1 : Theory of Matrices, Vector spaces and Linear Transformations

IT5506 – Mathematics for Computing II

Level III - Semester 5

Intended Learning Outcomes

At the end of this lesson, you will be able to;

- analyze the solution set of a system of m linear equations in n unknowns using multiple methods with the required theoretical background.

List of sub topics

1.3 Solving systems of linear equations using elementary row operations (Gaussian Elimination) and backward substitution in matrix form considering different cases (2 hours)

- 1.3.1 existence of unique solution
- 1.3.2 existence of infinitely many solution
- 1.3.3 no solution

1.3 Systems of Linear Equations

A system of **m** linear equations in **n** unknowns is of the form

$$\left.\begin{aligned} a_{11}x_1 + a_{12}x_2 + \dots\dots\dots a_{1n}x_n &= y_1 \\ a_{21}x_1 + a_{22}x_2 + \dots\dots\dots a_{2n}x_n &= y_2 \\ \dots\dots\dots\dots\dots\dots\dots\dots&\dots\dots\dots \\ a_{m1}x_1 + a_{m2}x_2 + \dots\dots\dots a_{mn}x_n &= y_m \end{aligned}\right\} \text{-------(1)}$$

Where $y_1, y_2, \dots\dots y_m$ and a_{ij} $1 \leq i \leq m$, $1 \leq j \leq n$ are real numbers

and $x_1, x_2, \dots\dots x_n$ are n unknowns.

Note: If $y_1 = y_2 = \dots\dots = y_m = 0$, the system is called a **homogeneous** system.

1.3 Systems of Linear Equations

We write this system in matrix form as:

$$\begin{pmatrix} a_{11} & a_{12} & . & . & a_{1n} \\ a_{21} & a_{22} & . & . & a_{2n} \\ . & . & . & . & . \\ . & . & . & . & . \\ . & . & . & . & . \\ a_{m1} & a_{m2} & . & . & a_{mn} \end{pmatrix} \begin{pmatrix} x_1 \\ x_2 \\ . \\ . \\ . \\ x_n \end{pmatrix} = \begin{pmatrix} y_1 \\ y_2 \\ . \\ . \\ . \\ y_m \end{pmatrix}$$

AX = Y

A is called the **matrix of coefficients** of the system.

Note: if Y is zero (zero matrix), the system is called a **homogeneous** system.

1.3 Systems of Linear Equations

Characteristics of the systems of Linear Equation

1. A solution of the system of linear equations is a set of values $x_1, x_2, ...x_n$ which satisfy the m equations in (1).
2. If the system is homogeneous then, $x_1 = x_2 = ... = x_n = 0$ is a solution of the system.
3. If the system is not homogeneous, it is possible that no set of values will satisfy all the equations in the system. If this is the case the system is said to be **inconsistent**.
4. If there exists a solution which satisfies all the equations of the system, the system is said to be **consistent**.

1.3 Systems of Linear Equations

Characteristics of the systems of Linear Equation

5. A homogeneous system is always consistent since it has the trivial **solution** $x_1 = x_2 = \ldots\ldots = x_n = 0$
6. There are two possible types of solutions to a consistent system of linear equations. Either the system will have a unique solution, or it will have infinitely many solutions.
7. If a homogeneous system has a unique solution then, since the trivial solution is always a solution, the trivial solution will be its unique solution.

1.3 Systems of Linear Equations

Examples: solutions for Consistent systems

(1)

$$2x + y = 5$$
$$x - y = 4.$$

This system has a unique solution, $x = 3$, $y = -1$.

(2)

$$2x + 3y + 4z = 5$$
$$x + 6y + 7z = 3$$

This system has infinitely many solutions of the form

$x = k$,

$y = (10k - 23) / 3$

$z = (7 - 3k)$ where k is any scalar.

1.3 Systems of Linear Equations

Examples: inconsistent system

(3)

$$
\begin{aligned}
x + 2y - 3z &= -1 \\
3x - y + 2z &= 7 \\
5x + 3y - 4z &= 2
\end{aligned}
$$

This system has no solution.

