Control Engineering Module-3

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1 Important Question CE Module-3

1. Describe transient vs. steady-state response in control systems and show both on a graph of output versus time.

Transient vs. Steady-State Response in Control Systems

Definitions

Transient response. The portion of the output that occurs immediately after an input or parameter change (e.g., a step command) and before the system settles. It describes how the system moves from its initial condition to the final value. The metrics include:

- **Rise time** (T_r): time to go from a lower to an upper percentage of the final value (e.g., 10% to 90%).
- **Peak time** (T_p): time to reach the first maximum (peak) of the response.
- **Percent overshoot** (M_p): the amount the response exceeds the final value, as a percentage of the final value.
- **Settling time** (T_s): time after which the response remains within a tolerance band around the final value (e.g., $\pm 2\%$).

Steady-state response. The portion of the output as time becomes large, after transients have decayed. It reflects long-term behavior and accuracy. A primary metric is the *steady-state error*

$$e_{ss} = \lim_{t \to \infty} \big(r(t) - y(t)\big),$$

when the limit exists, where $r(t)$ is the reference and $y(t)$ is the output.

Typical Step-Response Behavior (Qualitative)

- For a stable system, $y(t)$ moves from its initial value toward the final value after a step input.
- Underdamped systems may exhibit overshoot and oscillation before settling.
- Overdamped or first-order systems approach the final value monotonically (no oscillation).
- The steady-state region begins once $y(t)$ remains within a chosen tolerance band (e.g., $\pm 2\%$) of the final value.

How to Label on a Graph (Output vs. Time)

- X-axis: time t; Y-axis: output $y(t)$.
- Draw the reference step as a horizontal line at the final value.
- Highlight the initial portion of the response as the *Transient response* up to the settling time T_s.
- Mark the region beyond T_s as the *Steady-state response.*
- Annotate T_r, T_p (if applicable), M_p, and T_s, and draw the $\pm 2\%$ steady-state band around the final value.

graph

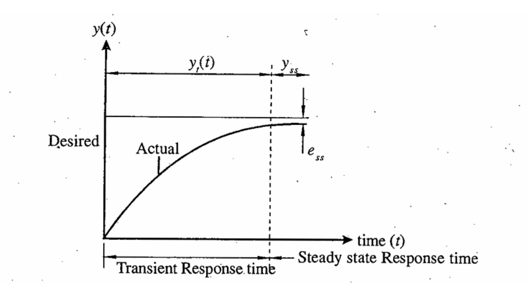

Figure 1: Response of the system and steady state error.

2. List and define the standard test signals (step, impulse, ramp, parabolic, sinusoidal) employed in time/frequency response analysis. State the purpose of each.

Standard Test Signals in Time/Frequency Response Analysis

Below are the commonly used test signals, each with a clear definition (including both dimensional and unit versions where applicable), key transforms, and their primary purposes in analysis and design.

1) Impulse and Unit Impulse (Dirac Delta)

Words: An idealized instantaneous "kick" at $t = 0$ with finite area. The unit impulse has area 1.
Time-domain:

$$Impulse of area A: \; A\,\delta(t), \qquad Unit impulse: \delta(t), \quad \int_{-\infty}^{\infty} \delta(t)\,dt = 1.$$

Transforms:

$$\mathcal{L}\{\delta(t)\} = 1, \qquad \mathcal{F}\{\delta(t)\} = 1.$$

Purpose:

- Reveals the impulse response $h(t)$, the intrinsic kernel of an LTI system.
- Basis for convolution: $y(t) = h(t) * x(t)$.
- Direct link to the transfer function and frequency response.

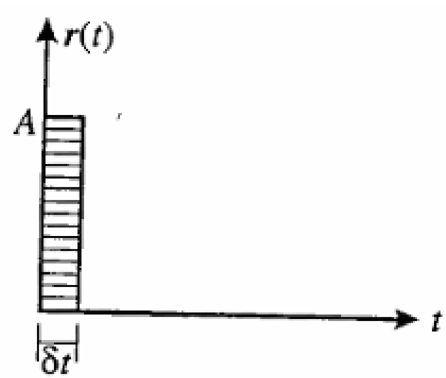

Figure 2: Impulse Input.

2) Step and Unit Step

Words: A sudden change in input from 0 to a constant level. The unit step has amplitude 1.
Time-domain:

$$Step of amplitude A: \; A\,u(t), \qquad Unit step: u(t) = \{\, 0\ , t < 0, 1, t \geq 0.$$

Transform:

$$\mathcal{L}\{u(t)\} = \frac{1}{s}.$$

Purpose:

- Canonical test for time-domain performance: rise time, peak time, overshoot, settling time.
- Assesses steady-state error to constant commands (position error constant).
- Most widely used benchmark in control.

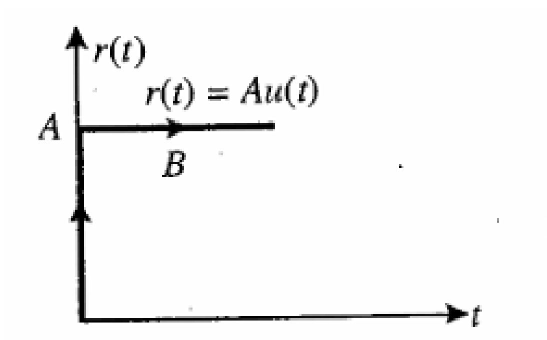

Figure 3: Step Input.

3) Ramp and Unit Ramp

Words: A linearly increasing input starting at $t = 0$ with slope A. The unit ramp has slope 1.
Time-domain:

$$Ramp of slope A: \; A\,t\,u(t), \qquad Unit ramp: r(t) = t\,u(t).$$

Transform:

$$\mathcal{L}\{t\,u(t)\} = \frac{1}{s^2}.$$

Purpose:

- Assesses tracking of steadily changing references.
- Determines velocity error constant; indicates system “type”.
- Highlights the role of integral action for ramp tracking.

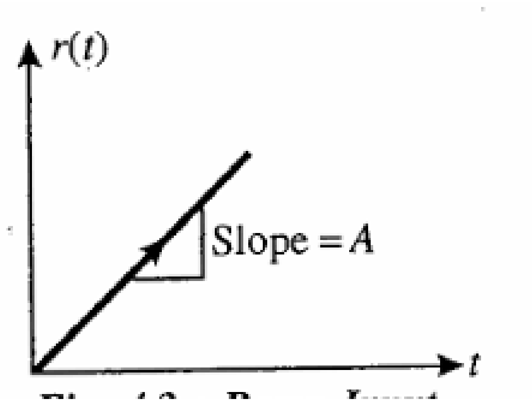

Figure 4: Ramp Input.

