20CYS111 Digital Signal Processing

Differential and Difference Equation Representations of LTI Systems and Solutions

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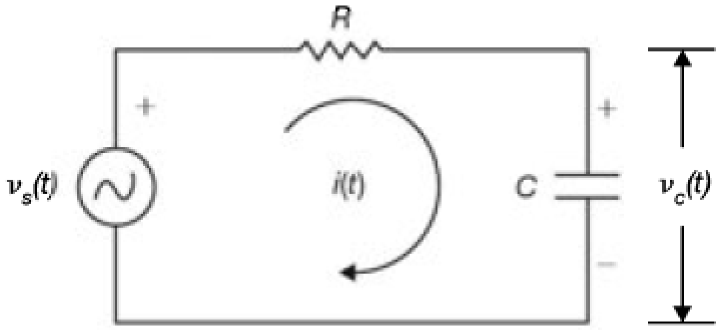
Differential and Difference Equation Representations of LTI Systems

Example: RC Circuit

Find the relationship between the input $x(t)$ and output $y(t)$:

- *(a) If the input is $x(t) = v_s(t)$ and the output is $y(t) = v_c(t)$.*
- *(b) If the input is $x(t) = v_s(t)$ and the output is $y(t) = i(t)$.*



Example: RC Circuit

(a) Applying KVL, we get

$$\begin{aligned} v_s(t) &= Ri(t) + v_c(t) = RC\frac{dv_c(t)}{dt} + v_c(t) \\ \Rightarrow \quad & \frac{dv_c(t)}{dt} + \frac{1}{RC}v_c(t) = \frac{1}{RC}v_s(t) \\ \Rightarrow \quad & \frac{dy(t)}{dt} + \frac{1}{RC}y(t) = \frac{1}{RC}x(t) \end{aligned}$$

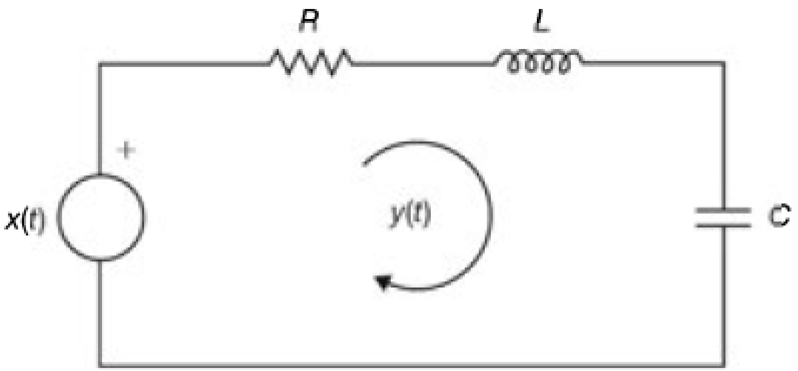
Example: RC Circuit

(b) Applying KVL, we get

$$
\begin{aligned}
v_s(t) &= Ri(t) + v_c(t) = Ri(t) + \frac{1}{C}\int_{-\infty}^{t} i(t')dt' \\
\Rightarrow \quad & R\frac{di(t)}{dt} + \frac{1}{C}i(t) = \frac{dv_s(t)}{dt} \\
\Rightarrow \quad & \frac{dy(t)}{dt} + \frac{1}{RC}y(t) = \frac{1}{R}\frac{dx(t)}{dt}
\end{aligned}
$$

Example: RLC Circuit

Find the relationship between the input voltage $x(t)$ *and output current* $y(t)$.



