Random Numbers

Nitya Seshagiri Bhamidipaty - CS21BTECH11041

Contents

1 Uniform Random Numbers

Let U be a uniform random variable between 0 and 1.

1.1 Generate 10^6 samples of U using a C program and save into a file called uni.dat .
Solution:

```
wget https://github.com/NityaBhamidipaty/
    RandAssig/blob/main/codes/coeffs.h

wget https://github.com/NityaBhamidipaty/
    RandAssig/blob/main/codes/main.c
```

1.2 Load the uni.dat file into python and plot the empirical CDF of U using the samples in uni.dat. The CDF is defined as

$$F_U(x) = \Pr(U \leq x) \tag{1.1}$$

Solution: The fig **??** is plotted using the code:

```
wget https://github.com/NityaBhamidipaty/
    RandAssig/blob/main/codes/cdf_plot.py
```

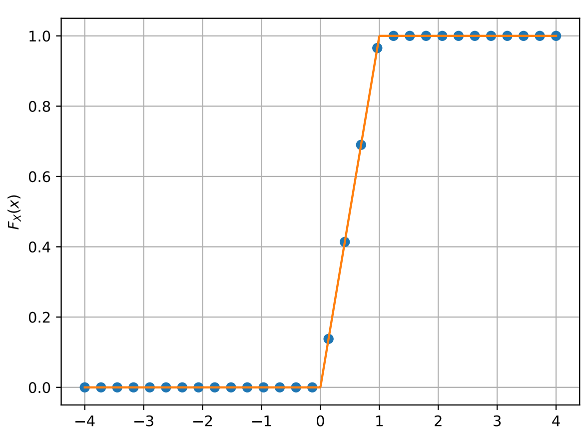


Fig. 1: CDF of U

1.3 Find a theoretical expression for $F_U(x)$.
Solution:

$$f_U(x) = \begin{cases} 1 & x \in (0,1) \\ 0 & \text{otherwise} \end{cases} \tag{1.2}$$

$$F_U(x) = \int_{-\infty}^{x} f_U(u)du \tag{1.3}$$

$$\tag{1.4}$$

if $x \leq 0$

$$F_U(x) = \int_{-\infty}^{x} f_U(u)dx \tag{1.5}$$

$$= \int_{-\infty}^{x} 0dx \tag{1.6}$$

$$= 0 \tag{1.7}$$

if $x \in (0,1)$

$$F_U(x) = \int_{-\infty}^{x} f_U(u)dx \tag{1.8}$$

$$= \int_{-\infty}^{x} 1dx \tag{1.9}$$

$$= x \tag{1.10}$$

if $x \geq 1$

$$F_U(x) = \int_{-\infty}^{x} f_U(u)dx \tag{1.11}$$

$$= \int_{-\infty}^{x} 0dx \tag{1.12}$$

$$= 0 \tag{1.13}$$

Hence,

$$F_U(x) = \begin{cases} 0 & x \leq 0 \\ x & x \in (0,1) \\ 1 & x \geq 1 \end{cases} \tag{1.14}$$

1.4 The mean of U is defined as

$$E[U] = \frac{1}{N}\sum_{i=1}^{N} U_i \tag{1.15}$$

and its variance as

$$\text{var}\,[U] = E\,[U - E\,[U]]^2 \tag{1.16}$$

Write a C program to find the mean and variance of U.
Solution:

$$E[X] = 0.500007 \tag{1.17}$$
$$Var[X] = 0.083301 \tag{1.18}$$

```
wget https://github.com/NityaBhamidipaty/
    RandAssig/blob/main/codes/coeffs.h
wget https://github.com/NityaBhamidipaty/
    RandAssig/blob/main/codes/main.c
```

