# Discrete Time Fourier Series (DTFS) and Fourier Transform (DTFT)

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# Recall: Four Classes of Fourier Representations

| Time Property | Periodic | Nonperiodic |
| --- | --- | --- |
| Continuous (t) | Fourier Series (FS) [Chapter 3.5] | Fourier Transform (FT) [Chapter 3.7] |
| Discrete (n) | Discrete Time Fourier Series (DTFS) [Chapter 3.4] | Discrete Time Fourier Transform (DTFT) [Chapter 3.6] |

# Recall: Frequency Response of DT Systems

Consider a discrete-time LTI system with impulse response $h[n]$.

- Suppose a **complex sinusoid** is applied as input. i.e., $\boxed{x[n] = e^{j\Omega n}}$.
- Then, the output $y[n]$ of the DT LTI system is given by

$$y[n] = \sum_{k=-\infty}^{\infty} h[k]x[n-k] = \sum_{k=-\infty}^{\infty} h[k]e^{j\Omega(n-k)}$$

$$= e^{j\Omega n} \sum_{k=-\infty}^{\infty} h[k]e^{-j\Omega k} = H(e^{j\Omega})e^{j\Omega n}, \quad \text{where}$$

$$\boxed{H(e^{j\Omega}) = \sum_{k=-\infty}^{\infty} h[k]e^{-j\Omega k}} = \text{Frequency Response.}$$

# Recall: Frequency Response of DT Systems

Substituting $\boxed{H(e^{j\Omega}) = |H(e^{j\Omega})|e^{j\arg\{H(e^{j\Omega})\}}}$, we obtain the output of the DT LTI system corresponding to a complex sinusoid $e^{j\Omega n}$ as

$$\boxed{y[n] = H(e^{j\Omega})e^{j\Omega n} = |H(e^{j\Omega})|e^{j(\Omega n + \arg\{H(e^{j\Omega})\})}},$$

which implies that **in steady state** the DT LTI system modifies the magnitude of the input by a factor $|H(e^{j\Omega})|$ and modifies the phase by a shift of $\arg\{H(e^{j\Omega})\}$.

Therefore, $|H(e^{j\Omega})|$ is called the **magnitude response** and $\arg\{H(e^{j\Omega})\}$ is called the **phase response** of the DT LTI system.

# Discrete-Time Fourier Series (DTFS) for Periodic Signals

# Harmonically-Related DT Complex Sinusoids

- A set of periodic complex sinusoids is said to be **harmonically related** if there exists some constant $2\pi/N$ such that the fundamental frequency of each complex sinusoid is an integer multiple of $2\pi/N$.
- Consider the set of harmonically-related complex sinusoids given by

$$\phi_k(n) = e^{j(2\pi/N)kn} \quad \text{for all integer } k.$$

- In the above set $\{\phi_k\}$, only $N$ elements are distinct, since

$$\phi_k = \phi_{k+N} \quad \text{for all integer } k.$$

- Since the fundamental frequency of each of the harmonically-related complex sinusoids is an integer multiple of $\frac{2\pi}{N}$, a linear combination of these complex sinusoids must be $N$-periodic.

# DTFS Definition

- A periodic complex-valued sequence $x$ with fundamental period $N$ can be represented as a linear combination of harmonically-related complex sinusoids as

$$x(n) = \sum_{k=\langle N \rangle} a_k e^{j(2\pi/N)kn},$$

where $\sum_{k=\langle N \rangle}$ denotes summation over any $N$ consecutive integers (e.g., $0, 1, \ldots, N-1$). (The summation can be taken over any $N$ consecutive integers, due to the $N$-periodic nature of $x$ and $e^{j(2\pi/N)kn}$.)

- The above representation of $x$ is known as the (DT) **Fourier series** and the $a_k$ are called **Fourier series coefficients**.

- **Notation:** Here, $\boxed{\Omega_0 = (2\pi/N)}$ denotes the **fundamental (angular) frequency** of the signal $x[n]$, and $\boxed{\Omega_k = k\Omega_0 = (2k\pi/N)}$ denotes its **k-th harmonic frequency**.

## DTFS Definition

- A periodic sequence $x$ with fundamental period $N$ has the Fourier series coefficient sequence $a$ given by

$$a_k = \frac{1}{N} \sum_{n=\langle N \rangle} x(n) e^{-j(2\pi/N)kn}.$$

(The summation can be taken over any $N$ consecutive integers due to the $N$-periodic nature of $x$ and $e^{-j(2\pi/N)kn}$.)

**Notation:**

- Here, $\boxed{a_k = X[k]}$, $k = 0, \ldots, N-1$, denote the DTFS coefficients.
- **Since both $x[n]$ and $e^{j(2\pi/N)kn} = e^{j\Omega_0 kn}$ are periodic functions with period $N$, the DTFS coefficients $a_k$ or $X[k]$, as a function of index $k$, is also periodic with period $N$.**

# DTFS Definition

- To denote that the DT signal (or, sequence) $x[n]$ has the Fourier series coefficient sequence $X[k]$, we write

$$x[n] \stackrel{\text{DTFS}}{\longleftrightarrow} X[k].$$

- The DTFS is the only Fourier representation that can be numerically evaluated and manipulated in a computer.

