20CYS111 Digital Signal Processing

Time Domain Representation of LTI Systems: Impulse Response and Convolution

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Linear and Time-Invariant (LTI) Systems

We classified systems in accordance with a number of characteristic properties or categories, namely, linearity, causality, stability, and time-invariance, etc.

*In the remainder of this course, we will be mostly concerned with an important class of systems, namely, the **Linear and Time-Invariant (LTI)** systems.*

LTI systems are mathematically easy to characterize and analyze, and consequently, easy to design.

Highly useful signal processing algorithms have been developed for this class of systems over the last several decades.

Linear and Time-Invariant (LTI) Systems

In this module, we are interested in characterizing the input-output relationship of LTI systems in ***time domain****, i.e., when both the input and output signals are represented as functions of time.*

We shall see that:

- *An LTI system is characterized in the time domain simply by its response to an unit impulse function, called its* ***impulse response****.*
- *The output of an LTI system is given by the* ***convolution*** *of the input and the impulse response of the LTI system.*

LTI Systems in Discrete Time

Consequence of Linearity

Consider an arbitrary input signal, $x[n]$, applied to an LTI system $\mathcal{H}$.

Suppose the input signal can be decomposed as a ***weighted sum of elementary signals****, $x_k[n]$, $k = 1, 2, \ldots$, as*

$$\boxed{x[n] = \sum_k a_k x_k[n]},$$

where a_k, $k = 1, 2, \ldots$, are the ***weighting coefficients****.*

Consequence of Linearity

Let the response of the LTI system to the elementary input $x_k[n]$ *be* $y_k[n]$*, i.e.,*

$$\boxed{\mathcal{H}\{x_k[n]\} = y_k[n]}.$$

Then, the output of the LTI system to the input $x[n]$ *is given by*

$$\boxed{\begin{aligned} y[n] = \mathcal{H}\{x[n]\} &= \mathcal{H}\left\{\sum_k a_k x_k[n]\right\} \\ &= \sum_k a_k \mathcal{H}\{x_k[n]\} = \sum_k a_k y_k[n]. \end{aligned}}$$