1.5 Verify your result theoretically given that

$$E\left[U^k\right] = \int_{-\infty}^{\infty} x^k dF_U(x) \tag{1.19}$$

Solution:

$$E[U^k] = \int_{-\infty}^{\infty} x^k dF_U(x) \tag{1.20}$$
$$= \int_{-\infty}^{0} 0 + \int_{0}^{1} x^k dx + \int_{1}^{\infty} 0 \tag{1.21}$$
$$= \frac{1}{k+1} \tag{1.22}$$
$$\implies E[U] = \frac{1}{2} \tag{1.23}$$

2 Central Limit Theorem

2.1 Generate 10^6 samples of the random variable

$$X = \sum_{i=1}^{12} U_i - 6 \tag{2.1}$$

using a C program, where $U_i, i = 1, 2, \ldots, 12$ are a set of independent uniform random variables between 0 and 1 and save in a file called gau.dat
Solution:

```
wget https://github.com/NityaBhamidipaty/
    RandAssig/blob/main/codes/coeffs.h
wget https://github.com/NityaBhamidipaty/
    RandAssig/blob/main/codes/main.c
```

2.2 Load gau.dat in python and plot the empirical CDF of X using the samples in gau.dat. What properties does a CDF have?
Solution: The fig**??** is plotted using:

```
wget https://github.com/NityaBhamidipaty/
    RandAssig/blob/main/codes/cdf_plot.py
```

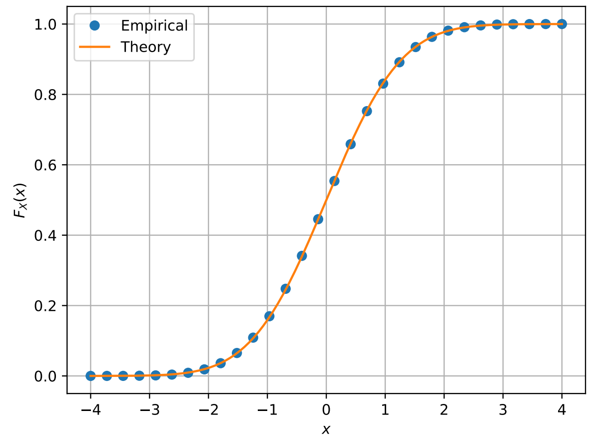


Fig. 2: CDF of X

Properties of CDF

a) $x \to -\infty$ $F_U(x) \to 0$
b) $x \to \infty$ $F_U(x) \to 1$
c) $F_U(x)$ is non-decreasing.
d) $F_U(x)$ is non-negative.

2.3 Load gau.dat in python and plot the empirical PDF of X using the samples in gau.dat. The PDF of X is defined as

$$p_X(x) = \frac{d}{dx} F_X(x) \tag{2.2}$$

What properties does the PDF have?
Solution: The PDF of X (**??**)is plotted using the code below:

```
wget https://github.com/NityaBhamidipaty/
    RandAssig/blob/main/codes/pdf_plot.py
```

Properties of PDF

a) $x \to -\infty$ $f_U(x) \to 0$
b) $x \to \infty$ $f_U(x) \to 0$
c) $f_U(x)$ is non-negative.

2.4 Find the mean and variance of X by writing a C program.
Solution:

$$E[X] = 0.000326 \tag{2.3}$$
$$Var[X] = 1.000906 \tag{2.4}$$

```
wget https://github.com/NityaBhamidipaty/
    RandAssig/blob/main/codes/coeffs.h
```