- This is because both the time and the frequency domain representations of the signal are exactly characterized by a finite set of $N$ numbers.

- DTFS is used for approximating the other three Fourier representations for the purpose of implementation on a computer.

# DTFS Examples and Exercises

**Example: (Method of Inspection) Determine the DTFS coefficients of $x[n] = \cos(\pi n/3 + \phi)$. Sketch the magnitude and phase spectrum.**

***Solution:*** *The period is $N = 6$. The fundamental frequency is $\Omega_0 = 2\pi/N = \pi/3$. The DTFS coefficients are:*

$$X[1] = e^{j\phi}/2, \qquad X[-1] = e^{-j\phi}/2.$$

***Can you draw the magnitude and phase plot?***

**Exercise: Determine the DTFS coefficients of $x[n] = 1 + \sin(\pi n/12 + 3\pi/8)$. Sketch the magnitude and phase spectrum.**

# DTFS Examples and Exercises

**Example:** **Determine the DTFS coefficients of the $N$-periodic impulse train given by**

$$x[n] = \sum_{\ell=-\infty}^{\infty} \delta[n - \ell N].$$

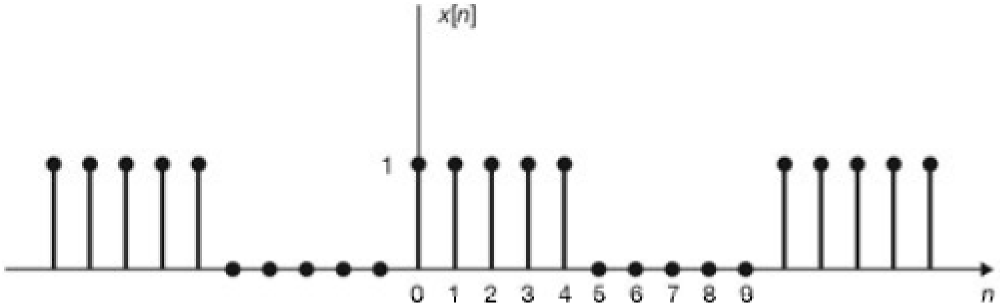
**Sketch the magnitude and phase spectrum.**

***Solution:*** *We have*

$$X[k] = \frac{1}{N} \sum_{n=0}^{N-1} x[n] e^{-jkn2\pi/N} = \frac{1}{N} \sum_{n=0}^{N-1} \delta[n] e^{-jkn2\pi/N} = \frac{1}{N}.$$

# DTFS Examples and Exercises

**Example:** **Determine the DTFS coefficients of the signal shown below. Sketch the magnitude and phase spectrum.**




***Solution:*** *We have* $N = 10$ *and* $\Omega_0 = 2\pi/N = \pi/5$*. Therefore, the DTFS coefficients are given by*

# DTFS Examples and Exercises

***Solution: (continued..)***

$$X[k] = \frac{1}{N} \sum_{n=0}^{N-1} x[n] e^{-jkn2\pi/N} = \frac{1}{10} \sum_{n=0}^{4} e^{-jkn\pi/5}$$

$$= \frac{1}{10} \frac{1 - e^{-jk\pi}}{1 - e^{-jk(\pi/5)}} = \frac{1}{10} \frac{e^{-jk(\pi/2)}}{e^{-jk(\pi/10)}} \frac{e^{jk(\pi/2)} - e^{-jk(\pi/2)}}{e^{jk(\pi/10)} - e^{-jk(\pi/10)}} \cdot$$

$$= \frac{1}{10} e^{-jk(2\pi/5)} \frac{\sin(k(\pi/2))}{\sin(k(\pi/10))}, \qquad k = 0, 1, \ldots, 9.$$

# DTFS Trigonometric Form

Suppose $N$ is even; then, we can rewrite the DTFS by separating the terms corresponding to $k = 0$ and $k = N/2$ from the remaining terms as

$$
\begin{aligned}
x[n] &= \sum_{k=-(N/2)+1}^{N/2} X[k]e^{jk\Omega_0 n} \\
&= X[0] + X[N/2]e^{j\pi n} + \sum_{k=1}^{(N/2)-1} X[k]e^{jk\Omega_0 n} + X[-k]e^{-jk\Omega_0 n} \\
&= X[0] + X[N/2]\cos(\pi n) + \sum_{k=1}^{(N/2)-1} \{B[k]\cos(k\Omega_0 n) + A[k]\sin(k\Omega_0 n)\},
\end{aligned}
$$

where $\boxed{B[k] = X[k] + X[-k]}$ and $\boxed{A[k] = j(X[k] - X[-k])}$.

# DTFS Trigonometric Form

Suppose $N$ is odd; then, we can rewrite the DTFS by separating the term corresponding to $k = 0$ from the remaining terms as

$$
\begin{aligned}
x[n] &= \sum_{k=-(N-1)/2}^{(N-1)/2} X[k] e^{jk\Omega_0 n} \\
&= X[0] + \sum_{k=1}^{(N-1)/2} X[k] e^{jk\Omega_0 n} + X[-k] e^{-jk\Omega_0 n} \\
&= X[0] + \sum_{k=1}^{(N-1)/2} \{ B[k] \cos(k\Omega_0 n) + A[k] \sin(k\Omega_0 n) \},
\end{aligned}
$$

where $\boxed{B[k] = X[k] + X[-k]}$ and $\boxed{A[k] = j(X[k] - X[-k])}$.

