Basics of Counting

Combinatorics

The study of arrangements of objects -

- is an important part of discrete mathematics

Enumeration

The counting of objects with certain properties -

- is an important part of combinatorics

Basics of Counting

Counting problems arise throughout mathematics and computer science.

- Example: We must count the successful outcomes of experiments and all the possible outcomes of these experiments to determine probabilities of discrete events.
- We need to count the number of operations used by an algorithm to study its time complexity.

Basic Counting Principles

Product rule and the sum rule

THE PRODUCT RULE Suppose that a procedure can be broken down into a sequence of two tasks. If there are n_1 ways to do the first task and for each of these ways of doing the first task, there are n_2 ways to do the second task, then there are $n_1 n_2$ ways to do the procedure.

Example 1

A new company with just two employees, Sanchez and Patel, rents a floor of a building with 12 offices. How many ways are there to assign different offices to these two employees?

Example 2

The chairs of an auditorium are to be labeled with an uppercase English letter followed by a positive integer not exceeding 100. What is the largest number of chairs that can be labeled differently?

Basic Counting Principles

Product rule and the sum rule

THE PRODUCT RULE Suppose that a procedure can be broken down into a sequence of two tasks. If there are n_1 ways to do the first task and for each of these ways of doing the first task, there are n_2 ways to do the second task, then there are $n_1 n_2$ ways to do the procedure.

Example 3

How many different bit strings of length seven are there?

Example 4

How many different license plates can be made if each plate contains a sequence of three uppercase English letters followed by three digits

Example 5 (Counting Functions)

How many functions are there from a set with m elements to a set with n elements?

Example 6 (Counting One-to-One Functions)

How many one-to-one functions are there from a set with m elements to one with n elements?

Basic Counting Principles

Product rule and the sum rule

Example 7

The Telephone Numbering Plan The North American numbering plan (NANP) specifies the format of telephone numbers in the U.S., Canada, and many other parts of North America. A telephone number in this plan consists of 10 digits, which are split into a three-digit area code, a three-digit office code, and a four-digit station code. Because of signaling considerations, there are certain restrictions on some of these digits. To specify the allowable format, let X denote a digit that can take any of the values 0 through 9, let N denote a digit that can take any of the values 2 through 9, and let Y denote a digit that must be a 0 or a 1. Two numbering plans, which will be called the old plan, and the new plan, will be discussed.

In the old plan, the formats of the area code, office code, and station code are NYX, NNX, and XXXX, respectively, so that telephone numbers had the form NYX-NNX-XXXX. In the new plan, the formats of these codes are NXX, NXX, and XXXX, respectively, so that telephone numbers have the form NXX-NXX-XXXX. How many different North American telephone numbers are possible under the old plan and under the new plan?

Basic Counting Principles

Product rule and the sum rule

Example 8

What is the value of k after the following code, where $n_1, n_2, \ldots, n_m$ are positive integers, has been executed?

$$
\begin{array}{l}
k := 0 \\
\textbf{for } i_1 := 1 \textbf{ to } n_1 \\
\quad \textbf{for } i_2 := 1 \textbf{ to } n_2 \\
\qquad \qquad \cdot \\
\qquad \qquad \cdot \\
\qquad \qquad \cdot \\
\qquad \textbf{for } i_m := 1 \textbf{ to } n_m \\
\qquad \quad k := k + 1
\end{array}
$$

Basic Counting Principles

Product rule and the sum rule

Example 9

Use the product rule to show that the number of different subsets of a finite set S is $2^{|S|}$

The product rule is often phrased in terms of sets in this way: If $A_1, A_2, \ldots, A_m$ are finite sets, then the number of elements in the Cartesian product of these sets is the product of the number of elements in each set. To relate this to the product rule, note that the task of choosing an element in the Cartesian product $A_1 \times A_2 \times \cdots \times A_m$ is done by choosing an element in A_1, an element in $A_2, \ldots,$ and an element in A_m. By the product rule it follows that

$$|A_1 \times A_2 \times \cdots \times A_m| = |A_1| \cdot |A_2| \cdot \cdots \cdot |A_m|.$$

We now introduce the SUM RULE

THE SUM RULE If a task can be done either in one of n_1 ways or in one of n_2 ways, where none of the set of n_1 ways is the same as any of the set of n_2 ways, then there are $n_1 + n_2$ ways to do the task.

Example 10

Suppose that either a member of the mathematics faculty or a student who is a mathematics major is chosen as a representative to a university committee. How many different choices are there for this representative if there are 37 members of the mathematics faculty and 83 mathematics majors and no one is both a faculty member and a student?

Example 11

A student can choose a computer project from one of three lists. The three lists contain 23, 15, and 19 possible projects, respectively. No project is on more than one list. How many possible projects are there to choose from?

We now introduce the SUM RULE

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Example 12

What is the value of k after the following code, where $n_1, n_2, \ldots, n_m$ are positive integers, has been executed?

$k := 0$
for $i_1 := 1$ **to** n_1
    $k := k + 1$
for $i_2 := 1$ **to** n_2
    $k := k + 1$
    .
    .
    .
for $i_m := 1$ **to** n_m
    $k := k + 1$

The sum rule can be phrased in terms of sets as: If $A_1, A_2, \ldots A_m$ are pairwise disjoint finite sets, then the number of elements in the union of these sets is the sum of the numbers of elements in the sets.

To relate this to our statement of the sum rule, note there are $|Ai|$ ways to choose an element from A_i for $i = 1, 2, \ldots, m$.

$$|A_1 \cup A_2 \cup \cdots \cup A_m| = |A_1| + |A_2| + \cdots + |A_m| \text{ when } A_i \cap A_j = \text{ for all } i, j.$$

Problems using both Product rule and SUM RULE

- Many counting problems cannot be solved using just the sum rule or just the product rule.
- Many complicated counting problems can be solved using both of these rules in combination.

Example 13

In a version of the computer language BASIC, the name of a variable is a string of one or two alphanumeric characters, where uppercase and lowercase letters are not distinguished. (An *alphanumeric* character is either one of the 26 English letters or one of the 10 digits.) Moreover, a variable name must begin with a letter and must be different from the five strings of two characters that are reserved for programming use. How many different variable names are there in this version of BASIC?