Discrete Mathematics

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Relations and Their Properties

Let A and B be sets. A *binary relation from A to B* is a subset of $A \times B$

A binary relation from *A* to *B* is a set *R* of ordered pairs

***a R b* to denote that $(a, b) \in R$**

a* is said to be related to *b* by *R

$a \not R\, b$ to denote that $(a, b) \notin R$

Relations and Their Properties

EXAMPLE 1

Let A be the set of students in your school, and let B be the set of courses. Let R be the relation that consists of those pairs (a, b), where a is a student enrolled in course b.

For instance, if **Shree** and **Mano** are enrolled in CS518, the pairs (**Shree**, CS518) and (**Mano**, CS518) belong to R.

If **Shree** is also enrolled in CS510, then the pair (**Shree**, CS510) is also in R. **However, if Mano is not enrolled in CS510, then the pair (Mano, CS510) is not in *R*.**

EXAMPLE

Let *A* be the set of cities in the U.S.A., and let *B* be the set of the 50 states in the U.S.A. Define the relation *R* by specifying that *(a, b)* belongs to *R* if a city with name *a* is in the state *b*.

EXAMPLE

Let A = {0, 1, 2} and B = {a, b}.
Then {(0, a), (0, b), (1, a), (2, b)} is a relation from A to B.

Relations can be represented graphically and by tables (see next slide)

Let A = {0, 1, 2} and B = {a, b}. Then {(0, a), (0, b), (1, a), (2, b)} is a relation from A to B.

R	a	b
0	×	×
1	×	
2		×

Functions as Relations

1. A function ***f*** from a set A to a set B assigns exactly one element of B to each element of A.
2. The graph of f is the set of ordered pairs (a, b) such that $b = f(a)$.

(Graph of the function *f* from A to B is the set of ordered pairs (a, *f*(a)) for a ∈ A)

- The graph of f is a subset of $A \times B$, it is a relation from A to B.
- **The graph of a function has the property that every element of A is the first element of exactly one ordered pair of the graph.**

Relations on a Set

Relations from a set A to itself are of special interest

- A relation on a set A is a relation from A to A.
- That is a relation on a set A is a subset of A × A.

Example1

Let A be the set $\{1, 2, 3, 4\}$. Which ordered pairs are in the relation $R = \{(a, b) \mid a \text{ divides } b\}$?

Solution: Because (a, b) is in R if and only if a and b are positive integers not exceeding 4 such that a divides b, we see that

R = {(1, 1), (1, 2), (1, 3), (1, 4), (2, 2), (2, 4), (3, 3), (4, 4)}.

The pairs in this relation are displayed both graphically and in tabular

R = {(1, 1), (1, 2), (1, 3), (1, 4), (2, 2), (2, 4), (3, 3), (4, 4)}.

The pairs in this relation are displayed both graphically and in tabular

R	1	2	3	4
1	×	×	×	×
2		×		×
3			×	
4				×

Example2

Consider these relations on the set of integers

$$R_1 = \{(a, b) \mid a \leq b\},$$
$$R_2 = \{(a, b) \mid a > b\},$$
$$R_3 = \{(a, b) \mid a = b \text{ or } a = -b\},$$
$$R_4 = \{(a, b) \mid a = b\},$$
$$R_5 = \{(a, b) \mid a = b + 1\},$$
$$R_6 = \{(a, b) \mid a + b \leq 3\}.$$

Which of these relations contain each of the pairs (1, 1), (1, 2), (2, 1), (1,–1), and (2, 2)?

Solution: The pair (1, 1) is in R_1, R_3, R_4, and R_6; (1, 2) is in R_1 and R_6; (2, 1) is in R_2, R_5, and R_6; (1, −1) is in R_2, R_3, and R_6; and finally, (2, 2) is in R_1, R_3, and R_4.

Example problem

List the ordered pairs in the relation R from $A = \{0, 1, 2, 3, 4\}$ to $B = \{0, 1, 2, 3\}$, where $(a, b) \in R$ if and only if

a) $a = b$.

b) $a + b = 4$.

c) $a > b$.

d) $a \mid b$.

e) $\gcd(a, b) = 1$.

f) $\operatorname{lcm}(a, b) = 2$.

Example problem

2. a) List all the ordered pairs in the relation $R = \{(a, b) \mid a \text{ divides } b\}$ on the set $\{1, 2, 3, 4, 5, 6\}$.

b) Display this relation graphically, as was done in Example 4.

c) Display this relation in tabular form, as was done in Example 4.

How many relations are there on a set with *n* elements?

1. A relation on a set A is a subset of A × A
2. The set A × A has n^2 elements when A has *n* elements
3. A set with *m* elements has 2^m subsets
4. There are 2^{n^2} subsets of A × A
5. For example, there are 512 relations on the set {a, b, c}

Properties of Relations

Reflexive

let R be the relation on the set of all people consisting of pairs (x, y) where x and y have the same mother and the same father.

