Operations Research

Theory and Applications

4th Edition

J K SHARMA

OPERATIONS RESEARCH
THEORY AND APPLICATIONS

Fourth Edition

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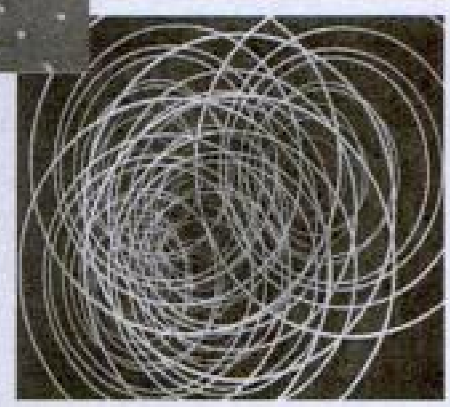
Contents

chapter 1

Operations Research: An Introduction



"The first rule of any technology used in a business is that automation applied to an efficient operation will magnify the efficiency. The second is that automation applied to an inefficient operation will magnify the inefficiency."

– Bill Gates

Preview This chapter presents a framework of a possible structural analysis of problems pertaining to an organization in order to arrive at an optimal solution using operations research approach.

Learning Objectives After studying this chapter you should be able to

- understand the need of using operations research – a quantitative approach for effective decision-making.
- know the historical perspective of operations research approach.
- know the various definitions of operations research, its characteristics and various phases of scientific study.
- recognize, classify and use various models for solving a problem under consideration.
- be familiar with several computer software available for solving an operations research model.

Chapter Outline

1.1 OPERATIONS RESEARCH – A QUANTITATIVE APPROACH TO DECISION-MAKING

Decision-making in today's social and business environment has become a complex task. High costs of technology, materials, labour, competitive pressures and so many different economic, social as well as political factors and viewpoints, greatly increase the difficulty of managerial decision-making. Knowledge and technology are changing rapidly and are continuously giving rise to problems with little or no precedents. Well-structured problems are routinely optimized at the operational level of organizations, and increased attention is now focussed on broader tactical and strategic issues. To effectively address the arising problems and to provide leadership in the advancing global age, decision-makers cannot afford to make decisions by simply applying their personal experiences, guesswork or intuition, because the consequences of wrong decisions can prove to be serious and costly. Hence, an understanding of the applicability of quantitative methods to decision-making is of fundamental importance to decision-makers. For example, entering the wrong markets, producing the wrong products, providing inappropriate services, etc., may cause disastrous consequences for organizations.

Quantitative analysis is the scientific approach to decision-making

In the global age, the practice of the operations research (OR) approach must maintain stride with the abovementioned trends. Some people claim that the OR approach does not adequately meet the needs of business and industry. The reasons for its failure are often behavioural in nature. Lack of implementation of results or findings is one of the major reasons. The implementation process presumes that the definition, analysis, modeling, and solution phases of a project have been adequately performed. Among the reasons for implementation failure is the lack of creative problem solving abilities of the decision-maker.

Operations research facilitates the comparison of all possible *alternatives* (*courses of action* or *acts*). This helps to know the potential outcomes and permits examination of the sensitivity of the solution to changes or errors in numerical values. It also encourages rational decision-making based on the best available approaches and/or techniques. However, the fact that timely and competent decisions should be an aid to the decision-makers's judgement, not a substitute for it, should be emphasized. That is not to say that the management decision-making is simply about the application of operations research techniques and/or approaches.

While solving a real-life problem, the decision-maker must examine a problem from both the quantitative as well as the qualitative perspective. This should be done so that data so obtained, be analyzed from both perspectives in order to suggest a solution to the problem. For example, consider the problem of an investor seeking advice for investments in three alternatives: Stock Market, Real Estate and Bank Deposit. To suggest an acceptable solution, we need to consider certain quantitative factors that should be examined in the light of the problem. For instance, factors like financial ratios from the balance sheets of several companies whose stocks are under consideration; real estate companies' cash flows and rates of return for investment in property; and how much the investment will be worth in the future when deposited at a bank at a given interest rate for a certain number of years, are factors that need to be examined. However, before reaching a conclusion, certain other qualitative factors, such as weather conditions, state and central policies, new technology, the political situation, etc., also need to be considered.

Decision maker should consider both qualitative and quantitative factors while solving a problem

The evaluation of each alternative can be extremely difficult or time consuming for two reasons: First, due to the amount and complexity of information that must be processed, second due to the number of alternative solutions available. The number of solutions can be so large that a decision-maker simply would not be able to evaluate all of them in order to select an appropriate one. For these reasons, when there is a lack of qualitative factors, decision-makers increasingly turn to quantitative factors and use computers to arrive at the optimal solution for problems that involve a large number of alternatives.

There is an imperative need for an analytical research for presenting a *structural analysis*. This can be done by critically examining the levels of interaction between the *application process* of operations research, various systems and organizations. Figure 1.1 summarizes the operations research approach by sketching the conceptual framework of the main elements of the analysis.

This book introduces a set of operations research techniques that would help decision-makers in making rational and effective decisions. It also gives a basic knowledge of mathematics and statistics, as well as of the use of computer software needed for computational purposes.

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- *Operations research is a scientific approach to problem-solving for executive management.*
– H M Wagner

As the discipline of operations research grew, numerous names such as *Operations Analysis, Systems Analysis, Decision Analysis, Management Science, Quantitative Analysis, Decision Science* were given to it. This is because of the fact that the types of problems encountered are always concerned with 'effective decision', but the solution of these problems do not always involve research into operations or aspects of the science of management.

1.4 FEATURES OF OPERATIONS RESEARCH APPROACH

The broad based definition of OR, with the additional features is as follows: *OR utilizes a planned approach following a scientific method and an interdisciplinary team, in order to represent complex functional relationship as mathematical models, for the purpose of providing a quantitative basis for decision-making and uncovering new problems for quantitative analysis.* The broad features of OR approach to any decision problem are summarized as follows:

Operations research uses:
(i) interdisciplinary,
(ii) scientific,
(iii) holistic, and
(iv) objective-oriented approaches to decision making

Interdisciplinary Approach For solving a problem interdisciplinary teamwork is essential. This is because while attempting to solve a complex management problem, one person may not have the complete knowledge of all its aspects (such as economic, social, political, psychological, engineering, etc.). This means we should not expect one person to find a desirable solution to all managerial problems. Therefore, a team of individuals specializing in mathematics, statistics, economics, engineering, computer science, psychology, etc., should be organized in a way that each aspect of the problem can be analysed by a particular specialist in that field. This would help to arrive to an appropriate and desirable solution of the problem. However, there are certain problem situations that can be analysed by even one individual.

Scientific Approach *Operations research is the application of scientific methods, techniques and tools to problems involving the operations of systems so as to provide those in control of operations with optimum solutions to the problems* (Churchman et al.). The scientific method consists of observing and defining the problem; formulating and testing the hypothesis; and analysing the results of the test. The data so obtained is then used to decide whether the hypothesis should be accepted or not. If the hypothesis is accepted, the results should be implemented otherwise an alternative hypothesis has to be formulated.

Holistic Approach While arriving at a decision, an operations research team examines the relative importance of all conflicting and multiple objectives. It also examines the validity of claims of various departments of the organization from the perspective of its implications to the whole organization.

Operations research attempts to resolve the conflicts of interest among various sections of the organization and seeks to find the optimal solution that is in the interest of the organization as a whole

Objective-Oriented Approach An operations research approach seeks to obtain an optimal solution to the problem under analysis. For this, a measure of desirability (or effectiveness) is defined, based on the objective(s) of the organization. A measure of desirability so defined is then used to compare alternative courses of action with respect to their possible outcomes.

Illustration The OR approach attempts to find global optimum by analysing interrelationships among the system components involved in the problem. One such situation is described below.

Consider the case of a large organization that has a number of management specialists but the organization is not exactly very well-coordinated. For example its inability to properly deal with the basic problem of maintaining stocks of finished goods. To the marketing manager, stocks of a large variety of products are purely a means of supplying the company's customers with what they want and when they want it. Clearly, according to a marketing manager, a fully stocked warehouse is of prime importance to the company. But the production manager argues for long production runs, preferably on a smaller product range, particularly if a significant amount of time is lost when production is switched from one variety to another. The result would again be a tendency to increase the amount of stock carried but it is, of course, vital that the plant should be kept running. On the other hand, the finance manager sees stocks in terms of capital that is unproductively tied up and argues strongly for its reduction. Finally, there appears the personnel manager for whom a steady level of production is advantageous for having better labour relations. Thus, all these people would claim to uphold the interests of the organization, but they do so only from their own specialized points of view. They may come up with contradictory solutions and obviously, all of them cannot be right.

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The relationship among velocity, distance and acceleration is an example of a mathematical model. In accounting, the cost-volume-profit model is also an example of a mathematical model.

Symbolic models are precise and abstract and can be analysed and manipulated by using laws of mathematics. The models are more explanatory rather than descriptive.

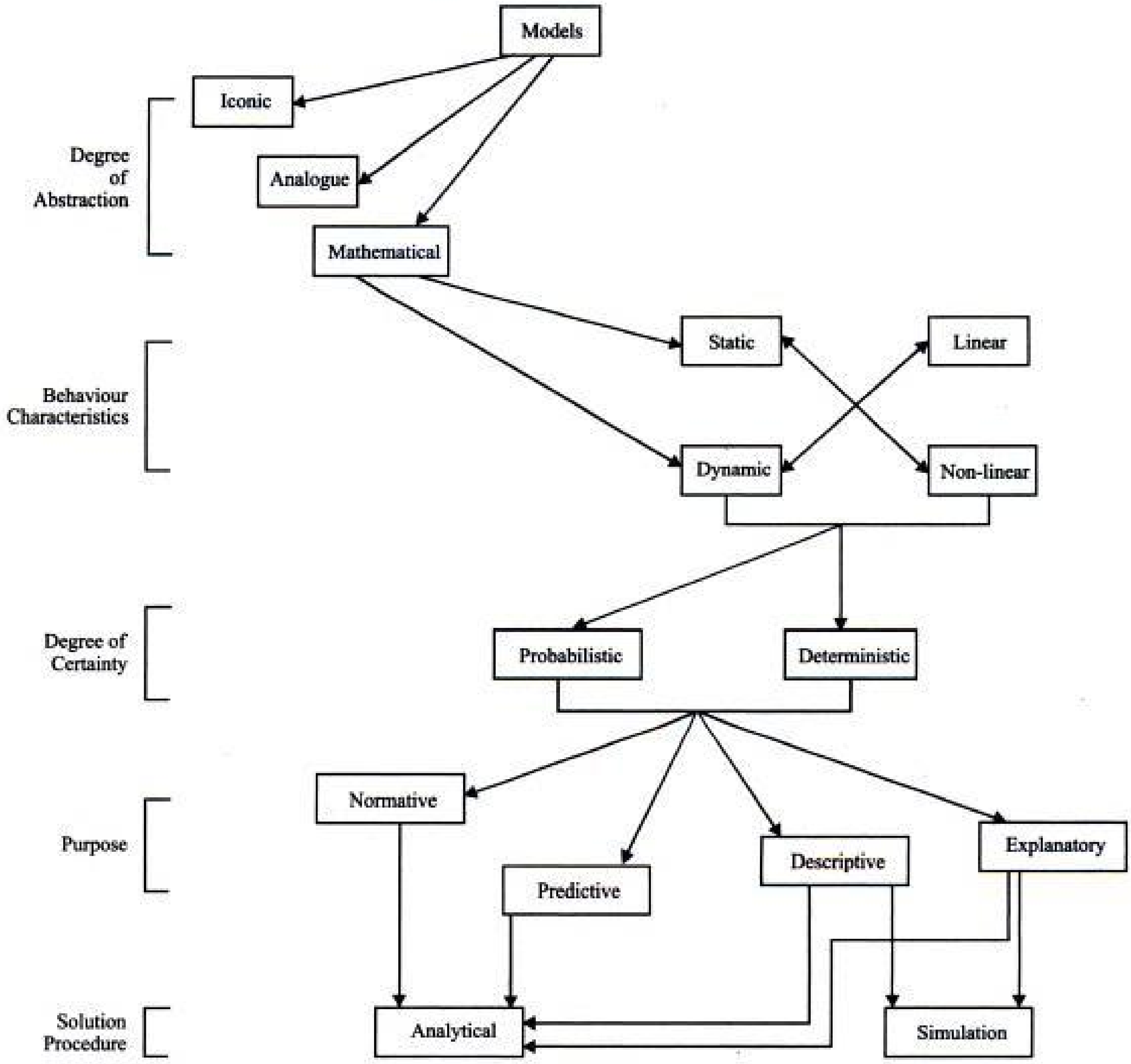

Fig. 1.2 Classification of Models

1.6.2 Classification Based on Function or Purpose

Models based on the purpose of their utility include the following types:

Descriptive models Descriptive models characterize things as they are. The major use of these models is to investigate the outcomes or consequences of various alternative courses of action. Since these models check the consequence only for a given condition (or alternative) rather than for all conditions, there is no guarantee that an alternative selected with the aid of descriptive analysis is optimal. These models are usually applied in decision situations where optimizing models are not applicable. They are also used when the final objective is to define the problem or to assess its seriousness rather than to select the best alternative. These models are especially used for predicting the behaviour of a particular system under various conditions.

Simulation is an example of a descriptive technique for conducting experiments with the systems.

Predictive models These models indicate the consequence – 'if this occurs, then that will follow'. They relate dependent and independent variables and permit the trying out, of the 'what if' questions.

In other words, these models are used to predict the outcomes of a given set of alternatives for the problem. These models do not have an objective function as a part of the model of evaluating decision alternatives.

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Model construction consists of hypothesizing relationships between variables subject to and not subject to control by decision-maker. Certain basic components required in every decision problem model are:

Controllable (Decision) Variables These are the issues or factors in the problem whose values are to be determined (in the form of numerical values) by solving the model. The possible values assigned to these variables are called decision alternatives (strategies or courses of action). For example, in *queuing theory*, the number of service facilities is the decision variable.

Uncontrollable (Exogenous) Variables These are the factors. The values of these variables are not under the control of the decision-maker and are also termed as state of nature.