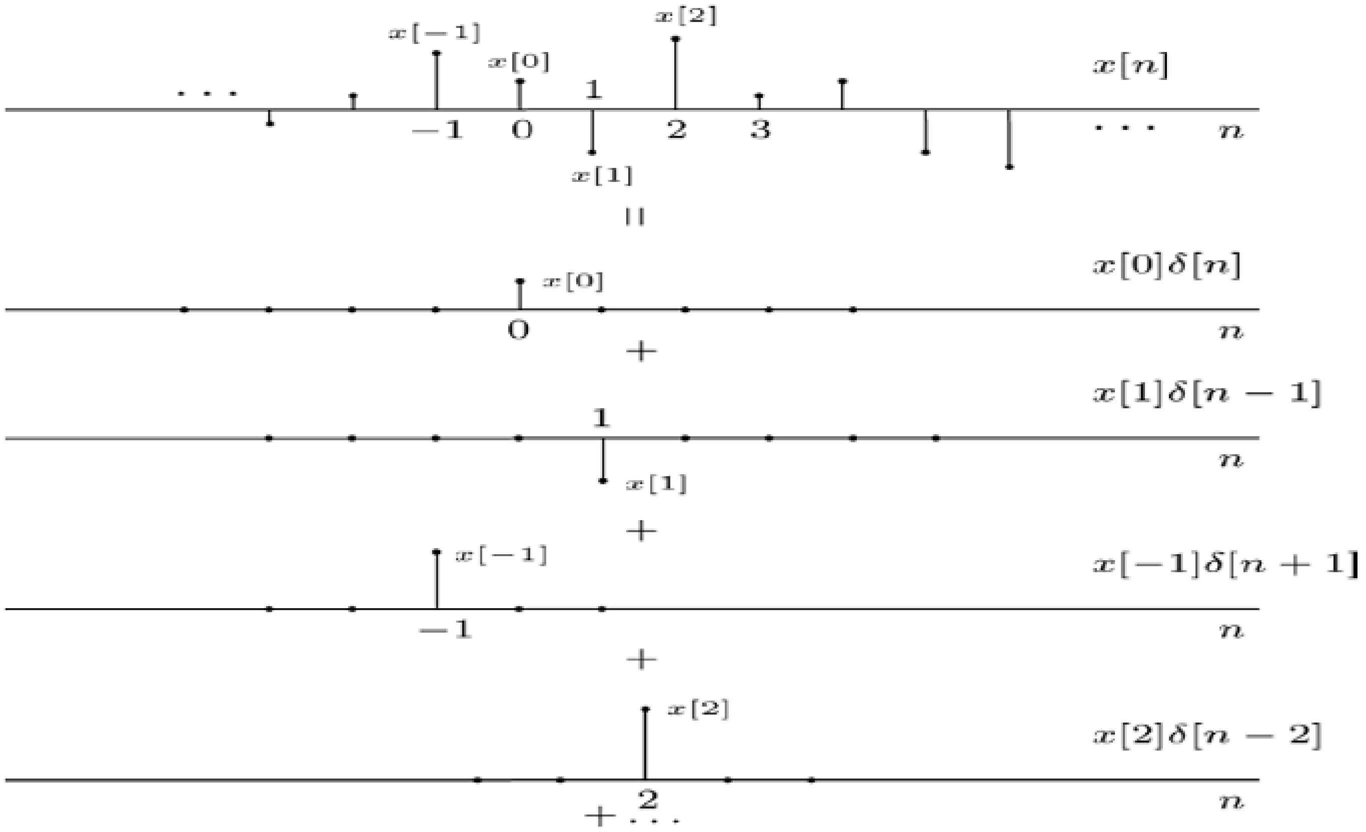
Choice of Elementary Signals

The choice of elementary signals is dependent on the class of input signals that we wish to consider.

- ***Unit impulses as elementary signals:*** *are mathematically convenient and completely general (i.e., they apply to any input signal).*
- ***Complex exponentials as elementary signals:*** *are more convenient mathematically if the input signal is periodic.*

Example: *Consider a finite-duration input sequence, given by* $x[n] = \{2, \underset{\uparrow}{4}, 0, 3\}$*, where the location of the time origin has been indicated with an upward arrow. Decompose* $x[n]$ *as a weighted sum of impulses.*

Decomposition of Input into Unit Impulses



Decomposition of Input into Unit Impulses

For any discrete-time signal $x[n]$, we have

$$\boxed{x[n] = \sum_{k=-\infty}^{\infty} x[k]\delta[n-k]},$$

which can be verified by substituting $n = 0, \pm 1, \pm 2, \ldots$.

Here,

- *The signals, $\delta[n-k], k = 0, \pm 1, \pm 2, \ldots$, can be viewed as the elementary signals.*
- *The values, $x[k], k = 0, \pm 1, \pm 2, \ldots$, can be viewed as the weighting coefficients.*

Impulse Response of Discrete-Time LTI Systems

The output of a discete-time LTI system $\mathcal{H}$, when a discrete-time unit impulse $\delta[n]$ is applied as input, is called its ***impulse response*** *$h[n]$, i.e.,*

- *The impulse response $h[n]$ of a discete-time system $\mathcal{H}$ is defined by*

$$\boxed{\mathcal{H}\{\delta[n]\} = h[n]}.$$

Since the discrete-time LTI system $\mathcal{H}$ is ***time-invariant****, the output of the system, when a time-shifted unit impulse $\delta[n-k]$ is applied as input, is given by*

$$\boxed{\mathcal{H}\{\delta[n-k]\} = h[n-k]}.$$

Output = Convolution of the Input and Impulse Response

The output $y[n]$ of the discrete-time LTI system $\mathcal{H}$ for any arbitrary discrete-time input signal $x[n]$ is given by

$$y[n] = \mathcal{H}\{x[n]\} = \mathcal{H}\left\{\sum_{k=-\infty}^{\infty} x[k]\delta[n-k]\right\}$$

$$= \sum_{k=-\infty}^{\infty} x[k]\mathcal{H}\{\delta[n-k]\} = \sum_{k=-\infty}^{\infty} x[k]h[n-k].$$

$$\boxed{\Rightarrow y[n] = \sum_{k=-\infty}^{\infty} x[k]h[n-k] = x[n] * h[n]},$$

where '$*$' denotes the ***convolution*** *operation.*

Finding the Impulse Response of an LTI System

Example: *Let the input-output relation of a discrete-time LTI system be given by*

$$y[n] = \alpha_0 x[n] + \alpha_1 x[n-1] + \alpha_2 x[n-2] + \alpha_3 x[n-3].$$

To obtain the impulse response $h[n]$ *of the system, set* $x[n] = \delta[n]$*. Then, the impulse response of the system is*

$$h[n] = \alpha_0 \delta[n] + \alpha_1 \delta[n-1] + \alpha_2 \delta[n-2] + \alpha_3 \delta[n-3].$$

Hence, the impulse response of the system is a sequence of length four, given by

$$\{\underset{\uparrow}{\alpha_0}, \alpha_1, \alpha_2, \alpha_3\},$$

where the location of the time origin has been indicated with an upward arrow.

