Math 369 Exam #2 Practice Problem Solutions

1. Is $\left\{ \begin{bmatrix} 1 \\ -2 \\ -1 \end{bmatrix}, \begin{bmatrix} 2 \\ -3 \\ 1 \end{bmatrix}, \begin{bmatrix} 5 \\ -8 \\ 1 \end{bmatrix} \right\}$ a basis for $\mathbb{R}^3$?

   Answer: No, it is not. To show that it is not a basis, it suffices to show that this is not a linearly independent set. To see that, we need to find coefficients a, b, c not all zero so that

$$a \begin{bmatrix} 1 \\ -2 \\ -1 \end{bmatrix} + b \begin{bmatrix} 2 \\ -3 \\ 1 \end{bmatrix} + c \begin{bmatrix} 5 \\ -8 \\ 1 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$$

   Equivalently, we can solve the matrix equation

$$\begin{bmatrix} 1 & 2 & 5 \\ -2 & -3 & -8 \\ -1 & 1 & 1 \end{bmatrix} \begin{bmatrix} a \\ b \\ c \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}.$$

   So form the augmented matrix

$$\left[\begin{array}{ccc|c} 1 & 2 & 5 & 0 \\ -2 & -3 & -8 & 0 \\ -1 & 1 & 1 & 0 \end{array}\right]$$

   and row-reduce: add twice row 1 to row 2 and add row 1 to row 3:

$$\left[\begin{array}{ccc|c} 1 & 2 & 5 & 0 \\ 0 & 1 & 2 & 0 \\ 0 & 3 & 6 & 0 \end{array}\right]$$

   Now subtract three times row 2 from row 3:

$$\left[\begin{array}{ccc|c} 1 & 2 & 5 & 0 \\ 0 & 1 & 2 & 0 \\ 0 & 0 & 0 & 0 \end{array}\right]$$

   So we see that any a, b, c satisfying the equations

$$\begin{aligned} a + 2b + 5c &= 0 \\ b + 2c &= 0 \end{aligned}$$

   will yield a non-trivial linear combination of the given vectors that is zero. For example, if $a = -1$, $b = -2$, and $c = 1$, then we have

$$-1 \begin{bmatrix} 1 \\ -2 \\ -1 \end{bmatrix} - 2 \begin{bmatrix} 2 \\ -3 \\ 1 \end{bmatrix} + 1 \begin{bmatrix} 5 \\ -8 \\ 1 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix},$$

   so the set is linearly dependent and hence cannot be a basis.

2. In each part, V is a vector space and S is a subset of V. Determine whether S is a *subspace* of V.

(a) $V = \mathbb{R}^3$

$S = \{ \begin{bmatrix} x \\ 12 \\ 3x \end{bmatrix} : x \in \mathbb{R}\}.$

Answer: S is not a subspace, because the zero vector $\begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$ cannot be written in the form $\begin{bmatrix} x \\ 12 \\ 3x \end{bmatrix}$ for any possible value of x, so $\begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix} \notin S$ and S cannot be a subspace.

(b) $V = \mathbb{R}^2$

$S = \{ \begin{bmatrix} x \\ y \end{bmatrix} : 2x - 5y = 11\}$

Answer: No, this is not a subspace. After all, the zero vector $\begin{bmatrix} 0 \\ 0 \end{bmatrix}$ is not in S since $2(0) - 5(0) = 0 \neq 11$.

(c) $V = \mathbb{R}^n$

$S = \{\vec{x} \in \mathbb{R}^n : A\vec{x} = 2\vec{x}\}$, where A is a particular $n \times n$ matrix.

Answer: Yes, this is a subspace. To prove it, suppose $\vec{x}_1$ and $\vec{x}_2$ are in this set, meaning that

$$A\vec{x}_1 = 2\vec{x}_1 \quad \text{and} \quad A\vec{x}_2 = 2\vec{x}_2$$

(such vectors are called *eigenvectors* of A; we'll learn more about them later). Then

$$A(\vec{x}_1 + \vec{x}_2) = A\vec{x}_1 + A\vec{x}_2 = 2\vec{x}_1 + 2\vec{x}_2 = 2(\vec{x}_1 + \vec{x}_2),$$

meaning that $\vec{x}_1 + \vec{x}_2$ is in this set as well.