Objective Function (or Performance Measures) This is a representation of (i) the criterion that expresses the decision-maker's manner of evaluating the desirability of alternative values of the decision variables, and (ii) how that criterion is to be optimized (minimized or maximized). For example, in *queuing theory* the decision-maker may consider several criteria such as minimizing the average waiting time of customers, or the average number of customers in the system at any given time.

Exogenous variables are those whose numerical value depends upon the external environment that influences an organization

Policies and Constraints (or Limitations) These are the restrictions on the values of the decision variables. These restrictions can arise due to organization policy, legal restraints or limited resources such as space, money, manpower, material, etc. The constraints may be in the form of equations or inequalities.

Functional Relationships In a decision problem, the decision variables in the objective function and in the constraints, are connected by a specific functional relationship. A general decision problem model might take the following form:

Optimize (Max. or Min.) $Z = f(\mathbf{x})$

subject to the constraints

$$g_i(\mathbf{x}) \{\leq, =, \geq\} b_i; \quad i = 1, 2, \ldots, m$$

and $\mathbf{x} \geq 0$

where, $\mathbf{x}$ = a vector of decision variables $(x_1, x_2, \ldots, x_n)$

$f(\mathbf{x})$ = criterion or objective function to be optimized

$g_i(\mathbf{x})$ = the ith constraint

b_i = fixed amount of the ith resource

A model is referred to as a linear model if all functional relationships among decision variables $x_1, x_2, \ldots, x_n$ in $f(x)$ and $g(x)$ are of a linear form. But if one or more of the relationships are non-linear, the model is said to be a non-linear model.

Parameters These are constants in the functional relationships. Parameters can either be deterministic or probabilistic in nature. A deterministic parameter is one whose value is assumed to occur with certainty. However, if constants are considered directly or explicitly as random variables, they are probabilistic parameters.

Optimization methods yield the best values for the decision variables both for unconstrained and constrained problems

Step 3: Solving the Mathematical Model Once a mathematical model of the problem has been formulated, the next step is to solve it, that is, to obtain numerical values of decision variables. Obtaining these values depends on the specific form or type, of mathematical model. Solving the model requires the use of various mathematical tools and numerical procedures. In general, the following two categories of methods are used for solving an OR model.

(i) In constrained problems, these values simultaneously satisfy all of the constraints and provide an optimal or acceptable value for the objective function or measure of effectiveness.

(ii) However, these values provide an acceptable value for the objective function.

Heuristic methods are sometimes described as *rules of thumb which work*. An example of a commonly used heuristic is 'stand in the shortest line'. Although using this rule may not work if everyone in the shortest line requires extra time, in general, it is not a bad rule to follow. These methods are used when obtaining optimal solution is either very time consuming or the model is too complex.

Sometimes difficulties in problem solving arise due to lack of an appropriate methodology for it and psychological perceptions on the part of the problem solver. The major difficulties in problem solving can be grouped into the following categories:

(i) Failure to recognize the existence of a problem
- Some people tend to personalize problems
- Information is not received to signal the fact that a problem exists

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If the measure of effectiveness such as profit, cost, etc., is represented as a linear function of several variables and if limitations on resources (constraints) can be expressed as a system of linear equalities or inequalities, the allocation problem is classified as a linear programming problem. But if the objective function of any or all of the constraints cannot be expressed as a system of linear equalities or inequalities, the allocation problem is classified as a non-linear programming problem.

When the solution values or decision variables of a problem are restricted to being integer values or just zero-one values, the problem is classified as an integer programming problem or a zero-one programming problem, respectively.

A problem having multiple, conflicting and incommensurable objective functions (goals) subject to linear constraints is called a goal programming problem. If the decision variables in the linear programming problem depend on chance the problem is called a stochastic programming problem.

If resources such as workers, machines or salesmen have to be assigned to perform a certain number of activities such as jobs or territories on a one-to-one basis so as to minimize total time, cost or distance involved in performing a given activity, such problems are classified as assignment problems. But if the activities require more than one resource and conversely if the resources can be used for more than one activity, the allocation problem is classified as a transportation problem.

- **Inventory models** Inventory models deal with the problem of determination of how much to order at a point in time and when to place an order. The main objective is to minimize the sum of three conflicting inventory costs: The cost of holding or carrying extra inventory, the cost of shortage or delay in the delivery of items when it is needed and the cost of ordering or set-up. These are also useful in dealing with quantity discounts and selective inventory control.

- **Waiting line (or Queuing) models** These models have been developed to establish a trade-off between costs of providing service and the waiting time of a customer in the queuing system. Constructing a model entails describing the components of the system: Arrival process, queue structure and service process and solving for the measure of performance – average length of waiting time, average time spent by the customer in the line, traffic intensity, etc., of the waiting system.

- **Competitive (Game Theory) models** These models are used to characterize the behaviour of two or more opponents (called players) who compete for the achievement of conflicting goals. These models are classified according to several factors such as number of competitors, sum of loss and gain, and the type of strategy which would yield the best or the worst outcomes.

- **Network models** These models are applied to the management (planning, controlling and scheduling) of large-scale projects. PERT/CPM techniques help in identifying potential trouble spots in a project through the identification of the critical path. These techniques improve project coordination and enable the efficient use of resources. Network methods are also used to determine time-cost trade-off, resource allocation and help in updating activity time.

- **Sequencing models** The sequencing problem arises whenever there is a problem in determining the sequence (order) in which a number of tasks can be performed by a number of service facilities such as hospital, plant, etc., in such a way that some measure of performance, for example, total time to process all the jobs on all the machines, is optimized.

- **Replacement models** These models are used when one must decide the optimal time to replace an equipment for one reason or the other – for instance, in the case of the equipment whose efficiency deteriorates with time or fails immediately and completely. For example, in case of an automobile, the user has his own measure of effectiveness. So there will not be one single optimal answer for everyone, even if each automobile gives exactly the same service.

- **Dynamic programming models** Dynamic programming may be considered as an outgrowth of mathematical programming, involving the optimization of multistage (sequence of interrelated decisions) decision processes. The method starts by dividing a given problem into stages or sub-problems and then solves those sub-problems sequentially until the solution to the original problem is obtained.

- **Markov-chain models** These models are used for analysing a system which changes over a period of time among various possible outcomes or states. The model, while dealing with such systems, describes transitions in terms of transition probabilities of various states. These models have been used to test brand-loyalty and brand-switching tendencies of consumers, where each system state is considered to be a particular brand purchase.

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Letter Values

A -	P -	E -	Total — Group 1 = ______
R -	F -	S -	Total — Group 2 = ______
H -	T -	O -	Score = ______

Hint The solution of this equal-group-total-assumption formulation has the letter values A – 6, E – 7, F – 2, H – 1, O – 8, P – 4, R – 5, S – 3, T – 9 with equal-group totals of 139. If this formulation had no feasible solution, then it would be necessary to solve a sequence of problems in which the right-hand side of the first constraint is varied with increasing positive and negative integer values until a feasible solution is found.

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5. Linear programming also helps in the re-evaluation of a basic plan for changing conditions. If conditions change when the plan is partly carried out, they can be determined so as to adjust the remainder of the plan for best results.

2.4 LIMITATIONS OF LINEAR PROGRAMMING

In spite of having many advantages and wide areas of applications, there are some limitations associated with this technique. These are given below:

1. Linear programming treats all relationships among decision variables as linear. However, generally, neither the objective functions nor the constraints in real-life situations concerning business and industrial problems are linearly related to the variables.
2. While solving an LP model, there is no guarantee that we will get integer valued solutions. For example, in finding out how many men and machines would be required to perform a particular job, a non-integer valued solution will be meaningless. Rounding off the solution to the nearest integer will not yield an optimal solution. In such cases, integer programming is used to ensure integer value to the decision variables.
3. The linear programming model does not take into consideration the effect of time and uncertainty. Thus, the LP model should be defined in such a way that any change due to internal as well as external factors can be incorporated.
4. Sometimes large-scale problems can be solved with linear programming techniques even when the assistance of a computer is available. For this, the main problem can be divided into several smal' problems and each one of them can be solved separately.
5. Parameters appearing in the model are assumed to be constant but in real-life situations, they are frequently neither known nor constant.
6. It deals with only single objective, whereas in real-life situations we may come across conflicting multi-objective problems. In such cases, instead of the LP model, a goal programming model is used to get satisfactory values of these objectives.

2.5 APPLICATION AREAS OF LINEAR PROGRAMMING

Linear programming is the most widely used technique of decision-making in business and industry and in various other fields. In this section, we will discuss a few of the broad application areas of linear programming.

Agricultural Applications

These applications fall into categories of farm economics and farm management. The former deals with the agricultural economy of a nation or a region, while the latter is concerned with the problems of the individual farm.

The study of farm economics deals with interregional competition and optimum allocation of crop production. Efficient production patterns can be specified by a linear programming model under regional land resources and national demand constraints.

Linear programming can be applied in agricultural planning, e.g. allocation of limited resources such as acreage, labour, water supply, working capital, etc., in a way so as to maximize the net revenue.

Military Applications

Military applications include the problem of selecting an air weapon system against the enemy so as to keep them pinned down, and at the same time ensuring that the minimum amount of aviation gasoline is used. The applications are also used for varying the transportation in such a way that it maximizes the total tonnage of bombs dropped on a set of targets and takes care of the problem of community defence against disaster, the solution of which yields the number of defence units that should be used in a given attack in order to provide the required level of protection at the lowest possible cost.

Production Management

- *Product Mix* A company can produce several different products, each of which requires the use of limited production resources. In such cases, it is essential to determine the quantity of each product to be produced, knowing its marginal contribution and amount of available resource used by it. The objective is to maximize the total contribution, subject to all constraints.

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to prepare and pack 1,000 gallons of A and Rs 28,000 to prepare and pack a quintal of B, whereas in plant 2 these costs are Rs 18,000 and Rs 26,000, respectively. The company is obliged to produce daily at least 10 thousand gallons of A and 8 quintals of B.

Formulate this problem as an LP model to find out as to how the company should organize its production so that the required amounts of the two products be obtained at the minimum cost.

LP model formulation The data of the problem is summarized as follows:

Resources/Constraints	*Product*		*Total Availability (hrs)*
	A	*B*	
Preparation time (hrs)	Plant 1: 3 hrs/thousand gallons	1 hr/quintal	16
	Plant 2: 2 hrs/thousand gallons	1.5 hr/quintal	16
Minimum daily production	10 thousand gallons	8 quintals	
Cost of production (Rs)	Plant 1: 15,000/thousand gallons	28,000/quintals	
	Plant 2: 18,000/thousand gallons	26,000/quintals	

Decision variables Let

x_1, x_2 = quantity of product A (in '000 gallons) to be produced in plants 1 and 2, respectively.

x_3, x_4 = quantity of product B (in quintals) to be produced in plants 1 and 2, respectively.

The LP model

Minimize (total cost) $Z = 15{,}000x_1 + 18{,}000x_2 + 28{,}000x_3 + 26{,}000x_4$

subject to the constraints

(i) Preparation time

(a) $3x_1 + x_2 \le 16$, (b) $2x_3 + 1.5x_4 \le 16$

(ii) Minimum daily production requirement

(a) $x_1 + x_2 \ge 10$, (b) $x_3 + x_4 \ge 8$

and $x_1, x_2, x_3, x_4 \ge 0$.

Example 2.3 An electronic company is engaged in the production of two components C_1 and C_2 that are used in radio sets. Each unit of C_1 costs the company Rs 5 in wages and Rs 5 in material, while each of C_2 costs the company Rs 25 in wages and Rs 15 in material. The company sells both products on one-period credit terms, but the company's labour and material expenses must be paid in cash. The selling price of C_1 is Rs 30 per unit and of C_2 it is Rs 70 per unit. Because of the company's strong monopoly in these components, it is assumed that the company can sell, at the prevailing prices, as many units as it produces. The company's production capacity is, however, limited by two considerations. First, at the beginning of period 1, the company has an initial balance of Rs 4,000 (cash plus bank credit plus collections from past credit sales). Second, the company has available in each period 2,000 hours of machine time and 1,400 hours of assembly time. The production of each C_1 requires 3 hours of machine time and 2 hours of assembly time, whereas the production of each C_2 requires 2 hours of machine time and 3 hours of assembly time. Formulate this problem as an LP model so as to maximize the total profit to the company.

LP model formulation The data of the problem is summarized as follows:

Resources/Constraints	*Components*		*Total Availability*
	C_1	C_2	
Budget (Rs)	10/unit	40/unit	Rs 4,000
Machine time	3 hrs/unit	2 hrs/unit	2,000 hours
Assembly time	2 hrs/unit	3 hrs/unit	1,400 hours
Selling price	Rs 30	Rs 70	
Cost (wages + material) price	Rs 10	Rs 40	

Decision variables Let x_1 and x_2 = number of units of components C_1 and C_2 to be produced, respectively.

The LP model

Maximize (total profit) Z = Selling price – Cost price

$= (30 - 10)\,x_1 + (70 - 40)\,x_2 = 20x_1 + 30x_2$

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to maximize his profit, subject of course, to all the production and marketing restrictions. Formulate this problem as an LP model to maximize total profit.

LP model formulation The data of the problem is summarized as follows:

Constraints	*Production*		*Availability*
	8-ounce Bottles	*16-ounce Bottles*	
Machine A time	100/minute	40/minute	8 × 5 × 60 = 2,400 minutes
Machine B time	60/minute	75/minute	8 × 5 × 60 = 2,400 minutes
Production	1	1	3,00,000 units/week
Marketing	1	–	25,000 units/week
	–	1	7,000 units/week
Profit/unit (Rs)	0.15	0.25	

Decision variables Let x_1 and x_2 = units of 8-ounce and 16-ounce bottles, respectively to be produced weekly.

The LP model

Maximize (total profit) $Z = 0.15x_1 + 0.25x_2$

subject to the constraints

(i) Machine time : (a) $\dfrac{x_1}{100} + \dfrac{x_2}{40} \le 2{,}400$ and (b) $\dfrac{x_1}{60} + \dfrac{x_2}{75} \le 2{,}400$

(ii) Production : $x_1 + x_2 \le 3{,}00{,}000$

(iii) Marketing : (a) $x_1 \le 25{,}000$, (b) $x_2 \le 7{,}000$

and $x_1, x_2 \ge 0$.

