Assignment 4: CS 754, Advanced Image Processing

Shreyas Narahari-203050037
Nitish Gangwar-203050069

April 7, 2021

Answer 2:

link to paper: CLICK HERE
CoSaMP ALGORITHM:
Input:
Measurement matrix ϕ
y = Measurement vector
s= signal sparsity
step 1: $x_0 = 0$, $r = y$,$k = 0$
step 2: while terminating condition is false do
step 3: iteration counter:

$$k = k + 1$$

step 4: calculating the **proxy signal z** by correlating the residual vector with the columns of matrix ϕ:

$$z = \phi * r$$

step 5: Selecting the 2s columns of matrix ϕ that corresponds to the 2s largest absolute values of proxy signal and rest other entries of z are set to 0.

$$\Omega = supp(z^{2s})$$

step 6: The selected columns indices are then added(by taking union) to the currently estimated support of the unknown vector x

$$T = \Omega \cup supp(x_{k-1})$$

step 7: A 3s-sparse signal $\tilde{x}$ is obtained after solving the least square problem.

$$\bar{x}|_T = \phi_T^{\dagger} y$$

$$x|_{T^c} = 0$$

step 8: s-sparse vector x is obtained by pruning the 3s-sparse vector and setting the other values other than s-sparse values to 0.

$$x_k = \bar{x}^s$$

step 9: Updating the current sample

$$r = y - \phi x_k$$

step 10: end while
step 11: $\hat{x} = x_k$
step 12: return $\hat{x}$

Important theorem from the paper is **Theorem 4.1 (Iteration invariation: Sparse case)**:

$$||x - a^{k+1}||_2 \leq 0.5||x - a^k||_2 + 7.5||e||_2$$

In particular,

$$||x - a^k||_2 \leq 2^{-k}||x||_2 + 15||e||_2$$

where,
x is s-sparse signal.
a^k is s-sparse signal approximation.
e is noise vector of dimension R^m
and now using this Theorem 4.1 another Theorem 2.1 is derived which is also a main result i.e.
Theorem 2.1(Iteration invariant).

$$||x - a^{k+1}||_2 \leq 0.5||x - a^k||_2 + 10\nu$$

In particular,

$$||x - a^k||_2 \leq 2^{-k}||x||_2 + 20\nu$$

where,
x is s-sparse signal.
a is approximated s-sparse signal.
ν is the unrecoverable energy.

Answer 3:

(a.)

We know that,

$$X = D\theta$$

When we have a derivative filter applied over a set of images denoted by class S_1.Then,
let's denote the convolution operation of derivative filter using

$$Conv(X, D_v)$$

where Conv denotes the convolution operator using Derivative filter(D_v) by taking care of the boundary pixels through padding.
Now, we want to find the dictionary corresponding to the class S_1 images.

$$Conv(X, D) = Conv(D\theta, D_v)$$

here the dimension of D$\in R^{n\times k}$,$\theta \in R^{k\times 1}$
So, the dimension of X or $D\theta$ will be $R^{n\times 1}$.We have to reshape it to $\sqrt{n} \times \sqrt{n}$ and then we will apply the convolution operation which will provide us the dictionary which has been created using the images which have a derivative filter applied over them.

$$Conv((D\theta)_{\sqrt{n}\times\sqrt{n}}, D_v)$$

(b.)

We have two sets of images where the first subset contains the images which are rotated by α and the other set by β. for the first subset of images having rotation by α:
We know,

$$X = D\theta$$

Now the images have a rotation of α.We will denote this operation of rotation by R$_\theta$.

$$R_\alpha(X) = R_\alpha(D\theta)$$

We already have the the original dictionary denoted by D. Now we will perform the rotation over matrix $D\theta$ and if we need it to reshape it to two dimension then also we can perform that and this is how we will obtain the updated dictionary.
Same thing can be done with other subset which has images having a rotation of β.Here will have the following equation :

$$R_\beta(X) = R_\beta(D\theta)$$

same rotation operation can be performed over the matrix $D\theta$ by angle β.For that if we need to reshape the resultant matrix then we can do that and after rotation we can again flatten it to get back the $n \times 1$ dimension vector.

(c.)

Class S_3 which have images which are obtained by applying the intensity transformation of the type:

$$I^i_{new}(x,y) = \alpha(I^i_{old}(x,y))^2 + \beta(I^i_{old}(x,y)) + \gamma$$

where I(x,y) denotes the pixel intensity at location (x,y). So, the output is kind of combination of intensities as shown above.We know,

$$X = D\theta$$

Now, X will have intensities for all (x,y). We can use the old D as follows:

$$I(X)(x,y) = I(D\theta)(x,y)$$
$$I^i_{new}(D\theta)(x,y) = \alpha(I^i_{old}(D\theta)(x,y))^2 + \beta(I^i_{old}(D\theta)(x,y)) + \gamma$$

and with the linear combination of these dictionary elements we will get the updated dictionary, which is obtained after applying intensity transformation over the images.

