Discrete Mathematics

Course Code: 20MAT115
Course Credits: 4

Srinivasan, Assistant Professor

Amrita Vishwa Vidyapeetham, Coimbatore, INDIA

Syllabus

Unit-1

Logic, Mathematical Reasoning and Counting: Logic, Prepositional Equivalence, Predicate and Quantifiers, Theorem Proving, Functions, Mathematical Induction. Recursive Definitions, Recursive Algorithms, Basics of Counting, Pigeonhole Principle, Permutation and Combinations.

Unit-2

Relations and Their Properties: Representing Relations, Closure of Relations, Partial Ordering, Equivalence Relations and partitions.

Unit-3

Advanced Counting Techniques and Relations: Recurrence Relations, Solving Recurrence Relations, Generating Functions, Solutions of Homogeneous Recurrence Relations, Divide and Conquer Relations, Inclusion-Exclusion.

Unit-4

Graphs: Special types of graphs, connectivity, Euler and Hamiltonian Paths.

Trees: Applications of trees, Tree traversal, Spanning trees.

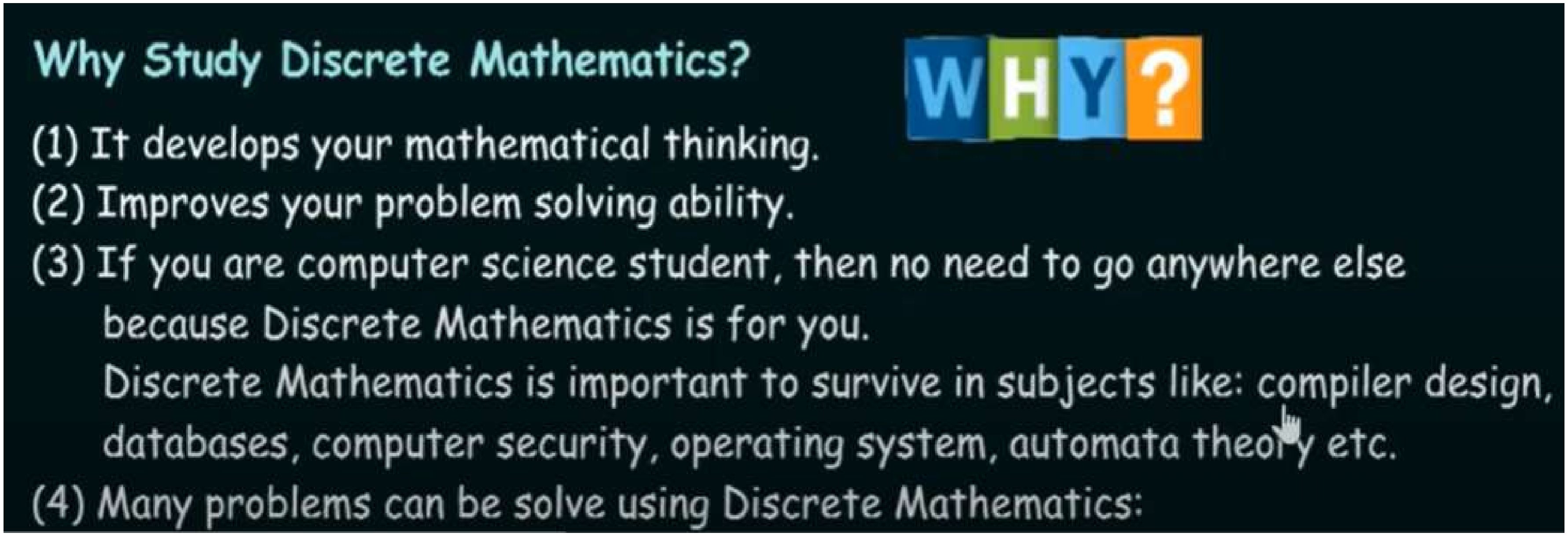
Textbook References

Textbook

1. Kenneth H. Rosen, Discrete Mathematics and its Applications, Tata McGraw-Hill Publishing Company Limited, New Delhi

Reference(s)

1. James Strayer, Elementary Number Theory, Waveland Press, 2002.
2. R.P. Grimaldi, Discrete and Combinatorial Mathematics, Pearson Education, Fifth Edition, 2007.
3. Thomas Koshy, Discrete Mathematics with Applications, Academic Press, 2005.Liu, Elements of Discrete Mathematics, Tata McGraw- Hill Publishing Company Limited, 2004.