Example 2.9 A company, engaged in producing tinned food has 300 trained employees on its rolls, each of whom can produce one can of food in a week. Due to the developing taste of public for this kind of food, the company plans to add to the existing labour force, by employing 150 people, in a phased manner, over the next five weeks. The newcomers would have to undergo a two-week training programme before being put to work. The training is to be given by employees from among the existing ones and it is a known fact that one employee can train three trainees. Assume that there would be no production from the trainers and the trainees during training period, as the training is off-the-job. However, the trainees would be remunerated at the rate of Rs 300 per week, the same rate would apply as for the trainers.

The company has booked the following orders to supply during the next five weeks:

Week :	1	2	3	4	5
No. of cans :	280	298	305	360	400

Assume that the production in any week would not be more than the number of cans ordered for so that every delivery of the food would be 'fresh'.

Formulate this problem as an LP model to develop a training schedule that minimizes the labour cost over the five-week period. [*Delhi Univ., MBA, Nov. 1998, 2000*]

LP model formulation The data of the problem is summarized as given below:

(i) Cans supplied

Week :	1	2	3	4	5
Number :	280	298	305	360	400

(ii) Each trainee has to undergo a two-week training.
(iii) One employee is required to train three trainees.
(iv) Every trained worker produces one can/week but there would be no production from trainers and trainees during training.
(v) Nur.ber of employees to be employed = 150
(vi) The production in any week is not to exceed the cans required.
(vii) Number of weeks for which newcomers would be employed: 5, 4, 3, 2, 1.

From the given information you may observe following facts:

(a) Workers employed at the beginning of the first week would get salary for all the five weeks; those employed at the beginning of the second week would get salary for four weeks and so on.
(b) The value of the objective function would be obtained by multiplying it by 300 because each person would get a salary of Rs 300 per week.

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LP model formulation Let

x_1 = quantity of Venus (kilolitres) produced in the company
x_2 = quantity of Diana (kilolitres) produced in the company
x_3 = quantity of Diana (kilolitres) produced by hired facilities
x_4 = quantity of Aurora (kilolitres) produced in the company

The LP model

Maximize (total profit) $Z = 4{,}000x_1 + 3{,}500x_2 + (3{,}500 - 1{,}000)x_3 + 2{,}000x_4$

subject to the constraints

(i) Special additive : $0.30x_1 + 0.15x_2 + 0.15x_3 + 0.75x_4 \le 600$

(ii) Own milling facility : $\dfrac{x_1}{2} + \dfrac{x_2}{3} + \dfrac{x_4}{5} \le 100$

(iii) Hired milling facility : $\dfrac{x_3}{3} \le 40$

(iv) Packing : $\dfrac{x_1}{12} + \dfrac{x_2 + x_3}{12} + \dfrac{x_4}{12} \le 80$

(v) Marketing:

(i) $x_1 \le 100$ (Venus); (ii) $x_2 + x_3 \le 400$ (Diana); (iii) $200 \le x_4 \le 600$ (Aurora)

and $x_1, x_2, x_3, x_4 \ge 0$.

Example 2.15 Four products have to be processed through a particular plant, the quantities required for the next production period are:

Product 1 :	2,000 units	Product 2 :	3,000 units
Product 3 :	3,000 units	Product 4 :	6,000 units

There are three production lines on which the products could be processed. The rates of production in units per day and the total available capacity in days are given in the following table. The corresponding cost of using the lines is Rs 600, Rs 500 and Rs 400 per day, respectively.

Production Line (days)	*Product*				*Maximum Line*
	1	*2*	*3*	*4*	
1	150	100	500	400	20
2	200	100	760	400	20
3	160	80	890	600	18
Total	2,000	3,000	3,000	6,000	

Formulate this problem as an LP model to minimize the cost of operation. [*Delhi Univ., MBA, 1994*]

LP model formulation Let x_{ij} = number of units of product i (i = 1, 2, 3, 4) produced on production line j (j = 1, 2, 3)

The LP model

Minimize (total cost) $Z = 600 \sum_{i=1}^{4} x_{i1} + 500 \sum_{i=1}^{4} x_{i2} + 400 \sum_{i=1}^{4} x_{i3}$

subject to the constraints

(i) Production: (a) $\sum_{i=1}^{3} x_{i1} = 2{,}000$, (b) $\sum_{i=1}^{3} x_{i2} = 3{,}000$

(c) $\sum_{i=1}^{3} x_{i3} = 3{,}000$, (d) $\sum_{i=1}^{3} x_{i4} = 6{,}000$

(ii) Line capacity

(a) $\dfrac{x_{11}}{150} + \dfrac{x_{12}}{100} + \dfrac{x_{13}}{500} + \dfrac{x_{14}}{400} \le 20$, (b) $\dfrac{x_{21}}{200} + \dfrac{x_{22}}{100} + \dfrac{x_{23}}{760} + \dfrac{x_{24}}{400} \le 20$

(c) $\dfrac{x_{31}}{160} + \dfrac{x_{32}}{80} + \dfrac{x_{33}}{890} + \dfrac{x_{34}}{600} \le 18$

and $x_{ij} \ge 0$ for all i and j.

Example 2.16 A manufacturer of biscuits is considering making four types of gift packs containing three types of biscuits: Orange cream (OC), Chocolate cream (CC) and Wafers (W). A market research study, recently conducted to assess the preference of the consumers, shows the following types of assortments to be in good demand:

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(iv) Maximum newspaper and magazine advertisement

$$\frac{x_3 + x_4}{x_1 + x_2 + x_3 + x_4} \leq 0.60 \quad \text{or} \quad -0.6x_1 - 0.6x_2 + 0.4x_3 + 0.4x_4 \leq 0$$

(v) Exposure to families with income over Rs 50,000

$$2,00,000x_1 + 5,00,000x_2 + 3,00,000x_3 + 1,00,000x_4 \geq 45,00,000$$

and $x_1, x_2, x_3, x_4 \geq 0$.

Example 2.20 An advertising agency is preparing an advertising campaign for a group of agencies. These agencies have decided that different characteristics of their target customers should be given different importance (weightage). The following table gives the characteristics with their corresponding importance (weightage).

	Characteristics	*Weightage (%)*
Age	25–40 years	20
Annual income	Above Rs 60,000	30
Female	Married	50

The agency has carefully analyzed of three media and has compiled the following data:

Data Item	*Media*		
	Women's Magazine (%)	*Radio (%)*	*Television (%)*
Reader characteristics			
(i) Age: 25–40 years	80	70	60
(ii) Annual income: Above Rs 60,000	60	50	45
(iii) Females/Married	40	35	25
Cost per advertisement (Rs)	9,500	25,000	1,00,000
Minimum number of advertisement allowed	10	5	5
Maximum number of advertisement allowed	20	10	10
Audience size (1000s)	750	1,000	1,500

The budget for launching the advertising campaign is Rs 5,00,000. Formulate this problem as an LP model for the agency to maximize the total expected effective exposure.

LP model formulation Let x_1, x_2 and x_3 = number of advertisements made using advertising media: women's magazines, radio and television, respectively.

The effectiveness coefficient corresponding to each of the advertising media is calculated as follows:

Media	*Effectiveness Coefficient*
Women's magazine	0.80 (0.20) + 0.60 (0.30) + 0.40 (0.50) = 0.54
Radio	0.70 (0.20) + 0.50 (0.30) + 0.35 (0.50) = 0.46
Television	0.60 (0.20) + 0.45 (0.30) + 0.25 (0.50) = 0.38

The coefficient of the objective function, i.e. effective exposure for all the three media employed, can be computed as follows:

Effective exposure = Effectiveness coefficient × Audience size

where effectiveness coefficient is a weighted average of audience characteristics. Thus, the effective exposure of each media is as follows:

$$\text{Women's magazine} = 0.54 \times 7,50,000 = 4,05,000$$

$$\text{Radio} = 0.46 \times 10,00,000 = 4,60,000$$

$$\text{Television} = 0.38 \times 15,00,000 = 5,70,000$$

The LP model

Maximize (effective exposure) $Z = 4,05,000x_1 + 4,60,000x_2 + 5,70,000x_3$

subject to the constraints

(i) Budget: $9,500x_1 + 25,000x_2 + 1,00,000x_3 \leq 5,00,000$

(ii) Minimum number of advertisements allowed

(a) $x_1 \geq 10$; (b) $x_2 \geq 5$; and (c) $x_3 \geq 5$

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Portfolio Data

Shares Under Consideration	*A*	*B*	*C*	*D*	*E*	*F*
Current price per share (Rs)	80.00	100.00	160.00	120.00	150.00	200.00
Projected annual growth rate	0.08	0.07	0.10	0.12	0.09	0.15
Projected annual dividend per share (Rs)	4.00	4.50	7.50	5.50	5.75	0.00
Projected risk return	0.05	0.03	0.10	0.20	0.06	0.08

The total amount available for investment is Rs 25 lakh and the following conditions are required to be satisfied:

(i) The maximum rupee amount to be invested in alternative F is Rs 2,50,000.
(ii) No more than Rs 5,00,000 should be invested in alternatives A and B combined.
(iii) Total weighted risk should not be greater than 0.10, where

$$\text{Total weighted risk} = \frac{(\text{Amount invested in alternative } j)\,(\text{Risk of alternative } j)}{\text{Total amount invested in all the alternatives}}$$

(iv) For the sake of diversity, at least 100 shares of each stock should be purchased.
(v) At least 10 per cent of the total investment should be in alternatives A and B combined.
(vi) Dividends for the year should be at least 10,000.

Rupee return per share of stock is defined as the price per share one year hence, less current price per share plus dividend per share. If the objective is to maximize total rupee return, formulate this problem as an LP model for determining the optimal number of shares to be purchased in each of the shares under consideration. You may assume that the time horizon for the investment is one year.

LP model formulation Let x_1, x_2, x_3, x_4, x_5 and x_6 = number of shares to be purchased in each of the six investment proposals A, B, C, D, E and F, respectively.

Rupee return per share = Price per share one year hence – Current price per share + Dividend per share

= Current price per share × Projected annual growth rate (i.e. Projected growth each year + Dividend per share).

Thus, we compute the following data:

Investment Alternatives	:	*A*	*B*	*C*	*D*	*E*	*F*
No. of shares purchased	:	x_1	x_2	x_3	x_4	x_5	x_6
Projected growth for each share (Rs)	:	6.40	7.00	16.00	14.40	13.50	30.00
Projected annual dividend per share (Rs)	:	4.00	4.50	7.50	5.50	5.75	0.00
Return per share (Rs)	:	10.40	11.50	23.50	19.90	19.25	30.00

The LP model

Maximize (total return) $R = 10.40x_1 + 11.50x_2 + 23.50x_3 + 19.90x_4 + 19.25x_5 + 30.00x_6$

subject to the constraints

(i) $80x_1 + 100x_2 + 160x_3 + 120x_4 + 150x_5 + 200x_6 \le 25{,}00{,}000$ (total fund available)

(ii) $200x_6 \le 2{,}50{,}000$ [from condition (i)]

(iii) $80x_1 + 100x_2 \le 5{,}00{,}000$ [from condition (ii)]

(iv) $$\frac{80x_1\,(0.05) + 100x_2\,(0.03) + 160x_3\,(0.10) + 120x_4\,(0.02) + 150x_5\,(0.06) + 200x_6\,(0.08)}{80x_1 + 100x_2 + 160x_3 + 120x_4 + 150x_5 + 200x_6} \le 1$$

or $4x_1 + 3x_2 + 16x_3 + 24x_4 + 9x_5 + 16x_6 \le 8x_1 + 10x_2 + 16x_3 + 12x_4 + 15x_5 + 20x_6$

or $-4x_1 - 7x_1 + 0x_3 + 12x_4 - 6x_5 - 4x_6 \le 0$

(v) $x_1 \ge 100,\ x_2 \ge 100,\ x_3 \ge 100,\ x_4 \ge 100,\ x_5 \ge 100,\ x_6 \ge 100$ [from condition (iv)]

(vi) $80x_1 + 100x_2 \ge 0.10\,(80x_1 + 100x_2 + 160x_3 + 120x_4 + 150x_5 + 200x_6)$ [from condition (v)]

or $80x_1 + 100x_2 \ge 8x_1 + 10x_2 + 16x_3 + 12x_4 + 15x_5 + 20x_6$

or $72x_1 + 90x_2 - 16x_3 - 12x_4 - 15x_5 - 20x_6 \ge 0$

(vii) $4x_1 + 4.5x_2 + 7.5x_3 + 5.5x_4 + 5.75x_5 \ge 10{,}000$ [from condition (vi)]

and $x_j \ge 0;\ j = 1, 2, 3, 4, 5$ and 6.

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$$x_1 + 0.5x_2 + 1.08s_1 = x_5 + s_2$$

Also, since the company will not commit an investment exceeding Rs 75,000 in any project, therefore the constraint becomes: $x_i \leq 75{,}000$ for $i = 1, 2, 3, 4, 5.$

and $\quad x_1, x_2, x_3, x_4, x_5, s_0, s_1, s_2 \geq 0.$

2.8.4 Examples on Agriculture

Example 2.29 A cooperative farm owns 100 acres of land and has Rs 25,000 in funds available for investment. The farm members can produce a total of 3,500 man-hours worth of labour during September–May and 4,000 man-hours during June – August. If any of these man-hours are not needed, some members of the firm would use them to work on a neighbouring farm for Rs 2 per hour during September – May and Rs 3 per hour during June – August. Cash income can be obtained from the three main crops and two types of livestock: dairy cows and laying hens. No investment funds are needed for the crops. However, each cow will require an investment outlay of Rs 3,200 and each hen will require Rs 15.

In additon each cow will also require 15 acres of land, 100 man-hours during the summer. Each cow will produce a net annual cash income of Rs 3,500 for the farm. The corresponding figures for each hen are: no acreage, 0.6 man-hours during September – May; 0.4 man-hours during June – August, and an annual net cash income of Rs 200. The chicken house can accommodate a maximum of 4,000 hens and the size of the cattle-shed limits the members to a maximum of 32 cows.