Algorithm to Compute Convolution Sum

The output at a particular time $n = n_0$ is

$$y[n_0] = \sum_{k=-\infty}^{\infty} x[k]h[n_0 - k].$$

- *Since the index in the summation is* k, *both the input signal* $x[k]$ *and the impulse response* $h[n_0 - k]$ *are functions of* k.

Hence, follow the procedure outlined below:

- *Obtain the sequence* $h[n_0 - k]$ *from* $h[k]$ *by*
    - ***Folding*** *or* ***reflecting*** $h[k]$ *about* $k = 0$ *(the time origin), which results in the sequence* $h[-k]$
    - ***Shifting*** *the sequence* $h[-k]$ *to the* ***right*** *by* n_0 *to yield* $h[n_0 - k]$.
- ***Multiply*** *the sequences* $x[k]$ *and* $h[n_0 - k]$ *to form a product sequence* $w_{n_0}[k] = x[k]h[n_0 - k]$.
- ***Sum*** *the product sequence for all* k *to get the output* $y[n_0]$ *at time* $n = n_0$.

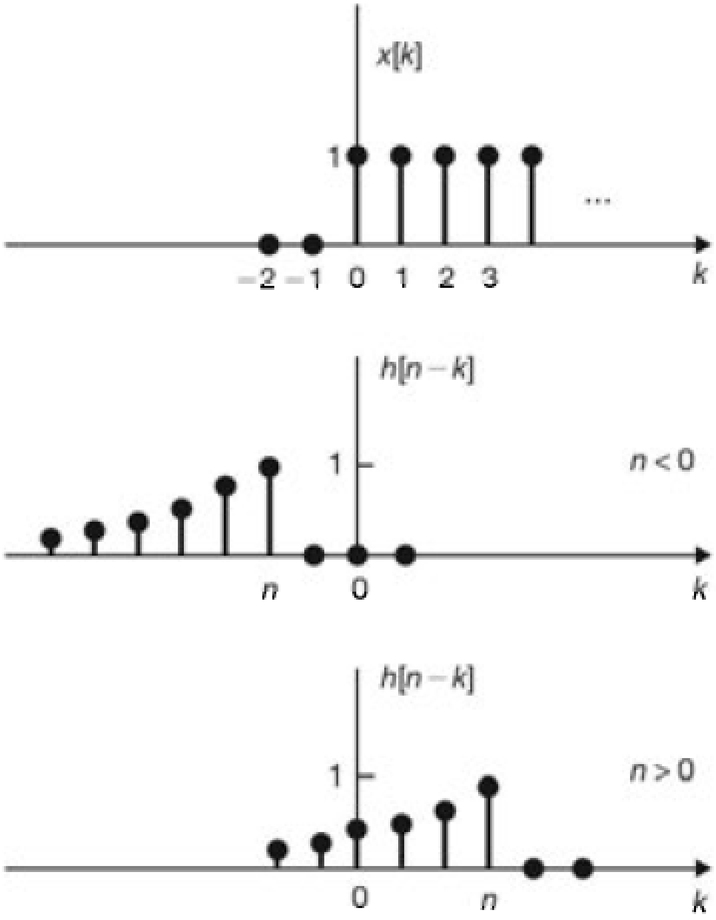
Example to Compute Convolution Sum

Problem: **Let the impulse response of a discrete-time LTI system be $h[n] = \alpha^n u[n]$, where $0 < \alpha < 1$. Find the output $y[n]$ of the system in response to an input $x[n] = u[n]$.**

