Assignment 2: CS 754, Advanced Image Processing

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Answer 1:

Q1:.

From the definition of the isometry constant we get that δ_s of a matrix ϕ as the smallest number such that:

$$(1-\delta_s)\|h\|_{l_2}^2 \leq \|\phi h\|_{l_2}^2 \leq (1+\delta_s)\|h\|_{l_2}^2$$

If we consider a 2s sparse vector h such that h=x-x' where x and x' are s sparse vectors we get,

$$(1-\delta_{2s})\|(x-x')\|_{l_2}^2 \leq \|\phi(x-x')\|_{l_2}^2 \leq (1+\delta_{2s})\|(x-x')\|_{l_2}^2$$

After taking $\delta_{2s} = 1$ we get the isometry property reduced to,

$$0 \leq \|\phi(x-x')\|_{l_2}^2 \leq (2)\|(x-x')\|_{l_2}^2$$

Q2:
Here we introduce a temporary variable y in the L.H.S,

$$\|\phi(x^*-x)\|_{l_2} = \|\phi x^* - \phi x\|_{l_2} = \|\phi x^* - y + y - \phi x\|_{l_2}$$

From triangle inequality,

$$\|(\phi x^* - y) + (y - \phi x)\|_{l_2} \leq \|\phi x^* - y\|_{l_2} + \|y - \phi x\|_{l_2} \tag{1}$$

From the definition of the convex optimization it is given that $\|y - \phi x\|_{l_2} \leq \epsilon$ and as x^* is a solution having the best sparse approximation we get,

$$\|\phi x^* - y\|_{l_2} + \|y - \phi x\|_{l_2} \leq 2\epsilon \tag{2}$$

Combining 1 and 2 we get the observation,

$$\|\phi(x^*-x)\|_{l_2} \leq \|\phi x^* - y\|_{l_2} + \|y - \phi x\|_{l_2} \leq 2\epsilon$$

Q3:
For each of i^{th} element in $h_{T_{ji}} \leq max(h_{T_j}) = \|h_{T_j}\|_{l_\infty}$ Squaring on both sides,

$$h^2_{T_{ji}} \leq \|h_{T_j}\|^2_{l_\infty}$$

Summation across s non-zero elements we get,

$$\sqrt{\sum_i h^2_{T_{ji}}} \leq \sqrt{\sum_i \|h_{T_j}\|^2_{l_\infty}}$$

$$= \|h_{T_j}\|_{l_2} \leq \sqrt{n\|h_{T_j}\|^2_{l_\infty}} = \sqrt{s}\|h_{T_j}\|_{l_\infty}$$

We can conclude that,

$$\|h_{T_j}\|_{l_2} \leq s^{1/2}\|h_{T_j}\|_{l_\infty} \tag{3}$$

For proving the second inequality we consider,

$$\|h_{T_j}\|_{l_\infty} \leq min(h_{T_{j-1}})$$

After taking summation on both sides,

$$\sum_i \|h_{T_j}\|_{l_\infty} \leq \sum_i h_{T_{j-1}}$$

$$= s\|h_{T_j}\|_{l_\infty} \leq \|h_{T_{j-1}}\|_{l_1}$$

$$= \sqrt{s}\sqrt{s}\|h_{T_j}\|_{l_\infty} \leq \|h_{T_{j-1}}\|_{l_1}$$

$$= \sqrt{s}\|h_{T_j}\|_{l_\infty} \leq \frac{1}{\sqrt{s}}\|h_{T_{j-1}}\|_{l_1}$$

$$= s^{1/2}\|h_{T_j}\|_{l_\infty} \leq s^{-1/2}\|h_{T_{j-1}}\|_{l_1} \tag{4}$$

Combining equations 3 and 4 we get.

$$\|h_{T_j}\|_{l_2} \leq s^{1/2}\|h_{T_j}\|_{l_\infty} \leq s^{-1/2}\|h_{T_{j-1}}\|_{l_1}$$

Q4:

From the previous question we know that, $\|h_{T_j}\|_{l_2} \leq s^{-1/2}\|h_{T_{j-1}}\|_{l_1}$ summing up both sides for $j \geq 2$ we get,

$$\sum_{j\geq 2}\|h_{T_j}\|_{l_2} \leq \sum_{j\geq 2} s^{-1/2}\|h_{T_{j-1}}\|_{l_1}$$

$$\sum_{j\geq 2}\|h_{T_j}\|_{l_2} \leq s^{-1/2}\sum_{j\geq 1}\|h_{T_j}\|_{l_1}$$

$$\sum_{j\geq 2}\|h_{T_j}\|_{l_2} \leq s^{-1/2}(\|h_{T_1}\| + \|h_{T_2}\| + \cdots) \tag{5}$$

From the definition we know that $h_{T_0^c} = h_{T_1} + h_{T_2} \cdots$ and we can extend the previous step taking this information as,

$$s^{-1/2}(\|h_{T_1}\| + \|h_{T_2}\| + \cdots) \leq s^{-1/2}\|h_{T_0^c}\|_{l_1} \tag{6}$$

From 5 and 6 we get the final inequality,

$$\sum_{j\geq 2}\|h_{T_j}\|_{l_2} \leq s^{-1/2}(\|h_{T_1}\| + \|h_{T_2}\| + \cdots) \leq s^{-1/2}\|h_{T_0^c}\|_{l_1}$$

Q5:

For proving the first inequality we need to prove that l_2 norm of sum is lesser than sum of l_2 norms, take the triangle inequality for the base case of induction,

$$\|x_1 + x_2\| \leq \|x_1\| + \|x_2\|$$

Let us assume that this inequality is true for n terms,

$$\|\sum_{j\geq 2}^{n} h_{T_j}\|_{l_2} = \sum_{j\geq 2}^{n}\|h_{T_j}\|_{l_2}$$

Then we need to prove that the inductive step is true ie.,

$$\|\sum_{j\geq 2}^{n+1} h_{T_j}\|_{l_2} \leq \sum_{j\geq 2}^{n+1}\|h_{T_j}\|_{l_2}$$

Using the previous equations we get,

$$\|\sum_{j\geq 2}^{n+1} h_{T_j}\|_{l_2} \leq \|\sum_{j\geq 2}^{n} h_{T_j}\|_{l_2} + \|h_{T_{n+1}}\|$$

$$\|\sum_{j\geq 2}^{n+1} h_{T_j}\|_{l_2} \leq \sum_{j\geq 2}^{n}\|h_{T_j}\|_{l_2} + \|h_{T_{n+1}}\| = \sum_{j\geq 2}^{n+1}\|h_{T_j}\|_{l_2}$$

So, the first inequality has been proved. Taking the second inequality has already indirectly been proven in the previous question so from there we can consider,

$$\sum_{j\geq 2}^{n+1}\|h_{T_j}\|_{l_2} \leq s^{-1/2}\|h_{T_0^c}\|_{l_1}$$

Q6:

Using reverse triangle inequality we get,

$$\|x_{T_0} + h_{T_0}\|_{l_1} \geq \left| \|x_{T_0}\|_{l_1} - \|h_{T_0}\|_{l_1} \right|$$

and,

$$\|x_{T_0^c} + h_{T_0^c}\|_{l_1} \geq \left| \|x_{T_0^c}\|_{l_1} - \|h_{T_0^c}\|_{l_1} \right| = \left| \|h_{T_0^c}\|_{l_1} - \|x_{T_0^c}\|_{l_1} \right|$$

Combining these inequalities we get the ineqality,

$$\|x_{T_0} + h_{T_0}\|_{l_1} + \|x_{T_0^c} + h_{T_0^c}\|_{l_1} \geq \|x_{T_0}\|_{l_1} - \|h_{T_0}\|_{l_1} + \|h_{T_0^c}\|_{l_1} - \|x_{T_0^c}\|_{l_1}$$

Q7:

From definition we get $\|x_{T_0^c}\|_{l_1} = \|x - x_s\|_{l_1}$ this can be written as $\|x_{T_0^c}\|_{l_1} = \|x - x_{T_0}\|_{l_1}$ from inverse triangle inequality we get,

$$\|x_{T_0^c}\|_{l_1} \geq \|x\|_{l_1} - \|x_{T_0}\|_{l_1}$$

Substituting $\|x\| \geq \|x + h\|$ we get

$$\|x_{T_0^c}\|_{l_1} \geq \|x + h\|_{l_1} - \|x_{T_0}\|_{l_1}$$

Substituting the inequality from the last derivation we get,

$$\|x_{T_0^c}\|_{l_1} \geq \|x_{T_0}\|_{l_1} - \|h_{T_0}\|_{l_1} + \|h_{T_0^c}\|_{l_1} - \|x_{T_0^c}\|_{l_1} - \|x_{T_0}\|_{l_1}$$

