Spectral Efficiency Analysis of HetNets Using Poisson Cluster Process

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1 Introduction

The aim of this project is to study the modelling of Heterogenous Networks (HetNets) using the Poisson Cluster Process (PCP). A key characteristics of a HetNet is the coupling between users and base stations (BSs). In a user-centric capacity-driven deployment, small cell base stations (SBSs) are used in regions of high user density or *hotspots* to provide additional capacity. Since SBSs are added to the network on ad-hoc basis, they tend to form clusters around user hotspots like malls, offices, residential complexes etc. PCP has been extensively used in the literature to model the tendency to form clusters as well as the correlation between users and SBSs.

The authors in [1] have proposed a comprehensive approach to the modelling and analysis of HetNets. They have derived exact expressions for coverage probability of a typical user and throughput of the whole network under the power-based association policy and distance-based association policy for a 2 tier HetNet. The authors in [6] generalize this framework to analyze coverage probability in a $\mathcal{K}$ tier HetNet under max-power association. Further, they also consider the effect of SBSs cluster on general users in the network who do not exhibit correlation with the BSs. These users do not show clustering tendency and are effectively modelled as PPP. The authors in [4] derive the spectral efficiency of the special case of PCP i.e Matern Cluster Process.

In this paper, we follow the approach used by [4] in calculation of the performance metric viz. spectral efficiency a.k.a average ergodic rate and generalize it for the computation of average ergodic rate for the system model of $\mathcal{K}$ tier HetNet under max-power association. We derive exact expressions for the Laplace transform of total received power and Laplace transform of interference for the typical users for both Type 1 and Type 2 as defined in [6]. The generalized expression of Laplace transform of total received power and interference opens new avenues for computation of performance metrics for the system model of PCP based $\mathcal{K}$ tier HetNet under max-power association proposed in [6]. One such performance metric which is derived in this paper is average ergodic rate which is based on the results of Laplace transform of total received power and interference.

Key index terms: Heterogeneous networks, Poisson Cluster Process, Average Ergodic Rate, Laplace transform.

2 System Model

2.1 Background

The Poisson Cluster Process as defined in [6] is as follows:

Definition 1. *(Poisson Cluster Process): A PCP, characterized by 3 parameters $\lambda_p, g, \bar{m}$, is defined in $\mathbb{R}^2$ as:*

$$\Phi(\lambda_p, g, \bar{m}) = \bigcup_{\mathbf{z}\in\Phi_{\mathrm{p}}(\lambda_p)} (\mathbf{z} + \mathcal{B}^{\mathbf{z}}) \tag{1}$$

where Φ_{p} is a parent poisson point process with density λ_p and $\mathcal{B}^{\mathbf{z}}$ is the offspring point process around the cluster center $\mathbf{z}$ which contains i.i.d random vectors $\boldsymbol{s}$ with probability density function (PDF) as $g(\boldsymbol{s})$. The number of points in a offspring point process, denoted by M, is a poisson RV with parameter $\bar{m}$.

We would use Φ to denote any point process. We use $\Phi(\lambda_p, g, \bar{m})$ to denote a PCP according to Definition (1) and $\Phi(\lambda)$ to denote a PPP with intensity λ. Note that the points $\{\mathbf{t}\} \equiv \mathbf{z} + \mathcal{B}^{\mathbf{z}}$ when conditioned on the cluster center $\mathbf{z} \in \Phi_{\mathrm{p}}$ are i.i.d. with PDF $f(\mathbf{t}|\mathbf{z}) = g(\mathbf{t}-\mathbf{z})$, i.e., the conditional distribution of the points given its cluster center $\mathbf{z}$ is equivalent to translating its offspring process clustered at origin to $\mathbf{z}$. The point process resulting from conditioning a PCP on its parent PPP is given in the following proposition.

Proposition 1. *The resulting point process when the PCP $\Phi(\lambda_p, g, \bar{m})$ is conditioned on its parent PPP Φ_{p} is an inhomogenous PPP with density*

$$\lambda(\mathbf{x}) = \bar{m} \sum_{\mathbf{z}\in\Phi_{\mathrm{p}}} f(\mathbf{x}|\mathbf{z}) \tag{2}$$

Proof. Let $\tilde{\Phi}$ denote the point process resulting from conditioning the PCP $\Phi(\lambda_p, g, \bar{m})$ on its parent PPP $\Phi_{\mathrm{p}}(\lambda_p)$. We start by computing the probability generating functional (PGFL) [2] of $\tilde{\Phi}$.

$$\begin{aligned}
&\mathbb{E}\left[\prod_{\mathbf{x}\in\tilde{\Phi}} \mu(\mathbf{x})\right] = \mathbb{E}\left[\prod_{\mathbf{z}\in\Phi_{\mathrm{p}}} \prod_{\mathbf{x}\in\mathcal{B}^{\mathbf{z}}} \mu(\mathbf{x})\right] \overset{(a)}{=} \prod_{\mathbf{z}\in\Phi_{\mathrm{p}}} \mathbb{E}\left[\prod_{\mathbf{x}\in\mathcal{B}^{\mathbf{z}}} \mu(\mathbf{x})\right] \overset{(b)}{=} \prod_{\mathbf{z}\in\Phi_{\mathrm{p}}} \mathbb{E}\left[\prod_{i=1}^{M} \mathbb{E}[\mu(\mathbf{x}_i)]\right] \\
&= \prod_{\mathbf{z}\in\Phi_{\mathrm{p}}} \mathbb{E}\left[\prod_{i=1}^{M} \left(\int_{\mathbb{R}^2} \mu(\mathbf{x}) f(\mathbf{x}|\mathbf{z}) \mathrm{d}\mathbf{x}\right)\right] = \prod_{\mathbf{z}\in\Phi_{\mathrm{p}}} \mathbb{E}\left[\left(\int_{\mathbb{R}^2} \mu(\mathbf{x}) f(\mathbf{x}|\mathbf{z}) \mathrm{d}\mathbf{x}\right)^M\right] \\
&\overset{(c)}{=} \prod_{\mathbf{z}\in\Phi_{\mathrm{p}}} \exp\left(-\bar{m}\left(1 - \int_{\mathbb{R}^2} \mu(\mathbf{x}) f(\mathbf{x}|\mathbf{z}) \mathrm{d}\mathbf{x}\right)\right) = \prod_{\mathbf{z}\in\Phi_{\mathrm{p}}} \exp\left(-\bar{m} \int_{\mathbb{R}^2} (1-\mu(\mathbf{x})) f(\mathbf{x}|\mathbf{z}) \mathrm{d}\mathbf{x}\right) \\
&= \exp\left(-\int_{\mathbb{R}^2} (1-\mu(\mathbf{x})) \left(\bar{m} \sum_{\mathbf{z}\in\Phi_{\mathrm{p}}} f(\mathbf{x}|\mathbf{z})\right) \mathrm{d}\mathbf{x}\right)
\end{aligned}$$

Here, (a) follows from the fact that conditioned on Φ_{p}, $\{\mathcal{B}^{\mathbf{z}}\}$ are i.i.d. offspring point processes, (b) follows from the fact that points in an offspring process are i.i.d. around the cluster center $\mathbf{z}$ and (c) follows from the fact that the number of points in a cluster $M \sim \text{Poisson}(\bar{m})$.

Since PGFL uniquely characterizes a point process, comparing the PGFL of $\tilde{\Phi}$ with that of PPP [2], we conclude that $\tilde{\Phi}$ is an inhomogenous PPP with densiy $\lambda(\mathbf{x}) = \bar{m} \sum_{\mathbf{z}\in\Phi_{\mathrm{p}}} f(\mathbf{x}|\mathbf{z})$. ☐

2.2 Spatial model of base stations

We follow the K-tier HetNet model, as described in [6], where BSs of each tier are distributed as PPP or PCP. The index set of the BS tiers modelled by PCPs is denoted as $\mathcal{K}_1$ and that of PPPs as $\mathcal{K}_2$ where $\mathcal{K}_1$ and $\mathcal{K}_2$ are mutually exclusive and $|\mathcal{K}| = |\mathcal{K}_1 \cup \mathcal{K}_2| = K$. The point process of the k^{th} tier BS is denoted by Φ_k where Φ_k can be either a PCP $\Phi(\lambda_{\mathrm{p}_k}, g, \bar{m})(\forall k \in \mathcal{K}_1)$ where λ_{p_k} is the density of the parent PPP Φ_{p_k} or it can be a PPP $\Phi(\lambda_k)(\forall k \in \mathcal{K}_2)$. Each BS of the k^{th} tier has a constant transmit power P_k.

2.3 Spatial model of users

We assume that the users are distributed according to stationary distribution Φ_u. Following [6], we have considered two types of users. For both type of user distributions, a *typical* user is considered for analysis. A *typical* user represents a point selected uniformly at random from Φ_u. Since Φ_u is stationary, the location of the *typical* user can be assumed to be located at the origin without loss of generality.

`TYPE 1`:*No user-BS coupling.* These users are uniformly distrbuted and their locations are independent of the locations of the BSs, for example pedestrians or users in transit. One way of modelling these users is by using a homogeneous PPP. Since Φ_u and $\Phi_k(\forall k \in \mathcal{K})$ are independent, the selection of the *typical* user does not bias the distribution of Φ_k.

`TYPE 2`:*User-BS coupling.* These users form clusters a.k.a *user hotspots* and their location can be modelled as a PCP $\Phi_u = \Phi(\lambda_{\mathrm{p}_u}, g_u, \bar{m_u})$. When the users are clustered, it is assumed that one of the BS tier belonging to the set $\mathcal{K}_1$ is deployed to serve the user hotspots. That is Φ_u and Φ_q are PCPs which share the same parent PPP Φ_{p_q} and thus, a coupling is introduced between user and basestation locations. Conditioned on the parent PPP, Φ_u and Φ_q are conditionally independent but not identically distributed. Due to this user-BS coupling, the selection of the *typical* user affects the distribution of Φ_q. Since the *typical* user belongs to a cluster centred at $\mathbf{z}_0$, the point process Φ_q is always conditioned to have a cluster centered at $\mathbf{z}_0$. So, by Slivnyak's theorem, from the perspective of the typical user, the palm version of Φ_q will be equivalent to a superposition of Φ_q and $\mathbf{z}_0 + \mathcal{B}^{\mathbf{z}_0}$ where Φ_q and $\mathbf{z} + \mathcal{B}^{\mathbf{z}}$ are independent.

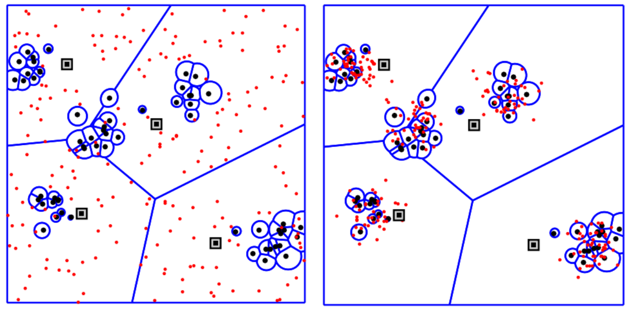


Figure 1: Illustration from [6] which depicts a two-tier HetNet. Black dots represent the PCP of SBSs, squares represent the PPP of MBSs and red dots represent the users.

So, for Type 2 users, the modified Φ_q is $\Phi(\lambda_{\mathrm{p}_q}, g_q, \bar{m}_q) \cup (\mathbf{z}_0 + \mathcal{B}^{\mathbf{z}_0})$ and the underlying parent PPP is modified as $\Phi(\lambda_{\mathrm{p}_q}) \cup \{\mathbf{z}_0\}$. The BS cluster $\mathcal{B}^{\mathbf{z}_0}$ is termed as the *representative cluster*.

