Advanced Counting Techniques

Applications of Recurrence Relations

we will show that recurrence relations can be used to study and to solve counting problems.

A **recursive definition/recurrence relation** of a sequence specifies one or more initial terms and a rule for determining subsequent terms from those that precede them.

A sequence is called a solution of a recurrence relation if its terms satisfy the recurrence relation.

Advanced Counting Techniques - **Applications of Recurrence Relations**

Problem 1

Suppose that the number of bacteria in a colony doubles every hour. If a colony begins with five bacteria, how many will be present in n hours?

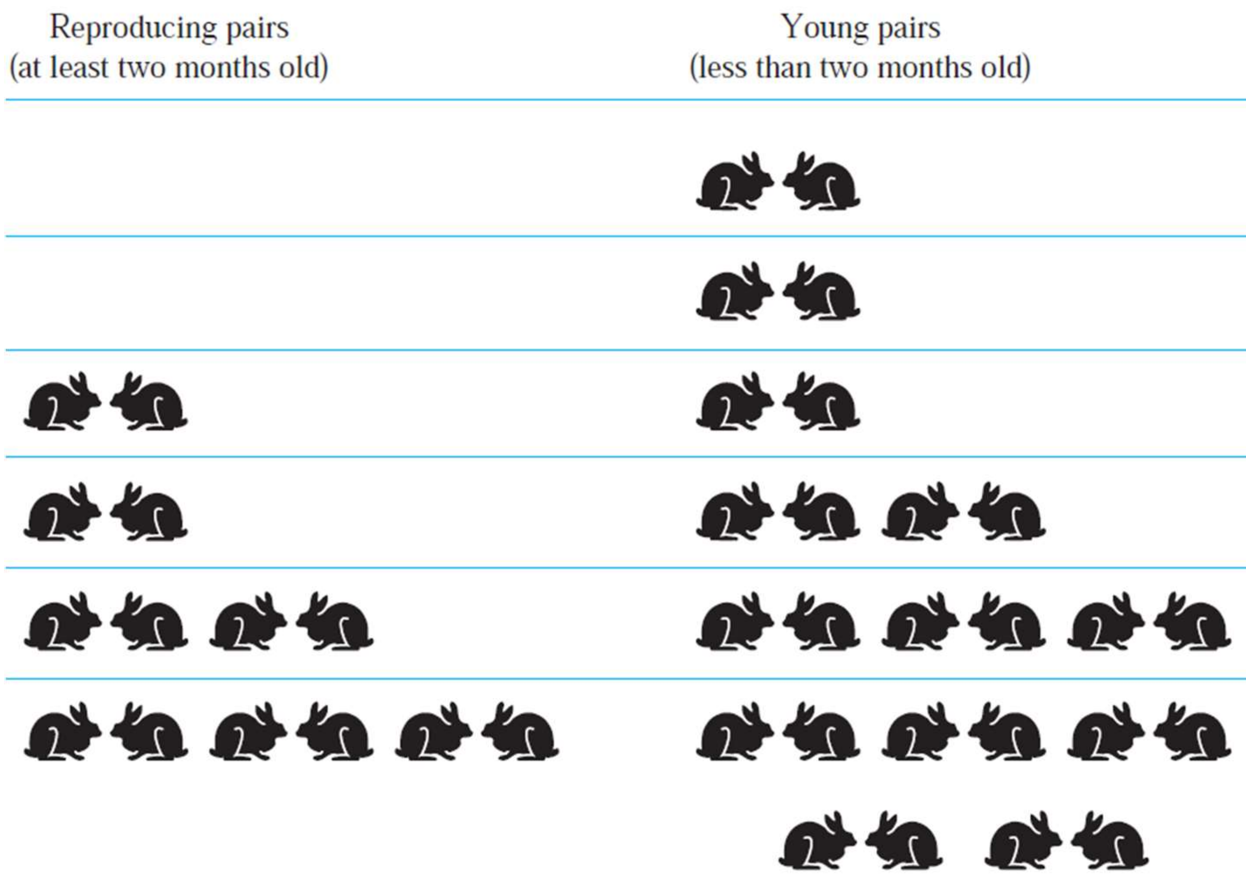
- Let a_n be the number of bacteria at the end of n hours. Because the number of bacteria doubles every hour
- The relationship $a_n = 2a_{n-1}$ holds whenever n is a positive integer.
- This recurrence relation, together with the initial condition $a_0 = 5$, uniquely determines a_n for all nonnegativeintegers n.

- Find a formula for sequence a_n using the iterative approach $a_n = 5 \cdot 2^n$ for all nonnegative integers n.