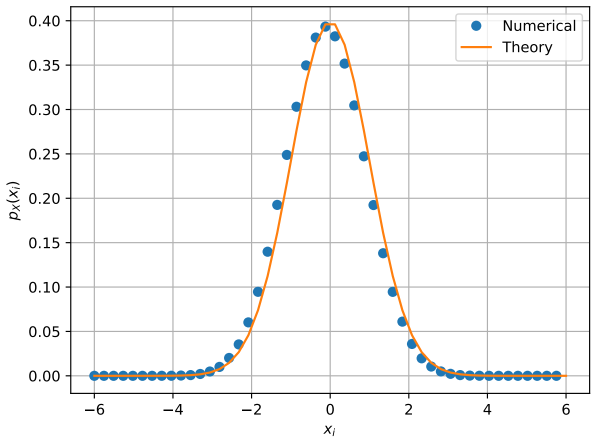


Fig. 2: PDF of X

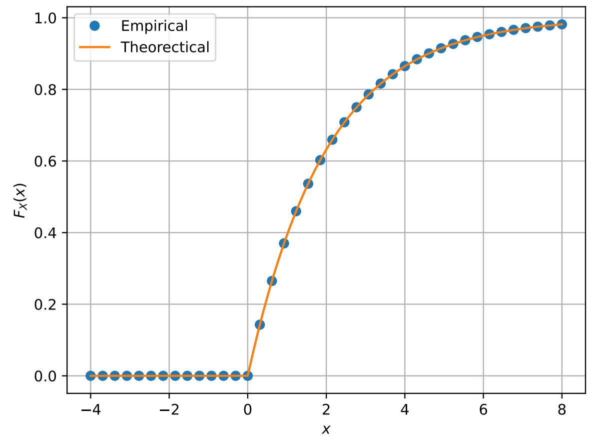


Fig. 3: CDF of V

```
wget https://github.com/NityaBhamidipaty/
    RandAssig/blob/main/codes/main.c
```

2.5 Given that

$$p_X(x) = \frac{1}{\sqrt{2\pi}} \exp\left(-\frac{x^2}{2}\right), -\infty < x < \infty, \quad (2.5)$$

repeat the above exercise theoretically.
Solution:

$$E(X) = \int_{-\infty}^{\infty} \frac{1}{\sqrt{2\pi}} \exp\left(-\frac{x^2}{2}\right) x\, dx = 0 \quad (2.6)$$

$$\text{Since the integrand is odd} \quad (2.7)$$

$$Var(X) = \int_{-\infty}^{\infty} x^2 \frac{1}{\sqrt{2\pi}} \exp\left(-\frac{x^2}{2}\right) dx \quad (2.8)$$

$$= \int_{-\infty}^{\infty} \frac{1}{\sqrt{2\pi}} x(xe^{-\frac{x^2}{2}})\, dx \quad (2.9)$$

$$= -\frac{1}{\sqrt{2\pi}} xe^{-\frac{x^2}{2}} |_{-\infty}^{\infty} + \int_{-\infty}^{\infty} \frac{1}{\sqrt{2\pi}} e^{-\frac{x^2}{2}}\, dx \quad (2.10)$$

$$-\frac{1}{\sqrt{2\pi}} xe^{-\frac{x^2}{2}} |_{-\infty}^{\infty} = \lim_{x\to\infty} -\frac{1}{\sqrt{2\pi}} xe^{-\frac{x^2}{2}} |_{-r}^{r} \quad (2.11)$$

$$= \lim_{x\to\infty} -\frac{1}{\sqrt{2\pi}} 2 \frac{r}{e^{\frac{r^2}{2}}} \quad (2.12)$$

$$= \lim_{x\to\infty} -\frac{1}{\sqrt{2\pi}} 2 \frac{1}{re^{\frac{r^2}{2}}} \quad (2.13)$$

$$= 0 \quad (2.14)$$

$$\int_{-\infty}^{\infty} \frac{1}{\sqrt{2\pi}} e^{-\frac{x^2}{2}}\, dx = \frac{1}{\sqrt{\pi}} \int_{-\infty}^{\infty} e^{-\frac{x^2}{2}}\, d\frac{x}{\sqrt{2}} \quad (2.15)$$

$$= \int_{-\infty}^{\infty} \frac{1}{\sqrt{\pi}} e^{-t^2}\, dt \quad (2.16)$$

$$= \frac{\sqrt{\pi}}{\sqrt{\pi}} = 1 \quad (2.17)$$

3 From Uniform to Other

3.1 Generate samples of

$$V = -2\ln(1-U) \quad (3.1)$$

and plot its CDF.
Solution: The CDF of V (**??**) is plotted using the code below:

```
wget https://github.com/NityaBhamidipaty/
    RandAssig/blob/main/codes/cdf_plot.py
```

