Assignment 2: CS 754, Advanced Image Processing

Due: 22nd Feb before 11:55 pm

Remember the honor code while submitting this (and every other) assignment. All members of the group should work on and understand all parts of the assignment. We will adopt a zero-tolerance policy against any violation.

Submission instructions: You should ideally type out all the answers in Word (with the equation editor) or using Latex. In either case, prepare a pdf file. Create a single zip or rar file containing the report, code and sample outputs and name it as follows: A2-IdNumberOfFirstStudent-IdNumberOfSecondStudent.zip. (If you are doing the assignment alone, the name of the zip file is A2-IdNumber.zip). Upload the file on moodle BEFORE 11:55 pm on 22nd Feb. No assignments will be accepted after a cutoff deadline of 10 am on 23rd Feb. Note that only one student per group should upload their work on moodle. Please preserve a copy of all your work until the end of the semester. If you have difficulties, please do not hesitate to seek help from me.

1. Refer to a copy of the paper 'The restricted isometry property and its implications for compressed sensing' in the homework folder. Your task is to open the paper and answer the question posed in each and every green-colored highlight. The task is the complete proof of Theorem 3 done in class. [24 points = 1.5 points for each of the 16 questions]
Solution:

   (a) Q1: We know that if $\delta_{2s}2$ is the order-$2s$ RIC of a matrix $\boldsymbol{\Phi}$, then we have $(1-\delta_{2s})\|\boldsymbol{x}\|^2 \leq \|\boldsymbol{\Phi x}\|^2 \leq (1+\delta_{2s})\|\boldsymbol{x}\|^2$ for any $2s$-sparse vector $\boldsymbol{x}$. If $\delta_{2s}=1$, then we could have $\boldsymbol{Ax}=\boldsymbol{0}$ for some $2s$-sparse $\boldsymbol{x}$, which would mean that there exist some set of $2s$ columns from $\boldsymbol{A}$ which are linearly]emphnot independent.

   (b) Q2: This holds due to the triangle inequality satisfied by the vector 2-norm and also because of the constraint imposed in the optimization problem.

   (c) Q3: This is true because $\|\mathbf{h}_{T_j}\|_2 = \sqrt{\sum_{i=1}^{s} h^2_{T_j,i}} \leq \sqrt{\sum_{i=1}^{s}\|\mathbf{h}_{T_j}\|^2_\infty} = s^{1/2}\|\mathbf{h}_{T_j}\|_\infty$. Also, notice that $s^{1/2}\|\mathbf{h}_{T_j}\|_\infty \leq \dfrac{s^{1/2}}{s}\sum_{i=1}^{s}\|\mathbf{h}_{T_j}\|_\infty \leq s^{-1/2}\|\mathbf{h}_{T_{j-1}}\|_1$. Also any element of $\mathbf{h}_{T_j}$ (including $\|\mathbf{h}_{T_j}\|_\infty$) is less than or equal to any element of $\mathbf{h}_{T_{j-1}}$.

   (d) Q4: $\sum_{j\geq 2}\|\mathbf{h}_{T_j}\|_2 \leq s^{-1/2}\sum_{j\geq 1}\|\mathbf{h}_{T_j}\|_1 = s^{-1/2}\|\mathbf{h}_{T_0^c}\|_1$. The last equality is because $T_0^c = T_1 \cup T_2 \cup .....$ The first inequality holds due to a simple summation starting from the previous relation $\|\boldsymbol{h}_{\boldsymbol{T_j}}\|_2 \leq s^{-1/2}\|\boldsymbol{h}_{\boldsymbol{T_{j-1}}}\|_1$.

   (e) Q5: We have $\|\mathbf{h}_{(T_0\cup T_1)^c}\|_2 = \|\sum_{j\geq 2}\mathbf{h}_{T_j}\|_2$ as $\forall j, \mathbf{h}_{T_j}$ have disjoint support. The next inequality follows by triangle inequality and the last one is because we earlier proved that $\sum_{j\geq 2}\|\mathbf{h}_{T_j}\|_2 \leq s^{-1/2}\|\boldsymbol{h}_{\boldsymbol{T_0^c}}\|_1$.