Then x R x for every person x.

A relation R on a set A is called reflexive if

$(a, a) \in R$ for every element $a \in A$.

OR A relation R is antisymmetric if either $(x,y) \notin R$ or $(y,x) \notin R$ whenever $x \neq y$.

Concept of a reflexive relation

Consider the following relations on {1, 2, 3, 4}:

$$R_1 = \{(1,1),(1,2),(2,1),(2,2),(3,4),(4,1),(4,4)\},$$
$$R_2 = \{(1,1),(1,2),(2,1)\},$$
$$R_3 = \{(1,1),(1,2),(1,4),(2,1),(2,2),(3,3),(4,1),(4,4)\},$$
$$R_4 = \{(2,1),(3,1),(3,2),(4,1),(4,2),(4,3)\},$$
$$R_5 = \{(1,1),(1,2),(1,3),(1,4),(2,2),(2,3),(2,4),(3,3),(3,4),(4,4)\},$$
$$R_6 = \{(3,4)\}.$$

Which of these relations are reflexive?

Is the “divides” relation on the set of positive integers reflexive?

Is the “divides” relation on the set of <u>all integers</u> reflexive?

Symmetric Relation

A relation R on a set A is called *symmetric* if $(b, a) \in R$ whenever $(a, b) \in R$, for all $a, b \in A$

The relation consisting of pairs (x, y), where x and y are students at your school with at least one common course has the property of symmetry

A relation R on a set A such that for all $a, b \in A$, if $(a, b) \in R$ and $(b, a) \in R$, then $a = b$ is called *antisymmetric*.

Which of the relations from this example are symmetric and which are antisymmetric?

Consider the following relations on {1, 2, 3, 4}:

$R_1 = \{(1, 1), (1, 2), (2, 1), (2, 2), (3, 4), (4, 1), (4, 4)\},$

$R_2 = \{(1, 1), (1, 2), (2, 1)\},$

$R_3 = \{(1, 1), (1, 2), (1, 4), (2, 1), (2, 2), (3, 3), (4, 1), (4, 4)\},$

$R_4 = \{(2, 1), (3, 1), (3, 2), (4, 1), (4, 2), (4, 3)\},$

$R_5 = \{(1, 1), (1, 2), (1, 3), (1, 4), (2, 2), (2, 3), (2, 4), (3, 3), (3, 4), (4, 4)\},$

$R_6 = \{(3, 4)\}.$

Which of the relations from this example are symmetric and which are antisymmetric?

$$R_1 = \{(a, b) \mid a \leq b\},$$
$$R_2 = \{(a, b) \mid a > b\},$$
$$R_3 = \{(a, b) \mid a = b \text{ or } a = -b\},$$
$$R_4 = \{(a, b) \mid a = b\},$$
$$R_5 = \{(a, b) \mid a = b + 1\},$$
$$R_6 = \{(a, b) \mid a + b \leq 3\}.$$

Consider these relations on the set of integers:

Is the “divides” relation on the set of positive integers symmetric? Is it antisymmetric?

Transitive Relation

Let R be the relation consisting of all pairs (x, y) of students at your school, where x has taken more credits than y. Suppose that x is related to y and y is related to z. This means that x has taken more credits than y and y has taken more credits than z.

A relation R on a set A is called *transitive* if whenever $(a, b) \in R$ and $(b, c) \in R$, then $(a, c) \in R$, for all $a, b, c \in A$.

Which of the relations in this example are <u>transitive</u>?

Consider the following relations on {1, 2, 3, 4}:

$R_1 = \{(1, 1), (1, 2), (2, 1), (2, 2), (3, 4), (4, 1), (4, 4)\},$

$R_2 = \{(1, 1), (1, 2), (2, 1)\},$

$R_3 = \{(1, 1), (1, 2), (1, 4), (2, 1), (2, 2), (3, 3), (4, 1), (4, 4)\},$

$R_4 = \{(2, 1), (3, 1), (3, 2), (4, 1), (4, 2), (4, 3)\},$

$R_5 = \{(1, 1), (1, 2), (1, 3), (1, 4), (2, 2), (2, 3), (2, 4), (3, 3), (3, 4), (4, 4)\},$

$R_6 = \{(3, 4)\}.$

Which of the relations in this example are <u>transitive</u>?

Consider the following relations on {1, 2, 3, 4}:

$$R_1 = \{(a, b) \mid a \leq b\},$$
$$R_2 = \{(a, b) \mid a > b\},$$
$$R_3 = \{(a, b) \mid a = b \text{ or } a = -b\},$$
$$R_4 = \{(a, b) \mid a = b\},$$
$$R_5 = \{(a, b) \mid a = b + 1\},$$
$$R_6 = \{(a, b) \mid a + b \leq 3\}.$$

Consider these relations on the set of integers:

Is the “divides” relation on the set of positive integers transitive?