3.2 Find a theoretical expression for $F_V(x)$.
Solution:

$$V = -2\ln(1-U) \quad (3.2)$$

$$F_V(x) = P\{V \le x\} \quad (3.3)$$

$$= P\{-2\ln(1-U) \le x\} \quad (3.4)$$

$$= P\{U \le 1 - e^{-\frac{x}{2}}\} \quad (3.5)$$

$$= F_U(1 - e^{-\frac{x}{2}}) \quad (3.6)$$

Case 1

$$F_V(x) = F_U(1 - e^{-\frac{x}{2}}) = 0 \tag{3.7}$$
$$1 - e^{-\frac{x}{2}} \leq 0 \tag{3.8}$$
$$1 \leq e^{-\frac{x}{2}} \tag{3.9}$$
$$0 \leq -\frac{x}{2} \tag{3.10}$$
$$x \leq 0 \tag{3.11}$$

Case 2

$$F_V(x) = F_U(1 - e^{-\frac{x}{2}}) = 1 - e^{-\frac{x}{2}} \tag{3.12}$$
$$0 < 1 - e^{-\frac{x}{2}} < 1 \tag{3.13}$$
$$\tag{3.14}$$

$1 - e^{-\frac{x}{2}} < 1$ is always true as $e^{-\frac{x}{2}} > 0$

$$0 < 1 - e^{-\frac{x}{2}} \tag{3.15}$$
$$-1 < -e^{-\frac{x}{2}} \tag{3.16}$$
$$1 > e^{-\frac{x}{2}} \tag{3.17}$$
$$0 > -\frac{x}{2} \tag{3.18}$$
$$0 < x \tag{3.19}$$

Case 3

$$1 - e^{-\frac{x}{2}} \geq 1 \tag{3.20}$$
$$0 \geq e^{-\frac{x}{2}} \tag{3.21}$$
$$x \in \phi \tag{3.22}$$

Hence,

$$F_V(x) = \begin{cases} 0 & x \leq 0 \\ 1 - e^{-\frac{x}{2}} & x > 0 \end{cases} \tag{3.23}$$

4 Triangular Distribution

4.1 Generate

$$T = U_1 + U_2 \tag{4.1}$$

Solution:

```
wget https://github.com/NityaBhamidipaty/
    RandAssig/blob/main/codes/main.c
```

4.2 Find the CDF of T
Solution:
The CDF (**??**) is plotted using

```
wget https://github.com/NityaBhamidipaty/
    RandAssig/blob/main/codes/cdf_plot.py
```