2.4 Propagation model

2.4.1 Received power

It is assumed that all links to a typical user located at the origin suffer a standard power-law path-loss with α as the path-loss exponent and $h_{\mathbf{x}}$ as the small scale fading coefficient. Further, each link is assumed to undergo Rayleigh fading. So $\{h_{\mathbf{x}}\}$ is a sequence of i.i.d random variables with $h_{\mathbf{x}} \sim \exp(1)$. The downlink received power from a BS located at $\mathbf{x} \in \Phi_k$ can be expressed as $P_k h_{\mathbf{x}} \|\mathbf{x}\|^{-\alpha}$.

2.4.2 Association event

It is assumed that the user connects to the BS providing the maximum received power averaged over fading. For a K-tier HetNet, the association can be found as follows.

$$\tilde{\mathbf{x}_k} = \underset{\mathbf{x}_k \in \Phi_k}{argmax}(P_k \|\mathbf{x}_k\|^{-\alpha}) = \underset{\mathbf{x} \in \Phi_k}{argmin}(\|\mathbf{x}\|) \tag{3}$$

$$\mathbf{x}^* = \underset{\{\tilde{\mathbf{x}_k}, k \in \mathcal{K}\}}{argmax}(P_k \|\tilde{\mathbf{x}_k}\|^{-\alpha}) \tag{4}$$

The *association event* $\mathcal{S}_i$ corresponding to the i^{th} tier is defined as the event that the serving BS belongs to the i^{th} tier, formally given as $\mathcal{S}_i = \{\mathbf{x}^* = \tilde{\mathbf{x}_i}\}$.

2.4.3 Signal to Interference Noise Ratio (SINR)

Conditioned on $\mathcal{S}_i$, the `SINR` for a typical user can be expressed as

$$\mathtt{SINR}(\mathbf{x}^*) = \frac{P_i h_{\mathbf{x}^*} \|\mathbf{x}^*\|^{-\alpha}}{N_0 + \sum\limits_{j \in \mathcal{K}} \left(\sum\limits_{\mathbf{x} \in \Phi_j \setminus \{\mathbf{x}^*\}} P_j h_{\mathbf{x}} \|\mathbf{x}\|^{-\alpha} \right)} \tag{5}$$

where N_0 is the thermal Noise power.

2.4.4 Spectral Efficiency a.k.a Average Ergodic Rate

The average ergodic rate is used to measure the spectral efficiency performance of a network. This metric is computed in nats/sec/Hz. The authors in [5] have defined average ergodic rate as follows.

$$\mathcal{R} = \sum_{i=1}^{K} \mathcal{R}_i \mathbb{P}[\mathcal{S}_i] \tag{6}$$

where $\mathbb{P}[\mathcal{S}_i]$ is the probability of the association event as defined in 2.4.2 and $\mathcal{R}_i$ is the average ergodic rate of the typical user connected to the i^{th} tier. The average ergodic rate is calculated in terms of the Shannon bound from Information Theory which is $\ln(1+\mathtt{SINR})$ for an instantaneous $\mathtt{SINR}$. For our analysis, we calculate the average ergodic rate of the typical user connected to the serving BS given by the max-power association criteria. We define the average ergodic rate of a typical user as follows.

$$\mathcal{R} = \mathbb{E}[\ln(1+\mathtt{SINR})] = \mathbb{E}_{\Phi_{\mathrm{p}_k}}[\mathbb{E}_{\mathcal{S}_i}[\mathbb{E}_{\mathtt{SINR}_i}[\ln(1+\mathtt{SINR}_i)|\mathcal{S}_i, \Phi_{\mathrm{p}_k}, \forall k \in \mathcal{K}_1]]] \tag{7}$$

3 Average Ergodic Rate Analysis

3.1 Solution Approach

The authors in [6] derive the exact expressions for the i^{th} tier coverage probability for Type 1 and Type 2 users. The first 3 steps mentioned below are common for analysis of any performance metric for the system model. Hence, we borrow these steps from [6] and state the results without proof. The key steps involved in the computation of average ergodic rate are as follows:

- Contact Distance Distribution: As defined in eqn 3, $\|\tilde{\mathbf{x}}_k\|$ is the nearest BS to the typical user located at the origin. Hence, the distribution of $\|\tilde{\mathbf{x}}_k\|$ is same as the contact distance distribution of Φ_k.
  - For $k \in \mathcal{K}_1$, the conditional contact distance distribution can be derived based on the property of PCP that conditioned on the parent PPP, the PCP can be viewed as inhomogeneous PPP with density as given in proposition 1. The result is stated in lemma 3.1, in equation (8a) and (9a).
  - For $k \in \mathcal{K}_2$, since Φ_k is a PPP, we directly use the result derived in class, which is the well-known Rayleigh distribution. The result is stated in lemma 3.1, in equation (8b) and (9b).
- Association Probability: As defined in 2.4.2, the probability of association to each tier in $\mathcal{K}$ conditioned on the parent PPP $\Phi_{p_k} \forall k \in \mathcal{K}_1$ can be derived based on the results of the contact distance distribution as in lemma3.1. The association takes place as per the max-power association policy given by eqn 4. The result is stated in lemma 3.2, in equation (10a) and (10b).
- PDF of serving distance given the association: Using the result of the association probability and contact distance distribution, the PDF of the serving distance $\|\mathbf{x}^*\|$ given the association of i^{th} tier can be derived. The result is stated in lemma3.3, in equation 11a and 11b.
- Laplace transform of total received power: The key steps involved in deriving the laplace transform are as follows:
  - We derive the laplace transform of total received power conditioned on $\mathcal{S}_i$ and $\Phi_{p_k} \forall k \in \mathcal{K}_1$.
  - We derive the laplace transform of total received power conditioned on $\Phi_{p_k} \forall k \in \mathcal{K}_1$ in equation (19a) and (19b)
  - We derive the laplace transform of total received power by deconditioning over $\Phi_{p_k} \forall k \in \mathcal{K}_1$. The deconditioning steps are different for Type 1 (eqn (22a), eqn (22b)) and Type 2 (eqn (24a), eqn(24b),eqn (24c)) users.
- Laplace transform of interference: The key steps involved in deriving the laplace transform are as follows:
  - We derive the laplace transform of interference conditioned on $\mathcal{S}_i$ and $\Phi_{p_k} \forall k \in \mathcal{K}_1$.
  - We derive the laplace transform of interference conditioned on $\Phi_{p_k} \forall k \in \mathcal{K}_1$ in equation (26a) and (26b)
  - We derive the laplace transform of interference by deconditioning over $\Phi_{p_k} \forall k \in \mathcal{K}_1$. The deconditioning steps are different for Type 1 (eqn (29a), eqn(29b)) and Type 2 (eqn (31a), eqn (31b), eqn
  - We derive the laplace transform of interference by deconditioning over $\Phi_{p_k} \forall k \in \mathcal{K}_1$. The deconditioning steps are different for Type 1 (eqn (29a), eqn(29b)) and Type 2 (eqn (31a), eqn (31b), eqn (31c)) users. (31c)) users.
- Average ergodic rate: The average ergodic rate is expressed in terms of Laplace transform of total received power and Laplace transform of interference in lemma 3.15. This result alongwith results of previous lemmas is used to derive the Average ergodic rate for Type 1 user in theorem 3.16 and Average ergodic rate for Type 2 user in theorem 3.17.

3.2 Some useful lemmas

We state the lemmas from [6] in 3.2 without proof. These lemmas are used in deriving the expression for Average Ergodic Rate. One can refer the original paper [6] for the proofs of these lemmas.

3.2.1 Contact Distance Distribution

Lemma 3.1. *(Lemma 2 in [6]) Conditioned on the parent PPP $\Phi_{\mathrm{p}_k} \forall k \in K_1$, the PDF and CDF of $\|\tilde{\mathbf{x}_k}\|$, where $\tilde{\mathbf{x}_k}$ was defined in eqn 3 is given as follows.*

$$f_{c_k}\left(r|\Phi_{\mathrm{p}_k}\right) = \bar{m}_k \sum_{\mathbf{z}\in\Phi_{\mathrm{p}_k}} f_{\mathrm{d}_k}(r|z) \prod_{\mathbf{z}\in\Phi_{\mathrm{p}_k}} \exp\left(-\bar{m}_k F_{\mathrm{d}_k}(r|z)\right), \quad r \geq 0, \forall k \in \mathcal{K}_1 \tag{8a}$$

$$f_{c_k}(r) = 2\pi\lambda_k r \exp\left(-\pi\lambda_k r^2\right) \quad r \geq 0, \forall k \in \mathcal{K}_2 \tag{8b}$$

$$F_{c_k}\left(r|\Phi_{\mathrm{p}_k}\right) = 1 - \exp\left(-\bar{m}_k \sum_{\mathrm{z}\in\Phi_{\mathrm{p}_k}} \int_0^r f_{\mathrm{d}_k}(y|z)\mathrm{d}y\right), \quad r \geq 0 \forall k \in \mathcal{K}_1 \tag{9a}$$

$$F_{c_k}(r) = 1 - \exp\left(-\pi\lambda_k r^2\right), r \geq 0, \forall k \in \mathcal{K}_2 \tag{9b}$$

where $f_{\mathrm{d}_k}(y|z)$ and $F_{\mathrm{d}_k}(r|z)$ is the PDF and CDF of the distance of a randomly selected point of $\Phi_{\mathrm{p}_k} \forall k \in \mathcal{K}_1$ given that its cluster centre is located at $\mathbf{z}$. As mentioned in Lemma 1 of [6], the conditional distance distribution depends only on the magnitude of $\mathbf{z}$.

3.2.2 Association Probability and Serving Distance Distribution

Lemma 3.2. *(Lemma 3 in [6]) Conditioned on the parent PPP $\Phi_{\mathrm{p}_k} \forall k \in \mathcal{K}_1$, the i^{th} tier association probability is given as follows.*

$$\mathcal{P}(\mathcal{S}_i|\Phi_{\mathrm{p}_k}, \forall k \in \mathcal{K}_1) =$$

$$\bar{m}_i \int_0^\infty \sum_{\mathbf{z}\in\Phi_{\mathbf{p}_i}} f_{\mathrm{d}_i}(r|z) \prod_{j_1\in\mathcal{K}_1} \prod_{\mathbf{z}\in\Phi_{\mathbf{p}_{j_1}}} \exp\left(-\bar{m}_{j_1} F_{\mathrm{d}_{j_1}}\left(\bar{P}_{j_1,i} r|z\right)\right) \prod_{j_2\in\mathcal{K}_2} \exp\left(-\pi\lambda_{j_2}\bar{P}_{j_2,i}^2 r^2\right)\mathrm{d}r \quad \forall i \in \mathcal{K}_1 \tag{10a}$$

$$2\pi\lambda_i \int_0^\infty \prod_{j_1\in\mathcal{K}_1} \prod_{\mathbf{z}\in\Phi_{\mathbf{p}_{j_1}}} \exp\left(-\bar{m}_{j_1} F_{\mathrm{d}_{j_1}}\left(\bar{P}_{j_1,i} r|z\right)\right) \prod_{j_2\in\mathcal{K}_2} \exp\left(-\pi\lambda_{j_2}\bar{P}_{j_2,i}^2 r^2\right) r\mathrm{d}r \quad \forall i \in \mathcal{K}_2 \tag{10b}$$

