Frequency Domain Representation of Signals and LTI Systems: Continuous Time Case

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Frequency Response of LTI Systems

Frequency Response of Continuous-Time Systems

Consider a continuous-time LTI system with impulse response $h(t)$.

- *Suppose a* ***complex sinusoid*** *is applied as input. i.e.,* $\boxed{x(t) = e^{j\omega t}}$.
- *Then, the otput $y(t)$ is given by*

$$
\begin{aligned}
y(t) &= \int_{-\infty}^{\infty} h(\tau)x(t-\tau)d\tau = \int_{-\infty}^{\infty} h(\tau)e^{j\omega(t-\tau)}d\tau \\
&= e^{j\omega t}\int_{-\infty}^{\infty} h(\tau)e^{-j\omega\tau}d\tau = H(j\omega)e^{j\omega t}, \quad \text{where}
\end{aligned}
$$

$$
\boxed{H(j\omega) = \int_{-\infty}^{\infty} h(\tau)e^{-j\omega\tau}d\tau} = \text{Frequency Response.}
$$

Frequency Response of Continuous-Time Systems

*In summary, **the output of a continuous-time LTI system for a complex sinusoid input is a complex sinusoid of the** **same** **frequency, multiplied by a complex number $H(j\omega)$, called the** **frequency response** **of the system, which is a function of only the frequency ω and not the time t.***

*Therefore, we say that the complex sinusoid $\psi(t) = e^{j\omega t}$ is an **eigenfunction** of the LTI system with an associated **eigenvalue** $\lambda = H(j\omega)$.*

Frequency Response of Continuous-Time Systems

Substituting $\boxed{H(j\omega) = |H(j\omega)|e^{j\,arg\{H(j\omega)\}}}$*, we obtain the output corresponding to a complex sinusoid* $e^{j\omega t}$ *as*

$$\boxed{y(t) = H(j\omega)e^{j\omega t} = |H(j\omega)|e^{j(\omega t + \arg\{H(j\omega)\})}},$$

*which implies that **in steady state** the LTI system modifies the magnitude of the input by a factor* $|H(j\omega)|$ *and modifies the phase by a shift of* $arg\{H(j\omega)\}$.

Therefore, $|H(j\omega)|$ *is called the **magnitude response** and* $arg\{H(j\omega)\}$ *is called the **phase response** of the LTI system.*

Example: RC Circuit

The impulse response is $\boxed{h(t) = \frac{1}{RC} e^{-\frac{t}{RC}} u(t)}$. *Then, we have*

$$\boxed{H(j\omega) = \frac{\frac{1}{RC}}{j\omega + \frac{1}{RC}}}.$$

$$\Rightarrow |H(j\omega)| = \frac{1/RC}{\sqrt{\omega^2 + (1/RC)^2}} \quad \text{and} \quad \arg\{H(j\omega)\} = -\tan^{-1}(\omega RC).$$

Taking $x(t) = e^{j\omega t} u(t)$, *it can be shown that*

$$y(t) = H(j\omega)\left(e^{j\omega t} - e^{-\frac{t}{RC}}\right) \longrightarrow H(j\omega)e^{j\omega t} \text{ in steady state.}$$

Frequency Response of Discrete-Time Systems

Consider a discrete-time LTI system with impulse response $h[n]$.

- *Suppose a **complex sinusoid** is applied as input. i.e.,* $\boxed{x[n] = e^{j\Omega n}}$.
- *Then, the otput $y[n]$ is given by*

$$y[n] = \sum_{k=-\infty}^{\infty} h[k]x[n-k] = \sum_{k=-\infty}^{\infty} h[k]e^{j\Omega(n-k)}$$

$$= e^{j\Omega n} \sum_{k=-\infty}^{\infty} h[k]e^{-j\Omega k} = H(e^{j\Omega})e^{j\Omega n}, \quad \text{where}$$

$$\boxed{H(e^{j\Omega}) = \sum_{k=-\infty}^{\infty} h[k]e^{-j\Omega k}} = \text{Frequency Response.}$$

Frequency Response of Discrete-Time Systems

*In summary, **the output of a discrete-time LTI system for a complex sinusoid input is a complex sinusoid of the** same **frequency, multiplied by a complex number $H(e^{j\Omega})$, called the** frequency response **of the system, which is a function of only the frequency Ω and not the time n.***

Similarly, we can write $\boxed{H(e^{j\Omega}) = |H(e^{j\Omega})| e^{j\, arg\{H(e^{j\Omega})\}}}$,

where

- $|H(e^{j\Omega})|$ *is called the **magnitude response** and*
- $arg\{H(e^{j\Omega})\}$ *is called the **phase response** of the LTI system.*

Fourier Representation of Signals

Motivation for Fourier Representations

Suppose we can decompose a general input signal $x(t)$ as a weighted sum of complex sinusoids as

$$\boxed{x(t) = \sum_k a_k e^{j\omega_k t}}.$$

Then, the output $y(t)$ is also a weighted sum of exponentials, given by

$$\boxed{y(t) = \sum_k a_k H(j\omega_k) e^{j\omega_k t}}.$$

*Note that **the operation of convolution, $h(t) * x(t)$, has been replaced by multiplication, $a_k H(j\omega_k)$, because $x(t)$ has been expressed as a sum of eigenfunctions**; analogous relationships hold in the discrete time as well.*

Four Classes of Fourier Representations

Time Property	Periodic	Nonperiodic
Continuous (t)	Fourier Series (FS) [Chapter 3.5]	Fourier Transform (FT) [Chapter 3.7]
Discrete (n)	Discrete Time Fourier Series (DTFS) [Chapter 3.4]	Discrete Time Fourier Transform (DTFT) [Chapter 3.6]

Fourier Series Representation of Continuous-Time Periodic Signals

Fundamental (Angular) Frequency and Harmonics

A sinusoid whose (angular) frequency ω is an integer multiple of a fundamental (angular) frequency ω_0 is said to be a ***harmonic*** *of the sinusoid of the fundamental frequency.*

- *The sinusoids $e^{\pm jk\omega_0}$ are the k-th harmonics of $e^{j\omega_0}$.*

Consider representing a continuous-time periodic signal $x(t)$ with fundamental period T_0 as a weighted sum of complex sinusoids as

$$\boxed{x(t) = \sum_k a_k e^{j\omega_k t}}.$$

How do we choose the frequencies ω_k and coefficients a_k?

Fundamental (Angular) Frequency and Harmonics

The weighted sum must have the same fundamental period T_0.

- *Therefore, each complex sinusoid in the weighted sum must have a fundamental period* T_0/n, $n = 1, 2, 3, \ldots$, *etc., so that the LCM of the fundamenatal periods of the constituent complex sinusoids is* T_0.
- *Equivalently, the frequency of each constituent complex sinusoid must be an integer multiple of the signal* $x(t)$*'s fundamental frequency* $\omega_0 = \dfrac{2\pi}{T_0}$.

This implies that a continuous-time periodic signal $x(t)$ *with frequency* ω_0 *can be written as a weighted sum of the harmonics of* $e^{j\omega_0}$ *as*

$$\boxed{x(t) = \sum_k a_k e^{j\omega_k t} = \sum_{k=-\infty}^{\infty} a_k e^{jk\omega_0 t}}.$$

Fourier Series Coefficients

The weighted sum representation of a continuous-time periodic signal $x(t)$ with fundamental period T_0 can be re-written as

$$\boxed{x(t) = \sum_{k=-\infty}^{\infty} X[k] e^{jk\omega_0 t}},$$

where any coefficient $X[\ell]$ is given by (why?)

$$\boxed{X[\ell] = \frac{1}{T_0} \int_0^{T_0} x(t) e^{-j\ell\omega_0 t}\, dt}.$$

*The FS coefficients $X[k]$ are known as a **frequency-domain representation** of $x(t)$ because each FS coefficient is associated with a complex sinusoid of a different frequency.*

Validity of Fourier Series Expansion

*We say that $x(t)$ and $X[k]$ are a **FS pair** and denote this relationship as*

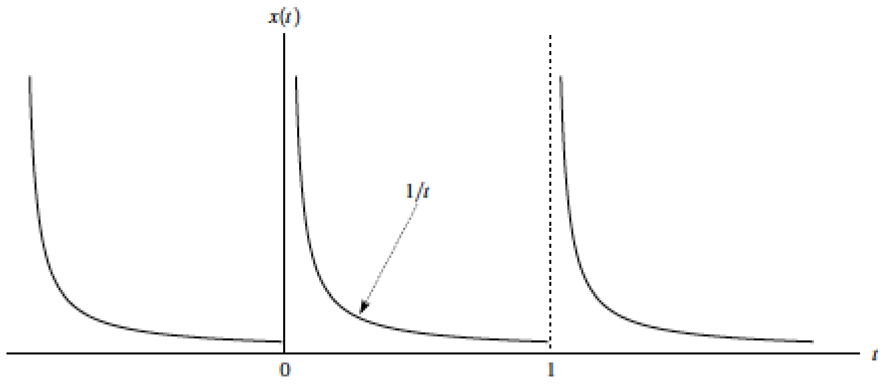
$$x(t) \overset{\text{CTFS}}{\longleftrightarrow} X[k].$$

*The foregoing equations for weighted sum decomposition of $x(t)$ and finding the corresponding Fourier Series coefficients $X[k]$ are valid only if the **Dirichlet conditions** are satisfied:*