4.3 Find the PDF of T
Solution: The PDF (**??**) is plotted using

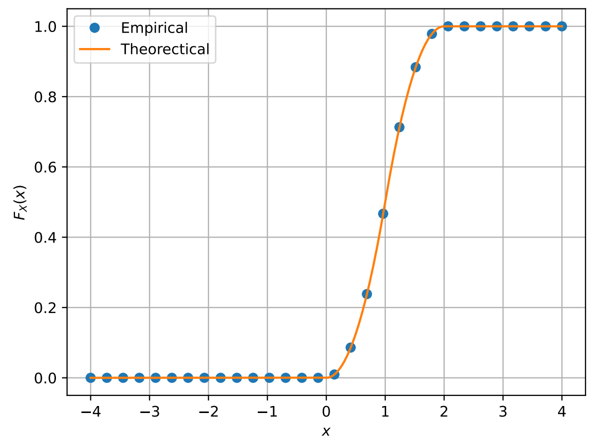


Fig. 4: CDF of T

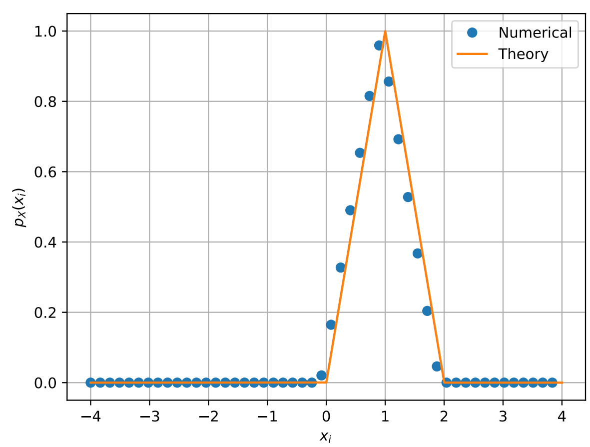


Fig. 4: PDF of T

```
weget https://github.com/NityaBhamidipaty/
    RandAssig/blob/main/codes/pdf_plot.py
```

4.4 Find the theoretical expressions of CDF an PDF of T.
Solution:

$$f_T(t) = \int_{-\infty}^{\infty} f_{U_2}(x) f_{U_1}(t - x)\, dx \tag{4.2}$$
$$= \int_0^1 f(x) f(t - x)\, dx \tag{4.3}$$

If $t \in (0, 1)$

$$f_T(t) = \int_0^t 1 \times 1\, dx \tag{4.4}$$
$$= t \tag{4.5}$$

If $t \in (1,2)$

$$f_T(t) = \int_{t-1}^{1} 1 \times 1 \, dx \tag{4.6}$$
$$= 2 - t \tag{4.7}$$

Hence,

$$f_T(t) = \begin{cases} t & t \in (0,1) \\ 2-t & t \in (1,2) \\ 0 & \text{otherwise} \end{cases} \tag{4.8}$$

$$F_T(t) = \int_{-\infty}^{t} f_T(t) \, dt \tag{4.9}$$
$$\tag{4.10}$$

$$F_T(t) = \begin{cases} 0 & t \le 0 \\ \frac{t^2}{2} & t \in (0,1) \\ \frac{-t^2}{2} + 2t - 1 & t \in (1,2) \\ 1 & t \ge 2 \end{cases} \tag{4.11}$$

5 Maximum Likelihood

5.1 Generate equiprobable $X \in \{1, -1\}$
Solution:

weget ...

5.2 Generate

$$Y = AX + N \tag{5.1}$$

where $A = 5$ dB and $N \sim \mathcal{N}(0,1)$
Solution:

weget ...

5.3 Plot Y using scatter plot
Solution:
The plot **??** is plotted using

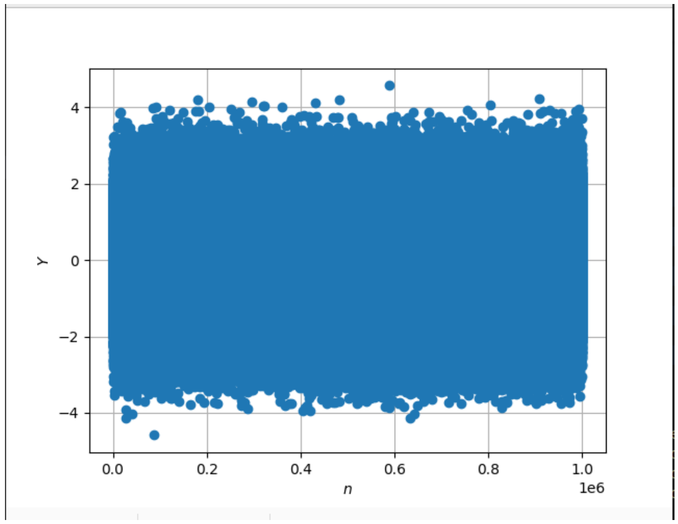
weget ...

