Step $k=2$:

$$\tilde{a}_2^{(2)} : \begin{array}{|c|c|c||c|}\hline 1 & 7 & 2 & -7 \\ \hline\end{array}$$

$$\underset{\tilde{A}^{(2)}}{\begin{array}{|c|c|c||c|}\hline 0 & 0 & 0 & 0 \\ \hline -4 & -27 & 1 & 27 \\ \hline -1 & -1 & 44 & 1 \\ \hline\end{array}} + \underset{w^{(2)}}{\begin{array}{|c|}\hline 1 \\ \hline 4 \\ \hline 1 \\ \hline\end{array}} \; \underset{w^{(2)} \otimes \tilde{a}_2^{(2)}}{\begin{array}{|c|c|c||c|}\hline 1 & 7 & 2 & -7 \\ \hline 4 & 28 & 8 & -28 \\ \hline 1 & 7 & 2 & -7 \\ \hline\end{array}} = \underset{A^{(3)}}{\begin{array}{|c|c|c||c|}\hline 1 & 7 & 2 & -7 \\ \hline 0 & 1 & 9 & -1 \\ \hline 0 & 6 & 46 & -6 \\ \hline\end{array}}$$

Step $k=3$:

$$\tilde{a}_3^{(3)} : \begin{array}{|c|c||c|}\hline 1 & 9 & -1 \\ \hline\end{array}$$

$$\underset{\tilde{A}^{(3)}}{\begin{array}{|c|c||c|}\hline 0 & 0 & 0 \\ \hline 6 & 46 & -6 \\ \hline\end{array}} + \underset{w^{(3)}}{\begin{array}{|c|}\hline 1 \\ \hline -6 \\ \hline\end{array}} \; \underset{w^{(3)} \otimes \tilde{a}_3^{(3)}}{\begin{array}{|c|c||c|}\hline 1 & 9 & -1 \\ \hline -6 & -54 & 6 \\ \hline\end{array}} = \underset{A^{(4)}}{\begin{array}{|c|c||c|}\hline 1 & 9 & -1 \\ \hline 0 & -8 & 0 \\ \hline\end{array}}$$

we obtain a triangular system $A^{(4)}X = b^{(4)}$ having X for solution:

$$A^{(4)} = \begin{pmatrix} 1 & 2 & 1 & 3 \\ 0 & 1 & 7 & 2 \\ 0 & 0 & 1 & 9 \\ 0 & 0 & 0 & -8 \end{pmatrix}; \qquad b^{(4)} = \begin{pmatrix} 0 \\ -7 \\ -1 \\ 0 \end{pmatrix};$$

$$X = \begin{pmatrix} 1 \\ 0 \\ -1 \\ 0 \end{pmatrix}.$$

Example 3. Let $A \in \mathbb{R}^{3\times 3}$ be invertible. Compute $B = A^{-1}$ by using the algorithm ($\mathscr{A}_{\mathrm{GJR}}$)

$$A = \begin{pmatrix} 1 & -1 & 0 \\ -2 & 1 & 1 \\ 2 & -1 & 4 \end{pmatrix}; \qquad B^{(1)} = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}.$$

Step $k=1$:
We calculate $v^{(1)} = B^{(1)}a_1 = a_1$

$$v^{(1)} = \begin{pmatrix} 1 \\ -2 \\ 2 \end{pmatrix} \leftarrow k=1; \qquad w^{(1)} = \begin{pmatrix} 1 \\ 2 \\ -2 \end{pmatrix},$$

we compute $B^{(2)} = \tilde{B}^{(1)} + w^{(1)} \otimes \tilde{b}_1^{(1)}$

$$\tilde{b}_1^{(1)} : \begin{array}{|c|c|c|}\hline 1 & 0 & 0 \\ \hline\end{array}$$

$$\underset{\tilde{B}^{(1)}}{\begin{array}{|c|c|c|}\hline \mathbf{0} & \mathbf{0} & \mathbf{0} \\ \hline 0 & 1 & 0 \\ \hline 0 & 0 & 1 \\ \hline\end{array}} + \underset{w^{(1)}}{\begin{array}{|c|}\hline 1 \\ \hline 2 \\ \hline -2 \\ \hline\end{array}} \; \underset{w^{(1)} \otimes \tilde{b}_1^{(1)}}{\begin{array}{|c|c|c|}\hline 1 & 0 & 0 \\ \hline 2 & 0 & 0 \\ \hline -2 & 0 & 0 \\ \hline\end{array}} = \underset{B^{(2)}}{\begin{array}{|c|c|c|}\hline 1 & 0 & 0 \\ \hline 2 & 1 & 0 \\ \hline -2 & 0 & 1 \\ \hline\end{array}}$$

This step is easy. It is considered as an initialization: it suffices to put $w^{(1)}$ in the first column of the matrix $B^{(1)}$ and to pass directly to iteration 2.

