Random Numbers

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Contents

1 Uniform Random Numbers

Let U be a uniform random variable between 0 and 1.

1.1 Generate 10^6 samples of U using a C program and save into a file called uni.dat .
Solution:

```
wget https://github.com/NityaBhamidipaty/
    RandAssig/blob/main/codes/coeffs.h

wget https://github.com/NityaBhamidipaty/
    RandAssig/blob/main/codes/main.c
```

1.2 Load the uni.dat file into python and plot the empirical CDF of U using the samples in uni.dat. The CDF is defined as

$$F_U(x) = \Pr(U \le x) \tag{1.1}$$

Solution: The fig 1 is plotted using the code:

```
wget https://github.com/NityaBhamidipaty/
    RandAssig/blob/main/codes/cdf_plot.py
```

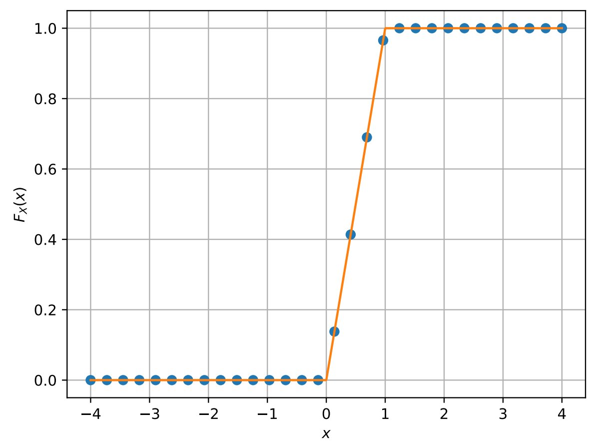

Fig. 1: CDF of U

1.3 Find a theoretical expression for $F_U(x)$.
Solution:

$$f_U(x) = \begin{cases} 1 & x \in (0,1) \\ 0 & \text{otherwise} \end{cases} \tag{1.2}$$

$$F_U(x) = \int_{-\infty}^{x} f_U(u)dx \tag{1.3}$$

$$\tag{1.4}$$

if $x \le 0$

$$F_U(x) = \int_{-\infty}^{x} f_U(u)dx \tag{1.5}$$
$$= \int_{-\infty}^{x} 0dx \tag{1.6}$$
$$= 0 \tag{1.7}$$

if $x \in (0,1)$

$$F_U(x) = \int_{-\infty}^{x} f_U(u)dx \tag{1.8}$$
$$= \int_{-\infty}^{x} 1dx \tag{1.9}$$
$$= x \tag{1.10}$$

if $x \ge 1$

$$F_U(x) = \int_{-\infty}^{x} f_U(u)dx \tag{1.11}$$
$$= \int_{-\infty}^{x} 0dx \tag{1.12}$$
$$= 0 \tag{1.13}$$

Hence,

$$F_U(x) = \begin{cases} 0 & x \le 0 \\ x & x \in (0,1) \\ 1 & x \ge 1 \end{cases} \tag{1.14}$$

1.4 The mean of U is defined as

$$E[U] = \frac{1}{N}\sum_{i=1}^{N} U_i \tag{1.15}$$

and its variance as

$$\text{var}[U] = E[U - E[U]]^2 \tag{1.16}$$

Write a C program to find the mean and variance of U.
Solution:

$$E[X] = 0.500007 \tag{1.17}$$
$$Var[X] = 0.083301 \tag{1.18}$$

```
wget https://github.com/NityaBhamidipaty/
    RandAssig/blob/main/codes/coeffs.h
wget https://github.com/NityaBhamidipaty/
    RandAssig/blob/main/codes/main.c
```

1.5 Verify your result theoretically given that

$$E\left[U^k\right] = \int_{-\infty}^{\infty} x^k dF_U(x) \tag{1.19}$$

Solution:

$$E[U^k] = \int_{-\infty}^{\infty} x^k dF_U(x) \tag{1.20}$$
$$= \int_{-\infty}^{0} 0 + \int_{0}^{1} x^k dx + \int_{1}^{\infty} 0 \tag{1.21}$$
$$= \frac{1}{k+1} \tag{1.22}$$
$$\implies E[U] = \frac{1}{2} \tag{1.23}$$

2 Central Limit Theorem

2.1 Generate 10^6 samples of the random variable

$$X = \sum_{i=1}^{12} U_i - 6 \tag{2.1}$$

using a C program, where $U_i, i = 1, 2, \dots, 12$ are a set of independent uniform random variables between 0 and 1 and save in a file called gau.dat
Solution:

```
wget https://github.com/NityaBhamidipaty/
    RandAssig/blob/main/codes/coeffs.h
wget https://github.com/NityaBhamidipaty/
    RandAssig/blob/main/codes/main.c
```

