Assignment 3: CS 663, Fundamentals of Digital Image Processing

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Answer 3 HW3:

we know that $\mathcal{F}(f(x,y)) = F(u,v)$,

$$F(u,v) = \sum_{x=0}^{M-1}\sum_{y=0}^{N-1} f(x,y)e^{-j2\pi(xu/M+yv/N)}$$

To prove that,

$$\mathcal{F}(f(x,y) * h(x,y)) = F(u,v)H(u,v) \tag{1}$$

i.e. ,

$$f(x,y) * h(x,y) \Leftrightarrow F(u,v)H(u,v)$$

and,

$$f(x,y)h(x,y) \Leftrightarrow F(u,v) * H(u,v)$$

Proof.

$$\begin{aligned}
\mathcal{F}(f(x,y) * h(x,y)) &= \sum_{x=0}^{M-1}\sum_{y=0}^{N-1}\sum_{i=0}^{M-1}\sum_{j=0}^{N-1} f(i,j)h(x-i,y-j)e^{-j2\pi(xu/M+yv/N)} \\
&= \sum_{i=0}^{M-1}\sum_{j=0}^{N-1} f(i,j)\sum_{x=0}^{M-1}\sum_{y=0}^{N-1} h(x-i,y-j)e^{-j2\pi(xu/M+yv/N)} \\
&= \sum_{i=0}^{M-1}\sum_{j=0}^{N-1} f(i,j)\sum_{x=0}^{M-1}\sum_{y=0}^{N-1} h(x-i,y-j)e^{-j2\pi((x-i)u/M+(y-j)v/N)}e^{-j2\pi(iu/M+jv/N)} \\
&= \sum_{i=0}^{M-1}\sum_{j=0}^{N-1} f(i,j)H(u,v)e^{-j2\pi(iu/M+jv/N)} \\
&= (\sum_{i=0}^{M-1}\sum_{j=0}^{N-1} f(i,j)e^{-j2\pi(iu/M+jv/N)})H(u,v) \\
&= F(u,v)H(u,v)
\end{aligned}$$

□

Hence proved $f(x,y) * h(x,y) \Leftrightarrow F(u,v)H(u,v)$,
Similarly,

Proof.

$$
\begin{aligned}
\mathcal{F}^{-1}(F(u,v) * H(u,v)) &= \sum_{u=0}^{M-1}\sum_{v=0}^{N-1}\sum_{i=0}^{M-1}\sum_{j=0}^{N-1} F(i,j)H(u-i,v-j)e^{j2\pi(xu/M+yv/N)} \\
&= \sum_{i=0}^{M-1}\sum_{j=0}^{N-1} F(i,j)\sum_{u=0}^{M-1}\sum_{v=0}^{N-1} H(u-i,v-j)e^{j2\pi(xu/M+yv/N)} \\
&= \sum_{i=0}^{M-1}\sum_{j=0}^{N-1} F(i,j)\sum_{u=0}^{M-1}\sum_{v=0}^{N-1} H(u-i,v-j)e^{j2\pi((u-i)x/M+(v-j)y/N)}e^{j2\pi(ix/M+jy/N)} \\
&= \sum_{i=0}^{M-1}\sum_{j=0}^{N-1} F(i,j)h(x,y)e^{j2\pi(ix/M+jy/N)} \\
&= (\sum_{i=0}^{M-1}\sum_{j=0}^{N-1} F(i,j)e^{j2\pi(ix/M+jy/N)})h(x,y) \\
&= f(x,y)h(x,y)
\end{aligned}
$$

□

Hence proved $F(u,v) * H(u,v) \Leftrightarrow f(x,y)h(x,y)$

Answer 4 HW3:

The answer is No,
Since we know that the Laplacian of the image is given as ,

$$\nabla^2 f(x,y) = \frac{\partial^2 f(x,y)}{\partial x^2} + \frac{\partial^2 f(x,y)}{\partial y^2} \tag{2}$$

And since the derivative is linear operator, therefore we can calculate the laplacian of the image at a pixel directly (Since laplacian is an Linear space invariant),
but, the gradient magnitude is given as ,

$$\|\nabla f\|_2 = \sqrt{f_x^2 + f_y^2} \tag{3}$$

The gradient magnitude is a non-linear operation on the image, as can be seen from the above formula. Canonical filtering operations however are linear operations, and so there cannot be an LSI filter that can get you the gradient directly. So it is not possible to calculate the grad-magnitude from the fourier transform directly.

Fourier transform of the $\nabla^2 f$,$\nabla_x f$ and $\nabla_y f$ can be given as ,

$$\mathcal{F}(\nabla^2 f) = F(u,v)((j2\pi u)^2 + (j2\pi v)^2) \tag{4}$$
$$\mathcal{F}(\nabla_x f) = j2\pi u F(u,v) \tag{5}$$
$$\mathcal{F}(\nabla_y f) = j2\pi v F(u,v) \tag{6}$$
$$\tag{7}$$

therefore, from equation (3)it is clear that as the grad magnitude is an Non-Linear operation, and The square and square roots involved in computing the gradient are nonlinear operations. We can use the FT to compute the derivative but the absolute value of the grad magnitude has to be calculated in spatial domain . Hence Proved .