# DTFS Properties

## Linearity

- Let $x$ and $y$ be $N$-periodic sequences. If $x(n) \stackrel{\text{DTFS}}{\longleftrightarrow} a_k$ and $y(n) \stackrel{\text{DTFS}}{\longleftrightarrow} b_k$, then

$$\alpha x(n) + \beta y(n) \stackrel{\text{DTFS}}{\longleftrightarrow} \alpha a_k + \beta b_k,$$

  where $\alpha$ and $\beta$ are complex constants.
- That is, a linear combination of sequences produces the same linear combination of their Fourier series coefficients.

## Translation (Time Shifting)

- Let $x$ denote a periodic sequence with period $N$. If $x(n) \overset{\text{DTFS}}{\longleftrightarrow} c_k$, then

$$x(n-n_0) \overset{\text{DTFS}}{\longleftrightarrow} e^{-jk(2\pi/N)n_0} c_k,$$

  where $n_0$ is an integer constant.
- In other words, time shifting a periodic sequence changes the argument (but not magnitude) of its Fourier series coefficients.

## Modulation (Frequency Shifting)

- Let $x$ denote a periodic sequence with period $N$. If $x(n) \overset{\text{DTFS}}{\longleftrightarrow} c_k$, then

$$e^{j(2\pi/N)k_0 n}x(n) \overset{\text{DTFS}}{\longleftrightarrow} c_{k-k_0},$$

  where $k_0$ is an integer constant.

- That is, multiplying a sequence by a complex sinusoid whose frequency is an integer multiple of $2\pi/N$ results in a translation of the corresponding Fourier series coefficient sequence.

## Reflection (Time Reversal)

- Let $x$ denote a periodic sequence with period $N$. If $x(n) \overset{\text{DTFS}}{\longleftrightarrow} c_k$, then

$$x(-n) \overset{\text{DTFS}}{\longleftrightarrow} c_{-k}.$$

- That is, time reversing a sequence results in a time reversal of the corresponding Fourier series coefficient sequence.

## Conjugation

- Let $x$ denote a periodic sequence with period $N$. If $x(n) \overset{\text{DTFS}}{\longleftrightarrow} c_k$, then

$$x^*(n) \overset{\text{DTFS}}{\longleftrightarrow} c^*_{-k}.$$

- In other words, conjugating a sequence has the effect of time reversing and conjugating the corresponding Fourier series coefficient sequence.

# Duality

- Let $x$ denote a periodic sequence with period $N$. If $x(n) \overset{\text{DTFS}}{\longleftrightarrow} a(k)$, then

$$a(n) \overset{\text{DTFS}}{\longleftrightarrow} \tfrac{1}{N} x(-k).$$

- This is known as the **duality property** of Fourier series.
- This property follows from the high degree of symmetry in the analysis and synthesis Fourier-series equations, which are respectively given by

$$x(m) = \sum_{\ell=\langle N \rangle} a(\ell) e^{j(2\pi/N)\ell m} \quad \text{and} \quad a(m) = \tfrac{1}{N} \sum_{\ell=\langle N \rangle} x(\ell) e^{-j(2\pi/N)m\ell}.$$

- That is, the analysis and synthesis equations are identical except for a *factor of $N$* and *different sign* in the parameter for the exponential function.

# Periodic Convolution

- Let $x$ and $y$ be $N$-periodic sequences. If $x(n) \overset{\text{DTFS}}{\longleftrightarrow} a_k$ and $y(n) \overset{\text{DTFS}}{\longleftrightarrow} b_k$, then

$$x \circledast y(n) \overset{\text{DTFS}}{\longleftrightarrow} N a_k b_k .$$

- That is, periodic convolution of two sequences multiplies their corresponding Fourier series coefficient sequences (up to a scale factor).

**Notation:** Here, the **circular convolution** of $x[n]$ and $y[n]$ is given by

$$x[n] \circledast h[n] = \sum_{m=0}^{N-1} x[m] y[n-m].$$

# Multiplication

- Let $x$ and $y$ be $N$-periodic sequences. If $x(n) \overset{\text{DTFS}}{\longleftrightarrow} a_k$ and $y(n) \overset{\text{DTFS}}{\longleftrightarrow} b_k$, then

$$x(n)y(n) \overset{\text{DTFS}}{\longleftrightarrow} a \circledast b(k).$$

- That is, multiplying two sequences results in a circular convolution of their corresponding Fourier series coefficient sequences.

**Notation:** Here, the **circular convolution** of $a_k = X[k]$ and $b_k = Y[k]$ is given by

$$X[k] \circledast Y[k] = \sum_{m=0}^{N-1} X[m]Y[k-m].$$

## Parseval's relation

- A sequence $x$ and its Fourier series coefficient sequence $a$ satisfy the following relationship:

$$\frac{1}{N}\sum_{n=\langle N\rangle}|x(n)|^2=\sum_{k=\langle N\rangle}|a_k|^2.$$

- The above relationship is simply stating that the amount of energy in a single period of $x$ and the amount of energy in a single period of $a$ are equal up to a scale factor.
- In other words, the transformation between a sequence and its Fourier series coefficient sequence preserves energy (up to a scale factor).