Rearranging,

$$\|h_{T_0^c}\|_{l_1} \leq \|h_{T_0}\|_{l_1} + 2\|x_{T_0^c}\|_{l_1}$$

Q8:
From previous derivation we consider $\|h^c_{T_0\cup T_1}\|_{l_1} \leq s^{-1/2}\|h_{T_0^s}\|_{l_1}$ substituting this in the equation derived from Q7 we get,

$$\|h^c_{(T_0\cup T_1)}\|_{l_1} \leq s^{-1/2}(\|h_{T_0}\|_{l_1} + 2\|x_{T_0^c}\|_{l_1})$$
$$\|h^c_{(T_0\cup T_1)}\|_{l_1} \leq s^{-1/2}\|h_{T_0}\|_{l_1} + 2s^{-1/2}\|x_{T_0^c}\|_{l_1}$$

Substituting this inequality that was derived $\|u\|_{l_1} \leq s^{-1/2}\|u\|_{l_2}$ and $\|x_{T_0^c}\|_{l_1} = \|x - x_s\|_{l_1}$ we get,

$$\|h^c_{(T_0\cup T_1)}\|_{l_1} \leq \|h_{T_0}\|_{l_2} + 2s^{-1/2}\|x - x_s\|_{l_1}$$
$$\|h^c_{(T_0\cup T_1)}\|_{l_1} \leq \|h_{T_0}\|_{l_2} + 2s^{-1/2}e_0, e_0 \equiv \|x - x_s\|_{l_1}$$

Q9:
To prove this inequality we have considered 2 separate inequalities and taken product of them. First we consider a previously proven inequality,

$$\|\phi(x^* - x)\|_{l_2} \leq 2\epsilon$$

From the definition we know that $x^* = x + h$ so, $x^* - x = h$ we get,

$$\|\phi h\|_{l_2} \leq 2\epsilon$$

Squaring both sides,

$$\|\phi h\|^2_{l_2} \leq (2\epsilon)^2 \tag{7}$$

Substituting vector $h_{(T_0\cup T_1)}$ into the restricted isometry property, (Here $(T_0 \cup T_1)$ is a 2s sparse vector)

$$\|\phi h_{(T_0\cup T_1)}\|^2_{l_2} \leq (1 + \delta_2 s)\|h_{(T_0\cup T_1)}\|^2_{l_2} \tag{8}$$

Multiplying the inequalities 7 and 8 we get,

$$\|\phi h_{(T_0\cup T_1)}\|^2_{l_2}\|\phi h\|^2_{l_2} \leq (2\epsilon)^2(1 + \delta_{2s})\|h_{(T_0\cup T_1)}\|^2_{l_2}$$

Taking square root on both sides,

$$\|\phi h_{(T_0\cup T_1)}\|_{l_2}\|\phi h\|_{l_2} \leq (2\epsilon)\sqrt{1 + \delta_{2s}}\|h_{(T_0\cup T_1)}\|_{l_2}$$

Q10:

$$|\langle \phi h_{T_0}, \phi h_{T_j} \rangle| \leq \delta_{2s} \|h_{T_0}\|_{l_2} \|h_{T_j}\|_{l_2}$$

For this inequality to be true the supports for the vectors h_{T_0} and h_{T_j} should be disjoint. From the definition of the s-sparse vectors $h_{T_0}, h_{T_1}, h_{T_2}, \cdots$ we know that none of the vectors have overlapping elements. Thus this inequality holds for the given set of vectors when $j \geq 1$

Q11:
Consider the norms of vectors $\|h_{T_0}\|_{l_2}$ and $\|h_{T_1}\|_{l_2}$ and the inequality $(u - v)^2 \geq 0$. Substituting we get,

$$(\|h_{T_0}\|_{l_2} - \|h_{T_1}\|_{l_2})^2 \geq 0$$

$$\|h_{T_0}\|_{l_2}^2 + \|h_{T_1}\|_{l_2}^2 - 2\|h_{T_0}\|\|h_{T_1}\| \geq 0$$

Adding terms on both sides and rearranging we get,

$$2(\|h_{T_0}\|^2 + \|h_{T_1}\|^2) \geq 2\|h_{T_0}\|\|h_{T_1}\| + \|h_{T_0}\|^2 + \|h_{T_1}\|^2$$

