Cheat Sheet DSP

DSP fundamentals (signals II) (Technische Universiteit Eindhoven)

Window	A [dB]	Transition Δ	Stopband [dB]
Rectangular	-13	$1 \times (2\pi/N)$	-21
Hanning	-31	$3.1 \times (2\pi/N)$	-44
Hamming	-41	$3.3 \times (2\pi/N)$	-53
Blackman	-57	$5.5 \times (2\pi/N)$	-74

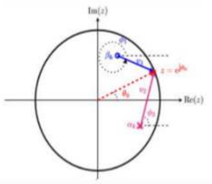
$20\log(\delta_s$

Frequency response from pole-zero plot

$$H(z)|_{z=e^{j\theta}} = |H(e^{j\theta})| \cdot e^{j\Phi(e^{j\theta})} \text{ with } H(z) = A \cdot z^{p-q} \frac{\prod_{k=1}^{q}(z-\beta_k)}{\prod_{k=1}^{p}(z-\alpha_k)}$$

$$|H(e^{j\theta})| = |A| \times \left(\prod_{k=1}^{q} \text{length}(e^{j\theta}-\beta_k)\right) / \left(\prod_{k=1}^{p} \text{length}(e^{j\theta}-\alpha_k)\right)$$

$$\Phi(e^{j\theta}) = (p-q)\cdot\theta + \sum_{k=1}^{q} \arg(e^{j\theta}-\beta_k) - \sum_{k=1}^{p} \arg(e^{j\theta}-\alpha_k)$$



$$|H(e^{j\theta_0})| = v_1/v_2$$
$$\Phi(e^{j\theta_0}) = \phi_1 - \phi_2$$

Circular shift: With DFT pair $x_p[n] \circ\!\!-\!\!\circ X_p[k]$

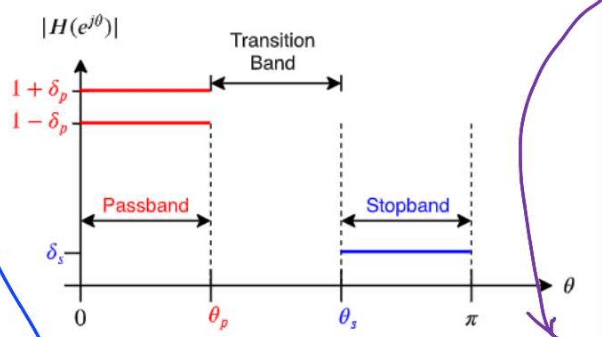
Time shift: $x_p[n-i] \circ\!\!-\!\!\circ e^{-j\frac{2\pi}{N}ki} \cdot X_p[k]$

Frequency shift: $e^{j\frac{2\pi}{N}ni} \cdot x_p[n] \circ\!\!-\!\!\circ X_p[k-i]$

δ_p passband ripple

δ_s stopband ripple

ω cutoff freq



$N = \frac{c}{\Delta f}$ $\quad \Delta\theta = \theta_s - \theta_p$

$\Delta f = \frac{\Delta\theta}{2\pi}$

$h_d[n] = \frac{\sin((n-\alpha)\omega_c)}{\pi(n-\alpha)}$ $\quad \alpha = \frac{N}{2}$

General specification LPF:

$0 \leq |\omega| \leq \omega_p \quad 1-\delta_p < |H(e^{j\theta})| \leq 1+\delta_p$

$\omega_s \leq |\omega| < \pi \quad |H(e^{j\theta})| \leq \delta_s$

Gain $= \frac{v_1}{v_2} = \frac{\sqrt{\text{real}^2 + \text{complex}^2}}{\sqrt{\text{real}^2 + \text{complex}^2}}$ to unit circle ← zeros ← pole