Moreover, for any $c \in \mathbb{R}$,

$$A(c\vec{x}_1) = c(A\vec{x}_1) = c(2\vec{x}_1) = 2(c\vec{x}_1),$$

so $c\vec{x}_1$ is in the set as well.

Therefore, this set is closed under addition and scalar multiplication, so it is indeed a subspace.

(d) $V = F(-\infty, \infty)$

$S = \{f : f(x) = a\cos x + b\sin x + c\}$

Answer: Yes, this is a subspace. If $a_1, a_2, b_1, b_2, c_1, c_2 \in \mathbb{R}$ and I define

$$f(t) = a_1 \cos t + b_1 \sin t + c_1$$

and

$$g(t) = a_2 \cos t + b_2 \sin t + c_2,$$

then f and g are in the given subset. The sum has the form

$$f(t)+g(t) = (a_1 \cos t+b_1 \sin t+c_1)+(a_2 \cos t+b_2 \sin t+c_2) = (a_1+a_2)\cos t+(b_1+b_2)\sin t+(c_1+c_2),$$

so $f + g$ is also in the subset, which is, therefore, closed under addition.

Also, if $r \in \mathbb{R}$, then

$$rf(t) = r(a_1 \cos t + b_1 \sin t + c_1) = (ra_1) \cos t + (rb_1) \sin t + (rc_1),$$

so rf is in the subset, which is, therefore, closed under scalar multiplication.

Hence, we can conclude that this subset is actually a subspace.

3. Let V be a vector space.

   (a) Define what it means for a set $\{u_1, \ldots, u_n\} \subset V$ to be linearly dependent.

   Answer: By definition, $\{u_1, \ldots, u_n\}$ is linearly dependent if there is a non-trivial way to write 0 as a linear combination of the u_i, meaning that there exist scalars $\lambda_1, \ldots, \lambda_n$ not all zero so that

   $$\lambda_1 u_1 + \lambda_2 u_2 + \ldots + \lambda_n u_n = 0.$$

   (b) Suppose $v \in V$. Is the set $\{0, v\}$ linearly dependent? Explain.

   Answer: Yes. Let λ be any nonzero scalar. Then

   $$\lambda 0 + 0v = 0$$

   is a nontrivial linear combination of 0 and v that yields 0, so this set is linearly *dependent*.

   (c) Define what it means for $u \in V$ to be in the span of a set $\{v_1, \ldots, v_n\}$.

   Answer: $u \in \text{span}(v_1, \ldots, v_n)$ means that there exist numbers $\lambda_1, \ldots, \lambda_n$ so that

   $$u = \lambda_1 v_1 + \lambda_2 v_2 + \ldots + \lambda_n v_n.$$

   (d) Suppose $\{v_1, \ldots, v_n\}$ is a set of vectors and $u \in \text{span}(v_1, \ldots, v_n)$. Show that $\{v_1, \ldots, v_n, u\}$ is linearly dependent.

   Answer: Since $u \in \text{span}(v_1, \ldots, v_n)$, we know there exist numbers $\lambda_1, \ldots, \lambda_n$ so that

   $$u = \lambda_1 v_1 + \lambda_2 v_2 + \ldots + \lambda_n v_n.$$

   But then

   $$\lambda_1 v_1 + \lambda_2 v_2 + \ldots + \lambda_n v_n - u = u - u = 0$$

   is a nontrivial linear combination of the vectors $\{v_1, \ldots, v_n, u\}$ (since the coefficient of u is $-1 \neq 0$) that produces 0, so this set is linearly dependent.

4. Let A be a 2×3 matrix.

   (a) Let $U = \{\vec{x} \in \mathbb{R}^3 : A\vec{x} = \vec{0}\}$. Show that U is a subspace of $\mathbb{R}^3$.