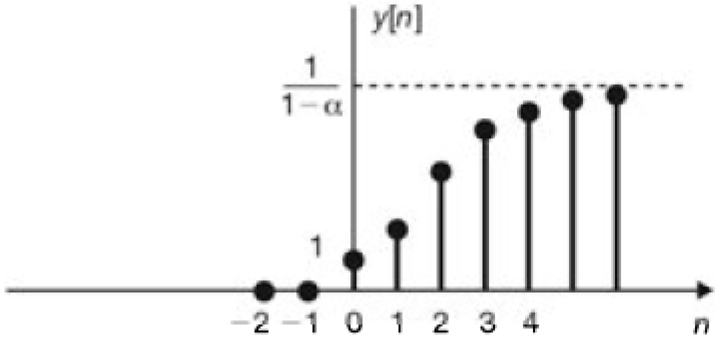
Solution: We have

$$y[n] = x[n] * h[n] = \sum_{k=-\infty}^{\infty} x[k]h[n-k] = \sum_{k=-\infty}^{\infty} u[k]\alpha^{n-k}u[n-k]$$

$$= \begin{cases} \sum_{k=0}^{n} \alpha^{n-k} = \sum_{k=0}^{n} \alpha^{k} = \dfrac{1-\alpha^{n+1}}{1-\alpha} & \text{for } n \geq 0 \\ 0 & \text{for } n < 0 \end{cases}$$



(a)



(b)

LTI Systems in Continuous Time

Decomposition of Input into Unit Impulses

In the continuous time, any arbitrary input signal $x(t)$ can be re-written as

$$\boxed{x(t) = \int_{-\infty}^{\infty} x(t')\delta(t - t')dt'},$$

*which is the **convolution property** satisfied by the delta function.*

- ***Remarks:** We can interpret this property just as in the discrete-time case by applying the following approximation:*

$$x(t) = \int_{-\infty}^{\infty} x(t')\delta(t - t')dt' \approx \sum_{k=-\infty}^{\infty} x(k\Delta t)\underbrace{\delta_{\Delta t}(t - k\Delta t)\Delta t}_{\text{unit area}}$$

$$\Rightarrow x(t_0) \approx x(k_0\Delta t), \text{ where } k_0\Delta t \leq t < (k_0 + 1)\Delta t$$

Consequences of Linearity and Time-Invariance

Consider a continuous-time LTI system $\mathcal{H}$ *to which we apply the input* $x(t) = \int_{-\infty}^{\infty} x(t')\delta(t - t')dt'$.

By linearity, we have

$$y(t) = \mathcal{H}\{x(t)\} = \mathcal{H}\left\{\int_{-\infty}^{\infty} x(t')\delta(t - t')dt'\right\} = \int_{-\infty}^{\infty} x(t')\mathcal{H}\left\{\delta(t - t')\right\} dt'$$

By time-invariance, we have

$$\mathcal{H}\{\delta(t)\} = h(t) \Rightarrow \mathcal{H}\{\delta(t - t')\} = h(t - t')$$