2.2 Load gau.dat in python and plot the empirical CDF of X using the samples in gau.dat. What properties does a CDF have?
Solution: The fig2 is plotted using:

```
wget https://github.com/NityaBhamidipaty/
    RandAssig/blob/main/codes/cdf_plot.py
```

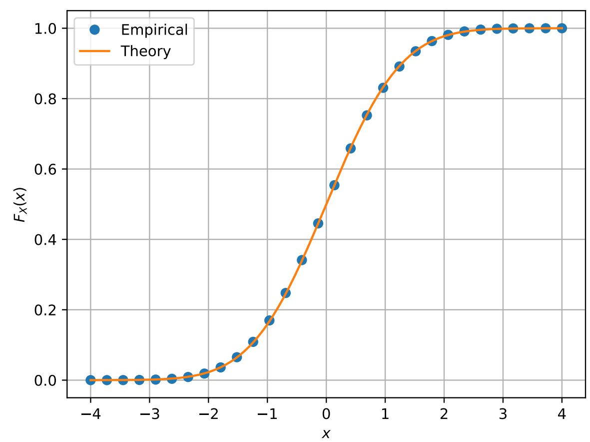

Fig. 2: CDF of X

Properties of CDF

a) $x \to -\infty$ $F_U(x) \to 0$
b) $x \to \infty$ $F_U(x) \to 1$
c) $F_U(x)$ is non-decreasing.
d) $F_U(x)$ is non-negative.

2.3 Load gau.dat in python and plot the empirical PDF of X using the samples in gau.dat. The PDF of X is defined as

$$p_X(x) = \frac{d}{dx} F_X(x) \tag{2.2}$$

What properties does the PDF have?
Solution: The PDF of X (2)is plotted using the code below:

```
wget https://github.com/NityaBhamidipaty/
    RandAssig/blob/main/codes/pdf_plot.py
```

Properties of PDF

a) $x \to -\infty$ $f_U(x) \to 0$

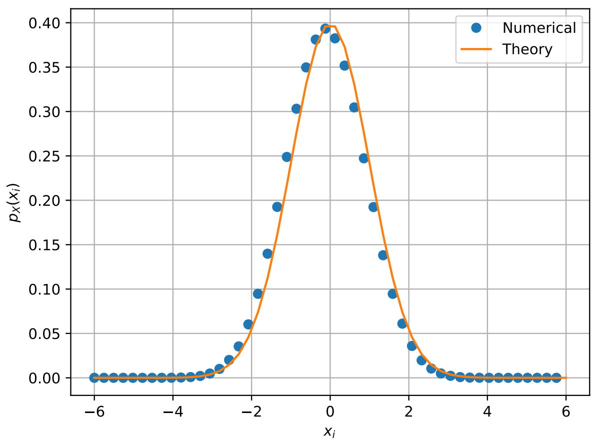

Fig. 2: PDF of X

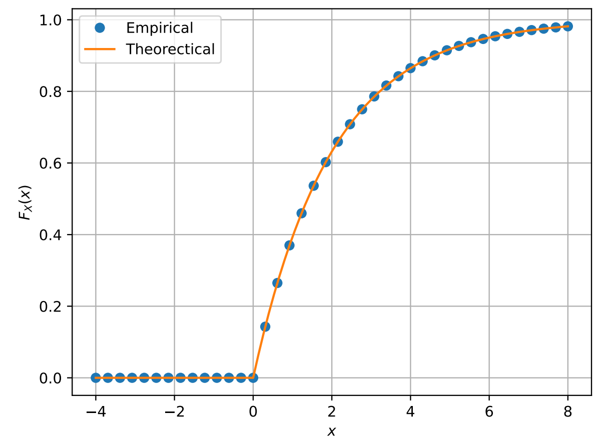

Fig. 3: CDF of V

b) $x \to \infty$ $f_U(x) \to 0$
c) $f_U(x)$ is non-negative.