1.3 Systems of Linear Equations

Elimination Method

The most fundamental method of finding solutions of systems of linear equations is the method of elimination.

Consider the following system.

$$2x + 3y = 3$$
$$x - y = 4.$$

$$\begin{pmatrix} 2 & 3 \\ 1 & -1 \end{pmatrix}\begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} 3 \\ 4 \end{pmatrix}$$

- We first eliminate y by multiplying the second equation by 3 and adding it to the first equation.

$$5x + 0y = 15$$
$$x - y = 4.$$

$$\begin{pmatrix} 5 & 0 \\ 1 & -1 \end{pmatrix}\begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} 15 \\ 4 \end{pmatrix}$$

1.3 Systems of Linear Equations

Elimination Method (contd…)

- Now multiplying the first row by 1/5 we obtain

$$\begin{aligned} x + 0y &= 3 \\ x - y &= 4 \end{aligned} \qquad \begin{pmatrix} 1 & 0 \\ 1 & -1 \end{pmatrix}\begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} 3 \\ 4 \end{pmatrix}$$

- By multiplying the first row by –1 and adding it to row 2 we obtain

$$\begin{aligned} 1x + 0y &= 3 \\ 0x + (-1)y &= 1 \end{aligned} \qquad \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}\begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} 3 \\ 1 \end{pmatrix}$$

- Finally by multiplying the second row by -1, we obtain

$$\begin{aligned} 1x + 0y &= 3 \\ 0x + 1y &= -1 \end{aligned} \qquad \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}\begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} 3 \\ -1 \end{pmatrix}$$

1.3 Systems of Linear Equations

Elimination Method (contd…)

Note that the process of solving the system of these two linear equation stops when the coefficient matrix becomes identity matrix.

From this we can directly obtain the solution,

$x = 3$

$y = -1$

We have used two operations here, namely

1. addition of a scalar multiple of a row to another row
2. multiplication of a row by a scalar.

1.3 Systems of Linear Equations

Elementary row operations

We use a similar method to find the solution of any system of linear equations. We apply 3 types of operations on the equations of the system to reduce them to another system of linear equation from which it will be possible to determine whether a system is consistent or not and if consistent to determine its solution. The operations we use are called **elementary row operations** and these are performed on the matrix of coefficients. The three operations are as follows:

1. Interchanging two rows of a matrix.
2. Multiplying a row by a non-zero constant.
3. Adding a multiple of one row to another row.

1.3.1 existence of unique solution

Example: **Consistent system with unique solution**

Consider the following system of linear equations.

$$2x + y + 3z = 5$$

$$3x - 2y + 2z = 5$$

$$5x - 3y - z = 16$$

This set of equations in matrix form

$$\begin{pmatrix} 2 & 1 & 3 \\ 3 & -2 & 2 \\ 5 & -3 & -1 \end{pmatrix} \begin{pmatrix} x \\ y \\ z \end{pmatrix} = \begin{pmatrix} 5 \\ 5 \\ 16 \end{pmatrix}$$

1.3.1 existence of unique solution

Step 1: Adding row 1 to row 2 we obtain

$$\begin{pmatrix} 2 & 1 & 3 \\ 5 & -1 & 5 \\ 5 & -3 & -1 \end{pmatrix}\begin{pmatrix} x \\ y \\ z \end{pmatrix} = \begin{pmatrix} 5 \\ 10 \\ 16 \end{pmatrix}$$

Step 2: Multiplying row 2 by –1 and adding to row 3 we obtain

$$\begin{pmatrix} 2 & 1 & 3 \\ 5 & -1 & 5 \\ 0 & -2 & -6 \end{pmatrix}\begin{pmatrix} x \\ y \\ z \end{pmatrix} = \begin{pmatrix} 5 \\ 10 \\ 6 \end{pmatrix}$$