4) Parabolic (Quadratic) and Unit Parabolic

Words: A smoothly accelerating input growing like $(A/2)\,t^2$ from $t = 0$. The unit parabolic has $A = 1$.
Time-domain:

$$Parabolic of scale A: \frac{A}{2}\,t^2\,u(t), \qquad Unit parabolic: p(t) = \frac{1}{2}\,t^2\,u(t).$$

Transform:

$$\mathcal{L}\{(1/2)\,t^2\,u(t)\} = \frac{1}{s^3}.$$

Purpose:

- Tests response to acceleration-like commands.
- Determines acceleration error constant.
- Probes need for higher system type (more integrators).

5) Sinusoidal (Sine/Cosine)

Words: A pure tone at angular frequency ω; a stable LTI system outputs the same frequency with modified amplitude and phase.
Time-domain (examples):

$$x(t) = A\sin(\omega t)\,u(t) \quad or \quad A\cos(\omega t)\,u(t).$$

Frequency response viewpoint:

$$Evaluate G(j\omega);\ steady-state output amplitude = |G(j\omega)|\,A,\ phase shift = \angle G(j\omega).$$

Purpose:

- Characterizes magnitude and phase across frequency (Bode, Nyquist, Nichols).
- Identifies resonance, bandwidth, and stability margins (gain/phase).
- Central for robustness and disturbance rejection design.

Notes

- "Unit" versions fix amplitude, slope, or area to 1 for normalization and comparison.
- Polynomial inputs (step, ramp, parabolic) map to $1/s$, $1/s^2$, $1/s^3$ in the Laplace domain; they diagnose steady-state tracking via error constants and system type.
- The impulse response fully characterizes an LTI system; sinusoidal tests fully characterize frequency response.

Summary Table

Signal (general, unit)	**Words**	**Time-domain expression**	**Laplace/Frequency representation**	**Primary pu**
Impulse, Unit impulse	Instantaneous kick (area A or 1)	$A\,\delta(t)$, $\delta(t)$	$\mathcal{L}\{\delta(t)\} = 1$; $\mathcal{F}\{\delta(t)\} = 1$	Obtain $h(t)$; c lution kernel to transfer fu
Step, Unit step	Sudden change to constant level	$A\,u(t)$, $u(t)$	$\mathcal{L}\{u(t)\} = 1/s$	Time m steady-state to constants tion constant
Ramp, Unit ramp	Linearly increasing input	$A\,t\,u(t)$, $t\,u(t)$	$\mathcal{L}\{t\,u(t)\} = 1/s^2$	Tracking of t velocity erro stant; system
Parabolic, Unit parabolic	Quadratically increasing (accelerating)	$A2\,t^2\,u(t)$, $12\,t^2\,u(t)$	$\mathcal{L}\{12\,t^2\,u(t)\} = 1/s^3$	Acceleration ing; accelerat ror constant; for higher typ
Sinusoidal	Pure tone at ω	$A\sin(\omega t)\,u(t)$ or $A\cos(\omega t)\,u(t)$	Evaluate $G(j\omega)$: $\lvert G(j\omega)\rvert$, $\angle G(j\omega)$	Frequency sponse; nance, band gain/phase m

3. Describe how to identify the order and type of a system by matching its transfer function to the standard form. Solve one example to demonstrate the steps.

Identifying System Order and Type from Its Transfer Function

Key Ideas

- **Order:** The order of a linear time-invariant (LTI) system is the highest power of s in the denominator of its transfer function when written as a proper rational function after any pole-zero cancellations.
- **Type:** The type is the number of pure integrators (i.e., poles at the origin $s = 0$) in the open-loop transfer function used for steady-state error analysis. In unity-feedback settings, the system type is determined from $G(s)H(s)$ (with $H(s) \equiv 1$ typically) by counting the multiplicity of the pole at $s = 0$.

Standard Forms to Match

- **General nth-order denominator (expanded):**

$$D(s) = s^n + a_{n-1}s^{n-1} + \cdots + a_1 s + a_0, \quad \mathit{Order} = n.$$

- **General factored form:**

$$D(s) = \prod_{i=1}^{n}(s - p_i), \quad \mathit{Order} = n.$$

- **Canonical second-order closed-loop model (Type 0):**

$$T(s) = \frac{\omega_n^2}{s^2 + 2\zeta\omega_n s + \omega_n^2}, \quad \mathit{Order} = 2,\ \mathit{Type} = 0.$$

- **Type determination (open-loop):** If $G(s)H(s) = Ks^{-v}\,G_p(s)$ where $G_p(s)$ has no pole at the origin, then $\mathit{Type} = v$ ($v = 0, 1, 2, \ldots$).

Procedure

1. Express the transfer function as a proper rational function; factor numerator/denominator if needed and cancel any common factors.
2. **Order:** Identify the highest power of s in the denominator (after cancellations).
3. **Type:** Inspect the open-loop transfer function $G(s)H(s)$ and count the multiplicity v of s in the denominator (i.e., poles at the origin). Then $\mathit{Type} = v$.
4. (Optional) If the transfer function resembles a standard second-order form, match coefficients to identify ω_n and ζ.

Example 1: Open-Loop Given, Unity Feedback

Given the open-loop plant

$$G(s) = \frac{K}{s(1+Ts)} = \frac{K}{Ts^2+s}.$$

Step A: Order (open-loop). The denominator is Ts^2+s; the highest power is $s^2 \Rightarrow$ open-loop order $= 2$.

Step B: Type. There is one explicit factor of s in the denominator (one pole at the origin) and the other factor is $(1+Ts)$, so the multiplicity at the origin is $1 \Rightarrow$ **Type** $= 1$.

Step C: Closed-loop transfer function (unity feedback).

$$T(s) \;=\; \frac{G(s)}{1+G(s)} \;=\; \frac{K}{Ts^2+s+K}.$$

The closed-loop denominator is quadratic $\Rightarrow$ closed-loop order $= 2$. The closed-loop denominator has no pole at the origin; however, *type* is defined from the *open-loop* transfer function, so Type remains 1.