   Proof. It suffices to show that (i) U is closed under addition and (ii) U is closed under scalar multiplication.

   i. Suppose $\vec{x}, \vec{y} \in U$. Then $A\vec{x} = \vec{0}$ and $A\vec{y} = \vec{0}$. But then

   $$A(\vec{x} + \vec{y}) = A\vec{x} + A\vec{y} = \vec{0} + \vec{0} = \vec{0},$$

   so $\vec{x} + \vec{y} \in U$ and U is closed under addition.

ii. Suppose $\vec{x} \in U$ and $\lambda \in \mathbb{R}$. Then $A\vec{x} = \vec{0}$ and so

$$A(\lambda\vec{x}) = \lambda A\vec{x} = \lambda\vec{0} = \vec{0},$$

so $\lambda\vec{x} \in U$ and U is closed under scalar multiplication.

Having proved (i) and (ii), we conclude that U is indeed a subspace of $\mathbb{R}^3$. □

(b) Is $W = \{\vec{x} \in \mathbb{R}^3 : A\vec{x} = \vec{b}\}$ a subspace when $\vec{b} = \begin{bmatrix} 1 \\ 2 \end{bmatrix}$? Explain.

Answer: No. In particular, $\vec{0} \notin W$ since

$$A\vec{0} = \vec{0} \neq \vec{b},$$

so W cannot be a subspace.

5. Let $V = P_\infty$ be the vector space of polynomials. Is the set

$$\{1 + x + x^2, 1 - x, 1 - x^3\}$$

linearly independent? Prove your claim.

Answer: This set is linearly independent. To see this, suppose $a, b, c \in \mathbb{R}$ are constants so that

$$a(1 + x + x^2) + b(1 - x) + c(1 - x^3) = 0.$$

Then, after distributing and combining terms, we see that

$$(a + b + c) + (a - b)x + ax^2 - cx^3 = 0 + 0x + 0x^2 + 0x^3.$$

Since the coefficients of x^2 and x^3 must be zero, we see that $a = 0$ and $c = 0$. But then the above reduces to

$$b - bx = 0 + 0x + 0x^2 + 0x^3,$$

so $b = 0$ as well. Therefore, the only way to write 0 as a linear combination of $1+x+x^2, 1-x$, and $1 - x^3$ is if all the coefficients are zero, which means the set is linearly independent.

6. Logan and Terry are both computing with the same 5×3 matrix. Logan determines that the nullspace of the matrix is 2-dimensional, while Terry computes that the column space is 2-dimensional. Can they both be right? Justify your answer.

Answer: No, they cannot both be correct. Since A has three columns, the rank-nullity theorem tells us that

$$\dim(\text{null}(A)) + \dim(\text{col}(A)) = 3.$$

There's no way that the two terms on the left hand side can both be 2.

7. For each of the following statements, say whether it is true or false. If the statement is true, prove it. If false, give a counterexample.

(a) If V is a vector space and S is a finite set of vectors in V, then some subset of S forms a basis for V.

Answer: False. Let $V = \mathbb{R}^2$, which is clearly a vector space, and let S be the singleton set $\{\begin{bmatrix} 1 \\ 0 \end{bmatrix}\}$. The single element of S does not span $\mathbb{R}^2$: since $\mathbb{R}^2$ is 2-dimensional, any spanning set must consist of at least two elements. Of course this means no subset of S can be a basis for $\mathbb{R}^2$. Hence, this provides a counterexample to the statement.

(b) Suppose A is an $m \times n$ matrix such that $A\vec{x} = \vec{b}$ can be solved for any choice of $\vec{b} \in \mathbb{R}^m$. Then the columns of A form a basis for $\mathbb{R}^m$.

Answer: False. Consider the matrix

$$A = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \end{bmatrix}.$$

Then A is already in reduced echelon form and clearly has 2 pivots, so $\operatorname{rank}(A) = 2$. This implies that $\dim \operatorname{col}(A) = 2$, so the column space of A consists of all of $\mathbb{R}^2$. Thus, the equation $A\vec{x} = \vec{b}$ can be solved for any $\vec{b} \in \mathbb{R}^2$ (since any $\vec{b}$ is in $\operatorname{col}(A)$). However, the columns of A are clearly not linearly independent (no set containing the zero vector can be linearly independent), so they cannot form a basis for $\mathbb{R}^2$.

