1 : Theory of Matrices, Vector spaces and Linear Transformations

IT5506 – Mathematics for Computing II

Level III - Semester 5

Intended Learning Outcomes

At the end of this lesson, you will be able to;

- Transform set of linear equations into matrix form
- Define elementary raw (column) operations applicable to the matrix algebra
- Apply elementary raw (column) operations to solve a system of linear equation

List of sub topics

1.4 Elementary row operations and their corresponding matrices (2 hours)

- 1.4.1 Finding row-echelon form of a matrix (Gaussian Method)
- 1.4.2 Finding row reduced-echelon form of a matrix (Gauss - Jordan Method)
- 1.4.3 Defining row rank and column rank of a matrix
- 1.4.4 Computing the inverse of a square matrix (if it exists) using Gauss - Jordan Method

1.4.1 Finding row-echelon form of a matrix (Gaussian Method)

ECHELON FORM

A rectangular matrix is in **echelon form** (or **row echelon form**) if it has the following three properties:

1. All nonzero rows are above any rows of all zeros.

2. Each leading entry of a row is in a column to the right of the leading entry of the row above it.

3. All entries in a column below a leading entry are zeros.

1.4.1 Finding row-echelon form of a matrix (Gaussian Method)

ECHELON FORM

The following matrix is in Echelon form

$$\begin{bmatrix} 1 & 4 & 5 & -9 & -7 \\ 0 & 2 & 4 & -6 & -6 \\ 0 & 0 & 0 & -5 & 0 \\ 0 & 0 & 0 & 0 & 0 \end{bmatrix}$$

1.4.1 Finding row-echelon form of a matrix (Gaussian Method)

ECHELON FORM

If a matrix in echelon form satisfies the following additional conditions, then it is in **reduced echelon form** (or **reduced row echelon form**):

4. The leading entry in each nonzero row is 1.

5. Each leading 1 is the only nonzero entry in its column.

An **echelon matrix** (respectively, **reduced echelon matrix**) is one that is in echelon form (respectively, reduced echelon form.)

The rows (columns) of a matrix which is in echelon form or reduced echelon form are said to be linearly independent.

1.4.1 Finding row-echelon form of a matrix (Gaussian Method)

ECHELON FORM

Matrix in Echelon form

$$\begin{bmatrix} 1 & 0 & 0 & 1 \\ 0 & 1 & 0 & 2 \\ 0 & 0 & 1 & 0 \end{bmatrix}$$

$$\begin{bmatrix} 1 & 0 & -2 & 3 & 0 & -24 \\ 0 & 1 & -2 & 2 & 0 & -7 \\ 0 & 0 & 0 & 0 & 1 & 4 \end{bmatrix}$$

1.4.2 Finding row reduced-echelon form of a matrix (Gauss - Jordan Method)

ECHELON FORM

Any nonzero matrix may be **row reduced** (i.e., transformed by elementary row operations) into more than one matrix in echelon form, using different sequences of row operations. However, the reduced echelon form one obtains from a matrix is unique.

Each matrix is row equivalent to one and only one reduced echelon matrix.

1.4.2 Finding row reduced-echelon form of a matrix (Gauss - Jordan Method)

ECHELON FORM- PIVOT POSITION

If a matrix A is row equivalent to an echelon matrix U, we call U **an echelon form** (or row echelon form) **of A**; if U is in reduced echelon form, we call U **the reduced echelon form of A**.

A **pivot position** in a matrix A is a location in A that corresponds to a leading 1 in the reduced echelon form of A. A **pivot column** is a column of A that contains a pivot position.

1.4.2 Finding row reduced-echelon form of a matrix (Gauss - Jordan Method)

ECHELON FORM- PIVOT POSITION

Example 1: Row reduce the matrix *A* below to echelon form, and locate the pivot columns of *A*.

$$A = \begin{bmatrix} 0 & -3 & -6 & 4 & 9 \\ -1 & -2 & -1 & 3 & 1 \\ -2 & -3 & 0 & 3 & -1 \\ 1 & 4 & 5 & -9 & -7 \end{bmatrix}$$

Solution: The top of the leftmost nonzero column is the first pivot position. A nonzero entry, or *pivot*, must be placed in this position.

1.4.2 Finding row reduced-echelon form of a matrix (Gauss - Jordan Method)

ECHELON FORM- PIVOT POSITION

Now, interchange rows 1 and 4.