Step D: Matching to the standard second-order form.

$$T(s) \;=\; \frac{K}{Ts^2+s+K} \;=\; \frac{K/T}{s^2+\frac{1}{T}s+\frac{K}{T}} \;\overset{!}{=}\; \frac{\omega_n^2}{s^2+2\zeta\omega_n s+\omega_n^2}.$$

By coefficient matching,

$$\omega_n^2 = \frac{K}{T}, \qquad 2\zeta\omega_n = \frac{1}{T} \;\;\Rightarrow\;\; \omega_n = \sqrt{\frac{K}{T}}, \qquad \zeta = \frac{1}{2\sqrt{KT}}.$$

Summary (Example 1).

$$Open-looporder = 2, \quad Type = 1, \quad \omega_n = \sqrt{\frac{K}{T}}, \quad \zeta = \frac{1}{2\sqrt{KT}}.$$

Example 2: Closed-Loop Given; Infer Order and Type via Open-Loop Form

Suppose the closed-loop (unity feedback) transfer function is

$$T(s) = \frac{\omega_n^2}{s^2+2\zeta\omega_n s+\omega_n^2}.$$

Step A: Order. The denominator is quadratic $\Rightarrow$ closed-loop order $= 2$.

Step B: Type. Type is defined from the open-loop $G(s)H(s)$. For unity feedback, $T(s) = \frac{G(s)}{1+G(s)} \Rightarrow G(s) = \frac{T(s)}{1-T(s)}$. The standard second-order closed-loop model above corresponds to an open-loop with *no* pole at the origin (finite DC gain), hence **Type** $= 0$.

Summary (Example 2).

$$Closed - looporder = 2, \qquad Type = 0.$$

4. Define steady-state error and list the standard system error metrics (e.g., position, velocity, and acceleration errors). Explain how each is computed.

Steady-State Error and Standard System Error Metrics

Definition: Steady-State Error

Let $r(t)$ be the reference and $y(t)$ the output of a feedback system. The tracking error is $e(t) = r(t) - y(t)$. The *steady-state error* (SSE) is

$$e_{ss} \;=\; \lim_{t\to\infty} e(t),$$

when the limit exists. For a stable closed-loop LTI system (with poles of $sE(s)$ in the open left half-plane and no cancelations that violate the final value theorem),

$$e_{ss} \;=\; \lim_{t\to\infty} e(t) \;=\; \lim_{s\to 0} sE(s).$$

For unity feedback ($H(s) = 1$) with open-loop $L(s) = G(s)$ and closed-loop transfer $T(s) = L(s)1 + L(s)$, the error transfer is

$$E(s) \;=\; \frac{R(s)}{1+L(s)} \quad\Longrightarrow\quad e_{ss} \;=\; \lim_{s\to 0} \frac{s\,R(s)}{1+L(s)}.$$

Error Constants and System Type

Define the *system type* v as the number of poles at the origin of $L(s) = G(s)H(s)$ (i.e., the number of pure integrators). The standard error constants are

$$K_p \;=\; \lim_{s\to 0} L(s), \qquad K_v \;=\; \lim_{s\to 0} s\,L(s), \qquad K_a \;=\; \lim_{s\to 0} s^2\,L(s).$$

These constants determine e_{ss} for the standard polynomial inputs.

Steady-State Error for Standard Inputs (Unity Feedback)

- **Unit step** $r(t) = u(t)$, $R(s) = 1s$:

$$e_{ss} \;=\; \lim_{s\to 0} \frac{1}{1+L(s)} \;=\; \frac{1}{1+K_p}.$$

  Type-0: finite nonzero; Type-1 or higher: $e_{ss} = 0$.

- **Unit ramp** $r(t) = t\,u(t)$, $R(s) = 1s^2$:

$$e_{ss} \;=\; \lim_{s\to 0} \frac{1}{s\,[1+L(s)]} \;=\; \frac{1}{K_v}.$$

  Type-0: $e_{ss} = \infty$; Type-1: finite $1/K_v$; Type-2 or higher: $e_{ss} = 0$.

- **Unit parabolic** $r(t) = 12t^2u(t)$, $R(s) = 1s^3$:

$$e_{ss} \;=\; \lim_{s\to 0} \frac{1}{s^2\,[1+L(s)]} \;=\; \frac{1}{K_a}.$$

  Type-0 or 1: $e_{ss} = \infty$; Type-2: finite $1/K_a$; Type-3 or higher: $e_{ss} = 0$.

Computing K_p, K_v, K_a in Practice

Let $L(s) = G(s)H(s)$ and factor out the integrators:

$$L(s) \;=\; \frac{K\,L_p(s)}{s^v}, \qquad L_p(0) \neq 0,$$

then

$$K_p \;=\; \{\,K\;\; L_p(0), v = 0, \infty, v \geq 1, \qquad K_v \;=\; \{\,0\;, v = 0, K\,L_p(0), v = 1, \infty, v \geq 2, \qquad K_a \;=\; \{\,0\;, v \leq 1, K\,L_p$$

Example

Open-loop $L(s) = Ks(1+Ts)$ (unity feedback).

- Type: one pole at the origin $\Rightarrow v = 1$ (Type-1).
- Constants:

$$K_p = \infty, \qquad K_v = \lim_{s\to 0} s\,\frac{K}{s(1+Ts)} = \frac{K}{1+0} = K, \qquad K_a = \lim_{s\to 0} s^2\,\frac{K}{s(1+Ts)} = 0.$$

- SSE:

$$Step: e_{ss} = 0, \qquad Ramp: e_{ss} = \frac{1}{K}, \qquad Parabolic: e_{ss} = \infty.$$

Summary Table

Input	**Transform $R(s)$**	**Error Constant**	**Steady-State Error**
Unit Step $u(t)$	$1/s$	$K_p = \lim_{s\to 0} L(s)$	$e_{ss} = \frac{1}{1+K_p}$
Unit Ramp $t\,u(t)$	$1/s^2$	$K_v = \lim_{s\to 0} s\,L(s)$	$e_{ss} = \frac{1}{K_v}$
Unit Parabolic $12t^2u(t)$	$1/s^3$	$K_a = \lim_{s\to 0} s^2\,L(s)$	$e_{ss} = \frac{1}{K_a}$

Type interpretation (unity feedback):

- Type-0: step $\Rightarrow$ finite e_{ss}; ramp $\Rightarrow \infty$; parabolic $\Rightarrow \infty$.
- Type-1: step $\Rightarrow 0$; ramp $\Rightarrow$ finite $1/K_v$; parabolic $\Rightarrow \infty$.
- Type-2: step $\Rightarrow 0$; ramp $\Rightarrow 0$; parabolic $\Rightarrow$ finite $1/K_a$.