We also know that an upper bound for $\|h_{T_0}\|_{l_2}^2$ and $\|h_{T_1}\|_{l_2}^2$ can be considered as $\|h_{T_0 \cup T_1}\|_{l_2}^2$ so we get,

$$2(\|h_{T_0 \cup T_1}\|^2) \geq (\|h_{T_0}\| + \|h_{T_1}\|)^2$$

Taking square root on both sized we get,

$$\sqrt{2}(\|h_{T_0 \cup T_1}\|) \geq (\|h_{T_0}\| + \|h_{T_1}\|)$$

Q12:
As mentioned earlier steps in the paper,

$$\|\phi h_{T_0 \cup T_1}\|_{l_2}^2 = |\langle \phi h_{T_0 \cup T_1}, \phi h \rangle| - |\langle \phi h_{T_0 \cup T_1}, \sum_{j \geq 2} \phi h_{T_j} \rangle|$$

The bound for the first term $|\langle \phi h_{T_0 \cup T_1}, \phi h \rangle|$ is justified in Q9 as $2\epsilon\sqrt{1 + \delta_{2s}}\|h_{(T_0 \cup T_1)}\|_{l_2}$. For deriving the bound for the second term we first divide it into sum of two terms as,

$$|\langle \phi h_{T_0 \cup T_1}, \sum_{j \geq 2} \phi h_{T_j} \rangle| = |\langle \phi h_{T_0} + \phi h_{T_1}, \sum_{j \geq 2} \phi h_{T_j} \rangle| = |\langle \phi h_{T_0}, \sum_{j \geq 2} \phi h_{T_j} \rangle| + |\langle \phi h_{T_1}, \sum_{j \geq 2} \phi h_{T_j} \rangle|$$

From the derivation given in Q10 we get,

$$|\langle \phi h_{T_0}, \sum_{j\geq 2} \phi h_{T_j}\rangle| \leq \sum_{j\geq 2} \delta_{2s}\|h_{T_0}\|_{l_2}\|h_{T_j}\|_{l_2} \text{ and } |\langle \phi h_{T_1}, \sum_{j\geq 2} \phi h_{T_j}\rangle| \leq \sum_{j\geq 2} \delta_{2s}\|h_{T_1}\|_{l_2}\|h_{T_j}\|_{l_2}$$

$$= |\langle \phi h_{T_0\cup T_1}, \sum_{j\geq 2} \phi h_{T_j}\rangle| \leq \sum_{j\geq 2} \delta_{2s}\|h_{T_0}\|_{l_2}\|h_{T_j}\|_{l_2} + \sum_{j\geq 2} \delta_{2s}\|h_{T_1}\|_{l_2}\|h_{T_j}\|_{l_2}$$

$$= |\langle \phi h_{T_0\cup T_1}, \sum_{j\geq 2} \phi h_{T_j}\rangle| \leq \sum_{j\geq 2} \delta_{2s}(\|h_{T_0}\|_{l_2} + \|h_{T_1}\|_{l_2})\|h_{T_j}\|_{l_2}$$

Applying the result from derivation in Q11 we get,

$$= |\langle \phi h_{T_0\cup T_1}, \sum_{j\geq 2} \phi h_{T_j}\rangle| \leq \sum_{j\geq 2} \delta_{2s}\sqrt{2}\|h_{T_0\cup T_1}\|_{l_2}\|h_{T_j}\|_{l_2}$$

Combining the terms we get,

$$\|\phi h_{T_0\cup T_1}\|_{l_2} \leq \; 2\epsilon\sqrt{1+\delta_{2s}}\|h_{(T_0\cup T_1)}\|_{l_2} - \sum_{j\geq 2} \delta_{2s}\sqrt{2}\|h_{T_0\cup T_1}\|_{l_2}\|h_{T_j}\|_{l_2}$$

$$= \|\phi h_{T_0\cup T_1}\|_{l_2}^2 \leq \|h_{(T_0\cup T_1)}\|_{l_2}(\; 2\epsilon\sqrt{1+\delta_{2s}} - \sum_{j\geq 2} \delta_{2s}\sqrt{2}\|h_{T_j}\|_{l_2})$$

