Course objectives

We will focus on two major goals:

- Basic tools and techniques in discrete mathematics
  - Propositional logic
  - Set Theory
  - Simple algorithms
  - Induction, recursion
  - Counting techniques (Combinatorics)
- Precise and rigorous mathematical reasoning
  - Writing proofs

To do well you should:

- Study with pen and paper
- Ask for help immediately
- Practice, practice, practice…
- Follow along in class rather than take notes
- Ask questions in class
- Keep up with the class
- Read the book, not just the slides

Reasoning about problems

- 0.9999999999999999….=1?
- There exists integers a,b,c that satisfy the equation $a^2+b^2 = c^2$
- The program below that I wrote works correctly for all possible inputs…..
- The program that I wrote never hangs (i.e. always terminates)…

Tools for reasoning: Logic

Ch. 1: Introduction to Propositional Logic

- Truth values, truth tables
- Boolean logic: $\vee \quad \wedge \quad \neg$
- Implications: $\rightarrow \quad \leftrightarrow$

Why study propositional logic?

- A formal mathematical “language” for precise reasoning.
- Start with propositions.
- Add other constructs like negation, conjunction, disjunction, implication etc.
- All of these are based on ideas we use daily to reason about things.

Propositions

- Declarative sentence
- Must be either True or False.

Propositions:

- York University is in Toronto
- York University is in downtown Toronto
- All students at York are Computer Sc. majors.

Not propositions:

- Do you like this class?
- There are x students in this class.

Propositions - 2

- Truth value: True or False
- Variables: p,q,r,s,…
- Negation:
- $\neg p$ (“not p”)
- Truth tables

p	$\neg p$
T	F
F	T

Caveat: negating propositions

$\neg$p: “it is not the case that p is true”

p: “it rained more than 20 inches in TO”

p: “John has many iPads”

Practice: Questions 1-7 page 12.

Q10 (a) p: “the election is decided”

Conjunction, Disjunction

- Conjunction: $p \wedge q$ [“and”]
- Disjunction: $p \vee q$ [“or”]

p	q	$p \wedge q$	$p \vee q$
T	T	T	T
T	F	F	T
F	T	F	T
F	F	F	F

Examples

- Q11, page 13

p: It is below freezing

q: It is snowing

(a) It is below freezing and snowing

(b) It is below freezing but now snowing

(d) It is either snowing or below freezing (or both)

Exclusive OR (XOR)

- $p \oplus q$ – T if p and q have different truth values, F otherwise
- Colloquially, we often use OR ambiguously – “an entrée comes with soup or salad” implies XOR, but “students can take MATH XXXX if they have taken MATH 2320 or MATH 1019” usually means the normal OR (so a student who has taken both is still eligible for MATH XXXX).

Conditional

- $p \rightarrow q$ ["if p then q"]
- p: *hypothesis*, q: *conclusion*
- E.g.: "If you turn in a homework late, it will not be graded"; "If you get 100% in this course, you will get an A+".
- <u>TRICKY</u>: Is $p \rightarrow q$ TRUE if p is FALSE?

YES!!

- Think of "If you get 100% in this course, you will get an A+" as a promise – is the promise violated if someone gets 50% and does not receive an A+?

Conditional - 2

- $p \to q$ ["if p then q"]
- Truth table:

p	q	$p \to q$	$\neg p \lor q$
T	T	T	T
T	F	F	F
F	T	T	T
F	F	T	T

Note the truth table of $\neg p \lor q$

Logical Equivalence

- $p \rightarrow q$ and $\neg p \vee q$ are logically equivalent
- Truth tables are the simplest way to prove such facts.
- We will learn other ways later.

Contrapositive

- Contrapositive of $p \rightarrow q$ is $\neg q \rightarrow \neg p$
- Any conditional and its contrapositive are logically equivalent (have the same truth table) – Check by writing down the truth table.
- E.g. The contrapositive of "If you get 100% in this course, you will get an A+" is "If you do not get an A+ in this course, you did not get 100%".

E.g.: Proof using contrapositive

Prove: If x^2 is even, x is even

- Proof 1: $x^2 = 2a$ for some integer a. Since 2 is prime, 2 must divide x.
- Proof 2: if x is not even, x is odd. Therefore x^2 is odd. This is the contrapositive of the original assertion.

Converse

- Converse of $p \rightarrow q$ is $q \rightarrow p$
- Not logically equivalent to conditional
- Ex 1: “If you get 100% in this course, you will get an A+” and “If you get an A+ in this course, you scored 100%” are not equivalent.
- Ex 2: If you won the lottery, you are rich.

Other conditionals

Inverse:

- inverse of $p \rightarrow q$ is $\neg p \rightarrow \neg q$
- How is this related to the converse?

Biconditional:

- "If and only if"
- True if p,q have same truth values, false otherwise. Q: How is this related to XOR?
- Can also be defined as $(p \rightarrow q) \wedge (q \rightarrow p)$

Example

- Q16(c) 1+1=3 if and only if monkeys can fly.

Readings and notes

- Read pages 1-12.
- Think about the notion of truth tables.
- Master the rationale behind the definition of conditionals.
- Practice translating English sentences to propositional logic statements.

Next

Ch. 1.2, 1.3: Propositional Logic - contd

– Compound propositions, precedence rules

– Tautologies and logical equivalences

– Read only the first section called "Translating English Sentences" in 1.2.

Compound Propositions

- Example: $p \wedge q \vee r$: Could be interpreted as $(p \wedge q) \vee r$ or $p \wedge (q \vee r)$
- precedence order: $\neg \;\wedge\; \vee \;\rightarrow\; \leftrightarrow$ (IMP!) (Overruled by brackets)
- We use this order to compute truth values of compound propositions.