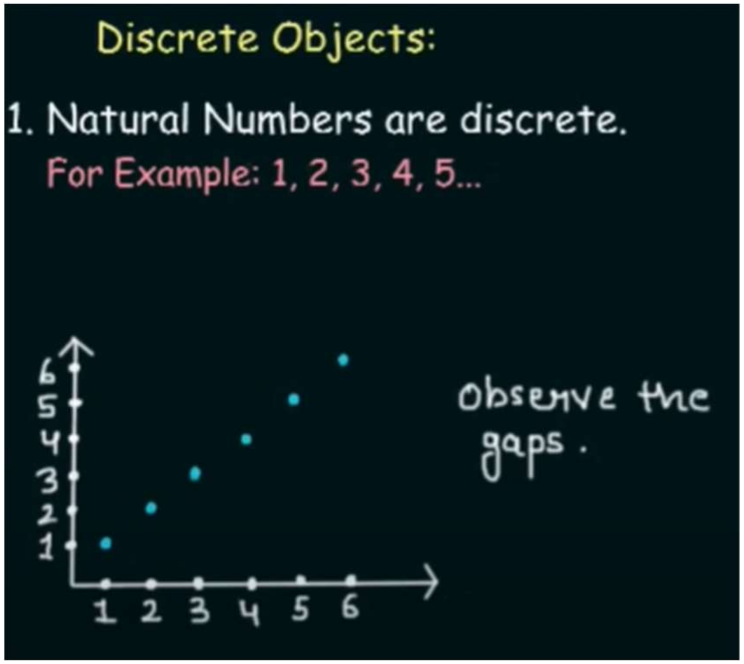
1. Sorting the list of integers
2. Finding the shortest path from a point to another point
3. Drawing a graph with given conditions
4. How many different combinations of password are possible with just 10 alphanumeric characters
5. Encrypt a message and deliver it your friend securely

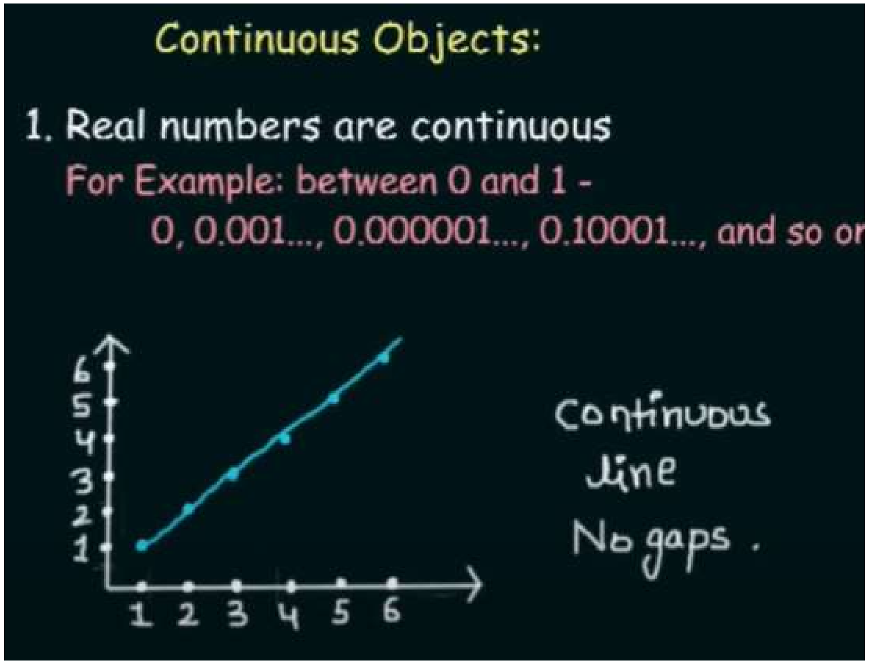
What is discrete mathematics?

Study of discrete objects

Discrete means: distinct or not connected

Discrete vs Continuous





SETS

Much of discrete mathematics is devoted to the study of discrete structures.

Discrete structures used to represent discrete objects.

Discrete objects

Separated from each other (oppo. to continues)

Exa: integers, people, house

Continuous objects are real numbers

Discrete structures

The abstract mathematical structures used to represent discrete objects and relation between them. Exa: sets, relations, graphs etc.