5. Explain the characteristics of a first-order system and derive its transfer function in the Laplace domain.

Characteristics of a First-Order System and Its Laplace-Domain Transfer Function

What is a First-Order System?

- A first-order (linear time-invariant) system is governed by a first-order differential equation relating its output and input.
- It has a single energy storage element (e.g., capacitor in an RC circuit, thermal capacitance in a thermal system, liquid level in a tank).
- Its unit-step response is a monotonic exponential approaching steady state (no oscillation).
- The key parameter is the *time constant* $\tau > 0$, which sets the speed of response.

Standard Time-Domain Model

For input $x(t)$ and output $y(t)$, a generic first-order system with static gain K and time constant τ is:

$$\tau \frac{dy(t)}{dt} + y(t) = K\,x(t).$$

- **Static (DC) gain:** For a constant input $x(t) = X$, as $t \to \infty$, $y(t) \to KX$.
- **Time constant τ:** Time to reach approximately 63.2% of the final value in response to a unit step.
- **Pole:** A single real pole at $s = -1/\tau$.

Derivation of the Transfer Function

Taking the Laplace transform (with zero initial conditions),

$$\tau\,[sY(s)] + Y(s) = K\,X(s).$$

Factor $Y(s)$:

$$Y(s)\,(\tau s + 1) = K\,X(s).$$

Therefore, the transfer function $G(s) = Y(s)X(s)$ is

$$G(s) = \frac{K}{\tau s + 1}.$$

This is the canonical first-order transfer function.

Key Characteristics from $G(s) = K\tau s + 1$

- **Pole:** $s = -1/\tau$ (real, negative for $\tau > 0$).
- **Stability:** Stable for $\tau > 0$ (pole in the left-half plane).
- **DC gain:** $G(0) = K$.
- **High-frequency behavior:** As $s \to \infty$, $G(s) \to 0$ (low-pass).
- **Unit-step response:**

$$y(t) = K\big(1 - e^{-t/\tau}\big)\,u(t).$$

- **Rise/settling time (rules of thumb):**

$$Rise time (10--90\%) \approx 2.2\,\tau, \qquad 2\% settling time \approx 4\,\tau.$$

- **Bandwidth (-3 dB cutoff):** $\omega_c = 1/\tau$ (for $|K|$ normalized to 1; more generally, the magnitude is $|K|/\sqrt{2}$ at $\omega = 1/\tau$).

Physical Examples

- **RC low-pass filter:** $K = 1$, $\tau = RC \Rightarrow G(s) = 1RC\,s + 1$.
- **Thermal system:** $K = 1/R_{th}$, $\tau = C_{th}R_{th}$.
- **Fluid level (linearized tank):** Gain set by inflow-to-level conversion; τ set by cross-sectional area and outflow conductance.

First-order LTI systems exhibit single-exponential dynamics characterized by a time constant τ and DC gain K. The Laplace-domain transfer function is

$$G(s) = \frac{K}{\tau s + 1},$$

which compactly captures stability, speed, and steady-state behavior.

6. Define a first-order system. Derive the transient response of a first-order system to a step input of amplitude A, and sketch the output showing key time constants on the response-time graph

First-Order System: Definition, Step Response, and Key Time-Constant Markers

Definition

A first-order linear time-invariant (LTI) system is governed by a first-order differential equation. For input $x(t)$ and output $y(t)$, the standard form is

$$\tau \frac{dy(t)}{dt} + y(t) = K\,x(t),$$

where $\tau > 0$ is the time constant and K is the static (DC) gain. The Laplace-domain transfer function is

$$G(s) \;=\; \frac{Y(s)}{X(s)} \;=\; \frac{K}{\tau s + 1}.$$

Step Response Derivation (Input Amplitude A)

Consider a step input of amplitude A applied at $t = 0$:

$$x(t) = A\,u(t), \qquad X(s) = \frac{A}{s}.$$

The output in the Laplace domain is

$$Y(s) \;=\; G(s)X(s) \;=\; \frac{K}{\tau s + 1} \cdot \frac{A}{s} \;=\; \frac{KA}{s(\tau s + 1)}.$$

Partial fraction decomposition:

$$\frac{KA}{s(\tau s + 1)} \;=\; KA\left(\frac{1}{s} - \frac{\tau}{\tau s + 1}\right).$$

Inverse Laplace transform gives the time response (for $t \geq 0$):

$$y(t) \;=\; KA\left[1 - e^{-t/\tau}\right] u(t).$$

Key properties:

$$y(0^+) = 0, \qquad \lim_{t \to \infty} y(t) = KA,$$

i.e., a monotone, non-oscillatory exponential rise to the final value KA (for $K, A > 0$).

Time-Constant Landmarks and Common Timing Metrics

y$(\tau) = KA\left(1 - e^{-1}\right) \approx 0.632\,KA$,
$y(2\tau) \approx 0.865\,KA$,
$y(3\tau) \approx 0.950\,KA$,
$y(4\tau) \approx 0.982\,KA \quad (within about 2\% of final value)$. Rules of thumb:

$$Rise time (10\% \to 90\%) \;\approx\; 2.2\,\tau, \qquad Settling time (2\%) \;\approx\; 4\,\tau.$$

Sketching Guidance (Response–Time Plot)

- Horizontal axis: time t; vertical axis: output $y(t)$.
- Plot the exponential $y(t) = KA\left(1 - e^{-t/\tau}\right)$ starting at 0 and asymptotically approaching KA.
- Draw a horizontal asymptote at $y = KA$ (steady-state).
- Mark $t = \tau, 2\tau, 3\tau, 4\tau$ with corresponding values $0.632\,KA$, $0.865\,KA$, $0.950\,KA$, $0.982\,KA$.
- Optionally annotate $T_r \approx 2.2\tau$ and $T_s \approx 4\tau$.