- *$x(t)$ is bounded or absolutely integrable over one time period, i.e., if*
$$\int_{T_0} |x(t)|dt \leq M_x < \infty.$$
- *$x(t)$ has a finite number of maxima and minima in one period.*
- *$x(t)$ has a finite number of discontinuities in one period.*

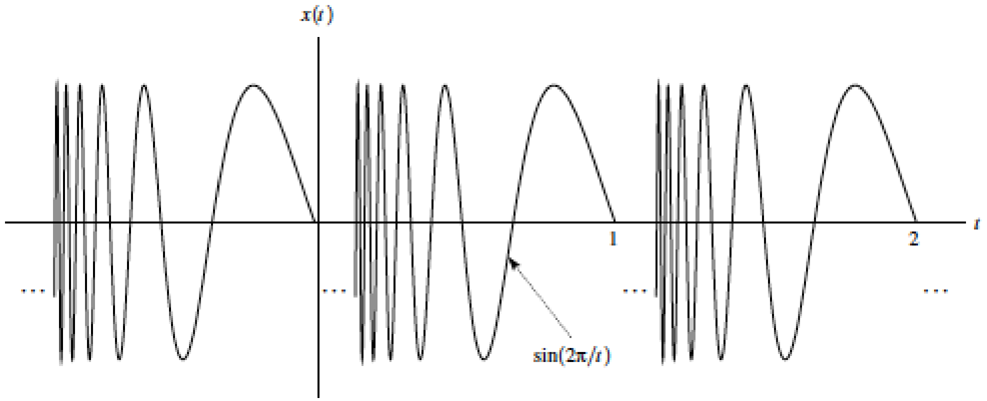
Validity of Fourier Series Expansion

*Here, $x(t)$ is **unbounded**.*



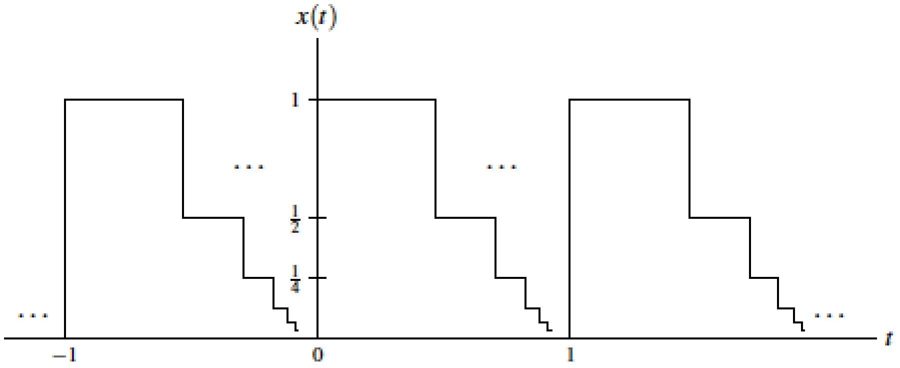
Validity of Fourier Series Expansion

Here, $x(t)$ has an ***infinite number of maxima and minima in one time period****.*



Validity of Fourier Series Expansion

Here, $x(t)$ has an ***infinite number of jump discontinuities in one time period.***



Simple Examples

Find the Fourier Series (FS) expansion of the following signals: (i) $x(t) = cos(\omega_0 t)$, (ii) $x(t) = sin(\omega_0 t)$, (iii) $x(t) = cos(2t + \frac{\pi}{4})$, (iv) $x(t) = cos(4t) + sin(6t)$, (v) $x(t) = sin^2(t)$.

Solution: *(i) We have* $cos(\omega_0 t) = \frac{1}{2}\left(e^{j\omega_0 t} + e^{-j\omega_0 t}\right)$*, which implies that* $X[1] = X[-1] = \frac{1}{2}$ *and* $X[k] = 0$ *for all* $k \neq \pm 1$.

(ii) We have $sin(\omega_0 t) = \frac{1}{2j}\left(e^{j\omega_0 t} - e^{-j\omega_0 t}\right)$*, which implies that* $X[1] = \frac{1}{2j}$, $X[-1] = -\frac{1}{2j}$ *and* $X[k] = 0$ *for all* $k \neq \pm 1$.

Simple Examples

Find the Fourier Series (FS) expansion of the following signals: (i) $x(t) = cos(\omega_0 t)$, (ii) $x(t) = sin(\omega_0 t)$, (iii) $x(t) = cos(2t + \frac{\pi}{4})$, (iv) $x(t) = cos(4t) + sin(6t)$, (v) $x(t) = sin^2(t)$.

Solution: *(iii) We have $\omega_0 = 2$, and*

$$\cos(2t + \frac{\pi}{4}) = \frac{1}{2}\left(e^{j\left(2t+\frac{\pi}{4}\right)} + e^{-j\left(2t+\frac{\pi}{4}\right)}\right)$$
$$= \frac{1}{2}e^{j\frac{\pi}{4}}e^{j2t} + \frac{1}{2}e^{-j\frac{\pi}{4}}e^{-j2t},$$

which implies that $X[1] = \frac{1}{2}e^{j\frac{\pi}{4}}$, $X[-1] = \frac{1}{2}e^{-j\frac{\pi}{4}}$ and $X[k] = 0$ for all $k \neq \pm 1$.

Simple Examples

Find the Fourier Series (FS) expansion of the following signals: (i) $x(t) = cos(\omega_0 t)$, (ii) $x(t) = sin(\omega_0 t)$, (iii) $x(t) = cos(2t + \frac{\pi}{4})$, (iv) $x(t) = cos(4t) + sin(6t)$, (v) $x(t) = sin^2(t)$.

Solution: *(iv) We have*

$$T_0 = LCM(T^1, T^2) = LCM\left(\frac{2\pi}{\omega^1}, \frac{2\pi}{\omega^2}\right)$$

$$= LCM\left(\frac{2\pi}{4}, \frac{2\pi}{6}\right) = \pi,$$

which implies that $\omega_0 = \frac{2\pi}{T_0} = 2$, *and*

$$\cos(4t) + \sin(6t) = \frac{1}{2}e^{j4t} + \frac{1}{2}e^{-j4t} + \frac{1}{2j}e^{j6t} - \frac{1}{2j}e^{-j6t}$$

$$= \frac{1}{2}e^{j2\omega_0 t} + \frac{1}{2}e^{-j2\omega_0 t} + \frac{1}{2j}e^{j3\omega_0 t} - \frac{1}{2j}e^{-j3\omega_0 t},$$

which implies that $X[2] = X[-2] = \frac{1}{2}$, $X[3] = \frac{1}{2j}$, $X[-3] = -\frac{1}{2j}$ *and* $X[k] = 0$ *for all* $k \neq \pm 2, \pm 3$.

Simple Examples

Find the Fourier Series (FS) expansion of the following signals: (i) $x(t) = cos(\omega_0 t)$, (ii) $x(t) = sin(\omega_0 t)$, (iii) $x(t) = cos(2t + \frac{\pi}{4})$, (iv) $x(t) = cos(4t) + sin(6t)$, (v) $x(t) = sin^2(t)$.

Solution: (v) We have $sin^2(t) = \frac{1}{2}(1 - cos(2t))$, which implies that $\omega_0 = 2$, and

$$\frac{1}{2}(1 - \cos(2t)) = \frac{1}{2} - \frac{1}{4}\left(e^{j2t} + e^{-j2t}\right)$$

which implies that $X[0] = \frac{1}{2}$, $X[1] = X[-1] = -\frac{1}{4}$, and $X[k] = 0$ for all $k \neq 0, \pm 1$.

Exponential and Trigonometric Forms

The decomposition of a continuous-time periodic signal $x(t)$ with period T_0 as a weighted sum of complex exponentials, as discussed earlier, and given by

$$\boxed{x(t) = \sum_{k=-\infty}^{\infty} X[k]e^{jk\omega_0 t}},$$

is called the ***exponential form*** *of FS representation.*

Applying Euler's identity, we get the ***trigonometric form*** *of FS representation as*

$$\boxed{\begin{aligned} x(t) &= X[0] + \sum_{k=1}^{\infty} \left(X[k]e^{jk\omega_0 t} + X[-k]e^{-jk\omega_0 t}\right) \\ &= X[0] + \sum_{k=1}^{\infty} B[k]\cos(k\omega_0 t) + A[k]\sin(k\omega_0 t) \end{aligned}},$$

Exponential and Trigonometric Forms

where for any $x(t)$ (real- or complex-valued) we have

$$\boxed{X[0] = \frac{1}{T_0}\int_0^{T_0} x(t)dt} = \text{time-averaged value of } x(t)$$

$$\boxed{B[k] = (X[k] + X[-k]) = \frac{2}{T_0}\int_0^{T_0} x(t)\cos(k\omega_0 t)dt}$$

$$\boxed{A[k] = j\,(X[k] - X[-k]) = \frac{2}{T_0}\int_0^{T_0} x(t)\sin(k\omega_0 t)dt}$$

Recall that any FS coefficient $X[\ell]$ is given by

$$\boxed{X[\ell] = \frac{1}{T_0}\int_0^{T_0} x(t)e^{-j\ell\omega_0 t}\,dt}.$$

Exponential and Trigonometric Forms

From the previous equation, it is easy to see that if $x(t)$ is real-valued, then $\boxed{X[-\ell] = X^*[\ell]}$, *where $X^*[\ell]$ denotes the complex conjugate of $X[\ell]$.*

