Mathematical Induction

- Mathematical statements assert that a property is true for all positive integers

Mathematical Induction is a technique of proving a statement, theorem or formula which is thought to be true, for each and every natural number n

- Examples of such statements are that for every positive integer

$$n : n! \leq n^n$$

$$n^3 - n \text{ is divisible by } 3$$

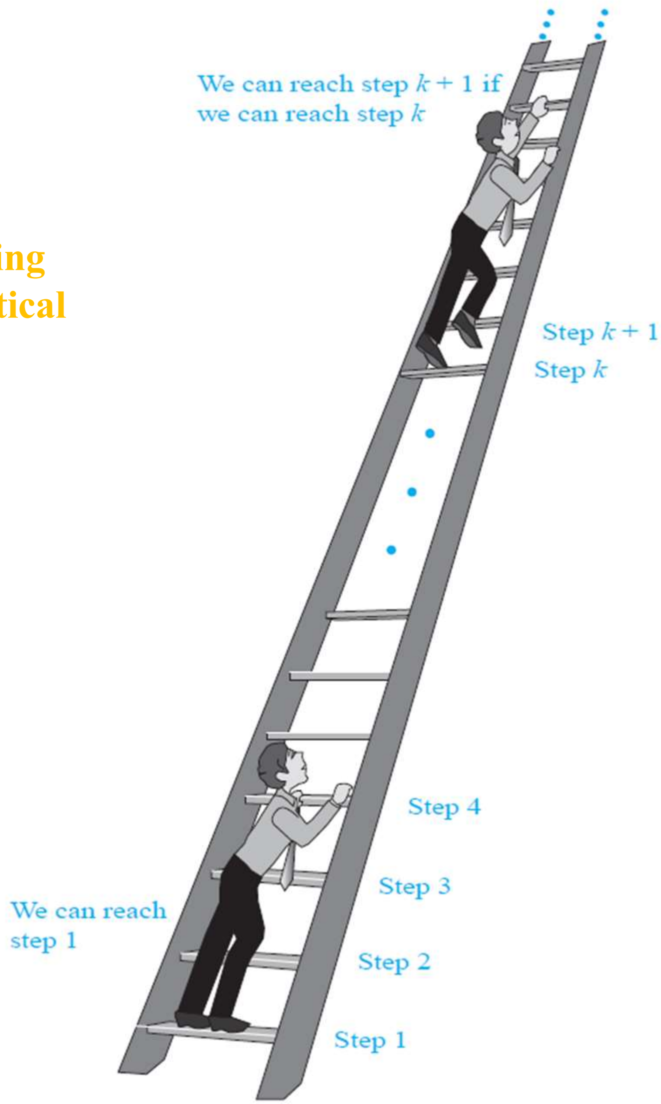
$$\text{A set with n elements has } 2^n \text{ subsets}$$

Proofs using mathematical induction have two parts.

✓First, they show that the statement holds for the positive integer 1.

✓Second, they show that if the statement holds for a positive integer. Then it must also hold for the next larger integer.

Proofs using mathematical induction



But can we conclude that we are able to reach every rung of this infinite ladder?

The answer is yes, something we can verify using an important proof technique called **mathematical induction**.

That is, we can show that $P(n)$ is true for every positive integer n, where $P(n)$ is the statement that we can reach the nth rung of the ladder.

Mathematical induction is an extremely important proof technique that can be used to prove assertions of this type.

Mathematical Induction

Assert that P(n) is true for all positive integers n, where P(n) is a propositional function

$$\text{example } P(n) : n! \leq n^n$$

A proof by mathematical induction has two parts, a **basis step**, where we show that $P(1)$ is true, and an **inductive step**, where we show that for all positive integers k, if $P(k)$ is true, then $P(k + 1)$ is true.

PRINCIPLE OF MATHEMATICAL INDUCTION To prove that $P(n)$ is true for all positive integers n, where $P(n)$ is a propositional function, we complete two steps:

BASIS STEP: We verify that $P(1)$ is true.

INDUCTIVE STEP: We show that the conditional statement $P(k) \rightarrow P(k+1)$ is true for all positive integers k.

EXAMPLE

Show that if n is a positive integer, then

$$1 + 2 + \cdots + n = \frac{n(n+1)}{2}.$$

Solution: Let $P(n)$ be the proposition that the sum of the first n positive integers, $1 + 2 + \cdots n = \frac{n(n+1)}{2}$, is $n(n+1)/2$. We must do two things to prove that $P(n)$ is true for $n = 1, 2, 3, \ldots$. Namely, we must show that $P(1)$ is true and that the conditional statement $P(k)$ implies $P(k+1)$ is true for $k = 1, 2, 3, \ldots$.

BASIS STEP: $P(1)$ is true, because $1 = \dfrac{1(1+1)}{2}$. (The left-hand side of this equation is 1 because 1 is the sum of the first positive integer. The right-hand side is found by substituting 1 for n in $n(n+1)/2$.)

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INDUCTIVE STEP: For the inductive hypothesis we assume that $P(k)$ holds for an arbitrary positive integer k. That is, we assume that

$$1 + 2 + \cdots + k = \frac{k(k+1)}{2}.$$

Under this assumption, it must be shown that $P(k + 1)$ is true

When we add $k + 1$ to both sides of the equation in $P(k)$, we obtain

EXAMPLE

Show that if n is a positive integer, then

$$1 + 2 + \cdots + n = \frac{n(n+1)}{2}.$$

This last equation shows that $P(k+1)$ is true under the assumption that $P(k)$ is true. This completes the inductive step.

We have completed the basis step and the inductive step, so by mathematical induction we know that $P(n)$ is true for all positive integers n. That is, we have proven that $1 + 2 + \cdots + n = n(n+1)/2$ for all positive integers n. ◀

EXAMPLE

Conjecture a formula for the sum of the first n positive odd integers. Then prove your conjecture using mathematical induction.

Solution: The sums of the first n positive odd integers for $n = 1, 2, 3, 4, 5$ are

$$\begin{aligned} &1 = 1, && 1+3 = 4, && 1+3+5 = 9, \\ &1+3+5+7 = 16, && 1+3+5+7+9 = 25. \end{aligned}$$

From these values it is reasonable to conjecture that the sum of the first n positive odd integers is n^2, that is, $1 + 3 + 5 + \cdots + (2n - 1) = n^2$. We need a method to *prove* that this *conjecture* is correct, if in fact it is.