where $\bar{P}_{j,i} = (\frac{P_j}{P_i})^{\frac{1}{\alpha}}$

Lemma 3.3. *(Lemma 4 in [6]) Conditioned on the parent PPP $\Phi_{\mathrm{p}_k} \forall k \in \mathcal{K}_1$ and the association to the i^{th} tier, the PDF of the serving distance $\|x^*\|$ defined in eqn4 is given as follows. $f_{s_i}(r|\mathcal{S}_i, \Phi_{\mathrm{p}_k} \forall k \in \mathcal{K}_1) =$*

$$\frac{\bar{m}_i}{\mathcal{P}(\mathcal{S}_i|\Phi_{\mathrm{p}_k}, \forall k \in \mathcal{K}_1)} \sum_{\mathbf{z}\in\Phi_{\mathbf{p}_i}} f_{\mathrm{d}_i}(r|z) \prod_{j_1\in\mathcal{K}_1} \prod_{\mathbf{z}\in\Phi_{\mathbf{p}_{j_1}}} \exp\left(-\bar{m}_{j_1} F_{\mathrm{d}_{j_1}}\left(\bar{P}_{j_1,i} r|z\right)\right) \prod_{j_2\in\mathcal{K}_2} \exp\left(-\pi\lambda_{j_2}\bar{P}_{j_2,i}^2 r^2\right),$$
$$r \geq 0, \forall i \in \mathcal{K}_1 \tag{11a}$$

$$\frac{2\pi\lambda_i}{\mathcal{P}(\mathcal{S}_i|\Phi_{\mathrm{p}_k}, \forall k \in \mathcal{K}_1)} \prod_{j_1\in\mathcal{K}_1} \prod_{\mathbf{z}\in\Phi_{\mathbf{p}_{j_1}}} \exp\left(-\bar{m}_{j_1} F_{\mathrm{d}_{j_1}}\left(\bar{P}_{j_1,i} r|z\right)\right) \prod_{j_2\in\mathcal{K}_2} \exp\left(-\pi\lambda_{j_2}\bar{P}_{j_2,i}^2 r^2\right) r, \quad r \geq 0, \forall i \in \mathcal{K}_2 \tag{11b}$$

3.2.3 Sum-Product Functional and PGFL of a PPP

Lemma 3.4. *For a homogeneous Poisson Point Process with density λ, the Sum-Product functional is given as follows:*

$$\mathbb{E}\left[\sum_{x\in\Phi} \nu(x) \prod_{y\in\Phi} \mu(y)\right] = 2\pi\lambda \int_0^\infty \nu(x)\mu(x)x\mathrm{d}x \exp\left(-2\pi\lambda \int_0^\infty (1-\mu(y))y\mathrm{d}y\right) \tag{12}$$

For a homogeneous Poisson Point Process with density λ, the Probability Generating Functional (PGFL) is given as follows:

$$\mathbb{E}\left[\prod_{\mathbf{y}\in\Phi} \mu(y)\right] = \exp\left(-2\pi\lambda \int_0^\infty (1-\mu(y))y\mathrm{d}y\right) \tag{13}$$

For a typical user belonging to Type 2, it was discussed in 2.3 that the underlying parent PPP is modified as $\Phi(\lambda_{\mathrm{p}_q}) \cup \{\mathbf{z}_0\}$. So when we compute the PGFL and the Sum-product functional of Φ_{p_q} for Type 2 users, we use the modified version of the underlying parent PPP.

Lemma 3.5. *For a homogeneous Poisson Point Process $\Phi_{\mathrm{p}_q} = \Phi(\lambda_{\mathrm{p}_q}) \cup \{\mathbf{z}_0\}$, the Probability Generating Functional (PGFL) is given as follows:*

$$\mathbb{E}\left[\prod_{\mathbf{z}\in\Phi_{\mathrm{p}_q}} \mu(\mathbf{z})\right] = \left(\int_0^\infty \mu(z_0) f_{d_u}(z_0|0) dz_0\right) \times \exp\left(-2\pi\lambda_{\mathrm{p}_q}\int_0^\infty (1-\mu(z)) z dz\right) \tag{14}$$

Here $f_{d_u}(z_0|0)$ denotes the PDF of the distance of a randomly selected point $\in \Phi_u$ given its cluster centre is located at the origin.

Proof.

$$\mathbb{E}\left[\prod_{\mathbf{z}\in\Phi_{\mathrm{p}_q}} \mu(\mathbf{z})\right] \overset{(a)}{=} \mathbb{E}\left[\mu(z_0)\right] \times \mathbb{E}\left[\prod_{\mathbf{z}\in\Phi(\lambda_{\mathrm{p}_q})} \mu(\mathbf{z})\right] \overset{(b)}{=} \mathbb{E}\left[\mu(z_0)\right] \times \exp\left(-2\pi\lambda_{\mathrm{p}_q}\int_0^\infty (1-\mu(z)) z dz\right)$$

where (a) follows from the fact that $\mathbf{z}_0$ and $\Phi(\lambda_{\mathrm{p}_q})$ are independent, (b) follows from 13. $\square$

Lemma 3.6. *For a homogeneous Poisson Point Process $\Phi_{\mathrm{p}_q} = \Phi(\lambda_{\mathrm{p}_q}) \cup \{\mathbf{z}_0\}$, the Sum-Product Functional is given as follows:*

$$\mathbb{E}\left[\sum_{\mathbf{z}\in\Phi_{\mathrm{p}_q}} \nu(z) \prod_{\mathbf{z}\in\Phi_{\mathrm{p}_q}} \mu(z)\right] = \exp\left(-2\pi\lambda_{\mathrm{p}_q}\int_0^\infty (1-\mu(z))\, z \mathrm{d}z\right)\left(\int_0^\infty \nu(z_0)\mu(z_0) f_{\mathrm{d}_u}(z_0|0)\mathrm{d}z_0 + \right.$$
$$\left. 2\pi\lambda_{\mathrm{p}_q}\int_0^\infty \nu(z)\mu(z) z \mathrm{d}z \int_0^\infty \mu(z_0) f_{\mathrm{d}_u}(z_0|0)\mathrm{d}z_0\right) \tag{15}$$

Proof.

$$\mathbb{E}\left[\sum_{\mathbf{z}\in\Phi_{\mathrm{p}_q}} \nu(z) \prod_{\mathbf{z}\in\Phi_{\mathrm{p}_q}} \mu(z)\right]$$
$$\overset{(a)}{=} \mathbb{E}\left[\nu(z_0)\mu(z_0)\right]\mathbb{E}_{\Phi(\lambda_{\mathrm{p}_q})}\left[\prod_{\mathbf{z}\in\Phi(\lambda_{\mathrm{p}_q})} \mu(z)\right] + \mathbb{E}_{\Phi(\lambda_{\mathrm{p}_q})}\left[\sum_{\mathbf{z}\in\Phi(\lambda_{\mathrm{p}_q})} \nu(z) \prod_{\mathbf{z}\in\Phi(\lambda_{\mathrm{p}_q})} \mu(z)\right]\mathbb{E}\left[\mu(z_0)\right]$$
$$\overset{(b)}{=} \int_0^\infty \nu(z_0)\mu(z_0) f_{\mathrm{d}_u}(z_0|0)\mathrm{d}z_0 \exp\left(-2\pi\lambda_{\mathrm{p}_q}\int_0^\infty (1-\mu(z))\right)$$
$$+ 2\pi\lambda_{\mathrm{p}_q}\int_0^\infty \nu(z)\mu(z) z\mathrm{d}z \exp\left(-2\pi\lambda_{\mathrm{p}_q}\int_0^\infty (1-\mu(z))\right)\int_0^\infty \mu(z_0) f_{\mathrm{d}_u}(z_0|0)\mathrm{d}z_0$$

Here (a) follows from the fact that $\mathbf{z}_0$ and $\Phi(\lambda_{\mathrm{p}_q})$ and (b) follows from the fact that $f_{d_u}(z_0|0)$ is the PDF of the distance of a randomly selected point in Φ_u given its cluster centre is located at the origin. The expressions of the PGFL and the Sum-Product functional are substituted from Equation (13) and (12). The final expression is obtained after algebraic simplifications. $\square$

3.3 Laplace transform of total received power

Lemma 3.7. *Conditioned on $\mathcal{S}_i$ and $\Phi_{\mathrm{p}_k} \forall k \in \mathcal{K}_1$, the laplace transform of total power P is given as:*

$$\mathcal{L}_{P|\mathcal{S}_i,\Phi_{\mathrm{p}_k}\forall k\in\mathcal{K}_1}(s) = \prod_{j_1\in\mathcal{K}_1}\prod_{\mathbf{z}\in\Phi_{\mathrm{p}_{j_1}}} \mathcal{D}'_{j_1}(s,z) \prod_{j_2\in\mathcal{K}_2} \exp\left(-\pi\lambda_{j_2}(sP_{j_2})^{2/\alpha}\zeta'(\alpha)\right) \tag{16}$$

where

$$\mathcal{D}'_i(s,z) = \exp\left(-\bar{m}_i\left(1-\int_0^\infty \frac{f_{\mathrm{d}_i}(y|z)}{1+sP_i y^{-\alpha}}dy\right)\right) \tag{17}$$

and, $\zeta'(\alpha) = \int_0^\infty \frac{1}{1+t^{\alpha/2}}dt = \frac{2}{\alpha}\beta(\frac{2}{\alpha}, 1-\frac{2}{\alpha})$, where $\beta(.,.)$ denotes the beta function.

Proof. The conditional laplace transform of total power P given $\Phi_{\mathrm{p}_k} \forall k \in \mathcal{K}_1$ and the association to the i^{th} tier is given by:

$$\mathcal{L}_{P|\mathcal{S}_i,\Phi_{\mathrm{p}_k}\forall k\in\mathcal{K}_1}(s) = \mathbb{E}[\exp(-sP)|\mathcal{S}_i, \Phi_{\mathrm{p}_k}, \forall k \in \mathcal{K}_1]$$

$$= \mathbb{E}\left[\exp\left(-s\sum_{j\in\mathcal{K}}\sum_{\mathbf{y}\in\Phi_j} P_j h_{\mathbf{y}}\|\mathbf{y}\|^{-\alpha}\right)\middle|\mathcal{S}_i, \Phi_{\mathrm{p}_k}, \forall k \in \mathcal{K}_1\right]$$

$$= \mathbb{E}\left[\prod_{j\in\mathcal{K}}\prod_{\mathbf{y}\in\Phi_j} \exp\left(-sP_j h_{\mathbf{y}}\|\mathbf{y}\|^{-\alpha}\right)\middle|\mathcal{S}_i, \Phi_{\mathrm{p}_k}, \forall k \in \mathcal{K}_1\right]$$

$$\overset{(a)}{=} \mathbb{E}\left[\prod_{j\in\mathcal{K}}\prod_{\mathbf{y}\in\Phi_j} \mathbb{E}_{h_{\mathbf{y}}}\left[\exp\left(-sP_j h_{\mathbf{y}}\|\mathbf{y}\|^{-\alpha}\right)\right]\middle|\mathcal{S}_i, \Phi_{\mathrm{p}_k}, \forall k \in \mathcal{K}_1\right]$$

$$\overset{(b)}{=} \mathbb{E}\left[\prod_{j\in\mathcal{K}}\prod_{\mathbf{y}\in\Phi_j} \frac{1}{1+sP_j\|\mathbf{y}\|^{-\alpha}}\middle|\mathcal{S}_i, \Phi_{\mathrm{p}_k}, \forall k \in \mathcal{K}_1\right]$$