Estimated man-hours and income per acre planted in each of the three crops are:

	Paddy	*Bajra*	*Jowar*
Man-hours			
September-May	40	20	25
June-August	50	35	40
Net annual cash income (Rs)	1,200	800	850

The cooperative farm wishes to determine how much acreage should be planted in each of the crops and how many cows and hens should be kept in order to maximize its net cash income. Formulate this problem as an LP model to maximize net annual cash income.

LP model formulation The data of the problem is summarized as follows:

Constraints	*Cows*	*Hens*	*Crop*			*Extra Hours*		*Total Availability*
			Paddy	*Bajra*	*Jowar*	*Sept–May*	*June–Aug*	
Man-hours								
Sept–May	100	0.6	40	20	25	1	–	3,500
June–Aug	50	0.4	50	35	40	–	1	4,000
Land	1.5	–	1	1	1	–	–	100
Cow	1	–	–	–	–	–	–	32
Hens	–	1	–	–	–	–	–	4,000
Net annual cash income (Rs)	3,500	200	1,200	800	850	2	3	

Decision variables Let

x_1 and x_2 = number of dairy cows and laying hens, respectively.
x_3, x_4 and x_5 = average of paddy crop, *bajra* crop and *jowar* crop, respectively.
x_6 = extra man-hours utilized in Sept–May.
x_7 = extra man-hours utilized in June–Aug.

The LP model

Maximize (net cash income) $Z = 3{,}500x_1 + 200x_2 + 1{,}200x_3 + 800x_4 + 850x_5 + 2x_6 + 3x_7$

subject to the constraints

(i) Man-hours: $100x_1 + 0.6x_2 + 40x_3 + 20x_4 + 25x_5 + x_6 = 3{,}500$ (Sept-May duration)
$50x_1 + 0.4x_2 + 50x_3 + 35x_4 + 40x_5 + x_7 = 4{,}000$ (June-Aug duration)

(ii) Land availability: $1.5x_1 + x_3 + x_4 + x_5 \leq 100$

(iii) Livestock: (a) $x_1 \leq 32$ (dairy cows), (b) $x_2 \leq 4{,}000$ (laying hens)

and $\quad x_1, x_2, x_3, x_4, x_5, x_6, x_7 \geq 0.$

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No more than 40 doctors can start their five working days on the same day. Formulate this problem as an LP model to minimize the number of doctors employed by the hospital.

[*Delhi Univ., MBA (HCA), 1999*]

LP model formulation Let x_j = number of doctors who start their duty on day j (j = 1, 2, . . ., 7) of the week.

The LP model

Minimize (total number of doctors) $Z = x_1 + x_2 + x_3 + x_4 + x_5 + x_6 + x_7$

subject to the constraints

(i) $x_1 + x_4 + x_5 + x_6 + x_7 \geq 35$, (ii) $x_2 + x_5 + x_6 + x_7 + x_1 \geq 55$

(iii) $x_3 + x_6 + x_7 + x_1 + x_2 \geq 60$, (iv) $x_4 + x_7 + x_1 + x_2 + x_3 \geq 50$

(v) $x_5 + x_1 + x_2 + x_3 + x_4 \geq 60$, (vi) $x_6 + x_2 + x_3 + x_4 + x_5 \geq 50$

(vii) $x_7 + x_3 + x_4 + x_5 + x_6 \geq 45$, (viii) $x_j \leq 40$

and $x_j \geq 0$ for all j.

Example 2.35 A machine tool company conducts on-the-job training programme for machinists. Trained machinists are used as teachers for the programme, in the ratio of one for every ten trainees. The training programme lasts for one month. From past experience it has been found that out of the ten trainees hired, only seven complete the programme successfully and the rest are released.

Trained machinists are also needed for machining. The company's requirement for machining for the next three months is as follows: January 100, February 150 and March 200. In addition, the company requires 250 machinists by April. There are 130 trained machinists available at the beginning of the year. Pays per month are:

Each trainee	:	Rs 4,400
Each trained machinist (machining and teaching)	:	Rs 4,900
Each trained machinist idle	:	Rs 4,700

Formulate this problem as an LP model to minimize the cost of hiring and training schedule and the company's requirements.

LP model formulation Let

x_1, x_2 = trained machinist teaching and idle in January, respectively

x_3, x_4 = trained machinist teaching and idle in February, respectively

x_5, x_6 = trained machinist teaching and idle in March, respectively

The LP model

Minimize (total cost) Z = Cost of training programme (teachers and trainees) + Cost of idle machinists + Cost of machinists doing machine work (constant)

$$= 4{,}400\,(10x_1 + 10x_3 + 10x_5) + 4{,}900\,(x_1 + x_3 + x_5) + 4{,}700\,(x_2 + x_4 + x_6)$$

subject to the constraints

(i) Total trained machinists available at the beginning of January

= Number of machinists doing machining + Teaching + Idle

$$130 = 100 + x_1 + x_2 \quad \text{or} \quad x_1 + x_2 = 30$$

(ii) Total trained machinists available at the beginning of February

= Number of machinists in January + Joining after training programme

$$130 + 7x_1 = 150 + x_3 + x_4 \quad \text{or} \quad 7x_1 - x_3 - x_4 = 20$$

In January there are $10x_1$ trainees in the programme and out of those only $7x_1$ will become trained machinists.

(iii) Total trained machinists available at the beginning of March

= Number of machinists in January + Joining after training programme in January and February

$$130 + 7x_1 + 7x_3 = 200 + x_5 + x_6$$
$$7x_1 + 7x_3 - x_5 - x_6 = 70$$

(iv) Company requires 250 trained machinists by April

$$130 + 7x_1 + 7x_3 + 7x_5 = 250$$
$$7x_1 + 7x_3 + 7x_5 = 120$$

and $x_1, x_2, x_3, x_4, x_5, x_6 \geq 0$.

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Price per kg (Rs)	Strength Index	Acidity Index	Per cent Caffeine	Supply Available (kg)
South Indian, 30	6	4.0	2.0	40,000
Assamese, 40	8	3.0	2.5	20,000
Imported, 35	5	3.5	1.5	15,000

The requirements for plains X and plains XX coffees are given in the following table:

Plains Coffee	Price per kg (Rs)	Minimum Strength Caffeine	Maximum Acidity (kg)	Maximum Per Cent	Quantity Demanded
X	45	6.5	3.8	2.2	35,000
XX	55	6.0	3.5	2.0	25,000

Assume that 35,000 kg of plains X and 25,000 kg of plains XX, are to be sold. Formulate this problem as an LP model to maximize sales.

15. A manufacturer of metal office equipments makes desks, chairs, cabinets and book cases. The work is carried out in the three major manufacturing departments: Metal stamping, Assembly and Finishing. Exhibits, A, B the and C give the requisite data of the problem.

Exhibit A

Department	Time Required per Unit of Product (hrs) Products				Available Time per Week (hrs)
	Desk	Chair	Cabinet	Bookcase	
Stamping	4	2	3	3	800
Assembly	10	6	8	7	1,200
Finishing	10	8	8	8	800

Exhibit B

Department	Cost (Rs) of Operation per Unit of Product			
	Desk	Chair	Cabinet	Bookcase
Stamping	15	8	12	12
Assembly	30	18	24	21
Finishing	35	28	25	21

Exhibit C: Selling price (Rs) per unit of product

Desk :	175	Chair :	95
Cabinet :	145	Bookcase :	130

In order to maximize weekly profits, what should be all production programme? Assuming that the items produced can be sold, which department needs to be expanded for increasing profits? Formulate this problem as an LP model.

16. The PQR stone company sells stone procured from any of three adjacent quarries. The stone sold by the company conforms to the following specification:

Material X equal to 30%
Material Y equal to or less than 40%
Material *Z* between 30% and 40%

Stone from quarry A costs Rs 10 per tonne and has the following properties:

Material X : 20%, Material Y : 60%, Material *Z* : 20%

Stone from quarry B costs Rs 12 per tonne and has the following properties:

Material X : 40%, Material Y : 30%, Material *Z* : 30%

Stone from quarry C costs Rs 15 per tonne and has the following properties:

Material X : 10%, Material Y : 40%, Material *Z* : 50%

From what quarries should the PQR stone company procure rocks in order to minimize cost per tonne of rock? Formulate this problem as an LP model. *[Delhi Univ., MBA, 1997]*

17. A manufacturer produces three models (I, II and III) of a certain product. He uses two types of raw material: A and B of which 4,000 and 6,000 units are available, respectively. The raw material requirements per unit of the three models are given below:

Raw Material	Requirements per Unit of Given Model		
	I	II	III
A	2	3	5
B	4	2	7

The labour time for each unit of model I is twice that of model II and three times that of model III. The entire labour force of the factory can produce the equivalent of 2,500 units of model I. A market survey indicates that the minimum demand for the three models is 500, 500 and 375 units, respectively. However, the ratios of the number of units produced must be equal to 3 : 2 : 5. Assume that the profit per unit of models I, II and III is Rs 60, Rs 40 and Rs 100, respectively. Formulate this problem as an LP model to determine the number of units of each product that will the maximize amount of profit.

18. A company manufactures two models of garden rollers: X and Y. When preparing the 2008 budget, it was found that the limitations on capacity were represented by the following weekly production maxima:

Model	Foundry	Machine-shop	Contribution per Model (Rs)
X	100	200	120
Y	240	150	90

In addition, the material required for model X was in short supply and sufficient only for 140 units per week, guaranteed for the year. Formulate this problem as an LP model to determine the optimal combination of output.

19. A company manufacturing television and radio sets has four major departments: chassis, cabinet, assembly and final testing. The monthly capacities of these are as follows:

	Television		Radio
Chassis	1,500	or	4,500
Cabinet	1,000	or	8,000
Assembly	2,000	or	4,000
Testing	3,000	or	9,000

The contribution of a television set is Rs 500 and that of a radio set Rs 250. Assume that the company can sell any quantity of either product. Formulate this problem as an LP model to determine the optimal combination of television and radio sets.

20. A company wants to plan production for the ensuing year so as to minimize the combined cost of production and inventory storage. In each quarter of the year, demand is anticipated to be 65, 80, 135 and 75 respectively. The product can be manufactured during regular time at a cost of Rs 16 per unit produced, or during overtime at a cost of Rs 20 per unit. The table given below gives data pertinent to production capacities. The cost of carrying one unit in inventory per quarter is Rs 2. The inventory level at the beginning of the first quarter is zero.

Quarter	Capacities (units)		Quarterly Demand
	Regular Time	Overtime	
1	80	10	65
2	90	10	80
3	95	20	135
4	70	10	75

Formulate this problem as an LP model so as to minimize the production plus storage costs for the entire year.

[Delhi Univ., MBA, 1998]

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HINTS AND ANSWERS

1. Let x_1, x_2 = number of units of products A and B to be produced, respectively.

$$\text{Max } Z = 40x_1 + 30x_2$$

subject to $3x_1 + x_2 \le 3{,}000$ (Man-hours)

$x_1 \le 8{,}000$; $x_2 \le 1{,}200$ (Marketing)

and $x_1, x_2 \ge 0$.

2. Let x_1, x_2 = number of productive runs of process 1 and 2, respectively.

$$\text{Max } Z = 300x_1 + 400x_2$$

subject to $\left.\begin{array}{l}5x_1 + 4x_2 \le 200\\ 3x_1 + 5x_2 \le 150\end{array}\right\}$ (Max amount of crude A and B)

$\left.\begin{array}{l}5x_1 + 4x_2 \ge 100\\ 8x_1 + 4x_2 \ge 80\end{array}\right\}$ (Market requirement of gasoline X and Y)

and $x_1, x_2 \ge 0$.

3. Let x_i = number of units purchased per month (i = 1, 2, 3 – April, May, June)

y_i = number of units sold per month (i = 1, 2, 3 – May, June, July)

$$\text{Max } Z = (90y_1 + 60y_2 + 75y_3) - (75x_1 + 75x_2 + 60x_3)$$

subject to $y_1 \le x_1 \le 150$

$y_2 \le x_1 + x_2 - y_1 \le 150$

$y_3 \le x_1 + x_2 + x_3 - y_1 - y_2 \le 150$

$x_1 + x_2 + x_3 = y_1 + y_2 + y_3$

and $x_i, y_i \ge 0$ for all i.

4. Let x_1, x_2 and x_3 = snumber of units of models I, II and III, respectively to be manufactured

$$\text{Max } Z = 60x_1 + 40x_2 + 100x_3$$

subject to $\left.\begin{array}{l}2x_1 + 3x_2 + 5x_3 \le 4{,}000\\ 4x_1 + 2x_2 + 7x_3 \le 6{,}000\end{array}\right\}$ (Raw material requirement)

$x_1 + \frac{1}{2}x_2 + \frac{1}{3}x_3 \le 2{,}500$ (Production limitation)

$x_1 \ge 500$; $x_2 \ge 500$; $x_3 \ge 375$ (Market demand)

$\frac{1}{3}x_1 = \frac{1}{2}x_2$; $\frac{1}{2}x_2 = \frac{1}{5}x_3$ (Ratios of production)

5. Let x_1 and x_2 = parts of A and B per hour manufactured, respectively.

$$\text{Max } Z = (5x_1 + 6x_2) - \left(\frac{20}{25} + \frac{14}{28} + \frac{17.50}{35} + 2.00\right)x_1 - \left(\frac{20}{24} + \frac{14}{35} + \frac{17.50}{25} + 3.00\right)x_2$$

$$= 1.20x_1 + 1.40x_2$$

subject to (i) $\frac{x_1}{25} + \frac{x_2}{24} \le 1$; (ii) $\frac{x_1}{28} + \frac{x_2}{35} \le 1$; (iii) $\frac{x_1}{35} + \frac{x_2}{25} \le 1$ (Manufacturing capacity)

and $x_1, x_2 \ge 0$.

