Assignment 8

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Outline

Problem

Let X and Y be independent random variables with common p.d.f $f_X(x) = \beta^{-\alpha}\alpha x^{\alpha-1}$, $0 < X < \beta$, and zero otherwise $(\alpha \geq 1)$. Let $Z = \min(X, Y)$ and $W = \max(X, Y)$. (a) Find the p.d.f. of $X + Y$. (b) Find the joint p.d.f. of Z and W. (c) Show that $\frac{Z}{W}$ and W are independent random variables.

Solution (a)

Let

$$U = X + Y \qquad , 0 < u < 2\beta \tag{1}$$

for $0 < u \leq \beta$, we have

$$F_U(u) = \int_0^u \int_0^{u-x} f_{XY}(x, y)\, dydx \tag{2}$$

which gives

$$f_U(u) = \int_0^u f_{XY}(x, u-x)\, dx \tag{3}$$

$$= \int_0^u f_X(x) f_Y(u-x)\, dx \tag{4}$$

$$= \alpha^2 \beta^{-2\alpha} \int_0^u x^{\alpha-1}(u-x)^{\alpha-1}\, dx \tag{5}$$

$$= \alpha^2 \beta^{-2\alpha} u^{2\alpha-1} \int_0^1 y^{\alpha-1}(1-y)^{\alpha-1}\, dy \tag{6}$$

Solution (a)

Similarly for $\beta < u \leq 2\beta$, we get

$$F_U(u) = 1 - \int_{u-\beta}^{\beta} \int_{u-x}^{\beta} f_{XY}(x, y)\, dydx \tag{7}$$

Hence

$$f_U(u) = \int_{u-\beta}^{\beta} f_{XY}(x, u-x)\, dx \tag{8}$$

$$= \alpha^2 \beta^{-2\alpha} \int_{u-\beta}^{\beta} x^{\alpha-1}(u-x)^{\alpha-1}\, dx \tag{9}$$

$$= \alpha^2 \beta^{-2\alpha} u^{2\alpha-1} \int_{1-\frac{\beta}{u}}^{\frac{\beta}{u}} y^{\alpha-1}(1-y)^{\alpha-1}\, dy \qquad \beta < u \leq 2\beta \tag{10}$$

Solution (b)

Given

$$Z = \min(X, Y), \qquad W = ma \tag{11}$$

$$F_{ZW}(z,w) = \begin{cases} F_{XY}(z,w) + F_{XY}(w,z) - F_{XY}(z,z) & w \geq z \\ F_{XY}(w,w) & w < z \end{cases} \tag{12}$$

$$\implies f_{ZW}(z,w) = f_X(z)f_Y(w) + f_X(w)f_Y(z) \qquad 0 \tag{13}$$

$$f_{ZW}(z,w) = \begin{cases} 2\alpha^2\beta^{-2\alpha}z^{\alpha-1}w^{\alpha-1} & 0 < z \leq \beta \\ 0 & \text{otherwise} \end{cases} \tag{14}$$

Solution (c)

Let

$$V = \frac{Z}{W} = \frac{\min(X, Y)}{\max(X, Y)} \tag{15}$$

$$= \begin{cases} \frac{Y}{X} & X \geq Y \\ \frac{X}{Y} & X < Y \end{cases} \tag{16}$$

and

$$W = \max(X, Y) \tag{17}$$

$$= \begin{cases} X & X \geq Y \\ Y & X < Y \end{cases} \tag{18}$$

Solution (c)

For $0 < v < 1$, $0 < w < \beta$

$$F_{VW} = P(V \leq v, W \leq w) \tag{19}$$
$$= P(V \leq v, W \leq w, (X \geq Y) \cup (X < Y)) \tag{20}$$
$$= P(Y \leq Xv, X \leq w, X \geq Y) + P(X < Yv, Y \leq w, X < Y) \tag{21}$$
$$= \int_0^w \int_0^{xv} f_{XY}(x, y) dydx + \int_0^w \int_0^{yw} f_{XY}(x, y) dxdy \tag{22}$$

$$f_{VW}(v,w) = \frac{\partial^2 F_{VW}(v,w)}{\partial v \partial w} \tag{23}$$

$$= \frac{\partial}{\partial v}\left\{\int_0^{vw} f_{XY}(w,y)dy + \int_0^{vw} f_{XY}(x,w)dx\right\} \tag{24}$$

$$= w f_{XY}(w,vw) + f_{XY}(vw,w) \tag{25}$$

$$= 2\alpha^2\beta^{-2\alpha}w^{2\alpha-1}v^{\alpha-1} \qquad 0 < v < 1, 0 < w \tag{26}$$

Hence,

$$f_V(v) = \int_0^{\beta} f_{VW}(v,w)dw \tag{27}$$

$$= \alpha v^{\alpha-1} \qquad 0 < v < 1 \tag{28}$$

$$f_W(w) = \int_0^{1} f_{VW}(v,w)dv \tag{29}$$

$$= 2\alpha\beta^{-2\alpha} w^{2\alpha-1} \qquad 0 < w < \beta \tag{30}$$

Hence,

$$f_{VW}(V,W) = f_V(v)f_W(w) \tag{31}$$

Thus V and W are independent random variables.