Output = Convolution of the Input and Impulse Response

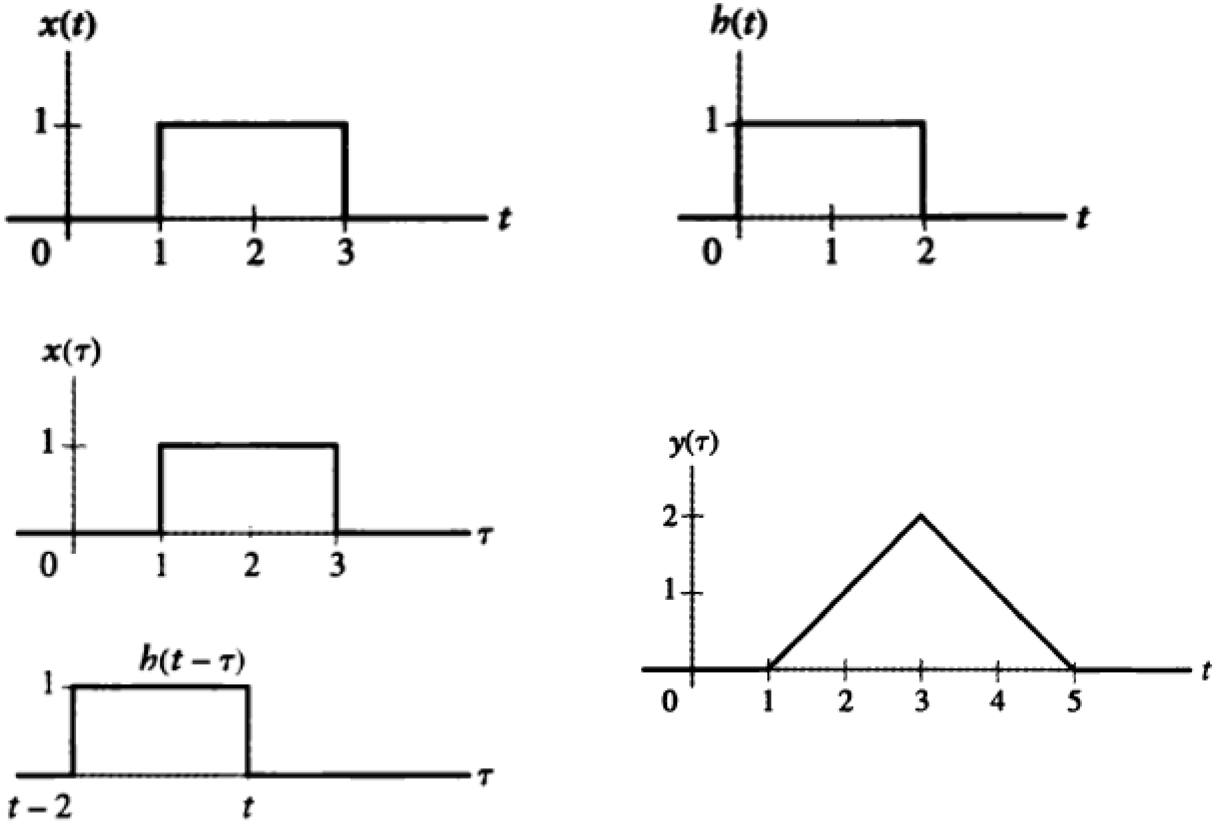
In summary, for the LTI system, we have

$$\boxed{y(t) = \int_{-\infty}^{\infty} x(t')h(t-t')dt' = x(t) * h(t)},$$

*where '$*$' denotes **convolution**.*

- *The convolution operation is a summation in the discrete-time, whereas it is an integration in the continuous time.*
- *Similar to the discrete-time case, the convolution in the continuous-time case can be obtained by performing the **folding** (or **reflecting**), **shifting**, **multiplying** and **integrating** operations, in that order.*

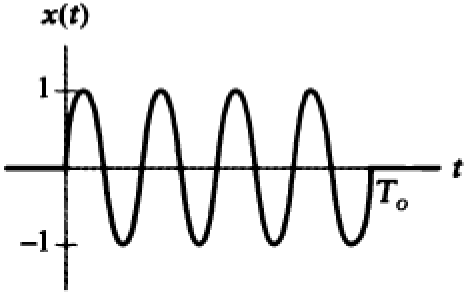
Example to Compute Convolution Integral

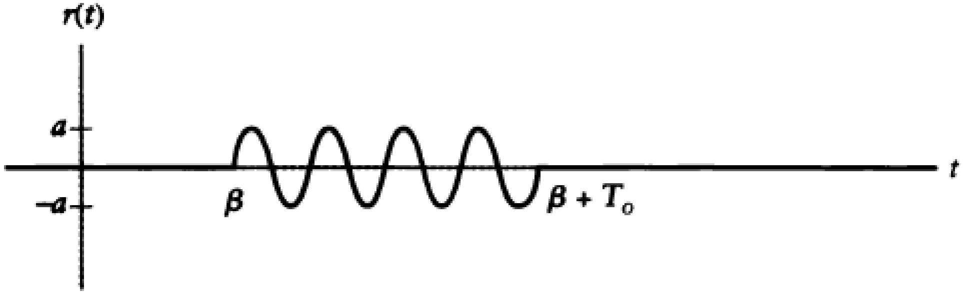


Example to Compute Convolution Integral

Radar Range Measurement: *Let $h(t) = a\delta(t - \beta)$. Find the output $r(t)$ if the input is given by*

$$x(t) = \begin{cases} \sin(\omega_c t) & 0 \leq t \leq T_0 \\ 0 & \text{otherwise} \end{cases}.$$