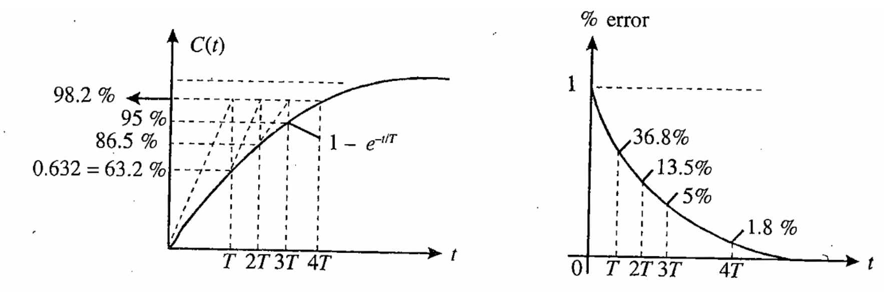

Figure 5: Step Response of First Orser System.

7. What is a first-order system? Obtain the time response of a first-order system to a unit impulse input and plot the response, clearly indicating the time constant and initial slope.

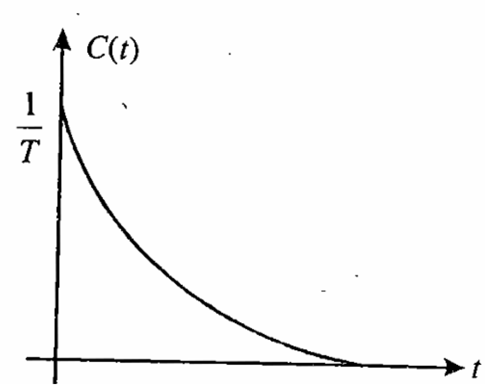

Figure 6: Impulse Response of First Order System.

First-Order System and Its Impulse Response

Definition

A first-order linear time-invariant (LTI) system is governed by a first-order differential equation. With input $x(t)$ and output $y(t)$, the standard form is

$$\tau \frac{dy(t)}{dt} + y(t) = K\,x(t),$$

where $\tau > 0$ is the time constant (sets the speed of response) and K is the static (DC) gain. The corresponding transfer function is

$$G(s) \;=\; \frac{Y(s)}{X(s)} \;=\; \frac{K}{\tau s + 1}.$$

Impulse Response to a Unit Impulse Input

For a unit impulse input $x(t) = \delta(t)$ (so $X(s) = 1$), the output in the Laplace domain is

$$Y(s) \;=\; G(s)X(s) \;=\; \frac{K}{\tau s + 1}.$$

Taking the inverse Laplace transform yields the impulse response

$$y(t) \;=\; \frac{K}{\tau}\, e^{-t/\tau}\, u(t), \qquad t \geq 0.$$

Key Properties

- **Initial value:** $y(0^+) = K\tau$.
- **Initial slope:** $\dfrac{dy}{dt}(t) = -\dfrac{K}{\tau^2}\, e^{-t/\tau} \;\Rightarrow\; \dfrac{dy}{dt}\Big|_{t=0^+} = -K\tau^2.$
- **Peak and monotonicity:** The response attains its maximum at $t = 0$ and decays monotonically thereafter.
- **Area under the impulse response:** $\displaystyle\int_0^\infty y(t)\,dt = \int_0^\infty \frac{K}{\tau} e^{-t/\tau}\,dt =$ K, consistent with $G(0) = K$.

Sketching Guidance (Response–Time Plot)

- Plot $y(t) = K\tau e^{-t/\tau}$ for $t \geq 0$: a decaying exponential starting at K/τ and asymptotically approaching 0.
- Mark the time constant τ on the time axis; at $t = \tau$, the response equals $y(\tau) = K\tau e^{-1}$.
- Indicate the initial value $y(0^+) = K\tau$ and draw the tangent at $t = 0$ with slope $-K\tau^2$.

8. Define a first-order system. Derive the transient response of a first-order system to a unit ramp input and provide a labeled graph showing the tracking error and the effect of the time constant.

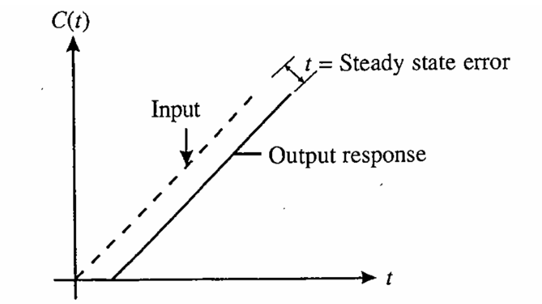

Figure 7: Ramp Response of First Order System.

First-Order System: Unit Ramp Response and Tracking Error

Definition

A first-order linear time-invariant (LTI) system with input $x(t)$ and output $y(t)$ is modeled by

$$\tau \frac{dy(t)}{dt} + y(t) = K\,x(t),$$

where $\tau > 0$ is the time constant (sets response speed) and K is the static (DC) gain. The transfer function is

$$G(s) \;=\; \frac{Y(s)}{X(s)} \;=\; \frac{K}{\tau s + 1}.$$

Unit Ramp Input and Output Response

For a unit ramp input $x(t) = t\,u(t)$ with Laplace transform $X(s) = 1/s^2$,

$$Y(s) \;=\; G(s)X(s) \;=\; \frac{K}{\tau s + 1} \cdot \frac{1}{s^2} \;=\; \frac{K}{s^2(\tau s + 1)}.$$

Perform a partial-fraction decomposition:

$$\frac{K}{s^2(\tau s + 1)} \;=\; \frac{A}{s} + \frac{B}{s^2} + \frac{C}{\tau s + 1}.$$

Multiplying through by $s^2(\tau s + 1)$ and matching coefficients yields

$$A = -K\tau, \qquad B = K, \qquad C = K\tau^2.$$

Hence

$$Y(s) \;=\; -\frac{K\tau}{s} \;+\; \frac{K}{s^2} \;+\; \frac{K\tau^2}{\tau s+1}.$$

Taking inverse Laplace transforms (for $t \geq 0$) gives

$$y(t) \;=\; -K\tau\cdot 1 \;+\; Kt \;+\; K\tau\, e^{-t/\tau} \;=\; K\Big[t-\tau+\tau e^{-t/\tau}\Big].$$

For unity gain $K=1$:

$$y(t) \;=\; t-\tau+\tau e^{-t/\tau}.$$

Tracking Error to a Ramp

With reference $r(t)=t$ (unit ramp), the tracking error is $e(t)=r(t)-y(t)$. For general K,

$$e(t) \;=\; t-K\Big[t-\tau+\tau e^{-t/\tau}\Big] \;=\; (1-K)\,t \;+\; K\tau\Big(1-e^{-t/\tau}\Big).$$

- **Unity DC gain ($K=1$):**

  $$e(t) \;=\; \tau\Big(1-e^{-t/\tau}\Big), \qquad e(0^+)=0, \quad \lim_{t\to\infty} e(t)=\tau.$$

  This is the classic Type-1 result: finite steady-state ramp error equal to τ.