   (f) Q6: Reverse Triangle inequality on $|x_i + h_i|$ in two different directions.

   (g) Q7: Directly uses the previous equation from the paper and re-arranges the terms.

   (h) Q8: This is almost directly given in the paper via equations 11 and 12.

   (i) Q9: This is due to the Cauchy Schwartz inequality.

   (j) Q10: This comes from Lemma 2.1 in the paper, but extended to vectors that have magnitude greater than 1. See lecture slides for more details.

(k) Q11: Consider a 2-element vector $\mathbf{w} = (\|\mathbf{h_{T_0}}\|_2 \|\mathbf{h_{T_1}}\|_2)$. Then $\|\mathbf{w}\|_1 = \|\mathbf{h_{T_0}}\|_2 + \|\mathbf{h_{T_1}}\|_2$. We know that $\|\mathbf{w}\|_1 \leq \sqrt{2}\|\mathbf{w}\|_2 = \sqrt{2}\|\mathbf{h_{T_0 \cup T_1}}\|_2$ since the support sets T_0 and T_1 are disjoint.

(l) Q12: We need to be very careful here as so many steps are involved! From the RIP of $\mathbf{\Phi}$ for order $2s$ with RIC δ_{2s}, we know that $(1-\delta_{2s})\|\mathbf{h}_{T_0\cup T_1}\|_2^2 \leq \|\mathbf{\Phi h}_{T_0 \cup T_1}\|_2^2 =$
$\langle \mathbf{\Phi h_{T_0\cup T_1}}, \mathbf{\Phi h}\rangle - \langle \mathbf{\Phi h_{T_0\cup T_1}}, \sum_{j\geq 2} \mathbf{\Phi h_{T_j}}\rangle$(as shown in Q14)
$\leq |\langle \mathbf{\Phi h_{T_0\cup T_1}}, \mathbf{\Phi h}\rangle| + |\langle \mathbf{\Phi h_{T_0\cup T_1}}, \sum_{j\geq 2} \mathbf{\Phi h_{T_j}}\rangle|$
$\leq 2\epsilon\sqrt{1+\delta_{2s}}\|\mathbf{h}_{T_0\cup T_1}\|_2 + |\langle \mathbf{\Phi h_{T_0\cup T_1}}, \sum_{j\geq 2} \mathbf{\Phi h_{T_j}}\rangle|$
(using Cauchy-Schwartz inequality, equation 9 from the paper and right side of RIP)
$\leq 2\epsilon\sqrt{1+\delta_{2s}}\|\mathbf{h}_{T_0\cup T_1}\|_2 + |\langle \mathbf{\Phi h_{T_0}} + \mathbf{\Phi h_{T_1}}, \sum_{j\geq 2} \mathbf{\Phi h_{T_j}}\rangle|$
(as T_0 and T_1 are disjoint sets and hence $\mathbf{\Phi h}_{T_0\cup T_1} = \mathbf{\Phi h}_{T_0} + \mathbf{\Phi h}_{T_1}$)
$\leq 2\epsilon\sqrt{1+\delta_{2s}}\|\mathbf{h}_{T_0\cup T_1}\|_2 + \delta_{2s}(\|\mathbf{h_{T_0}}\|_2 + \|\mathbf{h_{T_1}}\|_2)\|\sum_{j\geq 2}\|\mathbf{h_{T_j}}\|_2$ from Lemma 2.1 of the paper
$\leq 2\epsilon\sqrt{1+\delta_{2s}}\|\mathbf{h}_{T_0\cup T_1}\|_2 + \delta_{2s}\sqrt{2}(\|\mathbf{h_{T_0\cup T_1}}\|_2)\|\sum_{j\geq 2}\|\mathbf{h_{T_j}}\|_2$.

(m) Q13: This follows straightforwardly from the previous step. Just divide the leftmost and rightmost sides by $\|\boldsymbol{h}_{T_0\cup T_1}\|_2$, and from equation 10 of the paper.