Step $k=2$:
We calculate $v^{(2)} = B^{(2)}a_2$

$$v^{(2)} = \begin{pmatrix} -1 \\ -1 \\ 1 \end{pmatrix} \leftarrow k=2; \qquad w^{(2)} = \begin{pmatrix} -1 \\ -1 \\ 1 \end{pmatrix},$$

we compute $B^{(3)} = \tilde{B}^{(2)} + w^{(2)} \otimes \tilde{b}_2^{(2)}$

$$\tilde{b}_2^{(2)} : \begin{array}{|c|c|c|}\hline 2 & 1 & 0 \\ \hline\end{array}$$

$$\underset{\tilde{B}^{(2)}}{\begin{array}{|c|c|c|}\hline 1 & 0 & 0 \\ \hline \mathbf{0} & \mathbf{0} & \mathbf{0} \\ \hline -2 & 0 & 1 \\ \hline\end{array}} + \underset{w^{(2)}}{\begin{array}{|c|}\hline -1 \\ \hline -1 \\ \hline 1 \\ \hline\end{array}} \; \underset{w^{(2)} \otimes \tilde{b}_2^{(2)}}{\begin{array}{|c|c|c|}\hline -2 & -1 & 0 \\ \hline -2 & -1 & 0 \\ \hline 2 & 1 & 0 \\ \hline\end{array}} = \underset{B^{(3)}}{\begin{array}{|c|c|c|}\hline -1 & -1 & 0 \\ \hline -2 & -1 & 0 \\ \hline 0 & 1 & 1 \\ \hline\end{array}}$$

Step $k=3$:
We calculate $v^{(3)} = B^{(3)}a_3$

$$v^{(3)} = \begin{pmatrix} -1 \\ -1 \\ 5 \end{pmatrix} \leftarrow k=3; \qquad w^{(3)} = \begin{pmatrix} \frac{1}{5} \\ \frac{1}{5} \\ \frac{1}{5} \end{pmatrix},$$

we compute $B^{(4)} = \tilde{B}^{(3)} + w^{(3)} \otimes \tilde{b}_3^{(3)}$

$$\tilde{b}_3^{(3)} : \begin{array}{|c|c|c|}\hline 0 & 1 & 1 \\ \hline\end{array}$$

$$\underset{\tilde{B}^{(3)}}{\begin{array}{|c|c|c|}\hline -1 & -1 & 0 \\ \hline -2 & -1 & 0 \\ \hline \mathbf{0} & \mathbf{0} & \mathbf{0} \\ \hline\end{array}} + \underset{w^{(3)}}{\begin{array}{|c|}\hline \frac{1}{5} \\ \hline \frac{1}{5} \\ \hline \frac{1}{5} \\ \hline\end{array}} \; \underset{w^{(3)} \otimes \tilde{b}_3^{(3)}}{\begin{array}{|c|c|c|}\hline 0 & \frac{1}{5} & \frac{1}{5} \\ \hline 0 & \frac{1}{5} & \frac{1}{5} \\ \hline 0 & \frac{1}{5} & \frac{1}{5} \\ \hline\end{array}} = \underset{B^{(4)}}{\begin{array}{|c|c|c|}\hline -1 & -\frac{4}{5} & \frac{1}{5} \\ \hline -2 & -\frac{4}{5} & \frac{1}{5} \\ \hline 0 & \frac{1}{5} & \frac{1}{5} \\ \hline\end{array}}$$

$$A = \begin{pmatrix} 1 & -1 & 0 \\ -2 & 1 & 1 \\ 2 & -1 & 4 \end{pmatrix};$$

$$B^{(4)} = A^{-1} = \frac{1}{5} \begin{pmatrix} -5 & -4 & 1 \\ -10 & -4 & 1 \\ 0 & 1 & 1 \end{pmatrix}.$$

6. Conclusion

The idea of Proposition 2 gives rise to a new algorithm, which is easy, concise, clear and serves to calculate the inverse of the basic matrix of the current basis at iteration r, by starting from the identity matrix. In other words, this algorithm contributes to the amelioration of the product