Exercise Sheet 1
Linear Algebra II

Autumn semester 2018

Exercise 1. Determine whether the following sets are subspaces of $\mathbb{R}^3$ under the operations of addition and scalar multiplication defined on $\mathbb{R}^3$. Justify your answers.

a) $\mathcal{W}_1 = \{(a_1, a_2, a_3) \in \mathbb{R}^3 : a_1 = 3a_2, a_3 = -a_2\}$

b) $\mathcal{W}_2 = \{(a_1, a_2, a_3) \in \mathbb{R}^3 : 2a_1 - 7a_2 + a_3 = 0\}$

c) $\mathcal{W}_3 = \{(a_1, a_2, a_3) \in \mathbb{R}^3 : a_1^2 - 3a_3^2 = 0\}$

Remark During the tutorial lecture I also showed for a) that the zero vector is contained in $\mathcal{W}_1$. As was pointed out in class this is not necessary. It suffices to show that $\mathcal{W}_1$ is closed under scalar multiplication and vector addition. There was also a sign error in part c) that was corrected in these notes.

Solution to 1 a) Let $(a_1, a_2, a_3), (b_1, b_2, b_3) \in \mathcal{W}_1$. Then we have

$$a_1 = 3a_2, \qquad a_3 = -a_2\ , \tag{1.1}$$
$$b_1 = 3b_2, \qquad b_3 = -b_2\ . \tag{1.2}$$

Adding the equations on the second line to the ones on the first we obtain:

$$a_1 + b_1 = 3(a_2 + b_2), \qquad a_3 + b_3 = -(a_2 + b_2)\ .$$

Therefore the vector $(a_1 + b_1, a_2 + b_2, a_3 + b_3) \in \mathbb{R}^3$ is also an element of $\mathcal{W}_1$. Let $\lambda \in \mathbb{R}$ and multiply (1.1) by λ. This yields

$$\lambda a_1 = 3\lambda a_2, \qquad \lambda a_3 = -\lambda a_2$$

This means that if (a_1, a_2, a_3) is in $\mathcal{W}_1$, then so is $(\lambda a_1, \lambda a_2, \lambda a_3)$. Thus, $\mathcal{W}_1$ is a subspace of $\mathbb{R}^3$.

Solution to 1 b) We could use the same approach we used in 1a) to solve this question as well. A slightly different solution is based on the fact that $T : \mathbb{R}^3 \to \mathbb{R}$ given by $T(a_1, a_2, a_3) = 2a_1 - 7a_2 + a_3$ is a linear transformation. This follows from

$$\begin{aligned}
&T((a_1, a_2, a_3) + \lambda(b_1, b_2, b_3)) = T(a_1 + \lambda b_1, a_2 + \lambda b_2, a_3 + \lambda b_3) \\
&= 2(a_1 + \lambda b_1) - 7(a_2 + \lambda b_2) + (a_3 + \lambda b_3) \\
&= 2a_1 - 7a_2 + a_3 + \lambda(2b_1 - 7b_2 + b_3) \\
&= T(a_1, a_2, a_3) + \lambda T(b_1, b_2, b_3)
\end{aligned}$$

The set $\mathcal{W}_2$ is the kernel of the linear transformation T. In particular, $\mathcal{W}_2$ is a subspace of $\mathbb{R}^3$.

Solution to 1 c) The set $\mathcal{W}_3$ is not a subspace of $\mathbb{R}^3$, since $(\sqrt{3}, 0, 1)$ and $(3, 0, -\sqrt{3})$ are both elements of $\mathcal{W}_3$, but $(3 + \sqrt{3}, 0, 1 - \sqrt{3})$ is not, since

$$(3 + \sqrt{3})^2 - 3(1 - \sqrt{3})^2 = 12 \cdot \sqrt{3} \neq 0 \, .$$

Exercise 2. Consider the following definition:

DEFINITION **3.1.** If S_1 and S_2 are nonempty subsets of a vector space $\mathcal{V}$, then the **sum** of S_1 and S_2, denoted by $S_1 + S_2$, is the set $\{x + y : x \in S_1, y \in S_2\}$.