Why do we study discrete structures?

- ✓ Information is stored and manipulated by computers in a discrete fashion 0101101…
- ✓ it is used in programming languages, software development, cryptography, algorithms, software development.etc.
- ✓ Because of discrete mathematical applications in algorithms, today's computers run faster than ever before.

Set

A *set* is an unordered collection of objects, called *elements* or *members* of the set. A set is said to *contain* its elements. We write $a \in A$ to denote that a is an element of the set A. The notation $a \notin A$ denotes that a is not an element of the set A.

Members of the set are listed between braces

The notation {a, b, c, d} represents the set with the four elements a, b, c, and d.

EXAMPLE 1 The set V of all vowels in the English alphabet can be written as $V = \{a, e, i, o, u\}$.

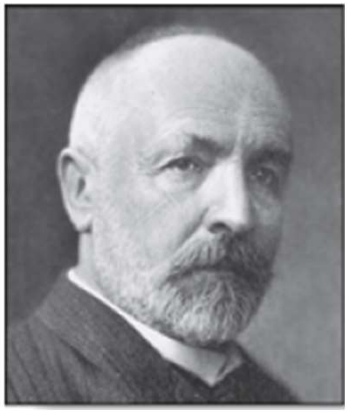
EXAMPLE 2 The set O of odd positive integers less than 10 can be expressed by $O = \{1, 3, 5, 7, 9\}$.

EXAMPLE 3 $\{a$, 2, sachin, messi$\}$

EXAMPLE 4 The set of positive integers less than 100 can be denoted by $\{1, 2, 3, \ldots, 99\}$

Another way to describe a set is to use **set builder notation**

GEORG CANTOR (1845–1918)



$O = \{x \mid x \text{ is an odd positive integer less than } 10\}.$

$O = \{x \in \mathbf{Z}^+ \mid x \text{ is odd and } x < 10\}.$

$\mathbf{Q}^+ = \{x \in \mathbf{R} \mid x = \frac{p}{q}, \text{ for some positive integers } p \text{ and } q\}.$

$\mathbf{N} = \{0, 1, 2, 3, \ldots\}$, the set of **natural numbers**
$\mathbf{Z} = \{\ldots, -2, -1, 0, 1, 2, \ldots\}$, the set of **integers**
$\mathbf{Z}^+ = \{1, 2, 3, \ldots\}$, the set of **positive integers**
$\mathbf{Q} = \{p/q \mid p \in \mathbf{Z}, q \in \mathbf{Z}, \text{ and } q \neq 0\}$, the set of **rational numbers**
$\mathbf{R}$, the set of **real numbers**
$\mathbf{R}^+$, the set of **positive real numbers**
$\mathbf{C}$, the set of **complex numbers**.

Recall the notation for intervals of real numbers, where a and b are real numbers with a < b, we write

$$[a, b] = \{x \mid a \leq x \leq b\}$$

$$[a, b) = \{x \mid a \leq x < b\}$$

$$(a, b] = \{x \mid a < x \leq b\}$$

$$(a, b) = \{x \mid a < x < b\}$$

Sets can have other sets as members

EXAMPLE 5 The set {**N**, **Z**, **Q**, **R**} is a set containing four elements, each of which is a set. The four elements of this set are **N**, the set of natural numbers; **Z**, the set of integers; **Q**, the set of rational numbers; and **R**, the set of real numbers. ◀

Remark:

- ❑ The concept of a **datatype, or type**, in computer science is built upon the concept of a set.
- ❑ A **datatype** or **type** is the name of a set, together with a set of operations that can be performed on objects from that set.

Equal sets

EXAMPLE 6 The sets $\{1, 3, 5\}$ and $\{3, 5, 1\}$ are equal, because they have the same elements. Note that the order in which the elements of a set are listed does not matter. Note also that it does not matter if an element of a set is listed more than once, so $\{1, 3, 3, 3, 5, 5, 5, 5\}$ is the same as the set $\{1, 3, 5\}$ because they have the same elements. ◀

THE EMPTY SET There is a special set that has no elements. This set is called the **empty set**, or **null set**, and is denoted by $\emptyset$.

The empty set can also be denoted by { }

The set of all positive integers that are greater than their squares is the null set.