Pivot

$$\begin{bmatrix} 1 & 4 & 5 & -9 & -7 \\ -1 & -2 & -1 & 3 & 1 \\ -2 & -3 & 0 & 3 & -1 \\ 0 & -3 & -6 & 4 & 9 \end{bmatrix}$$

Pivot column

Create zeros below the pivot, 1, by adding multiples of the first row to the rows below, and obtain the next matrix.

1.4.2 Finding row reduced-echelon form of a matrix (Gauss - Jordan Method)

ECHELON FORM- PIVOT POSITION

Choose 2 in the second row as the next pivot.

Pivot

$$\begin{bmatrix} 1 & 4 & 5 & -9 & -7 \\ 0 & 2 & 4 & -6 & -6 \\ 0 & 5 & 10 & -15 & -15 \\ 0 & -3 & -6 & 4 & 9 \end{bmatrix}$$

Next pivot column

Add $-5/2$ times row 2 to row 3, and add $3/2$ times row 2 to row 4.

1.4.2 Finding row reduced-echelon form of a matrix (Gauss - Jordan Method)

ECHELON FORM- PIVOT POSITION

$$\begin{bmatrix} 1 & 4 & 5 & -9 & -7 \\ 0 & 2 & 4 & -6 & -6 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & -5 & 0 \end{bmatrix}$$

There is no way a leading entry can be created in column 3. But, if we interchange rows 3 and 4, we can produce a leading entry in column 4.

1.4.2 Finding row reduced-echelon form of a matrix (Gauss - Jordan Method)

ECHELON FORM- PIVOT POSITION

Pivot

$$\begin{bmatrix} 1 & 4 & 5 & -9 & -7 \\ 0 & 2 & 4 & -6 & -6 \\ 0 & 0 & 0 & -5 & 0 \\ 0 & 0 & 0 & 0 & 0 \end{bmatrix}$$

Pivot columns

The matrix is in echelon form and thus reveals that columns 1, 2, and 4 of *A* are pivot columns.

The pivots in the example are 1, 2 and -5.

1.4.2 Finding row reduced-echelon form of a matrix (Gauss - Jordan Method)

ECHELON FORM- ROW REDUCTION ALGORITHM

Example 2: Apply elementary row operations to transform the following matrix first into echelon form and then into reduced echelon form.

$$\begin{bmatrix} 0 & 3 & -6 & 6 & 4 & -5 \\ 3 & -7 & 8 & -5 & 8 & 9 \\ 3 & -9 & 12 & -9 & 6 & 15 \end{bmatrix}$$

Solution:

STEP 1: Begin with the leftmost nonzero column. This is a pivot column. The pivot position is at the top.

1.4.2 Finding row reduced-echelon form of a matrix (Gauss - Jordan Method)

ECHELON FORM- ROW REDUCTION ALGORITHM

$$\begin{bmatrix} 0 & 3 & -6 & 6 & 4 & -5 \\ 3 & -7 & 8 & -5 & 8 & 9 \\ 3 & -9 & 12 & -9 & 6 & 15 \end{bmatrix}$$

Pivot column

STEP 2: Select a nonzero entry in the pivot column as a pivot. If necessary, interchange rows to move this entry into the pivot position.

1.4.2 Finding row reduced-echelon form of a matrix (Gauss - Jordan Method)

ECHELON FORM- ROW REDUCTION ALGORITHM

Interchange rows 1 and 3. (Rows 1 and 2 could have also been interchanged instead.)

Pivot

$$\begin{bmatrix} 3 & -9 & 12 & -9 & 6 & 15 \\ 3 & -7 & 8 & -5 & 8 & 9 \\ 0 & 3 & -6 & 6 & 4 & -5 \end{bmatrix}$$

STEP 3: Use row replacement operations to create zeros in all positions below the pivot.

1.4.2 Finding row reduced-echelon form of a matrix (Gauss - Jordan Method)

ECHELON FORM- ROW REDUCTION ALGORITHM

We could have divided the top row by the pivot, 3, but with two 3s in column 1, it is just as easy to add -1 times row 1 to row 2.

Pivot

$$\begin{bmatrix} 3 & -9 & 12 & -9 & 6 & 15 \\ 0 & 2 & -4 & 4 & 2 & -6 \\ 0 & 3 & -6 & 6 & 4 & -5 \end{bmatrix}$$

STEP 4: Cover the row containing the pivot position, and cover all rows, if any, above it. Apply steps 1–3 to the submatrix that remains. Repeat the process until there are no more nonzero rows to modify.