## Other Properties

- For an $N$-periodic sequence $x$ with Fourier-series coefficient sequence $a$, the following properties hold:

$$x \text{ is even} \Leftrightarrow a \text{ is even}; \quad \text{and}$$
$$x \text{ is odd} \Leftrightarrow a \text{ is odd}.$$

- In other words, the even/odd symmetry properties of $x$ and $a$ always match.

## Other Properties

- A sequence $x$ is ***real*** if and only if its Fourier series coefficient sequence $a$ satisfies

$$a_k = a^*_{-k} \text{ for all } k$$

(i.e., $a$ is ***conjugate symmetric***).

- From properties of complex numbers, one can show that $a_k = a^*_{-k}$ is equivalent to

$$|a_k| = |a_{-k}| \quad \text{and} \quad \arg a_k = -\arg a_{-k}$$

(i.e., $|a_k|$ is ***even*** and $\arg a_k$ is ***odd***).

- Note that $x$ being real does ***not*** necessarily imply that $a$ is real.

## Other Properties

- For an $N$-periodic sequence $x$ with Fourier-series coefficient sequence $a$, the following properties hold:
  1. $a_0$ is the average value of $x$ over a single period;
  2. $x$ is real and even $\Leftrightarrow$ $a$ is real and even; and
  3. $x$ is real and odd $\Leftrightarrow$ $a$ is purely imaginary and odd.

$$x(n) \overset{\text{DTFS}}{\longleftrightarrow} a_k \quad \text{and} \quad y(n) \overset{\text{DTFS}}{\longleftrightarrow} b_k$$

| Property | Time Domain | Fourier Domain |
|---|---|---|
| Linearity | $\alpha x(n) + \beta y(n)$ | $\alpha a_k + \beta b_k$ |
| Translation | $x(n - n_0)$ | $e^{-jk(2\pi/N)n_0} a_k$ |
| Modulation | $e^{j(2\pi/N)k_0 n} x(n)$ | $a_{k-k_0}$ |
| Reflection | $x(-n)$ | $a_{-k}$ |
| Conjugation | $x^*(n)$ | $a^*_{-k}$ |
| Duality | $a_n$ | $\frac{1}{N} x(-k)$ |
| Periodic Convolution | $x \circledast y(n)$ | $N a_k b_k$ |
| Multiplication | $x(n) y(n)$ | $a \circledast b_k$ |

| Property | |
|---|---|
| Parseval's Relation | $\frac{1}{N} \sum_{n=\langle N \rangle} \lvert x(n) \rvert^2 = \sum_{k=\langle N \rangle} \lvert a_k \rvert^2$ |
| Even Symmetry | $x$ is even $\Leftrightarrow$ $a$ is even |
| Odd Symmetry | $x$ is odd $\Leftrightarrow$ $a$ is odd |
| Real / Conjugate Symmetry | $x$ is real $\Leftrightarrow$ $a$ is conjugate symmetric |

# Discrete Time Fourier Transform (DTFT)

## DTFT from DTFS

- Recall that the Fourier series representation of a $N$-periodic sequence $x$ is given by

$$x(n) = \sum_{k=\langle N\rangle} \underbrace{\left(\frac{1}{N}\sum_{\ell=\langle N\rangle} x(\ell)e^{-j(2\pi/N)k\ell}\right)}_{c_k} e^{j(2\pi/N)kn}.$$

- In the above representation, if we take the limit as $N \to \infty$, we obtain

$$x(n) = \frac{1}{2\pi}\int_{2\pi} \underbrace{\left(\sum_{n=-\infty}^{\infty} x(n)e^{-j\Omega n}\right)}_{X(\Omega)} e^{j\Omega n}d\Omega$$

## FT and Inverse FT Equations

- The **Fourier transform** of the sequence $x$, denoted $\mathcal{F}x$ or $X$, is given by

$$\mathcal{F}x(\Omega) = X(\Omega) = \sum_{n=-\infty}^{\infty} x(n)e^{-j\Omega n}.$$

- The preceding equation is sometimes referred to as **Fourier transform analysis equation** (or **forward Fourier transform equation**).
- The **inverse Fourier transform** of $X$, denoted $\mathcal{F}^{-1}X$ or $x$, is given by

$$\mathcal{F}^{-1}X(n) = x(n) = \frac{1}{2\pi}\int_{2\pi} X(\Omega)e^{j\Omega n}d\Omega.$$

- The preceding equation is sometimes referred to as the **Fourier transform synthesis equation** (or **inverse Fourier transform equation**).

# DTFT Convergence

- For a sequence $x$, the Fourier transform analysis equation (i.e., $X(\Omega) = \sum_{-\infty}^{\infty} x(n) e^{-j\Omega n}$) converges ***uniformly*** if

$$\sum_{k=-\infty}^{\infty} |x(k)| < \infty$$

  (i.e., $x$ is ***absolutely summable***).

