20CYS111 Digital Signal Processing

Properties of Systems

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Memoryless Systems

A system is said to be ***memoryless*** *if the value of its output signal at any time* t_0 *may depend only on the value of the input signal at the same time* t_0.

In particular, for a memoryless system, the value of the output signal at any time t_0 *must be independent of the value of the input signal at times* $t \neq t_0$.

Examples:

- ***Resistance:***
$$v(t) = Ri(t).$$
- ***Amplifier / Attenuator:***
$$v_{out}(t) = cv_{in}(t).$$
- ***Squaring Circuit:***
$$y(t) = cx^2(t).$$

Systems with Memory

*A system is said to **possess memory** if it is not memoryless.*

The temporal extent of the past and/or future values on which the current output depends, defines the memory of the system.

***Examples:** A system with memory may have either **finite** or **infinite** memory.*

- ***Capacitance:***
$$v(t) = \frac{1}{C}\int_{-\infty}^{t} i(t')dt'.$$
- ***Inductance:***
$$i(t) = \frac{1}{L}\int_{-\infty}^{t} v(t')dt'.$$
- ***Moving Average (MA):***
$$y[n] = \frac{1}{3}(x[n] + x[n-1] + x[n-2])\,.$$

Causal Systems

A system is said to be ***causal*** *if the value of its output signal at any time t_0 may depend only on the value of the input signal at times $t \leq t_0$.*

In particular, for a causal system, the value of its output signal at any time t_0 must be independent of the value of the input signal at times $t > t_0$.

Examples: A memoryless system is causal, but the converse is not true*, e.g.,*

- *The Moving Average (MA) system defined below is causal but not memoryless:*

$$y[n] = \frac{1}{3}(x[n] + x[n-1] + x[n-2]) .$$

Noncausal or Anticipative Systems

*A system is said to be **noncausal** or **anticipative** if it is not causal.*

In particular, the output of a noncausal system depends on one or more future values of the input signal.

- *So, **noncausal systems cannot operate in real-time**, and **they are not physically realizable** if the independent variable indeed represents time.*

***Examples:** An alternative Moving Average (MA) system defined below is noncausal:*

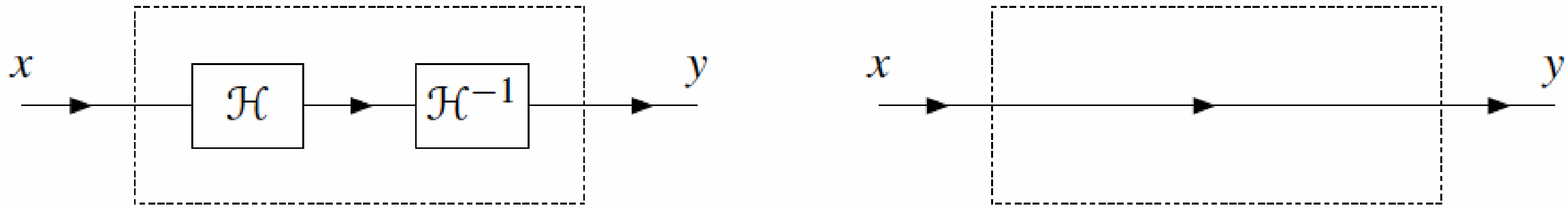
$$y[n] = \frac{1}{3}(x[n-1] + x[n] + x[n+1]) .$$

Invertible Systems

*The **inverse** of a system $\mathcal{H}$ is another system $\mathcal{H}^{-1}$ (if it exists) such that, for every input $x(t)$, we have*

$$\boxed{\mathcal{H}^{-1}\{\mathcal{H}\{x(t)\}\} = x(t)},$$

that is, the system formed by the cascade interconnection of $\mathcal{H}$ followed by $\mathcal{H}^{-1}$ is a system whose output is always equal to its input.