Substituting $\boxed{X[-\ell] = X^*[\ell]}$, *we get the coefficients of the* ***trigonometric form*** *of FS representation for a real-valued signal $x(t)$ as*

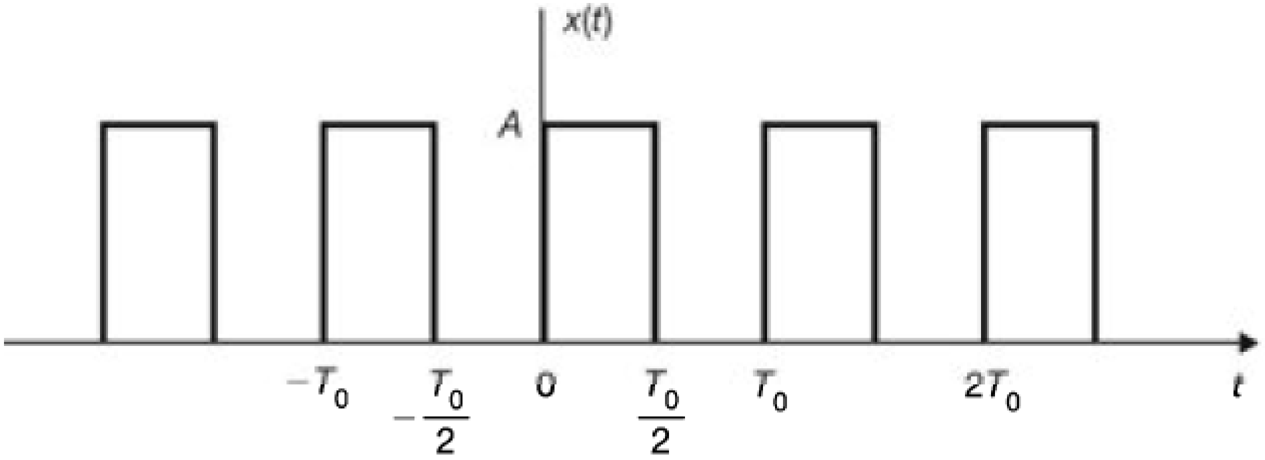
$$\boxed{B[k] = \left(X[k] + X^*[k]\right) \quad \text{and} \quad A[k] = j\left(X[k] - X^*[k]\right)}$$

If $x(t)$ is an even (resp. odd) signal, then the trigonometric form of FS expansion will consist of only cosine (resp. sine) terms.

Square Wave Example 1

$$X[0] = A/2,\ X[2m] = 0,\ X[2m+1] = A/\left[j\,(2m+1)\,\pi\right]$$

$$A[k] = ?\ B[k] = ?$$

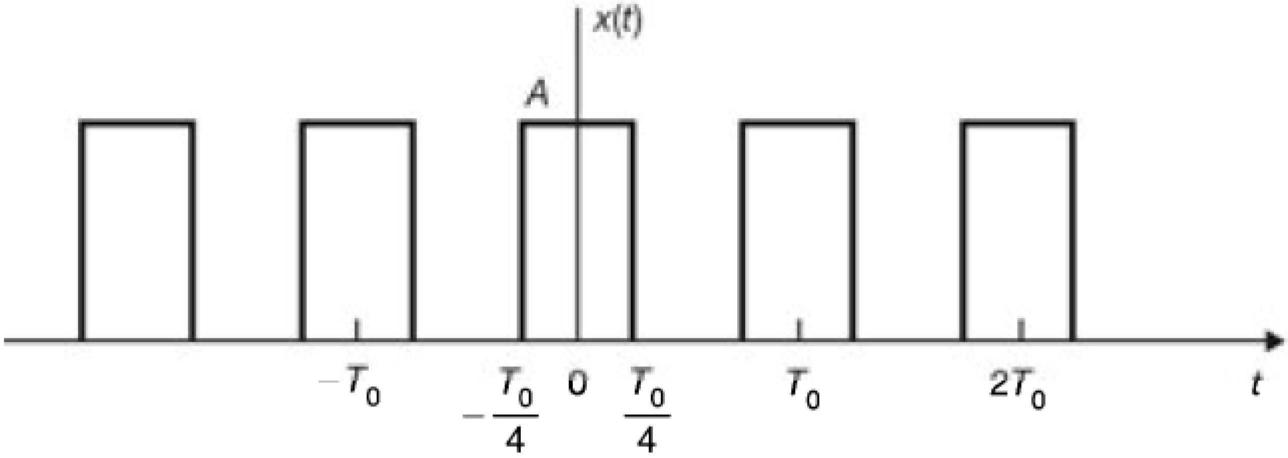


Square Wave Example 2

$$X[0] = A/2,\ X[2m] = 0,\ X[2m+1] = (-1)^m A/\left[(2m+1)\,\pi\right]$$

$$A[k] = ?\ B[k] = ?$$

Can we get the coefficients directly from Example 1?

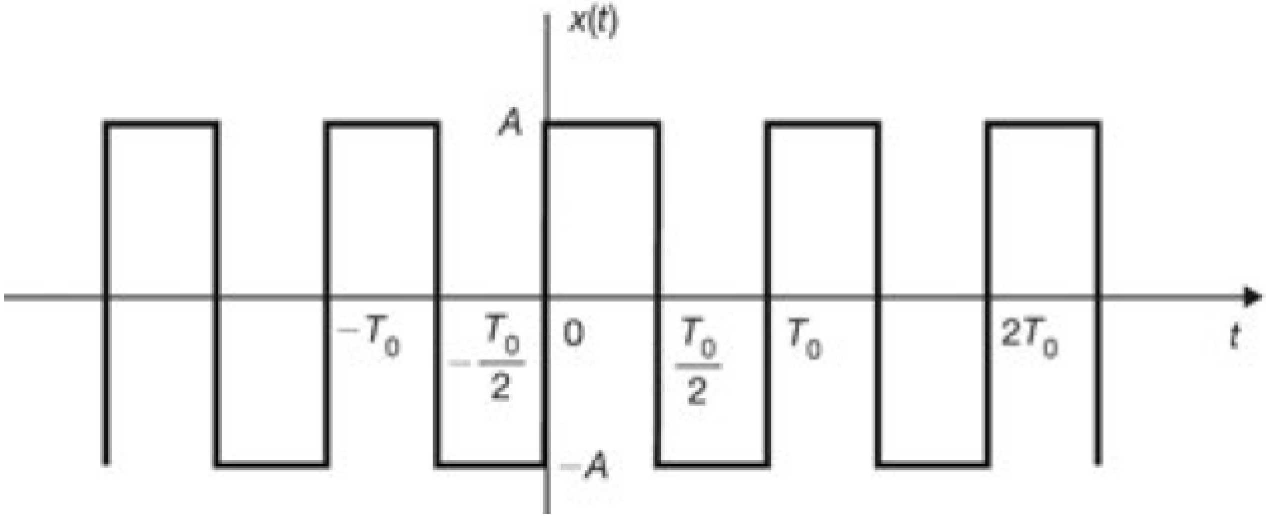


Square Wave Example 3

$$X[0] = 0,\ X[2m] = 0,\ X[2m+1] = 2A/\left[j\,(2m+1)\,\pi\right]$$

$$A[k] = ?\, B[k] = ?$$

Can we get the coefficients directly from Example 1?



Amplitude and Phase Spectra of a CT Periodic Signal

The FS coefficients can be written in polar form as

$$\boxed{X[k] = |X[k]|e^{j\theta_k}},$$

where $|X[k]|$ *and* θ_k *as functions of discrete frequency* $k\omega_0$ *are called the* ***magnitude spectrum*** *and* ***phase spectrum****, respectively.*

Recall that for a real-valued signal $x(t)$ *we have*
$\boxed{X[-k] = X^*[k]}$*, which implies that*

Amplitude and Phase Spectra of a CT Periodic Signal

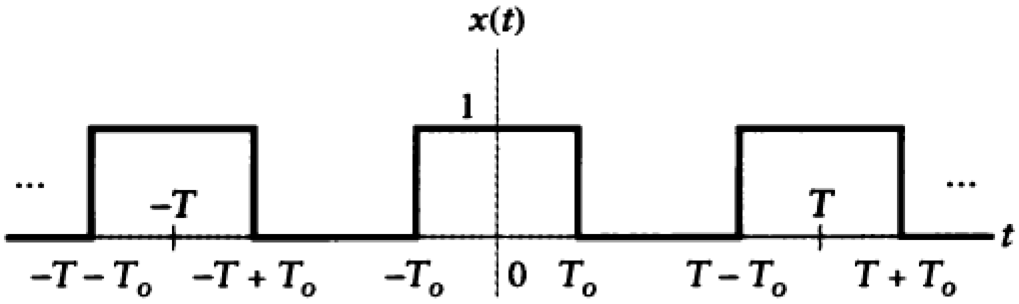
- (i) $\boxed{|X[-k]| = |X^*[k]| = |X[k]|}$, i.e., the magnitude spectrum is an even function of the discrete index k.
- (ii) $\boxed{\theta_{-k} = arg\{X^*[k]\} = -arg\{X[k]\} = -\theta_k}$, i.e., the phase spectrum is an odd function of the discrete index k.

For the above reason of symmetry, when dealing with real-valued signals $x(t)$, one can ignore the negative frequencies altogether.