Now consider two subspaces $\mathcal{W}_1$ and $\mathcal{W}_2$ of $\mathcal{V}$.

a) Prove that $\mathcal{W}_1 + \mathcal{W}_2$ is a subspace of $\mathcal{V}$ that contains both $\mathcal{W}_1$ and $\mathcal{W}_2$.

b) Prove that any subspace of $\mathcal{V}$ that contains both $\mathcal{W}_1$ and $\mathcal{W}_2$ must also contain $\mathcal{W}_1 + \mathcal{W}_2$.

Solution to 2 a) Let $x \in \mathcal{W}_1$. Since $\mathcal{W}_2$ is a subspace of $\mathcal{V}$, we have $0 \in \mathcal{W}_2$. By definition the vector $x = x + 0$ is contained in $\mathcal{W}_1 + \mathcal{W}_2$. This shows that $\mathcal{W}_1$ is contained in $\mathcal{W}_1 + \mathcal{W}_2$. A similar argument shows that the same is true for $\mathcal{W}_2$.

Note that $0 \in \mathcal{W}_1$ and $0 \in \mathcal{W}_2$, since both are subspaces of $\mathcal{V}$. Therefore $0 = 0 + 0$ is contained in $\mathcal{W}_1 + \mathcal{W}_2$. Let $x_1 + y_1$ and $x_2 + y_2$ be elements of $\mathcal{W}_1 + \mathcal{W}_2$ with $x_1, x_2 \in \mathcal{W}_1$ and $y_1, y_2 \in \mathcal{W}_2$. Let $\lambda \in F$. Since $\mathcal{W}_1$ and $\mathcal{W}_2$ are subspaces we have

$$\begin{aligned} \lambda x_1 \in \mathcal{W}_1 \quad &, \quad x_1 + x_2 \in \mathcal{W}_1 \ , \\ \lambda y_1 \in \mathcal{W}_2 \quad &, \quad y_1 + y_2 \in \mathcal{W}_2 \ . \end{aligned}$$

This implies that $\lambda(x_1 + y_1) = \lambda x_1 + \lambda y_1 \in \mathcal{W}_1 + \mathcal{W}_2$ and $x_1 + y_1 + x_2 + y_2 = (x_1 + x_2) + (y_1 + y_2) \in \mathcal{W}_1 + \mathcal{W}_2$. This shows that $\mathcal{W}_1 + \mathcal{W}_2$ is a subspace of $\mathcal{V}$.

Solution to 2 b) Let $\mathcal{W}$ be a subspace of $\mathcal{V}$ that contains $\mathcal{W}_1$ and $\mathcal{W}_2$. Let $x + y \in \mathcal{W}_1 + \mathcal{W}_2$ with $x \in \mathcal{W}_1$ and $y \in \mathcal{W}_2$. Since $\mathcal{W}$ contains $\mathcal{W}_1$, we have $x \in \mathcal{W}$. Likewise $y \in \mathcal{W}$. Since $\mathcal{W}$ is a subspace of $\mathcal{V}$, we obtain $x + y \in \mathcal{W}$. But $x + y$ was an arbitrary vector in $\mathcal{W}_1 + \mathcal{W}_2$. Therefore $\mathcal{W}_1 + \mathcal{W}_2 \subset \mathcal{W}$.

Exercise 3. Show that if

$$M_1 = \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix}, M_2 = \begin{pmatrix} 0 & 0 \\ 0 & 1 \end{pmatrix}, \text{ and } M_3 = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix},$$

then $\text{span}\{M_1, M_2, M_3\}$ is the set of all symmetric 2×2 matrices, when using the usual operations of matrix addition and scalar multiplication.

Solution to 3) Let $S \subset M_{2\times 2}(F)$ be the set of all symmetric 2×2-matrices over a field F. Let $A \in S$. Then A satisfies by definition $A^t = A$ and is therefore of the form

$$A = \begin{pmatrix} a & b \\ b & c \end{pmatrix} .$$

In particular, it can be written as a linear combination of the matrices M_1, M_2, M_3 as follows:

$$A = a \cdot \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix} + c \cdot \begin{pmatrix} 0 & 0 \\ 0 & 1 \end{pmatrix} + b \cdot \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} = aM_1 + cM_2 + bM_3 .$$

This shows that $A \in \text{span}\{M_1, M_2, M_3\}$. Since A was arbitrary, we obtain $S \subset \text{span}\{M_1, M_2, M_3\}$. Let $B \in \text{span}\{M_1, M_2, M_3\}$. Then there are scalars $a, b, c \in F$ such that

$$B = aM_1 + cM_2 + bM_3 = \begin{pmatrix} a & b \\ b & c \end{pmatrix} .$$

Since the result is a symmetric matrix and B was arbitrary, we have $\text{span}\{M_1, M_2, M_3\} \subset S$. Therefore $S = \text{span}\{M_1, M_2, M_3\}$.