$$= \prod_{j_1\in\mathcal{K}_1} \mathbb{E}_{\Phi_{j_1}}\left[\prod_{\mathbf{y}\in\Phi_{j_1}} \frac{1}{1+sP_{j_1}\|\mathbf{y}\|^{-\alpha}}\middle|\Phi_{\mathbf{p}_{j_1}}\right]\prod_{j_2\in\mathcal{K}_2} \mathbb{E}_{\Phi_{j_2}}\left[\prod_{\mathbf{y}\in\Phi_{j_2}} \frac{1}{1+sP_{j_2}\|\mathbf{y}\|^{-\alpha}}\right]$$

$$= \prod_{j_1\in\mathcal{K}_1}\prod_{\mathbf{z}\in\Phi_{\mathrm{p}_{j_1}}} \exp\left(-\int_0^\infty \bar{m}_{j_1} f_{\mathrm{d}_{j_1}}(y|z)\left(1-\frac{1}{1+sP_{j_1}y^{-\alpha}}\right)dy\right)\times$$
$$\prod_{j_2\in\mathcal{K}_2} \exp\left(-2\pi\lambda_{j_2}\int_0^\infty\left(1-\frac{1}{1+sP_{j_2}y^{-\alpha}}\right)ydy\right)$$

$$= \prod_{j_1\in\mathcal{K}_1}\prod_{\mathbf{z}\in\Phi_{\mathrm{p}_{j_1}}} \exp\left(-\bar{m_{j_1}}\left(1-\int_0^\infty \frac{f_{\mathrm{d}_{j_1}}(y|z)}{1+sP_{j_1}y^{-\alpha}}dy\right)\right)\prod_{j_2\in\mathcal{K}_2} \exp\left(-\pi\lambda_{j_2}(sP_{j_2})^{2/\alpha}\int_0^\infty \frac{1}{1+t^{\alpha/2}}dt\right)$$

Here (a) follows from the fact that all links are subjected to i.i.d. fading, (b) follows from the fact that $h_{\mathbf{y}} \sim \exp(1)$. The last step is obtained by substituting $sP_{j_2}y^{-\alpha} = t^{-\alpha/2}$. □

We would like to point out that the conditional laplace transform of total power P derived above is actually independent of $\mathcal{S}_i$ since total power P does not depend on the serving BS $\mathbf{x}^*$. But for purposes of future results, we express the result of Lemma 3.7 in the following form.

Lemma 3.8. *Conditioned on $\Phi_{\mathrm{p}_k}, \forall k \in \mathcal{K}_1$, the laplace transform of total power P is given as:*

$$\mathcal{L}_{P|\Phi_{\mathrm{p}_k},\forall k\in\mathcal{K}_1}(s) = \sum_{i=1}^{K}\mathcal{M}_i(s) \tag{18}$$

where $\mathcal{M}_i(s) =$

$$\bar{m}_i\int_0^\infty \prod_{j_1\in\mathcal{K}_1\setminus\{i\}}\prod_{\mathbf{z}\in\Phi_{\mathrm{p}_{j_1}}} \mathcal{D}_{j_1,i}(r,z,s)\prod_{j_2\in\mathcal{K}_2}\exp\left(-\pi\lambda_{j_2}\zeta_{j_2,i}(r,s)\right)\left(\sum_{\mathbf{z}\in\Phi_{\mathbf{p}_i}} f_{\mathrm{d}_i}(r|z)\prod_{\mathbf{z}\in\Phi_{\mathbf{p}_i}}\mathcal{D}_{i,i}(r,z,s)\right)\mathrm{d}r, \quad \textit{when } i\in\mathcal{K}_1 \tag{19a}$$

$$2\pi\lambda_i\int_0^\infty \prod_{j_1\in\mathcal{K}_1}\prod_{\mathbf{z}\in\Phi_{\mathrm{p}_{j_1}}} \mathcal{D}_{j_1,i}(r,z,s)\prod_{j_2\in\mathcal{K}_2}\exp\left(-\pi\lambda_{j_2}\zeta_{j_2,i}(r,s)\right)r\mathrm{d}r, \textit{when } i\in\mathcal{K}_2 \tag{19b}$$

Here

$$\mathcal{D}_{j,i}(r,z,s) = \mathcal{D}'_j(s,z)\exp\left(-\bar{m}_j F_{\mathrm{d}_j}\left(\bar{P}_{j,i}r|z\right)\right) = \exp\left(-\bar{m}_j\left(1+F_{\mathrm{d}_j}\left(\bar{P}_{j,i}r|z\right)-\int_0^\infty\frac{f_{\mathrm{d}_j}(y|z)}{1+sP_jy^{-\alpha}}dy\right)\right) \tag{20}$$

and $\zeta_{j,i}(r,s) = \bar{P}_{j,i}^2 r^2 + (sP_j)^{2/\alpha}\zeta'(\alpha)$.

Proof.

$$\mathcal{L}_{P|\Phi_{\mathrm{p}_k},\forall k\in\mathcal{K}_1}(s) \overset{(a)}{=} \mathcal{L}_{P|\mathcal{S}_i,\Phi_{\mathrm{p}_k},\forall k\in\mathcal{K}_1}(s) \overset{(b)}{=} \sum_{i=1}^{K}\mathbb{P}[\mathcal{S}_i|\Phi_{\mathrm{p}_k},\forall k\in\mathcal{K}_1]\mathcal{L}_{P|\mathcal{S}_i,\Phi_{\mathrm{p}_k},\forall k\in\mathcal{K}_1}(s) = \sum_{i=1}^{K}\mathcal{M}_i(s)$$

Here (a) follows from the fact that the conditional laplace transform of total power P is independent of $\mathcal{S}_i$ and (b) follows from writing "1" as the sum of all i^{th} tier association probability conditioned over the parent PPP $\Phi_{\mathrm{p}_k}\forall k\in\mathcal{K}_1$ Now we write $\mathcal{M}_i(s)$ as:

$$\begin{aligned}
&\mathcal{M}_i(s) = \mathbb{P}[\mathcal{S}_i|\Phi_{\mathrm{p}_k},\forall k\in\mathcal{K}_1]\mathcal{L}_{P|\mathcal{S}_i,\Phi_{\mathrm{p}_k},\forall k\in\mathcal{K}_1}(s)\\
&\overset{(a)}{=} \mathbb{P}[\mathcal{S}_i|\Phi_{\mathrm{p}_k},\forall k\in\mathcal{K}_1]\prod_{j_1\in\mathcal{K}_1}\prod_{\mathbf{z}\in\Phi_{\mathrm{p}_{j_1}}}\mathcal{D}'_{j_1}(s,z)\prod_{j_2\in\mathcal{K}_2}\exp\left(-\pi\lambda_{j_2}(sP_{j_2})^{2/\alpha}\zeta'(\alpha)\right)\\
&= \mathbb{P}[\mathcal{S}_i|\Phi_{\mathrm{p}_k},\forall k\in\mathcal{K}_1]\int_0^\infty\prod_{j_1\in\mathcal{K}_1}\prod_{\mathbf{z}\in\Phi_{\mathrm{p}_{j_1}}}\mathcal{D}'_{j_1}(s,z)\prod_{j_2\in\mathcal{K}_2}\exp\left(-\pi\lambda_{j_2}(sP_{j_2})^{2/\alpha}\zeta'(\alpha)\right)f_{s_i}(r|\mathcal{S}_i,\Phi_{\mathrm{p}_k},\forall k\in\mathcal{K}_1)\mathrm{d}r
\end{aligned}$$

Here (a) follows from Lemma 3.7. The last step follows from writing "1" as integration of $f_{s_i}(.)$ over r taking the terms independent of r inside the integration.

If $i\in\mathcal{K}_1$, using Equation 11a from Lemma 3.3, we get $\mathcal{M}_i(s) =$

$$\begin{aligned}
&\bar{m}_i\int_0^\infty\prod_{j_1\in\mathcal{K}_1\setminus\{i\}}\prod_{\mathbf{z}\in\Phi_{\mathrm{p}_{j_1}}}\mathcal{D}'_{j_1}(s,z)\exp\left(-\bar{m}_{j_1}F_{\mathrm{d}_{j_1}}\left(\bar{P}_{j_1,i}r|z\right)\right)\times\\
&\prod_{j_2\in\mathcal{K}_2}\exp\left(-\pi\lambda_{j_2}\left(\bar{P}_{j_2,i}^2r^2+(sP_{j_2})^{2/\alpha}\zeta'(\alpha)\right)\right)\left(\sum_{\mathbf{z}\in\Phi_{\mathbf{p}_i}}f_{\mathrm{d}_i}(r|z)\prod_{\mathbf{z}\in\Phi_{\mathbf{p}_i}}\mathcal{D}'_i(s,z)\exp\left(-\bar{m}_iF_{\mathrm{d}_i}\left(\bar{P}_{i,i}r|z\right)\right)\right)\mathrm{d}r
\end{aligned}$$

If $i\in\mathcal{K}_2$, using Equation 11b from Lemma 3.3, we get $\mathcal{M}_i(s) =$

$$2\pi\lambda_i\int_0^\infty\prod_{j_1\in\mathcal{K}_1}\prod_{\mathbf{z}\in\Phi_{\mathrm{p}_{j_1}}}\mathcal{D}'_{j_1}(s,z)\exp\left(-\bar{m}_{j_1}F_{\mathrm{d}_{j_1}}\left(\bar{P}_{j_1,i}r|z\right)\right)\prod_{j_2\in\mathcal{K}_2}\exp\left(-\pi\lambda_{j_2}\left(\bar{P}_{j_2,i}^2r^2+(sP_{j_2})^{2/\alpha}\zeta'(\alpha)\right)\right)r\mathrm{d}r$$

□

Lemma 3.9. (TYPE 1). *The laplace transform of total power P is given as:*

$$\mathcal{L}_P(s) = \sum_{i=1}^{K}\mathcal{N}_i(s) \tag{21}$$

where $\mathcal{N}_i(s) =$

$$\begin{aligned}
2\pi\lambda_{\mathrm{p}_i}\bar{m}_i\int_0^\infty\prod_{j_1\in\mathcal{K}_1}\exp\left(-2\pi\lambda_{\mathrm{p}_{j_1}}\int_0^\infty(1-\mathcal{D}_{j_1,i}(r,z,s))z\mathrm{d}z\right)\times\\
\prod_{j_2\in\mathcal{K}_2}\exp\left(-\pi\lambda_{j_2}\zeta_{j_2,i}(r,s)\right)\left(\int_0^\infty f_{\mathrm{d}_i}(r|z)\mathcal{D}_{i,i}(r,z,s)z\mathrm{d}z\right)\mathrm{d}r, \textit{when } i\in\mathcal{K}_1
\end{aligned} \tag{22a}$$

$$2\pi\lambda_i\int_0^\infty\prod_{j_1\in\mathcal{K}_1}\exp\left(-2\pi\lambda_{\mathrm{p}_{j_1}}\int_0^\infty(1-\mathcal{D}_{j_1,i}(r,z,s))z\mathrm{d}z\right)\prod_{j_2\in\mathcal{K}_2}\exp\left(-\pi\lambda_{j_2}\zeta_{j_2,i}(r,s)\right)r\mathrm{d}r, \textit{when } i\in\mathcal{K}_2 \tag{22b}$$

Proof. We decondition the conditional laplace transform of total power given in Lemma 3.8 over $\Phi_{\mathrm{p}_k}, \forall k \in \mathcal{K}_1$.