Answer 5 HW3:

Given : $f(x,y)$ is an real function,
$F(u,v) = \mathcal{F}(f(x,y))$ is the Fourier transform of the given function

we have to prove,: $F^*(u,v) = F(-u,-v)$
we have, $F(u,v) = \sum_{x=0}^{M-1} \sum_{y=0}^{N-1} f(x,y)e^{-j2\pi(xu/M+yv/N)}$
Since $f^*(x,y) = f(x,y)$ ....(f is real)

Proof.

$$\begin{aligned}
F^*(u,v) &= (\sum_{x=0}^{M-1} \sum_{y=0}^{N-1} f(x,y)e^{-j2\pi(xu/M+yv/N)})^* \\
&= \sum_{x=0}^{M-1} \sum_{y=0}^{N-1} f^*(x,y)e^{j2\pi(xu/M+yv/N)} \\
&= \sum_{x=0}^{M-1} \sum_{y=0}^{N-1} f(x,y)e^{-j2\pi(x(-u)/M+y(-v)/N)} \\
&= F(-u,-v)
\end{aligned}$$

□

Now,since given that $f(x,y)$ is real and even,
therefore we have, $f(x,y) = f^*(x,y) = f(-x,-y)$,
To prove that $F(u,v)$ is also real and even(and periodic with (M,N)),
i.e. we need to prove that $F(u,v) = F^*(u,v) = F(-u,-v)$

Proof.

$$F(u,v) = \mathcal{F}(f(x,y)) = \mathcal{F}(f(-x,-y)) = \sum_{x=0}^{M-1}\sum_{y=0}^{N-1} f(-x,-y)e^{-j2\pi(xu/M+yv/N)}$$

replace x with -x and y with -y

$$= \sum_{x=0}^{-M+1}\sum_{y=0}^{N+1} f(x,y)e^{-j2\pi((-x)u/M+(-y)v/N)}$$

$$= \sum_{x=M}^{1}\sum_{y=N}^{1} f(x,y)e^{-j2\pi((-x)u/M+(-y)v/N)}$$

$$= \sum_{x=M}^{1}\sum_{y=N}^{1} f(x,y)e^{-j2\pi((-u)x/M+(-v)y/N)}$$

$$= \sum_{x=0}^{-M+1}\sum_{y=0}^{-N+11} f(x,y)e^{-j2\pi((-u)x/M+(-v)y/N)}$$

$$= F(-u,-v) = F^*(u,v)$$

□

From first proof for real f and above proof we have $F(u,v) = F^*(u,v) = F(-u,-v) = \mathcal{F}(f(-x,-y))$
Hence proved

Answer 6HW3:

Given $f(t)$
we have

$$\mathcal{F}(f(t)) = \int_t f(t)e^{-j2\pi\mu t}dt$$

and we know the duality property of the function:,

$$f(t) \Leftrightarrow F(\mu)$$

$$F(t) \Leftrightarrow f(-\mu)$$

Proving above property, by taking the Fourier transform of the Fourier transform with respect to variable μ

Proof.

$$\begin{aligned}
\mathcal{F}(\mathcal{F}(f(t))) &= \mathcal{F}(F(\mu)) \\
&= \int_\mu \int_t f(t)e^{-j2\pi\mu t}dt e^{-j2\pi t\mu}dmu \\
&= \int_\mu F(\mu)e^{-j2\pi t\mu}dmu \\
&= \int_\mu F(\mu)e^{j2\pi(-t)\mu}dmu \\
&= f(-t)
\end{aligned}$$

□

Let $f(-t) = f(x)$

and from above proof $\mathcal{F}(\mathcal{F}(f(t))) = f(-t) = f(x)$ therefore the given expression implies as,

$$\mathcal{F}(\mathcal{F}(\mathcal{F}(\mathcal{F}(f(t))))) = \mathcal{F}(\mathcal{F}(f(x))) \tag{8}$$

Since as proved above, the a equation, $\mathcal{F}(\mathcal{F}(f(x)))$ will be equal to $f(-x)$
Since $f(-t) = f(x)$, therefore $f(-x) = f(t)$
therefore,

$$\mathcal{F}(\mathcal{F}(\mathcal{F}(\mathcal{F}(f(t))))) = \mathcal{F}(\mathcal{F}(f(x))) = f(-x) = f(t) \tag{9}$$

Hence proved

Answer 7HW3:

We have High Pass filter as H_{HP} as the function of All pass filter and Low Pass filter H_{LP}, i.e.

$$H_{HP}(u,v) = 1 - H_{LP}(u,v)$$

i.e. High Pass Filter can be expressed a 1 minus the transfer function of low Pass Filter (which we know do not have an impulse at the origin). And, we know that.

$$\begin{aligned}
H_{Ideal}(u,v) &= 1 - \begin{cases} 1 & \text{if } D(u,v) \leq D_0 \\ 0 & \text{if } D(u,v) > D_0 \end{cases} \\
H_{GaussHP}(u,v) &= 1 - e^{-D^2(u,v)/D_0^2} \\
H_{ButterHP}(u,v) &= 1 - \frac{1}{1 + [D_0/D(u,v)]^{2=}}
\end{aligned}$$

After taking inverse Fourier transform of the above functions we respectively get,

$$\begin{aligned}
h_{HP}(t) &= \delta(t) - 2\mu_0 sinc(2\mu_0 t) \\
h_{GauassHP}(t) &= \delta(t) - \sqrt{\frac{a}{\pi}} e^{-at^2} \\
h_{ButterHP}(t) &= \delta(t) - h_{ButterLP}(t)
\end{aligned}$$

From above it is clear that in spatial domain due to the presence of the delta Dirac function along with the negative of the lpw pass spatial function, the spike is available(due to presence of the dirac delta function in spatial domain) . Hence proved .