Exercise 4. Show that if S_1 and S_2 are subsets of a vector space $\mathcal{V}$, then $\text{span}(S_1 \cup S_2) = \text{span}(S_1) + \text{span}(S_2)$.

Solution to 4) First note that if U_1 and U_2 are both subsets of $\mathcal{V}$ with the property that $U_1 \subset U_2$, then $\text{span}(U_1) \subset \text{span}(U_2)$. Since $S_1 \subset S_1 \cup S_2$ and $S_2 \subset S_1 \cup S_2$, we obtain

$$\text{span}(S_1) \subset \text{span}(S_1 \cup S_2) ,$$
$$\text{span}(S_2) \subset \text{span}(S_1 \cup S_2) .$$

It follows from Exercise 2b) that

$$\text{span}(S_1) + \text{span}(S_2) \subset \text{span}(S_1 \cup S_2) .$$

Therefore it remains to show the other inclusion. Let $v \in \text{span}(S_1 \cup S_2)$. This means that there are finitely many vectors from $S_1 \cup S_2$ such that v is a linear combination of those. Let $s_1, \ldots, s_k \in S_1$, $a_1, \ldots, a_k \in F$, $t_1, \ldots, t_l \in S_2$ and $b_1, \ldots, b_l \in F$ such that

$$v = a_1 s_1 + \cdots + a_k s_k + b_1 t_1 + \cdots + b_l t_l .$$

Note that $a_1 s_1 + \cdots + a_k s_k \in \text{span}(S_1)$ and $b_1 t_1 + \cdots + b_l t_l \in \text{span}(S_2)$. Therefore $v \in \text{span}(S_1) + \text{span}(S_2)$. Since we started with an arbitrary vector $v \in \text{span}(S_1 \cup S_2)$, this shows $\text{span}(S_1 \cup S_2) \subset \text{span}(S_1) + \text{span}(S_2)$.

Exercise 5. Let S_1 and S_2 be subsets of a vector space $\mathcal{V}$. Prove that $\text{span}(S_1 \cap S_2) \subset \text{span}(S_1) \cap \text{span}(S_2)$. Give an example in which $\text{span}(S_1 \cap S_2)$ and $\text{span}(S_1) \cap \text{span}(S_2)$ are equal and one in which they are unequal.

Solution to 5) Note that if U_1 and U_2 are both subsets of $\mathcal{V}$ with the property that $U_1 \subset U_2$, then $\text{span}(U_1) \subset \text{span}(U_2)$. Since $S_1 \cap S_2 \subset S_1$ and $S_1 \cap S_2 \subset S_2$, we obtain

$$\text{span}(S_1 \cap S_2) \subset \text{span}(S_1) ,$$
$$\text{span}(S_1 \cap S_2) \subset \text{span}(S_2) .$$

Therefore $\text{span}(S_1 \cap S_2) \subset \text{span}(S_1) \cap \text{span}(S_2)$.

Let $\mathcal{W} \subset \mathcal{V}$ be a subspace of $\mathcal{V}$ and let $w \in \mathcal{W}$. Let $S_1 = \{w\}$ and let $S_2 = \mathcal{W}$. Then we have

$$\text{span}(S_1 \cap S_2) = \text{span}(\{w\}) .$$

Since $w \in \mathcal{W}$, we obtain that $\text{span}(\{w\})$ is a subspace of $\mathcal{W}$. Moreover, $\text{span}(\mathcal{W}) = \mathcal{W}$. Thus,

$$\text{span}(S_1) \cap \text{span}(S_2) = \text{span}(\{w\}) \cap \text{span}(\mathcal{W}) = \text{span}(\{w\}) \cap \mathcal{W} = \text{span}(\{w\}) .$$

Thus, the above is an example where the two subspaces are equal.