- For a sequence $x$, the Fourier transform analysis equation (i.e., $X(\Omega) = \sum_{-\infty}^{\infty} x(n) e^{-j\Omega n}$) converges in the ***MSE sense*** if

$$\sum_{k=-\infty}^{\infty} |x(k)|^2 < \infty$$

  (i.e., $x$ is ***square summable***).

- For a bounded Fourier transform $X$, the Fourier transform synthesis equation (i.e., $x(n) = \frac{1}{2\pi} \int_{2\pi} X(\Omega) e^{j\Omega n} d\Omega$) will always converge, since the integration interval is finite.

# DTFT Examples and Exercises

**Example:** **Find the DTFT of the signal $x[n] = \alpha^n u[n]$. Sketch the magnitude and phase spectrum assuming $\alpha$ to be real-valued.**

***Solution:*** *The DTFT is given by*

$$X(e^{j\Omega}) = \sum_{n=-\infty}^{\infty} \alpha^n u[n] e^{-j\Omega n} = \sum_{n=0}^{\infty} \alpha^n e^{-j\Omega n}$$

$$= \frac{1}{1-\alpha e^{-j\Omega}}, \qquad |\alpha| < 1.$$

*If $\alpha$ is real-valued and $|\alpha| < 1$, then we have*

$$X(e^{j\Omega}) = \frac{1}{1-\alpha\cos(\Omega) + j\alpha\sin(\Omega)}$$

# DTFT Examples and Exercises

$$\Rightarrow \quad |X(e^{j\Omega})| = \frac{1}{\left((1-\alpha\cos(\Omega))^2 + \alpha^2 \sin^2(\Omega)\right)^{1/2}}, \quad \text{and}$$

$$\arg\{X(e^{j\Omega})\} = -\tan^{-1}\left(\frac{\alpha\sin(\Omega)}{1-\alpha\cos(\Omega)}\right)$$

***Can you draw the magnitude and phase plot? (see next slide)***

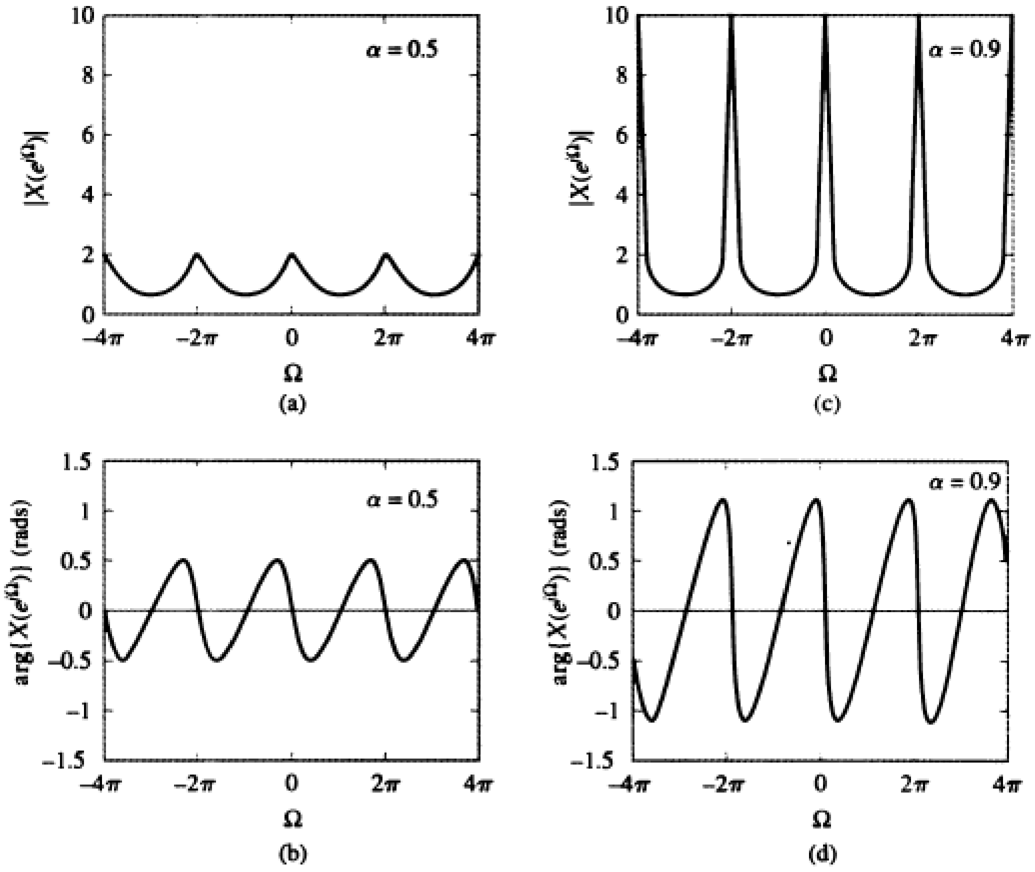
**Exercise:** Determine the DTFT of the following signals:

$$(i)\ x[n] = \delta[n],\ \ (ii)\ x[n] = 1,\ \ (iii)\ \cos(\Omega_0 n),\ \ (iv)\ \sin(\Omega_0 n).$$

**Exercise:** Determine the inverse DTFT of the spectrum

$$X(e^{j\Omega}) = \begin{cases} 1, & |\Omega| < W \\ 0, & W < |\Omega| < \pi \end{cases}.$$

