Discrete Mathematics

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Solving Linear Recurrence Relations

A wide variety of recurrence relations occur in models.

A *linear homogeneous recurrence relation of degree k with constant coefficients* is a recurrence relation of the form

$$a_n = c_1 a_{n-1} + c_2 a_{n-2} + \cdots + c_k a_{n-k},$$

where $c_1, c_2, \ldots, c_k$ are real numbers, and $c_k \neq 0$.

Examples

$P_n = (1.11) P_{n-1}$

$f_n = f_{n-1} + f_{n-2}$

$a_n = a_{n-5}$

Non-Examples

$a_n = a_{n-1} + a_{n-2}^2$ is not linear.

$2H_{n-1} + 1$ is not homogeneous

$B_n = nB_{n-1}$ does not have constant

The basic approach for solving linear homogeneous recurrence relations

$$a_n = c_1 a_{n-1} + c_2 a_{n-2} + \cdots + c_k a_{n-k}$$

Find the **characteristic equation** of the recurrence relation

$$r^k - c_1 r^{k-1} - c_2 r^{k-2} - \cdots - c_{k-1} r - c_k = 0.$$

- The solutions of this equation are called the **characteristic roots** of the recurrence relation.
- These characteristic roots can be used to give an explicit formula for all the solutions of the recurrence relation.

Theorem 1

Let c_1 and c_2 be real numbers. Suppose that $r^2 - c_1 r - c_2 = 0$ has two distinct roots r_1 and r_2. Then the sequence $\{a_n\}$ is a solution of the recurrence relation $a_n = c_1 a_{n-1} + c_2 a_{n-2}$ if and only if $a_n = \alpha_1 r_1^n + \alpha_2 r_2^n$ for $n = 0, 1, 2, \ldots,$ where α_1 and α_2 are constants.

Theorem 1

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Example 1

What is the solution of the recurrence relation

$$a_n = a_{n-1} + 2a_{n-2}$$

with $a_0 = 2$ and $a_1 = 7$?

$$\boxed{a_n = 3 \cdot 2^n - (-1)^n}$$

The characteristic equation of the recurrence relation is $r^2 - r - 2 = 0$

Its roots are $r = 2$ and $r = -1$.

Solution to the recurrence relation $a_n = \alpha_1 2^n + \alpha_2 (-1)^n$

From initial conditions

$$a_0 = 2 = \alpha_1 + \alpha_2,$$
$$a_1 = 7 = \alpha_1 \cdot 2 + \alpha_2 \cdot (-1)$$

Solving these two equations shows that $\alpha_1 = 3$ and $\alpha_2 = -1$.

Example 2

Find an explicit formula for the Fibonacci numbers.

- Recall that the sequence of Fibonacci numbers satisfies the recurrence relation $f_n = f_{n-1} + f_{n-2}$ initial conditions $f_0 = 0$ and $f_1 = 1$.

- The roots of the characteristic equation $r^2 - r - 1 = 0$ are

$$r_1 = (1 + \sqrt{5})/2 \text{ and } r_2 = (1 - \sqrt{5})/2.$$

from Theorem 1 it follows that the Fibonacci numbers are given by

$$f_n = \alpha_1 \left(\frac{1+\sqrt{5}}{2}\right)^n + \alpha_2 \left(\frac{1-\sqrt{5}}{2}\right)^n,$$

$$f_0 = \alpha_1 + \alpha_2 = 0,$$

$$f_1 = \alpha_1 \left(\frac{1+\sqrt{5}}{2}\right) + \alpha_2 \left(\frac{1-\sqrt{5}}{2}\right) = 1.$$

The solution to these simultaneous equations for α_1 and α_2 is $\alpha_1 = 1/\sqrt{5}, \qquad \alpha_2 = -1/\sqrt{5}.$

Theorem 2

Let c_1 and c_2 be real numbers with $c_2 \neq 0$. Suppose that $r^2 - c_1 r - c_2 = 0$ has only one root r_0. A sequence $\{a_n\}$ is a solution of the recurrence relation $a_n = c_1 a_{n-1} + c_2 a_{n-2}$ if and only if $a_n = \alpha_1 r_0^n + \alpha_2 n r_0^n$, for $n = 0, 1, 2, \ldots$, where α_1 and α_2 are constants.