A set with one element is called a singleton set. Exa: {1}, {b}

Do not get confused the set $\emptyset$ with the empty {$\emptyset$}

✓The empty set can be thought of as an empty folder in your computer

Subsets

The set A is a *subset* of B if and only if every element of A is also an element of B. We use the notation $A \subseteq B$ to indicate that A is a subset of the set B.

Showing that A is a Subset of B To show that $A \subseteq B$, show that if x belongs to A then x also belongs to B.

Showing that A is Not a Subset of B To show that $A \not\subseteq B$, find a single $x \in A$ such that $x \notin B$.

The set of all odd positive integers less than 10 is a subset of the set of all positive integers less than 10

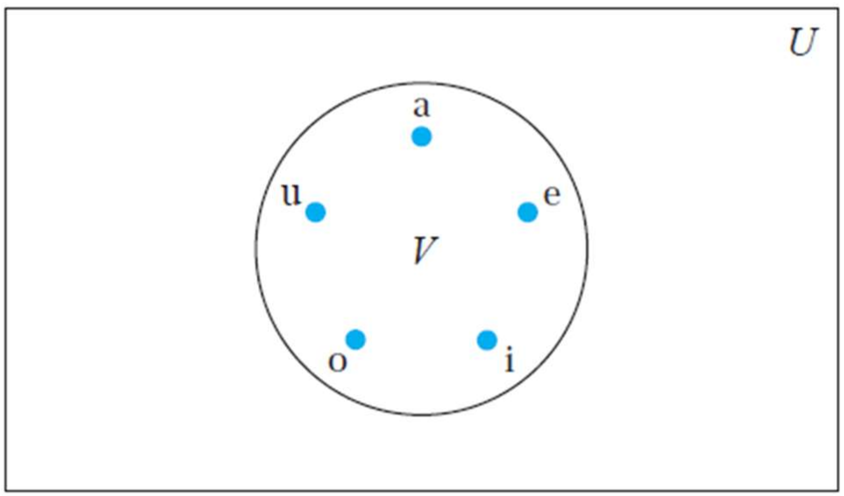
The set of all computer science majors at your school is a subset of the set of all students at your school

The set of all people in China is a subset of the set of all people in China

The set of people who have taken discrete mathematics at your school is not a subset of the set of all computer science majors at your school

Sets can be represented graphically using **Venn diagrams**, named after the English mathematician John Venn, who introduced their use in 1881

EXAMPLE 7 Draw a Venn diagram that represents V, the set of vowels in the English alphabet.



Showing Two Sets are Equal To show that two sets A and B are equal, show that $A \subseteq B$ and $B \subseteq A$.

Sets may have other sets as members. For instance, we have the sets

$$A = \{\emptyset, \{a\}, \{b\}, \{a, b\}\} \qquad \text{and} \qquad B = \{x \mid x \text{ is a subset of the set } \{a, b\}\}.$$

Note that these two sets are equal, that is, $A = B$. Also note that $\{a\} \in A$, but $a \notin A$.

Let S be a set. If there are exactly n distinct elements in S where n is a nonnegative integer, we say that S is a *finite set* and that n is the *cardinality* of S. The cardinality of S is denoted by $|S|$.

Let A be the set of odd positive integers less than 10. Then $|A| = 5$.

Let S be the set of letters in the English alphabet. Then $|S| = 26$.

Because the null set has no elements, it follows that $|\emptyset| = 0$.

Power Sets

Given a set S, the *power set* of S is the set of all subsets of the set S. The power set of S is denoted by $\mathcal{P}(S)$.

What is the power set of the set $\{0, 1, 2\}$?

Solution: The power set $\mathcal{P}(\{0, 1, 2\})$ is the set of all subsets of $\{0, 1, 2\}$. Hence,

$$\mathcal{P}(\{0, 1, 2\}) = \{\emptyset, \{0\}, \{1\}, \{2\}, \{0, 1\}, \{0, 2\}, \{1, 2\}, \{0, 1, 2\}\}.$$

Note that the empty set and the set itself are members of this set of subsets.

Cartesian Products

The *ordered n-tuple* $(a_1, a_2, \ldots, a_n)$ is the ordered collection that has a_1 as its first element, a_2 as its second element, $\ldots$, and a_n as its nth element.