- **Non-unity DC gain ($K \neq 1$):** The term $(1-K)\,t$ produces an unbounded error as $t\to\infty$ (slope mismatch).

Interpretation (for $K=1$)

- Output: $\quad y(t)=t-\tau+\tau e^{-t/\tau}$.
- Error: $\quad e(t)=\tau(1-e^{-t/\tau})$, increasing monotonically from 0 to τ.
- Larger τ implies a slower response and larger steady-state ramp error; smaller τ improves both.

Labeled Graph: Sketching Guidance

To produce a clear response–time plot (for $K=1$):

- Plot the reference $r(t)=t$ (straight line with slope 1 from the origin).
- Plot the output $y(t)=t-\tau+\tau e^{-t/\tau}$:
  - $y(0^+)=0$ and $y(t)\to t-\tau$ as $t\to\infty$ (a line parallel to $r(t)$ but offset downward by τ).
  - Initial slope: $\dfrac{dy}{dt}=1-e^{-t/\tau} \Rightarrow \left.\dfrac{dy}{dt}\right|_{0^+}=0$, and $\left.\frac{dy}{dt}\right|_{\infty}=1$.

- Plot the error $e(t) = \tau(1 - e^{-t/\tau})$:
  - $e(0^+) = 0$, $e(\tau) = \tau(1 - e^{-1}) \approx 0.632\,\tau$, and $e(\infty) = \tau$.
  - Draw the asymptote $e = \tau$ as a horizontal line indicating the steady-state tracking error.
- Mark the time constant τ on the time axis and annotate the values of $y(t)$ and $e(t)$ at $t = \tau$ if desired.

9.Given $G(s) = \frac{K}{s\,(1+Ts)}$ under unity feedback, obtain the overall closed-loop transfer function and match its denominator to $s^2+2\zeta\omega_n s+\omega_n^2$ to compute ω_n and ζ. Comment on how K and T influence ω_n and ζ.

Solution

Given a plant

$$G(s) = \frac{K}{s(1+Ts)} = \frac{K}{Ts^2+s},$$

in a unity-feedback configuration, derive the closed-loop transfer function

$$T(s) = \frac{G(s)}{1+G(s)},$$

and identify the natural frequency and damping ratio by matching its denominator to the standard second-order form

$$s^2 + 2\zeta\omega_n s + \omega_n^2.$$

Comment on how the parameters K and T influence ω_n and ζ.

Derivation

The unity-feedback closed-loop transfer function is

$$T(s) = \frac{G(s)}{1+G(s)} = \frac{KTs^2+s}{1+KTs^2+s} = \frac{K}{Ts^2+s+K}.$$

Factor out T in the denominator:

$$T(s) = \frac{K/T}{s^2+\frac{1}{T}s+\frac{K}{T}}.$$

Parameter Identification

Match the denominator to the standard second-order form:

$$s^2 + 2\zeta\omega_n s + \omega_n^2 \quad\Longleftrightarrow\quad s^2 + \frac{1}{T}s + \frac{K}{T}.$$

Therefore,

$$\omega_n^2 = \frac{K}{T}, \qquad 2\zeta\omega_n = \frac{1}{T}.$$

Hence,

$$\omega_n = \sqrt{\frac{K}{T}}, \qquad \zeta = \frac{1}{2\sqrt{KT}}.$$

Influence of K and T

- Natural frequency:

$$\omega_n = \sqrt{\frac{K}{T}} \quad \Rightarrow \quad \{\omega_n\, increaseswithK, \omega_n decreaseswithT.$$

- Damping ratio:

$$\zeta = \frac{1}{2\sqrt{KT}} \quad \Rightarrow \quad \{\zeta\, decreaseswithK, \zeta decreaseswithT.$$

- Damping regime (for $K > 0,\ T > 0$):

$$\zeta 1 \iff \frac{1}{2\sqrt{KT}} 1 \iff KT\frac{1}{4}.$$

Thus larger K yields faster but less damped dynamics; larger T slows the system and also reduces the damping ratio.

Hence the canonical parametrization is consistent with the derived closed-loop form.

10. Consider G(s) = $s^2 + 2\zeta\omega_n s + \omega_n^2$ Form Y(s) = G(s)U(s). Decompose Y(s) into partial fractions according to the roots of $s^2 + 2\zeta\omega_n s + \omega_n^2$ Apply the inverse Laplace transform to derive the time-domain response y(t). Present final expressions for each damping regime.

Derivation of the unit-step response (general ζ) with stepwise inverse Laplace transform.

Given the standard second-order transfer function

$$G(s) = \frac{\omega_n^2}{s^2 + 2\zeta\omega_n s + \omega_n^2},$$

the unit-step input has Laplace transform $U(s) = 1s$. Thus

$$Y(s) = \frac{G(s)}{s} = \frac{\omega_n^2}{s\,(s^2 + 2\zeta\omega_n s + \omega_n^2)}.$$

Step 1: Partial-fraction decomposition. Assume

$$\frac{\omega_n^2}{s\,(s^2 + 2\zeta\omega_n s + \omega_n^2)} = \frac{A}{s} + \frac{Bs + C}{s^2 + 2\zeta\omega_n s + \omega_n^2}.$$

Multiply both sides by the common denominator:

$$\omega_n^2 = A\big(s^2 + 2\zeta\omega_n s + \omega_n^2\big) + (Bs + C)\,s = (A + B)s^2 + (2A\zeta\omega_n + C)s + A\,\omega_n^2.$$

Match coefficients of like powers of s:

$\{ A+B=0, 2A\zeta\omega_n+C=0, A\,\omega_n^2=\omega_n^2. \quad \Longrightarrow \quad A=1,\ B=-1,\ C=-2\zeta\omega_n.$