Modeling with Recurrence Relations

Show how the population of rabbits on an island can be modeled using a recurrence relation

Rabbits and the Fibonacci Numbers Consider this problem, which was originally posed by Leonardo Pisano, also known as Fibonacci, in the thirteenth century in his book *Liber abaci*. A young pair of rabbits (one of each sex) is placed on an island. A pair of rabbits does not breed until they are 2 months old. After they are 2 months old, each pair of rabbits produces another pair each month, as shown in Figure 1. Find a recurrence relation for the number of pairs of rabbits on the island after n months, assuming that no rabbits ever die.



Month	Reproducing pairs	Young pairs	Total pairs
1	0	1	1
2	0	1	1
3	1	1	2
4	1	2	3
5	2	3	5
6	3	5	8

Solution: Denote by f_n the number of pairs of rabbits after n months. We will show that f_n $(n = 1, 2, 3, \ldots)$, are the terms of the Fibonacci sequence.

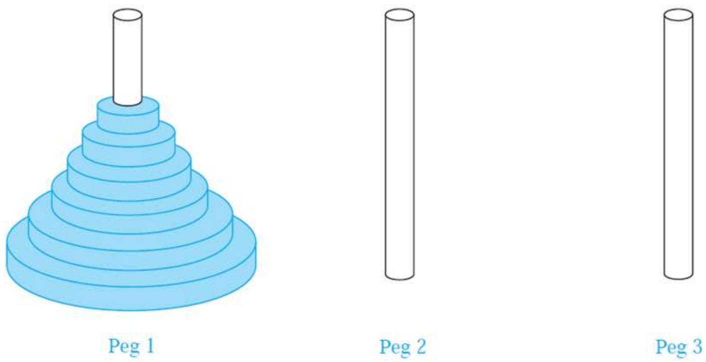
- The rabbit population can be modeled using a recurrence relation
- At the end of the first month, the number of pairs of rabbits on the island is $f_1 = 1$.
- Since this pair does not breed during the second month, $f_2 = 1$ also.
- To find the number of pairs after n months, add the number on the island the previous month, f_{n-1}, and the number of newborn pairs, which equals f_{n-2}
- The sequence $\{f_n\}$ satisfies the recurrence relation

Each newborn pair comes from a pair at least 2 months old.

$$f_n = f_{n-1} + f_{n-2}$$

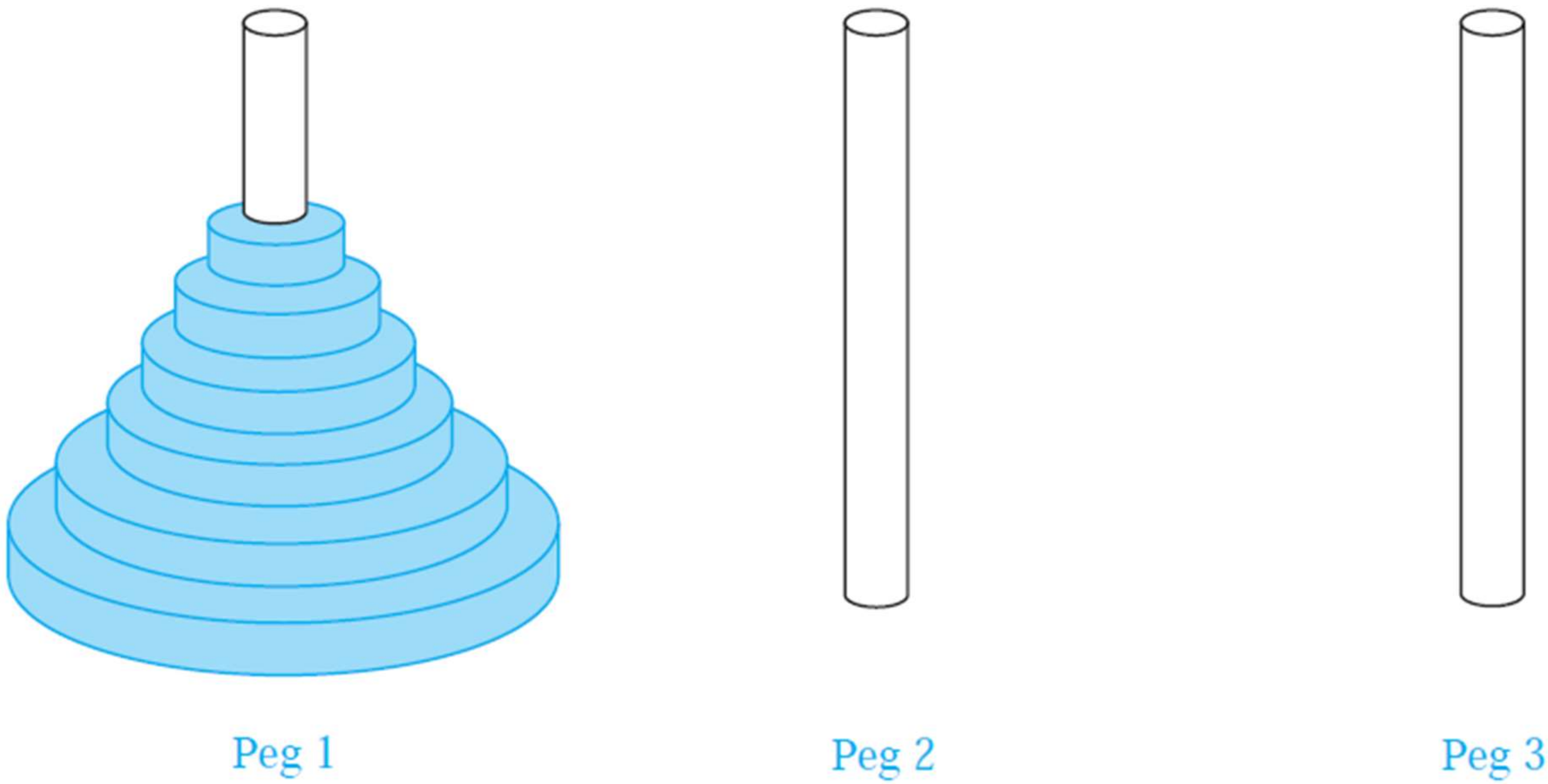
for $n \geq 3$ together with the initial conditions $f_1 = 1$ and $f_2 = 1$.