A related but true statement would be the following: "Suppose A is an $m \times n$ matrix such that $A\vec{x} = \vec{b}$ can be solved for any choice of $\vec{b} \in \mathbb{R}^m$. Then *some subset of* the columns of A forms a basis for $\mathbb{R}^m$."

(c) The set of polynomials of degree ≤ 5 forms a vector space.

Answer: True. You should check that the set of polynomials of degree ≤ 5 satisfies all the rules for being a vector space. Since this is a subset of the collection of all polynomials (which we know is a vector space) all you really need to check is that this collection is closed under addition and scalar multiplication.

8. Consider the system of equations

$$\begin{array}{rcrcrcrcl} x_1 & + & 2x_2 & + & x_3 & - & 3x_4 & = & b_1 \\ x_1 & + & 2x_2 & + & 2x_3 & - & 5x_4 & = & b_2 \\ 2x_1 & + & 4x_2 & + & 3x_3 & - & 8x_4 & = & b_3 \end{array}.$$

(a) Find all solutions when $b_1 = b_2 = b_3 = 0$. Find a basis for the space of solutions to the homogeneous system.

Answer: Convert the system into the augmented matrix

$$\left[\begin{array}{cccc|c} 1 & 2 & 1 & -3 & 0 \\ 1 & 2 & 2 & -5 & 0 \\ 2 & 4 & 3 & -8 & 0 \end{array}\right].$$

Now do elimination to get the reduced echelon form. First, subtract row 1 from row 2 and subtract twice row 1 from row 3:

$$\left[\begin{array}{cccc|c} 1 & 2 & 1 & -3 & 0 \\ 0 & 0 & 1 & -2 & 0 \\ 0 & 0 & 1 & -2 & 0 \end{array}\right].$$

Now, subtract row 2 from both row 1 and row 3:

$$\left[\begin{array}{cccc|c} 1 & 2 & 0 & -1 & 0 \\ 0 & 0 & 1 & -2 & 0 \\ 0 & 0 & 0 & 0 & 0 \end{array}\right].$$

Then this system is consistent provided that

$$\begin{aligned} x_1 &= -2x_2 + x_4 \\ x_3 &= 2x_4. \end{aligned}$$

. Hence, the solutions to the homogeneous equation are those vectors of the form

$$x_2 \begin{bmatrix} -2 \\ 1 \\ 0 \\ 0 \end{bmatrix} + x_4 \begin{bmatrix} 1 \\ 0 \\ 2 \\ 1 \end{bmatrix}$$

for $x_2, x_4 \in \mathbb{R}$. Then a basis for the space of solutions to the homogeneous system (i.e. nullspace of the corresponding matrix) is

$$\left\{ \begin{bmatrix} -2 \\ 1 \\ 0 \\ 0 \end{bmatrix}, \begin{bmatrix} 1 \\ 0 \\ 2 \\ 1 \end{bmatrix} \right\}.$$

(b) Let S be the set of vectors $\vec{b} = \begin{bmatrix} b_1 \\ b_2 \\ b_3 \end{bmatrix}$ such that the system can be solved. What is the dimension of S?

Answer: Letting A be the matrix of the system, we know that the set of vectors $\vec{b}$ for which the system can be solved is the column space of A: $S = \text{col}(A)$. Since A is 3×4, we know that

$$\dim \text{col}(A) + \dim \text{null}(A) = 4.$$

Since, from part (a), we know that the dimension of the nullspace is 2, this implies that $S = \text{col}(A)$ is two-dimensional.

(c) It's easy to check that the vector $\vec{v} = \begin{bmatrix} 1 \\ 0 \\ 2 \\ 0 \end{bmatrix}$ is a solution to the system that arises when $b_1 = 3$, $b_2 = 5$, and $b_3 = 8$. Find *all* the solutions to this system.