Hence

$$Y(s)=\frac{1}{s}+\frac{-s-2\zeta\omega_n}{s^2+2\zeta\omega_n s+\omega_n^2}.$$

Step 2: Complete the square. Write the quadratic as

$$s^2+2\zeta\omega_n s+\omega_n^2=(s+\zeta\omega_n)^2+\omega_n^2(1-\zeta^2).$$

Define the (possibly generalized) damped natural frequency

$$\omega_d=\omega_n\sqrt{1-\zeta^2}.$$

Step 3: Re-express the rational term for direct inversion. Rewrite the second term's numerator in terms of $(s+\zeta\omega_n)$:

$$-s-2\zeta\omega_n=-\big(s+\zeta\omega_n\big)-\zeta\omega_n.$$

Therefore

$$\frac{-s-2\zeta\omega_n}{(s+\zeta\omega_n)^2+\omega_d^2}=-\frac{s+\zeta\omega_n}{(s+\zeta\omega_n)^2+\omega_d^2}\ -\ \zeta\omega_n\,\frac{1}{(s+\zeta\omega_n)^2+\omega_d^2}.$$

Step 4: Apply inverse Laplace transforms term by term. Recall the standard transforms (with time-shift in the s-domain):

$$\mathcal{L}^{-1}\left\{\frac{1}{s}\right\}=1,$$

$$\mathcal{L}^{-1}\left\{\frac{s+a}{(s+a)^2+\omega^2}\right\}=e^{-at}\cos(\omega t),$$

$$\mathcal{L}^{-1}\left\{\frac{\omega}{(s+a)^2+\omega^2}\right\}=e^{-at}\sin(\omega t).$$

Using these with $a=\zeta\omega_n$ and $\omega=\omega_d$, we obtain

$$\mathcal{L}^{-1}\left\{-\frac{s+\zeta\omega_n}{(s+\zeta\omega_n)^2+\omega_d^2}\right\}=-\,e^{-\zeta\omega_n t}\cos(\omega_d t),$$

$$\mathcal{L}^{-1}\left\{-\,\zeta\omega_n\,\frac{1}{(s+\zeta\omega_n)^2+\omega_d^2}\right\}=-\,\zeta\omega_n\,\frac{1}{\omega_d}\,e^{-\zeta\omega_n t}\sin(\omega_d t).$$

Step 5: Assemble $y(t)$. Adding the inverse transforms of both partial-fraction terms:

$$y(t)=1\ -\ e^{-\zeta\omega_n t}\cos(\omega_d t)\ -\ \frac{\zeta\omega_n}{\omega_d}\,e^{-\zeta\omega_n t}\sin(\omega_d t),\qquad t\geq 0.$$

Equivalently, using $\omega_d = \omega_n\sqrt{1-\zeta^2}$,

$$y(t) = 1 - e^{-\zeta\omega_n t}\left[\cos\!\big(\omega_n\sqrt{1-\zeta^2}\,t\big) + \frac{\zeta}{\sqrt{1-\zeta^2}}\,\sin\!\big(\omega_n\sqrt{1-\zeta^2}\,t\big)\right], \quad t \geq 0.$$

"'latex

$$y(t) \;=\; 1 - e^{-\zeta\omega_n t}\left[\cos(\omega_d t) \;+\; \zeta\sqrt{1-\zeta^2}\,\sin(\omega_d t)\right]$$

$$y(t) \;=\; 1 - \frac{e^{-\zeta\omega_n t}}{\sqrt{1-\zeta^2}}\left[\sqrt{1-\zeta^2}\,\cos(\omega_d t) \;+\; \zeta\,\sin(\omega_d t)\right]$$

$$y(t) \;=\; 1 - \frac{e^{-\zeta\omega_n t}}{\sqrt{1-\zeta^2}}\left[\sin\theta\,\cos(\omega_d t) \;+\; \cos\theta\,\sin(\omega_d t)\right]$$

$$\textit{where} \quad \sin\theta = \sqrt{1-\zeta^2}, \qquad \cos\theta = \zeta, \qquad \theta = \tan^{-1}\!\Big(\frac{\sqrt{1-\zeta^2}}{\zeta}\Big).$$

$$y(t) \;=\; 1 - \frac{e^{-\zeta\omega_n t}}{\sqrt{1-\zeta^2}}\,\sin\big(\omega_d t + \theta\big).$$

"'

Step 6: Interpretation for all ζ. The above expression is valid for all ζ by analytic continuation:

- For $0 < \zeta < 1$, ω_d is real and the response is oscillatory with an exponentially decaying envelope.
- For $\zeta = 1$, take the limit $\omega_d \to 0$:

  $$\lim_{\zeta\to 1}\left[\cos(\omega_d t) + \frac{\zeta}{\sqrt{1-\zeta^2}}\sin(\omega_d t)\right] = 1 + \omega_n t,$$

  yielding the critically damped form

  $$y(t) = 1 - e^{-\omega_n t}\big(1 + \omega_n t\big).$$

- For $\zeta > 1$, $\omega_d = j\,\omega_n\sqrt{\zeta^2 - 1}$ and the sine/cosine evaluate to hyperbolic functions; the expression remains real and describes a monotone, non-oscillatory response.

11. Define what is peak time, maximum peak overshoot , rise time and settling time. Derive the peak time, maximum peak overshoot , rise time and settling time

Transient Specifications for a Standard Second-Order System: Definitions and Derivations

System Model and Step Response

Consider the standard underdamped ($0 < \zeta < 1$) second-order closed-loop transfer function subjected to a unit-step input:

$$T(s) = \frac{Y(s)}{R(s)} = \frac{\omega_n^2}{s^2 + 2\zeta\omega_n s + \omega_n^2}, \qquad R(s) = \frac{1}{s}.$$

Define the damped natural frequency

$$\omega_d \;=\; \omega_n\sqrt{1-\zeta^2}.$$

The unit-step response (with $y(0^-) = 0$) is

$$y(t) = 1 - \frac{e^{-\zeta\omega_n t}}{\sqrt{1-\zeta^2}}\sin\Big(\omega_d t + \phi\Big), \qquad \phi \;=\; \arctan\Big(\frac{\sqrt{1-\zeta^2}}{\zeta}\Big) \in (0, \pi 2),$$

equivalently,

$$y(t) = 1 - e^{-\zeta\omega_n t}\left[\cos(\omega_d t) + \frac{\zeta}{\sqrt{1-\zeta^2}}\sin(\omega_d t)\right].$$

Definitions (Step-Response Based)

- **Peak time** (T_p): the time at which $y(t)$ attains its first (maximum) peak after $t > 0$.
- **Maximum peak overshoot** (M_p): the relative amount by which $y(t)$ exceeds its final value (here 1) at $t = T_p$:

$$M_p \;=\; y(T_p) - 1 \quad (fraction); \qquad \%M_p = 100\, M_p \ (\%).$$

- **Rise time** (T_r): the time for the response to first reach the final value (for underdamped case, the first $t > 0$ such that $y(t) = 1$).
- **Settling time** (T_s): the time after which $|y(t) - 1|$ remains within a prescribed tolerance band around the final value for all subsequent time. Common bands: $\pm 2\%$ or $\pm 5\%$.