Let $\mathcal{V} = \mathbb{R}^2$ and let $S_1 = \{(1,0),(0,1)\}$, $S_2 = \{(2,0),(0,1)\}$. Then

$$\text{span}(S_1 \cap S_2) = \text{span}\{(0,1)\} ,$$

but $\text{span}(S_1) = \mathbb{R}^2$, $\text{span}(S_2) = \mathbb{R}^2$ and therefore

$$\text{span}(S_1) \cap \text{span}(S_2) = \mathbb{R}^2 .$$

This is an example where the two subspaces are not equal.

Exercise 6. Give an example of three linearly dependent vectors in $\mathbb{R}^3$ such that none of the three is a multiple of another.

Solution to 6) Let

$$u = \begin{pmatrix} 1 \\ -1 \\ 0 \end{pmatrix} \quad , \quad v = \begin{pmatrix} 0 \\ 1 \\ -1 \end{pmatrix} \quad , \quad w = \begin{pmatrix} -1 \\ 0 \\ 1 \end{pmatrix} .$$

These are three vectors in $\mathbb{R}^3$. We have $u + v + w = 0$. Therefore they are linearly dependent. However, none of them is a multiple of the others.

Exercise 7. Let $S = \{u_1, u_2, \dots, u_n\}$ be a finite set of vectors. Prove that S is linearly dependent if and only if $u_1 = 0$ or $u_{k+1} \in \operatorname{span}(\{u_1, \dots, u_k\})$ for some k where $1 \le k < n$.

Solution to 7) Suppose that S is linearly dependent. This means we can find $a_1, \dots, a_n \in F$ with the property that at least one of them is non-zero and

$$a_1u_1 + a_2u_2 + \cdots + a_nu_n = 0 .$$

If $a_1 \neq 0$, but $a_2 = a_3 = \cdots = a_n = 0$, then this equation boils down to $a_1u_1 = 0$, which implies $u_1 = 0$. Otherwise let $m \in \{1, \dots, n\}$ be the smallest number with the property that $a_r = 0$ for all $r > m$ and $a_m \neq 0$. Since we excluded the case $m = 1$ at the beginning, we must have $m \geq 2$ and we can write

$$u_m = -\frac{a_1}{a_m}u_1 - \frac{a_2}{a_m}u_2 - \cdots - \frac{a_{m-1}}{a_m}u_{m-1} .$$

Therefore $u_m \in \operatorname{span}(\{u_1, \dots, u_{m-1}\})$ and the statement is true with $k = m - 1$.

To show the other direction note that $u_1 = 0$ directly implies that S is linearly dependent. If we suppose the other condition, i.e. that $u_{k+1} \in \operatorname{span}(\{u_1, \dots, u_k\})$ for some k with $1 \le k < n$, then we can find $b_1, \dots, b_k \in F$ with the property

$$u_{k+1} = b_1u_1 + \cdots + b_ku_k \quad \Leftrightarrow \quad 0 = b_1u_1 + \cdots + b_ku_k - u_{k+1} .$$

Since the coefficient in front of u_{k+1} is -1 and in particular non-zero, this implies that S is linearly independent.

Exercise 8. The set of all $n \times n$ matrices having trace equal to 0 is a subspace $\mathcal{W}$ of $M_{n\times n}(F)$. Find a basis for $\mathcal{W}$. What is the dimension of $\mathcal{W}$?

Solution to 8) Let $\mathrm{tr}\colon M_{n\times n}(F) \to F$ be the trace map on the vector space of $n \times n$-matrices. Let $A, B \in M_{n\times n}(F)$ be matrices with entries a_{ij} and b_{ij} respectively and let $\lambda \in F$. Then we have

$$\mathrm{tr}(A + \lambda B) = \sum_{i=1}^{n}(a_{ii} + \lambda b_{ii}) = \sum_{i=1}^{n} a_{ii} + \lambda \cdot \left(\sum_{i=1}^{n} b_{ii}\right) = \mathrm{tr}(A) + \lambda \cdot \mathrm{tr}(B)\ .$$