Answer: All solutions $\vec{x}$ to the system $A\vec{x} = \vec{b}$ take the form $\vec{x} = \vec{x}_0 + \vec{x}_p$, where $\vec{x}_p$ is a particular solution and $\vec{x}_0$ is the homogeneous solution to the corresponding homogeneous problem. Thus, we can let $\vec{x}_p = \vec{v}$, which we're told solves the system and we see that, using part (a), the general solution is

$$\begin{bmatrix} 1 \\ 0 \\ 2 \\ 0 \end{bmatrix} + x_2 \begin{bmatrix} -2 \\ 1 \\ 0 \\ 0 \end{bmatrix} + x_4 \begin{bmatrix} 1 \\ 0 \\ 2 \\ 1 \end{bmatrix},$$

where $x_2, x_4 \in \mathbb{R}$.

9. Let $B = \{\begin{bmatrix} 2 \\ 1 \end{bmatrix}, \begin{bmatrix} 5 \\ 2 \end{bmatrix}\}$ and $B' = \{\begin{bmatrix} 1 \\ -2 \end{bmatrix}, \begin{bmatrix} -3 \\ 7 \end{bmatrix}\}$.

   (a) Show that B and B' are both bases for $\mathbb{R}^2$.

   Answer: If we form the matrices

$$A = \begin{bmatrix} 2 & 5 \\ 1 & 2 \end{bmatrix} \qquad \text{and} \qquad A' = \begin{bmatrix} 1 & -3 \\ -2 & 7 \end{bmatrix},$$

   then $\det(A) = -1$ and $\det(A') = 1$, so A and A' are both invertible. This means the columns of both matrices are bases for $\mathbb{R}^2$, but of course the columns of A are just the elements of B and the columns of A' are just the elements of B', so B and B' are bases.

   (b) Find the change-of-basis matrix $[M]_{B \to B'}$ for converting coordinate vectors with respect to B to coordinate vectors with respect to B'.

   Answer: We can follow the procedure for finding the change-of-basis matrix by forming the super-augmented matrix with the entries of B' on the left and the entries of B on the right.

$$\left[\begin{array}{cc|cc} 1 & -3 & 2 & 5 \\ -2 & 7 & 1 & 2 \end{array}\right]$$

   Now we row-reduce: add 2 times row 1 to row 2:

$$\left[\begin{array}{cc|cc} 1 & -3 & 2 & 5 \\ 0 & 1 & 5 & 12 \end{array}\right]$$

   Then add 3 times row 2 to row 1:

$$\left[\begin{array}{cc|cc} 1 & 0 & 17 & 41 \\ 0 & 1 & 5 & 12 \end{array}\right]$$

   Therefore, the change-of-basis matrix is

$$[M]_{B \to B'} = \begin{bmatrix} 17 & 41 \\ 5 & 12 \end{bmatrix}$$

   (c) Let $\vec{v} = \begin{bmatrix} 2 \\ 1 \end{bmatrix}$. What is the coordinate vector $[\vec{v}]_B$ for $\vec{v}$ with respect to the basis B?

   Answer: Notice that $\vec{v}$ is just the first basis vector in B, so

$$\vec{v} = 1 \begin{bmatrix} 2 \\ 1 \end{bmatrix} + 0 \begin{bmatrix} 5 \\ 2 \end{bmatrix},$$

   so the coordinate vector for $\vec{v}$ with respect to the basis B is simply $[\vec{v}]_B = \begin{bmatrix} 1 \\ 0 \end{bmatrix}$.

   (d) Use your answer to part (b) to determine $[\vec{v}]_{B'}$, the coordinate vector for B with respect to B'.

Answer: By definition of the change-of-basis matrix,

$$[\vec{v}]_{B'} = [M]_{B\to B'}[\vec{v}]_B = \begin{bmatrix} 17 & 41 \\ 5 & 12 \end{bmatrix} \begin{bmatrix} 1 \\ 0 \end{bmatrix} = \begin{bmatrix} 17 \\ 5 \end{bmatrix}.$$

We can check this:

$$17 \begin{bmatrix} 1 \\ -2 \end{bmatrix} + 5 \begin{bmatrix} -3 \\ 7 \end{bmatrix} = \begin{bmatrix} 2 \\ 1 \end{bmatrix},$$

which is indeed our vector $\vec{v}$.