System 1: $y = \mathcal{H}^{-1}\mathcal{H}x$

System 2: $y = x$

Invertible Systems

*A system $\mathcal{H}$ is said to be **invertible** if it has a corresponding inverse system $\mathcal{H}^{-1}$ (i.e., if its inverse exists).*

- *For an invertible system, we have $\boxed{\mathcal{H}^{-1}\mathcal{H} = \mathcal{I}}$, where $\mathcal{I}$ denotes the **identity** operator, i.e., $\boxed{\mathcal{I}\{x(t)\} = x(t)}$ for any signal $x(t)$.*

*Equivalently, a system is invertible if its input can always be **uniquely** determined from its output. In other words,*

- *For a system to be invertible, there must be a **one-to-one mapping** between input and output signals.*
- ***$\Rightarrow$ For an invertible system, distinct inputs must lead to distinct outputs**.*

Invertible Systems

Examples: *of invertible systems and their inverse systems are as follows:*

- ***An amplifier is the inverse system of an attenuator*** *and vice versa, if their amplitude scaling factors are chosen to be the inverse of one another.*
- ***An integrator is the inverse system of a differentiator*** *and vice versa.*
- ***Channel Equalizer:*** *undoes the impairments caused by a channel.*

Invertible systems are "nice" in the sense that their effects can be undone.

To show that a system is invertible, we simply have to find the inverse system.

Non-invertible Systems

To show that a system is not invertible, we find two distinct inputs that result in identical outputs.

Examples: *of non-invertible systems:*

- ***A squaring circuit:*** $y(t) = x^2(t)$.
- ***Sinusoid:*** $y(t) = sin(x(t))$

Stability of Systems

*A continuous-time real- or complex-valued signal $z(t)$ is said to be **bounded** if*

$$|z(t)| \leq M_z < \infty, \quad \text{for all } t,$$

*where M_z represents a **finite** and **positive** number that may depend on $z(t)$.*

*A system is said to be **bounded-input bounded-output (BIBO) stable** if every bounded input results in a bounded output, that is:*

- *For every input signal $x(t)$, satisfying the condition $|x(t)| \leq M_x < \infty, \ \forall t$, the output signal $y(t)$ also satisfies the condition $|y(t)| \leq M_y < \infty, \ \forall t$.*

Similar definitions hold for discrete-time signals and systems.

Stability of Systems

The output of a BIBO stable system does not diverge if the input does not diverge.

- *From an engineering perspective, it is an important property because the system of interest remain stable under all finite input operating conditions.*

Example of an Unstable System: *Consider the discrete-time system whose input-output relation is defined by $y[n] = r^n x[n]$, where $r > 1$. Assuming that $|x[n]| \leq M_x < \infty, \ \forall n$, we have*

$$\boxed{|y[n]| = |r^n x[n]| = |r^n||x[n]| \leq |r^n| M_x},$$

where, with $r > 1$, the factor r^n diverges for increasing n, and the system is unstable. Therefore, **the input being bounded is not sufficient to guarantee that the output will be bounded**.

Stability of Systems

Example of a BIBO Stable System: *Consider the MA system given by the input-output relation* $y[n] = \frac{1}{3}(x[n] + x[n-1] + x[n-2])$. *Assuming that* $|x[n]| \leq M_x < \infty, \ \forall n$, *we have*

$$\boxed{\begin{aligned} |y[n]| &= \left|\frac{1}{3}(x[n] + x[n-1] + x[n-2])\right| \\ &\leq \frac{1}{3}(|x[n]| + |x[n-1]| + |x[n-2]|) \\ &\leq \frac{1}{3}(M_x + M_x + M_x) = M_x. \end{aligned}}$$

Therefore, the system is BIBO stable.