- If the signal $x(t)$ is complex-valued, then the above symmetry will not hold and one has to consider both positive and negative frequencies separately.

Example: Periodic Rectangular Pulses

For $x(t)$ *shown below, we have* $\boxed{\omega_0 = \frac{2\pi}{T}}$.



Example: Periodic Rectangular Pulses

It can be shown that $\boxed{X[0] = \dfrac{2T_0}{T}}$ *and for all* $k \neq 0$*, we have*

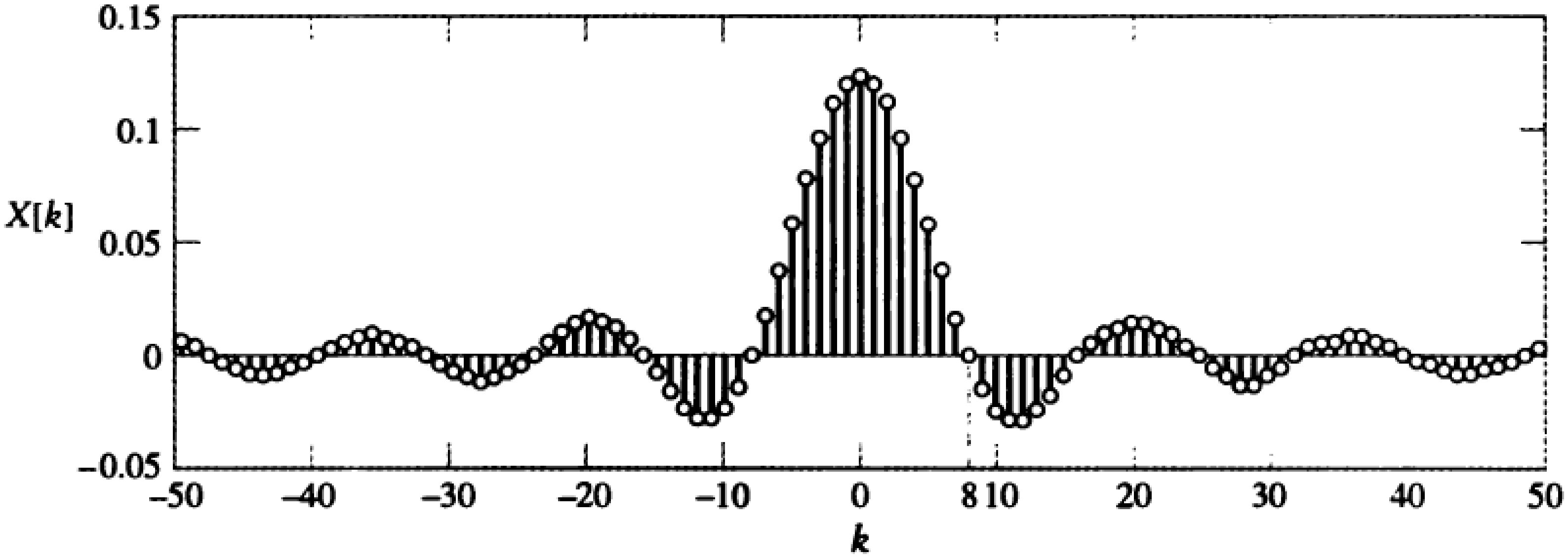
$$X[k] = \frac{2\sin(k\omega_0 T_0)}{k\omega_0 T} = \frac{2T_0}{T}\frac{\sin(k\omega_0 T_0)}{k\omega_0 T_0}$$
$$= \frac{2T_0}{T}\frac{\sin(k2\pi T_0/T)}{k2\pi T_0/T} = \frac{2T_0}{T}\operatorname{sinc}\left(k\frac{2T_0}{T}\right),$$

where

$\boxed{sinc(x) = \dfrac{sin(\pi x)}{\pi x}}$.

Example: Periodic Rectangular Pulses

Can you sketch the amplitude and phase spectra?



Output Computation of an LTI System using FS

Suppose a continuous-time periodic signal $x(t)$ is applied as input to an LTI system with impulse response $h(t)$.

From time-domain analysis, we know that the output $y(t)$ can be obtained by taking a convolution of the input $x(t)$ and impulse response $h(t)$.

An alternate method that does not require convolution:

- *Obtain the FS spectrum of $x(t)$. Let $x(t) \overset{CTFS}{\longleftrightarrow} X[k]$.*
- *Obtain the frequency response of the LTI system $H(j\omega)$.*
- *Obtain the FS spectrum of $y(t)$ from that of $x(t)$ as $Y[k] = H(jk\omega_0)X[k]$.*
- *Obtain the output $Y[k] \overset{CTFS}{\longleftrightarrow} y(t)$.*

Example: Square Wave Through an RC Circuit

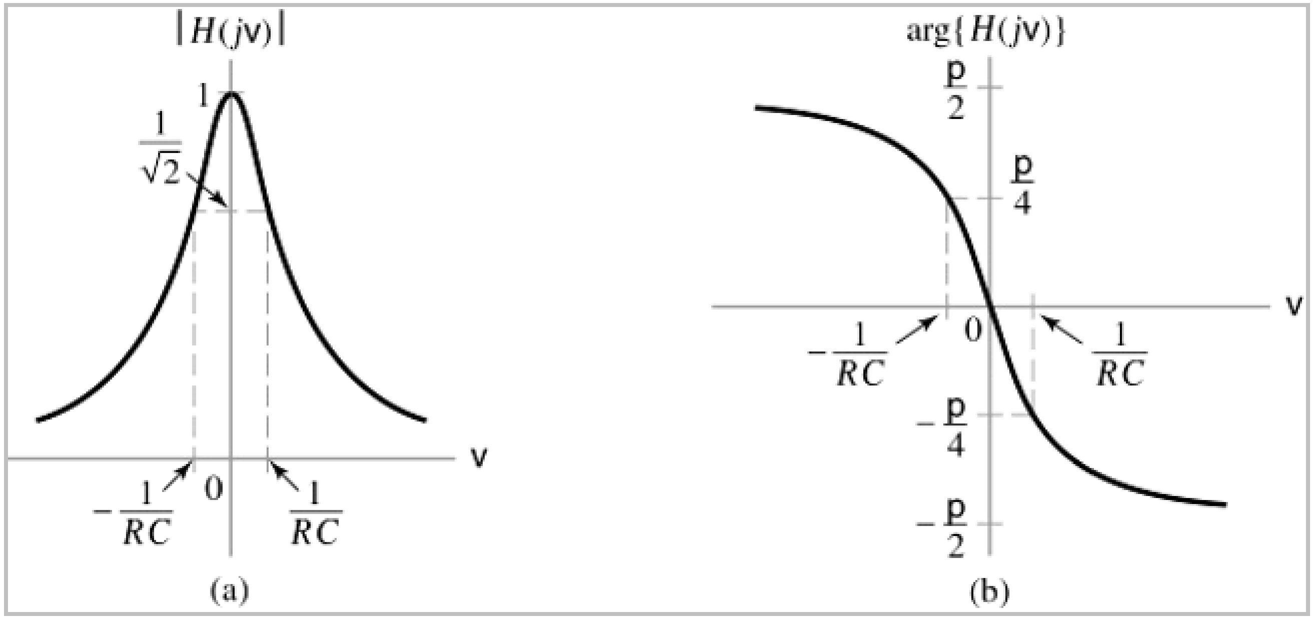
Recall that for the RC circuit, the frequency response is given by

$$\boxed{H(j\omega) = \frac{\frac{1}{RC}}{j\omega + \frac{1}{RC}}}.$$

The magnitude and phase responses are given by (plotted on the next slide)

$$|H(j\omega)| = \frac{1/RC}{\sqrt{\omega^2 + (1/RC)^2}} \quad \text{and} \quad \arg\{H(j\omega)\} = -\tan^{-1}(\omega RC).$$

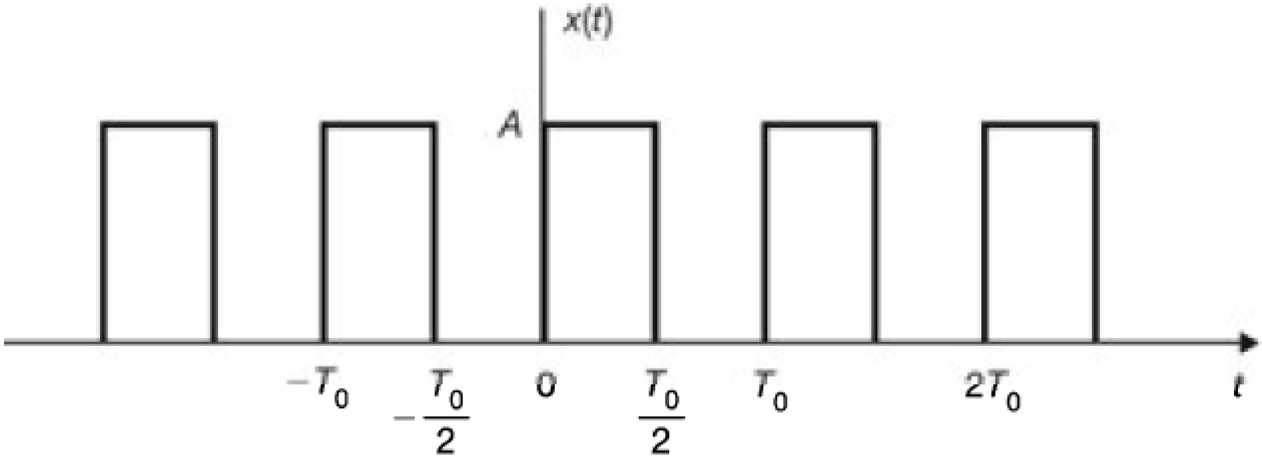
Example: Square Wave Through an RC Circuit



Example: Square Wave Through an RC Circuit

Suppose a square wave as shown below is applied as input $x(t)$.