How many reflexive relations are there on a set with n elements?

- ✓A relation R on a set A is a subset of $A \times A$.
- ✓Consequently, a relation is determined by specifying whether each of the n^2 ordered pairs in $A \times A$ is in R.
- ✓However, if R is reflexive, each of the n ordered pairs (a, a) for $a \in A$ must be in R.
- ✓Each of the other $n(n-1)$ ordered pairs of the form (a, b), where a not equal to b, may or may not be in R.

How many symmetric relations are there on a set with *n* elements?

- ✓ **A relation R on a set A is a subset of A × A.**
- ✓ **Consequently, a relation is determined by specifying whether each of the n^2 ordered pairs in A × A is in R.**

$$2^{\frac{n(n-1)}{2}} \text{ ordered pairs exit with aRb } a \neq b$$

- ✓ **We have 2^n pairs such that aRa**

$$\text{The no.of symmetric relations: } 2^n.2^{\frac{n(n-1)}{2}} = 2^{\frac{n(n+1)}{2}}$$

Combining Relations

Relations from A to B are subsets of A × B, two relations from A to B can be combined in any way two sets can be combined.

Let A = {1, 2, 3} and B = {1, 2, 3, 4}. The relations

R1 = {(1, 1), (2, 2), (3, 3)} and R2 = {(1, 1), (1, 2), (1, 3), (1, 4)} can be combined to obtain

$$R_1 \cup R_2 = \{(1, 1), (1, 2), (1, 3), (1, 4), (2, 2), (3, 3)\}.$$

$$R_1 \cap R_2 = \{(1, 1)\},$$

$$R_1 - R_2 = \{(2, 2), (3, 3)\},$$

$$R_2 - R_1 = \{(1, 2), (1, 3), (1, 4)\}.$$

More examples

1. **A relation R is defined on the set of all real numbers N by 'a R b' if and only if $|a-b| \leq b$, for $a, b \in N$. Show that the R is not reflexive relation.**
2. **Consider the set A in which a relation R is defined by 'x R y if and only if $x + 3y$ is divisible by 4, for $x, y \in A$. Show that R is a reflexive relation on set A.**
3. **A relation R on the set A by "x R y if $x - y$ is divisible by 5" for $x, y \in A$. Check if R is a reflexive relation on set A.**
4. **Which of these are antisymmetric?**

   (i) R = {(1,1),(1,2),(2,1),(2,2),(3,4),(4,1),(4,4)}

   (ii) R = {(1,1),(1,3),(3,1)}

   (iii) R = {(1,1),(1,2),(1,4),(2,1),(2,2),(3,3),(4,1),(4,4)}

More examples

If A = {1,2,3,4} and R is the relation on set A, then find the antisymmetric relation on set A.

The antisymmetric relation on set A = {1,2,3,4} is R = {(1,1), (2,2),(3,3),(4,4)}

Representing Relations

There are many ways to represent a relation between finite sets.

- ✓ One method uses zero–one matrices
- ✓ The other method uses pictorial representations called directed graphs

Representing Relations Using Matrices

Suppose that R is a relation from $A = \{a_1, a_2, \ldots, a_m\}$ to $B = \{b_1, b_2, \ldots, b_n\}$.

The relation R can be represented by the matrix $MR = [m_{ij}]$

$$m_{ij} = \begin{cases} 1 \text{ if } (a_i, b_j) \in R, \\ 0 \text{ if } (a_i, b_j) \notin R. \end{cases}$$

Representing Relations Using Matrices

Suppose that A = {1, 2, 3} and B = {1, 2}. Let R be the relation from A to B containing (a, b) if a ∈ A, b ∈ B, and a > b. What is the matrix representing R

$$m_{ij} = \begin{cases} 1 \text{ if } (a_i, b_j) \in R, \\ 0 \text{ if } (a_i, b_j) \notin R. \end{cases}$$

R = {(2, 1), (3, 1), (3, 2)},

$$\mathbf{M}_R = \begin{bmatrix} 0 & 0 \\ 1 & 0 \\ 1 & 1 \end{bmatrix}$$

Example: Let $A = \{a_1, a_2, a_3\}$ and $B = \{b_1, b_2, b_3, b_4, b_5\}$. Which ordered pairs are in the relation R represented by the matrix.

$$\mathbf{M}_R = \begin{bmatrix} 0 & 1 & 0 & 0 & 0 \\ 1 & 0 & 1 & 1 & 0 \\ 1 & 0 & 1 & 0 & 1 \end{bmatrix} ?$$

Reflexive Relation

R is reflexive if and only if $(a_i, a_i) \in R$ for $i = 1, 2, \ldots, n$.
Hence, R is reflexive if and only if $m_{ii} = 1$, for $i = 1, 2, \ldots, n$.

$$\begin{bmatrix} 1 & & & & & & \\ & 1 & & & & & \\ & & 1 & & & & \\ & & & \cdot & & & \\ & & & & \cdot & & \\ & & & & & \cdot & \\ & & & & & & 1 \\ & & & & & & & 1 \end{bmatrix}$$

Symmetric Relation

The relation R on the set $A = \{a_1, a_2, \ldots, a_n\}$ is symmetric if and only if $(a_j, a_i) \in R$ whenever $(a_i, a_j) \in R$

In terms of the entries of matrix M_R, R is symmetric if and only if $m_{ji} = 1$ whenever $m_{ij} = 1$.