1.4.2 Finding row reduced-echelon form of a matrix (Gauss - Jordan Method)

ECHELON FORM- ROW REDUCTION ALGORITHM

With row 1 covered, step 1 shows that column 2 is the next pivot column; for step 2, select as a pivot the “top” entry in that column.

Pivot

$$\begin{bmatrix} 3 & -9 & 12 & -9 & 6 & 15 \\ 0 & 2 & -4 & 4 & 2 & -6 \\ 0 & 3 & -6 & 6 & 4 & -5 \end{bmatrix}$$

New pivot column

For step 3, we could insert an optional step of dividing the “top” row of the submatrix by the pivot, 2. Instead, we add $-3/2$ times the “top” row to the row below.

1.4.2 Finding row reduced-echelon form of a matrix (Gauss - Jordan Method)

ECHELON FORM- ROW REDUCTION ALGORITHM

This produces the following matrix.

$$\begin{bmatrix} \boxed{3 & -9 & 12 & -9 & 6 & 15} \\ 0 & 2 & -4 & 4 & 2 & -6 \\ 0 & 0 & 0 & 0 & 1 & 4 \end{bmatrix}$$

When we cover the row containing the second pivot position for step 4, we are left with a new submatrix that has only one row.

$$\begin{bmatrix} 3 & -9 & 12 & -9 & 6 & 15 \\ 0 & 2 & -4 & 4 & 2 & -6 \\ 0 & 0 & 0 & 0 & 1 & 4 \end{bmatrix}$$

1.4.2 Finding row reduced-echelon form of a matrix (Gauss - Jordan Method)

ECHELON FORM- ROW REDUCTION ALGORITHM

Steps 1–3 require no work for this submatrix, and we have reached an echelon form of the full matrix. We perform one more step to obtain the reduced echelon form.

STEP 5: Beginning with the rightmost pivot and working upward and to the left, create zeros above each pivot. If a pivot is not 1, make it 1 by a scaling operation.

The rightmost pivot is in row 3. Create zeros above it, adding suitable multiples of row 3 to rows 2 and 1.

1.4.2 Finding row reduced-echelon form of a matrix (Gauss - Jordan Method)

ECHELON FORM- ROW REDUCTION ALGORITHM

$$\begin{bmatrix} 3 & -9 & 12 & -9 & 0 & -9 \\ 0 & 2 & -4 & 4 & 0 & -14 \\ 0 & 0 & 0 & 0 & 1 & 4 \end{bmatrix} \begin{array}{l} \leftarrow \text{Row } 1 + (-6) \times \text{row } 3 \\ \leftarrow \text{Row } 2 + (-2) \times \text{row } 3 \\ \end{array}$$

The next pivot is in row 2. Scale this row, dividing by the pivot.

$$\begin{bmatrix} 3 & -9 & 12 & -9 & 0 & -9 \\ 0 & 1 & -2 & 2 & 0 & -7 \\ 0 & 0 & 0 & 0 & 1 & 4 \end{bmatrix} \begin{array}{l} \\ \leftarrow \text{Row scaled by } \dfrac{1}{2} \\ \end{array}$$

1.4.2 Finding row reduced-echelon form of a matrix (Gauss - Jordan Method)

ECHELON FORM- ROW REDUCTION ALGORITHM

Create a zero in column 2 by adding 9 times row 2 to row 1.

$$\begin{bmatrix} 3 & 0 & -6 & 9 & 0 & -72 \\ 0 & 1 & -2 & 2 & 0 & -7 \\ 0 & 0 & 0 & 0 & 1 & 4 \end{bmatrix} \leftarrow \text{Row } 1+(9)\times\text{row } 2$$

Finally, scale row 1, dividing by the pivot, 3.

1.4.2 Finding row reduced-echelon form of a matrix (Gauss - Jordan Method)

ECHELON FORM- ROW REDUCTION ALGORITHM

$$\begin{bmatrix} 1 & 0 & -2 & 3 & 0 & -24 \\ 0 & 1 & -2 & 2 & 0 & -7 \\ 0 & 0 & 0 & 0 & 1 & 4 \end{bmatrix} \leftarrow \text{Row scaled by } \frac{1}{3}$$

This is the reduced echelon form of the original matrix.

The combination of steps 1–4 is called the **forward phase** of the row reduction algorithm. Step 5, which produces the unique reduced echelon form, is called the **backward phase**.