Example to Compute Convolution Integral

Solution: *We have* $h(t') = a\delta(t'.$
$- \beta)$

$$\Rightarrow h(-t') = a\delta(-t' - \beta) = a\delta(t' + \beta).$$

$$\Rightarrow h(t - t') = h(-(t' - t)) = a\delta((t' - t) + \beta) = a\delta(t' - (t - \beta))$$

$$\Rightarrow r(t) = x(t) * h(t) = \int_{-\infty}^{\infty} x(t')h(t - t')dt'$$

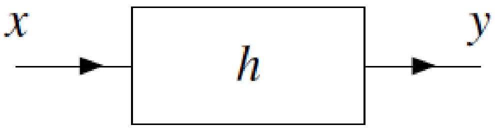
$$= \int_{-\infty}^{\infty} x(t')a\delta(t' - (t - \beta))dt' = ax(t - \beta)$$

$$= \begin{cases} a\sin(\omega_c(t - \beta)) & 0 \leq t \leq T_0 \\ 0 & \text{otherwise} \end{cases}.$$

Interconnection of LTI Systems

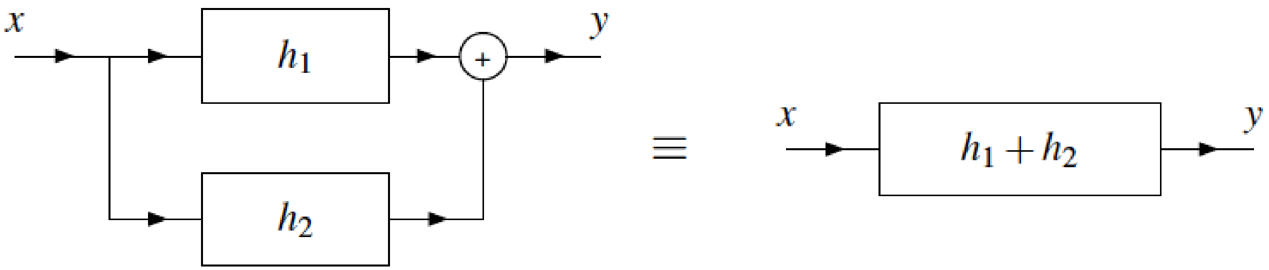
Block Representation

- Often, it is convenient to represent a (CT) LTI system in block diagram form.
- Since such systems are completely characterized by their impulse response, we often label a system with its impulse response.
- That is, we represent a system with input x, output y, and impulse response h, as shown below.



Parallel Connection of LTI Systems

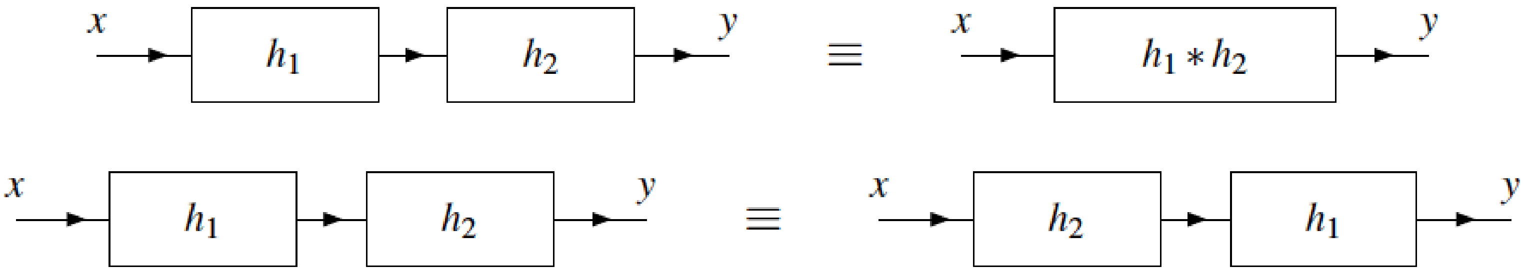
The ***parallel*** interconnection of the LTI systems with impulse responses h_1 and h_2 is a LTI system with the impulse response $h = h_1 + h_2$. That is, we have the equivalence shown below.



Distributive Property: $\boxed{x(t) * h_1(t) + x(t) * h_2(t) = x(t) * \{h_1(t) + h_2(t)\}}$.