6. Let x_{ijk} = number of units manufactured in month i (1, 2, 3 – Oct., Nov., Dec.) during shift j (j = 1, 2 – regular, overtime) and shipped in month k (k = 1, 2, 3 – Oct., Nov., Dec.)

$$\text{Min } Z = 3x_{111} + 5x_{121} + 4x_{112} + 6x_{122} + 5x_{113} + 7x_{123} + 3x_{212} + 5x_{222} + 4x_{213} + 6x_{223} + 3x_{313} + 5x_{323}$$

subject to $\left.\begin{array}{ll}x_{111} + x_{112} + x_{113} & \le 1{,}500\\ x_{212} + x_{213} & \le 1{,}500\\ x_{313} & \le 1{,}500\end{array}\right\}$ (Regular time)

$\left.\begin{array}{ll}x_{121} + x_{122} + x_{123} & \le 750\\ x_{222} + x_{223} & \le 750\\ x_{323} & \le 750\end{array}\right\}$ (Overtime)

$x_{111} + x_{121} = 1{,}000$

$x_{121} + x_{122} + x_{212} + x_{222} = 3{,}000$

$x_{113} + x_{123} + x_{213} + x_{223} + x_{313} + x_{323} = 2{,}000$

and $x_{ijk} \ge 0$ for all i, j, k.

7. Let x_1, x_2 = number of gallons of wine B and C in the blend, respectively.

$$\text{Max } Z = 20 + x_1 + x_2$$

subject to $30 \le \dfrac{(20 \times 27) + 33x_1 + 32x_2}{20 + x_1 + x_2} \le 31$ (Resultant degrees proof of blend)

$\dfrac{(20 \times 0.32) + 0.2x_1 + 0.3x_2}{20 + x_1 + x_2} \ge 0.25$ (Acidity)

$\dfrac{(20 \times 1.07) + 1.08x_1 + 1.04x_2}{20 + x_1 + x_2} \ge 1.06$ (Specific gravity)

$x_1 \le 34$ (Quality)

and $x_1, x_2 \ge 0$.

8. Let x_1, x_2 and x_3 = quantity of foods 1, 2 and 3 to be used, respectively.

$$\text{Min } Z = 1.50x_1 + 2.00x_1 + 1.20x_3$$

$x_1 + y_1 \le 12,$

subject to $350x_1 + 250x_2 + 200x_3 \ge 300$

$250x_1 + 300x_2 + 150x_3 \ge 200$

$100x_1 + 150x_2 + 75x_3 \ge 100$

$75x_1 + 125x_2 + 150x_3 \ge 100$

and $x_1, x_2, x_3 \ge 0$.

9. Let x_1 and x_2 = number of soccer balls of types X and Y, respectively.

Min Z = (2 hrs) (Rs 5.50/hr) x_1 + (4 hrs) (Rs 8.50/hr) x_1 + (3 hrs) (Rs 5.50/hr) x_2 + (6 hrs) (Rs 8.50/hr) x_2 = $45x_1 + 67.50x_2$

subject to $2x_1 + 3x_2 \le 80$ (Semi-skilled hours)

$4x_1 + 6x_2 \le 150$ (Skilled hours)

$x_1 \le 15$ (Ball X)

$x_2 \le 10$ (Ball Y)

and $x_1, x_2 \ge 0$.

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23. A constraint in an LP model restricts
    (a) value of objective function
    (b) value of a decision variable
    (c) use of the available resource
    (d) all of the above
24. The distinguishing feature of an LP model is
    (a) relationship among all variables is linear
    (b) it has single objective function and constraints
    (c) value of decision variables is non-negative
    (d) all of the above
25. Constraints in an LP model represents
    (a) limitations
    (b) requirements
    (c) balancing limitations and requirements
    (d) all of the above
26. Non-negativity condition is an important component of LP model because
    (a) variables value should remain under the control of the decision-maker
    (b) value of variables make sense and correspond to real-world problems
    (c) variables are interrelated in terms of limited resources
    (d) none of the above
27. Before formulating a formal LP model, it is better to
    (a) express each constraint in words
    (b) express the objective function in words
    (c) verbally identify decision variables
    (d) all of the above
28. Each constraint in an LP model is expressed as an
    (a) inequality with ³ sign
    (b) inequality with £ sign
    (c) equation with = sign
    (d) none of the above
29. Maximization of objective function in an LP model means
    (a) value occurs at allowable set of decisions
    (b) highest value is chosen among allowable decisions
    (c) neither of above
    (d) both (a) and (b)
30. Which of the following is not a characteristic of the LP model
    (a) alternative courses of action
    (b) an objective function of maximization type
    (c) limited amount of resources
    (d) non-negativity condition on the value of decision variables
31. The best use of linear programming technique is to find an optimal use of
    (a) money (b) manpower
    (c) machine (d) all of the above
32. Which of the following is not the characteristic of linear programming
    (a) resources must be limited
    (b) only one objective function
    (c) parameters value remains constant during the planning period
    (d) the problem must be of minimization type
33. Non-negativity condition in an LP model implies
    (a) a positive coefficient of variables in objective function
    (b) a positive coefficient of variables in any constraint
    (c) non-negative value of resources
    (d) none of the above
34. Which of the following is an assumption of an LP model
    (a) divisibility (b) proportionality
    (c) additivity (d) all of the above
35. Which of the following is a limitation associated with an LP Model
    (a) the relationship among decision variables in linear
    (b) no guarantee to get integer valued solutions
    (c) no consideration of effect of time and uncertainty on LP model
    (d) all of the above

Answers to Quiz

1. T	2. F	3. T	4. T	5. F	6. T	7. T
8. F	9. T	10. T	11. resources, objective		12. decisions	13. linear
14. linearly	15. parameters, unknown		16. objective function, constraints		17. inequalities	18. scarce resources
19. controllable and uncontrollable		20. certainty	21. (a)	22. (d)	23. (d)	24. (a)
25. (d)	26. (b)	27. (d)	28. (d)	29. (a)	30. (b)	31. (d)
32. (d)	33. (d)	34. (d)	35. (d)			

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These two points are then connected by a straight line as shown in Fig. 3.1(a). But the question is: *Where are these points satisfying $4x_1 + 6x_2 \leq 360$*. Any point above the constraint line violates the inequality condition. But any point below the line does not violate the constraint. Thus, the inequality and non-negativity condition can only be satisfied by the shaded area (feasible region) as shown in Fig. 3.1(a).

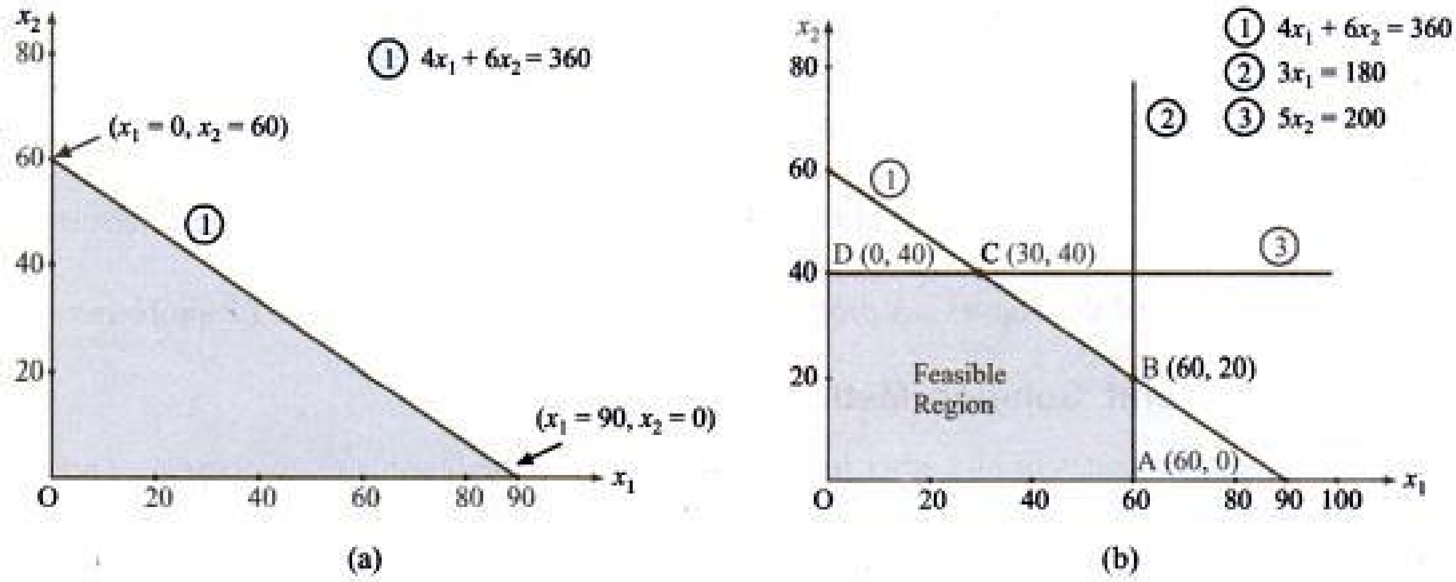

Fig. 3.1 Graphical Solution of LP Problem

Similarly, the constraints $3x_1 \leq 180$ and $5x_2 \leq 200$ are also plotted on the graph and are indicated by the shaded area as shown in Fig. 3.1(b).

Since all constraints have been graphed, the area which is bounded by all the constraints lines including all the boundary points is called the *feasible region* (*or solution space*). The feasible region is shown in Fig. 3.1(b) by the shaded area OABCD.

3. (i) Since the optimal value of the objective function occurs at one of the extreme points of the feasible region, it is necessary to determine their coordinates. The coordinates of extreme points of the feasible region are: O = (0, 0), A = (60, 0), B = (60, 20), C = (30, 40), D = (0, 40).

(ii) Evaluate objective function value at each extreme point of the feasible region as shown in the table:

Extreme Point	*Coordinates* (x_1, x_2)	*Objective Function Value* $Z = 15x_1 + 10x_2$
O	(0, 0)	15(0) + 10(0) = 0
A	(60, 0)	15(60) + 10(0) = 900
B	(60, 20)	15(60) + 10(20) = 1,100
C	(30, 40)	15(30) + 10(40) = 850
D	(0, 40)	15(0) + 10(40) = 400

(iii) Since we desire Z to be maximum, from 3(ii), we conclude that maximum value of $Z = 1{,}100$ is achieved at the point extreme B (60, 20). Hence the optimal solution to the given LP problem is: $x_1 = 60, x_2 = 20$ and Max $Z = 1{,}100$.

Feasible region is the overlapping area of constraints that satisfies all of the constraints on resources.

Remark To determine which side of a constraint equation is in the feasible region, examine whether the origin (0, 0) satisfies the constraints. If it does, then all points on and below the constraint equation towards the origin are feasible points. If it does not, then all points on and above the constraint equation away from the origin are feasible points.

Example 3.2 Use the graphical method to solve the following LP problem.

Maximize $Z = 2x_1 + x_2$

subject to the constraints

(i) $x_1 + 2x_2 \leq 10$, (ii) $x_1 + x_2 \leq 6$,

(iii) $x_1 - x_2 \leq 2$, (iv) $x_1 - 2x_2 \leq 1$

and $x_1, x_2 \geq 0$.

Solution Plot on a graph each constraint by first treating it as a linear equation in the same way as discussed earlier. Use the inequality condition of each constraint to mark the feasible region as shown in Fig. 3.2. The feasible region is shown by the shaded area. Here it may be noted that we have not considered the area below

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3.3.3 Examples on Minimization LP Problem

Example 3.6 Use the graphical method to solve the following LP problem.

Minimize $Z = 3x_1 + 2x_2$

subject to the constraints

(i) $5x_1 + x_2 \geq 10$, (ii) $x_1 + x_2 \geq 6$, (iii) $x_1 + 4x_2 \geq 12$

and $x_1, x_2 \geq 0$.

Solution Plot on a graph each constraint by first treating them as a linear equation in the same way as discussed earlier. Use the inequality condition of each constraint to mark the feasible region as shown in Fig. 3.6.

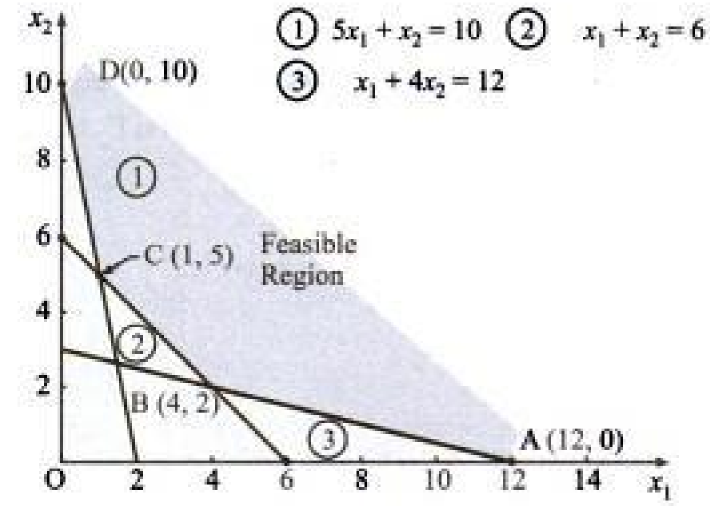

Fig. 3.6
Graphical Solution of LP Problem

The coordinates of the extreme points of the feasible region (bounded from below) are: A = (12, 0), B = (4, 2), C = (1, 5) and D = (0, 10). The value of objective function at each of these extreme points is as follows:

Extreme Point	*Coordinates* (x_1, x_2)	*Objective Function Value* $Z = 3x_1 + 2x_2$
A	(12, 0)	3(12) + 2(0) = 36
B	(4, 2)	3(4) + 2(2) = 16
C	(1, 5)	3(1) + 2(5) = 13
D	(0, 10)	3(0) + 2(10) = 20

The minimum value of the objective function $Z = 13$ occurs at the extreme point C (1, 5). Hence, the optimal solution to the given LP problem is: $x_1 = 1$, $x_2 = 5$, and Min $Z = 13$.

Example 3.7 Use the graphical method to solve the following LP problem.

Minimize $Z = -x_1 + 2x_2$

subject to the constraints

(i) $-x_1 + 3x_2 \leq 10$, (ii) $x_1 + x_2 \leq 6$, (iii) $x_1 - x_2 \leq 2$

and $x_1, x_2 \geq 0$.