Here as δ_{2s} and $\sum_{j\geq 2}\|h_{T_j}\|_{l_2}$ are both positive terms so changing from difference of the two terms to sum of the two terms the inequality will still hold and from RIP we get the other inequality also so we get the final expression as,

$$= (1-\delta_{2s})\|h_{T_0\cup T_1}\|_{l_2}^2 \leq \|\phi h_{T_0\cup T_1}\|_{l_2}^2 \leq \|h_{(T_0\cup T_1)}\|_{l_2}(\; 2\epsilon\sqrt{1+\delta_{2s}} + \sqrt{2}\delta_{2s}\sum_{j\geq 2}\|h_{T_j}\|_{l_2})$$

Q13:

From last derivation we get that,

$$(1-\delta_{2s})\|h_{T_0\cup T_1}\|_{l_2}^2 \leq \|h_{(T_0\cup T_1)}\|_{l_2}(\; 2\epsilon\sqrt{1+\delta_{2s}} + \sqrt{2}\delta_{2s}\sum_{j\geq 2}\|h_{T_j}\|_{l_2})$$

Cancelling out $\|h_{T_0\cup T_1}\|_{l_2}$ on both sides we get,

$$(1-\delta_{2s})\|h_{T_0\cup T_1}\|_{l_2} \leq (2\epsilon\sqrt{1+\delta_{2s}} + \sqrt{2}\delta_{2s}\sum_{j\geq 2}\|h_{T_j}\|_{l_2}$$

Substituting the result $\sum_{j\geq 2}\|h_{T_j}\|_{l_2} \leq s^{-1/2}\|h_{T_0^c}\|_{l_1}$ we get,

$$(1-\delta_{2s})\|h_{T_0\cup T_1}\|_{l_2} \leq (2\epsilon\sqrt{1+\delta_{2s}} + \sqrt{2}\delta_{2s}s^{-1/2}\|h_{T_0^c}\|$$

This can be equivalently written as,

$$\|h_{T_0\cup T_1}\|_{l_2} \leq \alpha\epsilon + \rho s^{-1/2}\|h_{T_0^c}\|_{l_1}, \alpha \equiv \frac{2\sqrt{1+\delta_{2s}}}{1-\delta_{2s}}, \rho \equiv \frac{\sqrt{2}\delta_{2s}}{1-\delta_{2s}}$$

Q14:
We consider this $\|h_{T_0^c}\|_{l_1} \leq \|h_{T_0}\|_{l_1} + 2\|x_{T_0^c}\|_{l_1}$ equation from before and substituted in the previously derived equation to get,

$$\|h_{T_0\cup T_1}\|_{l_2} \leq \alpha\epsilon + \rho s^{-1/2}\|h_{T_0}\|_{l_1} + 2\rho s^{-1/2}\|x_{T_0^c}\|_{l_1}$$

From the inequality relationship $s^{-1/2}\|h_{T_0}\|_{l_1} \leq \|h_{T_0}\|_{l_2}$ we get,

$$\|h_{T_0\cup T_1}\|_{l_2} \leq \alpha\epsilon + \rho\|h_{T_0}\|_{l_2} + 2\rho s^{-1/2}\|x_{T_0^c}\|_{l_1}$$

We already know that $\|h_{T_0}\|_{l_2} \leq \|h_{T_0\cup T_1}\|_{l_2}$ and $e_0 \equiv s^{-1/2}\|x_{T_0^c}\|_{l_2}$ substituting these we get,

$$\|h_{T_0\cup T_1}\|_{l_2} \leq \alpha\epsilon + \rho\|h_{T_0\cup T_1}\|_{l_2} + 2\rho e_0$$