Common Z-transform pairs

Sequence	Z-transform	ROC
$\delta[n]$	1	all z
$\delta[n-i]$	z^{-i}	$z \neq 0, \infty$
$a^n u[n]$	$\frac{1}{1-az^{-1}}$	$\lvert z\rvert > a$
$-a^n u[-n-1]$	$\frac{1}{1-az^{-1}}$	$\lvert z\rvert < a$
$na^n u[n]$	$\frac{az^{-1}}{(1-az^{-1})^2}$	$\lvert z\rvert > a$
$-na^n u[-n-1]$	$\frac{az^{-1}}{(1-az^{-1})^2}$	$\lvert z\rvert < a$
$a^n \cos(n\theta_0)u[n]$	$\frac{1-az^{-1}\cos(\theta_0)}{1-2az^{-1}\cos(\theta_0)+a^2z^{-2}}$	$\lvert z\rvert > a$
$a^n \sin(n\theta_0)u[n]$	$\frac{az^{-1}\sin(\theta_0)}{1-2az^{-1}\cos(\theta_0)+a^2z^{-2}}$	$\lvert z\rvert > a$
$a^n \sin(n\theta_0+\phi)u[n]$	$\frac{\sin(\phi)+az^{-1}\sin(\theta_0-\phi)}{1-2az^{-1}\cos(\theta_0)+a^2z^{-2}}$	$\lvert z\rvert > a$

$$y[n] = \sum_{k=0}^{q} b_k x[n-k] - \sum_{k=1}^{p} a_k y[n-k]$$

$$H(z) = \frac{\sum_{k=0}^{q} b_k z^{-k}}{1+\sum_{k=1}^{p} a_k z^{-k}} = b_0 \frac{\prod_{k=1}^{q}(1-\beta_k z^{-1})}{\prod_{k=1}^{p}(1-\alpha_k z^{-1})} = b_0 \cdot z^{p-q} \frac{\prod_{k=1}^{q}(z-\beta_k)}{\prod_{k=1}^{p}(z-\alpha_k)}$$

β_k values are the zeros of $H(z)$

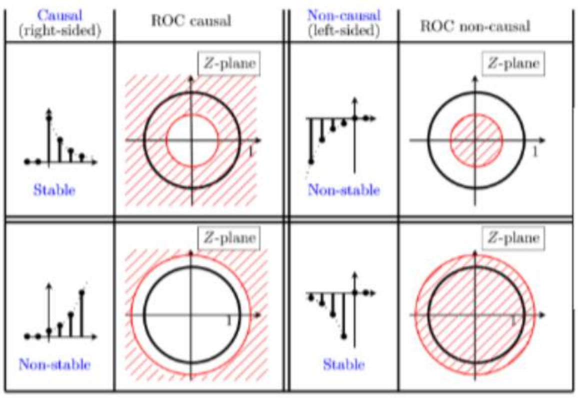
α_k are the poles of $H(z)$

Definitions 2^{nd} – order statistics: Real and WSS

- Mean: $\mu_x[n] = E\{x[n]\}$
- Variance: $\sigma_x^2 = E\{(x[n]-\mu_x)^2\} = E\{x^2[n]\} - \mu_x^2$
- Autocorrelation: $r_x[l] = E\{x[n]\cdot x[n-l]\} = E\{x[n+l]\cdot x[n]\}$
  - Notes: Power $E\{x^2[n]\} = r_x[0] \geq 0$ and $r_x[0] \geq r_x[l] \quad \forall l$
  - Symmetry $r_x[l] = r_x[-l]$
- Autocovariance: $\gamma_x[l] = E\{(x[n]-\mu_x)\cdot(x[n-l]-\mu_x)\}$ $= E\{(x[n+l]-\mu_x)\cdot(x[n]-\mu_x)\} = r_x[l] - \mu_x^2$
- Cross-correlation: $r_{xy}[l] = E\{x[n]\cdot y[n-l]\} = E\{x[n+l]\cdot y[n]\}$
- Cross-covariance: $\gamma_{xy}[l] = E\{(x[n]-\mu_x)\cdot(y[n-l]-\mu_y)\}$ $= r_{xy}[l] - \mu_x \cdot \mu_y$
- Normalized γ_{xy}: $\rho_{xy}[l] = \frac{\gamma_{xy}[l]}{\sigma_x \cdot \sigma_y}$

()DFT pair:

$$x_p[n] = \frac{1}{N}\sum_{k=0}^{N-1} X[k] e^{j\frac{2\pi}{N}kn} \circ\!\!-\!\!\circ X[k] = \sum_{n=0}^{N-1} x_p[n] e^{-j\frac{2\pi}{N}kn}$$



Exercise 8 From $P_x(e^{j\theta}) = \frac{5}{4} + \cos(\theta)$ it follows: $r_x[l] = \frac{1}{2}\delta[l+1] + \frac{5}{4}\delta[l] + \frac{1}{2}\delta[l-1]$. Furthermore from the given impulse response we obtain: $y[n] = x[n] - \frac{1}{2}x[n-1]$. Thus:

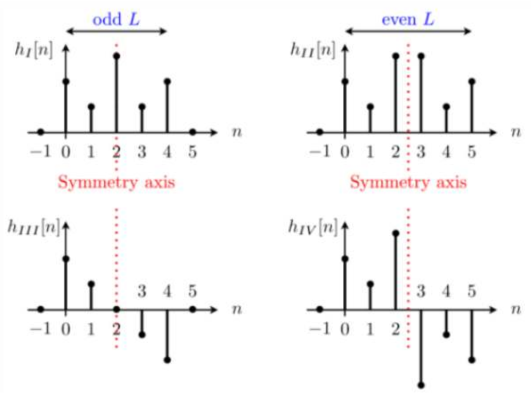
change to freq

using $h[n]$, shift

$$r_y[l] = E\{y[n]y[n-l]\} = E\{(x[n] - \frac{1}{2}x[n-1])(x[n-l] - \frac{1}{2}x[n-1-l])\}$$
$$= r_x[l] - \frac{1}{2}r_x[l-1] - \frac{1}{2}r_x[l+1] + \frac{1}{4}r_x[l] = \frac{17}{16}\delta[l] - \frac{1}{4}\delta[l-2] - \frac{1}{4}\delta[l+2]$$

$l =$	-4	-3	-2	-1	0	1	2	3	4
$r_y[l] =$	0	0	$-\frac{1}{4}$	0	$\frac{17}{16}$	0	$-\frac{1}{4}$	0	0

$$= P_x(e^{j\theta}) \cdot \frac{1}{9}(e^{3j\theta} + 2e^{j\theta} + 3 + 2e^{-j\theta} + e^{-2j\theta})$$



The definition of the Hanning window:

$$w[n] = \begin{cases} 0.5 - 0.5\cos\left(\frac{2\pi n}{N}\right), & \text{if } 0 \leq n \leq N \\ 0, & \text{otherwise} \end{cases}$$

4.1 a

$$r_x[l] = E\{x[n]x[n-l]\}$$
$$r_y[l] = E\{y[n]y[n-l]\}$$

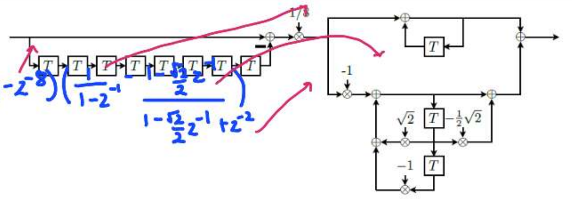
$$r_y[l] = E\{y[n]y[n-l]\}$$
$$= E\{\frac{1}{3}(x[n]+x[n-1]+x[n-2]) \cdot \frac{1}{3}(x[n-l]+x[n-1-l]+x[n-2-l])\}$$
$$= (E\{x[n]x[n-l]\} + E\{x[n-1]x[n-l]\} + E\{x[n-2]x[n-l]\}$$
$$+ E\{x[n]x[n-1-l]\} + E\{x[n-1]x[n-1-l]\} + E\{x[n-2]x[n-1-l]\}$$
$$+ E\{x[n]x[n-2-l]\} + E\{x[n-1]x[n-2-l]\} + E\{x[n-2]x[n-2-l]\})$$

Since $r_x[l] = E\{x[n]x[n-l]\}$ we obtain:

$$r_y[l] = \frac{1}{9}(r_x[l] + r_x[l-1] + r_x[l-2]$$
$$+ r_x[l+1] + r_x[l] + r_x[l-1]$$
$$+ r_x[l+2] + r_x[l+1] + r_x[l])$$
$$= \frac{1}{9}(3r_x[l] + 2r_x[l-1] + 2r_x[l+1] + r_x[l+2] + r_x[l-2])$$
$$= \frac{1}{9}(\delta[l+2] + 2\delta[l+1] + 3\delta[l] + 2\delta[l-1] + \delta[l-2])$$