Let A and B be sets. The *Cartesian product* of A and B, denoted by $A \times B$, is the set of all ordered pairs (a, b), where $a \in A$ and $b \in B$. Hence,

$$A \times B = \{(a, b) \mid a \in A \wedge b \in B\}.$$

EXAMPLE

Let A represent the set of all students at a university, and let B represent the set of all courses offered at the university. What is the Cartesian product $A \times B$ and how can it be used?

One way to use the set $A \times B$ is to represent all possible enrollments of students in courses at the university.

Cartesian Products

EXAMPLE

What is the Cartesian product of A = {1, 2} and B = {a, b, c}?
Solution: The Cartesian product A × B is
A × B = {(1, a), (1, b), (1, c), (2, a), (2, b), (2, c)}

Note that the Cartesian products A × B and B × A are not equal

The *Cartesian product* of the sets $A_1, A_2, \ldots, A_n$, denoted by $A_1 \times A_2 \times \cdots \times A_n$, is the set of ordered n-tuples $(a_1, a_2, \ldots, a_n)$, where a_i belongs to A_i for $i = 1, 2, \ldots, n$.

$$A \times B \times C = \{(0, 1, 0), (0, 1, 1), (0, 1, 2), (0, 2, 0), (0, 2, 1), (0, 2, 2), (1, 1, 0), (1, 1, 1), (1, 1, 2), (1, 2, 0), (1, 2, 1), (1, 2, 2)\}.$$

Cartesian product and Relation

Suppose that $A = \{1, 2\}$. It follows that $A^2 = \{(1, 1), (1, 2), (2, 1), (2, 2)\}$ and $A^3 = \{(1, 1, 1), (1, 1, 2), (1, 2, 1), (1, 2, 2), (2, 1, 1), (2, 1, 2), (2, 2, 1), (2, 2, 2)\}$.

A subset R of the Cartesian product $A \times B$ is called a **relation** from the set A to the set B. The elements of R are ordered pairs, where the first element belongs to A and the second to B. For example, $R = \{(a, 0), (a, 1), (a, 3), (b, 1), (b, 2), (c, 0), (c, 3)\}$ is a relation from the set $\{a, b, c\}$ to the set $\{0, 1, 2, 3\}$. A relation from a set A to itself is called a relation on A.

EXAMPLE

What are the ordered pairs in the less than or equal to relation, which contains (a, b) if $a \leq b$, on the set $\{0, 1, 2, 3\}$?

Solution: The ordered pair (a, b) belongs to R if and only if both a and b belong to $\{0, 1, 2, 3\}$ and $a \leq b$. Consequently, the ordered pairs in R are (0,0), (0,1), (0,2), (0,3), (1,1), (1,2), (1,3), (2,2), (2, 3), and (3, 3).

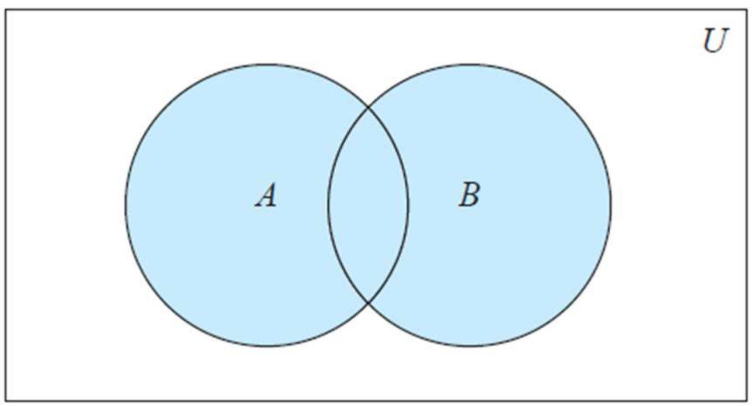
Set Operations

Let A and B be sets. The *union* of the sets A and B, denoted by $A \cup B$, is the set that contains those elements that are either in A or in B, or in both.

$$A \cup B = \{x \mid x \in A \vee x \in B\}$$

EXAMPLE

The union of the sets $\{1, 3, 5\}$ and $\{1, 2, 3\}$ is the set $\{1, 2, 3, 5\}$; that is, $\{1, 3, 5\} \cup \{1, 2, 3\} = \{1, 2, 3, 5\}$.



$A \cup B$ is shaded.

Venn Diagram of the Union of A and B.