1.4.2 Finding row reduced-echelon form of a matrix (Gauss - Jordan Method)

ECHELON FORM- Solution of Linear Equation

The row reduction algorithm leads to an explicit description of the solution set of a linear system when the algorithm is applied to the augmented matrix of the system.

Suppose that the augmented matrix of a linear system has been changed into the equivalent *reduced* echelon form.

$$\begin{bmatrix} 1 & 0 & -5 & 1 \\ 0 & 1 & 1 & 4 \\ 0 & 0 & 0 & 0 \end{bmatrix}$$

1.4.2 Finding row reduced-echelon form of a matrix (Gauss - Jordan Method)

ECHELON FORM- Solution of Linear Equation

There are 3 variables because the augmented matrix has four columns. The associated system of equations is

$$\begin{aligned} x_1 - 5x_3 &= 1 \\ x_2 + x_3 &= 4 \\ 0 &= 0 \end{aligned} \qquad \text{----(1)}$$

The variables x_1 and x_2 corresponding to pivot columns in the matrix are called **basic variables**. The other variable, x_3, is called a **free variable**.

1.4.2 Finding row reduced-echelon form of a matrix (Gauss - Jordan Method)

ECHELON FORM- Solution of Linear Equation

Whenever a system is consistent, as in (1), the solution set can be described explicitly by solving the *reduced* system of equations for the basic variables in terms of the free variables.

This operation is possible because the reduced echelon form places each basic variable in one and only one equation.

In (1), solve the first and second equations for x_1 and x_2. (Ignore the third equation; it offers no restriction on the variables.)

1.4.2 Finding row reduced-echelon form of a matrix (Gauss - Jordan Method)

ECHELON FORM- Solution of Linear Equation

$$\begin{aligned} x_1 &= 1 + 5x_3 \\ x_2 &= 4 - x_3 \qquad \text{----(2)} \\ x_3 &\text{ is free} \end{aligned}$$

The statement "x_3 is free" means that you are free to choose any value for x_3. Once that is done, the formulas in (2) determine the values for x_1 and x_2. For instance, when $x_3 = 0$, the solution is (1,4,0); when $x_3 = 1$, the solution is (6,3,1).

Each different choice of x_3 determines a (different) solution of the system, and every solution of the system is determined by a choice of x_3.

1.4.2 Finding row reduced-echelon form of a matrix (Gauss - Jordan Method)

ECHELON FORM- Parametric Description of Solution Sets

The description in (2) is a *parametric description* of solutions sets in which the free variables act as parameters.

Solving a system amounts to finding a parametric description of the solution set or determining that the solution set is empty.

Whenever a system is consistent and has free variables, the solution set has many parametric descriptions.

1.4.2 Finding row reduced-echelon form of a matrix (Gauss - Jordan Method)

ECHELON FORM- Parametric Description of Solution Sets

For instance, in system (1), add 5 times equation 2 to equation 1 and obtain the following equivalent system.

$$x_1 + 5x_2 = 21$$

$$x_2 + x_3 = 4$$

We could treat x_2 as a parameter and solve for x_1 and x_3 in terms of x_2, and we would have an accurate description of the solution set.

When a system is inconsistent, the solution set is empty, even when the system has free variables. In this case, the solution set has no parametric representation.

1.4.2 Finding row reduced-echelon form of a matrix (Gauss - Jordan Method)

ECHELON FORM- Existence and Uniqueness

A linear system is consistent if and only if the rightmost column of the augmented matrix is *not* a pivot column—i.e., if and only if an echelon form of the augmented matrix has *no* row of the form

$$[0 \; \ldots \; 0 \quad b] \text{ with } b \text{ nonzero.}$$

If a linear system is consistent, then the solution set contains either (i) a unique solution, when there are no free variables, or (ii) infinitely many solutions, when there is at least on free variable.

1.4.2 Finding row reduced-echelon form of a matrix (Gauss - Jordan Method)

Using Row Reduction to Solve a Linear System

- Write the augmented matrix of the system.
- Use the row reduction algorithm to obtain an equivalent augmented matrix in echelon form. Decide whether the system is consistent. If there is no solution, stop; otherwise, go to the next step.
- Continue row reduction to obtain the reduced echelon form.
- Write the system of equations corresponding to the matrix obtained in step 3.

