SOME COOL PROBLEMS AND THEIR SOLUTIONS

NIVAS

1. Probability

1.1. **Handling Infinite Chains.** What is the mean number of times we need to roll a fair die to land a 6?

1.1.1. *Summing an AGP.* This is a simple problem with a few different ways to solve it. The first way is straightforward, involving a sort of insight that is a bit removed from the problem itself. Suppose m is the average number of rolls to get a 6. Clearly, the chance of getting a 6 on the first roll is $1/6$. The chance of getting it on the second roll is $\frac{5}{6}\frac{1}{6}$, and on roll i, is $\frac{5}{6}^{i-1}\frac{1}{6}$. Averaging,

$$m = \sum_{i=0}^{\infty}(i+1)\frac{5}{6}^{i}\frac{1}{6}.$$

We note that m is the scalar product between an arithmetic and geometric progression, or succinctly: AGP. The problem reduces to summing this infinite AGP. For this, we employ the same trick we use for summing a GP, twice over: the first time to obtain a GP and the second time to obtain the sum of this GP. Let $r = 5/6$ and $a = 1/6$.

$$m' := mr = \sum(i+1)r^{i+1}a, \quad \Rightarrow m' - m = a + \sum_{i=1} r^i a = a\left(1 + \sum_{i=1} r^i\right) = a\sum_{i=0} r^i.$$

Since the final sum is just a GP, we employ the standard trick to get

$$m - m' = (1-r)m = \frac{a}{1-r},$$

whence

$$m = \frac{a}{(1-r)^2} = 36/6 = 6.$$

1.1.2. *Pinching The Beginning.* There is another, more beautiful way to solve the problem. Note that, if we don't get a 6 on the first throw, we are in a position exactly as we were in the beginning, before the first throw, and it'll take m throws on average to land a 6. If we do land a 6 on the first throw, then we're done. Weighting these events with the corresponding probabilities,

$$m = \frac{1}{6} + (m+1)\frac{5}{6} = \frac{5}{6}m + 1,$$

implying that

$$\frac{m-1}{m} = \frac{5}{6} \Rightarrow m = 6.$$