(n) Q14: Follows from straightforward algebra using equation 12 of the paper.

(o) Q15: Follows from triangle inequality.

(p) Q16: This follows in a very straightforward way using Lemma 2.2 from the paper which shows that $\|\boldsymbol{h_{T_0}}\|_1 \leq \rho\|\boldsymbol{h_{T_0^c}}\|_1$. The paper has already derived a bound for $\|\boldsymbol{h_{T_0^c}}\|_1$. This produces the bound for $\|\boldsymbol{h}\|_1$. (Food for thought: can this be easily extended for the noisy case as well? Why (not?))

2. Your task here is to implement the ISTA algorithm for the following three cases:

(a) Consider the image from the homework folder. Add iid Gaussian noise of mean 0 and variance 4 (on a [0,255] scale) to it, using the 'randn' function in MATLAB. Thus $\boldsymbol{y} = \boldsymbol{x} + \boldsymbol{\eta}$ where $\boldsymbol{\eta} \sim \mathcal{N}(0,4)$. You should obtain $\boldsymbol{x}$ from $\boldsymbol{y}$ using the fact that patches from $\boldsymbol{x}$ have a sparse or near-sparse representation in the 2D-DCT basis.

(b) Divide the image shared in the homework folder into patches of size 8×8. Let $\boldsymbol{x_i}$ be the vectorized version of the i^{th} patch. Consider the measurement $\boldsymbol{y_i} = \mathbf{\Phi}\boldsymbol{x_i}$ where $\mathbf{\Phi}$ is a 32×64 matrix with entries drawn iid from $\mathcal{N}(0,1)$. Note that $\boldsymbol{x_i}$ has a near-sparse representation in the 2D-DCT basis $\boldsymbol{U}$ which is computed in MATLAB as 'kron(dctmtx(8)',dctmtx(8)')'. In other words, $\boldsymbol{x_i} = \boldsymbol{U\theta_i}$ where $\boldsymbol{\theta_i}$ is a near-sparse vector. Your job is to reconstruct each $\boldsymbol{x_i}$ given $\boldsymbol{y_i}$ and $\mathbf{\Phi}$ using ISTA. Then you should reconstruct the image by averaging the overlapping patches. You should choose the α parameter in the ISTA algorithm judiciously. Choose $\lambda = 1$ (for a [0,255] image). Display the reconstructed image in your report. State the RMSE given as $\|X(:) - \hat{X}(:)\|_2/\|X(:)\|_2$ where $\hat{X}$ is the reconstructed image and X is the true image. [16 points]

(c) Repeat the reconstruction task using the Haar wavelet basis via the MATLAB command 'dwt2' with the option 'db1'. Display the reconstructed image in your report. State the RMSE. Use MATLAB function handles carefully. [8 points]

(d) Consider a 100-dimensional sparse signal $\boldsymbol{x}$ containing 10 non-zero elements. Let this signal be convolved with a kernel $\boldsymbol{h} = [1,2,3,4,3,2,1]/16$ followed by addition of Gaussian noise of standard deviation equal to 5% of the magnitude of $\boldsymbol{x}$ to yield signal $\boldsymbol{y}$, i.e. $\boldsymbol{y} = \boldsymbol{h} * \boldsymbol{x} + \boldsymbol{\eta}$. Your job is to reconstruct $\boldsymbol{x}$ from $\boldsymbol{y}$ given $\boldsymbol{h}$. Be careful of how you create the matrix $\boldsymbol{A}$ in the ISTA algorithm. [8 points]