1.4.2 Finding row reduced-echelon form of a matrix (Gauss - Jordan Method)

Using Row Reduction to Solve a Linear System

- Write the augmented matrix of the system.
- Use the row reduction algorithm to obtain an equivalent augmented matrix in echelon form. Decide whether the system is consistent. If there is no solution, stop; otherwise, go to the next step.
- Continue row reduction to obtain the reduced echelon form.
- Write the system of equations corresponding to the matrix obtained in step 3.
- Rewrite each nonzero equation from step 4 so that its one basic variable is expressed in terms of any free variables appearing in the equation.

1.4.3 Defining row rank and column rank of a matrix

Rank of a Matrix

The maximum number of its linearly independent rows (or columns) of a matrix is called the rank of a matrix. The rank of a matrix cannot exceed the number of its rows or columns.

If we consider a square matrix, the rows (columns) are linearly independent only if the matrix is nonsingular. In other words, the rank of any nonsingular matrix ($|A| \neq 0$) of order m is m. The rank of a matrix A is denoted by $\rho(A)$

The rank of a null matrix is zero.

Note that the rows (columns) of a matrix which is in echelon form or reduced echelon forms are linearly independent.

1.4.3 Defining row rank and column rank of a matrix

Raw rank of a Matrix

To find the raw rank of a matrix, we will transform that matrix into its echelon form.

Then determine the raw rank by the number of non-zero rows.

1.4.3 Defining row rank and column rank of a matrix

Raw Rank of a Matrix

Consider the following matrix.

$$\begin{bmatrix} 2 & 4 & 6 \\ 4 & 8 & 12 \end{bmatrix}$$

Here we have two rows. But the second row is two times the first row.

The raw rank is considered as 1

1.4.3 Defining row rank and column rank of a matrix

Raw Rank of a Matrix

Consider the Identity matrix.

$$\begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

All the rows are independent.

The rank of this matrix is 3.

The rank of an identity matrix of order m is m

1.4.3 Defining row rank and column rank of a matrix

Raw Rank of a Matrix

If A matrix is of order $m \times n$,
then $\rho(A) \leq \min\{m, n\}$ = minimum of m, n..

If A is of order $n \times n$ and $|A| \neq 0$, then the rank of A = n..

If A is of order $n \times n$ and $|A| = 0$, then the raw rank of A will be less than n

1.4.3 Defining row rank and column rank of a matrix

Raw Rank of a Matrix by Row-Echelon Form

We can transform a given non-zero matrix to a simplified form called a Row-echelon form, using the row elementary operations

In this form, we may have rows all of whose entries are zero. Such rows are called zero rows. A non-zero row is one in which at least one of the elements is not zero

$$\begin{bmatrix} 1 & 0 & 2 \\ 0 & 0 & 1 \\ 0 & 0 & 0 \end{bmatrix}$$

Here R_1 and R_2 are non-zero rows, R_3 is a zero row

1.4.3 Defining row rank and column rank of a matrix

Raw Rank of a Matrix by Row-Echelon Form

A non-zero matrix A is said to be in a row-echelon form if:

(i) All zero rows of A occur below every non-zero row of A.

(ii) The first non-zero element in any row i of A occurs in the j^{th} column of A, and then all other elements in the j^{th} column of A below the first non-zero element of row i are zeros.

(iii) The first non-zero entry in the i^{th} row of A lies to the left of the first non-zero entry in $(i + 1)^{th}$ row of A.

Note: A non-zero matrix is said to be in a row-echelon form if all zero rows occur as bottom rows of the matrix and if the first non-zero element in any lower row occurs to the right of the first non-zero entry in the higher row.

If a matrix is row-echelon, all elements below the leading diagonal are zeros

1.4.3 Defining row rank and column rank of a matrix

Raw Rank of a Matrix by Row-Echelon Form

Consider the Identity matrix.

$$\begin{bmatrix} 0 & 0 & 1 \\ 0 & 0 & 5 \\ 0 & 0 & 0 \end{bmatrix}$$

Check the rows from the last row of the matrix. The third row is a zero row.

The first non-zero element in the second row occurs in the third column, and it lies to the right of the first non-zero element in the first row, which occurs in the second column.