-2 $\quad$ +2

$H(z)=\frac{1}{8}(1-z^{-8})\left(\frac{1}{1-z^{-1}}\ \frac{\frac{\sqrt{2}}{2}z^{-1}}{1-\frac{\sqrt{2}}{2}z^{-1}+z^{-2}}\right)$



$$H(z) = \frac{1}{N}\left(1-z^{-N}\right)\cdot\left(\frac{1}{1-z^{-1}} - \frac{\frac{1}{2}}{1-e^{j\frac{2\pi}{N}}z^{-1}} - \frac{\frac{1}{2}}{1-e^{j\frac{2\pi}{N}(N-1)}z^{-1}}\right)$$
$$= \frac{1}{N}\left(1-z^{-N}\right)\cdot\left(\frac{1}{1-z^{-1}} - \frac{1-\cos(\frac{2\pi}{N})z^{-1}}{1-2\cos(\frac{2\pi}{N})z^{-1}+z^{-2}}\right)$$

Linear phase filter $h[n] = h[0]\delta[n] + h[1]\delta[n-1] + h[2]\delta[n-2] + h[3]\delta[n-3]$. Even length $L = 4$, thus two cases even- and odd-symmetry resp.:

Case II $\quad h[0] = h[3] = a\ ;\ h[1] = h[2] = b \quad \circ\!-\!\circ \quad H_{II}(e^{j\theta}) = 2\left(b\cos(\frac{1}{2}\theta) + a\cos(\frac{3}{2}\theta)\right)\cdot e^{-j\frac{3}{2}\theta}$

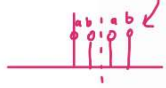
Case IV $\quad h[0] = -h[3] = a\ ;\ h[1] = -h[2] = b \quad \circ\!-\!\circ \quad H_{IV}(e^{j\theta}) = 2\left(b\sin(\frac{1}{2}\theta) + a\sin(\frac{3}{2}\theta)\right)\cdot e^{-j(\frac{3}{2}\theta-\frac{\pi}{2})}$

a) Since $H(e^{j\theta})|_{\theta=0} = 0 \Rightarrow$ Case IV (odd-symmetry):

$$H_{IV}(e^{j\theta})|_{\theta=\frac{\pi}{3}} = -1 \equiv 2\left(b\frac{1}{2}+a\right)\cdot e^{-j0}$$
$$H_{IV}(e^{j\theta})|_{\theta=\frac{2\pi}{3}} = -j\sqrt{3} \equiv 2\left(b\frac{1}{2}\sqrt{3}+a\cdot 0\right)\cdot e^{-j\frac{\pi}{2}}$$

$\Rightarrow a = -1, b = 1$. Thus $\boxed{h[n] = -\delta[n] + \delta[n-1] - \delta[n-2] + \delta[n-3]}$

a b a b



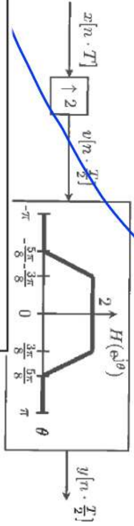
Discrete-time systems

$x[n] \rightarrow \boxed{T_r\{\cdot\}} \rightarrow y[n] = T_r\{x[n]\}$

* Properties:

LTI	:	Linear Time-Invariance (book: "Shift"-Invariance) Additive, Homogeneous and Time-Invariant
Causality	:	Response at n_0 depends on input up to $n = n_0$. In practice we cannot predict sample values, therefore it is important to design causal filters
(BIBO) Stability	:	For $A, B < \infty$, $\lvert x[n]\rvert < A \Rightarrow \lvert y[n]\rvert < B$. Input bounded by some number A will yield output boundedby some number B
Invertibility	:	Input may be uniquely determined from output. Only one input can be traced back from the output

TU/e — Fac. EE, SPS — DSP Fundamentals (Signals II) / 5ESC0 / Discrete-time Signals and Systems



The filter changes only the phase (amplitude stays the same). At frequency $\frac{\pi}{3}$ the phase is changed by $-\frac{\pi}{2}$, while at frequency $\frac{2\pi}{3}$ the phase is changed with $-\pi$. From the general expressions as given above it follows that this can only be achieved in case II.

amplitude change — phase change

$$H_{II}(e^{j\theta})|_{\theta=\frac{\pi}{3}} = 1\cdot e^{-j\frac{\pi}{2}} \equiv 2\left(b\frac{1}{2}\sqrt{3}+a\cdot 0\right)\cdot e^{-j\frac{\pi}{2}}$$
$$H_{II}(e^{j\theta})|_{\theta=\frac{2\pi}{3}} = 1\cdot e^{-j\pi} \equiv 2\left(b\frac{1}{2}+a\cdot(-1)\right)\cdot e^{-j\pi}$$

$\Rightarrow b = \frac{1}{3}\sqrt{3}\ (\cong 0.6)$ and $a = (\frac{1}{6}\sqrt{3}-\frac{1}{2})\ (\cong -0.2)$. Thus $\boxed{h[n] = a\delta[n] + b\delta[n-1] + b\delta[n-2] + a\delta[n-3]}$.