R is symmetric if and only if $M_R = (M_R)^t$

Antisymmetric

The matrix of an antisymmetric relation has the property that if $m_{ij} = 1$ with $i \neq j$, then $m_{ji} = 0$. Or, in other words, either $m_{ij} = 0$ or $m_{ji} = 0$ when $i \neq j$.

(a) Symmetric (b) Antisymmetric

Example

Suppose that the relation R on a set is represented by the matrix

$$\mathbf{M}_R = \begin{bmatrix} 1 & 1 & 0 \\ 1 & 1 & 1 \\ 0 & 1 & 1 \end{bmatrix}.$$

Is R reflexive, symmetric, and/or antisymmetric?

The **Boolean operations** **join** and **meet** can be used to find the matrices representing the union and the intersection of two relations.

Suppose that R1 and R2 are relations on a set A represented by the matrices M_{R1} and M_{R2}, respectively.

$$\mathbf{M}_{R_1 \cup R_2} = \mathbf{M}_{R_1} \vee \mathbf{M}_{R_2} \qquad \text{and} \qquad \mathbf{M}_{R_1 \cap R_2} = \mathbf{M}_{R_1} \wedge \mathbf{M}_{R_2}.$$

$$\mathbf{M}_{R_1 \cup R_2} = \mathbf{M}_{R_1} \vee \mathbf{M}_{R_2} \qquad \text{and} \qquad \mathbf{M}_{R_1 \cap R_2} = \mathbf{M}_{R_1} \wedge \mathbf{M}_{R_2}.$$

Problem

Suppose that the relations R_1 and R_2 on a set A are represented by the matrices

$$\mathbf{M}_{R_1} = \begin{bmatrix} 1 & 0 & 1 \\ 1 & 0 & 0 \\ 0 & 1 & 0 \end{bmatrix} \qquad \text{and} \qquad \mathbf{M}_{R_2} = \begin{bmatrix} 1 & 0 & 1 \\ 0 & 1 & 1 \\ 1 & 0 & 0 \end{bmatrix}.$$

What are the matrices representing $R_1 \cup R_2$ and $R_1 \cap R_2$?

Solution: The matrices of these relations are

$$\mathbf{M}_{R_1 \cup R_2} = \mathbf{M}_{R_1} \vee \mathbf{M}_{R_2} = \begin{bmatrix} 1 & 0 & 1 \\ 1 & 1 & 1 \\ 1 & 1 & 0 \end{bmatrix}$$

$$\mathbf{M}_{R_1 \cap R_2} = \mathbf{M}_{R_1} \wedge \mathbf{M}_{R_2} = \begin{bmatrix} 1 & 0 & 1 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}$$

Composition of relations

Suppose that R is a relation from A to B and S is a relation from B to C.

The ordered pair (a_i, c_j) belongs to $S \circ R$ if and only if there is an element b_k such that (a_i, b_k) belongs to R and (b_k, c_j) belongs to S.

From the definition of the <u>Boolean product</u>, this means that

$$\mathbf{M}_{S \circ R} = \mathbf{M}_R \odot \mathbf{M}_S.$$

matrices for $S \circ R$, R, and S be $\mathbf{M}_{S \circ R} = [t_{ij}]$, $\mathbf{M}_R = [r_{ij}]$, and $\mathbf{M}_S = [s_{ij}]$,

$t_{ij} = 1$ if and only if $r_{ik} = s_{kj} = 1$ for some k.

Composition of relations

$$\mathbf{M}_{S \circ R} = \mathbf{M}_R \odot \mathbf{M}_S.$$

Problem

Find the matrix representing the relations $S \circ R$, where the matrices representing R and S are

$$\mathbf{M}_R = \begin{bmatrix} 1 & 0 & 1 \\ 1 & 1 & 0 \\ 0 & 0 & 0 \end{bmatrix} \quad \text{and} \quad \mathbf{M}_S = \begin{bmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ 1 & 0 & 1 \end{bmatrix}.$$

Solution: The matrix for $S \circ R$ is

$$\mathbf{M}_{S \circ R} = \mathbf{M}_R \odot \mathbf{M}_S = \begin{bmatrix} 1 & 1 & 1 \\ 0 & 1 & 1 \\ 0 & 0 & 0 \end{bmatrix}.$$