Q15:
For the first inequality we know that,

$$\|h\|_{l_2} = \|h_{T_0\cup T_1} + h^c_{T_0\cup T_1}\|_{l_2}$$

Applying triangle inequality we get,

$$\|h\|_{l_2} = \|h_{T_0\cup T_1}\|_{l_2} + \|h^c_{T_0\cup T_1}\|_{l_2}$$

From one of the previously proven inequalities we get,

$$\|h_{(T_0\cup T_1)^c}\| \leq \|h_{T_0}\|_{l_2} + 2e_0$$

As we already know $\|h_{T_0}\|_{l_2} \leq \|h_{T_0\cup T_1}\|_{l_2}$ we get,

$$\|h_{T_0\cup T_1^c}\| \leq 2\|h_{(T_0\cup T_1)^c}\|_{l_2} + 2e_0$$

Rearranging the terms from the previous result we can get $\|h_{T_0\cup T_1}\|_{l_2} \leq (1-\rho)^{-1}(\alpha\epsilon + 2\rho e_0)$ substituting this,

$$2\|h_{(T_0\cup T_1)}\|_{l_2} + 2e_0 \leq 2(1-\rho)^{-1}(\alpha\epsilon + (1+\rho)e_0)$$

Q16:
As mentioned in **Lemma 2.2** we consider the inequality $\|h_{T_0}\|_{l_1} \leq s^{1/2}\|h_{T_0}\|_{l_2} \leq s^{1/2}\|h_{T_0\cup T_1}\|_{l_2}$ and as we consider a noiseless case we can take $\epsilon = 0$ and with the result proved in the previous part we get,

$$s^{1/2}\|h_{(T_0\cup T_1)^c}\|_{l_2} \leq 2(1-\rho)^{-1}(1+\rho)\|x_{T_0^c}\|_{l_1}$$

It is given that $\|h_{T_0^c}\|_{l_1} \leq 2(1-\rho)^{-1}\|x_{T_0^c}\|_{l_1}$. So we combine the two terms,

$$\|h\|_{l_1} = \|h_{T_0}\|_{l_1} + \|h_{h_0^c}\|_{l_1} \leq 2(1-\rho)^{-1}(1+\rho)\|x_{T_0^c}\|_{l_1} + 2(1-\rho)^{-1}\|x_{T_0^c}\|_{l_1}$$

$$\|h\|_{l_1} \leq 2(1-\rho)^{-1}\|x_{T_0^c}\|_{l_1}(1+\rho+1)$$
$$\|h\|_{l_1} \leq 2(1+\rho)(1-\rho)^{-1}\|x_{T_0^c}\|_{l_1}$$

Answer 3:

(a.) We have,

$$y = \phi x + \eta$$

we need to recover the signal x from the given data y and ϕ is the measurement matrix of size m$\times$ n. Which obeys a "restricted isometry property
If oracular solution is defined as the solution, then we have a information about the non zero entries inside the signal x.So,with this additional information we can approach the problem by solving the Least squares problem. which is formulated as:

$$J(x) = ||y - \phi_S\tilde{x}||_2^2$$

$$= (y - \phi_S\tilde{x})^T(y - \phi_S\tilde{x})$$

we can differentiate the above given equation with respect to x and then can equate to 0.We will get

$$-2(y - \phi_s\tilde{x})(\phi_S) = 0$$

$$-2\phi_S^T y + 2\phi_S^T \phi_s \tilde{x} = 0$$
$$\tilde{x} = (\phi_s^T \phi_s)^{-1} \phi_s^T y$$

The pseudo inverse of matrix ϕ_S is

$$piv(\phi_S) = (\phi_S^T \phi_S)^{-1} \phi_S^T$$

As it has already been stated in the problem that the inverse of $\phi_s^T \phi_s$ exists.

(b.) As we have derived in the previous part that if we know in advance the S indices of maximum magnitude non-zero elements in x then using least square method. For those elements which belongs to this S set,value of is

$$\tilde{x} = (\phi_s^T \phi_s)^{-1} \phi_s^T y$$

and for other elements which does not belongs to this set S in $\tilde{x}$ is 0.
So, clearly we can also write as,

$$\tilde{x} = x_0 + (\phi_s^T \phi_s)^{-1} \phi_s^T \eta$$

$$\tilde{x} - x_0 = (\phi_s^T \phi_s)^{-1} \phi_s^T \eta$$

we can take the l2 norm of these equations as shown below:

$$||\tilde{x} - x_0||_2 = ||(\phi_s^T \phi_s)^{-1} \phi_s^T \eta||_2$$

Now,we can use Cauchy Schwartz inequality.Which is:

$$|v.w| \leq ||v||_2 ||w||_2$$

We can write the above equation as:

$$||\tilde{x} - x_0||_2 = ||(\phi_S^T \phi_S)^{-1} \phi_S^T \eta||_2 \leq ||(\phi_S^T \phi_S)^{-1} \phi_S^T||_2 ||\eta||_2$$

and if we use

$$\phi_S^\dagger \widehat{=} (\phi_S^T \phi_S)^{-1} \phi_S^T$$

$$||\tilde{x} - x_0||_2 = ||\phi_S^\dagger \eta||_2 \leq ||\phi_S^\dagger||_2 ||\eta||_2$$