Solution: See sample code in the homework folder. (b) For correct construction of DCT matrix - 2 points; for the ISTA algorithm correctly implemented with DCT, there are 10 points; 2 points for displaying the reconstructed images in the report (one point for each image); 2 points for the RMSE values. Please ensure that the stepsize $1/\alpha$ is chosen correctly: either α should be chosen to be greater than the maximum eigenvalue of $\boldsymbol{A}^t\boldsymbol{A}$ where $\boldsymbol{A} = \mathbf{\Phi}\boldsymbol{U}$, or else the stepsize should be chosen adaptively to ensure that the gradient update (to $\boldsymbol{\theta}$ causes a decrease in the cost function). 3 points to be deducted if proper stepsize is not chosen. (c) For correct implementation via MATLAB function handles, 6 points. 1 point for displaying the images, and 1 point for the RMSE values. Some students may use a Haar wavelet matrix, and full credit will be given provided the results are acceptable. (d) For the $\boldsymbol{A}$ matrix, there are 3 points. Note that it is a circulant matrix created from the filter kernel h. For correct implementation of ISTA, there will be 4 points, and 1 point for displaying the reconstructed signal and stating the RMSE value. The code for this is almost readily available at `http://eeweb.poly.edu/iselesni/lecture_notes/sparse_signal_restoration.pdf`.

3. One of the questions that came up in a live session was the notion of an oracle. Consider compressive measurements $\boldsymbol{y} = \boldsymbol{\Phi x} + \boldsymbol{\eta}$ of a purely sparse signal $\boldsymbol{x}$, where $\|\boldsymbol{\eta}\|_2 \leq \epsilon$. When we studied Theorem 3 in class, I had made a statement that the solution provided by the basis pursuit problem for a purely sparse signal comes very close (i.e. has an error that is only a constant factor worse than) an oracular solution. An oracular solution is defined as the solution that we could obtain if we knew in advance the indices (set S) the non-zero elements of the signal $\boldsymbol{x}$. This homework problem is to understand my statement better. For this, do as follows. In the following, we will assume that the inverse of $\boldsymbol{\Phi_S^T \Phi_S}$ exists, where $\boldsymbol{\Phi_S}$ is a submatrix of $\boldsymbol{\Phi}$ with columns belonging to indices in S.

   (a) Express the oracular solution $\tilde{\boldsymbol{x}}$ using a pseudo-inverse of the sub-matrix $\boldsymbol{\Phi_S}$. [5 points]
   Solution: $\tilde{\boldsymbol{x}} = \boldsymbol{\Phi_S}^{\dagger}\boldsymbol{y}$.

   (b) Now, show that $\|\tilde{\boldsymbol{x}} - \boldsymbol{x}\|_2 = \|\boldsymbol{\Phi}_{\boldsymbol{S}}^{\dagger}\boldsymbol{\eta}\|_2 \leq \|\boldsymbol{\Phi}_{\boldsymbol{S}}^{\dagger}\|_2\|\boldsymbol{\eta}\|_2$. Here $\boldsymbol{\Phi}_{\boldsymbol{S}}^{\dagger} \triangleq (\boldsymbol{\Phi_S^T \Phi_S})^{-1}\boldsymbol{\Phi_S^T}$ is standard notation for the pseudo-inverse of $\boldsymbol{\Phi_S}$. The largest singular value of matrix $\boldsymbol{X}$ is denoted as $\|\boldsymbol{X}\|_2$. [3 points]
   Solution: We have $\|\boldsymbol{x} - \tilde{\boldsymbol{x}}\|_2 = \|(\boldsymbol{\Phi_S}^T\boldsymbol{\Phi_S})^{-1}\boldsymbol{\Phi_S}^T(\boldsymbol{\Phi x} + \boldsymbol{e}) - \boldsymbol{x}\|_2 = \|\boldsymbol{\Phi_S}^{\dagger}\boldsymbol{\eta}\|_2 \leq \|\boldsymbol{\Phi_S}^{\dagger}\|_2\|\boldsymbol{\eta}\|_2$. The last inequality follows by the definition of the matrix operator norm (i.e. the largest singular value of the matrix).