1.3.1 existence of unique solution

Step 3: Multiplying row 2 by 1/5 and then multiplying row 3 by –1/2 we obtain

$$\begin{pmatrix} 2 & 1 & 3 \\ 1 & -1/5 & 1 \\ 0 & 1 & 3 \end{pmatrix}\begin{pmatrix} x \\ y \\ z \end{pmatrix} = \begin{pmatrix} 5 \\ 2 \\ -3 \end{pmatrix}$$

Step 4: Multiplying row 3 by –1 and adding to row1 we obtain

$$\begin{pmatrix} 2 & 0 & 0 \\ 1 & -1/5 & 1 \\ 0 & 1 & 3 \end{pmatrix}\begin{pmatrix} x \\ y \\ z \end{pmatrix} = \begin{pmatrix} 8 \\ 2 \\ -3 \end{pmatrix}$$

1.3.1 existence of unique solution

Step 5: Multiplying row 1 by ½ we obtain

$$\begin{pmatrix} 1 & 0 & 0 \\ 1 & -1/5 & 1 \\ 0 & 1 & 3 \end{pmatrix} \begin{pmatrix} x \\ y \\ z \end{pmatrix} = \begin{pmatrix} 4 \\ 2 \\ -3 \end{pmatrix}$$

Step 6: Multiplying row 1 by –1 and adding to row 2 and then multiplying row 3 by 1/5 and adding to row 2 we obtain

$$\begin{pmatrix} 1 & 0 & 0 \\ 0 & 0 & 8/5 \\ 0 & 1 & 3 \end{pmatrix} \begin{pmatrix} x \\ y \\ z \end{pmatrix} = \begin{pmatrix} 4 \\ -13/5 \\ -3 \end{pmatrix}$$

1.3.1 existence of unique solution

Step 7: Multiplying row 2 by 5/8 we obtain

$$\begin{pmatrix} 1 & 0 & 0 \\ 0 & 0 & 1 \\ 0 & 1 & 3 \end{pmatrix}\begin{pmatrix} x \\ y \\ z \end{pmatrix} = \begin{pmatrix} 4 \\ -13/8 \\ -3 \end{pmatrix}$$

Step 8: Multiplying row 2 by -3 and adding to row 3 we obtain

$$\begin{pmatrix} 1 & 0 & 0 \\ 0 & 0 & 1 \\ 0 & 1 & 0 \end{pmatrix}\begin{pmatrix} x \\ y \\ z \end{pmatrix} = \begin{pmatrix} 4 \\ -13/8 \\ 15/8 \end{pmatrix}$$

1.3.1 existence of unique solution

Step 9: Finally interchanging row 2 and row 3 we obtain

$$\begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}\begin{pmatrix} x \\ y \\ z \end{pmatrix} = \begin{pmatrix} 4 \\ 15/8 \\ -13/8 \end{pmatrix}$$

Thus the system is reduced to

x = 4

y = 15/8

z = -13/8

This is the unique solution to the system.

1.3.2 existence of infinitely many solution

Example: **consistent system with infinitely many solutions**

Consider the following system of linear equations.

$$x + 2y - 3z = 6$$

$$2x - y + 4z = 2$$

$$4x + 3y - 2z = 14$$

This set of equations in matrix form is

$$\begin{pmatrix} 1 & 2 & -3 \\ 2 & -1 & 4 \\ 4 & 3 & -2 \end{pmatrix}\begin{pmatrix} x \\ y \\ z \end{pmatrix} = \begin{pmatrix} 6 \\ 2 \\ 14 \end{pmatrix}$$

1.3.2 existence of infinitely many solution

Step 1: Multiplying row 2 by –2 and adding to row 3 we obtain

$$\begin{pmatrix} 1 & 2 & -3 \\ 2 & -1 & 4 \\ 0 & 5 & -10 \end{pmatrix}\begin{pmatrix} x \\ y \\ z \end{pmatrix} = \begin{pmatrix} 6 \\ 2 \\ 10 \end{pmatrix}$$

