NUMBER THEORY AND ALGEBRA

ANSWER

① $1 + \frac{1}{2} + \frac{1}{3} + \dots + \frac{1}{23} = \frac{a}{23!}$

$a = 23! + \frac{23!}{2} + \frac{23!}{3} + \dots + \frac{23!}{23}$

$= \frac{23!}{13 \times 13}$

$= \frac{(23)\cdot(22)\cdot(21)(20)\dots(14)\cdot(12)(11)\cdot(10)\dots 1}{13}$

$= (26-3)(26-4)\dots 19 \dots 2 / 13$

$= (26-3)(26-4)\dots(13+6)\dots(13+1)(13-1)(13-2)\dots$
$(13-6)\cdot 6 \cdot 5 \cdot 4 \cdot 3 \cdot 2$

$= (-3)(-4)(-5)(-6)(6)(5)(4)(3)(2)(-1)(-2)(-3)$
$(-4)(-5)(-6)(6)(5)(4)(3)(2)(1) / 13$

$= (3 \cdot 4 \cdot 5 \cdot 6)^4 (2)^3 / 13$

$= (60 \times 6)^4 \times \frac{8}{13}$

$= (8 \times 6)^4 \times 8 / 13$

$= (48)^4 \times 8 / 13$

$= (39+9)^4 \times 8/13 \quad = (9)^4 \times 8/13$

$= (81)^2 \times 8/13 \quad = (3)^2 \times 8/13 \quad = 72/13$

So the final Remainder is 7.

(2) $(\cancel{42})\ (47)^{2022} \pmod{113}$

$[(47)^2]^{1011} \pmod{113}$

$= (2209)^{1011}$

$= (62)^{1011}$

$= (62)\left[(62)^2\right]^{505}$

$= [3844]^{505}\ (62)$

$= [2]^{505}\ (62)$

$= (62)(32)\ (16)^{125}$

$= (1984)\ (16)^{125}$

$= (63)(16)\left[(16)^2\right]^{62}$

$= (104)\ [30]^{62}$

$= (104)\ (900)^{31}$

$= (109)(104)\left[(109)^2\right]^{15}$

$= (36)\ (16)^{15}$

$= (256)^7\ (36)(16)$

$= (30)^7\ 11$

$= (900)^3\ 330$

$= (109)^3\ 104$

$= 16 \times 36$

$= 1$

(3) $2026! - 2022! \mod (2027)$

$(2022!)\left[(2023 \times 2024 \times 2025 \times 2026) - 1\right]$

$(2022!)\ (16798513762800 - 1) \mod 2027$

$(2022!)\ (23) \mod (2027)$

(4) $2x \equiv 6 \mod 14$ $\quad b_1 \mod m_1$

$3x \equiv 9 \mod 15$ $\quad b_2 \mod m_2$

$5x \equiv 20 \mod 60$ $\quad b_3 \mod m_3$

$\gcd(14, 15, 60) = 1$

$M = 14 \times 15 \times 60 = 12600$

$M_1 = \frac{12600}{14} = 15 \times 60 = 900$

$M_2 = 14 \times 60 = 840$

$M_3 = 14 \times 15 = 210$

$\cancel{M_1^{-1} \mod m_1}$ $\quad$ $\cancel{M_2^{-1} \mod m_2}$ $\quad$ $\cancel{M_3^{-1} \mod m_3}$

$\cancel{900^{-1} \mod 14}$ $\quad$ $\cancel{840 \mod 15}$ $\quad$ $\cancel{210 \mod 2}$

$\cancel{2}$ $\quad$ $\cancel{0}$ $\quad$ $\cancel{30}$

$\cancel{x = 2 \times 900 \times 6 + 30 \times 210 \times 20}$

$\cancel{= 10{,}800 + 126000}$

$\cancel{= 136800}$

$x \equiv 3 \mod 7$

$x \equiv 3 \mod 5$

$x \equiv 4 \mod 12$

$m_1 = 5 \times 12 = 60 = 4 \pmod 7$

$m_2 = 7 \times 12 = 84 \equiv 4 \pmod 5$

$m_3 = 7 \times 5 = 35 = 11 \pmod{12}$

$y_1 = 2^5 \equiv 1024 \equiv 2 \pmod 7$
$y_2 \equiv 4^3 \equiv 64 \equiv 4 \pmod 5$
$y_3 = 11^3 \equiv (-1)^3 \equiv -1 \equiv 11 \pmod{12}$

$x = y_1 m_1 b_1 + y_2 m_2 b_2 + y_3 m_3 b_3$
$= 2 \times 60 \times 3 + 4 \times 84 \times 3 + 11 \times 35 \times 4$
$= 2908 \pmod{420}$
$= 388 \pmod{420}$