- *We had obtained the FS coefficients $X[k]$ earlier.*



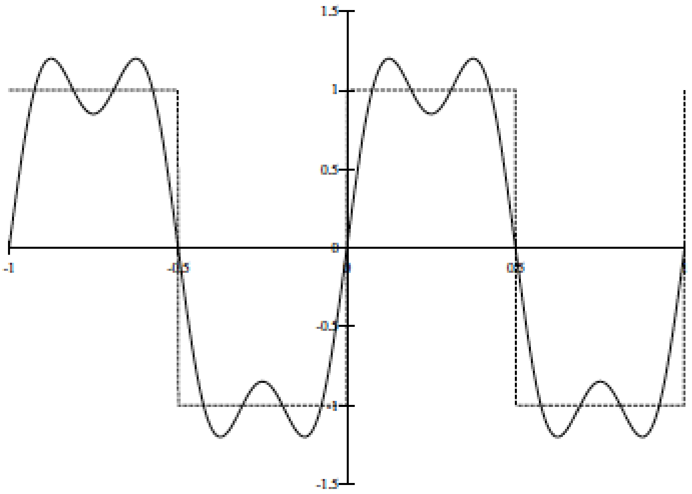
Example: Square Wave Through an RC Circuit

The output is then given by

$$
\begin{aligned}
y(t) &= \sum_{k=-\infty}^{\infty} Y[k]e^{j\omega_0 t} = \sum_{k=-\infty}^{\infty} H(jk\omega_0)X[k]e^{j\omega_0 t} \\
&\approx \sum_{k=-K}^{K} H(jk\omega_0)X[k]e^{j\omega_0 t}
\end{aligned}
$$

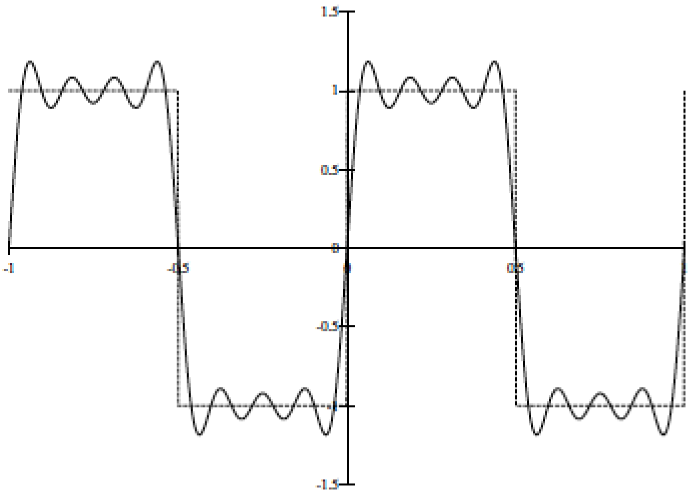
Can you sketch the output (at least approximately)?

Ex: Square Wave Through An Ideal Lowpass Filter



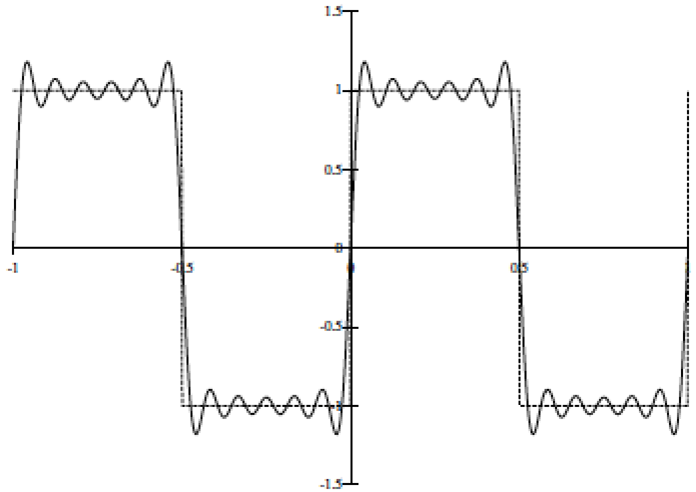
Fourier series truncated after the
3rd harmonic components

Ex: Square Wave Through An Ideal Lowpass Filter



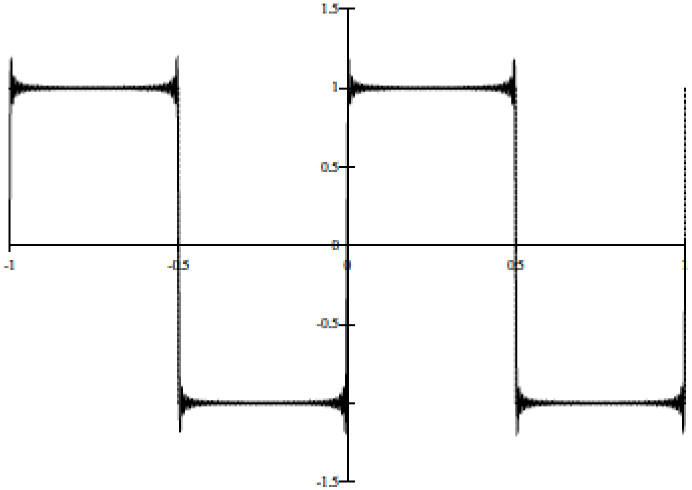
Fourier series truncated after the
7th harmonic components

Ex: Square Wave Through An Ideal Lowpass Filter



Fourier series truncated after the
11th harmonic components

Ex: Square Wave Through An Ideal Lowpass Filter



Fourier series truncated after the
101st harmonic components

CTFS Properties

Linearity

- Let x and y be two periodic functions with the same period. If $x(t) \overset{\text{CTFS}}{\longleftrightarrow} a_k$ and $y(t) \overset{\text{CTFS}}{\longleftrightarrow} b_k$, then

$$\alpha x(t) + \beta y(t) \overset{\text{CTFS}}{\longleftrightarrow} \alpha a_k + \beta b_k,$$

  where α and β are complex constants.

- That is, a linear combination of functions produces the same linear combination of their Fourier series coefficients.

Time Shifting (Translation)

- Let x denote a periodic function with period T and the corresponding frequency $\omega_0 = 2\pi/T$. If $x(t) \overset{\text{CTFS}}{\longleftrightarrow} c_k$, then

$$x(t - t_0) \overset{\text{CTFS}}{\longleftrightarrow} e^{-jk\omega_0 t_0} c_k = e^{-jk(2\pi/T)t_0} c_k,$$

  where t_0 is a real constant.
- In other words, time shifting a periodic function changes the argument (but not magnitude) of its Fourier series coefficients.

Frequency Shifting (Modulation)

- Let x denote a periodic function with period T and the corresponding frequency $\omega_0 = 2\pi/T$. If $x(t) \overset{\text{CTFS}}{\longleftrightarrow} c_k$, then

$$e^{jM(2\pi/T)t}x(t) = e^{jM\omega_0 t}x(t) \overset{\text{CTFS}}{\longleftrightarrow} c_{k-M},$$

  where M is an integer constant.

- In other words, multiplying a periodic function by $e^{jM\omega_0 t}$ shifts the Fourier-series coefficient sequence.

Time Reversal (Reflection)

- Let x denote a periodic function with period T and the corresponding frequency $\omega_0 = 2\pi/T$. If $x(t) \overset{\text{CTFS}}{\longleftrightarrow} c_k$, then

$$x(-t) \overset{\text{CTFS}}{\longleftrightarrow} c_{-k}.$$

- That is, time reversal of a function results in a time reversal of its Fourier series coefficients.

Conjugation

- For a T-periodic function x with Fourier series coefficient sequence c, the following property holds:

$$x^*(t) \stackrel{\text{CTFS}}{\longleftrightarrow} c^*_{-k}$$

- In other words, conjugating a function has the effect of time reversing and conjugating the Fourier series coefficient sequence.

Periodic Convolution

- Let x and y be two periodic functions with the same period T. If $x(t) \overset{\text{CTFS}}{\longleftrightarrow} a_k$ and $y(t) \overset{\text{CTFS}}{\longleftrightarrow} b_k$, then

$$x \circledast y(t) \overset{\text{CTFS}}{\longleftrightarrow} Ta_k b_k.$$

- In other words, periodic convolution of two functions corresponds to the multiplication (up to a scale factor) of their Fourier-series coefficient sequences.

Multiplication

- Let x and y be two periodic functions with the same period. If $x(t) \overset{\text{CTFS}}{\longleftrightarrow} a_k$ and $y(t) \overset{\text{CTFS}}{\longleftrightarrow} b_k$, then

$$x(t)y(t) \overset{\text{CTFS}}{\longleftrightarrow} \sum_{n=-\infty}^{\infty} a_n b_{k-n}$$

- As we shall see later, the above summation is the DT convolution of a and b.
- In other words, the multiplication of two periodic functions corresponds to the DT convolution of their corresponding Fourier-series coefficient sequences.