Composition of R with R

Example: Find the matrix representing the relation R^2, where the matrix representing R is

$$\mathbf{M}_R = \begin{bmatrix} 0 & 1 & 0 \\ 0 & 1 & 1 \\ 1 & 0 & 0 \end{bmatrix}$$

Solution: The matrix for R^2 is

$$\mathbf{M}_{R^2} = \mathbf{M}_R^{[2]} = \begin{bmatrix} 0 & 1 & 1 \\ 1 & 1 & 1 \\ 0 & 1 & 0 \end{bmatrix}$$

Exercise problems

1. Represent each of these relations on $\{1, 2, 3\}$ with a matrix (with the elements of this set listed in increasing order).

a) $\{(1, 1), (1, 2), (1, 3)\}$
b) $\{(1, 2), (2, 1), (2, 2), (3, 3)\}$
c) $\{(1, 1), (1, 2), (1, 3), (2, 2), (2, 3), (3, 3)\}$
d) $\{(1, 3), (3, 1)\}$

2. Represent each of these relations on $\{1, 2, 3, 4\}$ with a matrix (with the elements of this set listed in increasing order).

a) $\{(1, 2), (1, 3), (1, 4), (2, 3), (2, 4), (3, 4)\}$
b) $\{(1, 1), (1, 4), (2, 2), (3, 3), (4, 1)\}$
c) $\{(1, 2), (1, 3), (1, 4), (2, 1), (2, 3), (2, 4), (3, 1), (3, 2), (3, 4), (4, 1), (4, 2), (4, 3)\}$
d) $\{(2, 4), (3, 1), (3, 2), (3, 4)\}$

Exercise problems

Let R_1 and R_2 be relations on a set A represented by the matrices

$$\mathbf{M}_{R_1} = \begin{bmatrix} 0 & 1 & 0 \\ 1 & 1 & 1 \\ 1 & 0 & 0 \end{bmatrix} \quad \text{and} \quad \mathbf{M}_{R_2} = \begin{bmatrix} 0 & 1 & 0 \\ 0 & 1 & 1 \\ 1 & 1 & 1 \end{bmatrix}.$$

Find the matrices that represent

a) $R_1 \cup R_2$. **b)** $R_1 \cap R_2$. **c)** $R_2 \circ R_1$.

Relations represented using directed graphs

The directed graph with vertices a, b, c, and d, and edges or arcs (a, b), (a, d), (b, b), (b, d), (c, a), (c, b), and (d, b)

Example: **The directed graph of the relation**

R = {(1, 1), (1, 3), (2, 1), (2, 3), (2, 4), (3, 1), (3, 2), (4, 1)}

on the set {1, 2, 3, 4} is shown in Figure.

Example: What are the ordered pairs in the relation *R* represented by the directed graph shown in Figure below

Problem:

Determine whether the relations for the directed graphs shown in the below Figure are reflexive, symmetric, antisymmetric, and/or transitive.

Closures of Relations

1. Reflexive closure of R.
2. Symmetric closure of R.
3. Transitive closure of R.

Reflexive closure of R

The relation R = {(1, 1), (1, 2), (2, 1), (3, 2)} on the set A = {1, 2, 3} is not reflexive.

How can we produce a reflexive relation containing R that is as small as possible?

This can be done by adding (2, 2) and (3, 3) to R called the reflexive closure of R

Example: What is the **reflexive closure** of the relation $R = \{(a, b) \mid a < b\}$ on the set of integers?

Solution: The reflexive closure of R is

$$R \cup \Delta = \{(a, b) \mid a < b\} \cup \{(a, a) \mid a \in \mathbf{Z}\} = \{(a, b) \mid a \leq b\}.$$

Symmetric closure of R

The relation $\{(1, 1), (1, 2), (2, 2), (2, 3), (3, 1), (3, 2)\}$ on set $\{1, 2, 3\}$ is not symmetric

We need only add (2, 1) and (1, 3)

Problem: What is the <u>symmetric closure</u> of the relation? $R = \{(a, b) \mid a > b\}$ on the set of positive integers?

Solution: The <u>symmetric closure</u> of R is the relation $R \cup R^{-1} = \{(a, b) \mid a > b\} \cup \{(b, a) \mid a > b\} = \{(a, b) \mid a \neq b\}$.

Note: The <u>symmetric closure</u> of a relation can be constructed by taking the union of relation with its inverse

Transitive closure of R

Can the transitive closure of a relation R be produced by adding all the pairs of the form (a, c), where (a, b) and (b, c) are already in the relation?

Example:

- ✓ R = {(1, 3), (1, 4), (2, 1), (3, 2)} on the set {1, 2, 3, 4}.
- ✓ This relation is not transitive because it does not contain all pairs of the form (a, c) where (a, b) and (b, c) are in R.
- ✓ The pairs of this form not in R are (1, 2), (2, 3), (2, 4), and (3, 1).
- ✓ Adding these pairs does not produce a transitive relation, because the resulting relation contains (3, 1) and (1, 4) but does not contain (3, 4).