5.4 Guess how to estimate X from Y
Solution:
Estimated value of $X = \hat{X}$

$$\hat{X} = \begin{cases} -1 & Y < 0 \\ 1 & Y \ge \end{cases} \tag{5.2}$$

5.5 Find

$$P_{e|0} = \Pr\left(\hat{X} = -1 | X = 1\right) \tag{5.3}$$



and

$$P_{e|1} = \Pr\left(\hat{X} = 1 | X = -1\right) \tag{5.4}$$

Solution:

$$P_{e|0} = \Pr\left(\hat{X} = -1 | X = 1\right) = 0.49999 \tag{5.5}$$
$$P_{e|1} = \Pr\left(\hat{X} = 1 | X = -1\right) = 0.50075 \tag{5.6}$$

weget ...

5.6 Find P_e assuming that X has equiprobable symbols.
Solution:

$$P_e = P(X = 1)P_{e|0} + P(X = -1)P_{e|1} \tag{5.7}$$
$$= \frac{1}{2}(P_{e|0} + P_{e|1}) \tag{5.8}$$
$$= \frac{1}{2}(0.49999 + 0.50075) \tag{5.9}$$
$$= 0.50037 \tag{5.10}$$

5.7 Verify by plotting the theoretical P_e with respect to A from 0 to 10 dB.
Solution:

$$P_e = P_{e|0}P(X = 1) + P_{e|1}P(X = -1) \tag{5.11}$$
$$= \frac{1}{2}(P_{e|0} + P_{e|1}) \tag{5.12}$$

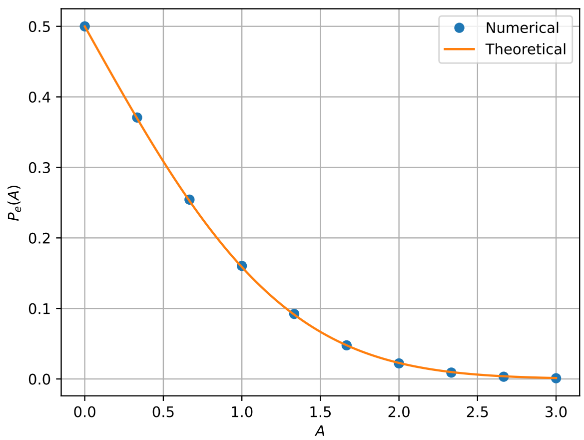


Fig. 5: $P_e(A)$ vs A

$$\begin{aligned} P_{e|0} &= P(Y \le 0|X = 1) && (5.13)\\ &= P(AX + N \le 0|X = 1) && (5.14)\\ &= P(A + N \le 0) && (5.15)\\ &= P(N \le -A) && (5.16)\\ &= P(N > A) && (5.17) \end{aligned}$$

$$\begin{aligned} P_{e|1} &= P(Y > 0|X = -1) && (5.18)\\ &= P(AX + N > 0|X = -1) && (5.19)\\ &= P(-A + N > 0) && (5.20)\\ &= P(N > A) && (5.21) \end{aligned}$$

$$P_e = P(N > A) = Q(A) \qquad (5.22)$$

Plotted using code

```
weget ...
```

5.8 Now, consider a threshold δ while estimating X from Y. Find the value of δ that maximizes the theoretical P_e.
Solution:

$$\hat{X} = \begin{cases} 1 & Y > \delta \\ -1 & Y < \delta \end{cases} \qquad (5.23)$$

$$\begin{aligned} P_{e|0} &= P(Y < \delta|X = 1) && (5.24)\\ &= P(AX + N < \delta|X = 1) && (5.25)\\ &= P(A + N < \delta) && (5.26)\\ &= P(N < \delta - A) && (5.27)\\ &= P(N > A - \delta) && (5.28)\\ &= Q(A - \delta) && (5.29) \end{aligned}$$