Hence, matrix is in row echelon form**.** Number of non-zero rows = 2 and the rank of matrix is 2

1.4.3 Defining row rank and column rank of a matrix

Rank of a Matrix by Row-Echelon Form

Ex 1: Find the rank of matrix A by using the row echelon form.

$$A = \begin{bmatrix} 1 & 2 & 3 \\ 2 & 1 & 4 \\ 3 & 0 & 5 \end{bmatrix}$$

$R_2 \rightarrow R_2 - 2R_1$

$R_3 \rightarrow R_3 - 3R_1$

$$\begin{bmatrix} 1 & 2 & 3 \\ 0 & -3 & -2 \\ 0 & -6 & -4 \end{bmatrix}$$

1.4.3 Defining row rank and column rank of a matrix

Rank of a Matrix by Row-Echelon Form

$R_3 \rightarrow R_3 - 2R_2$

$$\begin{bmatrix} 1 & 2 & 3 \\ 0 & -3 & -2 \\ 0 & 0 & 0 \end{bmatrix}$$

The above matrix is in row echelon form

Number of non-zero rows = 2

Hence the rank of matrix A = 2

1.4.3 Defining row rank and column rank of a matrix

Rank of a Matrix by Row-Echelon Form

Ex 2: Find the rank of matrix A by using the row echelon form.

$$A = \begin{bmatrix} 1 & 2 & 3 \\ 2 & 3 & 4 \\ 3 & 5 & 7 \end{bmatrix}$$

$R_2 \rightarrow R_2 - 2R_1$

$R_3 \rightarrow R_3 - 3R_1$

$$\begin{bmatrix} 1 & 2 & 3 \\ 0 & -1 & -2 \\ 0 & -1 & -2 \end{bmatrix}$$

1.4.3 Defining row rank and column rank of a matrix

Rank of a Matrix by Row-Echelon Form

$R_3 \rightarrow R_3 - R_2$

$$\begin{bmatrix} 1 & 2 & 3 \\ 0 & -1 & -2 \\ 0 & 0 & 0 \end{bmatrix}$$

The above matrix is in row echelon form

Number of non-zero rows = 2

Hence the rank of matrix A = 2

1.4.3 Defining row rank and column rank of a matrix

Rank of a Matrix by Row-Echelon Form

Ex 3: Find the rank of matrix A by using the row echelon form.

$$A = \begin{bmatrix} 1 & 1 & 1 \\ 1 & 1 & 1 \\ 1 & 1 & 1 \end{bmatrix}$$

$R_2 \rightarrow R_2 - R_1$
$R_3 \rightarrow R_3 - R_1$

$$\begin{bmatrix} 1 & 1 & 1 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}$$

The above matrix is in row echelon form

Number of non-zero rows = 1

Hence the rank of matrix A = 1

1.4.3 Defining row rank and column rank of a matrix

Rank of a Matrix by Row-Echelon Form

Ex 4: Find the rank of matrix A by using Row-Echelon.

B = $\begin{bmatrix} 4 & 7 \\ 8 & 14 \end{bmatrix}$

By observing the rows, we can see that the elements of the second row are twice the elements of the first row.

$R_1 \rightarrow 2R_1 - R_2$ $\begin{bmatrix} 0 & 0 \\ 8 & 14 \end{bmatrix}$

Number of non-zero rows = 1.

The rank of matrix A = 1.

1.4.3 Defining row rank and column rank of a matrix

Rank of a Matrix by Row-Echelon Form

Ex 5:

Let $A = \begin{bmatrix} 1 & 1 & -1 \\ 2 & -3 & 4 \\ 3 & -2 & 3 \end{bmatrix}$ and $B = \begin{bmatrix} -1 & -2 & -1 \\ 6 & 12 & 6 \\ 5 & 10 & 5 \end{bmatrix}$ be two matrices

Then the rank of A+B=

(a) 1
(b) 0
(c) 2
(d) 3

1.4.3 Defining row rank and column rank of a matrix

Rank of a Matrix by Row-Echelon Form

Solution

$$A = \begin{bmatrix} 1 & 1 & -1 \\ 2 & -3 & 4 \\ 3 & -2 & 3 \end{bmatrix} \quad B = \begin{bmatrix} -1 & -2 & -1 \\ 6 & 12 & 6 \\ 5 & 10 & 5 \end{bmatrix}$$

$$A+B = \begin{bmatrix} 0 & -1 & -2 \\ 8 & 9 & 10 \\ 8 & 8 & 8 \end{bmatrix}$$

1.4.3 Defining row rank and column rank of a matrix

Rank of a Matrix by Row-Echelon Form

$$\begin{bmatrix} 0 & -1 & -2 \\ 8 & 9 & 10 \\ 8 & 8 & 8 \end{bmatrix}$$

Interchange C_1 and C_2 $\begin{bmatrix} -1 & 0 & -2 \\ 9 & 8 & 10 \\ 8 & 8 & 8 \end{bmatrix}$