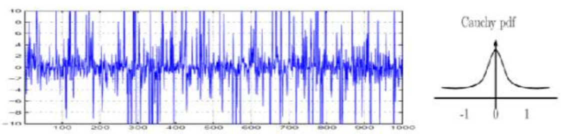
$h[n] = \frac{1}{N}\left\{1-\cos(\frac{2\pi}{N}n)\right\}$ DFT

$\frac{1}{N}$ for $k = N\ell = 0 \rightarrow \delta[k]$ | $\frac{1}{2N}$ for $k+1 = N\ell \hookrightarrow \delta[k-N+1]$

$\frac{1}{2N}$ for $k-1 = N\ell \rightarrow \delta[k-1]$ | N given

$$= \frac{1}{N}\sum_{n=0}^{N-1} e^{-j\frac{2\pi}{N}kn} - \frac{1}{2N}\sum_{n=0}^{N-1} e^{-j\frac{2\pi}{N}(k-1)n} - \frac{1}{2N}\sum_{n=0}^{N-1} e^{-j\frac{2\pi}{N}(k+1)n}$$
$$= \delta[k] - \frac{1}{2}\delta[k-1] - \frac{1}{2}\delta[k-N+1]$$

* Cauchy: Containing spikes and therefore has high variability



Now we use 4-point DFT's. From the 4-point DFT values $X[k]$ of one period of $x[n]$ we are given:

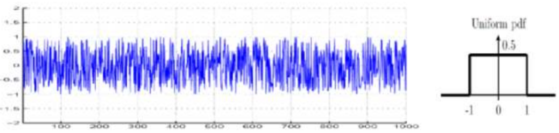
$$X[0] = 0\ ;\ X[1] = \sqrt{2} + j\sqrt{2} \text{ and } X[2] = 0$$

while from the 4-point DFT values of $H[k]$ of the length 4 impulse response $h[n]$ we are given:

$$H[0] = 2\ ;\ H[1] = 1 + j \text{ and } H[2] = 0$$

Evaluate $x[n]$, $h[n]$ and the **linear** convolution result $y[n] = x[n] \star h[n]$ all in the range for $n = -1, 0, 1, 2, 3, 4, 5, 6$.

* Uniform
* White noise: uniformly distributed between -1 and 1



Time domain — Frequency domain

Averaging filter $\circ\!-\!\circ$ Dirichlet function

$h[n] = u[n] - u[n-N]$ $\quad\circ\!-\!\circ\quad$ $H(e^{j\theta}) = e^{-j\frac{N-1}{2}\theta}\cdot\frac{\sin(\frac{N}{2}\theta)}{\sin(\frac{1}{2}\theta)}$

Examples: Low Pass Filter

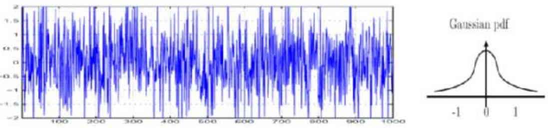
Time domain: Sinc function $\frac{\theta_c}{\pi}\frac{\sin(\theta_c n)}{\theta_c n}$ $\circ\!-\!\circ$ Frequency domain: Ideal Low Pass Filter (LPF) $H(e^{j\theta}) = \begin{cases}1, & \lvert\theta\rvert \le \theta_c\\ 0, & \text{elsewhere}\end{cases}$

* If we want to filter out higher frequencies, we use a LPF
* An ideal Low Pass Filter is shown in the figure below

$$X(e^{j\theta}) \overset{FTD}{\cong} \sum_{n=-\infty}^{\infty} x[n]e^{-jn\theta} \quad\circ\!-\!\circ\quad x[n] \overset{IFTD}{\cong} \frac{1}{2\pi}\int_{-\pi}^{\pi} X(e^{j\theta})e^{jn\theta}d\theta$$