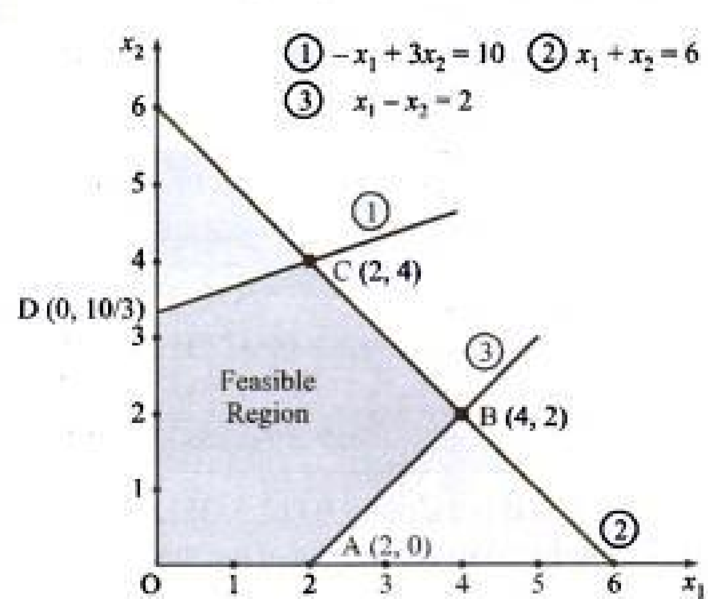

Fig. 3.7
Graphical Solution of LP Problem

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Example 3.11 Use the graphical method to solve the following LP problem.

Maximize $Z = 7x_1 + 3x_2$

subject to the constraints

(i) $x_1 + 2x_2 \geq 3$ (ii) $x_1 + x_2 \leq 4$

(iii) $0 \leq x_1 \leq 5/2$ (iv) $0 \leq x_2 \leq 3/2$

and $x_1, x_2 \geq 0$.

Solution Plot on a graph each constraint by first treating it as a linear equation. Then use the inequality condition of each constraint to mark the feasible region as shown in Fig. 3.11.

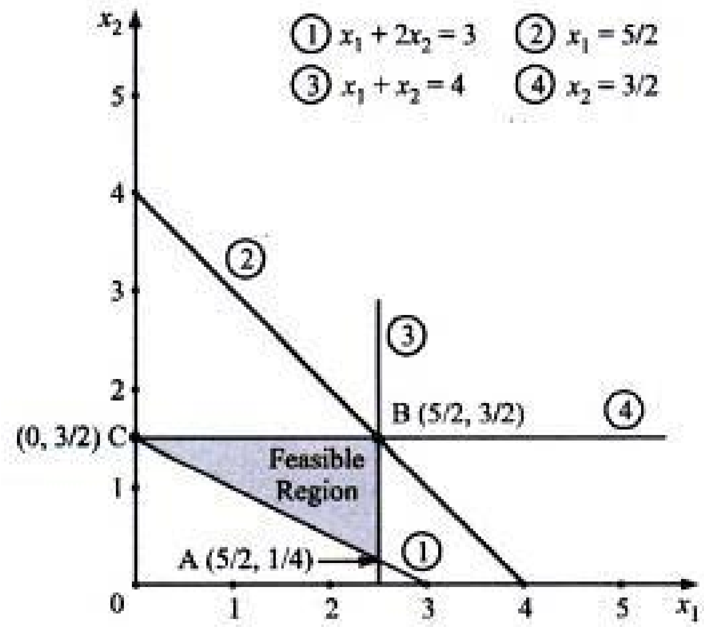

Fig. 3.11 Graphical Solution of LP Problem

The coordinates of the extreme points of the feasible region are: A = (5/2, 1/4), B = (5/2, 3/2), and C = (0, 3/2). The value of the objective function at each of these extreme points is as follows:

Extreme Point	*Coordinates* (x_1, x_2)	*Objective Function Value* $Z = 7x_1 + 3x_2$
A	(5/2, 1/4)	7(5/2) + 3(1/4) = 73/4
B	(5/2, 3/2)	7(5/2) + 3(3/2) = 22
C	(0, 3/2)	7(0) + 3(3/2) = 9/2

The maximum value of the objective function $Z = 22$ occurs at the extreme point B (5/2, 3/2). Hence, the optimal solution to the given LP problem is: $x_1 = 5/2$, $x_2 = 3/2$ and Max $Z = 22$.

Example 3.12 Use the graphical method to solve the following LP problem.

Minimize $Z = 20x_1 + 10x_2$

subject to the constraints

(i) $x_1 + 2x_2 \leq 40$, (ii) $3x_1 + x_2 \geq 30$, (iii) $4x_1 + 3x_2 \geq 60$

and $x_1, x_2 \geq 0$.

Solution Plot on a graph each constraint by first treating it as a linear equation. Then use the inequality condition of each constraint to mark the feasible region as shown in Fig. 3.12.

The coordinates of the extreme points of the feasible region are: A = (15, 0), B = (40, 0), C = (4, 18) and D = (6, 12). The value of the objective function at each of these extreme points is as follows:

Extreme Point	*Coordinates* (x_1, x_2)	*Objective Function Value* $Z = 20x_1 + 10x_2$
A	(15,0)	20(15) + 10(0) = 300
B	(40,0)	20(40) + 10(0) = 800
C	(4,18)	20(4) + 10(18) = 260
D	(6,12)	20(6) + 10(12) = 240

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subject to the constraints

(i) $\left.\begin{array}{l} 5x_1 + 4x_2 \leq 200 \\ 3x_1 + 5x_2 \leq 150 \end{array}\right\}$ (Input)

(ii) $\left.\begin{array}{l} 5x_1 + 4x_2 \geq 100 \\ 8x_1 + 4x_2 \geq 80 \end{array}\right\}$ (Output)

and $x_1, x_2 \geq 0$

For solving this LP problem graphically, let us graph each constraint by treating it as a linear equation in the same way as discussed earlier. Then use the inequality condition of each constraint to mark the feasible region as shown in Fig. 3.16.

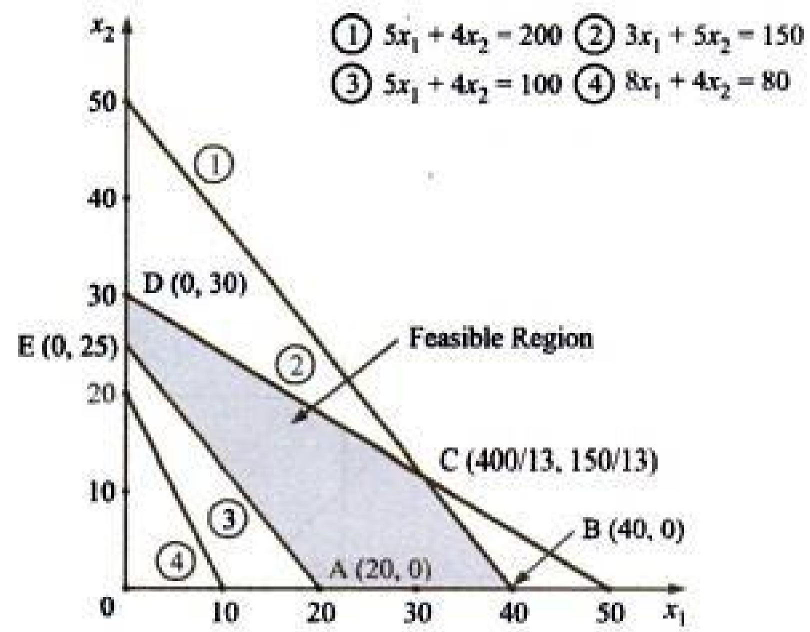

Fig. 3.16 Graphical Solution of LP Problem

The coordinates of extreme points of the feasible region are: A = (20, 0), B = (40, 0), C = (400/13, 150/13), D = (0, 30) and E = (0, 25). The value of the objective function at each of these extreme points is as follows:

Extreme Point	*Coordinates* (x_1, x_2)	*Objective Function Value* $Z = 300x_1 + 400x_2$
A	(20, 0)	300(20) + 400(0) = 6,000
B	(40, 0)	300(40) + 400(0) = 12,000
C	(400/13, 150/13)	300(400/13) + 400(150/13) = 1,80,000/13
D	(0, 30)	300(0) + 400(30) = 12,000
E	(0, 25)	300(0) + 400(25) = 10,000

The maximum value of the objective function occurs at the extreme point (400/13, 150/13). Hence, the manager of the oil refinery should produce, x_1 = 400/13 units under process 1 and x_2 = 150/13 units under process 2 in order to achieve the maximum profit of Rs 1,80,000/13.

Example 3.17 A manufacturer produces two different models – X and Y – of the same product. Model X makes a contribution of Rs 50 per unit and model Y, Rs 30 per unit, towards total profit. Raw materials r_1 and r_2 are required for production. At least 18 kg of r_1 and 12 kg of r_2 must be used daily. Also at most 34 hours of labour are to be utilized. A quantity of 2 kg of r_1 is needed for model X and 1 kg of r_1 for model Y. For each of X and Y, 1 kg of r_2 is required. It takes 3 hours to manufacture model X and 2 hours to manufacture model Y. How many units of each model should be produced in order to maximize the profit?

Solution Let us define the following decision variables:

x_1 and x_2 = number of units of model X and Y to be produced, respectively.

Then the LP model of the given problem can be written as:

Maximize (total profit) $Z = 50x_1 + 30x_2$

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Example 3.19 Consider the LP problem

Maximize $Z = 15x_1 + 10x_2$

subject to the constraints

(i) $4x_1 + 6x_2 \leq 360$, (ii) $3x_1 + \leq 180$, (iii) $5x_2 \leq 200$

and $x_1, x_2 \geq 0$.

Solution: To begin with, inequality constraints are considered equations, as shown in Fig. 3.19. The feasible area is formed by considering the area that lies on the lower left side of each equation (towards the origin).

A family of lines that represents various levels of objective function is drawn (broken lines in Fig. 3.19). These lines are iso-profit lines.

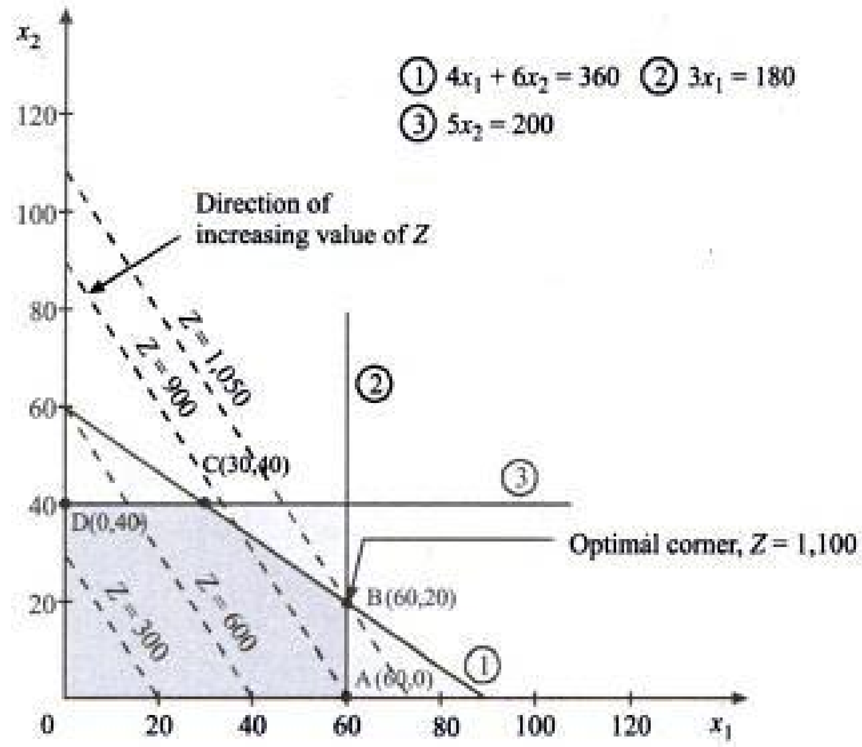

Fig. 3.19
Optimal Solution (Iso-profit Function Approach)

In Fig. 3.19, a value of $Z = 300$ is arbitrarily selected. The iso-profit function equation then becomes: $15x_1 + 10x_2 = 300$. This equation can be plotted in the same manner as the equality constraints were previously plotted. This line is then moved upward until it first intersects a corner (or corners) in the feasible region (corner B). The coordinates of corner point B can be read from the graph or can be computed as the intersection of the two linear equations.

The coordinates $x_1 = 60$ and $x_2 = 0$ of corner point B satisfy the given constraints and the total profit obtained is $Z = 1,100$.

3.3.3 Comparison of Two Graphical Solution Methods

After having plotted the constraints of the given LP problem and after having identified the feasible solution space, select one of the two graphical solution methods and proceed to solve the given LP problem.

Extreme Point Method	*Iso-Profit (or Cost) Method*
(i) Identify coordinates of each of the extreme (or corner) points of the feasible region by either drawing perpendiculars on the x-axis and the y-axis or by solving two intersecting equations.	(i) Determine the slope (x_1, x_2) of the objective function and then join intercepts to reveal the profit (or cost) line.
(ii) Compute the profit (or cost) at each extreme point by substituting that point's coordinates into the objective function.	(ii) In case of maximization, maintain the same slope through a series of parallel lines, and move the line up and towards the right until it touches the feasible region at only one point. But in case of minimization, move down and towards left until it touches only one point in the feasible region.
(iii) Identify the optimal solution at that extreme point with highest profit in a maximization problem or lowest cost in a minimization problem.	(iii) Compute the coordinates of the point touched by the iso-profit (or iso-cost) line on the feasible region.
	(iv) Compute the profit or cost.

3.4 SPECIAL CASES IN LINEAR PROGRAMMING

3.4.1 Alternative (or Multiple) Optimal Solutions

Alternative optimal solution is arrived at when the angle or slope of the objective function is the same as any in the LP Problem slope of the constraint

So far, we have seen that the optimal solution of any linear programming problem occurs at an extreme point of the feasible region and that the solution is unique, i.e. no other solution yields the same value of the objective function. However, in certain cases, a given LP problem may have more than one solution yielding the same optimal objective function value. Each of such optimal solutions is termed as *alternative optimal solution.*

There are two conditions that should be satisfied for an alternative optimal solution to exist:

(i) The given objective function should be parallel to a constraint that forms the boundary (or edge) of the feasible solutions region. In other words, the slope of the objective function should be the same as that of the constraint forming the boundary of the feasible solutions region, and

(ii) The constraint should form a boundary on the feasible region in the direction of optimal movement of the objective function. In other words, the constraint should be an active constraint.