Differentiation in Time

We have

$$\frac{d}{dt}x(t) \stackrel{\text{CTFS}}{\longleftrightarrow} jk\omega_0 X[k].$$

Scaling in Time

We have

$$x(at) \overset{\text{CTFS}}{\longleftrightarrow} X[k],$$

but the spacing between the Fourier coefficients changes from ω_0 *to* $a\omega_0$.

Even and Odd Symmetry

- For a T-periodic function x with Fourier series coefficient sequence c, the following properties hold:

$$x \text{ is even} \Leftrightarrow c \text{ is even}; \quad \text{and}$$
$$x \text{ is odd} \Leftrightarrow c \text{ is odd}.$$

- In other words, the even/odd symmetry properties of x and c always match.

Real-Valued Functions

- A function x is ***real*** if and only if its Fourier series coefficient sequence c satisfies

$$c_k = c^*_{-k} \text{ for all } k$$

(i.e., c is ***conjugate symmetric***).

- Thus, for a real-valued function, the negative-indexed Fourier series coefficients are ***redundant***, as they are completely determined by the nonnegative-indexed coefficients.

Real-Valued Functions

- From properties of complex numbers, one can show that $c_k = c^*_{-k}$ is equivalent to

$$|c_k| = |c_{-k}| \quad \text{and} \quad \arg c_k = -\arg c_{-k}$$

(i.e., $|c_k|$ is ***even*** and $\arg c_k$ is ***odd***).

- Note that x being real does ***not*** necessarily imply that c is real.

Parseval's Relation

- A function x and its Fourier series coefficient sequence a satisfy the following relationship:

$$\frac{1}{T}\int_T |x(t)|^2\,dt = \sum_{k=-\infty}^{\infty} |a_k|^2.$$

- The above relationship is simply stating that the amount of energy in x (i.e., $\frac{1}{T}\int_T |x(t)|^2\,dt$) and the amount of energy in the Fourier series coefficient sequence a (i.e., $\sum_{k=-\infty}^{\infty} |a_k|^2$) are equal.
- In other words, the transformation between a function and its Fourier series coefficient sequence preserves energy.

Other Properties

- For a T-periodic function x with Fourier-series coefficient sequence c, the following properties hold:
  1. c_0 is the average value of x over a single period;
  2. x is real and even $\Leftrightarrow$ c is real and even; and
  3. x is real and odd $\Leftrightarrow$ c is purely imaginary and odd.

Table of CTFS Properties

$$x(t) \overset{\text{CTFS}}{\longleftrightarrow} a_k \quad \text{and} \quad y(t) \overset{\text{CTFS}}{\longleftrightarrow} b_k$$

Property	Time Domain	Fourier Domain
Linearity	$\alpha x(t) + \beta y(t)$	$\alpha a_k + \beta b_k$
Translation	$x(t - t_0)$	$e^{-jk(2\pi/T)t_0} a_k$
Modulation	$e^{jM(2\pi/T)t} x(t)$	a_{k-M}
Reflection	$x(-t)$	a_{-k}
Conjugation	$x^*(t)$	a^*_{-k}
Periodic Convolution	$x \circledast y(t)$	$T a_k b_k$
Multiplication	$x(t)y(t)$	$\sum_{n=-\infty}^{\infty} a_n b_{k-n}$

Table of CTFS Properties 2

Property	
Parseval's Relation	$\frac{1}{T}\int_T \lvert x(t)\rvert^2\,dt = \sum_{k=-\infty}^{\infty} \lvert a_k\rvert^2$
Even Symmetry	x is even $\Leftrightarrow$ a is even
Odd Symmetry	x is odd $\Leftrightarrow$ a is odd
Real / Conjugate Symmetry	x is real $\Leftrightarrow$ a is conjugate symmetric

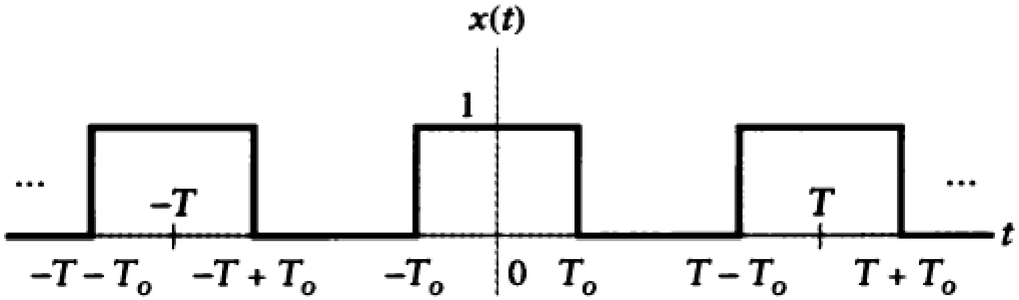
Continuous Time Fourier Transform (CTFT)

Motivation for CTFT

- The (CT) Fourier series provide an extremely useful representation for periodic functions.
- Often, however, we need to deal with functions that are not periodic.
- A more general tool than the Fourier series is needed in this case.
- The (CT) Fourier transform can be used to represent both periodic and aperiodic functions.
- Since the Fourier transform is essentially derived from Fourier series through a limiting process, the Fourier transform has many similarities with Fourier series.

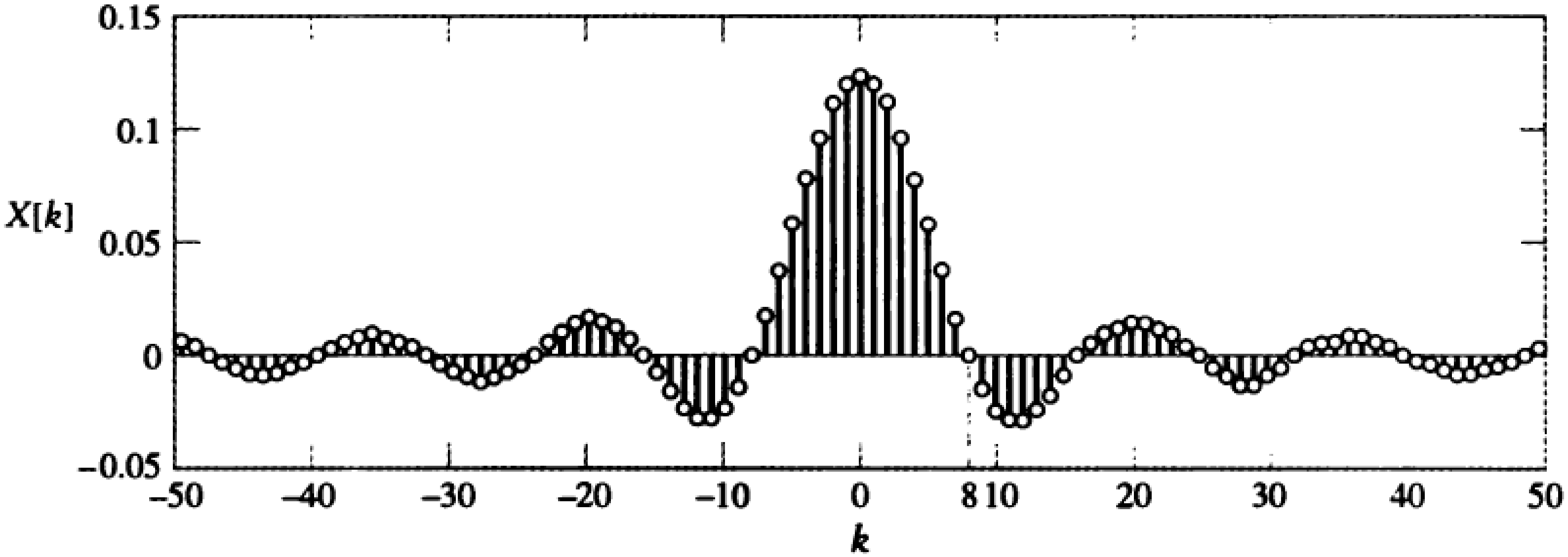
CTFT from CTFS

Recall the example of periodic rectangular pulses $x(t)$ discussed earlier.



CTFT from CTFS

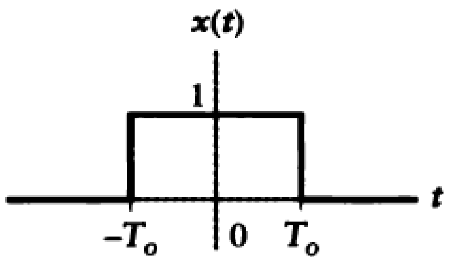
Recall the corresponding CTFS coefficients $X[k]$ obtained earlier.

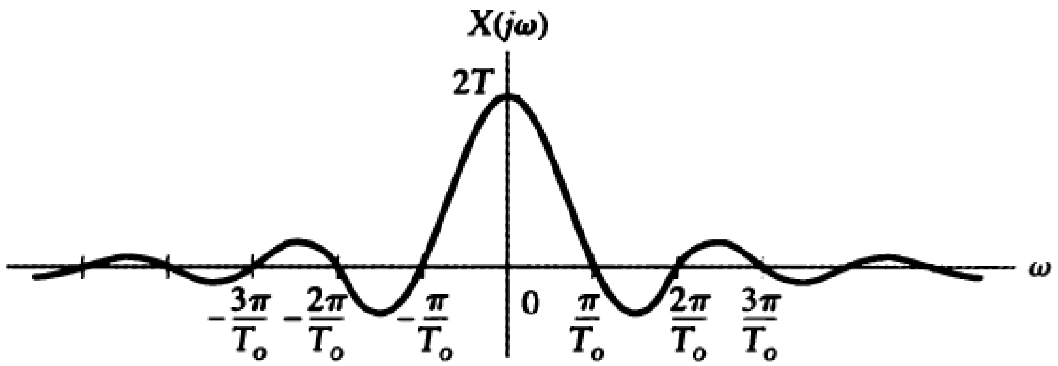


CTFT from CTFS

If we make the period T grow very large, then $x(t)$ becomes a non-periodic signal and the fundamental frequency $\omega_0 = \dfrac{2\pi}{T}$ reduces to $\Delta\omega \approx 0$.