Sequence	FTD
$\delta[n]$	1
$\delta[n-n_0]$	$e^{-jn_0\theta}$
1	$2\pi\delta(\theta)$
$e^{jn\theta_0}$	$2\pi\delta(\theta-\theta_0)$
$a^n u[n], \quad \lvert a\rvert < 1$	$\frac{1}{1-ae^{-j\theta}}$
$-a^n u[-n-1], \quad \lvert a\rvert > 1$	$\frac{1}{1-ae^{-j\theta}}$
$\cos(n\theta_0)$	$\pi\delta(\theta+\theta_0) + \pi\delta(\theta-\theta_0)$

* Gaussian: Allowing for any possible (confined) value

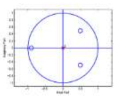


$$H(z) = z^{-1} + 2z^{-2} + z^{-3} = z^{-1}(1+z^{-1})^2 \equiv (1-z^{-4})\cdot G(z) \Rightarrow$$
$$G(z) = \frac{z^{-1}(1+z^{-1})^2}{(1-z^{-4})} = \frac{z^{-1}(1+z^{-1})}{(1-z^{-1})\cdot(1-jz^{-1})\cdot(1+jz^{-1})}$$
$$= \frac{A}{(1-z^{-1})} + \frac{B}{(1-jz^{-1})} + \frac{C}{(1+jz^{-1})} = \frac{1}{(1-z^{-1})} + \frac{-\frac{1}{2}}{(1-jz^{-1})} + \frac{-\frac{1}{2}}{(1+jz^{-1})}$$

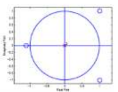
Ergodicity

* An ensemble of measurements may not be available
* A process is ergodic if the statistics can be found from one single realization
* Only stationary signals can be ergodic
* Stationarity ensures time invariance of statistics of random signal
* Ergodicity implies that the statistics can be calculated by time-averaging over a single representative member of the ensemble. Practice: Number of measured samples is limited to, say, N
  → "Replace" ensemble-averaging by time-averaging:
  $E\{\cdot\} = \frac{1}{N}\sum_{n=0}^{N-1}(\cdot)$

* **Minimum phase:** Zeros inside unit circle



* **Maximum phase:** Mirror zeros of minimum phase



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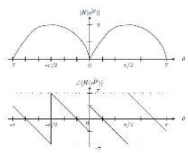
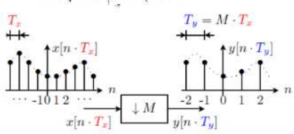
$2\sin(\theta)\cdot e^{-j(3\theta-\frac{\pi}{2})}$



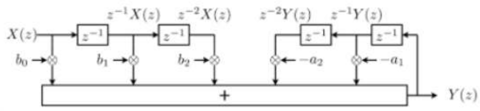
⇒ Interpolation formula (time domain):

$$x_a(t) = \sum_{n=-\infty}^{\infty} x[n]\left(\frac{\sin(\frac{\pi}{T_s}\cdot(t-nT_s))}{\frac{\pi}{T_s}\cdot(t-nT_s)}\right)$$



$$y[n\cdot T_y] = x[n\cdot(M\cdot T_x)]$$
$$X(e^{j\theta}) = \frac{1}{T_x}\sum_{k=-\infty}^{\infty} X_a\left(\frac{\theta}{T_x} - k\frac{2\pi}{T_x}\right)$$
$$Y(e^{j\theta}) = \frac{1}{T_y}\sum_{r=-\infty}^{\infty} X_a\left(\frac{\theta}{T_y} - r\frac{2\pi}{T_y}\right)$$

Example 1



Let's define the system output in Z-domain:

$$Y(z) = b_0X(z) + b_1z^{-1}X(z) + b_2z^{-2}X(z) - a_1z^{-1}Y(z) - a_2z^{-2}Y(z)$$

Split last sum with $r = p + k\cdot M$ and use $T_y = M\cdot T_x$

$$Y(e^{j\theta}) = \frac{1}{M}\sum_{p=0}^{M-1}\left\{\frac{1}{T_x}\sum_{k=-\infty}^{\infty} X_a\left(\frac{(\theta-p\cdot 2\pi)}{M\cdot T_x} - k\cdot\frac{2\pi}{T_x}\right)\right\} \Rightarrow$$

$$\boxed{Y(e^{j\theta}) = \frac{1}{M}\sum_{p=0}^{M-1} X(e^{j(\frac{\theta}{M}-p\cdot\frac{2\pi}{M})})}$$