   (c) Argue that the largest singular value of $\boldsymbol{\Phi}_{\boldsymbol{S}}^{\dagger}$ lies between $\dfrac{1}{\sqrt{1+\delta_{2k}}}$ and $\dfrac{1}{\sqrt{1-\delta_{2k}}}$ where $k = |S|$ and δ_{2k} is the RIC of $\boldsymbol{\Phi}$ of order $2k$. [4 points]
   Solution: If $\boldsymbol{\Phi}$ obeys RIP of order $2k$ with RIC δ_{2k}, then the singular values of $\boldsymbol{\Phi_S}$ must satisfy $\sqrt{1-\delta_{2k}} \leq \lambda_{min} \leq \lambda_{max} \leq \sqrt{1+\delta_{2k}}$. (This is by the very definition of singular value.) Hence the largest singular value of $\boldsymbol{\Phi_S}^{\dagger}$ (which is equal to $\dfrac{1}{\lambda_{max}}$) must lie in the range from $\dfrac{1}{\sqrt{1+\delta_{2k}}}$ to $\dfrac{1}{\sqrt{1-\delta_{2k}}}$.

   (d) This yields $\dfrac{\epsilon}{\sqrt{1+\delta_{2k}}} \leq \|\boldsymbol{x} - \tilde{\boldsymbol{x}}\|_2 \leq \dfrac{\epsilon}{\sqrt{1-\delta_{2k}}}$. Argue that the solution given by Theorem 3 is only a constant factor worse than this solution. [3 points]
   Solution: The (upper) error bound given by Theorem 3 for purely sparse signals is $\dfrac{4\sqrt{1+\delta_{2s}}}{1-\delta_{2s}(\sqrt{2}+1)}$. This upper bound is worse than the oracle solutions. To see this, notice that the denominator of the oracle solution is larger (since it involves a square root of a value between 0 and 1). A comment about the lower bound: Just as we have $\|\boldsymbol{x} - \tilde{\boldsymbol{x}}\|_2 \leq \|\boldsymbol{\Phi_S}^{\dagger}\|_2\varepsilon$, we also have $\|\boldsymbol{x} - \tilde{\boldsymbol{x}}\|_2 = \|\boldsymbol{\Phi}_{\boldsymbol{S}}^{\dagger}\boldsymbol{\eta}\|_2 \geq \|\boldsymbol{\Phi}_{\boldsymbol{S}}^{\dagger}\|_{min}\epsilon$, where $\|\boldsymbol{\Phi}_{\boldsymbol{S}}^{\dagger}\|_{min}$ represents the least singular value of $\boldsymbol{\Phi}_{\boldsymbol{S}}^{\dagger}$. The lower bound argument is not required and no points to be deducted for missing out on it.

4. If $s < t$ where s and t are positive integers, prove that $\delta_s \leq \delta_t$ where δ_s, δ_t stand for the restricted isometry constant (of any sensing matrix) of order s and t respectively. [8 points]
Solution: We have $\delta_s < \delta_t$ for $s < t$ because $\delta_s = \max_{\Gamma_1 \subset [n], |\Gamma_1| \leq s}\|\boldsymbol{A_{\Gamma_1}}^T\boldsymbol{A_{\Gamma_1}} - \boldsymbol{I_s}\|_2$, where $[n] = \{1, 2, ..., n\}$ and $\|\boldsymbol{B}\|_2$ for a matrix means its operator norm (i.e. its largest singular value). In English, this means that δ_s is the largest singular value of any sub-matrix of $\boldsymbol{A}$ with at the most s columns. Likewise, δ_t is the largest singular value of any sub-matrix of $\boldsymbol{A}$ with at the most t columns. Since the maximum is being taken over a larger set of sub-matrices, it will potentially be greater than the earlier one (or equal to the earlier one, but never lower than the earlier one). Hence $\delta_s \leq \delta_t$.
Marking scheme: Correct expression of the RIC in terms of eigenvalues of $\boldsymbol{A_{\Gamma_1}}^T\boldsymbol{A_{\Gamma_1}}$ will fetch 4 points. 4 points for the rest of the argument.