- The **Tower of Hanoi :** A popular puzzle of the late nineteenth century invented by the French mathematician Édouard Lucas, called the Tower of Hanoi.
- Consists of three pegs mounted on a board together with disks of different sizes. Initially these disks are placed on the first peg in order of size, with the largest on the bottom.

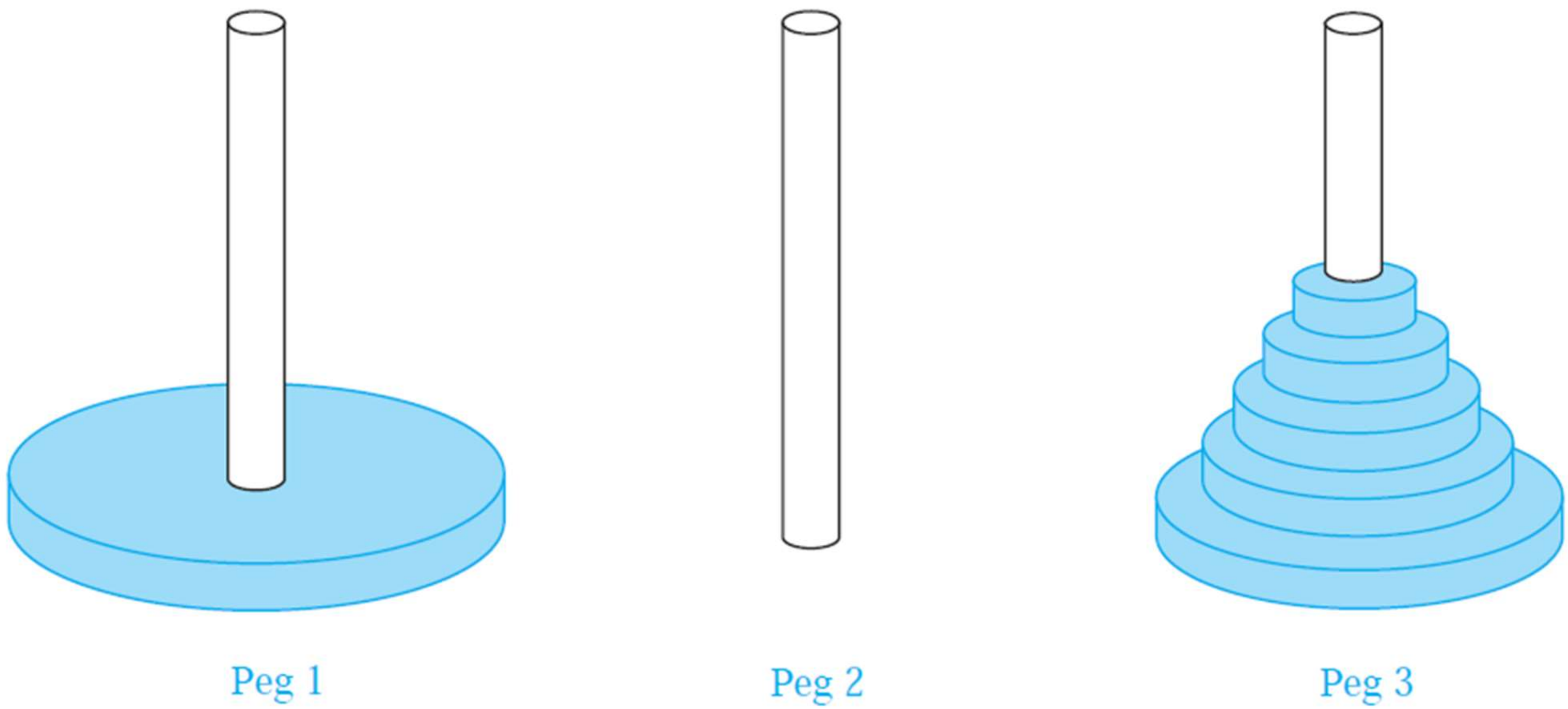


- The rules of the puzzle allow disks to be moved one at a time from one peg to another as long as a disk is never placed on top of a smaller disk.

- The goal of the puzzle is to have all the disks on the second peg in order of size, with the largest on the bottom.



The Initial Position in the Tower of Hanoi.



An Intermediate Position in the Tower of Hanoi.

- ✓ Begin with n disks on peg 1.
- ✓ We can transfer the top $n-1$ disks, following the rules of the puzzle, to peg 3 using H_{n-1} moves.
- ✓ We keep the largest disk fixed during these moves.
- ✓ Then, we use one move to transfer the largest disk to the second peg. We can transfer the n − 1 disks on peg 3 to peg 2 using H_{n-1} additional moves.
- ✓ This shows that $H_n = 2H_{n-1} + 1$, The initial condition is $H_1 = 1$.

Example Problem:

Find a recurrence relation and give initial conditions for the number of bit strings of length ***n*** that do not have two consecutive 0s. How many such bit strings are there of length five?

Let a_n denote the number of bit strings of length n that do not have two consecutive 0s.

To obtain a recurrence relation for $\{a_n\}$, note that by the sum rule, the number of bit strings of length n that do not have two consecutive 0s equals the number of such bit strings ending with a 0 plus the number of such bit strings ending with a 1.

The bit strings of length n ending with 1 that do not have two consecutive 0s are precisely the bit strings of length $n - 1$ with no two consecutive 0s with a 1 added at the end. Consequently, there are a_{n-1} such bit strings.