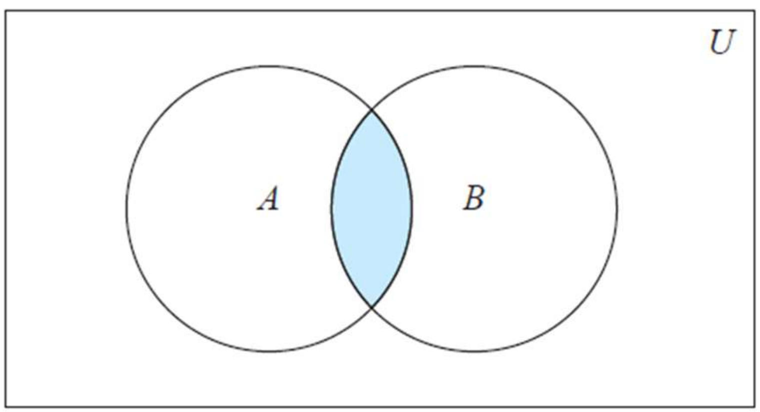
Set Operations

Let A and B be sets. The *intersection* of the sets A and B, denoted by $A \cap B$, is the set containing those elements in both A and B.

$$A \cap B = \{x \mid x \in A \wedge x \in B\}$$

EXAMPLE

The intersection of the sets $\{1, 3, 5\}$ and $\{1, 2, 3\}$ is the set $\{1, 3\}$; that is, $\{1, 3, 5\} \cap \{1, 2, 3\} = \{1, 3\}$.



$A \cap B$ is shaded.

Venn Diagram of the Intersection of A and B.

Set Operations continues

Two sets are called *disjoint* if their intersection is the empty set.

EXAMPLE

Let $A = \{1, 3, 5, 7, 9\}$ and $B = \{2, 4, 6, 8, 10\}$. Because $A \cap B = \emptyset$, A and B are disjoint.

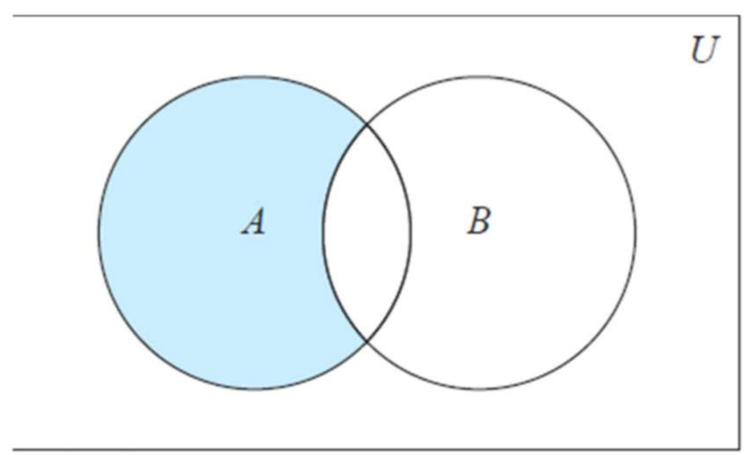
Cardinality of a Union of finite sets

$$|A \cup B| = |A| + |B| - |A \cap B|.$$

Difference of *A* and *B*

$$A - B = \{x \mid x \in A \wedge x \notin B\}$$

Difference of *A* and *B*

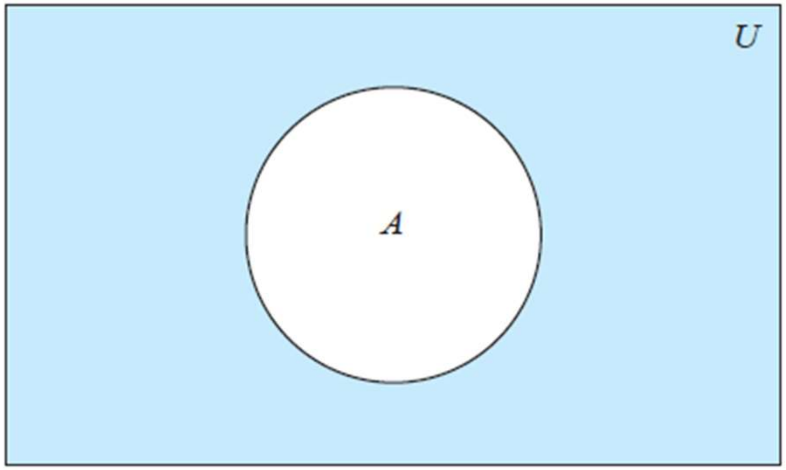


$A - B$ is shaded.