Example 3

What is the solution of the recurrence relation

$$a_n = 6a_{n-1} - 9a_{n-2}$$

with initial conditions $a_0 = 1$ and $a_1 = 6$?

Solution

The only root of $r^2 - 6r + 9 = 0$ is $r = 3$

The solution to this recurrence relation is

$$a_n = \alpha_1 3^n + \alpha_2 n 3^n$$

Using the initial conditions

$$a_0 = 1 = \alpha_1,$$
$$a_1 = 6 = \alpha_1 \cdot 3 + \alpha_2 \cdot 3.$$

Solving these two equations shows that $\alpha_1 = 1$ and $\alpha_2 = 1$.

Solution to this recurrence relation and the initial conditions is

$$a_n = 3^n + n3^n.$$

THEOREM 3

Let $c_1, c_2, \ldots, c_k$ be real numbers. Suppose that the characteristic equation

$$r^k - c_1 r^{k-1} - \cdots - c_k = 0$$

has k distinct roots $r_1, r_2, \ldots, r_k$. Then a sequence $\{a_n\}$ is a solution of the recurrence relation

$$a_n = c_1 a_{n-1} + c_2 a_{n-2} + \cdots + c_k a_{n-k}$$

if and only if

$$a_n = \alpha_1 r_1^n + \alpha_2 r_2^n + \cdots + \alpha_k r_k^n$$

for $n = 0, 1, 2, \ldots$, where $\alpha_1, \alpha_2, \ldots, \alpha_k$ are constants.

Example 4

Find the solution to the recurrence relation

$$a_n = 6a_{n-1} - 11a_{n-2} + 6a_{n-3}$$

with the initial conditions $a_0 = 2$, $a_1 = 5$, and $a_2 = 15$.

Solution

The characteristic polynomial of this recurrence relation is $\boxed{r^3 - 6r^2 + 11r - 6}$

Characteristic roots are $r = 1$, $r = 2$, and $r = 3$

$$\boxed{a_n = \alpha_1 \cdot 1^n + \alpha_2 \cdot 2^n + \alpha_3 \cdot 3^n}$$

Linear Nonhomogeneous Recurrence Relations with Constant Coefficients

Is there a relatively simple technique for solving a linear, but not homogeneous, recurrence relation with constant coefficients?

$$a_n = 3a_{n-1} + 2n$$

General form of linear nonhomogeneous recurrence relation

$$a_n = c_1 a_{n-1} + c_2 a_{n-2} + \cdots + c_k a_{n-k} + F(n),$$

where $c_1, c_2, \ldots, c_k$ are real numbers and $F(n)$ is a function of n

Associated homogeneous recurrence relation

$$a_n = c_1 a_{n-1} + c_2 a_{n-2} + \cdots + c_k a_{n-k}$$

Linear Nonhomogeneous Recurrence Relations

$$a_n = a_{n-1} + 2^n$$

$$a_n = a_{n-1} + a_{n-2} + n^2 + n + 1$$

$$a_n = 3a_{n-1} + n3^n$$

$$a_n = a_{n-1} + a_{n-2} + a_{n-3} + n!$$

Associated linear homogeneous recurrence relations

$$a_n = a_{n-1}$$

$$a_n = a_{n-1} + a_{n-2}$$

$$a_n = 3a_{n-1}$$

$$a_n = a_{n-1} + a_{n-2} + a_{n-3}$$

The key fact about linear nonhomogeneous recurrence relations with constant coefficients is

Every solution is the sum of a particular solution and a solution of the associated linear homogeneous recurrence relation

Theorem 5

If $\{a_n^{(p)}\}$ is a particular solution of the nonhomogeneous linear recurrence relation with constant coefficients

$$a_n = c_1 a_{n-1} + c_2 a_{n-2} + \cdots + c_k a_{n-k} + F(n),$$

then every solution is of the form $\{a_n^{(p)} + a_n^{(h)}\}$, where $\{a_n^{(h)}\}$ is a solution of the associated homogeneous recurrence relation

$$a_n = c_1 a_{n-1} + c_2 a_{n-2} + \cdots + c_k a_{n-k}.$$

Example

Find all solutions of the recurrence relation $a_n = 3a_{n-1} + 2n$. What is the solution with $a_1 = 3$?