Series Connection of LTI Systems

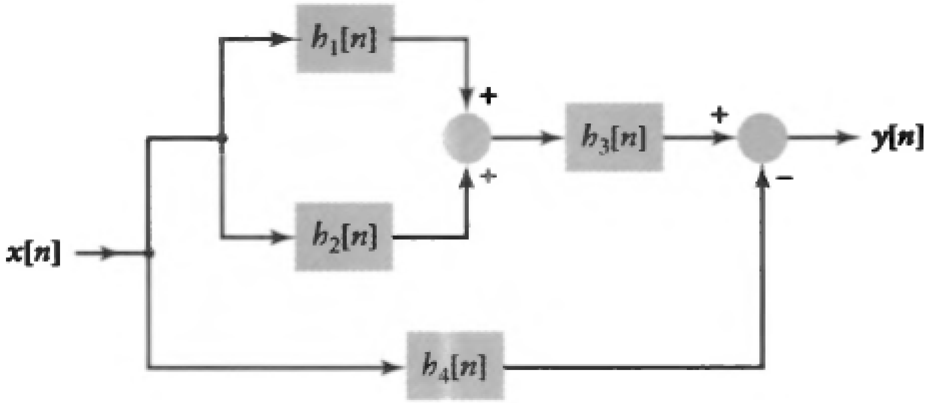
The *series* interconnection of the LTI systems with impulse responses h_1 and h_2 is the LTI system with impulse response $h = h_1 * h_2$. That is, we have the equivalences shown below.



Associative Property: $\boxed{\{x(t) * h_1(t)\} * h_2(t) = x(t) * \{h_1(t) * h_2(t)\}}$.

Commutative Property: $\boxed{h_1(t) * h_2(t) = h_2(t) * h_1(t)}$.

Example to Find Equivalent System



Properties of LTI Systems

Memoryless LTI Systems

For a discrete-time LTI system, we have

$$y[n] = \sum_{k=-\infty}^{\infty} x[k]h[n-k] = \sum_{k=-\infty}^{\infty} h[k]x[n-k]$$

$$\Rightarrow y[n_0] = h[0]x[n_0] + h[1]x[n_0-1] + h[2]x[n_0-2] + \dots$$
$$+ h[-1]x[n_0+1] + h[-2]x[n_0+2] + \dots$$

If the system is memoryless, then $y[n_0]$ can depend only on $x[n_0]$ and no other value of $x[n]$ $\Rightarrow \boxed{y[n_0] = h[0]x[n_0]}$.

- *Therefore, for a discrete-time memoryless LTI system, we must have*

$$\Rightarrow \boxed{h[n] = 0, \quad \forall n \neq 0} \Rightarrow \boxed{h[n] = c\delta[n]}.$$

Memoryless LTI Systems

Similarly, for a continuous-time memoryless LTI system, we must have

$$\boxed{h(t) = c\delta(t)}.$$

Remarks: *The memoryless condition places severe restrictions on the form of the impulse response.*

- ***All memoryless LTI systems simply perform scalar multiplication of the input.***

Causal LTI Systems

For a discrete-time LTI system, we have

$$y[n] = \sum_{k=-\infty}^{\infty} x[k]h[n-k] = \sum_{k=-\infty}^{\infty} h[k]x[n-k]$$

$$\Rightarrow\ y[n_0] = h[0]x[n_0] + h[1]x[n_0-1] + h[2]x[n_0-2] + \dots$$
$$+\, h[-1]x[n_0+1] + h[-2]x[n_0+2] + \dots$$

If the system is causal, then $y[n_0]$ can depend only on $x[n]$, $n \leq n_0$.

- *Therefore, for a discrete-time causal LTI system, we must have*

$$\Rightarrow \boxed{h[n] = 0, \quad \forall n < 0}$$

Causal LTI Systems

The output of a discrete-time causal LTI system is given by

$$\Rightarrow \boxed{y[n] = \sum_{k=-\infty}^{n} x[k]h[n-k] = \sum_{k=0}^{\infty} h[k]x[n-k]}.$$

Similarly, for a continuous-time memoryless LTI system, we must have

$$\boxed{h(t) = 0, \quad \forall t < 0}.$$

$$\Rightarrow \boxed{y(t) = \int_{-\infty}^{t} x(t')h(t-t')dt' = \int_{0}^{\infty} h(t')x(t-t')dt'}.$$

Stable LTI Systems

Assume that the input $x[n]$ applied to a discrete-time LTI system is bounded, i.e., suppose

$$|x[n]| \leq M_x < \infty, \qquad \forall n.$$

Let the impulse response of the discrete-time LTI system be $h[n]$.