5. Here is our obligatory Google search question :-). Your task is to find out any one paper from within the last 5 years, apart from our Tapestry pooling paper from `https://arxiv.org/abs/2005.07895`, which applied compressed sensing for group testing (not necessarily for COVID-19 pooling, but other applications or even theoretical papers are fine). You may look for references in the Tapestry pooling paper as well. Unpublished papers from arxiv are allowed as well. Answer the following questions: (1) Mention the title of the paper

and a link to it. (2) Mention the key objective function being minimized in the paper with the meaning of all symbols clearly explained. (3) Enlist any three differences between the proposed approach and the Tapestry pooling approach. [8+7=15 points]

Solution: There are many answers to the question. I will quote the paper 'Efficient high throughput SARS-CoV-2 testing to detect asymptomatic carriers' by the group of Noam Shental in Israel, which is reference 37 from the aforementioned paper. A few differences between Tapestry and the technique in the paper from Israel are already mentioned in Section 5 ('Relation to previous work') of the Tapestry paper. The paper from Israel runs a LASSO optimization procedure which minimizes an objective function of the form $J(\boldsymbol{x}) = \|\boldsymbol{y} - \boldsymbol{A}\boldsymbol{x}\|^2 + \lambda\|\boldsymbol{x}\|_1$. To quote from the paper, the following post-processing step was performed to convert the fractional output of LASSO the discrete case: 'The 20 samples with highest scores were selected, and only subsets of these 20 were further considered. In total, 2^{20} subsets of samples were tested. Each subset corresponds to a vector $\boldsymbol{x}$ of length 384, in which the entries of the selected samples were equal to 1, and all others were set to zero. Nonzero entries of the product of $\boldsymbol{M}\boldsymbol{x}$, where $\boldsymbol{M}$ is the pooling matrix, were replaced by the value 1 and compared to the (binary) measurement vector $\boldsymbol{y}$. The vector $\tilde{\boldsymbol{x}}$ for which $\|\boldsymbol{M}\tilde{\boldsymbol{x}} - \boldsymbol{y}\|_1$ achieved its minimum was selected.' Thus their approach considers binary measurement vectors, whereas Tapestry considers real-valued measurements. Tapestry runs other algorithms besides just LASSO and does not do the later post-processing step.

6. Consider the problem P1: $\min_{\boldsymbol{x}} \|\boldsymbol{x}\|_1$ s. t. $\|\boldsymbol{y} - \boldsymbol{\Phi}\boldsymbol{x}\|_2 \leq \epsilon$. Also consider the LASSO problem which seeks to minimize the cost function $J(\boldsymbol{x}) = \|\boldsymbol{y} - \boldsymbol{\Phi}\boldsymbol{x}\|_2^2 + \lambda\|\boldsymbol{x}\|_1$. If $\boldsymbol{x}$ is a minimizer of $J(.)$ for some value of $\lambda > 0$, then show that there exists some value of ϵ for which $\boldsymbol{x}$ is also the minimizer of the problem P1. [6 points] (Hint: Consider $\epsilon' = \|\boldsymbol{y} - \boldsymbol{\Phi}\boldsymbol{x}\|_2$. Now use the fact that $\boldsymbol{x}$ is a minimizer of $J(.)$ to show that it is also a minimizer of P1 subject to an appropriate constraint involving ϵ'.)

   Solution: Consider $\epsilon' = \|\boldsymbol{y} - \boldsymbol{\Phi}\boldsymbol{x}\|_2$. Consider some vector $\boldsymbol{z}$ with n elements such that $\|\boldsymbol{A}\boldsymbol{z} - \boldsymbol{y}\|_2 \leq \epsilon'$. As $\boldsymbol{x}$ is the minimizer of $J(.)$, we have $\lambda\|\boldsymbol{x}\|_1 + \|\boldsymbol{y} - \boldsymbol{A}\boldsymbol{x}\|^2 \leq \lambda\|\boldsymbol{z}\|_1 + \|\boldsymbol{y} - \boldsymbol{A}\boldsymbol{z}\|^2 \leq \lambda\|\boldsymbol{z}\|_1 + \|\boldsymbol{y} - \boldsymbol{A}\boldsymbol{x}\|^2$ (3 points for this step). This yields $\|\boldsymbol{x}\|_1 \leq \|\boldsymbol{z}\|_1$ (1.5 points for this step). Hence $\boldsymbol{x}$ is a minimizer of P1 subject to the constraint involving ϵ' (1.5 points for this step).