# Periodic Magnitude and Phase Spectra



# DTFT Properties

# Periodicity

- Recall the definition of the Fourier transform $X$ of the sequence $x$:

$$X(\Omega) = \sum_{n=-\infty}^{\infty} x(n)e^{-j\Omega n}.$$

- For all integer $k$, we have that

$$\begin{aligned}
X(\Omega + 2\pi k) &= \sum_{n=-\infty}^{\infty} x(n)e^{-j(\Omega+2\pi k)n} \\
&= \sum_{n=-\infty}^{\infty} x(n)e^{-j(\Omega n+2\pi kn)} \\
&= \sum_{n=-\infty}^{\infty} x(n)e^{-j\Omega n} \\
&= X(\Omega).
\end{aligned}$$

- Thus, the Fourier transform $X$ of the sequence $x$ is always $2\pi$-***periodic***.

## Linearity

- If $x_1(n) \xleftrightarrow{\text{DTFT}} X_1(\Omega)$ and $x_2(n) \xleftrightarrow{\text{DTFT}} X_2(\Omega)$, then

$$a_1x_1(n) + a_2x_2(n) \xleftrightarrow{\text{DTFT}} a_1X_1(\Omega) + a_2X_2(\Omega),$$

  where $a_1$ and $a_2$ are arbitrary complex constants.
- This is known as the **linearity property** of the Fourier transform.

## Translation or Time Shifting

- If $x(n) \stackrel{\text{DTFT}}{\longleftrightarrow} X(\Omega)$, then

$$x(n-n_0) \stackrel{\text{DTFT}}{\longleftrightarrow} e^{-j\Omega n_0} X(\Omega),$$

  where $n_0$ is an arbitrary integer.
- This is known as the **translation (or time-domain shifting) property** of the Fourier transform.

## Modulation or Frequency Shifting

- If $x(n) \xleftrightarrow{\text{DTFT}} X(\Omega)$, then

$$e^{j\Omega_0 n}x(n) \xleftrightarrow{\text{DTFT}} X(\Omega - \Omega_0),$$

  where $\Omega_0$ is an arbitrary real constant.

- This is known as the **modulation (or frequency-domain shifting) property** of the Fourier transform.

## Conjugation and Time Reversal

- If $x(n) \overset{\text{DTFT}}{\longleftrightarrow} X(\Omega)$, then

$$x^*(n) \overset{\text{DTFT}}{\longleftrightarrow} X^*(-\Omega).$$

- This is known as the **conjugation property** of the Fourier transform.

- If $x(n) \overset{\text{DTFT}}{\longleftrightarrow} X(\Omega)$, then

$$x(-n) \overset{\text{DTFT}}{\longleftrightarrow} X(-\Omega).$$

- This is known as the **time-reversal property** of the Fourier transform.

## Convolution

- If $x_1(n) \overset{\text{DTFT}}{\longleftrightarrow} X_1(\Omega)$ and $x_2(n) \overset{\text{DTFT}}{\longleftrightarrow} X_2(\Omega)$, then

$$x_1 * x_2(n) \overset{\text{DTFT}}{\longleftrightarrow} X_1(\Omega)X_2(\Omega).$$

- This is known as the **convolution (or time-domain convolution) property** of the Fourier transform.
- In other words, a convolution in the time domain becomes a multiplication in the frequency domain.
- This suggests that the Fourier transform can be used to avoid having to deal with convolution operations.

## Parseval's Relation

- If $x(n) \overset{\text{DTFT}}{\longleftrightarrow} X(\Omega)$, then

$$\sum_{n=-\infty}^{\infty} |x(n)|^2 = \frac{1}{2\pi} \int_{2\pi} |X(\Omega)|^2 \, d\Omega$$

  (i.e., the energy of $x$ and energy of $X$ are equal up to a factor of $2\pi$).
- This is known as **Parseval's relation**.
- Since energy is often a quantity of great significance in engineering applications, it is extremely helpful to know that the Fourier transform ***preserves energy*** (up to a scale factor).

## Other Properties

- For a sequence $x$ with Fourier transform $X$, the following assertions hold:
  1. $x$ is even $\Leftrightarrow X$ is even; and
  2. $x$ is odd $\Leftrightarrow X$ is odd.
- In other words, the forward and inverse Fourier transforms preserve even/odd symmetry.

## Other Properties

- A sequence $x$ is ***real*** if and only if its Fourier transform $X$ satisfies

$$X(\Omega) = X^*(-\Omega) \text{ for all } \Omega$$

  (i.e., $X$ is ***conjugate symmetric***).
- Thus, for a real-valued sequence, the portion of the graph of a Fourier transform for negative values of frequency $\Omega$ is ***redundant***, as it is completely determined by symmetry.
- From properties of complex numbers, one can show that $X(\Omega) = X^*(-\Omega)$ is equivalent to

$$|X(\Omega)| = |X(-\Omega)| \quad \text{and} \quad \arg X(\Omega) = -\arg X(-\Omega)$$

  (i.e., $|X(\Omega)|$ is ***even*** and $\arg X(\Omega)$ is ***odd***).
- Note that $x$ being real does ***not*** necessarily imply that $X$ is real.

