Assignment 9

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Outline

Papoulis Chapter 7 16

Problem

Given n independent $N(\eta_i, 1)$ random variables Z_i, we form the random variable $W = Z_1^2 + ... + Z_n^2$. This random variable is called a noncentral chi-square with n degrees of freedom and eccentricity $e = \eta_1^2 + ... + \eta_n^2$. Show that its moment generating function equals

$$\Phi_w(s) = \frac{1}{\sqrt{(1-2s)^n}} exp\left\{\frac{es}{1-2s}\right\} \tag{1}$$

Solution

$$\Phi_W(s) = E(e^{sW}) \tag{2}$$
$$= E(e^{s\sum_i Z_i^2}) \tag{3}$$
$$= E\left(\Pi_i e^{sZ_i^2}\right) \tag{4}$$

Since Z_i(s) are independent

$$\Phi_W(s) = \Pi_i E(e^{sZ_i^2}) \tag{5}$$

$$E(e^{sZ_i^2}) = \int_{-\infty}^{\infty} e^{sZ_i^2} f_{Z_i}(z)\, dz \tag{6}$$

$$= \int_{-\infty}^{\infty} e^{sZ_i^2} \frac{1}{\sqrt{2\pi}} e^{-\frac{(z-\eta_i)^2}{2}}\, dz \tag{7}$$

$$= \int_{-\infty}^{\infty} \frac{1}{\sqrt{2\pi}} exp \left\{ -\left(\sqrt{\frac{1-2s}{2}} z - \frac{\eta_i}{\sqrt{2(1-2s)}} \right)^2 \right\} e^{\frac{\eta_i^2 s}{1-2s}}\, dz \tag{8}$$

$$\tag{9}$$

Let

$$u = \sqrt{\frac{1-2s}{2}} z - \frac{\eta_i}{\sqrt{2(1-2s)}} \tag{10}$$

$$du = \sqrt{\frac{1-2s}{2}} dz \tag{11}$$

Substituting we get

$$E(e^{sZ_i^2}) = \int_{-\infty}^{\infty} \frac{e^{\frac{\eta_i^2 s}{1-2s}}}{\sqrt{2\pi}} e^{-u^2} \sqrt{\frac{2}{1-2s}}\, du \tag{12}$$

$$= \frac{e^{\frac{\eta_i^2 s}{1-2s}}}{\sqrt{2\pi}} \sqrt{\frac{2}{1-2s}} \sqrt{\pi} \tag{13}$$

$$= e^{\frac{\eta_i^2 s}{1-2s}} \sqrt{\frac{1}{1-2s}} \tag{14}$$

$$\tag{15}$$

Substituting in (5), we get

$$
\begin{aligned}
\Phi_W(s) &= \Pi_i E(e^{sZ_i^2}) && (16)\\
&= \Pi_i exp\left\{\frac{\eta_i^2 s}{1-2s}\right\}\frac{1}{\sqrt{1-2s}} && (17)\\
&= exp\left\{\frac{\sum_i \eta_i^2 s}{1-2s}\right\}\frac{1}{\sqrt{(1-2s)^n}} && (18)\\
&= exp\left\{\frac{es}{1-2s}\right\}\frac{1}{\sqrt{(1-2s)^n}} && (19)\\
&= RHS && (20)
\end{aligned}
$$

Hence, proved.