Example

What is the least number of area codes needed to guarantee that the 25 million phones in a state can be assigned distinct 10-digit telephone numbers? (Assume that telephone numbers are of the form NXX-NXX-$XXXX$, where the first three digits form the area code, N represents a digit from 2 to 9 inclusive, and X represents any digit.)

solution

There are eight million different phone numbers of the form NXX-XXXX

Hence, by the generalized pigeonhole principle, among 25 million telephones, at least

$$\lceil \frac{25000000}{8000000} \rceil = 4$$

Counting using Permutations and Combinations

Many counting problems can be solved by

- Finding the number of ways to arrange a specified number of distinct elements of a set of a particular size, where the order of these elements matters.
- Finding the number of ways to select a particular number of elements from a set of a particular size, where the order of the elements selected does not matter.

Problem 1 (Permutations)

In how many ways can we select three students from a group of five students to stand in line for a picture? In how many ways can we arrange all five of these students in a line for a picture?

Here the order in which we select the students matters.

A **permutation** of a set of distinct objects is an ordered arrangement of these objects.

We also are interested in ordered arrangements of some of the elements of a set. An ordered arrangement of r elements of a set is called an r-permutation.

Problem 2 (Permutations)

Let $S = \{1, 2, 3\}$. The ordered arrangement 3, 1, 2 is a permutation of S. The ordered arrangement 3, 2 is a 2-permutation of S.

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The number of r-permutations of a set with n elements is denoted by $P(n, r)$. We can find $P(n, r)$ using the product rule.

Example

Let $S = \{a, b, c\}$. The 2-permutations of S are the ordered arrangements a, b; a, c; b, a; b, c; c, a; and c, b. Consequently, there are six 2-permutations of this set with three elements. There are always six 2-permutations of a set with three elements. There are three ways to choose the first element of the arrangement. There are two ways to choose the second element of the arrangement, because it must be different from the first element. Hence, by the product rule, we see that $P(3, 2) = 3 \cdot 2 = 6$. the first element. By the product rule, it follows that $P(3, 2) = 3 \cdot 2 = 6$. ◀

Permutations defined

If n is a positive integer and r is an integer with $1 \leq r \leq n$, then there are

$$P(n, r) = n(n-1)(n-2)\cdots(n-r+1)$$

r-permutations of a set with n distinct elements.

If n and r are integers with $0 \leq r \leq n$, then $P(n, r) = \dfrac{n!}{(n-r)!}$.

Problem 3

How many ways are there to select a first-prize winner, a second-prize winner, and a third-prize winner from 100 different people who have entered a contest?

Problem 4

Suppose that there are eight runners in a race. The winner receives a gold medal, the second place finisher receives a silver medal, and the third-place finisher receives a bronze medal. How many different ways are there to award these medals, if all possible outcomes of the race can occur and there are no ties?

Problem 5

Suppose that a saleswoman has to visit eight different cities. She must begin her trip in a specified city, but she can visit the other seven cities in any order she wishes. How many possible orders can the saleswoman use when visiting these cities?

Problem 6

How many permutations of the letters ABCDEFGH contain the string ABC ?

Combinations - unordered selections of objects.

Problem 7

How many different committees of three students can be formed from a group of four students?

we need only find the number of subsets with three elements from the set containing the four students.

An r-combination of elements of a set is an unordered selection of r elements from the set. Thus, an r-combination is simply a subset of the set with r elements.

The number of r-combinations of a set with n distinct elements is denoted by C(n, r). Note that C(n, r) is also denoted by

Problem 8

Let S be the set {1, 2, 3, 4}. Then {1, 3, 4} is a 3-combination from S. (Note that {4, 1, 3} is the same 3-combination as {1, 3, 4}, because the order in which the elements of a set are listed does not matter.)

Problem 9

We see that C(4, 2) = 6, because the 2-combinations of {a, b, c, d} are the six subsets {a, b}, {a, c}, {a, d}, {b, c}, {b, d}, and {c, d}.

The number of r-combinations of a set with n elements, where n is a nonnegative integer and r is an integer with $0 \leq r \leq n$, equals

$$C(n, r) = \frac{n!}{r!\,(n-r)!}.$$

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The *P(n, r) r*-permutations of the set can be obtained by forming the *C(n, r) r*-combinations of the set, and then ordering the elements in each *r*-combination

by the product rule $P(n, r) = C(n, r) \cdot P(r, r)$.

Problem 8

How many poker hands of five cards can be dealt from a standard deck of 52 cards? Also, how many ways are there to select 47 cards from a standard deck of 52 cards?

Let n and r be nonnegative integers with $r \leq n$. Then $C(n, r) = C(n, n-r)$.

Few more problems

1. How many ways are there to select five players from a 10-member tennis team to make a trip to a match at another school?

2. A group of 30 people have been trained as astronauts to go on the first mission to Mars. How many ways are there to select a crew of six people to go on this mission (assuming that all crew members have the same job)?

3. How many bit strings of length n contain exactly r 1s?

Problem

Suppose that there are 9 faculty members in the mathematics department and 11 in the computer science department. How many ways are there to select a committee to develop a discrete mathematics course at a school if the committee is to consist of three faculty members from the mathematics department and four from the computer science department?