2.4 Find the mean and variance of X by writing a C program.
Solution:

$$E[X] = 0.000326 \tag{2.3}$$
$$Var[X] = 1.000906 \tag{2.4}$$

```
wget https://github.com/NityaBhamidipaty/
    RandAssig/blob/main/codes/coeffs.h
wget https://github.com/NityaBhamidipaty/
    RandAssig/blob/main/codes/main.c
```

2.5 Given that

$$p_X(x) = \frac{1}{\sqrt{2\pi}} \exp\left(-\frac{x^2}{2}\right), -\infty < x < \infty, \tag{2.5}$$

repeat the above exercise theoretically.
Solution:

$$E(X) = \int_{-\infty}^{\infty} \frac{1}{\sqrt{2\pi}} \exp\left(-\frac{x^2}{2}\right) x\,dx = 0 \tag{2.6}$$
$$\text{Since the integrand is odd} \tag{2.7}$$

$$Var(X) = \int_{-\infty}^{\infty} x^2 \frac{1}{\sqrt{2\pi}} \exp\left(-\frac{x^2}{2}\right) dx \tag{2.8}$$
$$= \int_{-\infty}^{\infty} \frac{1}{\sqrt{2\pi}} x(xe^{-\frac{x^2}{2}})\,dx \tag{2.9}$$
$$= -\frac{1}{\sqrt{2\pi}} xe^{-\frac{x^2}{2}}|_{-\infty}^{\infty} + \int_{-\infty}^{\infty} \frac{1}{\sqrt{2\pi}} e^{-\frac{x^2}{2}}\,dx \tag{2.10}$$

$$-\frac{1}{\sqrt{2\pi}} xe^{-\frac{x^2}{2}}|_{-\infty}^{\infty} = \lim_{x\to\infty} -\frac{1}{\sqrt{2\pi}} xe^{-\frac{x^2}{2}}|_{-r}^{r} \tag{2.11}$$
$$= \lim_{x\to\infty} -\frac{1}{\sqrt{2\pi}} 2\frac{r}{e^{\frac{r^2}{2}}} \tag{2.12}$$
$$= \lim_{x\to\infty} -\frac{1}{\sqrt{2\pi}} 2\frac{1}{re^{\frac{r^2}{2}}} \tag{2.13}$$
$$= 0 \tag{2.14}$$

$$\int_{-\infty}^{\infty} \frac{1}{\sqrt{2\pi}} e^{-\frac{x^2}{2}}\,dx = \frac{1}{\sqrt{\pi}} \int_{-\infty}^{\infty} e^{-\frac{x^2}{2}}\,d\frac{x}{\sqrt{2}} \tag{2.15}$$
$$= \int_{-\infty}^{\infty} \frac{1}{\sqrt{\pi}} e^{-t^2}\,dt \tag{2.16}$$
$$= \frac{\sqrt{\pi}}{\sqrt{\pi}} = 1 \tag{2.17}$$

3 From Uniform to Other

3.1 Generate samples of

$$V = -2\ln(1-U) \tag{3.1}$$

and plot its CDF.
Solution: The CDF of V (3) is plotted using the code below:

```
wget https://github.com/NityaBhamidipaty/
    RandAssig/blob/main/codes/cdf_plot.py
```

3.2 Find a theoretical expression for $F_V(x)$.

Solution:

$$V = -2\ln(1-U) \tag{3.2}$$
$$F_V(x) = P\{V \le x\} \tag{3.3}$$
$$= P\{-2\ln(1-U) \le x\} \tag{3.4}$$
$$= P\{U \le 1 - e^{-\frac{x}{2}}\} \tag{3.5}$$
$$= F_U(1 - e^{-\frac{x}{2}}) \tag{3.6}$$

Case 1

$$F_V(x) = F_U(1 - e^{-\frac{x}{2}}) = 0 \tag{3.7}$$
$$1 - e^{-\frac{x}{2}} \le 0 \tag{3.8}$$
$$1 \le e^{-\frac{x}{2}} \tag{3.9}$$
$$0 \le -\frac{x}{2} \tag{3.10}$$
$$x \le 0 \tag{3.11}$$

Case 2

$$F_V(x) = F_U(1 - e^{-\frac{x}{2}}) = 1 - e^{-\frac{x}{2}} \tag{3.12}$$
$$0 < 1 - e^{-\frac{x}{2}} < 1 \tag{3.13}$$
$$\tag{3.14}$$

$1 - e^{-\frac{x}{2}} < 1$ is always true as $e^{-\frac{x}{2}} > 0$

$$0 < 1 - e^{-\frac{x}{2}} \tag{3.15}$$
$$-1 < -e^{-\frac{x}{2}} \tag{3.16}$$
$$1 > e^{-\frac{x}{2}} \tag{3.17}$$
$$0 > -\frac{x}{2} \tag{3.18}$$
$$0 < x \tag{3.19}$$

Case 3

$$1 - e^{-\frac{x}{2}} \ge 1 \tag{3.20}$$
$$0 \ge e^{-\frac{x}{2}} \tag{3.21}$$
$$x \in \phi \tag{3.22}$$

Hence,

$$F_V(x) = \begin{cases} 0 & x \le 0 \\ 1 - e^{-\frac{x}{2}} & x > 0 \end{cases} \tag{3.23}$$