(c.) If ϕ follows the RIP and δ_{2k} is the RIC of ϕ of order 2k then using RIP:

$$(1 - \delta_{2k})||x||_2^2 \leq ||\phi x||_2^2 \leq (1 + \delta_{2k})||x||_2^2$$

Using the RIP for a matrix ϕ_{2k}. we can write the above equation as:

$$(1-\delta_{2k}) \leq \frac{||\phi x||_2^2}{||x||_2^2} \leq (1+\delta_{2k})$$

we have,

$$||\phi x||_2 \leq \sqrt{1+\delta_{2k}}||x||_2$$

then on taking the inverse,

$$||\phi_S^\dagger x||_2 \leq \frac{1}{\sqrt{1-\delta_{2k}}}||x||_2$$

similarly for the other side,

$$||\phi_S^\dagger x||_2 \geq \frac{1}{\sqrt{1+\delta_{2k}}}||x||_2$$

here, $0 \leq \delta_{2k} < 1$ on manipulating the above given equation by using the properties of singular value decomposition, we can say that the singular values of $pinv(\phi_{2k})$ will lie in the range $\frac{1}{\sqrt{1+\delta_{2k}}}$ and $\frac{1}{\sqrt{1-\delta_{2k}}}$

(d.) We have the following error bound:

$$\frac{\epsilon}{\sqrt{1+\delta_{2k}}} \leq ||x-\tilde{x}||_2 \leq \frac{\epsilon}{\sqrt{1-\delta_{2k}}}$$

Now we have to argue that the solution obtained by solving the basis pursuit lies very close to the oracular solution.So,

$$||x-\tilde{x}||_2 \leq \frac{C_0||x-x_k||_1}{\sqrt{s}} + C_1\epsilon$$

where,

$$C_0 = \frac{1+\delta_{2k}(\sqrt{2}-1)}{1-\delta_{2k}(\sqrt{2}+1)} \qquad C_1 = \frac{2\sqrt{1+\delta_{2k}}}{1-\delta_{2k}(1+\sqrt{2})}$$

So, if we compare the above given two equations. We can say that the bound obtained after solving the theorem 3 is just worse by a constant factor or we can say that if we know the true support of x then also the oracular solution cannot improve more than a constant factor.

Answer 4:

We know through Restricted Isometry Property,

$$(1-\delta)||x||_2^2 \leq ||\phi x||_2^2 \leq (1+\delta)||x||_2^2$$

from the above given equation we can write it as

$$||\phi_S x||_2^2 \leq ||x||_2^2 + \delta||x||_2^2$$

where $\phi \in R^{m\times N}$
for all S $\subseteq [N], |S| \leq$ s for all x $\in C^{|S|}$

$$||\phi_S x||_2^2 - ||x||_2^2 \leq \delta||x||_2^2$$

Now, we can see that,

$$||\phi_S x||_2^2 - ||x||_2^2 = ((\phi_S^T\phi_S - I)x, x)$$

where, the term $(\phi_S^T\phi_S - I)$ denotes the hermitian matrix. Now,

$$max_{x\in C^{|S|}\setminus 0}\frac{((\phi_S^T\phi_S - I)x, x)}{||x||_2} = ||\phi_S^T\phi_S - I||_{2->2} \leq \delta$$

So, if we consider two integers s and t denoting the restricted isometry constant of order s and t them from the obtained equation

$$max_{S\subseteq[N],|S|\leq s}||\phi_S^T\phi_S - I||_{2->2} \leq \delta$$

we can say that if s<t then we will have $\delta_s \leq \delta_t$ as those indices of t will increase and that can be seen in $\phi^T\phi$ matrix.