Step 2: Multiplying row 1 by –2 and adding to row 2 we obtain

$$\begin{pmatrix} 1 & 2 & -3 \\ 0 & -5 & 10 \\ 0 & 5 & -10 \end{pmatrix}\begin{pmatrix} x \\ y \\ z \end{pmatrix} = \begin{pmatrix} 6 \\ -10 \\ 10 \end{pmatrix}$$

1.3.2 existence of infinitely many solution

Step 3: Adding row 2 to row 3 we obtain

$$\begin{pmatrix} 1 & 2 & -3 \\ 0 & -5 & 10 \\ 0 & 0 & 0 \end{pmatrix}\begin{pmatrix} x \\ y \\ z \end{pmatrix} = \begin{pmatrix} 6 \\ -10 \\ 0 \end{pmatrix}$$

Step 4: Multiplying row 2 by -1/5 we obtain

$$\begin{pmatrix} 1 & 2 & -3 \\ 0 & 1 & -2 \\ 0 & 0 & 0 \end{pmatrix}\begin{pmatrix} x \\ y \\ z \end{pmatrix} = \begin{pmatrix} 6 \\ 2 \\ 0 \end{pmatrix}$$

1.3.2 existence of infinitely many solution

Step 5: Multiplying row 2 by –2 and adding to row 1 we obtain

$$\begin{pmatrix} 1 & 0 & 1 \\ 0 & 1 & -2 \\ 0 & 0 & 0 \end{pmatrix}\begin{pmatrix} x \\ y \\ z \end{pmatrix} = \begin{pmatrix} 2 \\ 2 \\ 0 \end{pmatrix}$$

Thus, the system reduces to

$$x + z = 2$$

$$y - 2z = 2$$

$$0 = 0$$

This system is consistent and has infinitely many solutions given by

$$x = k$$

$$y = 6 - 2k$$

$$z = 2 - k \quad \text{where } k \text{ is a scalar.}$$

1.3.3 no solution

Example: **Inconsistent system**

Consider the following system of three linear equations.

$$x + 2y - 3z = -1$$

$$3x - y + 2z = 7$$

$$5x + 3y - 4z = 2$$

This system in matrix form is

$$\begin{pmatrix} 1 & 2 & -3 \\ 3 & -1 & 2 \\ 5 & 3 & -4 \end{pmatrix} \begin{pmatrix} x \\ y \\ z \end{pmatrix} = \begin{pmatrix} -1 \\ 7 \\ 2 \end{pmatrix}$$

1.3.3 no solution

Step 1: Multiplying the first row by –3 and adding it to the second row we obtain

$$\begin{pmatrix} 1 & 2 & -3 \\ 0 & -7 & 11 \\ 5 & 3 & -4 \end{pmatrix}\begin{pmatrix} x \\ y \\ z \end{pmatrix} = \begin{pmatrix} -1 \\ 10 \\ 2 \end{pmatrix}$$

Step 2: Multiplying the first row by –5 and adding it to the third row we obtain

$$\begin{pmatrix} 1 & 2 & -3 \\ 0 & -7 & 11 \\ 0 & -7 & 11 \end{pmatrix}\begin{pmatrix} x \\ y \\ z \end{pmatrix} = \begin{pmatrix} -1 \\ 10 \\ 7 \end{pmatrix}$$

1.3.3 no solution

Step 3: Multiplying row 2 by –1 and adding to row 3 we obtain

$$\begin{pmatrix} 1 & 2 & -3 \\ 0 & -7 & 11 \\ 0 & 0 & 0 \end{pmatrix} \begin{pmatrix} x \\ y \\ z \end{pmatrix} = \begin{pmatrix} -1 \\ 10 \\ -3 \end{pmatrix}$$

Thus the system reduces to

$$\begin{aligned} x + 2y - 3z &= -1 \\ -7y + 11z &= 10 \\ 0 &= -3 \end{aligned}$$

This shows that the system is **inconsistent** since the third equation is false. Thus, this system has no solution.