- *Then, it is intuitively clear that in the limit $T \to \infty$ the discrete-index function $X[k]$ becomes a function $X(\omega)$ of continuous frequency ω.*





CTFT from CTFS

The FS coefficient $X[k]$ becomes

$$X[k] = \frac{2T_0}{T}\frac{\sin(k2\pi T_0/T)}{k2\pi T_0/T} = \frac{2T_0}{T}\frac{\sin(k\Delta\omega T_0)}{k\Delta\omega T_0} = \frac{1}{2\pi}\frac{2\sin(k\Delta\omega T_0)}{k\Delta\omega}\Delta\omega,$$

and in the limit $T \to \infty$, we have $\Delta\omega \to d\omega$ and $k\Delta\omega \to \omega$, and the FS representation of $x(t)$ becomes

$$x(t) = \sum_{k=-\infty}^{\infty} X[k]e^{jk\omega_0 t} \xrightarrow[\mathrm{T}\to\infty]{} \frac{1}{2\pi}\int_{-\infty}^{\infty} \underbrace{\frac{2\sin(\omega T_0)}{\omega}}_{X(\omega)} e^{j\omega t}\, d\omega$$

CTFT from CTFS

In general, for any periodic signal $x(t)$ with period T, we have

$$x(t) = \sum_{k=-\infty}^{\infty} X[k] e^{jk\omega_0 t} = \sum_{k=-\infty}^{\infty} \left(\frac{1}{T} \int_{-T/2}^{T/2} x(t) e^{-jk(2\pi/T)t} \, dt \right) e^{jk\omega_0 t}$$

$$\xrightarrow[\mathrm{T}\to\infty]{} \frac{1}{2\pi} \int_{-\infty}^{\infty} \left(\int_{-\infty}^{\infty} x(t) e^{-j\omega t} \, dt \right) e^{j\omega t} \, d\omega = \frac{1}{2\pi} \int_{-\infty}^{\infty} X(\omega) e^{j\omega t} \, d\omega,$$

where $\boxed{X(\omega) = \int_{-\infty}^{\infty} x(t) e^{-j\omega t} \, dt}$ *is called the* ***Fourier Transform*** *of the* ***non-periodic*** *signal $x(t)$.*

Dirichlet Conditions for the Existence of CTFT

*The foregoing equation representing $x(t)$ as an integration (weighted sum) of a continuum of frequency components and the equation representing the corresponding Fourier Transform $X(\omega)$ are valid at all points, except at discontinuities, only if the **Dirichlet conditions** are satisfied:*

- *$x(t)$ is absolutely integrable, i.e., if*

$$\int_{-\infty}^{\infty} |x(t)|dt \leq M_x < \infty.$$

- *$x(t)$ has a finite number of maxima, minima and discontinuities in any finite interval, and the size of each discontinuity is finite.*

Dirichlet Conditions for the Existence of CTFT

If the FT exists, then we say that $x(t)$ and $X(\omega)$ form a ***FT pair*** *and denote this relationship as*

$$x(t) \overset{\text{CTFT}}{\longleftrightarrow} X(\omega).$$

Example: ***Consider the signal $x(t) = e^{-at}u(t)$. Does this signal have a Fourier Transform? If yes, find the FT.***

- ***Solution:*** *FT exists only if $a > 0$. If FT exists, then we have*

$$X(\omega) = \frac{1}{a + j\omega}$$

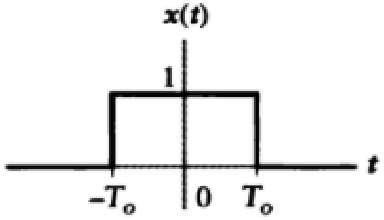
$$\Rightarrow |X(\omega)| = \left(a^2 + \omega^2\right)^{-1/2} \text{ and } \arg\{X(\omega)\} = -\tan^{-1}(\omega/a).$$

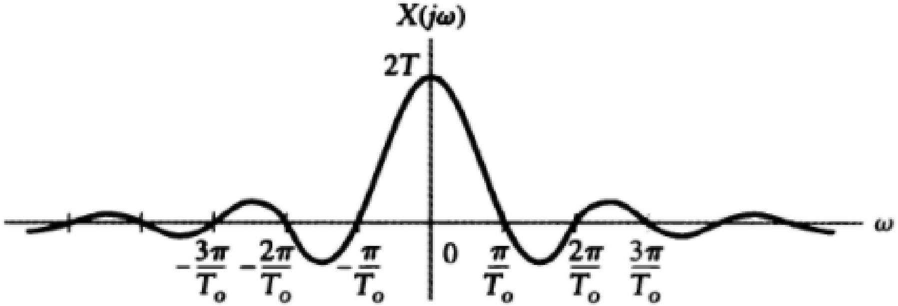
CTFT Example

Example: Consider the (single) rectangular pulse again. We have

$$X(\omega) = \int_{-\infty}^{\infty} x(t)e^{-j\omega t}\,dt = \int_{-T_0}^{T_0} e^{-j\omega t}\,dt = \begin{cases} \dfrac{2}{\omega}\sin(\omega T_0) & \omega \neq 0 \\ 2T_0 & \omega = 0. \end{cases}$$

- **Write $X(\omega)$ in terms of $\mathrm{sinc}(.)$ function.**
- **Sketch the magnitude and phase spectra.**

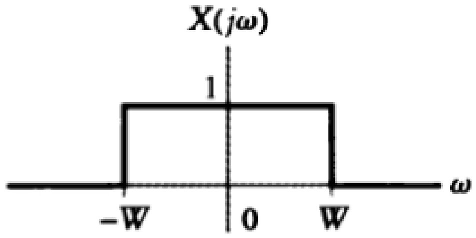


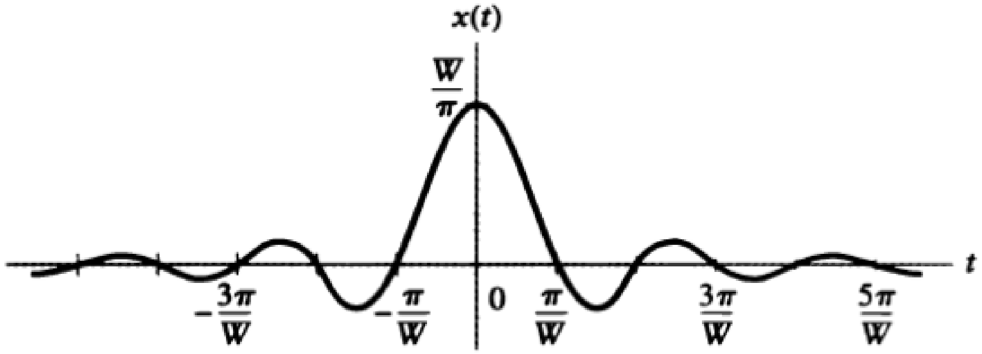


Inverse Fourier Transform Example

Example: Consider a rectangular spectrum in the frequency domain shown below. We have

$$x(t) = \frac{1}{2\pi}\int_{-\infty}^{\infty} X(\omega)e^{j\omega t}\,d\omega = \int_{-W}^{W} e^{j\omega t}\,d\omega = \begin{cases} \dfrac{\sin(Wt)}{\pi t} & t \neq 0 \\ W/\pi & t = 0. \end{cases}$$





Fourier Transforms of Some Important Signals

> **Problem:** **Find the Fourier Transform of each of the following signals: (i) A unit impulse function, (ii) A unit DC signal, (iii) A unit step function, (iv) $x(t) = e^{j\omega_0 t}$, (v) $x(t) = \cos(\omega_0 t)$, (vi) $x(t) = \sin(\omega_0 t)$**

> **Solution:** (i) 1, (ii) $2\pi\delta(\omega)$ (requires duality property, to be discussed later), (iii) $\pi\delta(\omega) + (1/j\omega)$ (requires differentiation property, to be discussed later), (iv) $2\pi\delta(\omega - \omega_0)$, (v) $\pi[\delta(\omega - \omega_0) + \delta(\omega + \omega_0)]$, (vi) $j\pi[\delta(\omega - \omega_0) - \delta(\omega + \omega_0)]$.

CTFT Properties

Linearity

- If $x_1(t) \xleftrightarrow{\text{CTFT}} X_1(\omega)$ and $x_2(t) \xleftrightarrow{\text{CTFT}} X_2(\omega)$, then

$$a_1 x_1(t) + a_2 x_2(t) \xleftrightarrow{\text{CTFT}} a_1 X_1(\omega) + a_2 X_2(\omega),$$

  where a_1 and a_2 are arbitrary complex constants.
- This is known as the **linearity property** of the Fourier transform.