$$\begin{aligned} P_{e|1} &= P(Y > \delta|X = -1) && (5.30)\\ &= P(-A + N > \delta|X = -1) && (5.31)\\ &= P(-A + N > \delta) && (5.32)\\ &= P(N > \delta + A) && (5.33)\\ &= Q(A + \delta) && (5.34) \end{aligned}$$

$$P_e = \frac{1}{2}(Q(A + \delta) + Q(A - \delta)) \qquad (5.35)$$

Differentiating wrt δ to find maximum

$$\frac{dQ(x)}{dx} = -\frac{1}{\sqrt{2\pi}}e^{\frac{-x^2}{2}} \qquad (5.36)$$

$$\begin{aligned} \frac{dP_e}{d\delta} &= \frac{1}{2}\left(\frac{1}{\sqrt{2\pi}}e^{-\frac{(A-\delta)^2}{2}} - \frac{1}{\sqrt{2\pi}}e^{-\frac{(A+\delta)^2}{2}}\right) && (5.37)\\ &= 0 && (5.38)\\ \implies \delta &= 0 \text{ or} && (5.39)\\ A &= 0, \delta \in \mathbb{R} && (5.40) \end{aligned}$$

Verifying it is maximum

5.9 Repeat the above exercise when

$$p_X(0) = p \qquad (5.41)$$

Solution:

$$P_e = (1 - p)Q(A + \delta) + Q(A - \delta)p \qquad (5.42)$$

Differentiating

$$\begin{aligned} \frac{dP_e}{d\delta} &= p\frac{1}{\sqrt{2\pi}}e^{-\frac{(A-\delta)^2}{2}} - \frac{1}{\sqrt{2\pi}}e^{-\frac{(A+\delta)^2}{2}}(1 - p) && (5.43)\\ &= 0 && (5.44)\\ & && (5.45) \end{aligned}$$

$$pe^{-\frac{(\delta-A)^2}{2}} = (1-p)e^{-\frac{(A+\delta)^2}{2}} \tag{5.46}$$

$$\implies \delta = \frac{1}{2A}\ln\left(\frac{1-p}{p}\right) \tag{5.47}$$

5.10 Repeat the above exercise using the MAP criterion.

Solution:

$$P(X=1|Y=y) = \frac{P(Y=y, X=1)P(X=1)}{P(Y=y)} \tag{5.48}$$

$$= \frac{P(A+N=y)p}{f_Y(y)} \tag{5.49}$$

$$= \frac{f_N(y-A)p}{f_Y(y)} \tag{5.50}$$

$$P(X=-1|Y=y) = \frac{P(Y=y, X=-1)P(X=-1)}{P(Y=y)} \tag{5.51}$$

$$= \frac{P(-A+N=y)(1-p)}{f_Y(y)} \tag{5.52}$$

$$= \frac{f_N(y+A)(1-p)}{f_Y(y)} \tag{5.53}$$

For max error

$$\frac{f_N(y+A)(1-p)}{f_Y(y)} = \frac{f_N(y-A)p}{f_Y(y)} \tag{5.54}$$

$$e^{-Ay}(1-p) = e^{Ay}p \tag{5.55}$$

$$\implies y = \frac{1}{2A}\ln\left(\frac{1-p}{p}\right) \tag{5.56}$$

6 Gaussian to Other

6.1 Let $X_1 \sim \mathcal{N}(0,1)$ and $X_2 \sim \mathcal{N}(0,1)$. Plot the CDF and PDF of

$$V = X_1^2 + X_2^2 \tag{6.1}$$

Solution: The code for plotting the cdf and pdf

```
weget ...
```

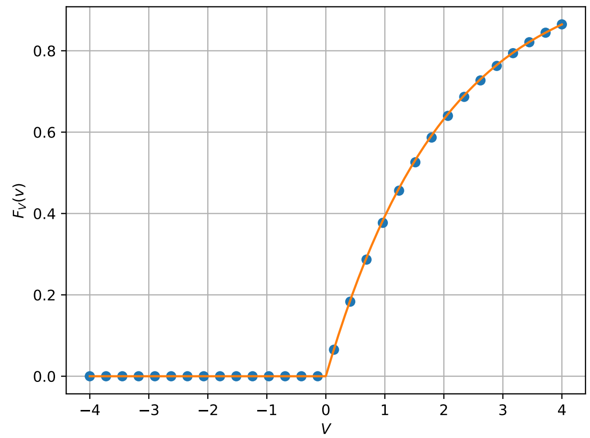
6.2 If

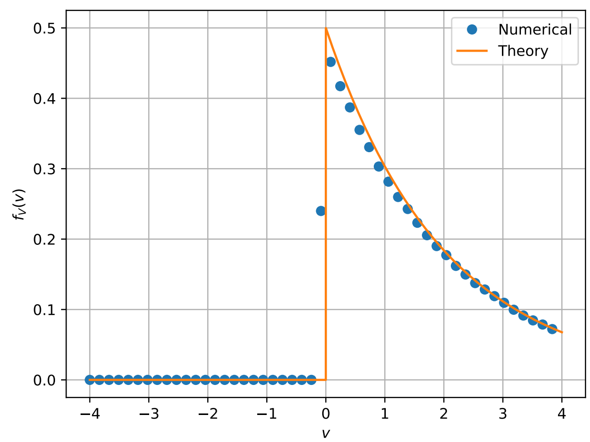
$$F_V(x) = \begin{cases} 1 - e^{-\alpha x} & x \geq 0 \\ 0 & x < 0, \end{cases} \tag{6.2}$$

find α.

6.3 Plot the CDF and PDf of

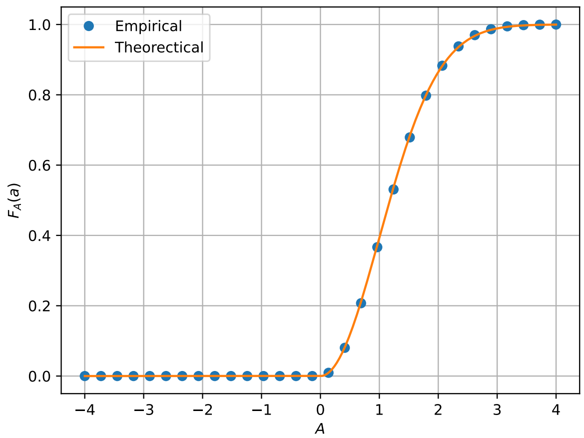
$$A = \sqrt{V} \tag{6.3}$$

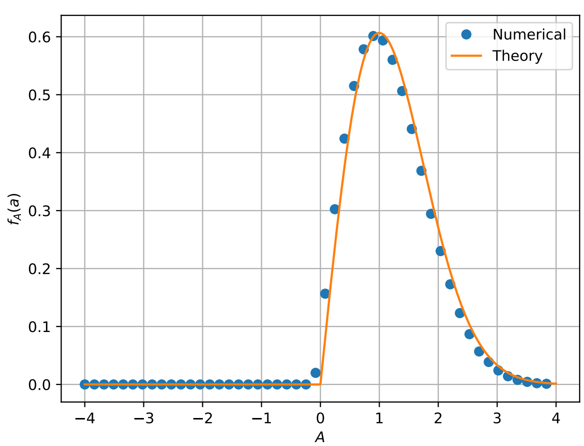




Solution: The code to plot the cdf and pdf

```
weget ...
```


Empirical
Theorectical
$F_A(a)$
A
1.0
0.8
0.6
0.4
0.2
0.0
−4
−3
−2
−1
0
1
2
3
4


Numerical
Theory
$f_A(a)$
A
0.6
0.5
0.4
0.3
0.2
0.1
0.0
−4
−3
−2
−1
0
1
2
3
4