$R_2 \longrightarrow R_2 + 9R_1$

$R_3 \longrightarrow R_3 + 8R_1$

$$\begin{bmatrix} -1 & 0 & -2 \\ 0 & 8 & -8 \\ 0 & 8 & -8 \end{bmatrix}$$

1.4.3 Defining row rank and column rank of a matrix

Rank of a Matrix by Row-Echelon Form

$R_3 \rightarrow R_3 - R_2$

$$\begin{bmatrix} -1 & 0 & -2 \\ 0 & 8 & -8 \\ 0 & 0 & 0 \end{bmatrix}$$

$R_2 \rightarrow R_2/8$

$$\begin{bmatrix} -1 & 0 & -2 \\ 0 & 1 & -1 \\ 0 & 0 & 0 \end{bmatrix}$$

number of non-zero rows=2. So the rank = 2

Hence option (c) is the answer.

1.4.4 Computing the inverse of a square matrix (if it exists) using Gauss - Jordan Method.

To obtain the inverse of a $n \times n$ matrix A :

- Create the partitioned matrix , (A|I) where I is the identity matrix.
- Perform Gauss-Jordan Elimination on the partitioned matrix with the objective of converting the first part of the matrix to reduced-row echelon form.
- The resulting partitioned matrix will take the form of $(I|A^{-1})$

1.4.4 Computing the inverse of a square matrix (if it exists) using Gauss - Jordan Method.

Find the inverse of the matrix A using Gauss-Jordan elimination

$$A = \begin{bmatrix} 5 & 8 & 6 \\ 7 & 10 & 9 \\ 3 & 11 & 12 \end{bmatrix}$$

Solution:

We write matrix A on the left and the Identity matrix I on its right separated with a dotted line, as follows. The result is called an **augmented** matrix.

1.4.4 Computing the inverse of a square matrix (if it exists) using Gauss - Jordan Method.

$$\begin{array}{cc} \left(\begin{array}{ccc|ccc} 5 & 8 & 6 & 1 & 0 & 0 \\ 7 & 10 & 9 & 0 & 1 & 0 \\ 3 & 11 & 12 & 0 & 0 & 1 \end{array}\right) & \begin{array}{c} R1 \\ R2 \\ R3 \end{array} \end{array}$$

Row numbers are included for convenient.

Our row operations procedure is as follows:

1. get a "1" in the top left corner by dividing the first row
2. Then get "0" in the rest of the first column
3. Then need to get "1" in the second row, second column
4. Then make all the other entries in the second column "0" and etc.

1.4.4 Computing the inverse of a square matrix (if it exists) using Gauss - Jordan Method.

$$\begin{array}{c} \left(\begin{array}{ccc|ccc} 5 & 8 & 6 & 1 & 0 & 0 \\ 7 & 10 & 9 & 0 & 1 & 0 \\ 3 & 11 & 12 & 0 & 0 & 1 \end{array}\right) \end{array} \begin{array}{l} R1 \\ R2 \\ R3 \end{array}$$

R1← R1/5

$$\left(\begin{array}{ccc|ccc} 1 & 1.6 & 1.2 & 0.2 & 0 & 0 \\ 7 & 10 & 9 & 0 & 1 & 0 \\ 3 & 11 & 12 & 0 & 0 & 1 \end{array}\right) \begin{array}{l} R1 \\ R2 \\ R3 \end{array}$$

R2← R2 -7*R1

$$\left(\begin{array}{ccc|ccc} 1 & 1.6 & 1.2 & 0.2 & 0 & 0 \\ 0 & -1.2 & 0.6 & -1.4 & 1 & 0 \\ 3 & 11 & 12 & 0 & 0 & 1 \end{array}\right) \begin{array}{l} R1 \\ R2 \\ R3 \end{array}$$

1.4.4 Computing the inverse of a square matrix (if it exists) using Gauss - Jordan Method.

$$\begin{array}{cc} \left(\begin{array}{ccc|ccc} 1 & 1.6 & 1.2 & 0.2 & 0 & 0 \\ 0 & -1.2 & 0.6 & -1.4 & 1 & 0 \\ 3 & 11 & 12 & 0 & 0 & 1 \end{array}\right) & \begin{array}{c} \text{R1} \\ \text{R2} \\ \text{R3} \end{array} \end{array}$$