$$\mathcal{L}_P(s) = \mathbb{E}\left[\mathcal{L}_{P|\Phi_{\mathrm{p}_k},\forall k\in\mathcal{K}_1}(s)\right] = \mathbb{E}\left[\sum_{i=1}^{K}\mathcal{M}_i(s)\right] = \sum_{i=1}^{K}\mathbb{E}\left[\mathcal{M}_i(s)\right] = \sum_{i=1}^{K}\mathcal{N}_i(s)$$

If $i \in \mathcal{K}_1$, using Equation 19a from Lemma 3.8, we get $\mathcal{N}_i(s) =$

$$\bar{m}_i \int_0^\infty \prod_{j_1\in\mathcal{K}_1\setminus\{i\}} \mathbb{E}_{\Phi_{\mathrm{p}_{j_1}}}\left[\prod_{\mathbf{z}\in\Phi_{\mathrm{p}_{j_1}}} \mathcal{D}_{j_1,i}(r,z,s)\right] \prod_{j_2\in\mathcal{K}_2} \exp\left(-\pi\lambda_{j_2}\zeta_{j_2,i}(r,s)\right) \times$$
$$\mathbb{E}_{\Phi_{\mathrm{p}_i}}\left[\sum_{\mathbf{z}\in\Phi_{\mathbf{p}_i}} f_{\mathrm{d}_i}(r|z) \prod_{\mathbf{z}\in\Phi_{\mathbf{p}_i}} \mathcal{D}_{i,i}(r,z,s)\right] \mathrm{d}r$$

This step follows from the fact that all Φ_j-s are assumed to be independent for $\forall j \in \mathcal{K}$. The final expression (**??**) is obtained by substituting the PGFL of $\Phi_{\mathrm{p}_k} \forall k \in \mathcal{K}_1 \setminus \{i\}$ (13) and the sum-product functional of Φ_{p_i} (12) followed by algebraic simplifications.

If $i \in \mathcal{K}_2$, using Equation 19b from Lemma 3.8, we get $\mathcal{N}_i(s) =$

$$2\pi\lambda_i \int_0^\infty \prod_{j_1\in\mathcal{K}_1} \mathbb{E}_{\Phi_{\mathrm{p}_{j_1}}}\left[\prod_{\mathbf{z}\in\Phi_{\mathrm{p}_{j_1}}} \mathcal{D}_{j_1,i}(r,z,s)\right] \prod_{j_2\in\mathcal{K}_2} \exp\left(-\pi\lambda_{j_2}\zeta_{j_2,i}(r,s)\right) r\mathrm{d}r$$

The final expression is obtained by substituting the PGFL of $\Phi_{\mathrm{p}_k} \forall k \in \mathcal{K}_1$ (13). ☐

Lemma 3.10. `(TYPE 2).` *The laplace transform of total power P is given as:*

$$\mathcal{L}_P(s) = \sum_{i=1}^{K}\mathcal{N}_i(s) \tag{23}$$

where $\mathcal{N}_i(s) =$

$$2\pi\lambda_{\mathrm{p}_i}\bar{m}_i \int_0^\infty \prod_{j_1\in\mathcal{K}_1} \exp\left(-2\pi\lambda_{\mathrm{p}_{j_1}} \int_0^\infty (1-\mathcal{D}_{j_1,i}(r,z,s))z\mathrm{d}z\right) \int_0^\infty \mathcal{D}_{q,i}(r,z_0,s) f_{\mathrm{d}_u}(z_0|0)\mathrm{d}z_0 \times$$
$$\prod_{j_2\in\mathcal{K}_2} \exp\left(-\pi\lambda_{j_2}\zeta_{j_2,i}(r,s)\right) \left(\int_0^\infty f_{\mathrm{d}_i}(r|z)\mathcal{D}_{i,i}(r,z,s)z\mathrm{d}z\right) \mathrm{d}r, \textit{when } i \in \mathcal{K}_1 \setminus \{q\} \tag{24a}$$

$$\bar{m}_q \int_0^\infty \prod_{j_1\in\mathcal{K}_1} \exp\left(-2\pi\lambda_{\mathrm{p}_{j_1}} \int_0^\infty (1-\mathcal{D}_{j_1,i}(r,z,s))z\mathrm{d}z\right) \times$$
$$\prod_{j_2\in\mathcal{K}_2} \exp\left(-\pi\lambda_{j_2}\zeta_{j_2,i}(r,s)\right) \left(\int_0^\infty f_{\mathrm{d}_q}(r|z_0)\mathcal{D}_{q,q}(r,z_0,s) f_{\mathrm{d}_u}(z_0|0)\mathrm{d}z_0 + \right.$$
$$\left. 2\pi\lambda_{\mathrm{p}_q} \int_0^\infty f_{\mathrm{d}_q}(r|z)\mathcal{D}_{q,q}(r,z,s)z\mathrm{d}z \int_0^\infty \mathcal{D}_{q,q}(r,z_0,s) f_{\mathrm{d}_u}(z_0|0)\mathrm{d}z_0\right) \mathrm{d}r, \textit{when } i = q \tag{24b}$$

$$2\pi\lambda_{\mathrm{p}_i} \int_0^\infty \prod_{j_1\in\mathcal{K}_1} \exp\left(-2\pi\lambda_{\mathrm{p}_{j_1}} \int_0^\infty (1-\mathcal{D}_{j_1,i}(r,z,s))z\mathrm{d}z\right) \times$$
$$\int_0^\infty \mathcal{D}_{q,i}(r,z_0,s) f_{\mathrm{d}_u}(z_0|0)\mathrm{d}z_0 \prod_{j_2\in\mathcal{K}_2} \exp\left(-\pi\lambda_{j_2}\zeta_{j_2,i}(r,s)\right) \mathrm{d}r, \textit{when } i \in \mathcal{K}_2 \tag{24c}$$

Proof. We decondition the conditional laplace transform of total power given in Lemma 3.8 over $\Phi_{\mathrm{p}_k}, \forall k \in \mathcal{K}_1$.

$$\mathcal{L}_P(s) = \mathbb{E}\left[\mathcal{L}_{P|\Phi_{\mathrm{p}_k},\forall k\in\mathcal{K}_1}(s)\right] = \mathbb{E}\left[\sum_{i=1}^{K}\mathcal{M}_i(s)\right] = \sum_{i=1}^{K}\mathbb{E}\left[\mathcal{M}_i(s)\right] = \sum_{i=1}^{K}\mathcal{N}_i(s)$$

If $i \in \mathcal{K}_1 \setminus \{q\}$, using Equation 19a from Lemma 3.8, we get $\mathcal{N}_i(s) =$

$$\bar{m}_i \int_0^\infty \prod_{j_1\in\mathcal{K}_1\setminus\{i,q\}} \mathbb{E}_{\Phi_{\mathrm{p}_{j_1}}}\left[\prod_{\mathbf{z}\in\Phi_{\mathrm{p}_{j_1}}}\mathcal{D}_{j_1,i}(r,z,s)\right]\mathbb{E}_{\Phi_{\mathrm{p}_q}}\left[\prod_{\mathbf{z}\in\Phi_{\mathrm{p}_q}}\mathcal{D}_{q,i}(r,z,s)\right]\prod_{j_2\in\mathcal{K}_2}\exp\left(-\pi\lambda_{j_2}\zeta_{j_2,i}(r,s)\right)\times$$
$$\mathbb{E}_{\Phi_{\mathrm{p}_i}}\left[\sum_{\mathbf{z}\in\Phi_{\mathbf{p}_i}} f_{\mathrm{d}_i}(r|z)\prod_{\mathbf{z}\in\Phi_{\mathbf{p}_i}}\mathcal{D}_{i,i}(r,z,s)\right]\mathrm{d}r$$

The final result is obtained by using the results in (12), (13) and (14).

If $i = q$, using Equation 19a from Lemma 3.8, we get $\mathcal{N}_i(s) =$

$$\bar{m}_q \int_0^\infty \prod_{j_1\in\mathcal{K}_1\setminus\{q\}} \mathbb{E}_{\Phi_{\mathrm{p}_{j_1}}}\left[\prod_{\mathbf{z}\in\Phi_{\mathrm{p}_{j_1}}}\mathcal{D}_{j_1,q}(r,z,s)\right]\prod_{j_2\in\mathcal{K}_2}\exp\left(-\pi\lambda_{j_2}\zeta_{j_2,q}(r,s)\right)\times$$
$$\mathbb{E}_{\Phi_{\mathrm{p}_q}}\left[\sum_{\mathbf{z}\in\Phi_{\mathbf{p}_q}} f_{\mathrm{d}_q}(r|z)\prod_{\mathbf{z}\in\Phi_{\mathbf{p}_q}}\mathcal{D}_{q,q}(r,z,s)\right]\mathrm{d}r$$

The final result can be obtained by using the results in (13) and (15).

If $i \in \mathcal{K}_2$, using Equation 19b from Lemma 3.8, we get $\mathcal{N}_i(s) =$

$$2\pi\lambda_i \int_0^\infty \prod_{j_1\in\mathcal{K}_1\setminus\{q\}} \mathbb{E}_{\Phi_{\mathrm{p}_{j_1}}}\left[\prod_{\mathbf{z}\in\Phi_{\mathrm{p}_{j_1}}}\mathcal{D}_{j_1,i}(r,z,s)\right]\mathbb{E}_{\Phi_{\mathrm{p}_q}}\left[\prod_{\mathbf{z}\in\Phi_{\mathrm{p}_q}}\mathcal{D}_{q,i}(r,z,s)\right]\prod_{j_2\in\mathcal{K}_2}\exp\left(-\pi\lambda_{j_2}\zeta_{j_2,i}(r,s)\right)r\mathrm{d}r$$

The final result is obtained by using the results in (13) and (14). □

3.4 Laplace transform of interference

Lemma 3.11. *Conditioned on $\mathcal{S}_i$ and $\Phi_{\mathrm{p}_k}\forall k \in \mathcal{K}_1$, the laplace transform of interference I is given as:*

$$\mathcal{L}_{I|\mathcal{S}_i,\Phi_{\mathrm{p}_k}\forall k\in\mathcal{K}_1}(s) = \int_0^\infty \prod_{j_1\in\mathcal{K}_1}\prod_{\mathbf{z}\in\Phi_{\mathrm{p}_{j1}}}\exp\left(-\int_{\bar{P}_{j_1,i}r}^\infty \bar{m}_{j_1} f_{\mathrm{d}_{j_1}}(y|z)\left(1-\frac{1}{1+sP_{j_1}y^{-\alpha}}\right)dy\right)\times$$
$$\prod_{j_2\in\mathcal{K}_2}\exp\left(-2\pi\lambda_{j_2}\int_{\bar{P}_{j_2,i}r}^\infty\left(1-\frac{1}{1+sP_{j_2}y^{-\alpha}}\right)ydy\right)f_{s_i}(r|\mathcal{S}_i,\Phi_{\mathrm{p}_k}\forall k\in\mathcal{K}_1)\mathrm{d}r$$