Example **Find all solutions of the recurrence relation $a_n = 3a_{n-1} + 2n$. What is the solution with $a_1 = 3$?**

Solution:

The associated linear homogeneous equation $a_n = 3a_{n-1}$,

Its solutions are $a_n^{(h)} = \alpha 3^n$ where α is a constant.

To find a particular solution.

Since $F(n) = 2n$, which is a polynomial of degree 1, we consider the particular solutions to be $p_n = cn + d$ *(where c and d are constants).*

Substitute p_n in the given non-homogeneous recurrence relation and find the values of c and d.

This shows that $cn + d$ is a solution if and only if $c = -1$ and $d = -3/2$.

Hence $a_n^{(p)} = -n - \frac{3}{2}$ is a particular

All solutions are of the form $a_n = a_n^{(p)} + a_n^{(h)} = -n - \frac{3}{2} + \alpha 3^n$, where α is a constant.

To find the solution with $a_1 = 3$, put $n = 1$

Example 2

Find all solutions of the recurrence relation $a_n = 5a_{n-1} - 6a_{n-2} + 7^n$

Solution

Associated homogeneous recurrence relation $a_n = 5a_{n-1} - 6a_{n-2}$

The solution is $a_n^{(h)} = \alpha_1.3^n + \alpha_2.2^n$, where α_1 and α_2 are constants

Since $F(n) = 7^n$,
the particular solution of non-homogeneous relation is $a_n^{(p)} = C.7^n$ (C is constant).

Substituting the terms of this sequence into the recurrence relation

$$C.7^n = 5C.7^{n-1} - 6C.7^{n-2} + 7^n$$

Simplify to $49C = 35C - 6C + 49$
Implies $C = 49/20$

- Hence all solutions are of the form

$$a_n = \alpha_1 \cdot 3^n + \alpha_2 \cdot 2^n + (49/20)7^n.$$

Theorem 6

Suppose that $\{a_n\}$ satisfies the linear nonhomogeneous recurrence relation

$$a_n = c_1 a_{n-1} + c_2 a_{n-2} + \cdots + c_k a_{n-k} + F(n),$$

where $c_1, c_2, \ldots, c_k$ are real numbers, and

$$F(n) = (b_t n^t + b_{t-1} n^{t-1} + \cdots + b_1 n + b_0) s^n,$$

where $b_0, b_1, \ldots, b_t$ and s are real numbers. When s is not a root of the characteristic equation of the associated linear homogeneous recurrence relation, there is a particular solution of the form

$$(p_t n^t + p_{t-1} n^{t-1} + \cdots + p_1 n + p_0) s^n.$$

When s is a root of this characteristic equation and its multiplicity is m, there is a particular solution of the form

$$n^m (p_t n^t + p_{t-1} n^{t-1} + \cdots + p_1 n + p_0) s^n.$$

Example

What form does a particular solution of the linear nonhomogeneous recurrence relation $a_n = 6a_{n-1} - 9a_{n-2} + F(n)$ have when $F(n) = 3^n$, $F(n) = n3^n$, $F(n) = n^2 2^n$, and $F(n) = (n^2 + 1)3^n$?

Solution: The associated linear homogeneous recurrence relation is $a_n = 6a_{n-1} - 9a_{n-2}$. Its characteristic equation, $r^2 - 6r + 9 = (r - 3)^2 = 0$, has a single root, 3, of multiplicity two. To apply Theorem 6, with $F(n)$ of the form $P(n)s^n$, where $P(n)$ is a polynomial and s is a constant, we need to ask whether s is a root of this characteristic equation.

Because $s = 3$ is a root with multiplicity $m = 2$ but $s = 2$ is not a root, Theorem 6 tells us that a particular solution has the form $p_0 n^2 3^n$ if $F(n) = 3^n$, the form $n^2(p_1 n + p_0)3^n$ if $F(n) = n3^n$, the form $(p_2 n^2 + p_1 n + p_0)2^n$ if $F(n) = n^2 2^n$, and the form $n^2(p_2 n^2 + p_1 n + p_0)3^n$ if $F(n) = (n^2 + 1)3^n$. ◀