R3← R3 -3*R1

$$\begin{array}{cc} \left(\begin{array}{ccc|ccc} 1 & 1.6 & 1.2 & 0.2 & 0 & 0 \\ 0 & -1.2 & 0.6 & -1.4 & 1 & 0 \\ 0 & 6.2 & 8.4 & -0.6 & 0 & 1 \end{array}\right) & \begin{array}{c} \text{R1} \\ \text{R2} \\ \text{R3} \end{array} \end{array}$$

R2← R2/(-1.2)

$$\begin{array}{cc} \left(\begin{array}{ccc|ccc} 1 & 1.6 & 1.2 & 0.2 & 0 & 0 \\ 0 & 1 & -0.5 & -1.1667 & -0.8333 & 0 \\ 0 & 6.2 & 8.4 & -0.6 & 0 & 1 \end{array}\right) & \begin{array}{c} \text{R1} \\ \text{R2} \\ \text{R3} \end{array} \end{array}$$

1.4.4 Computing the inverse of a square matrix (if it exists) using Gauss - Jordan Method.

$$\begin{array}{c} \left(\begin{array}{ccc|ccc} 1 & 1.6 & 1.2 & 0.2 & 0 & 0 \\ 0 & 1 & -0.5 & -1.1667 & -0.8333 & 0 \\ 0 & 6.2 & 8.4 & -0.6 & 0 & 1 \end{array}\right) \end{array} \begin{array}{l} \text{R1} \\ \text{R2} \\ \text{R3} \end{array}$$

R1← R1 -1.6*R2

$$\left(\begin{array}{ccc|ccc} 1 & 0 & 2 & -1.6667 & 1.3333 & 0 \\ 0 & 1 & -0.5 & -1.1667 & -0.8333 & 0 \\ 0 & 6.2 & 8.4 & -0.6 & 0 & 1 \end{array}\right) \begin{array}{l} \text{R1} \\ \text{R2} \\ \text{R3} \end{array}$$

R3← R3 -6.2*R2

$$\left(\begin{array}{ccc|ccc} 1 & 0 & 2 & -1.6667 & 1.3333 & 0 \\ 0 & 1 & -0.5 & -1.1667 & -0.8333 & 0 \\ 0 & 0 & 11.5 & -7.8333 & 5.16670 & 1 \end{array}\right) \begin{array}{l} \text{R1} \\ \text{R2} \\ \text{R3} \end{array}$$

1.4.4 Computing the inverse of a square matrix (if it exists) using Gauss - Jordan Method.

$$\left(\begin{array}{ccc|ccc} 1 & 0 & 2 & -1.6667 & 1.3333 & 0 \\ 0 & 1 & -0.5 & -1.1667 & -0.8333 & 0 \\ 0 & 0 & 11.5 & -7.8333 & 5.16670 & 1 \end{array}\right) \begin{array}{c} R1 \\ R2 \\ R3 \end{array}$$

R3← R3 /11.5

$$\left(\begin{array}{ccc|ccc} 1 & 0 & 2 & -1.6667 & 1.3333 & 0 \\ 0 & 1 & -0.5 & -1.1667 & -0.8333 & 0 \\ 0 & 0 & 1 & -0.6812 & 0.4493 & 0.087 \end{array}\right) \begin{array}{c} R1 \\ R2 \\ R3 \end{array}$$

R2← R2 -0.5*R3

$$\left(\begin{array}{ccc|ccc} 1 & 0 & 2 & -1.6667 & 1.3333 & 0 \\ 0 & 1 & 0 & 0.8261 & -0.6087 & 0.0435 \\ 0 & 0 & 1 & -0.6812 & 0.4493 & 0.087 \end{array}\right) \begin{array}{c} R1 \\ R2 \\ R3 \end{array}$$

1.4.4 Computing the inverse of a square matrix (if it exists) using Gauss - Jordan Method.

$$\left(\begin{array}{ccc|ccc} 1 & 0 & 2 & -1.6667 & 1.3333 & 0 \\ 0 & 1 & 0 & 0.8261 & -0.6087 & 0.0435 \\ 0 & 0 & 1 & -0.6812 & 0.4493 & 0.087 \end{array}\right)$$

It can be concluded that the inverse of the matrix A is the right-hand portion of the augmented matrix:

$$A^{-1} = \begin{pmatrix} -1,6667 & 1.3333 & 0 \\ 0.8261 & -0.6087 & 0.0435 \\ -0.6812 & 0.4493 & 0.087 \end{pmatrix}$$