Time-Invariant Systems

*A system is said to be **time-invariant** if a time shift of the input signal by t_0 units results only in an identical time shift of t_0 units of the output signal.*

- *For a time-invariant system $\mathcal{H}$, we have*

$$\boxed{\text{If} \quad \mathcal{H}\{x(t)\} = y(t), \quad \text{then} \quad \mathcal{H}\{x(t-t_0)\} = y(t-t_0)}.$$

- *This implies that **the system responds in the same way no matter when the input signal is applied**.*
- *In other words, **the characteristics of the system does not change with time**.*

Time-Invariant Systems

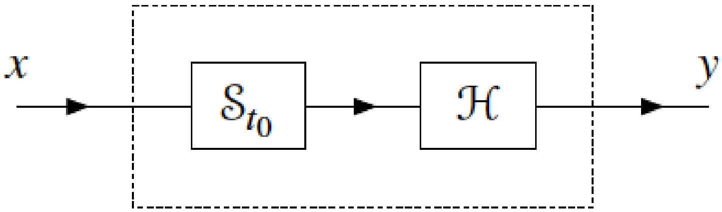
For any signal $z(t)$*, we have* $\boxed{z(t - t_0) = S_{t_0}\{z(t)\}}$,

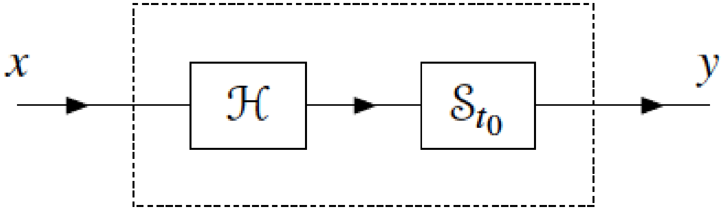
where S_{t_0} *represents a system operator that shifts the input signal by* t_0 *units.*

- *For a time-invariant system* $\mathcal{H}$*, we have*
  $\mathcal{H}\{x(t - t_0)\} = y(t - t_0)$

$$\boxed{\Rightarrow \mathcal{H}\{S_{t_0}\{x(t)\}\} = S_{t_0}\{\mathcal{H}\{x(t)\}\}},$$

i.e., ***the system operator*** $\mathcal{H}$ ***of a time-invariant system commutes with a time shift operator*** S_{t_0} ***for all*** t_0.





Time-Invariant Systems

Examples: *of time-invariant and time-variant systems are as follows:*

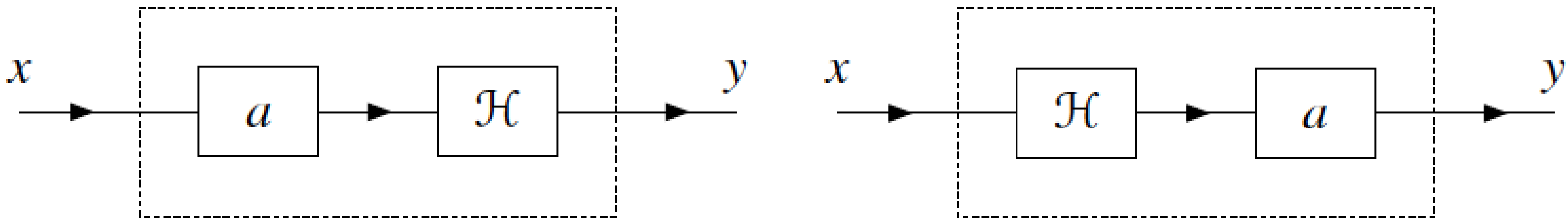
- ***Time Shifter:*** $y(t) = x(t-b)$. $\longleftarrow$ ***time-invariant***
- ***Thermister:*** $i(t) = \frac{v(t)}{R(t)}$ $\longleftarrow$ ***time-variant***
- ***Inductor:*** $i_L(t) = \frac{1}{L}\int_{-\infty}^{t} v_L(t')dt'$. $\longleftarrow$ ***time-invariant***
- ***Discrete-Time Exponential:*** $y[n] = r^n x[n]$ $\longleftarrow$ ***time-variant***