Example: RLC Circuit

Applying KVL, we get

$$x(t) = Ry(t) + L\frac{dy(t)}{dt} + \frac{1}{C}\int_{-\infty}^{t} y(t')dt'$$

$$\Rightarrow \quad L\frac{d^2y(t)}{dt^2} + R\frac{dy(t)}{dt} + \frac{1}{C}y(t) = \frac{dx(t)}{dt}$$

$$\Rightarrow \quad \frac{d^2y(t)}{dt^2} + \frac{R}{L}\frac{dy(t)}{dt} + \frac{1}{LC}y(t) = \frac{1}{L}\frac{dx(t)}{dt}$$

Discrete-Time Examples

Moving Average:

$$y[n] = \frac{1}{3}(x[n] + x[n-1] + x[n+1])\,.$$

Accumulaor:

$$y[n] = \sum_{k=-\infty}^{n} x[k] = y[n-1] + x[n].$$

Differential Equation Representation of Continuous-Time LTI Systems

Continuous-time LTI systems can be represented by constant-coefficient linear differential equations of the form

$$\boxed{\sum_{k=0}^{N} a_k \frac{d^k}{dt^k} y(t) = \sum_{k=0}^{M} b_k \frac{d^k}{dt^k} x(t)},$$

where $x(t)$ is the input applied to the system, $y(t)$ is the output, and a_ks and b_ks are the constant coefficients describing the system.

*The **order** of the differential equation is (N, M); often, $N \geq M$, and the order is described simply using N.*

The order represents the number of energy storage devices in the system.

Difference Equation Representation of Discrete-Time LTI Systems

Discrete-time LTI systems can be represented by constant-coefficient linear difference equations of the form

$$\boxed{\sum_{k=0}^{N} a_k y[n-k] = \sum_{k=0}^{M} b_k x[n-k]},$$

where $max(N, M)$ is the ***order*** *of the system representing the* ***memory****.*

Difference equations are easily rearranged to obtain recursive formulas for computing the current output of the system from input signal and past outputs:

$$\boxed{y[n] = \frac{1}{a_0} \sum_{k=0}^{M} b_k x[n-k] - \frac{1}{a_0} \sum_{k=1}^{N} a_k y[n-k]}.$$

Initial Conditions

*We require **initial conditions** to compute the current output of the system from input signal and past outputs.*

Initial conditions summarize all the information about the system's past that is needed to determine the future outputs.

It is customary to choose $n = 0$ or $t = 0$ as the starting time for solving a difference or differential equation, respectively.

*In general, the number of initial conditions required to determine the output is equal to the **order** of the system.*

Initial Conditions: Example

Consider a discrete-time system whose input-output relation is given by

$$y[n] + y[n-1] + \frac{1}{4}y[n-2] = x[n] + 2x[n-1].$$

Rearranging, we get

$$y[n] = -y[n-1] - \frac{1}{4}y[n-2] + x[n] + 2x[n-1].$$

Taking the starting time as $n = 0$, we would like to determine the output at times $n \geq 0$.

If we know the input $x[n]$ at all times and the initial conditions $y[-1]$ and $y[-2]$, then we can compute the output at times $n \geq 0$.

Initial Conditions: General Case

For an N-th order differential equation, the intial conditions are values of the first N derivatives of the output, viz.,

$$y(t)\big|_{t=0^-},\ \frac{d}{dt}y(t)\Big|_{t=0^-},\ \frac{d^2}{dt^2}y(t)\Big|_{t=0^-},\ \ldots,\ \frac{d^{N-1}}{dt^{N-1}}y(t)\Big|_{t=0^-}.$$

- *These initial conditions are directly related to the initial states of the energy storage devices in the system, such as initial voltages on capacitors and initial currents through inductors.*

For an N-th order difference equation, the intial conditions are the N values $y[-1], \ldots, y[-N]$.

- *These initial conditions summarize all information about the past history of the system that can affect future outputs.*

Solving Differential and Difference Equations

References:

[1] Simon Haykin and Barry Van Veen, Signals and Systems, Second Edition, John Wiley and Sons, 2003.

[2] Lecture Notes by Michael D. Adams.
https://www.ece.uvic.ca/~frodo/sigsysbook/downloads/lecture_slides_for_signals_and_systems-2.0.pdf
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[3] Lecture Notes by Richard Baraniuk.
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