In particular, tr is a linear transformation. The subspace $\mathcal{W} \subset M_{n\times n}(F)$ is the kernel of tr. Since tr is surjective, we obtain from the rank-nullity theorem

$$n^2 = \dim(M_{n\times n}(F)) = \dim(\ker(\mathrm{tr})) + \dim(\mathrm{Im}(\mathrm{tr})) = \dim(\mathcal{W}) + 1\ ,$$

which gives $\dim(\mathcal{W}) = n^2 - 1$. To construct a basis of $\mathcal{W}$ it therefore suffices to find $n^2 - 1$ linearly independent vectors in $\mathcal{W}$. Let $E_{ij} \in M_{n\times n}(F)$ be the matrix which has a 1 in the ith row and the jth column and zeroes everywhere else. The set $\{E_{ij} \mid 1 \leq i \leq n, 1 \leq j \leq n\}$ is a basis for the vector space $M_{n\times n}(F)$. Note that $\mathrm{tr}(E_{ij}) = 0$ for all $i, j \in \{1, \dots, n\}$ with $i \neq j$. Moreover, let $F_i = E_{ii} - E_{nn}$ for $1 \leq i \leq n-1$. Then $F_i \neq 0$ and $\mathrm{tr}(F_i) = 0$. Now consider the set

$$S = \{E_{ij} \mid i, j \in \{1, \dots, n\} \text{ with } i \neq j\} \cup \{F_i \mid 1 \leq i \leq n-1\}$$

It contains $(n^2 - n) + (n - 1) = n^2 - 1$ vectors. Let $a_{ij} \in F$ and consider the linear combination

$$\sum_{i\neq j} a_{ij}E_{ij} + \sum_{i=1}^{n-1} a_{ii}F_i = \begin{pmatrix} a_{11} & \cdots & a_{1(n-1)} & a_{1n} \\ \vdots & \ddots & \vdots & \vdots \\ a_{(n-1)1} & \cdots & a_{(n-1)(n-1)} & a_{(n-1)n} \\ a_{n1} & \cdots & a_{n(n-1)} & x \end{pmatrix}$$

with $x = -a_{11} - a_{22} - \cdots - a_{(n-1)(n-1)}$. If we set this matrix equal to zero, we obtain that all a_{ij} have to be zero. Therefore S is linearly independent.

Exercise 9. The set of all skew-symmetric $n \times n$ matrices is a subspace $\mathcal{W}$ of $M_{n\times n}(\mathbb{R})$. Find a basis for $\mathcal{W}$. What is the dimension of $\mathcal{W}$?

Solution to 9) Let $A \in M_{n\times n}(\mathbb{R})$ be a skew-symmetric matrix with entries a_{ij}. By definition it satisfies $A = -A^t$, which we can express in terms of the entries as $a_{ij} = -a_{ji}$ for all $i, j \in \{1, \dots, n\}$. In particular, the diagonal entries have to satisfy $a_{ii} = -a_{ii}$, which implies that they are zero and the matrix is completely fixed by knowing the entries a_{ij} for $i < j$. Let E_{ij} be the matrices from Exercise 8 and consider

$$S = \{E_{ij} - E_{ji} \mid i, j \in \{1, \dots, n\} \text{ with } i < j\} .$$

Note that $(E_{ij} - E_{ji})^t = E_{ji} - E_{ij} = -(E_{ij} - E_{ji})$. Therefore $S \subset \mathcal{W}$. Let $a_{ij} \in F$ for $i < j$ and consider the linear combination

$$\sum_{i<j} a_{ij}(E_{ij} - E_{ji}) = \begin{pmatrix} 0 & a_{12} & \cdots & a_{1n} \\ -a_{12} & 0 & \cdots & a_{2n} \\ \vdots & \vdots & \ddots & \vdots \\ -a_{1n} & -a_{2n} & \cdots & 0 \end{pmatrix}$$