Homogeneity, Additivity and Linearity

A system $\mathcal{H}$ is said to be ***homogeneous*** *if, for any input signal $x(t)$, we have*

$$\boxed{\mathcal{H}\{ax(t)\} = a\mathcal{H}\{x(t)\}},$$

i.e., ***the system operator $\mathcal{H}$ of a homogeneous system commutes with a scalar multiplication (or amplitude scaling) operator a for all a.***

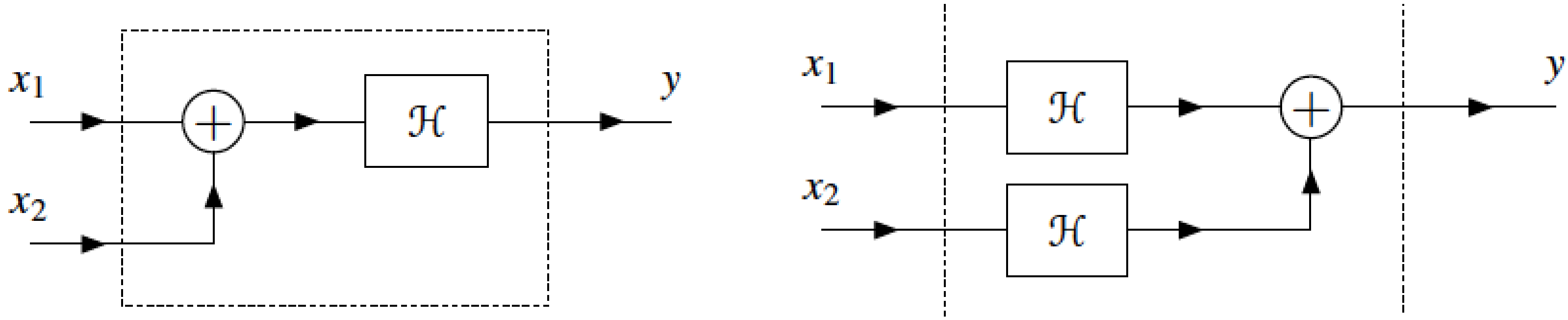


Homogeneity, Additivity and Linearity

A system $\mathcal{H}$ is said to be ***additive*** *(or to be satisfying the* ***principle of superposition****) if, for any input signals $x_1(t)$ and $x_2(t)$, we have*

$$\boxed{\mathcal{H}\{x_1(t) + x_2(t)\} = \mathcal{H}\{x_1(t)\} + \mathcal{H}\{x_2(t)\}},$$

i.e., ***the system operator $\mathcal{H}$ of a homogeneous system commutes with the addition operator****.*

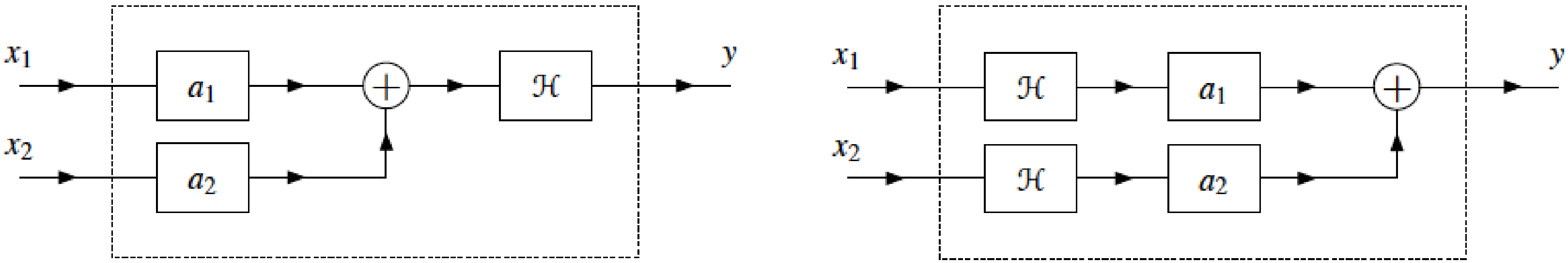


Homogeneity, Additivity and Linearity

A system $\mathcal{H}$ is said to be ***linear*** *if it is both homogeneous and additive, i.e., for any input signals $x_1(t)$ and $x_2(t)$ and any scalars a_1 and a_2, we have*