Proof. The conditional laplace transform of interference I given $\Phi_{\mathrm{p}_k}\forall k \in \mathcal{K}_1$ and the association to the i^{th} tier is given by:

$$\mathcal{L}_{I|\mathcal{S}_i,\Phi_{\mathrm{p}_k}\forall k\in\mathcal{K}_1}(s) = \mathbb{E}[\exp(-sI)|\mathcal{S}_i,\Phi_{\mathrm{p}_k},\forall k\in\mathcal{K}_1]$$
$$= \mathbb{E}\left[\exp\left(-s\sum_{j\in\mathcal{K}}\sum_{\mathbf{y}\in\Phi_j\setminus\{\mathbf{x}^*\}}P_j h_{\mathbf{y}}\|\mathbf{y}\|^{-\alpha}\right)\middle|\mathcal{S}_i,\Phi_{\mathrm{p}_k},\forall k\in\mathcal{K}_1\right]$$
$$= \mathbb{E}\left[\prod_{j\in\mathcal{K}}\prod_{\mathbf{y}\in\Phi_j\setminus\{\mathbf{x}^*\}}\exp\left(-sP_j h_{\mathbf{y}}\|\mathbf{y}\|^{-\alpha}\right)\middle|\mathcal{S}_i,\Phi_{\mathrm{p}_k},\forall k\in\mathcal{K}_1\right]$$

$$\overset{(a)}{=} \mathbb{E}\left[\prod_{j\in\mathcal{K}}\prod_{\mathbf{y}\in\Phi_j\setminus\{\mathbf{x}^*\}}\mathbb{E}_{h_{\mathbf{y}}}\left[\exp\left(-sP_jh_{\mathbf{y}}\|\mathbf{y}\|^{-\alpha}\right)\right]\middle|\mathcal{S}_i,\Phi_{\mathrm{p}_k},\forall k\in\mathcal{K}_1\right]$$

$$\overset{(b)}{=} \mathbb{E}\left[\prod_{j\in\mathcal{K}}\prod_{\mathbf{y}\in\Phi_j\cap b(0,\bar{P}_{j,i}\|\mathbf{x}^*\|)^c}\frac{1}{1+sP_j\|\mathbf{y}\|^{-\alpha}}\middle|\mathcal{S}_i,\Phi_{\mathrm{p}_k},\forall k\in\mathcal{K}_1\right]$$

$$=\int_0^\infty\prod_{j_1\in\mathcal{K}_1}\mathbb{E}_{\Phi_{j_1}}\left[\prod_{\mathbf{y}\in\Phi_{j_1}\cap b(0,\bar{P}_{j_1,i}r)^c}\frac{1}{1+sP_{j_1}\|\mathbf{y}\|^{-\alpha}}\middle|\Phi_{\mathrm{p}_{j_1}}\right]\times$$

$$\prod_{j_2\in\mathcal{K}_2}\mathbb{E}_{\Phi_{j_2}}\left[\prod_{\mathbf{y}\in\Phi_{j_2}\cap b(0,\bar{P}_{j_2,i}r)^c}\frac{1}{1+sP_{j_2}\|\mathbf{y}\|^{-\alpha}}\right]f_{s_i}(r|\mathcal{S}_i,\Phi_{\mathrm{p}_k}\forall k\in\mathcal{K}_1)\mathrm{d}r$$

$$=\int_0^\infty\prod_{j_1\in\mathcal{K}_1}\prod_{\mathbf{z}\in\Phi_{\mathrm{p}_{j_1}}}\exp\left(-\int_{\bar{P}_{j_1,i}r}^\infty\bar{m}_{j_1}f_{\mathrm{d}_{j_1}}(y|z)\left(1-\frac{1}{1+sP_{j_1}y^{-\alpha}}\right)dy\right)\times$$

$$\prod_{j_2\in\mathcal{K}_2}\exp\left(-2\pi\lambda_{j_2}\int_{\bar{P}_{j_2,i}r}^\infty\left(1-\frac{1}{1+sP_{j_2}y^{-\alpha}}\right)ydy\right)f_{s_i}(r|\mathcal{S}_i,\Phi_{\mathrm{p}_k}\forall k\in\mathcal{K}_1)\mathrm{d}r$$

where (a) follows from the fact that all links are subjected to i.i.d fading, (b) follows from the fact that $h_y\sim\exp(1)$ and $\mathbf{y}\in\Phi_j\setminus\{\mathbf{x}^*\}$ implies that there are no points inside the exclusion disc $b(0,\bar{P}_{j,i}\|\mathbf{x}^*\|)$. $\square$

Lemma 3.12. *Conditioned on $\Phi_{\mathrm{p}_k},\forall k\in\mathcal{K}_1$, the laplace transform of interference I is given as:*

$$\mathcal{L}_{I|\Phi_{\mathrm{p}_k},\forall k\in\mathcal{K}_1}(s)=\sum_{i=1}^{K}\mathcal{I}_i(s) \tag{25}$$

where $\mathcal{I}_i(s)=$

$$\bar{m}_i\int_0^\infty\prod_{j_1\in\mathcal{K}_1\setminus\{i\}}\prod_{\mathbf{z}\in\Phi_{\mathrm{p}_{j_1}}}\mathcal{C}_{j_1,i}(r,s,z)\prod_{j_2\in\mathcal{K}_2}\exp(-\pi r^2\lambda_{j_2}\bar{P}_{j_2,i}^2\rho_i(s,r,\alpha))\left(\sum_{\mathbf{z}\in\Phi_{\mathbf{p}_i}}f_{\mathrm{d}_i}(r|z)\prod_{\mathbf{z}\in\Phi_{\mathrm{p}_i}}\mathcal{C}_{i,i}(r,s,z)\right)\mathrm{d}r,\ \textit{when } i\in\mathcal{K}_1 \tag{26a}$$

$$2\pi\lambda_i\int_0^\infty\prod_{j_1\in\mathcal{K}_1}\prod_{\mathbf{z}\in\Phi_{\mathrm{p}_{j_1}}}\mathcal{C}_{j_1,i}(r,s,z)\prod_{j_2\in\mathcal{K}_2}\exp(-\pi r^2\lambda_{j_2}\bar{P}_{j_2,i}^2\rho_i(s,r,\alpha))r\mathrm{d}r\ \textit{when } i\in\mathcal{K}_2 \tag{26b}$$

Here

$$\mathcal{C}_{i,k}(r,s,z)=\exp\left(-\bar{m}_i\left(1-\int_{\bar{P}_{i,k}r}^\infty\frac{f_{\mathrm{d}_i}(y|z)}{1+sP_iy^{-\alpha}}dy\right)\right) \tag{27}$$

and $\rho_i(s,r,\alpha)=1+_2\mathcal{F}_1\left[1,1-\frac{2}{\alpha};2-\frac{2}{\alpha};-sP_ir^{-\alpha}\right]\left(\frac{2sP_i}{(\alpha-2)r^\alpha}\right)$ where ${}_2\mathcal{F}_1$ is the Gauss Hypergeometric Function.

Proof.

$$\mathcal{L}_{I|\Phi_{\mathrm{p}_k},\forall k\in\mathcal{K}_1}(s)\overset{(a)}{=}\sum_{i=1}^{K}\mathbb{P}[\mathcal{S}_i|\Phi_{\mathrm{p}_k},\forall k\in\mathcal{K}_1]\mathcal{L}_{I|\mathcal{S}_i,\Phi_{\mathrm{p}_k},\forall k\in\mathcal{K}_1}(s)=\sum_{i=1}^{K}\mathcal{I}_i(s)$$

where (a) follows by deconditioning the laplace transform from lemma3.11 over the association event $\mathcal{S}_i$ given the parent PPP $\Phi_{\mathrm{p}_k}\forall k\in\mathcal{K}_1$. We express $\mathcal{I}_i(s)$ as:

$\mathcal{I}_i(s)=\mathbb{P}[\mathcal{S}_i|\Phi_{\mathrm{p}_k},\forall k\in\mathcal{K}_1]\mathcal{L}_{I|\mathcal{S}_i,\Phi_{\mathrm{p}_k},\forall k\in\mathcal{K}_1}(s)$

$$= \mathbb{P}[\mathcal{S}_i|\Phi_{\mathrm{p}_k}, \forall k \in \mathcal{K}_1] \int_0^\infty \prod_{j_1 \in \mathcal{K}_1} \prod_{\mathbf{z} \in \Phi_{\mathrm{p}_{j_1}}} \exp\left(-\int_{\bar{P}_{j_1,i} r}^\infty \bar{m}_{j_1} f_{\mathrm{d}_{j_1}}(y|z) \left(1 - \frac{1}{1 + sP_{j_1} y^{-\alpha}}\right) dy \right) \times$$

$$\prod_{j_2 \in \mathcal{K}_2} \exp\left(-2\pi\lambda_{j_2} \int_{\bar{P}_{j_2,i} r}^\infty \left(1 - \frac{1}{1 + sP_{j_2} y^{-\alpha}}\right) y dy \right) f_{s_i}(r|\mathcal{S}_i, \Phi_{\mathrm{p}_k} \forall k \in \mathcal{K}_1) \mathrm{d}r$$

When $i \in \mathcal{K}_1$, by substituting $f_{s_i}(r|\mathcal{S}_i, \Phi_{\mathrm{p}_k} \forall k \in \mathcal{K}_1)$ from 11a, we get:

$$\mathcal{I}_i(s) = \bar{m}_i \int_0^\infty \prod_{j_1 \in \mathcal{K}_1} \prod_{\mathbf{z} \in \Phi_{\mathrm{p}_{j_1}}} \exp\left(-\int_{\bar{P}_{j_1,i} r}^\infty \bar{m}_{j_1} f_{\mathrm{d}_{j_1}}(y|z) \left(1 - \frac{1}{1 + sP_{j_1} y^{-\alpha}}\right) dy \right) \times$$

$$\prod_{j_2 \in \mathcal{K}_2} \exp\left(-2\pi\lambda_{j_2} \int_{\bar{P}_{j_2,i} r}^\infty \left(1 - \frac{1}{1 + sP_{j_2} y^{-\alpha}}\right) y dy \right) \times$$

$$\sum_{\mathbf{z} \in \Phi_{\mathbf{p}_i}} f_{\mathrm{d}_i}(r|z) \prod_{j_1 \in \mathcal{K}_1} \prod_{\mathbf{z} \in \Phi_{\mathbf{p}_{j_1}}} \exp\left(-\bar{m}_{j_1} F_{\mathrm{d}_{j_1}}\left(\bar{P}_{j_1,i} r|z\right)\right) \prod_{j_2 \in \mathcal{K}_2} \exp\left(-\pi\lambda_{j_2} \bar{P}_{j_2,i}^2 r^2 \right) \mathrm{d}r$$

$$= \bar{m}_i \int_0^\infty \prod_{j_1 \in \mathcal{K}_1} \prod_{\mathbf{z} \in \Phi_{\mathrm{p}_{j_1}}} \exp\left(-\bar{m}_{j_1} \int_0^{\bar{P}_{j_1,i} r} f_{\mathrm{d}_{j_1}}(y|z) dy - \bar{m}_{j_1} \int_{\bar{P}_{j_1,i} r}^\infty f_{\mathrm{d}_{j_1}}(y|z) \left(1 - \frac{1}{1 + sP_{j_1} y^{-\alpha}}\right) dy \right) \times$$

$$\exp\left(-2\pi \sum_{j_2 \in \mathcal{K}_2} \lambda_{j_2} \int_{\bar{P}_{j_2,i} r}^\infty \left(1 - \frac{1}{1 + sP_{j_2} y^{-\alpha}}\right) y dy - \pi \sum_{j_2 \in \mathcal{K}_2} \lambda_{j_2} \bar{P}_{j_2,i}^2 r^2 \right) \sum_{\mathbf{z} \in \Phi_{\mathbf{p}_i}} f_{\mathrm{d}_i}(r|z) \mathrm{d}r$$