Remark The constraint is said to be *active* or *binding* or *tight,* if at the point of optimality, the left-hand side of a constraint equals the right-hand side. In other words, an equality constraint is always active. An inequality constraint may or may not be active.

Geometrically, an *active* constraint is one that passes through one of the extreme points of the feasible solution space.

Example 3.20 Use the graphical method to solve the following LP problem.

Maximize $Z = 10x_1 + 6x_2$

subject to the constraints

(i) $5x_1 + 3x_2 \leq 30$, (ii) $x_1 + 2x_2 \leq 18$

and $x_1, x_2 \geq 0$.

Solution The constraints are plotted as usual on graph as shown in Fig. 3.20. The feasible region is shaded. The extreme points of the region are O, A, B and C.

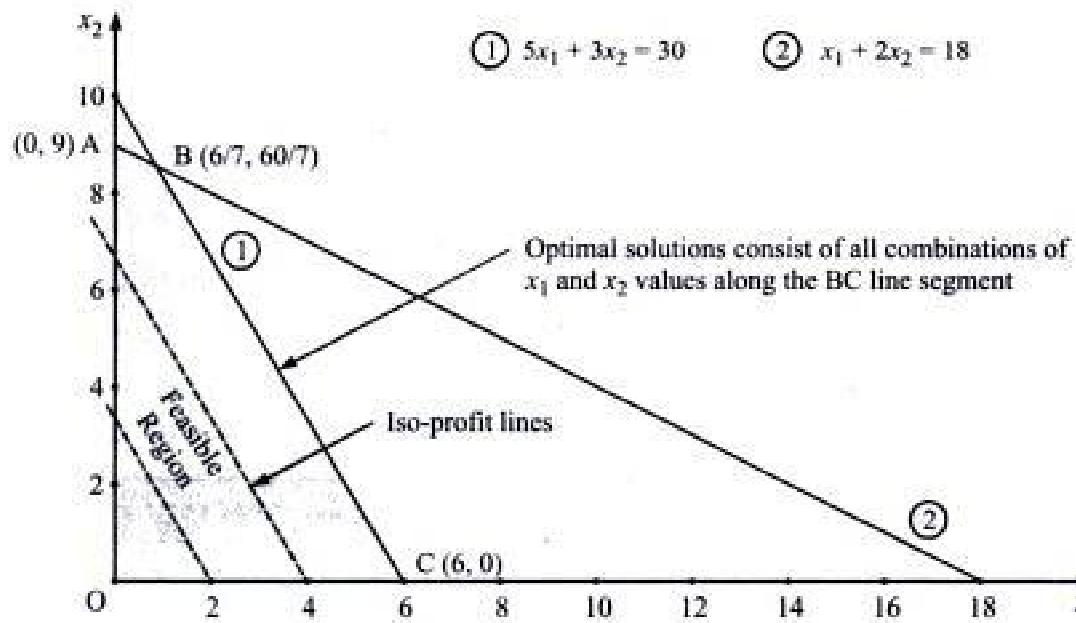

Fig. 3.20 Graphical Solution — Multiple Optima

We observe that the objective function (iso-profit line) is parallel to the line BC (or the first constraint), which forms the boundary of the feasible region. Thus, as the iso-profit line moves away from the origin, it coincides with the portion BC of the constraint line that forms the boundary of the feasible region. This implies that any point including extreme points B and C on the same line between B and C is an optimal solution. Therefore, several combinations of values of x_1 and x_2, infact, give the same value of objective function.

We may disregard all other solutions obtained on the line segment BC and consider only those obtained at extreme points B and C to establish that the solution to an LP problem will always lie at an extreme point of the feasible region.

The evaluation of four extreme points is shown as below:

Extreme Point	*Coordinates* (x_1, x_2)	*Objective Function Value* $Z = 10x_1 + 6x_2$
O	(0, 0)	$10(0) + 6(0) = 0$
A	(0, 9)	$10(0) + 6(9) = 54$
B	(6/7, 60/7)	$10(6/7) + 6(60/7) = 60$
C	(6, 0)	$10(6) + 6(0) = 60$

Since on two different extreme points B and C the value of objective function is same, i.e. Max Z = 60, two alternative solutions: $x_1 = 6/7$, $x_2 = 60/7$ and $x_1 = 6$, $x_2 = 0$ exist.

Remark If a constraint to which the objective function is parallel does not form the boundary of the feasible region, the multiple solutions will not exist. This type of a constraint is called *redundant constraint,* i.e. a redundant constraint is one whose removal does not change the feasible region.

3.4.2 Unbounded Solution

Sometimes an LP problem will not have a finite solution. This means that when one or more decision variable values and the value of the objective function (maximization case) are permitted to increase infinitely, without violating the feasibility condition, then the solution will be infinite. This is known as an *unbounded* solution. It is important here to note that there is a difference between a *feasible region* being unbounded and an LP problem being unbounded. It is possible that in a particular problem the feasible region may be unbounded but LP problem may not be unbounded, i.e. an unbounded feasible region may yield some definite value of the objective function. The general cause for an unbounded LP problem is an improper formulation of the real-life problem.

Unbounded solution exists when the solution variable and the profit can be made infinitely large without violating any of the maximization LP problem constraints

Example 3.21 Use the graphical method to solve the following LP problem:

Maximize $Z = 3x_1 + 4x_2$

subject to the constraints

(i) $x_1 - x_2 = -1$ (ii) $-x_1 + x_2 \leq 0$

and $x_1, x_2 \geq 0$.

Solution Plot on graph each constraint by first treating it as a linear equation in the same way as discussed earlier. Then use the inequality condition of each constraint to mark the feasible region as shown in Fig. 3.21.

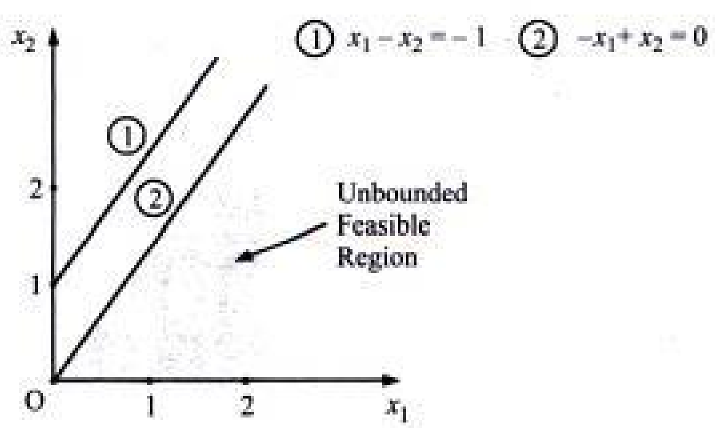

Fig. 3.21 Unbounded Solution

It may be noted from Fig. 3.21 that there exist an infinite number of points in the convex region for which the value of the objective function increases as we move from the extreme point (origin), to the right. That is, both the variables x_1 and x_2 can be made arbitrarily large and according the value of objective function Z will also increase. Thus, the problem has an unbounded solution.

Example 3.22 Use graphical method to solve the following LP problem:

Maximize $Z = 3x_1 + 2x_2$

subject to the constraints

(i) $x_1 - x_2 \geq 1$ (ii) $x_1 + x_2 \geq 3$

and $x_1, x_2 \geq 0$.

Solution The constraints are plotted on graph as usual as shown in Fig. 3.22. The solution space is shaded and is *bounded from below*.

It is noted here that the shaded convex region (solution space) is unbounded from above. The two corners of the region are, A = (0, 3) and B = (2, 1). The value of the objective function at these corners is: Z(A) = 6 and Z(B) = 8.

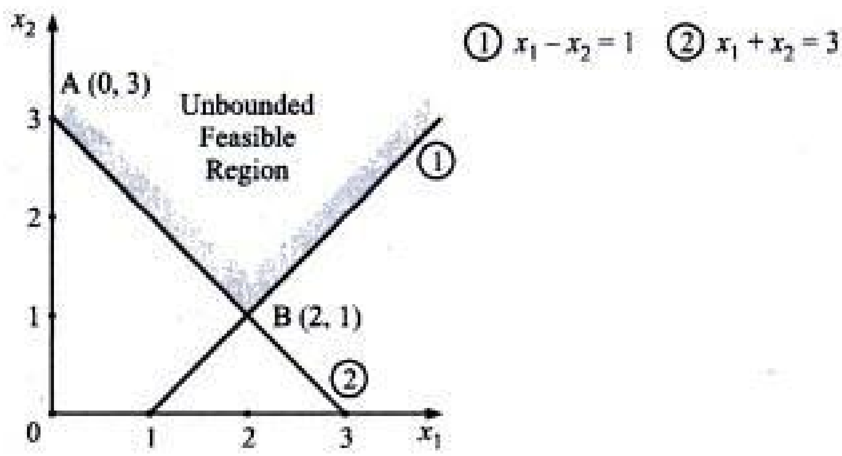

Fig. 3.22 Graphical Solution of LP Problem

Since the given LP problem is of maximization, there exist a number of points in the shaded region for which the value of the objective function is more than 8. For example, the point (2, 3) lies in the region and the function value at this point is 12 which is more than 8. Thus, both the variables x_1 and x_2 can be made arbitrarily large and accordingly the value of Z will also increase. Hence, the problem has an unbounded solution.

Example 3.23 Use graphical method to solve the following LP problem

Maximize $Z = 5x_1 + 4x_2$

subject to the constraints

(i) $x_1 - 2x_2 \leq 1$, (ii) $x_1 + 2x_2 \geq 3$

and $x_1, x_2 \geq 0$ *[PT Univ., BE, 2001]*

Solution Constraints are plotted on a graph as usual as shown in Fig. 3.23. The solution space is shown shaded and is bounded from below. This shaded convex region (solution space) is unbounded from above.

The two extreme points of the solution space are, A(0, 3/2) and B(2, 1/2). The value of objective function at these points is Z(A) = 6 and Z(B) = 12. Since the given LP problem is of maximization, these exists a number of points in the solution space where the value of objective function is much more than 12. Hence, the unique value of Z cannot be found as it occurs at infinity only. The problem, therefore, has an unbounded solution.

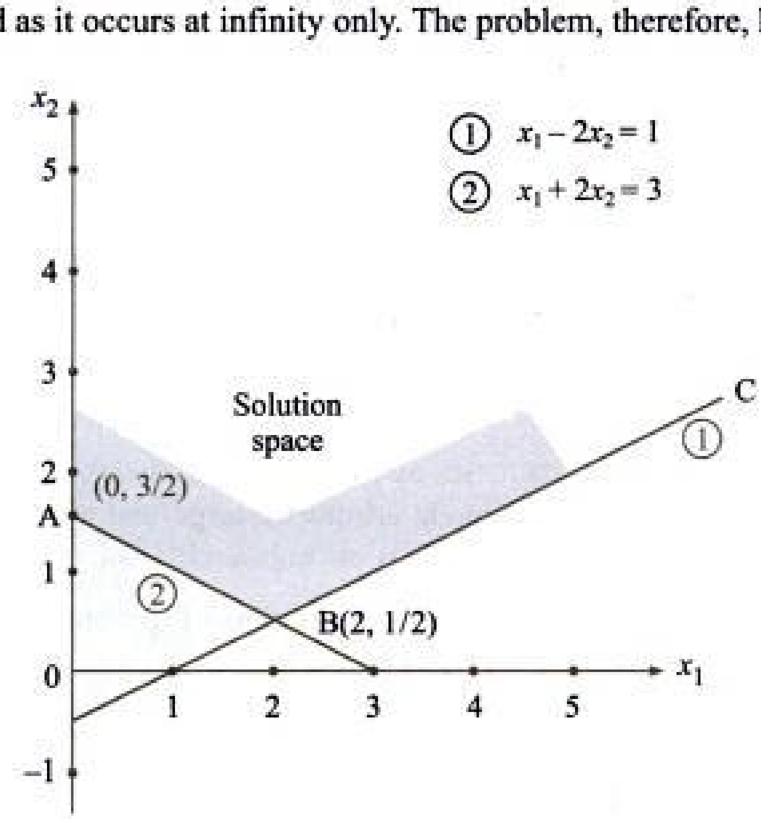

Fig. 3.23 Unbounded Solution

Example 3.24 Solve the following LP problem graphically

Maximize $Z = -4x_1 + 3x_2$

subject to the constraints

(i) $x_1 - x_2 \leq 0$, (ii) $x_1 \leq 4$

and $x_1, x_2 \geq 0$. *[Punjab Univ., B Com, 2005]*

Solution The solution space satisfying the constraints and the non-negativity restrictions is shown shaded in Fig 3.24. The line $x_1 - x_2 = 0$ is drawn by joining origin point (0, 0) and has a slope of 45°. Also as $x_1 - x_2 \leq 0$, i.e. $x_1 \leq x_2$, the solution space due to this line is in the upward direction.

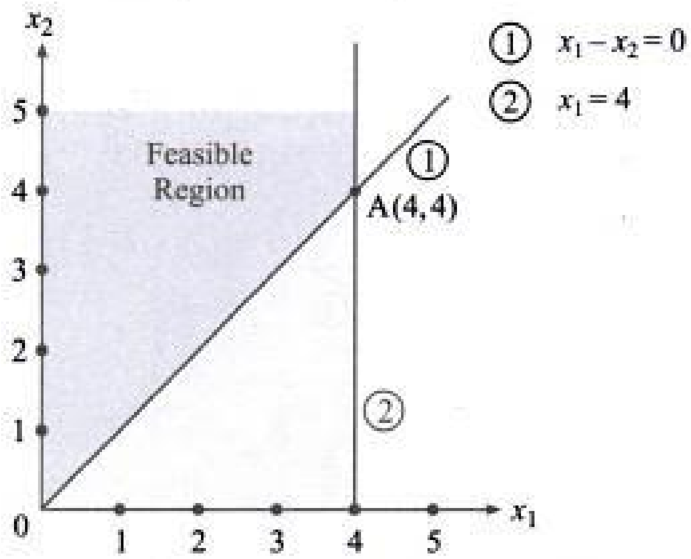

Fig. 3.24 Unbounded Solution

Since objective function is of maximization, therefore the value of Z can be made arbitrarily large. Hence, this LP problem has an unbounded solution. Value of variable x_1 is limited to 4, while value of variable x_2 can be increased indefinitely.