# DT LTI Systems Given By Difference Equations

- Many LTI systems of practical interest can be represented using an ***Nth-order linear difference equation with constant coefficients***.
- Consider a system with input $x$ and output $y$ that is characterized by an equation of the form

$$\sum_{k=0}^{N} b_k y(n-k) = \sum_{k=0}^{M} a_k x(n-k).$$

- Let $h$ denote the impulse response of the system, and let $X$, $Y$, and $H$ denote the Fourier transforms of $x$, $y$, and $h$, respectively.
- One can show that $H(\Omega)$ is given by

$$H(\Omega) = \frac{Y(\Omega)}{X(\Omega)} = \frac{\sum_{k=0}^{M} a_k (e^{j\Omega})^{-k}}{\sum_{k=0}^{N} b_k (e^{j\Omega})^{-k}} = \frac{\sum_{k=0}^{M} a_k e^{-jk\Omega}}{\sum_{k=0}^{N} b_k e^{-jk\Omega}}.$$

- Each of the numerator and denominator of $H$ is a ***polynomial*** in $e^{-j\Omega}$.
- Thus, $H$ is a ***rational function*** in the variable $e^{-j\Omega}$.

# DTFT Properties: Examples and Exercises

**Example:** **Find the output of a DT LTI system whose impulse response is given by $h[n] = (1/(\pi n))\sin(\pi n/2)$ if the input is $x[n] = (1/2)^n u[n]$.**

***Solution:*** *The DTFT of the input is given by*

$$X(e^{j\Omega}) = \frac{1}{1-(1/2)e^{-j\Omega}}.$$

*We have*

$$\frac{\sin(\pi n)}{\pi n} = \mathrm{sinc}(n) \overset{\text{DTFT}}{\longleftrightarrow} \mathrm{rect}(\Omega/(2\pi))$$

$$\Rightarrow \quad h[n] = \frac{\sin(\pi n/2)}{\pi n} = (1/2)\mathrm{sinc}(n/2) \overset{\text{DTFT}}{\longleftrightarrow} \mathrm{rect}(\Omega/\pi) = H(e^{j\Omega})$$

# DTFT Properties: Examples and Exercises

*Therefore, the DTFT of the output is given by*

$$Y(e^{j\Omega}) = X(e^{j\Omega})H(e^{j\Omega}) = \frac{\mathrm{rect}(\Omega/\pi)}{1-(1/2)e^{-j\Omega}}$$
$$= \begin{cases} \dfrac{1}{1-(1/2)e^{-j\Omega}} & |\Omega| \leq \pi/2 \\ 0, & \pi/2 < |\Omega| \leq \pi \end{cases}$$

**Example:** **Consider the two-path communication channel given by the input-output relationship**

$$y[n] = x[n] + ax[n-1], \quad |a| < 1.$$

**Find the impulse response of the inverse system by the convolution property of DTFT.**

# DTFT Properties: Examples and Exercises

***Solution:*** *The impulse response of the system is*

$$h[n] = \delta[n] + a\delta[n-1].$$

*The frequency response of the system is*

$$H(e^{j\Omega}) = 1 + ae^{-j\Omega},$$

*which implies that*

$$H^{inv}(e^{j\Omega}) = \frac{1}{1 + ae^{-j\Omega}} \quad \Rightarrow \quad h^{inv}[n] = (-a)^n u[n].$$

# DTFT Properties: Examples and Exercises

**Example:** Use the frequency-differentiation property and find out the DTFT of the signal

$$x[n] = (n+1)\alpha^n u[n], \qquad |\alpha| < 1.$$

***Solution:*** *We have*

$$\alpha^n u[n] \overset{DTFT}{\longleftrightarrow} \frac{1}{1-\alpha e^{-j\Omega}}$$

$$\Rightarrow \quad n\alpha^n u[n] \overset{\text{DTFT}}{\longleftrightarrow} j\frac{d}{d\Omega}\left(\frac{1}{1-\alpha e^{-j\Omega}}\right) = \frac{\alpha e^{-j\Omega}}{(1-\alpha e^{-j\Omega})^2}$$

$$\Rightarrow \quad (n+1)\alpha^n u[n] \overset{\text{DTFT}}{\longleftrightarrow} \frac{1}{1-\alpha e^{-j\Omega}} + \frac{\alpha e^{-j\Omega}}{(1-\alpha e^{-j\Omega})^2} = \frac{1}{(1-\alpha e^{-j\Omega})^2}$$