Bit strings of length n ending with a 0 that do not have two consecutive 0s must have 1 as their $(n - 1)$st bit; otherwise they would end with a pair of 0s.

It follows that the bit strings of length n ending with a **0** that have no two consecutive 0s are precisely the bit strings of length ***n*** **– 2** with no two consecutive 0s with 10 added at the end.

Hence, $\mathbf{a_n = a_{n-1} + a_{n-2}}$ for $\boldsymbol{n \geq 3}$.

$a_1 = 2$, because both bit strings of length one, 0 and 1 do not have consecutive 0s, and $a_2 = 3$

Codeword Enumeration A computer system considers a string of decimal digits a valid codeword if it contains an even number of 0 digits. For instance, 1230407869 is valid, whereas 120987045608 is not valid. Let a_n be the number of valid n-digit codewords. Find a recurrence relation for a_n.

Note that $\boldsymbol{a_1 = 9}$ because there are 10 one-digit strings, and only one, namely, the string 0, is not valid.

A valid string with n digits can be formed in this manner in $9a_{n-1}$ ways.

First, a valid string of n digits can be obtained by appending a valid string of $n - 1$ digits with a digit other than 0.

Second, a valid string of n digits can be obtained by appending a 0 to a string of length $n - 1$ that is not valid.

$$\begin{aligned} a_n &= 9a_{n-1} + (10^{n-1} - a_{n-1}) \\ &= 8a_{n-1} + 10^{n-1} \end{aligned}$$

Solving Linear Recurrence Relations

A wide variety of recurrence relations occur in models.

A *linear homogeneous recurrence relation of degree k with constant coefficients* is a recurrence relation of the form

$$a_n = c_1 a_{n-1} + c_2 a_{n-2} + \cdots + c_k a_{n-k},$$

where $c_1, c_2, \ldots, c_k$ are real numbers, and $c_k \neq 0$.

Examples

$$P_n = (1.11) P_{n-1}$$

$$f_n = f_{n-1} + f_{n-2}$$

$$a_n = a_{n-5}$$

Non-Examples

$a_n = a_{n-1} + a_{n-2}^2$ is not linear.

$2H_{n-1} + 1$ is not homogeneous

$B_n = nB_{n-1}$ does not have constant