If we set this equal to zero, it follows that all a_{ij} with $i < j$ have to be zero. This implies that S is linearly independent. Since we are free to choose the entries a_{ij}, we obtain $\operatorname{span}(S) = \mathcal{W}$. We can read off the dimension from this:

$$\dim(\mathcal{W}) = (n-1) + (n-2) + \cdots + 1 = \frac{n(n-1)}{2} .$$

Exercise 10. Prove that if $\mathcal{W}_1$ and $\mathcal{W}_2$ are finite-dimensional subspaces of a vector space $\mathcal{V}$, then the subspace $\mathcal{W}_1+\mathcal{W}_2$ is finite-dimensional, and $\dim(\mathcal{W}_1+\mathcal{W}_2) = \dim(\mathcal{W}_1)+\dim(\mathcal{W}_2)-\dim(\mathcal{W}_1\cap\mathcal{W}_2)$. (Hint: Consider the vector space $\mathcal{W}_1 \cap \mathcal{W}_2$, its finite basis, and extending it to ones for $\mathcal{W}_1$ and $\mathcal{W}_2$)

Solution to 10) Let $S_0 = \{w_1, \dots, w_k\}$ be a (finite) basis for the vector space $\mathcal{W}_1 \cap \mathcal{W}_2$. Let S_1 be an extension of S_0 to a basis of $\mathcal{W}_1$, i.e. $S_1 = \{w_1, \dots, w_k, u_1, \dots, u_r\}$ for vectors $u_1, \dots, u_r \in \mathcal{W}_1 \setminus (\mathcal{W}_1 \cap \mathcal{W}_2)$. Let $S_2 = \{w_1, \dots, w_k, v_1, \dots, v_s\}$ with $v_1, \dots, v_s \in \mathcal{W}_2 \setminus (\mathcal{W}_1 \cap \mathcal{W}_2)$ be a similar extension of S_0 to a basis of $\mathcal{W}_2$. I claim that

$$S = \{w_1, \dots, w_k, u_1, \dots, u_r, v_1, \dots, v_s\}$$

is a basis for $\mathcal{W}_1 + \mathcal{W}_2$. We first have to show that S is linearly independent. Let $c_1, \dots, c_k \in F$, $a_1, \dots, a_r \in F$ and $b_1, \dots b_s \in F$ be scalars such that

$$\sum_{i=1}^{k} c_i w_i + \sum_{j=1}^{r} a_j u_j + \sum_{m=1}^{s} b_m v_m = 0 \, .$$

This can be rewritten as

$$\sum_{i=1}^{k} c_i w_i + \sum_{j=1}^{r} a_j u_j = -\sum_{m=1}^{s} b_m v_m \, .$$

The vector on the left hand side is in $\mathcal{W}_1$, the vector on the right hand side is in $\mathcal{W}_2$. Therefore both sides must denote vectors in $\mathcal{W}_1 \cap \mathcal{W}_2$. Now since $\{w_1, \dots, w_k\}$ is a basis for $\mathcal{W}_1 \cap \mathcal{W}_2$, there must be scalars $d_1, \dots, d_k \in F$ with the property that

$$-\sum_{m=1}^{s} b_m v_m = \sum_{l=1}^{k} d_l w_l \qquad \Leftrightarrow \qquad \sum_{l=1}^{k} d_l w_l + \sum_{m=1}^{s} b_m v_m = 0 \, .$$

But $\{w_1, \dots, w_k, v_1, \dots, v_s\}$ is a basis of $\mathcal{W}_2$. Hence, $d_1 = \cdots = d_k = 0$ and $b_1 = \cdots = b_s = 0$. Since $\{w_1, \dots, w_k, u_1, \dots, u_r\}$ is a basis of $\mathcal{W}_1$, the first equation then implies that $c_1 = \cdots = c_k = 0$ and $a_1 = \cdots = a_r = 0$ as well. Thus, S is linearly independent. Moreover, it is not too difficult to check that

$$\mathrm{span}(S) = \mathcal{W}_1 + \mathcal{W}_2 \, .$$

Since S is a basis for $\mathcal{W}_1 + \mathcal{W}_2$, we have $\dim(\mathcal{W}_1 + \mathcal{W}_2) = k + r + s$. With $\dim(\mathcal{W}_1 \cap \mathcal{W}_2) = k$, $\dim(\mathcal{W}_1) = k + r$ and $\dim(\mathcal{W}_2) = k + s$ we obtain

$$\dim(\mathcal{W}_1 + \mathcal{W}_2) = k + r + s = (k + r) + (k + s) - k = \dim(\mathcal{W}_1) + \dim(\mathcal{W}_2) - \dim(\mathcal{W}_1 \cap \mathcal{W}_2) .$$