$$\boxed{\mathcal{H}\{a_1 x_1(t) + a_2 x_2(t)\} = a_1 \mathcal{H}\{x_1(t)\} + a_2 \mathcal{H}\{x_2(t)\}},$$

i.e., ***the system operator $\mathcal{H}$ of a homogeneous system commutes with linear combinations****.*



Homogeneity, Additivity and Linearity

Examples: *of linear and nonlinear systems:*

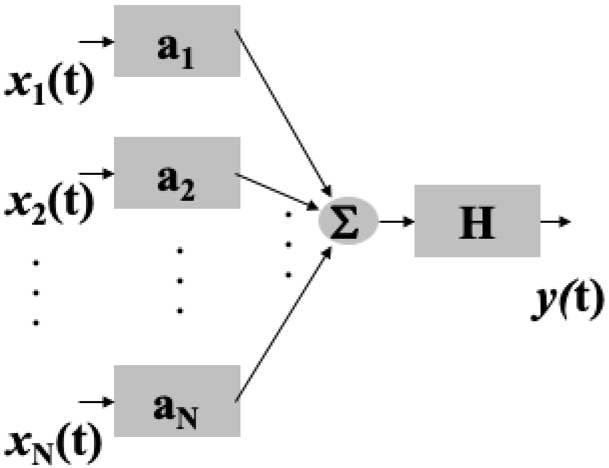
- ***Squaring Circuit:*** $y(t) = x^2(t)$.
- ***Multiplication by a Ramp:*** $y[n] = r[n]x[n]$

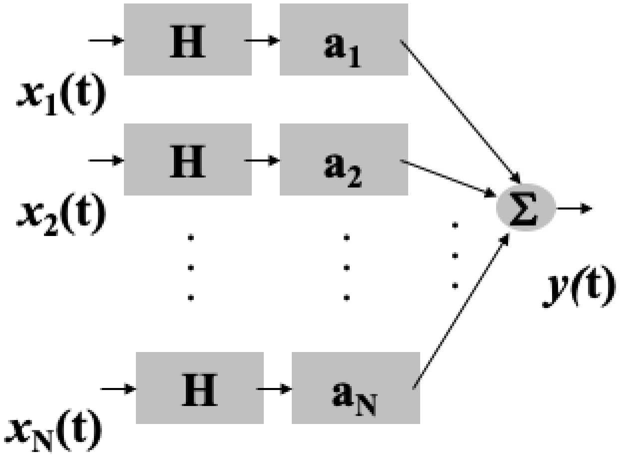
$$= nx[n]u[n]$$

Homogeneity, Additivity and Linearity

In general, if an input $x(t) = \sum_{i=1}^{N} a_i x_i(t)$ *is applied to a linear system, then the output* $y(t)$ *is given by*