(d.)

For class S_4 images obtained by applying a known blur kernel to the images.Then, this part is quite analogous to running a kernel or filter over the image or matrix and averaging out the values like for 3×3 kernel the averaging will be done by multiplying with $\frac{1}{9}$.

$$Conv(X, Blur) = Conv(D\theta, Blur)$$

and this operation will provide us the dictionary which is updated using the images which have blur.

(e.)

For class S_5 we have set of images which are obtained after applying linear combination of blur kernel that belongs to some known set B.
So, say

$$B = a_1 \times B_1 + a_2 \times B_2 + ... + a_n \times B_n.$$

we know that,

$$x = D\theta$$

So,

$$Conv(X, B) = Conv(D\theta, B)$$

where the B denotes the linear combination of blur kernels.As we know that,D$\in R^{n\times k}$ and $\theta \in R^{k\times 1}$. Resultant matrix will have a size of $n \times 1$ which can be reshaped and this linear combination of blur kernel can be convolved which will provide us the required dictionary.

Answer 4:

Problem 1: For solving the minimization of the objective function,

$$J(\boldsymbol{A_r}) = \|\boldsymbol{A} - \boldsymbol{A}_r\|_F^2$$

where $\boldsymbol{A}$ is a $m \times n$ matrix of rank greater than r and $\boldsymbol{A}_r$ is a rank r matrix.
This is low-rank approximation of matrices is a minimization problem where the objective function measures the best matrix that fits the given matrix. The proof for minimization is given by the Ekart-Young-Mirsky theorem for Forbenius Norm.

This objective function is minimized by using the k largest entries from the SVD of A. We know that,

$$A = U\Sigma V^T$$

is the Singular Value Decomposition of A where U and V are orthogonal matrices and Σ is an $m \times n$ rectangular diagonal matrix with non-negative real numbers on the diagonal with entries $(\sigma_1, \sigma_2, \ldots, \sigma_n)$ such that $\sigma_1 \geq \sigma_2 \geq \cdots \geq \sigma_n \geq 0$. We get the best rank-k approximation to A by choosing the largest k terms and k corresponding columns in the matrices U and V as,

$$A_k = \sum_{i=1}^{k} \sigma_i u_i v_i^T$$

Here u_i and v_i denote the i^{th} column of U and V.

Application in Image Processing: Low-rank approximation can be used for Matrix completion which has application in Image In-painting. In such images a lot of image values are missing because of the In-Painting, matrix completion can be used to infer the missing pixel values just using the defined pixel values. In this we can consider vectorized patches of m pixels each and such K columns. This $m \times K$ matrix will have many missing entries which can be filled using Matrix Completion.

Problem 2:

$$J(R) = ||A - RB||_F^2$$

where,

$$A \in R^{n\times m}, B \in R^{n\times m}, R \in R^{n\times n}, m > n$$

R is constrained to be orthonormal.
This optimization problem is used in Method of Optimal Directions(MOD).

$$J(R) = ||Y - AS||_F^2 \tag{1}$$

where Y and S is known, we need to find dictionary A. Which we can compute by taking the derivative of J(R) with respect to R.

$$J'(R) = 0$$

$$(Y - AS)S^T = 0$$
$$YS^T - ASS^T = 0$$
$$A = YS^T(SS^T)^{-1}$$

Here, the columns of A are updated and independently scaled to unit norm which helps in acquiring the orthonormality of A.

Application in Image Processing: Sparse dictionary learning can be used for Image Denoising. Here, the dictionary which is used is learned before hand from patches from images. We can also use the noisy patches itself to train the dictionary to help the dictionary understand the underlying structure. In this we first estimate the sparse codes and update the dictionary. Finally image can be estimated by averaging over overlapping estimated patches.

Answer 5:

(a.)