Tautology

- A compound proposition that is always TRUE, e.g. $q \vee \neg q$
- <u>Logical equivalence redefined</u>: p,q are logical equivalences if $p \leftrightarrow q$ is a tautology. Symbolically $p \equiv q$.
- Intuition: $p \leftrightarrow q$ is true precisely when p,q have the same truth values.

Manipulating Propositions

- Compound propositions can be simplified by using simple rules.
- Read page 25 - 28.
- Some are obvious, e.g. Identity, Domination, Idempotence, double negation, commutativity, associativity
- Less obvious: Distributive, De Morgan's laws, Absorption

Distributive Laws

$$p \wedge (q \vee r) \equiv (p \wedge q) \vee (p \wedge r)$$

Intuition (not a proof!) – For the LHS to be true: p must be true and q or r must be true. This is the same as saying p and q must be true or p and r must be true.

$$p \vee (q \wedge r) \equiv (p \vee q) \wedge (p \vee r)$$

Intuition (less obvious) – For the LHS to be true: p must be true or both q and r must be true. This is the same as saying p or q must be true and p or r must be true.

Proof: use truth tables.

De Morgan's Laws

$\neg(q \vee r) \equiv \neg q \wedge \neg r$

Intuition – For the LHS to be true: neither q nor r can be true. This is the same as saying q and r must be false.

$\neg(q \wedge r) \equiv \neg q \vee \neg r$

Intuition – For the LHS to be true: $q \wedge r$ must be false. This is the same as saying q or r must be false.

Proof: use truth tables.

Using the laws

- Q: Is $p \rightarrow (p \rightarrow q)$ a tautology?
- Can use truth tables
- Can write a compound proposition and simplify

Limitations of Propositional Logic

- What can we NOT express using predicates?

  Ex: How do you make a statement about all even integers?

  If $x > 2$ then $x^2 > 4$

- A more general language: Predicate logic (Sec 1.4)

Next: Predicate Logic

Ch 1.4

- Predicates and quantifiers
- Rules of Inference

Predicate Logic

- A predicate is a proposition that is a function of one or more variables.

  E.g.: P(x): x is an even number. So P(1) is false, P(2) is true,….
- Examples of predicates:
  - Domain ASCII characters - IsAlpha(x) : TRUE iff x is an alphabetical character.
  - Domain floating point numbers - IsInt(x): TRUE iff x is an integer.
  - Domain integers: Prime(x) - TRUE if x is prime, FALSE otherwise.

Quantifiers

- describes the values of a variable that make the predicate true. E.g. $\exists x\ P(x)$
- Domain or universe: range of values of a variable (sometimes implicit)

Two Popular Quantifiers

- Universal: $\forall x\, P(x)$ – "P(x) for all x in the domain"
- Existential: $\exists x\, P(x)$ – "P(x) for some x in the domain" or "there exists x such that P(x) is TRUE".
- Either is meaningless if the domain is not known/specified.
- Examples (domain real numbers)
  - $\forall x\, (x^2 >= 0)$
  - $\exists x\, (x > 1)$
  - $(\forall x > 1)\, (x^2 > x)$ – quantifier with restricted domain

Using Quantifiers

Domain integers:

- Using implications: The cube of all negative integers is negative.
  $\forall x\ (x < 0) \rightarrow (x^3 < 0)$
- Expressing sums :

$$\forall n\ \left(\sum_{i=1}^{n} i = n(n+1)/2\right)$$

Aside: summation notation

Scope of Quantifiers

- $\forall$ $\exists$ have higher precedence than operators from Propositional Logic; so $\forall x\ P(x) \vee Q(x)$ is not logically equivalent to $\forall x\ (P(x) \vee Q(x))$
- $\exists x\ (P(x) \wedge Q(x)) \vee \forall x\ R(x)$

  Say P(x): x is odd, Q(x): x is divisible by 3, R(x): $(x=0) \vee (2x > x)$
- Logical Equivalence: $P \equiv Q$ iff they have same truth value no matter which domain is used and no matter which predicates are assigned to predicate variables.

Negation of Quantifiers

- “There is no student who can …”
- “Not all professors are bad….”
- “There is no Toronto Raptor that can dunk like Vince …”
- $\neg \forall x\, P(x) \equiv \exists x\, \neg P(x)$ why?
- $\neg \exists x\, P(x) \equiv \forall x\, \neg P(x)$
- Careful: The negation of “Every Canadian loves Hockey” is NOT “No Canadian loves Hockey”! <u>Many, many students make this mistake!</u>

Nested Quantifiers

- Allows simultaneous quantification of many variables.
- E.g. – domain integers,

  $\exists x \exists y \exists z \; x^2 + y^2 = z^2$
- $\forall n \exists x \exists y \exists z \; x^n + y^n = z^n$ (Fermat's Last Theorem)
- Domain real numbers:

  $\forall x \forall y \exists z \; (x < z < y) \vee (y < z < x)$

Is this true?

Nested Quantifiers - 2

$\forall x\, \exists y\, (x + y = 0)$ is true over the integers

- Assume an arbitrary integer x.
- To show that there exists a y that satisfies the requirement of the predicate, choose y = -x. Clearly y is an integer, and thus is in the domain.
- So $x + y = x + (-x) = x - x = 0$.
- Since we assumed nothing about x (other than it is an integer), the argument holds for any integer x.
- Therefore, the predicate is TRUE.

Nested Quantifiers - 3

- Caveat: In general, order matters! Consider the following propositions over the integer domain:

  $\forall x\ \exists y\ (x < y)$ and $\exists y\ \forall x\ (x < y)$
- $\forall x\ \exists y\ (x < y)$: "there is no maximum integer"
- $\exists y\ \forall x\ (x < y)$: "there is a maximum integer"
- Not the same meaning at all!!!