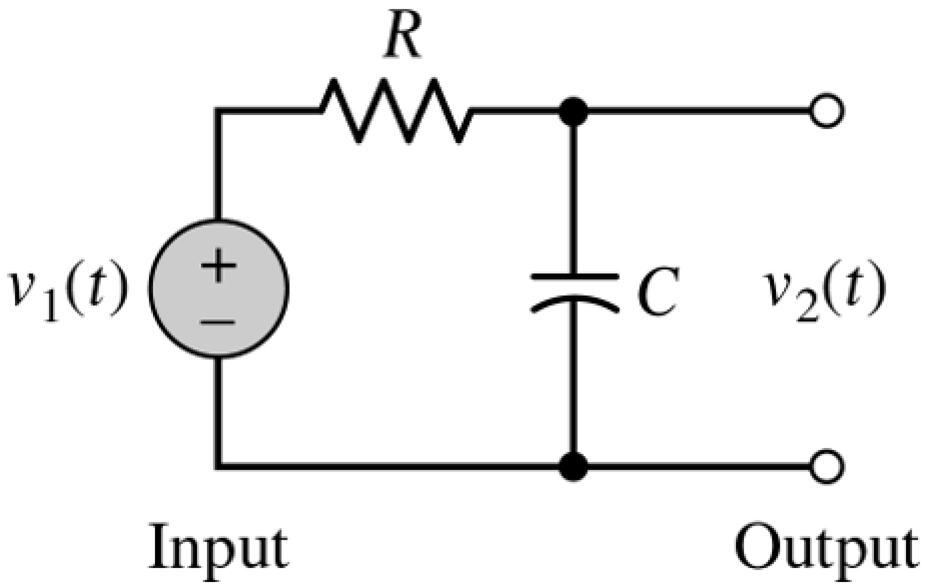
$$\boxed{y(t) = \mathcal{H}\{x(t)\} = \mathcal{H}\{\sum_{i=1}^{N} a_i x_i(t)\} = \sum_{i=1}^{N} a_i \mathcal{H}\{x_i(t)\}}.$$