- *Then, we have*

$$
\begin{aligned}
|y[n]| &= \left| \sum_{k=-\infty}^{\infty} h[k]x[n-k] \right| \\
&\leq \sum_{k=-\infty}^{\infty} |h[k]x[n-k]| = \sum_{k=-\infty}^{\infty} |h[k]||x[n-k]| \\
&\leq M_x \sum_{k=-\infty}^{\infty} |h[k]|.
\end{aligned}
$$

Stable LTI Systems

Therefore, the discrete-time LTI system is BIBO stable, i.e., its output is bounded for bounded input if the impulse response is ***absolutely summable****, that is, if*

$$\boxed{\sum_{k=-\infty}^{\infty} |h[k]| < \infty}.$$

Similarly, the continuous-time LTI system is BIBO stable, i.e., its output is bounded for bounded input if the impulse response is ***absolutely summable****, that is, if*

$$\boxed{\int_{-\infty}^{\infty} |h(t')|dt' < \infty}.$$

Invertible LTI Systems and Deconvolution

An LTI system may or may not have an inverse system, e.g., a ***lowpass filter****.*

An inverse of an LTI system, if it exists, may or may not be an LTI system itself.

Here, we restrict our discussion to inverse systems that are LTI.

- *For physical realizability and practical considerations, one should restrict to inverse systems that are also causal and stable.*

Consider a continuous-time LTI system with impulse response $h(t)$*. Assume that the inverse system exists and is itself an LTI system with impulse response* $h^{inv}(t)$*.*

- *This inverse LTI system may be noncausal or unstable.*

Invertible LTI Systems and Deconvolution

By the definition of the inverse system, we must have

$$\begin{aligned} & \{x(t) * h(t)\} * h^{inv}(t) = x(t) \\ \Rightarrow \quad & x(t) * \{h(t) * h^{inv}(t)\} = x(t) \\ \Rightarrow \quad & \boxed{h(t) * h^{inv}(t) = \delta(t)}. \end{aligned}$$

For a discrete-time LTI system with impulse response $h[n]$, if the inverse exists and is itself an LTI system with impulse response $h^{inv}[n]$, then we must have

$$\boxed{h[n] * h^{inv}[n] = \delta[n]}.$$

Example of Finding Inverse LTI System

Problem: *Consider a multipath communication system given by the input-output relationship in the discrete-time $y[n] = x[n] + ax[n-1]$. Find a causal LTI inverse system. Check if the inverse system is stable.*

Solution: *Setting $x[n] = \delta[n]$, we find the impulse response as*

$$h[n] = \begin{cases} 1 & n = 0 \\ a & n = 1 \\ 0 & \text{otherwise} \end{cases}.$$

Assume that an LTI inverse system exists whose impulse response is $h^{inv}[n]$. Then, we must have

$$h[n] * h^{inv}[n] = \delta[n] \Rightarrow h^{inv}[n] + ah^{inv}[n-1] = \delta[n].$$

Example of Finding Inverse LTI System

For $n = 0$, we have $\delta[n] = \delta[0] = 1$, which implies that

$$h^{inv}[0] + ah^{inv}[-1] = \delta[0] = 1.$$

If the LTI inverse system has to be causal, we must have $h^{inv}[n] = 0, \ \forall n < 0$. This implies that $h^{inv}[-1] = 0$, and hence $h^{inv}[0] = 1$.

For $n \neq 0$, we have $\delta[n] = 0$, which implies that

$$h^{inv}[n] + ah^{inv}[n-1] = 0 \Rightarrow h^{inv}[n] = -ah^{inv}[n-1], \forall n \neq 0.$$

Therefore, we have $h^{inv}[n]$.

$= (-a)^n u[n]$

Example of Finding Inverse LTI System

To check if the inverse system is stable, we need to check if $h^{inv}[n]$ *is absolutely summable.*

Since $h^{inv}[n]$ *is causal, it is stable if*

$$\sum_{k=-\infty}^{\infty} |h^{inv}[k]| = \sum_{k=0}^{\infty} |a|^k < \infty,$$

which is a geometric series that converges if $|a| < 1$.

References:

[1] Simon Haykin and Barry Van Veen, Signals and Systems, Second Edition, John Wiley and Sons, 2003.

[2] Lecture Notes by Michael D. Adams.
https://www.ece.uvic.ca/~frodo/sigsysbook/downloads/lecture_slides_for_signals_and_systems
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[3] Lecture Notes by Richard Baraniuk.
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