Paths in Directed Graphs

A path from ***a*** to ***b*** in the directed graph G is a sequence of edges $(x_0, x_1), (x_1, x_2), (x_2, x_3), \ldots, (x_{n-1}, x_n)$ in G, where n is a non-negative integer.

$$x_0 = a \text{ and } x_n = b$$

A sequence of edges where the terminal vertex of an edge is the same as the initial vertex in the next edge in the path.

This path is denoted by $x_0, x_1, x_2, \ldots, x_{n-1}, x_n$ and has length n.

EXAMPLE Which of the following are paths in the directed graph : a, 1) **b, e, d;** 2) **a, e, c, d, b;** 3) **b, a, c, b, a, a, b;** 4) **d, c;** 5) **c, b, a;** **6) e, b, a, b, a, b, e?**

- **What are the lengths of those that are paths?**
- Which of the paths in this list are circuits?

The term path also applies to relations

Carrying over the definition from directed graphs to relations, there is a path from a to b in R if there is a sequence of elements $a, x_1, x_2, \ldots, x_{n-1}, b$ with $(a, x_1) \in \mathrm{R}$, $(x_1, x_2) \in \mathrm{R}$, . . . , and $(x_{n-1}, b) \in \mathrm{R}$.

Let R be a relation on a set A. There is a path of length *n*, where *n* is a positive integer, from *a* to *b* if and only if $(a, b) \in \mathrm{R}^n$.

Transitive Closures

Let **R** be a relation on a set **A**. The connectivity relation **R*** consists of the pairs ***(a, b)*** such that there is a path of length at least one from ***a*** to ***b*** in **R**.

Because $\mathbf{R}^n$ consists of the pairs ***(a, b)*** such that there is a path of length ***n*** from ***a*** to ***b***, it follows that **R*** is the union of all the sets $\mathbf{R}^n$. In other words,

$$R^* = \bigcup_{n=1}^{\infty} R^n$$

Transitive Closures

Example to understand R^n and R^*

Let **R** be the relation on the set of all people in the world that contains **(*a*, *b*)** if **a** has met **b**.

What is $\mathbf{R}^n$, where ***n*** is a positive integer greater than one? What is **R***?

Solution: The relation R^2 contains (a, b) if there is a person c such that $(a, c) \in R$ and $(c, b) \in R$, that is, if there is a person c such that a has met c and c has met b. Similarly, R^n consists of those pairs (a, b) such that there are people $x_1, x_2, \ldots, x_{n-1}$ such that a has met x_1, x_1 has met $x_2, \ldots,$ and x_{n-1} has met b.

Transitive Closures

Example to understand $\mathbf{R^n}$ and $\mathbf{R^*}$. Let **R** be the relation on the set of all states in the United States that contains **(a, b)** if state ***a*** and state ***b*** have a common border. What is $\mathbf{R}^n$, where n is a positive integer? What is $\mathbf{R^*}$?

Solution: The relation $\mathbf{R^n}$ consists of the pairs **(a, b)**, where it is possible to go from state ***a*** to state ***b*** by crossing exactly ***n*** state borders. $\mathbf{R^*}$ consists of the ordered pairs **(a, b)**, where it is possible to go from state ***a*** to state ***b*** crossing as many borders as necessary.

Transitive Closures

The transitive closure of a relation R equals the connectivity relation R^*

Let A be a set with n elements, and let R be a relation on A. The transitive closure of R is the union of R, R^2, R^3, . . . ,and R^n.

$$R^* = R \cup R^2 \cup R^3 \cup \cdots \cup R^n$$

Transitive Closures

$$R^* = R \cup R^2 \cup R^3 \cup \cdots \cup R^n$$

Result

Let $\mathbf{M}_R$ be the zero–one matrix of the relation R on a set with n elements. Then the zero–one matrix of the transitive closure R^* is

$$\mathbf{M}_{R^*} = \mathbf{M}_R \vee \mathbf{M}_R^{[2]} \vee \mathbf{M}_R^{[3]} \vee \cdots \vee \mathbf{M}_R^{[n]}.$$

Transitive Closures

Result

Let $\mathbf{M}_R$ be the zero–one matrix of the relation R on a set with n elements. Then the zero–one matrix of the transitive closure R^* is

$$\mathbf{M}_{R^*} = \mathbf{M}_R \vee \mathbf{M}_R^{[2]} \vee \mathbf{M}_R^{[3]} \vee \cdots \vee \mathbf{M}_R^{[n]}.$$

Find the zero–one matrix of the transitive closure of the relation R where

$$\mathbf{M}_R = \begin{bmatrix} 1 & 0 & 1 \\ 0 & 1 & 0 \\ 1 & 1 & 0 \end{bmatrix}.$$