# DTFT Table of Properties

| Property | Time Domain | Frequency Domain |
|---|---|---|
| Linearity | $a_1x_1(n)+a_2x_2(n)$ | $a_1X_1(\Omega)+a_2X_2(\Omega)$ |
| Translation | $x(n-n_0)$ | $e^{-j\Omega n_0}X(\Omega)$ |
| Modulation | $e^{j\Omega_0 n}x(n)$ | $X(\Omega-\Omega_0)$ |
| Conjugation | $x^*(n)$ | $X^*(-\Omega)$ |
| Time Reversal | $x(-n)$ | $X(-\Omega)$ |
| Upsampling | $(\uparrow M)x(n)$ | $X(M\Omega)$ |
| Downsampling | $(\downarrow M)x(n)$ | $\frac{1}{M}\sum_{k=0}^{M-1}X\left(\frac{\Omega-2\pi k}{M}\right)$ |
| Convolution | $x_1 * x_2(n)$ | $X_1(\Omega)X_2(\Omega)$ |
| Multiplication | $x_1(n)x_2(n)$ | $\frac{1}{2\pi}\int_{2\pi}X_1(\theta)X_2(\Omega-\theta)d\theta$ |
| Freq.-Domain Diff. | $nx(n)$ | $j\frac{d}{d\Omega}X(\Omega)$ |
| Differencing | $x(n)-x(n-1)$ | $\left(1-e^{-j\Omega}\right)X(\Omega)$ |
| Accumulation | $\sum_{k=-\infty}^{n}x(k)$ | $\frac{e^{j\Omega}}{e^{j\Omega}-1}X(\Omega)+\pi X(0)\sum_{k=-\infty}^{\infty}\delta(\Omega-2\pi k)$ |

# DTFT Table of Properties

| Property | |
|---|---|
| Periodicity | $X(\Omega) = X(\Omega + 2\pi)$ |
| Parseval's Relation | $\sum_{n=-\infty}^{\infty} \lvert x(n)\rvert^2 = \frac{1}{2\pi}\int_{2\pi} \lvert X(\Omega)\rvert^2 \, d\Omega$ |
| Even Symmetry | $x$ is even $\Leftrightarrow X$ is even |
| Odd Symmetry | $x$ is odd $\Leftrightarrow X$ is odd |
| Real / Conjugate Symmetry | $x$ is real $\Leftrightarrow X$ is conjugate symmetric |

# DTFT of Common DT Signals

| Pair | $x(n)$ | $X(\Omega)$ |
|---|---|---|
| 1 | $\delta(n)$ | $1$ |
| 2 | $1$ | $2\pi\sum_{k=-\infty}^{\infty}\delta(\Omega-2\pi k)$ |
| 3 | $u(n)$ | $\frac{e^{j\Omega}}{e^{j\Omega}-1}+\sum_{k=-\infty}^{\infty}\pi\delta(\Omega-2\pi k)$ |
| 4 | $a^n u(n), \lvert a\rvert<1$ | $\frac{e^{j\Omega}}{e^{j\Omega}-a}$ |
| 5 | $-a^n u(-n-1), \lvert a\rvert>1$ | $\frac{e^{j\Omega}}{e^{j\Omega}-a}$ |
| 6 | $a^{\lvert n\rvert}, \lvert a\rvert<1$ | $\frac{1-a^2}{1-2a\cos\Omega+a^2}$ |
| 7 | $\cos\Omega_0 n$ | $\pi\sum_{k=-\infty}^{\infty}[\delta(\Omega-\Omega_0-2\pi k)+\delta(\Omega+\Omega_0-2\pi k)]$ |
| 8 | $\sin\Omega_0 n$ | $j\pi\sum_{k=-\infty}^{\infty}[\delta(\Omega+\Omega_0-2\pi k)-\delta(\Omega-\Omega_0-2\pi k)]$ |
| 9 | $(\cos\Omega_0 n)u(n)$ | $\frac{e^{j2\Omega}-e^{j\Omega}\cos\Omega_0}{e^{j2\Omega}-2e^{j\Omega}\cos\Omega_0+1}+\frac{\pi}{2}\sum_{k=-\infty}^{\infty}[\delta(\Omega-2\pi k-\Omega_0)+\delta(\Omega-2\pi k+\Omega_0)]$ |
| 10 | $(\sin\Omega_0 n)u(n)$ | $\frac{e^{j\Omega}\sin\Omega_0}{e^{j2\Omega}-2e^{j\Omega}\cos\Omega_0+1}+\frac{\pi}{2j}\sum_{k=-\infty}^{\infty}[\delta(\Omega-2\pi k-\Omega_0)-\delta(\Omega-2\pi k+\Omega_0)]$ |
| 11 | $\frac{B}{\pi}\operatorname{sinc}Bn, 0<B<\pi$ | $\sum_{k=-\infty}^{\infty}\operatorname{rect}\left(\frac{\Omega-2\pi k}{2B}\right)$ |
| 12 | $u(n)-u(n-M)$ | $e^{-j\Omega(M-1)/2}\left(\frac{\sin(M\Omega/2)}{\sin(\Omega/2)}\right)$ |

# References:


*[1] Simon Haykin and Barry Van Veen, Signals and Systems, Second Edition, John Wiley and Sons, 2003.*

*[2] Lecture Notes by Michael D. Adams.*
*https://www.ece.uvic.ca/~frodo/sigsysbook/downloads/lecture_slides_for_signals_and_systems*
*2.0.pdf*
*(https://www.ece.uvic.ca/~frodo/sigsysbook/downloads/lecture_slides_for_signals_and_system*
*2.0.pdf)*

*[3] Lecture Notes by Richard Baraniuk.*
*https://www.di.univr.it/documenti/OccorrenzaIns/matdid/matdid018094.pdf*
*(https://www.di.univr.it/documenti/OccorrenzaIns/matdid/matdid018094.pdf)*