Answer 5:

Title: Low-Cost and High-Throughput Testing of COVID-19 Viruses and Antibodies via Compressed Sensing: System Concepts and Computational Experiments.
Link: https://arxiv.org/pdf/2004.05759.pdf
Objective function The objective function that is being minimized is:
Without noise case,

$$minimize||x||_0$$

subject to

$$y = Ax, x \geq 0$$

where,
$\|x\|_0$ denotes the number of non-zero entries in the vector x.
y is the measurement result, where a entry denotes the quantity of DNA present in that person's sample.
A is the measurement matrix of dimension $R^{m \times n}$ which is designed as

$$A = E \odot W$$

Operator $\odot$ denotes the element wise multiplication.
E is the mixing matrix which contain only 0's and 1's.Mostly 0's as 1's are nearly very small in number.So, this limitation on 1's helps in achieving the sparse E matrix.
$E_{i,j} = 1$ if sample j participates in testing i.
$E_{i,j} = 0$,otherwise

where,

$$1 \leq i \leq m \quad and \quad 1 \leq j \leq n$$

W denotes the allocation matrix.As a person's sample is mixed with other samples also for testing.That's why we need to allocate the portion of that person's sample to each of the involved testings of that person.

Since, l_0 is an NP-hard problem but we know that the optimal solution of l_0 minimization can be obtained if Restricted Isometry Property if followed by matrix A then by solving the l_1 minimization we can obtain x. So, l_1 minimization is used which acts as the closest approximation and hence, the problem statement becomes:

$$minimize||x||_1$$

subject to,

$$y = Ax, x \geq 0$$

here, $||x||_1$ denotes the sum of absolute values of all the elements in x. After solving the required x, a threshold value is used so if the value of an element in x is greater than equal to that threshold then that person is considered as positive and if value is smaller than that threshold, person is said to be tested negative.

Along with noise:

They have also checked their method's efficiency along with noise:

$$min_{x \in R^n} ||x||_1 s.t. ||Ax - y||_2 \leq \epsilon, x \geq 0,$$

where,
$\epsilon > 0$ is a parameter which has been tuned to noise magnitude. Everything is similar to the earlier defined methodology the only difference is that a randomly generated noise v is added to the measurements at every iteration like,

$$y = Ax + v$$

Differences in the proposed approach and Tapestry pooling approach are:

1. **Tapestry pooling approach** uses kirkman triples which are steiner triple system to generate the A matrix whereas the **proposed approach** uses two types of pooling matrix i.e. **Bernoulli random matrix** where every entry of this matrix has a probability of 0.5 whether that entry is 0 or 1 and the other is measurement matrix that is obtained from **expander graph**.

2. In **Tapestry pooling approach** along with P1, LASSO(Least Absolute Shrinkage and Selection Operator) is being used as a cost function but in **proposed approach** no cost function like LASSO is being used.

3. **Tapestry pooling approaoch** is a two step procedure where, COMP(Combinatorial Orthogonal Matching Pursuit) is being used as the first stage for identifying the sure negatives that are present in x and the second stage is basically the compressed sensing step where we estimate the x.whereas in **proposed approach** no such stages are being used.

4. Inside **Tapestry pooling approach** the number of ones in any column of A is 3 according to kirman's triples whereas in the **proposed approach** the result has been demonstrated by keeping the number of ones as k=3 and for k=5(where k denotes the number of ones in any column of A).

Answer 6:

Given:
Problem P1: $\min_x ||x||_1$ s.t. $||y - \phi x||_2 \leq \epsilon$

LASSO problems cost function:

$$J(x) = ||y - \phi x||_2^2 + \lambda ||x||_1$$

If x is a minimizer of this cost function J for some $\lambda \geq 0$.
To prove: Show that their exist some value of ϵ for which x minimizes the problem P1.
Proof:
Let's consider

$$\epsilon' = ||y - \phi x||_2$$

.Let's take some q such that from problem P1 we can say that:

$$||y - \phi q||_2 \leq \epsilon$$

Now, we can use the fact that x is a minimizer of cost function J.Then, we have

$$||y - \phi x||_2^2 + \lambda ||x||_1 \leq ||y - \phi q||_2^2 + \lambda ||q||_1 \leq ||y - \phi x||_2^2 + \lambda ||q||_1$$

On solving the above inequality we obtain:

$$||x||_1 \leq ||q||_1$$

hence,we started with a fact that x is a minimizer of cost function J but on solving the inequality we obtain that x is the minimizer of problem P1.

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