Transitive Closures

$$\mathbf{M}_{R^*} = \mathbf{M}_R \vee \mathbf{M}_R^{[2]} \vee \mathbf{M}_R^{[3]} \vee \cdots \vee \mathbf{M}_R^{[n]}.$$

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$$\mathbf{M}_{R^*} = \mathbf{M}_R \vee \mathbf{M}_R^{[2]} \vee \mathbf{M}_R^{[3]}.$$

$$\mathbf{M}_R^{[2]} = \begin{bmatrix} 1 & 1 & 1 \\ 0 & 1 & 0 \\ 1 & 1 & 1 \end{bmatrix} \quad \text{and} \quad \mathbf{M}_R^{[3]} = \begin{bmatrix} 1 & 1 & 1 \\ 0 & 1 & 0 \\ 1 & 1 & 1 \end{bmatrix}$$

Transitive Closures

$$\mathbf{M}_{R^*} = \mathbf{M}_R \vee \mathbf{M}_R^{[2]} \vee \mathbf{M}_R^{[3]}.$$

Find the zero–one matrix of the transitive closure of the relation R where

$$\mathbf{M}_R = \begin{bmatrix} 1 & 0 & 1 \\ 0 & 1 & 0 \\ 1 & 1 & 0 \end{bmatrix}.$$

$$\mathbf{M}_{R^n} = \mathbf{M}_R^{[n]},$$

$$\mathbf{M}_R^{[2]} = \begin{bmatrix} 1 & 1 & 1 \\ 0 & 1 & 0 \\ 1 & 1 & 1 \end{bmatrix} \quad \text{and} \quad \mathbf{M}_R^{[3]} = \begin{bmatrix} 1 & 1 & 1 \\ 0 & 1 & 0 \\ 1 & 1 & 1 \end{bmatrix}$$

$$\mathbf{M}_{R^*} = \begin{bmatrix} 1 & 0 & 1 \\ 0 & 1 & 0 \\ 1 & 1 & 0 \end{bmatrix} \vee \begin{bmatrix} 1 & 1 & 1 \\ 0 & 1 & 0 \\ 1 & 1 & 1 \end{bmatrix} \vee \begin{bmatrix} 1 & 1 & 1 \\ 0 & 1 & 0 \\ 1 & 1 & 1 \end{bmatrix} = \begin{bmatrix} 1 & 1 & 1 \\ 0 & 1 & 0 \\ 1 & 1 & 1 \end{bmatrix}$$

ALGORITHM 1 A Procedure for Computing the Transitive Closure.

procedure *transitive closure* ($\mathbf{M}_R$: zero–one $n \times n$ matrix)
$\mathbf{A} := \mathbf{M}_R$
$\mathbf{B} := \mathbf{A}$
for $i := 2$ **to** n
    $\mathbf{A} := \mathbf{A} \odot \mathbf{M}_R$
    $\mathbf{B} := \mathbf{B} \vee \mathbf{A}$
return $\mathbf{B}\{\mathbf{B}$ is the zero–one matrix for $R^*\}$

Warshall's Algorithm

Assignment

Warshall's algorithm, named after Stephen Warshall, who described this algorithm in 1960, is an efficient method for ***computing the transitive closure of a relation***

Equivalence Relations

- A relation on a set **A** is called an equivalence relation if it is **reflexive, symmetric, and transitive**.

- Two elements **a and b** that are related by an equivalence relation are called equivalent.

- The notation **a ~ b** is often used to denote that a and b are equivalent elements

Equivalence Relations – verify these examples

1. Let R be the relation on the set of integers such that aRb if and only if $a = b$ or $a = -b$.
2. Let R be the relation on the set of real numbers such that aRb if and only if a − b is an integer.
3. R = {(a, b) | a ≡ b (mod m)} is an equivalence relation on the set of integers.
4. Suppose that R is the relation on the set of strings of English letters such that aRb if and only if $l(a) = l(b)$, where $l(x)$ is the length of the string x. Is R an equivalence relation?

Equivalence Classes – definition & notations

Let **R** be an ***equivalence relation*** on a set **A**. The set of all elements that are related to an element **a** of **A** is called the equivalence class of **a**.

The equivalence class of a with respect to **R** is denoted by $[a]_R$ or simple denote $[a]$

If R is an equivalence relation on a set A, the equivalence class of the element a is

$$[a]_R = \{s \mid (a, s) \in R\}.$$

Equivalence Classes – examples

$[a]_R = \{s \mid (a, s) \in R\}$.

1. Let R be the relation on the set of integers such that $a R b$ if and only if $a = b$ or $a = -b$.

Solution: here we get

$$[a] = \{-a, a\} \text{ for all non-zero } a$$

Equivalence Classes – examples

$[a]_R = \{s \mid (a, s) \in R\}$

2. What are the equivalence classes of 0 and 1 for congruence modulo 4?

Solution:

The equivalence class of 0 contains all integers *a* such that $a \equiv 0 \pmod 4$

$[0] = \{\ldots, -8, -4, 0, 4, 8, \ldots\}$.