Complement of a Set

Let U be the universal set. The *complement* of the set A, denoted by $\overline{A}$, is the complement of A with respect to U. Therefore, the complement of the set A is $U - A$.

$$\overline{A} = \{x \in U \mid x \notin A\}.$$



$\overline{A}$ is shaded.

Is this true

$A - B = A \cap \overline{B}.$

EXAMPLE

Let $A = \{a, e, i, o, u\}$ (where the universal set is the set of letters of the English alphabet).

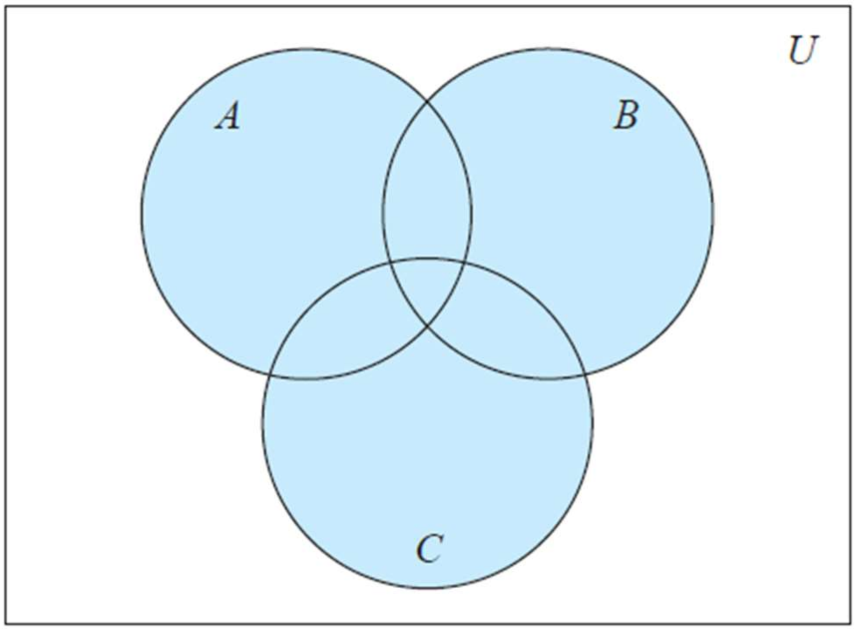
Let A be the set of positive integers greater than 10 (with universal set the set of all positive integers).

Set Identities

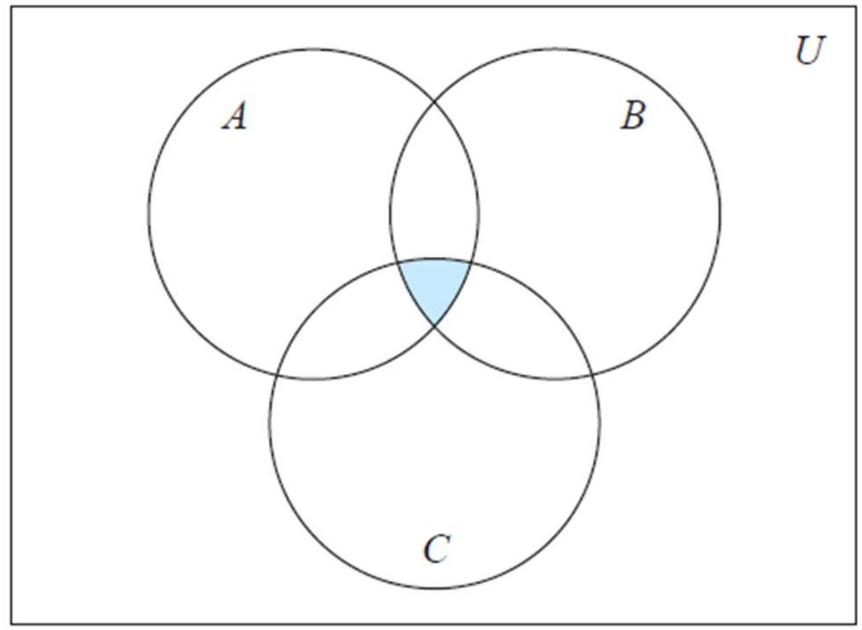
Identity	*Name*
$A \cap U = A$ $A \cup \emptyset = A$	Identity laws
$A \cup U = U$ $A \cap \emptyset = \emptyset$	Domination laws
$A \cup A = A$ $A \cap A = A$	Idempotent laws
$\overline{(\overline{A})} = A$	Complementation law
$A \cup B = B \cup A$ $A \cap B = B \cap A$	Commutative laws