$$\overset{(a)}{=} \bar{m}_i \int_0^\infty \prod_{j_1 \in \mathcal{K}_1} \prod_{\mathbf{z} \in \Phi_{\mathrm{p}_{j_1}}} \exp\left(-\bar{m}_{j_1} \left(1 - \int_{\bar{P}_{j_1,i} r}^\infty \frac{f_{\mathrm{d}_{j_1}}(y|z)}{1 + sP_{j_1} y^{-\alpha}} dy \right)\right) \times$$

$$\exp\left(-2\pi \sum_{j_2 \in \mathcal{K}_2} \lambda_{j_2} \frac{(sP_{j_2})^{\frac{2}{\alpha}}}{2} \int_{(sP_i r^{-\alpha})^{\frac{-2}{\alpha}}}^\infty \frac{1}{1 + t^{\frac{\alpha}{2}}} dt - \pi \sum_{j_2 \in \mathcal{K}_2} \lambda_{j_2} \bar{P}_{j_2,i}^2 r^2 \right) \sum_{\mathbf{z} \in \Phi_{\mathbf{p}_i}} f_{\mathrm{d}_i}(r|z) \mathrm{d}r$$

$$= \bar{m}_i \int_0^\infty \prod_{j_1 \in \mathcal{K}_1 \setminus \{i\}} \prod_{\mathbf{z} \in \Phi_{\mathrm{p}_{j_1}}} \mathcal{C}_{j_1,i}(r, s, z) \prod_{j_2 \in \mathcal{K}_2} \exp(-\pi r^2 \lambda_{j_2} \bar{P}_{j_2,i}^2 \rho_i(s, r, \alpha)) \times$$

$$\left(\sum_{\mathbf{z} \in \Phi_{\mathbf{p}_i}} f_{\mathrm{d}_i}(r|z) \prod_{\mathbf{z} \in \Phi_{\mathrm{p}_i}} \mathcal{C}_{i,i}(r, s, z) \right) \mathrm{d}r$$

where (a) follows from substituting $sP_{j_2} y^{-\alpha} = t^{\frac{-\alpha}{2}}$

When $i \in \mathcal{K}_2$, by substituting $f_{s_i}(r|\mathcal{S}_i, \Phi_{\mathrm{p}_k} \forall k \in \mathcal{K}_1)$ from 11b, we get:

$$\mathcal{I}_i(s) = 2\pi\lambda_i \int_0^\infty \prod_{j_1 \in \mathcal{K}_1} \prod_{\mathbf{z} \in \Phi_{\mathrm{p}_{j_1}}} \exp\left(-\int_{\bar{P}_{j_1,i} r}^\infty \bar{m}_{j_1} f_{\mathrm{d}_{j_1}}(y|z) \left(1 - \frac{1}{1 + sP_{j_1} y^{-\alpha}}\right) dy \right) \times$$

$$\prod_{j_2 \in \mathcal{K}_2} \exp\left(-2\pi\lambda_{j_2} \int_{\bar{P}_{j_2,i} r}^\infty \left(1 - \frac{1}{1 + sP_{j_2} y^{-\alpha}}\right) y dy \right) \times$$

$$\prod_{j_1 \in \mathcal{K}_1} \prod_{\mathbf{z} \in \Phi_{\mathbf{p}_{j_1}}} \exp\left(-\bar{m}_{j_1} F_{\mathrm{d}_{j_1}}\left(\bar{P}_{j_1,i} r|z\right)\right) \prod_{j_2 \in \mathcal{K}_2} \exp\left(-\pi\lambda_{j_2} \bar{P}_{j_2,i}^2 r^2 \right) r\mathrm{d}r$$

$$= 2\pi\lambda_i \int_0^\infty \prod_{j_1 \in \mathcal{K}_1} \prod_{\mathbf{z} \in \Phi_{\mathrm{p}_{j_1}}} \exp\left(-\bar{m}_{j_1} \left(1 - \int_{\bar{P}_{j_1,i} r}^\infty \frac{f_{\mathrm{d}_{j_1}}(y|z)}{1 + sP_{j_1} y^{-\alpha}} dy \right)\right) \times$$

$$\exp\left(-2\pi \sum_{j_2 \in \mathcal{K}_2} \lambda_{j_2} \frac{(sP_{j_2})^{\frac{2}{\alpha}}}{2} \int_{(sP_i r^{-\alpha})^{\frac{-2}{\alpha}}}^\infty \frac{1}{1 + t^{\frac{\alpha}{2}}} dt - \pi \sum_{j_2 \in \mathcal{K}_2} \lambda_{j_2} \bar{P}_{j_2,i}^2 r^2 \right) r\mathrm{d}r$$

$$= 2\pi\lambda_i \int_0^\infty \prod_{j_1\in\mathcal{K}_1} \prod_{\mathbf{z}\in\Phi_{\mathrm{p}_{j_1}}} \mathcal{C}_{j_1,i}(r,s,z) \prod_{j_2\in\mathcal{K}_2} \exp(-\pi r^2 \lambda_{j_2} \bar{P}^2_{j_2,i}\rho_i(s,r,\alpha)) r\mathrm{d}r$$

□

Lemma 3.13. (TYPE 1). *The laplace transform of interference I is given as:*

$$\mathcal{L}_I(s) = \sum_{i=1}^{K} \mathcal{J}_i(s) \tag{28}$$

where $\mathcal{J}_i(s) =$

$$2\pi\lambda_{\mathrm{p}_i}\bar{m}_i \int_0^\infty \prod_{j_1\in\mathcal{K}_1} \exp\left(-2\pi\lambda_{\mathrm{p}_{j_1}} \int_0^\infty (1-\mathcal{C}_{j_1,i}(r,s,z))zdz\right) \times \prod_{j_2\in\mathcal{K}_2} \exp(-\pi r^2\lambda_{j_2}\bar{P}^2_{j_2,i}\rho_i(s,r,\alpha)) \left(\int_0^\infty f_{\mathrm{d}_i}(r|z)\mathcal{C}_{i,i}(r,s,z)zdz\right)\mathrm{d}r \quad \text{,when } i\in\mathcal{K}_1 \tag{29a}$$

$$2\pi\lambda_i \int_0^\infty \prod_{j_1\in\mathcal{K}_1} \exp\left(-2\pi\lambda_{\mathrm{p}_{j_1}} \int_0^\infty (1-\mathcal{C}_{j_1,i}(r,s,z))zdz\right) \prod_{j_2\in\mathcal{K}_2} \exp(-\pi r^2\lambda_{j_2}\bar{P}^2_{j_2,i}\rho_i(s,r,\alpha)) r\mathrm{d}r \text{ ,when } i\in\mathcal{K}_2 \tag{29b}$$

Proof. We decondition the conditional laplace transform of interference given in Lemma 3.12 over $\Phi_{\mathrm{p}_k}, \forall k\in\mathcal{K}_1$.

$$\mathcal{L}_I(s) = \mathbb{E}\left[\mathcal{L}_{I|\Phi_{\mathrm{p}_k},\forall k\in\mathcal{K}_1}(s)\right] = \mathbb{E}\left[\sum_{i=1}^{K}\mathcal{I}_i(s)\right] = \sum_{i=1}^{K}\mathbb{E}\left[\mathcal{I}_i(s)\right] = \sum_{i=1}^{K}\mathcal{J}_i(s)$$

If $i\in\mathcal{K}_1$, using Equation 26a from Lemma 3.12, we get $\mathcal{J}_i(s) =$

$$\bar{m}_i \int_0^\infty \prod_{j_1\in\mathcal{K}_1\setminus\{i\}} \mathbb{E}_{\Phi_{\mathrm{p}_{j_1}}}\left[\prod_{\mathbf{z}\in\Phi_{\mathrm{p}_{j_1}}} \mathcal{C}_{j_1,i}(r,s,z)\right] \prod_{j_2\in\mathcal{K}_2} \exp(-\pi r^2\lambda_{j_2}\bar{P}^2_{j_2,i}\rho_i(s,r,\alpha)) \times \mathbb{E}_{\Phi_{\mathrm{p}_i}}\left[\sum_{\mathbf{z}\in\Phi_{\mathbf{p}_i}} f_{\mathrm{d}_i}(r|z) \prod_{\mathbf{z}\in\Phi_{\mathrm{p}_i}} \mathcal{C}_{i,i}(r,s,z)\right]\mathrm{d}r$$

This step follows from the fact that all Φ_j-s are assumed to be independent for $\forall j\in\mathcal{K}$. The final expression (29a) is obtained by substituting the PGFL of $\Phi_{\mathrm{p}_k}\forall k\in\mathcal{K}_1\setminus\{i\}$ (13) and the sum-product functional of Φ_{p_i} (12) followed by algebraic simplifications.

If $i\in\mathcal{K}_2$, using Equation 26b from Lemma 3.12, we get $\mathcal{J}_i(s) =$

$$2\pi\lambda_i \int_0^\infty \prod_{j_1\in\mathcal{K}_1} \mathbb{E}_{\Phi_{\mathrm{p}_{j_1}}}\left[\prod_{\mathbf{z}\in\Phi_{\mathrm{p}_{j_1}}} \mathcal{C}_{j_1,i}(r,s,z)\right] \prod_{j_2\in\mathcal{K}_2} \exp(-\pi r^2\lambda_{j_2}\bar{P}^2_{j_2,i}\rho_i(s,r,\alpha)) r\mathrm{d}r$$

The final expression is obtained by substituting the PGFL of $\Phi_{\mathrm{p}_k}\forall k\in\mathcal{K}_1$ (13). □

Lemma 3.14. (TYPE 2). *The laplace transform of interference I is given as:*

$$\mathcal{L}_I(s) = \sum_{i=1}^{K} \mathcal{J}_i(s) \tag{30}$$

where $\mathcal{J}_i(s) =$

$$2\pi\lambda_{\mathrm{p}_i}\bar{m}_i\int_0^\infty\prod_{j_1\in\mathcal{K}_1}\exp\left(-2\pi\lambda_{\mathrm{p}_{j_1}}\int_0^\infty(1-\mathcal{C}_{j_1,i}(r,s,z))zdz\right)\int_0^\infty\mathcal{C}_{q,i}(r,s,z_0))f_{d_u}(z_0|0)dz_0\times$$
$$\prod_{j_2\in\mathcal{K}_2}\exp(-\pi r^2\lambda_{j_2}\bar{P}_{j_2,i}^2\rho_i(s,r,\alpha))\left(\int_0^\infty f_{\mathrm{d}_i}(r|z)\mathcal{C}_{i,i}(r,s,z)zdz\right)\mathrm{d}r\quad,\textit{when } i\in\mathcal{K}_1\setminus\{q\}\tag{31a}$$

$$\bar{m}_q\int_0^\infty\prod_{j_1\in\mathcal{K}_1}\exp\left(-2\pi\lambda_{\mathrm{p}_{j_1}}\int_0^\infty(1-\mathcal{C}_{j_1,q}(r,s,z))zdz\right)\times$$
$$\prod_{j_2\in\mathcal{K}_2}\exp(-\pi r^2\lambda_{j_2}\bar{P}_{j_2,q}^2\rho_q(s,r,\alpha))\left(\int_0^\infty f_{\mathrm{d}_q}(r|z_0)\mathcal{C}_{q,q}(r,s,z_0)f_{\mathrm{d}_u}(z_0|0)\mathrm{d}z_0+\right.$$
$$\left.2\pi\lambda_{\mathrm{p}_q}\int_0^\infty f_{\mathrm{d}_q}(r|z)\mathcal{C}_{q,q}(r,s,z)z\mathrm{d}z\int_0^\infty\mathcal{C}_{q,q}(r,s,z_0)f_{\mathrm{d}_u}(z_0|0)\mathrm{d}z_0\right)\mathrm{d}r\ ,\textit{when } i=q\tag{31b}$$

$$2\pi\lambda_i\int_0^\infty\prod_{j_1\in\mathcal{K}_1}\exp\left(-2\pi\lambda_{\mathrm{p}_{j_1}}\int_0^\infty(1-\mathcal{C}_{j_1,i}(r,s,z))zdz\right)\times$$
$$\int_0^\infty\mathcal{C}_{q,i}(r,s,z_0))f_{d_u}(z_0|0)dz_0\prod_{j_2\in\mathcal{K}_2}\exp(-\pi r^2\lambda_{j_2}\bar{P}_{j_2,i}^2\rho_i(s,r,\alpha))r\mathrm{d}r\ ,\textit{when } i\in\mathcal{K}_2\tag{31c}$$

