NUMBER THEORY AND ALGEBRA

① If $a|b$ then $b = ax$ and if $b|c$ then $c = bs$
Then $7b - 5c = 7ax - 5bs = 7ax - 5(ax)s = a(7x - 5xs)$
Since $7x - 5xs$ is an Integer then $a \mid (7b - 5c)$

② $g.c.d\ (a,b) = 1$
Let $g.c.d$ of $(a+b, a-b) = d$
$\Rightarrow d \mid (a-b)$ and $d \mid (a+b)$
$a - b = cd$
$a + b = ed$

$2a = d(e+c) \quad \Rightarrow d \mid 2a$
$2b = d(e-c) \quad \Rightarrow d \mid 2b$

$gcd\ (a,b) = 1$
This is only possible if either $d = 2$ or $d = 1$
$\therefore$ d can not divide both a and b
$\therefore gcd\ (a+b, a-b) = 1$ or 2.

③ $3587 = 1819 \cdot 1 + 1768$
$1819 = 1768 \cdot 1 + 51$
$1768 = 51 \cdot 34 + 34$
$51 = 34 \cdot 1 + 17$
$34 = 17 \cdot 2 + 0$

$17 = 51 - 34 \cdot 1$
$= 51 - (1768 - 51 \cdot 34) \cdot 1$
$= 51 \cdot 35 - 1768$
$= (1819 - 1768) \cdot 35 - (3587 - 1819)$
$= 1819 \cdot 35 - 1768 \cdot 36$
$= 1819 \cdot 35 - (3587 - 1819) \cdot 36$
$= 1819 \cdot 71 - 3587 \cdot 36$
$\therefore g = 17 = 1819 \cdot 71 - 3587 \cdot 36 = 1819x + 3587y \quad x = 71 \quad y =$

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4) i) $a^3+b^3 = (a+b)(a^2-ab+b^2)$

so $n^3+1 = (n+1)(n^2-n+1)$

Let $(n+1) \to a$, $(n^2-n+1) \to m$

$a \mid am$

$\Rightarrow (n+1) \mid (n+1)(n^2-n+1)$

$(n+1) \mid (n^3+1) \quad \forall \; n \geq 2$

ii) We can take $n = k$

$P(k) = (k^2-1) \mid (k^3+1)$

For $k = 2 \to P(2) = 3 \mid 9 \Rightarrow$ True

So for $k \to k+1$

$P(k+1) = ((k+1)^2-1) \mid ((k+1)^3+1)$

$= k^2+2k \mid k^3+3k^2+3k+2$

$P(3) = 15 \mid 65 \Rightarrow$ False

$P(4) = 16+16+3 \mid 64+96+48+9 \Rightarrow$ False

So, we can say that $(n^2-1) \mid (n^3+1)$ only works for $n=2$

5) $3^{n-1}+5^{n-1} \mid 3^n+5^n$

$3^{n-1}+5^{n-1} \mid 3^n+5^n - 5(3^{n-1}+5^{n-1}) = -2 \cdot 3^{n-1}$

$3^{n-1}+5^{n-1} \mid 3^n+5^n - 3(3^{n-1}+5^{n-1}) = 2 \cdot 5^{n-1}$

$2 \cdot 3^{n-1} = x(3^{n-1}+5^{n-1}) \quad — ①$

$2 \cdot 5^{n-1} = y(3^{n-1}+5^{n-1}) \quad — ②$

① + ②

$2 \cdot 5^{n-1} = (2 \cdot x)(3^{n-1}+5^{n-1})$

$x = \dfrac{2 \cdot 3^{n-1}}{3^{n-1}+5^{n-1}}$

x is odd for the $n=1$

There can be only 1 solution which is 1.

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⑥ Given $3m + n = 3\,LCM(m,n) + gcd(m,n)$ — ①
Let us assume $gcd = (m,n) = d$

$m = d \cdot K \quad , \quad n = d \cdot X$

Eq ① $\div d$

$$\frac{3m}{d} + \frac{n}{d} = \frac{3\, dKX}{d} + \frac{d}{d}$$

$3K + X = 3KX + 1$
$(3K - 1)(X - 1) = 0$
$K = 1/3$ or $X = 1$
K cannot be $1/3$ because K is an integer
So $X = 1$ or $n = d$
If $gcd(m,n) = n$
then n divides m.

⑦ Given $T_1 = 2$
$T_{n+1} = T_n^2 - T_n + 1$
$T_2 = T_1^2 - T_1 + 1 = 4 - 2 + 1 = 3$
$T_3 = T_2^2 - T_2 + 1 = 9 - 3 + 1 = 4$
$T_4 = T_3^2 - T_3 + 1 = 49 - 7 + 1 = 43$
$T_{n+1} = T_n^2 - T_n + 1$
$T_2 = 3 = 2 + 1 = T_1 + 1$
$T_3 = 7 = 6 + 1 = T_1 T_2 + 1$
$T_4 = 43 = 42 + 1 = T_1 T_2 T_3 + 1$
$T_n = T_1 T_2 \dots T_n + 1$

$gcd(T_i, T_j) = 1 \quad \forall i \neq j$

Lets take an example
$gcd(7, 43) = 1$
$gcd(3, 7) = 1$
$gcd(3, 43) = 1$

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(8) abcd.bc 1001 = $1000000a + 100000b + 1000c$

$100a + 10b + c$

$= 100100a + 10010b + 1002c$

$= 1001(100a + 10b + c)$

$= 1001$ "abc" $= 7 \cdot 11 \cdot 13$