} 20 & 0 & 0 \\ 0 & 22.22 & 0 \\ 0 & 0 & 20 \end{bmatrix} \begin{bmatrix} -2 & 0 \\ 0 & -10 \\ -4 & 4 \end{bmatrix} \right)^{-1}$$

$$=> \left(\begin{bmatrix} -40 & 0 & -80 \\ 0 & -222.2 & 80 \end{bmatrix} \begin{bmatrix} -2 & 0 \\ 0 & -10 \\ -4 & 4 \end{bmatrix} \right)^{-1}$$

$$=> \begin{bmatrix} 400 & -320 \\ -320 & 2542 \end{bmatrix}^{-1}$$

$$=> \begin{bmatrix} 0.00278 & 0.00034 \\ 0.00034 & 0.00043 \end{bmatrix}$$

The state variable $\hat{\mathbf{x}} = (\mathbf{H}^{\mathrm{T}}\mathbf{R}^{-1}\mathbf{H})^{-1}\mathbf{H}^{\mathrm{T}}\mathbf{R}^{-1}\mathbf{z}$

$$=> \begin{bmatrix} 0.00278 & 0.00034 \\ 0.00034 & 0.00043 \end{bmatrix} \begin{bmatrix} -2 & 0 \\ 0 & -10 \\ -4 & 4 \end{bmatrix}^{T} \begin{bmatrix} 0.05 & 0 & 0 \\ 0 & 0.045 & 0 \\ 0 & 0 & 0.05 \end{bmatrix}^{-1} \begin{bmatrix} 0.21 \\ 2 \\ 1.18 \end{bmatrix}$$

$$=> \begin{bmatrix} 0.00278 & 0.00034 \\ 0.00034 & 0.00043 \end{bmatrix} \begin{bmatrix} -2 & 0 & -4 \\ 0 & -10 & 4 \end{bmatrix} \begin{bmatrix} 20 & 0 & 0 \\ 0 & 22.22 & 0 \\ 0 & 0 & 20 \end{bmatrix} \begin{bmatrix} 0.21 \\ 2 \\ 1.18 \end{bmatrix}$$

$$=> \begin{bmatrix} -0.00556 & -0.0034 & -0.00976 \\ -0.00068 & -0.0043 & 0.00036 \end{bmatrix} \begin{bmatrix} 20 & 0 & 0 \\ 0 & 22.22 & 0 \\ 0 & 0 & 20 \end{bmatrix} \begin{bmatrix} 0.21 \\ 2 \\ 1.18 \end{bmatrix}$$

$$=> \begin{bmatrix} -0.1112 & -0.075548 & -0.1952 \\ -0.0136 & -0.095546 & 0.0072 \end{bmatrix} \begin{bmatrix} 0.21 \\ 2 \\ 1.18 \end{bmatrix}$$

$$=> \begin{bmatrix} -0.404768 \\ -0.18544 \end{bmatrix}$$