In Hyperspectral unmixing each pixel consists of multiple spectral measurements mixed during the capture of the reflectance spectral data. In the paper in order to represent the mixing equation a linear mixing model (LMM) has been considered. LMM is not considered to always true as some situation's may exhibit strong non-linearity. But in real-world scenarios it is considered to be an acceptable model. We consider the equation,

$$\boldsymbol{y}[n] = \sum_{i=1}^{N} \boldsymbol{a}_i s_i[n] + \boldsymbol{\nu}[n] = \boldsymbol{A}\boldsymbol{s}[n] + \boldsymbol{\nu}[n]$$

$\boldsymbol{y}[n] = [y_1[n], y_2[n], \ldots, y_m[n]]^T$ Here $y_m[n]$ denotes the hyperspectral cameras measurement at spectral band m and at pixel n.
$\boldsymbol{s}[n] = [s_1[n], \ldots, s_N[n]]$ denotes the abundance vector at pixel n.
$\mathbf{a}_i \in R^M$ denotes the endmember signature vector.
$\mathbf{A}$=$[a_1, a_2, ..., a_N] \in R^{M\times N}$ denotes the endmember matrix.
$\nu[n]$ denotes the noise.

(b.)

We know,
from Sparse Regression(SR) problem

$$min_{s[n]}||s[n]||_0 \quad s.t. \quad y[n] = As[n]$$

and it is known that the above equation has a unique solution if s[n] satisfies

$$||s[n]||_0 < \frac{1}{2}.spark(A)$$

where spark(A) denotes the smallest number of linearly dependent columns of A.But, the SR problem is a NP hard in general.
Hence, we consider a SR(CSR) problem which can be solved using approximation:

$$min_S||S||_{row-0} \quad s.t. \quad Y = AS,$$

where $||S||_{row-0}$ denotes the number of endmembers. A convex relaxation is applied to CSR(Constrained Sparse Regression) by replacing $||S||_{row-0}$ by $||||_{2,1}$.Through the literature we know that if we increase the number of measurements then probability of recovery failure reduces significantly and the non negativity constrained applied over the problem helps in improving the solution.So, the optimization problem now becomes which also considers the noise:

$$min_{S\geq 0}||||_F^2 + \lambda||S||_{2,1}$$

By using the idea of subspace method we have

$$R(Y) = R(A_s)$$

where R denotes the range space of its argument.
Y is measured data.
A_s is a submatrix of A.
and hence finally we have equation 40.

$$||S||_{row-0} < spark(A) - 1$$

so by using the non negative matrix factorization we can obtain the matrix A and S which can be used as estimates of the endmembers and abundances.This is where we are using non negative matrix factorization for hyperspectral unmixing.

(c.)

Improvement to non-negative matrix factorization in equation 41 of the paper are:
equation 41 is:

$$min_{A\geq 0,S\geq 0}||Y - AS||_F^2$$

we know that factors obtained using Non negative matrix factorization acts as the estimates of endmembers and abundances.But there are two problems associated in solving the equation 41 and they are:
1. It is NP hard problem.
2. There are no guarantees for obtaining the unique solution.
Hence, In blind hyperspectral unmixing ,Non negative matrix factorization certain changes are made to overcome such problems i.e.

$$min_{A\geq 0,S\in S^L}||Y - AS||_F^2 + \lambda g(A) + \mu.h(S)$$

So, we use some regularizers like g and h along with some constants like λ and μ.There are wide range of functions g and h available but they are selected with reference to the type of work we are going to do.Some of the examples of function g and h are :

In case of MVC-NMF(Minimum Volume Constrained - Non-negative Matrix Factorization)

Here, In this method we have a variation of the Volmin formulation in noisy case.which incorporates the nonnegativity of endmembers.

$$min_{A\geq 0, S\in S^L} ||Y - AS||_F^2 + \lambda.(vol(B))^2$$

but here, vol(B) is non convex type.

$$g = vol^2(C^\dagger(A - d1^T))$$

$$h = 0$$

In case of ICE(Iterated Constrained Endmember)

$$g = \sum_{i=1}^{N-1}\sum_{j=i+1}^{N} ||a_i - a_j||_2^2$$

$$h = 0$$

ICE contains function g which is convex in nature.So, ICE overcomes the non convex problem that we had with MVC_NMF.

$$g(A) = \sum_{i=1}^{N-1}\sum_{j=i+1}^{N} ||a_i - a_j||_2^2$$

References:

1. http://dsp-book.narod.ru/TAH/ch08.pdf
2. https://www.sciencedirect.com/science/article/pii/S1063520308000638?via%3Dihub
3. https://www.ee.cuhk.edu.hk/ wkma/publications/HU_SPM2014.pdf
4. https://en.wikipedia.org/wiki/Low-rank$_{a}pproximation$
5. https://www.sjsu.edu/faculty/guangliang.chen/Math253S20/lec7matrixnorm.pdf
6. https://www1.icsi.berkeley.edu/ stellayu/publication/doc/2020ocnnCVPR.pdf
7. https://www.hindawi.com/journals/mpe/2016/7616393/
8. https://www.hindawi.com/journals/mpe/2016/7616393/alg3/