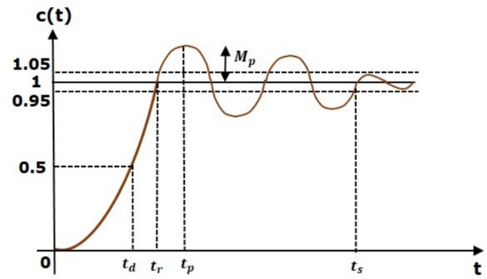

Figure 8: Time Domain Specification of Second Order system.

Derivation of Peak Time T_p

Differentiate $y(t)$ and set $\dot{y}(t) = 0$ for $t > 0$:

$$\dot{y}(t) = e^{-\zeta\omega_n t}\left[\omega_d \sin(\omega_d t) - \zeta\omega_n \cos(\omega_d t)\right] = 0.$$

Cancel the nonzero factor $e^{-\zeta\omega_n t}$ and rearrange:

$$\omega_d \sin(\omega_d t) = \zeta\omega_n \cos(\omega_d t) \quad \Longrightarrow \quad \tan(\omega_d t) = \frac{\zeta\omega_n}{\omega_d} = \frac{\zeta}{\sqrt{1-\zeta^2}}.$$

The first extremum after $t > 0$ that corresponds to a maximum occurs when

$$\omega_d T_p = \pi \quad \Longrightarrow \quad T_p = \frac{\pi}{\omega_d} = \frac{\pi}{\omega_n\sqrt{1-\zeta^2}}\,.$$

Derivation of Maximum Peak Overshoot M_p

Evaluate $y(t)$ at $t = T_p = \pi/\omega_d$. Using the cosine–sine form,

$$y(T_p) = 1 - e^{-\zeta\omega_n T_p}\left[\cos(\pi) + \frac{\zeta}{\sqrt{1-\zeta^2}}\sin(\pi)\right] = 1 - e^{-\zeta\omega_n T_p}(-1) = 1 + e^{-\zeta\omega_n T_p}.$$

Since $\omega_n T_p = \frac{\omega_n}{\omega_d}\pi = \frac{\pi}{\sqrt{1-\zeta^2}}$,

$$M_p = y(T_p) - 1 = \exp\left(-\frac{\zeta\pi}{\sqrt{1-\zeta^2}}\right).$$

Derivation of Rise Time T_r (First Crossing of 1)

Solve $y(t) = 1$ for the first $t > 0$. Using the cosine–sine form,

$$1 - e^{-\zeta\omega_n t}\left[\cos(\omega_d t) + \frac{\zeta}{\sqrt{1-\zeta^2}}\sin(\omega_d t)\right] = 1 \quad \Longrightarrow \quad \cos(\omega_d t) + \frac{\zeta}{\sqrt{1-\zeta^2}}\sin(\omega_d t) = 0.$$

Hence

$$\tan(\omega_d t_r) = -\frac{\sqrt{1-\zeta^2}}{\zeta}.$$

With $0 < \zeta < 1$, the principal solution in $(0, \pi)$ is

$$\omega_d t_r = \pi - \arctan\Big(\frac{\sqrt{1-\zeta^2}}{\zeta}\Big).$$

Using $\arccos(\zeta) = \arctan\big(\sqrt{1-\zeta^2}\zeta\big)$ for $0 < \zeta < 1$, we get

$$T_r = \frac{\pi - \arccos(\zeta)}{\omega_d} = \frac{\pi - \arccos(\zeta)}{\omega_n\sqrt{1-\zeta^2}}.$$

Remark: other engineering definitions (e.g. 10–90% rise time) yield different formulas; the above is the 0% $\to$ 100% first-crossing definition for underdamped systems.

Derivation of Settling Time T_s

For a tolerance band $\pm\delta$ around the final value (e.g. $\delta = 0.02$ for 2%), we require

$$|y(t) - 1| \;\le\; \delta \qquad \forall\, t \ge T_s.$$

The oscillatory part is enveloped by $e^{-\zeta\omega_n t}/\sqrt{1-\zeta^2}$. A conservative bound is

$$\frac{e^{-\zeta\omega_n t}}{\sqrt{1-\zeta^2}} \;\le\; \delta \quad\Longrightarrow\quad e^{-\zeta\omega_n t} \;\le\; \delta\sqrt{1-\zeta^2}.$$

Thus,

$$-\zeta\omega_n t \;\le\; \ln\big(\delta\sqrt{1-\zeta^2}\big) \quad\Longrightarrow\quad t \;\ge\; \frac{-\ln\big(\delta\sqrt{1-\zeta^2}\big)}{\zeta\omega_n}.$$

Hence a rigorous envelope-based estimate is

$$T_s(\delta) = \frac{-\ln\big(\delta\sqrt{1-\zeta^2}\big)}{\zeta\omega_n}.$$

Common engineering rules of thumb ignore the factor $\sqrt{1-\zeta^2}$ in the envelope and use

$$T_s \approx \frac{4}{\zeta\omega_n} \quad (for \delta = 0.02), \qquad T_s \approx \frac{3}{\zeta\omega_n} \quad (for \delta = 0.05).$$

Summary of Results

$$T_p = \frac{\pi}{\omega_n\sqrt{1-\zeta^2}}, \qquad M_p = \exp\Big(-\frac{\zeta\pi}{\sqrt{1-\zeta^2}}\Big), \qquad T_r = \frac{\pi - \arccos(\zeta)}{\omega_n\sqrt{1-\zeta^2}},$$

$$T_s(\delta) = \frac{-\ln\big(\delta\sqrt{1-\zeta^2}\big)}{zeta\omega_n} \;(exact envelope - based), \qquad T_s \approx \frac{4}{\zeta\omega_n}\;(2\%), \;\; T_s \approx \frac{3}{\zeta\omega_n}\;(5\%)\;(engineering).$$

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