The equivalence class of 1 contains all integers *a* such that $a \equiv 1 \pmod 4$

$[1] = \{\ldots, -7, -3, 1, 5, 9, \ldots\}$.

Equivalence Classes (Points To Note)

Let **R** be an equivalence relation on a set **A**. These statements for elements **a** and **b** of **A** are *equivalent*:

(i) **aRb**

(ii) **[a] = [b]**

(i) **[a] ∩ [b] ≠ ∅**

Equivalence Relation partitions the set A

Equivalence Classes (Points To Note)

Equivalence Relation partitions the set A

Consider the equivalence relation R on a set S

Find the equivalence classes?

S = {1, 2, 3, 4, 5}

R = {(1,1), (2, 2), (3, 3), (4, 4), (5, 5), (1, 3), (3, 1)}

$[1] = \{1, 3\}$ $[2] = \{2\}$

$[3] = \{3, 1\}$ $[4] = \{4\}$

$[5] = \{5\}$

Partial Orderings

A relation **R** on a set **S** is called a partial ordering or partial order if it is **reflexive**, **antisymmetric**, and **transitive**.

A set **S** together with a partial ordering **R** is called a **partially ordered set**, or **poset**, and is denoted by **(S, R)**.

Members of **S** are called elements of the **poset**.

Partial Orderings – few examples

1. Show that the "**greater than or equal**" relation ($\geq$) is a partial ordering on the set of integers.

2. The **divisibility** relation is a partial ordering on the set of positive integers, because it is *reflexive, antisymmetric,* and *transitive*.

3. Show that the ***inclusion*** relation $\subseteq$ is a partial ordering on the power set of a set **S**.

Partial Orderings – points to note

In different posets different symbols such as $\leq$, $\subseteq$, and $|$, are used for a partial ordering.

The notation $\mathbf{a \preceq b}$ is used to denote that $\mathbf{(a, b) \in R}$ in an arbitrary poset $\mathbf{(S, R)}$

The elements a and b of a poset $(S, \preceq)$ are called *comparable* if either $a \preceq b$ or $b \preceq a$.

TOTALLY ORDER SET

A total order (or "totally ordered set," or "linearly ordered set") is a ***set*** plus a *relation on the set* (called a total order) that satisfies the conditions for a partial order plus an additional condition ***known as the comparability condition***.

1. **Reflexivity**
2. **Antisymmetry**
3. **Transitivity**
4. **Comparability:**

Relation on set {1, 2, 3, 4}

R = {(1, 2), (2, 3), (1, 3), (4, 3), (1, 1), (2, 2), (3, 3), (4, 4)}

for all a, b in set S, either $a \preceq b$ or $b \preceq a$

find the solution of these problems

Which relation on the set {1, 2, 3, 4} is a partial ordering?

- a. { (1, 2), (2, 3), (1, 3), (4, 3) }
- b. { (1, 2), (2, 3), (1, 3), (3, 4) }
- c. { (1, 2), (2, 3), (1, 3), (3, 4), (1, 1), (2, 2), (3, 3), (4, 4) }
- d. { (1, 2), (2, 3), (1, 3), (4, 3), (1, 1), (2, 2), (3, 3), (4, 4) }

Examples

1. **The poset (Z, ≤) is <u>totally ordered</u>**
2. **The poset (Z+ , |) is <u>not totally ordered</u> because it contains elements that are incomparable, such as 5 and 7.**

More examples

1. Show that the inclusion relation $\subseteq$ is a partial ordering on the power set of a set S.

2. Let R be the relation on the set of people such that xRy if x and y are people and x is older than y. Show that R is not a partial ordering.

Hasse Diagrams

1. Constructing the Hasse Diagram for ({1, 2, 3, 4}, ≤)

Step by Step procedure for Hasse diagram

1. **Start with the directed graph for this relation.**
2. **A loop (a, a) is present at every vertex a. Remove these loops.**
3. **Next, remove all transitivity edges.**

   (*That is, remove all edges* (x, y) *for which there is an element* $z \in S$ *such that* $x \prec z$ *and* $z \prec y$)

4. **Finally, arrange each edge so that its initial vertex is below its terminal vertex**
5. **Remove all the arrows on the directed edges**

   because all edges point "upward" toward their terminal vertex.

Problem: Draw the Hasse diagram representing the partial ordering {(a, b) | a divides b} on {1, 2, 3, 4, 6, 8, 12}.

More problems

1. **Draw the Hasse diagram for the "less than or equal to" relation on {0, 2, 5, 10, 11, 15}.**
2. **List all ordered pairs in the partial ordering with the accompanying Hasse diagram.**

3. **Draw the Hasse diagram for divisibility on the set {1, 2, 3, 4, 5, 6}.**