Theme Examples

RC Circuit

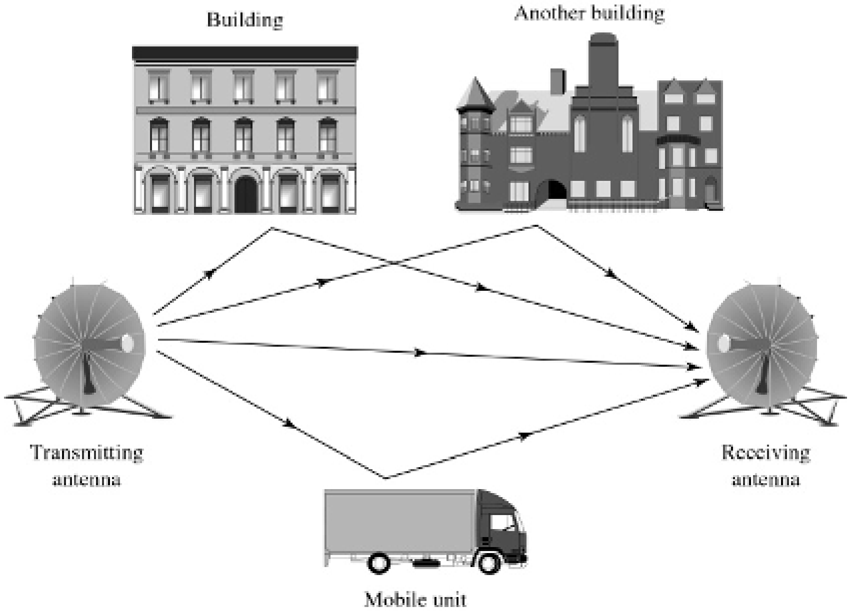


RC Circuit

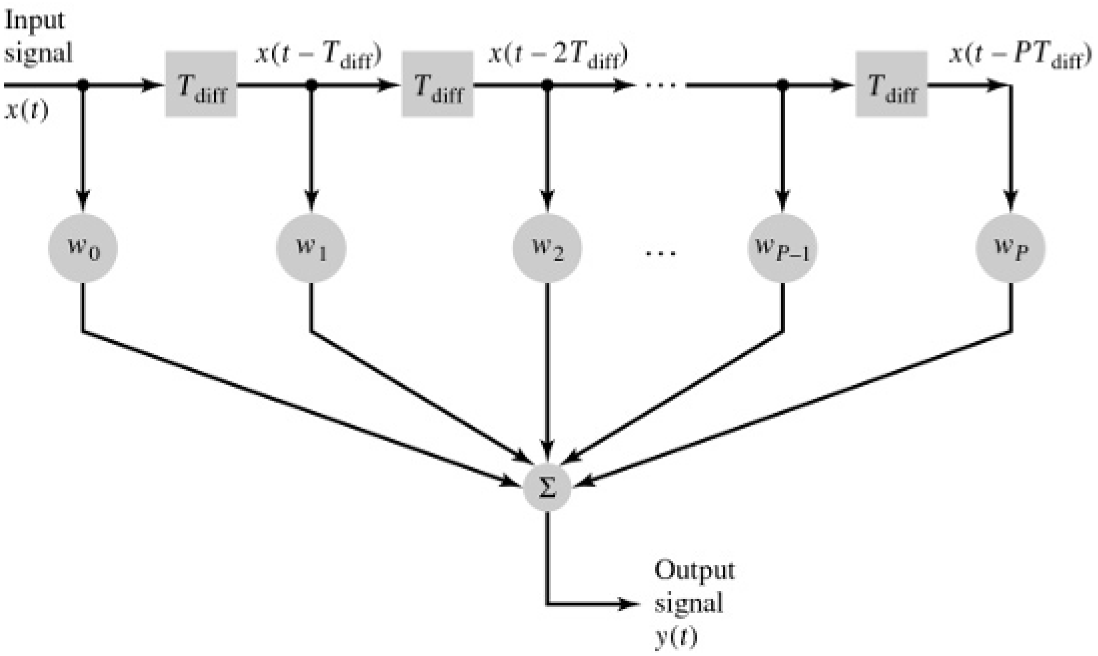
Applying KVL, we get $v_1(t) = Ri(t) + v_2(t) = Ri(t) + \frac{1}{C}\int_{-\infty}^{t} i(t')dt'$

$$\Rightarrow \frac{d}{dt}v_1(t) = R\frac{d}{dt}i(t) + \frac{1}{C}i(t)$$
$$\Rightarrow \frac{d}{dt}i(t) + \frac{1}{RC}i(t) = \frac{1}{R}\frac{d}{dt}v_1(t)$$

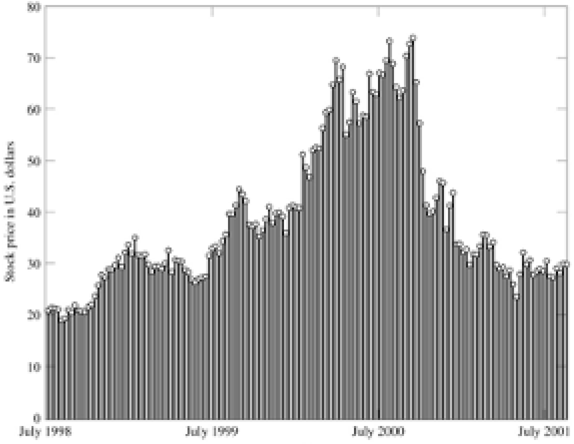
Wireless Channel



Wireless Channel



(Smoothening) of Stock Price



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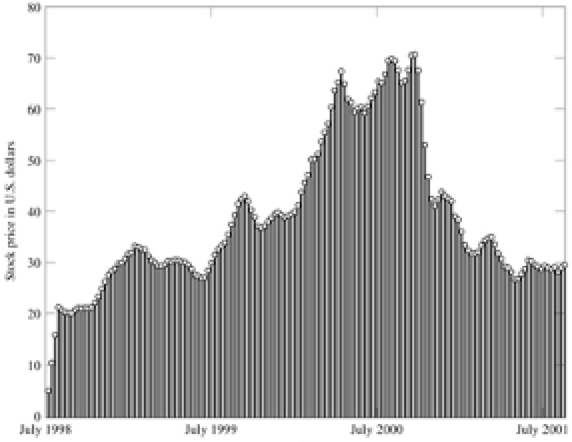

(b)

References:

[1] Simon Haykin and Barry Van Veen, Signals and Systems, Second Edition, John Wiley and Sons, 2003.

[2] Lecture Notes by Michael D. Adams.
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