Proof. We decondition the conditional laplace transform of interference given in Lemma 3.12 over $\Phi_{\mathrm{p}_k},\forall k\in\mathcal{K}_1$.

$$\mathcal{L}_I(s)=\mathbb{E}\left[\mathcal{L}_{I|\Phi_{\mathrm{p}_k},\forall k\in\mathcal{K}_1}(s)\right]=\mathbb{E}\left[\sum_{i=1}^K\mathcal{I}_i(s)\right]=\sum_{i=1}^K\mathbb{E}\left[\mathcal{I}_i(s)\right]=\sum_{i=1}^K\mathcal{J}_i(s)$$

If $i\in\mathcal{K}_1\setminus\{q\}$, using Equation 26a from Lemma 3.12, we get $\mathcal{J}_i(s)=$

$$\bar{m}_i\int_0^\infty\prod_{j_1\in\mathcal{K}_1\setminus\{i,q\}}\mathbb{E}_{\Phi_{\mathrm{p}_{j_1}}}\left[\prod_{\mathbf{z}\in\Phi_{\mathrm{p}_{j_1}}}\mathcal{C}_{j_1,i}(r,s,z)\right]\mathbb{E}_{\Phi_{\mathrm{p}_q}}\left[\prod_{\mathbf{z}\in\Phi_{\mathrm{p}_q}}\mathcal{C}_{q,i}(r,s,z)\right]\times$$
$$\prod_{j_2\in\mathcal{K}_2}\exp(-\pi r^2\lambda_{j_2}\bar{P}_{j_2,i}^2\rho_i(s,r,\alpha))\mathbb{E}_{\Phi_{\mathrm{p}_i}}\left[\sum_{\mathbf{z}\in\Phi_{\mathbf{p}_i}}f_{\mathrm{d}_i}(r|z)\prod_{\mathbf{z}\in\Phi_{\mathrm{p}_i}}\mathcal{C}_{i,i}(r,s,z)\right]\mathrm{d}r$$

The final result can be obtained by using the results in (13), (14) and (12).
If $i=q$, using Equation 26a from Lemma 3.12, we get

$$\bar{m}_q\int_0^\infty\prod_{j_1\in\mathcal{K}_1\setminus\{q\}}\mathbb{E}_{\Phi_{\mathrm{p}_{j_1}}}\left[\prod_{\mathbf{z}\in\Phi_{\mathrm{p}_{j_1}}}\mathcal{C}_{j_1,q}(r,s,z)\right]\prod_{j_2\in\mathcal{K}_2}\exp(-\pi r^2\lambda_{j_2}\bar{P}_{j_2,q}^2\rho_q(s,r,\alpha))\times$$
$$\mathbb{E}_{\Phi_{\mathrm{p}_q}}\left[\sum_{\mathbf{z}\in\Phi_{\mathbf{p}_q}}f_{\mathrm{d}_q}(r|z)\prod_{\mathbf{z}\in\Phi_{\mathrm{p}_q}}\mathcal{C}_{q,q}(r,s,z)\right]\mathrm{d}r$$

The final result can be obtained by using the results in (13) and (15).
If $i\in\mathcal{K}_2$, using Equation 26b from Lemma 3.12, we get $\mathcal{J}_i(s)=$

$$2\pi\lambda_i\int_0^\infty\prod_{j_1\in\mathcal{K}_1\setminus\{q\}}\mathbb{E}_{\Phi_{\mathrm{p}_{j_1}}}\left[\prod_{\mathbf{z}\in\Phi_{\mathrm{p}_{j_1}}}\mathcal{C}_{j_1,i}(r,s,z)\right]\mathbb{E}_{\Phi_{\mathrm{p}_q}}\left[\prod_{\mathbf{z}\in\Phi_{\mathrm{p}_q}}\mathcal{C}_{q,i}(r,s,z)\right]\prod_{j_2\in\mathcal{K}_2}\exp(-\pi r^2\lambda_{j_2}\bar{P}_{j_2,i}^2\rho_i(s,r,\alpha))r\mathrm{d}r$$

The final result can be obtained by using the results in (14) and (13). □

3.5 Average Ergodic Rate in terms of Laplace Transform of Interference and total received power

The key-step in obtaining an expression for the average ergodic rate is expressing it in terms of Laplace transform of total received power and Laplace transform of interference [4]. We state the result in the following lemma.

Lemma 3.15.

$$\mathcal{R} = \mathbb{E}_{\texttt{SINR}}[\ln(1+\texttt{SINR})] = \int_0^\infty \frac{\exp(-sN_0)}{s} \left(\mathcal{L}_I(s) - \mathcal{L}_P(s)\right) \mathrm{d}s \tag{32}$$

Proof. We use the following lemma from [3].

$$\ln(1+x) = \int_0^\infty \frac{1}{t}\left(e^{-t} - e^{-(x+1)t}\right) \mathrm{d}t, \forall x > 0 \tag{33}$$

The average ergodic rate can then be expressed as:

$$\mathcal{R} = \mathbb{E}_{\texttt{SINR}}[\ln(1+\texttt{SINR})] \overset{(a)}{=} \mathbb{E}_{P,I}\left[\int_0^\infty \frac{1}{t}\left(e^{-t} - \exp\left(-\left(\frac{P+N_0}{I+N_0}\right)t\right)\right) \mathrm{d}t\right]$$
$$\overset{(b)}{=} \mathbb{E}_{P,I}\left[\int_0^\infty \frac{\exp(-sN_0)}{s}\left(e^{-sI} - e^{-sP}\right) ds\right]$$
$$= \int_0^\infty \frac{\exp(-sN_0)}{s} \left(\mathcal{L}_I(s) - \mathcal{L}_P(s)\right) \mathrm{d}s$$

where (a) follows by substituting $x = \texttt{SINR}$ in Equation (33) and (b) follows from substituting $t = s(I + N_0)$. □

Theorem 3.16. `(TYPE 1).` *The average ergodic rate for a typical user of Type 1 is given as:*

$$\mathcal{R} = \int_0^\infty \frac{\exp(-sN_0)}{s} \left(\mathcal{L}_I(s) - \mathcal{L}_P(s)\right) \mathrm{d}s = \int_0^\infty \frac{\exp(-sN_0)}{s} \left((\sum_{i=1}^{K} \mathcal{J}_i(s) - \sum_{i=1}^{K} \mathcal{N}_i(s)\right) \mathrm{d}s$$
$$= \int_0^\infty \frac{\exp(-sN_0)}{s} \left(\sum_{i\in\mathcal{K}_1} \underbrace{\mathcal{J}_i(s)}_{(29a)} + \sum_{i\in\mathcal{K}_2} \underbrace{\mathcal{J}_i(s)}_{(29b)} - \sum_{i\in\mathcal{K}_1} \underbrace{\mathcal{N}_i(s)}_{(22a)} - \sum_{i\in\mathcal{K}_2} \underbrace{\mathcal{N}_i(s)}_{(22b)}\right) \mathrm{d}s \tag{34}$$

Theorem 3.17. `(TYPE 2).` *The average ergodic rate for a typical user of Type 2 is given as:*

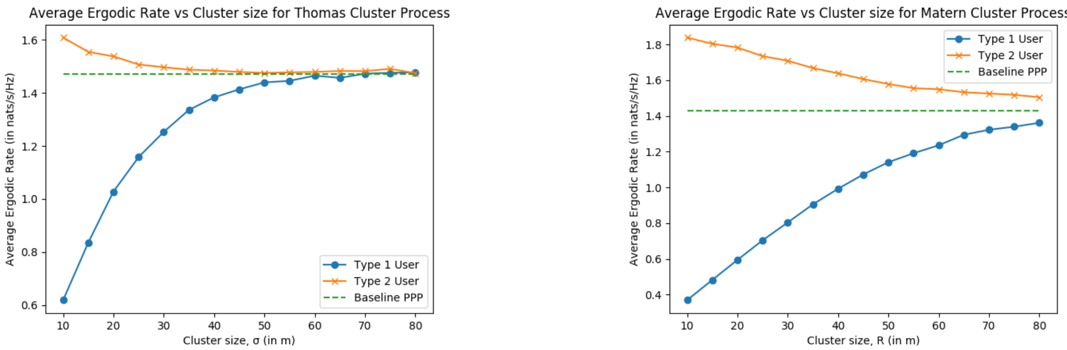
$$\mathcal{R} = \int_0^\infty \frac{\exp(-sN_0)}{s} \left(\mathcal{L}_I(s) - \mathcal{L}_P(s)\right) \mathrm{d}s = \int_0^\infty \frac{\exp(-sN_0)}{s} \left(\sum_{i=1}^{K} \mathcal{J}_i(s) - \sum_{i=1}^{K} \mathcal{N}_i(s)\right) \mathrm{d}s$$
$$= \int_0^\infty \frac{\exp(-sN_0)}{s} \left(\sum_{i\in\mathcal{K}_1\setminus\{q\}} \underbrace{\mathcal{J}_i(s)}_{(31a)} + \underbrace{\mathcal{J}_q(s)}_{(31b)} + \sum_{i\in\mathcal{K}_2} \underbrace{\mathcal{J}_i(s)}_{(31c)} - \sum_{i\in\mathcal{K}_1\setminus\{q\}} \underbrace{\mathcal{N}_i(s)}_{(24a)} - \underbrace{\mathcal{N}_q(s)}_{(24b)} - \sum_{i\in\mathcal{K}_2} \underbrace{\mathcal{N}_i(s)}_{(24c)}\right) \mathrm{d}s \tag{35}$$

4 Results

In this section, we perform the Monte Carlo Simulations of the network to compute the Average Ergodic Rate of Type 1 and Type 2 users. For the purpose of simulation we choose a 2-tier HetNet model where $\mathcal{K}_1 = \{1\}$ and $\mathcal{K}_2 = \{2\}$. We performed simulations for 2 special cases of Poisson Cluster Process viz, Thomas Cluster Process and Matern Cluster Process. We make the assumption of interference-limited network $N_o = 0$. We choose the following parameters. $\alpha = 4, \bar{m} = 10, \lambda_{p_1} = 100\text{km}^{-2}, \lambda_2 = 1\text{km}^{-2}, P1 = 1W, P2 = 1000W$.

4.1 Variation of Cluster Size

In fig 2, we observe the variation of the average ergodic rate as function of the cluster size in terms of σ for TCP and R, cluster radius for MCP. We note different trends in average ergodic rate for Type 1 and Type 2 user as the cluster size increases. For Type 1 users, the average ergodic rate increases with increase in cluster size for both MCP and TCP. For Type 2 users, the average ergodic rate decreases with increase in cluster size for both MCP and TCP.



(a) Thomas Cluster Process

(b) Matern Cluster Process

Figure 2: Average Ergodic rate as a function of cluster size

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