Time Shifting (Translation)

- If $x(t) \overset{\text{CTFT}}{\longleftrightarrow} X(\omega)$, then

$$x(t - t_0) \overset{\text{CTFT}}{\longleftrightarrow} e^{-j\omega t_0} X(\omega),$$

  where t_0 is an arbitrary real constant.
- This is known as the **translation (or time-domain shifting) property** of the Fourier transform.

Frequency Shifting (Modulation)

- If $x(t) \overset{\text{CTFT}}{\longleftrightarrow} X(\omega)$, then

$$e^{j\omega_0 t}x(t) \overset{\text{CTFT}}{\longleftrightarrow} X(\omega - \omega_0),$$

  where ω_0 is an arbitrary real constant.
- This is known as the **modulation (or frequency-domain shifting) property** of the Fourier transform.

Time and Frequency Scaling

- If $x(t) \stackrel{\text{CTFT}}{\longleftrightarrow} X(\omega)$, then

$$x(at) \stackrel{\text{CTFT}}{\longleftrightarrow} \frac{1}{|a|} X\left(\frac{\omega}{a}\right),$$

  where a is an arbitrary nonzero real constant.
- This is known as the **dilation (or time/frequency-scaling) property** of the Fourier transform.

Conjugation

- If $x(t) \xleftrightarrow{\text{CTFT}} X(\omega)$, then

$$x^*(t) \xleftrightarrow{\text{CTFT}} X^*(-\omega).$$

- This is known as the **conjugation property** of the Fourier transform.

Duality

- If $x(t) \overset{\text{CTFT}}{\longleftrightarrow} X(\omega)$, then
$$X(t) \overset{\text{CTFT}}{\longleftrightarrow} 2\pi x(-\omega)$$
- This is known as the **duality property** of the Fourier transform.
- This property follows from the high degree of symmetry in the forward and inverse Fourier transform equations, which are respectively given by
$$X(\lambda) = \int_{-\infty}^{\infty} x(\theta) e^{-j\theta\lambda} d\theta \quad \text{and} \quad x(\lambda) = \frac{1}{2\pi} \int_{-\infty}^{\infty} X(\theta) e^{j\theta\lambda} d\theta.$$
- That is, the forward and inverse Fourier transform equations are identical except for a *factor of* 2π and *different sign* in the parameter for the exponential function.

Convolution

- If $x_1(t) \overset{\text{CTFT}}{\longleftrightarrow} X_1(\omega)$ and $x_2(t) \overset{\text{CTFT}}{\longleftrightarrow} X_2(\omega)$, then

$$x_1 * x_2(t) \overset{\text{CTFT}}{\longleftrightarrow} X_1(\omega)X_2(\omega).$$

- This is known as the **convolution (or time-domain convolution) property** of the Fourier transform.
- In other words, a convolution in the time domain becomes a multiplication in the frequency domain.
- This suggests that the Fourier transform can be used to avoid having to deal with convolution operations.

Multiplication

- If $x_1(t) \overset{\text{CTFT}}{\longleftrightarrow} X_1(\omega)$ and $x_2(t) \overset{\text{CTFT}}{\longleftrightarrow} X_2(\omega)$, then

$$x_1(t)x_2(t) \overset{\text{CTFT}}{\longleftrightarrow} \tfrac{1}{2\pi}X_1 * X_2(\omega) = \frac{1}{2\pi}\int_{-\infty}^{\infty} X_1(\theta)X_2(\omega-\theta)d\theta.$$

- This is known as the **(time-domain) multiplication (or frequency-domain convolution) property** of the Fourier transform.
- In other words, multiplication in the time domain becomes convolution in the frequency domain (up to a scale factor of 2π).
- Do not forget the factor of $\frac{1}{2\pi}$ in the above formula!
- This property of the Fourier transform is often tedious to apply (in the forward direction) as it turns a multiplication into a convolution.

Differentiation in Time Domain

- If $x(t) \overset{\text{CTFT}}{\longleftrightarrow} X(\omega)$, then

$$\frac{dx(t)}{dt} \overset{\text{CTFT}}{\longleftrightarrow} j\omega X(\omega).$$

- This is known as the **(time-domain) differentiation property** of the Fourier transform.
- Differentiation in the time domain becomes multiplication by $j\omega$ in the frequency domain.
- Of course, by repeated application of the above property, we have that $\left(\frac{d}{dt}\right)^n x(t) \overset{\text{CTFT}}{\longleftrightarrow} (j\omega)^n X(\omega)$.
- The above suggests that the Fourier transform might be a useful tool when working with differential (or integro-differential) equations.

Differentiation in Frequency Domain

- If $x(t) \xleftrightarrow{\text{CTFT}} X(\omega)$, then

$$tx(t) \xleftrightarrow{\text{CTFT}} j\frac{d}{d\omega}X(\omega).$$

- This is known as the **frequency-domain differentiation property** of the Fourier transform.

Integration in Time Domain

- If $x(t) \overset{\text{CTFT}}{\longleftrightarrow} X(\omega)$, then

$$\int_{-\infty}^{t} x(\tau)d\tau \overset{\text{CTFT}}{\longleftrightarrow} \frac{1}{j\omega}X(\omega) + \pi X(0)\delta(\omega).$$

- This is known as the **(time-domain) integration property** of the Fourier transform.
- Whereas differentiation in the time domain corresponds to ***multiplication*** by $j\omega$ in the frequency domain, integration in the time domain is associated with ***division*** by $j\omega$ in the frequency domain.
- Since integration in the time domain becomes division by $j\omega$ in the frequency domain, integration can be easier to handle in the frequency domain.
- The above property suggests that the Fourier transform might be a useful tool when working with integral (or integro-differential) equations.

Even and Odd Symmetry

- For a function x with Fourier transform X, the following assertions hold:

$$x \text{ is even} \Leftrightarrow X \text{ is even}; \quad \text{and}$$
$$x \text{ is odd} \Leftrightarrow X \text{ is odd}.$$

- In other words, the forward and inverse Fourier transforms preserve even/odd symmetry.

Real-Valued Functions

- A function x is ***real*** if and only if its Fourier transform X satisfies

$$X(\omega) = X^*(-\omega) \text{ for all } \omega$$

(i.e., X is ***conjugate symmetric***).

- Thus, for a real-valued function, the portion of the graph of a Fourier transform for negative values of frequency ω is ***redundant***, as it is completely determined by symmetry.
- From properties of complex numbers, one can show that $X(\omega) = X^*(-\omega)$ is equivalent to

$$|X(\omega)| = |X(-\omega)| \quad \text{and} \quad \arg X(\omega) = -\arg X(-\omega)$$

(i.e., $|X(\omega)|$ is ***even*** and $\arg X(\omega)$ is ***odd***).

- Note that x being real does ***not*** necessarily imply that X is real.

Parseval's Relation

- Recall that the energy of a function x is given by $\int_{-\infty}^{\infty} |x(t)|^2 \, dt$.
- If $x(t) \overset{\text{CTFT}}{\longleftrightarrow} X(\omega)$, then

$$\int_{-\infty}^{\infty} |x(t)|^2 \, dt = \frac{1}{2\pi} \int_{-\infty}^{\infty} |X(\omega)|^2 \, d\omega$$

  (i.e., the energy of x and energy of X are equal up to a factor of 2π).
- This relationship is known as **Parseval's relation**.
- Since energy is often a quantity of great significance in engineering applications, it is extremely helpful to know that the Fourier transform ***preserves energy*** (up to a scale factor).

Other Properties

Time-Bandwidth Product

Table of CTFT Properties

Property	Time Domain	Frequency Domain
Linearity	$a_1x_1(t)+a_2x_2(t)$	$a_1X_1(\omega)+a_2X_2(\omega)$
Time-Domain Shifting	$x(t-t_0)$	$e^{-j\omega t_0}X(\omega)$
Frequency-Domain Shifting	$e^{j\omega_0 t}x(t)$	$X(\omega-\omega_0)$
Time/Frequency-Domain Scaling	$x(at)$	$\frac{1}{\lvert a\rvert}X\left(\frac{\omega}{a}\right)$
Conjugation	$x^*(t)$	$X^*(-\omega)$
Duality	$X(t)$	$2\pi x(-\omega)$
Time-Domain Convolution	$x_1 * x_2(t)$	$X_1(\omega)X_2(\omega)$
Time-Domain Multiplication	$x_1(t)x_2(t)$	$\frac{1}{2\pi}X_1 * X_2(\omega)$
Time-Domain Differentiation	$\frac{d}{dt}x(t)$	$j\omega X(\omega)$
Frequency-Domain Differentiation	$tx(t)$	$j\frac{d}{d\omega}X(\omega)$
Time-Domain Integration	$\int_{-\infty}^{t}x(\tau)d\tau$	$\frac{1}{j\omega}X(\omega)+\pi X(0)\delta(\omega)$

Table of CTFT Properties

Property	
Parseval's Relation	$\int_{-\infty}^{\infty} \lvert x(t)\rvert^2 dt = \frac{1}{2\pi}\int_{-\infty}^{\infty} \lvert X(\omega)\rvert^2 d\omega$
Even Symmetry	x is even $\Leftrightarrow$ X is even
Odd Symmetry	x is odd $\Leftrightarrow$ X is odd
Real / Conjugate Symmetry	x is real $\Leftrightarrow$ X is conjugate symmetric

References:

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