$A \cup (B \cup C) = (A \cup B) \cup C$ $A \cap (B \cap C) = (A \cap B) \cap C$	Associative laws
$A \cup (B \cap C) = (A \cup B) \cap (A \cup C)$ $A \cap (B \cup C) = (A \cap B) \cup (A \cap C)$	Distributive laws
$\overline{A \cap B} = \overline{A} \cup \overline{B}$ $\overline{A \cup B} = \overline{A} \cap \overline{B}$	De Morgan's laws
$A \cup (A \cap B) = A$ $A \cap (A \cup B) = A$	Absorption laws
$A \cup \overline{A} = U$ $A \cap \overline{A} = \emptyset$	Complement laws

Generalized Unions and Intersections



(a) $A \cup B \cup C$ is shaded.



(b) $A \cap B \cap C$ is shaded.

Let $A = \{0, 2, 4, 6, 8\}$, $B = \{0, 1, 2, 3, 4\}$, and $C = \{0, 3, 6, 9\}$. What are $A \cup B \cup C$ and $A \cap B \cap C$?

De Morgan's Law

Prove that $\overline{A \cap B} = \overline{A} \cup \overline{B}$.

Solution:

$$
\begin{aligned}
\overline{A \cap B} &= \{x \mid x \notin A \cap B\} \\
&= \{x \mid x \notin A \vee x \notin B\} \\
&= \{x \mid x \in \overline{A} \vee x \in \overline{B}\} \\
&= \{x \mid x \in \overline{A} \cup \overline{B}\} \\
&= \overline{A} \cup \overline{B}
\end{aligned}
$$

Set identities can also be proved using membership tables

Use a membership table to show that $A \cap (B \cup C) = (A \cap B) \cup (A \cap C)$.

A Membership Table for the Distributive Property

A	B	C	$B \cup C$	$A \cap (B \cup C)$	$A \cap B$	$A \cap C$	$(A \cap B) \cup (A \cap C)$
1	1	1	1	1	1	1	1
1	1	0	1	1	1	0	1
1	0	1	1	1	0	1	1
1	0	0	0	0	0	0	0
0	1	1	1	0	0	0	0
0	1	0	1	0	0	0	0
0	0	1	1	0	0	0	0
0	0	0	0	0	0	0	0

Computer Representation of Sets

There are various ways to represent sets using a computer

Assume that the universal set U is finite

First, specify an arbitrary ordering of the elements of U

EXAMPLE

Let $U = \{1, 2, 3, 4, 5, 6, 7, 8, 9, 10\}$, and the ordering of elements of U has the elements in increasing order; that is, $a_i = i$. What bit strings represent the subset of all odd integers in U, the subset of all even integers in U, and the subset of integers not exceeding 5 in U?

Solution: The bit string that represents the set of odd integers in U, namely, $\{1, 3, 5, 7, 9\}$, has a one bit in the first, third, fifth, seventh, and ninth positions, and a zero elsewhere. It is

01 0101 0101

Computer Representation of Sets

The set of all integers in U that do not exceed 5, namely, $\{1, 2, 3, 4, 5\}$, is represented by the string

11 1110 0000

Using bit strings to represent sets, it is easy to find complements of sets and unions, intersections, and differences of sets.

EXAMPLE

The bit strings for the sets $\{1, 2, 3, 4, 5\}$ and $\{1, 3, 5, 7, 9\}$ are 11 1110 0000 and 10 1010 1010, respectively. Use bit strings to find the union and intersection of these sets.

11 1110 0000 $\vee$ 10 1010 1010 = 11 1110 1010 **Union of above sets is {1, 2, 3, 4, 5, 7, 9}**

11 1110 0000 $\wedge$ 10 1010 1010 = 10 1010 0000 **Intersection of above sets {1, 3, 5}.**