3.4.3 Infeasible Solution

Infeasibility is condition that arises when there is no solution to an LP problem that satisfies all the constraints simultaneously. This means, there would be no unique (single) feasible region. Such a problem arises when a wrong model is formulated that has conflicting constraints. Any point lying out side the feasible region. It violates one or more of the given constraints.

An infeasible solution lies outside the feasible region, it violates one or more of the given constraints

Infeasibility solely depends on the constraints and has nothing to do with the objective function. This type of a situation requires the ability of the decision-maker to resolve the conflicting requirements of resources so that a decision that is acceptable to all sections of the organization can be made.

Example 3.25 (*Problem with inconsistence system of constraints*) Use the graphical method to solve the following LP problem:

Maximize $Z = 6x_1 - 4x_2$

subject to the constraints

(i) $2x_1 + 4x_2 \leq 4$ (ii) $4x_1 + 8x_2 \geq 16$

and $x_1, x_2 \geq 0$

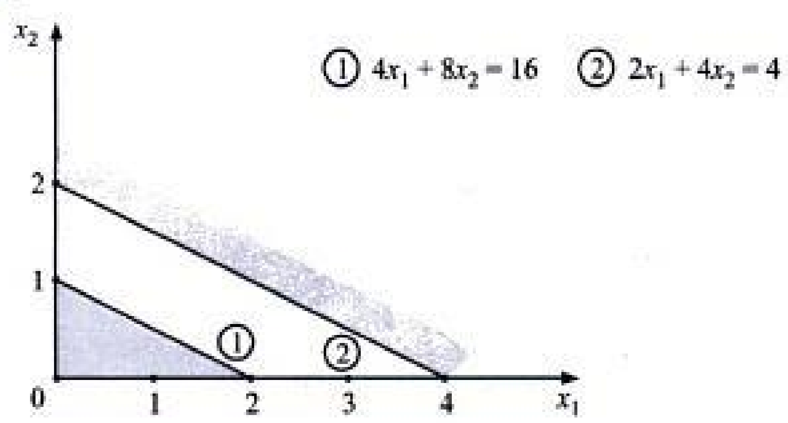

Fig. 3.25 An Infeasible Solution

Solution The constraints are plotted on graph as usual as shown in Fig. 3.25. As you can see there is no unique feasible solution space that allows us to obtain a unique set of values of variables x_1 and x_2 that satisfy all the constraints. Hence, there is no feasible solution to this problem because of the conflicting constraints.

Example 3.26 Use the graphical method to solve the following LP problem:

Maximize $Z = x_1 + \frac{x_2}{2}$

subject to the constraints

(i) $3x_1 + 2x_2 \leq 12$ (ii) $5x_1 = 10$

(iii) $x_1 + x_2 \geq 8$ (iv) $-x_1 + x_2 \geq 4$

and $x_1, x_2 \geq 0$

Solution The constraints are plotted on graph as usual as shown in Fig. 3.26. The feasible region is shaded. As shown in Fig. 3.26, the three shaded areas indicate non-overlapping regions. All of these can be considered feasible solution areas because they all satisfy some subsets of the constraints.

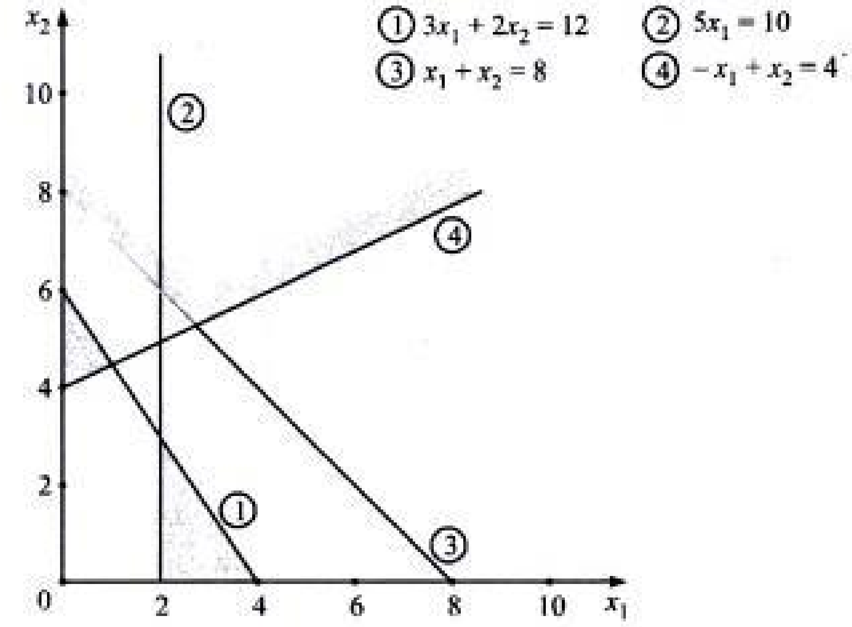

Fig. 3.26 An Infeasible Solution

However, we cannot find any particular point (x_1, x_2) in these shaded regions that can satisfy all the constraints simultaneously. Thus, the LP problem has an infeasible solution.

Example 3.27 Solve the following LP problem graphically

Maximize $Z = 3x + 2y$

subject to the constraints

(i) $-2x + 3y \leq 9$, (ii) $3x - 2y \leq -20$

and $x, y \geq 0$. *[Punjab Univ., BE (Elect.) 1996]*

Solution There are two solution spaces (shaded areas) shown in the Fig. 3.27. One of these solution space is satisfying the constraint $-2x + 3y \leq 9$ while the other is satisfying the constraint $3x - 2y \leq -20$. These two shaded regions in the first quadrant do not overlap and hence there is no point (x, y) common to both the shaded regions. This implies that the feasible solution to the problem does not exist. Consequently, this LP problem cannot solved graphically.

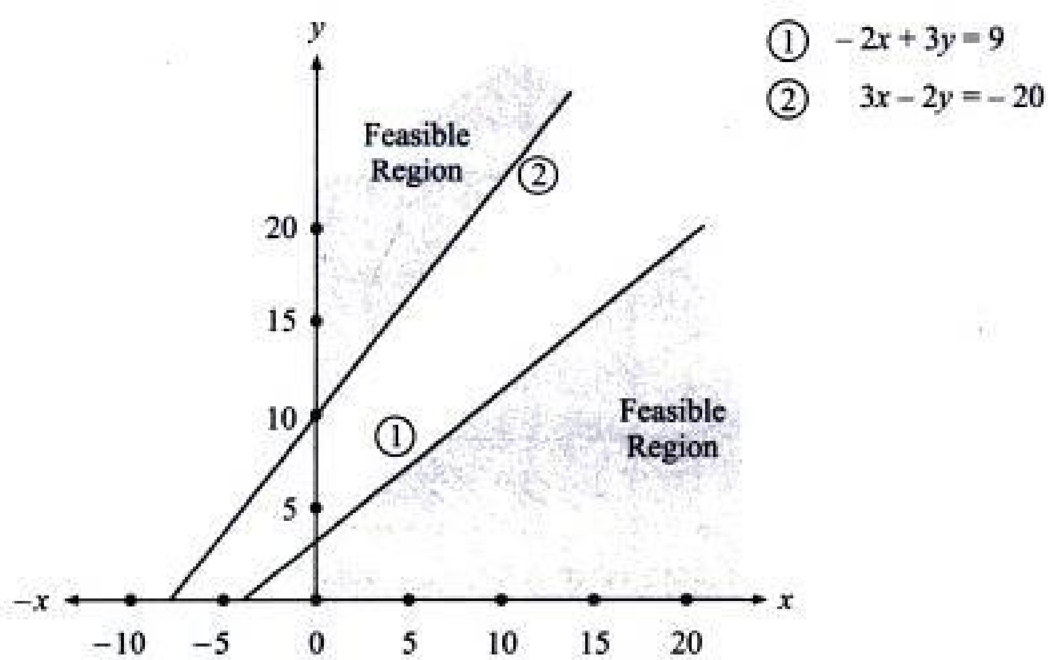

Fig. 3.27 No Feasible Solution

3.4.4 Redundancy

A redundant constraint is one that does not affect the feasible solution region (or space) and thus redundancy of any constraint does not cause any difficulty in solving an LP problem graphically. As previously shown in Fig. 3.12 constraint $8x_1 + 4x_2 \geq 80$ is redundant. In other words, a constraint is said to be redundant when it may be more binding (restrictive) than the another.

Redundancy is a situation in which one or more constraints do not affect the feasible solution region

CONCEPTUAL QUESTIONS

1. Explain the graphical method of solving an LP problem.
2. Give a brief description of an LP problem with illustrations. How can it be solved graphically?
3. What is meant by the term 'feasible region'? Why must this be a well-defined boundary for the maximization problem?
4. What is a feasibility region? Is necessary that it should always be a convex set?
5. Define iso-profit and iso-cost lines. How do these help us to obtain a solution to an LP problem?
6. Define the concept of convexity. Why must the feasible region exhibit the property of convexity in an LP problem?
7. Explain the procedure of generating extreme point solutions to an LP problem, pointing out the assumptions made, if any.
8. It has been said that each LP problem that has a feasible region has an infinite number of solutions. Explain.
9. You have just formulated a maximization LP problem and are preparing to solve it graphically. What criteria should you consider in deciding whether it would be easier to solve the problem by the extreme point enumeration method or the iso-profit line method.
10. Under what condition is it possible for an LP problem to have more than one optimal solution? What do these alternative optimal solutions represent?
11. What is an infeasible solution, and how does it occur? How is this condition recognized in the graphical method?
12. What is an unbounded solution, and how is this condition recognized in the graphical method?

SELF PRACTICE PROBLEMS

1. Comment on the solution of the following LP problems

(i) Max $Z = 4x_1 + 4x_2$
subject to $x_1 + 2x_2 \leq 10$
$6x_1 + 6x_2 \leq 36$
$x_1 \leq 6$
and $x_1, x_2 \geq 0$

(ii) Max $Z = 5x_1 + 3x_2$
subject to $4x_1 + 2x_2 \leq 8$
$x_1 \geq 3$
$x_2 \geq 7$
and $x_1, x_2 \geq 0$

(iii) Max $Z = 4x_1 + 2x_2$
subject to $-x_1 + 2x_2 \leq 6$
$-x_1 + x_2 \leq 2$
and $x_1, x_2 \geq 0$

(iv) Min $Z = x_1 - 2x_2$
subject to $-2x_1 + x_2 \leq 8$
$-x_1 + 2x_2 \leq -24$
and $x_1, x_2 \geq 0$

(v) Max $Z = 6x_1 + 12x_2$
subject to $x_1 + 2x_2 \leq 10$
$2x_1 - 5x_2 \leq 20$
$x_1 + x_2 \leq 15$
and $x_1, x_2 \geq 0$

(vi) Max $Z = 3x_1 + 5x_2$
subject to $x_1 + x_2 \geq 100$
$5x_1 + 10x_2 \leq 400$
$6x_1 + 8x_2 \leq 440$
and $x_1, x_2 \geq 0$

(vii) Max $Z = 1.75\,x_1 + 1.5\,x_2$
subject to $8x_1 + 5x_2 \leq 320$
$4x_1 + 5x_2 \leq 20$
$x_1 \geq 15;\ x_2 \geq 10$
and $x_1, x_2 \geq 0$

(viii) Max $Z = 3x_1 + 2x_2$
subject to $-2x_1 + 3x_2 \leq 9$
$x_1 - 5x_2 \geq -20$
and $x_1, x_2 \geq 0$

(ix) Max $Z = 5x_1 + 3x_2$
subject to $3x_1 + 5x_2 \leq 15$
$5x_1 + 2x_2 \leq 10$
and $x_1, x_2 \geq 0$
[*Kerala, BSc (Engg.), 1995*]

(x) Max $Z = x_1 + x_2$
subject to $x_1 + x_2 \leq 1$
$-3x_1 + x_2 \geq 3$
and $x_1, x_2 \geq 0$

2. Solve the following LP problems graphically and state what your solution indicates.

(i) Min $Z = 4x_1 - 2x_2$
subject to $x_1 + x_2 \leq 14$
$3x_1 + 2x_2 \geq 36$
$2x_1 + x_2 \leq 24$
and $x_1, x_2 \geq 0$

(ii) Min $Z = 3x_1 + 5x_2$
subject to $-3x_1 + 4x_2 \leq 12$
$2x_1 - x_2 \geq -2$
$2x_1 + 3x_2 \geq 12$
$x_1 \leq 4;\ x_2 \geq 2$
and $x_1, x_2 \geq 0$

(iii) Min $Z = 20x_1 + 10x_2$
subject to $x_1 + 2x_2 \leq 40$
$3x_1 + x_2 \geq 30$
$4x_1 + 3x_2 \geq 60$
and $x_1, x_2 \geq 0$

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available

12. A company produces two types of presentation goods A and B that require gold and silver. Each unit of type A requires 3g of silver and 1g of gold while B requires 1g of silver and 2g of gold. The company can produce 9 g of silver and 8 g of gold. If each unit of type A brings a profit of Rs 40 and that of type B Rs 50, determine the number of units of each type that should be produced in order to maximize the profit.

13. A firm makes two types of furniture: chairs and tables. The contribution to profit by each product as calculated by the accounting department is – Rs 20 per chair and Rs 30 per table. Both products are to be processed on three machines M_1, M_2 and M_3. The time required in hours by each product and total time available in hours per week on each machine are as follows:

Machine	*Chair*	*Table*	*Available Time (hrs)*
M_1	3	3	36
M_2	5	2	50
M_3	2	6	60

How should the manufacturer schedule his production in order to maximize profit?

14. The ABC company has been a producer of picture tubes for television sets and of certain printed circuits for radios. The company has just expanded into full-scale production and marketing of AM and AM-FM radios. It has built a new plant that can operate for 48 hours per week. Production of an AM radio in the new plant will require 2 hours and production of an AM-FM radio will require 3 hours. Each AM radio will contribute Rs 40 towards the profits while an AM-FM radio will contribute Rs 80 towards the profits. The marketing department, after extensive research, has determined that a maximum of 15 AM radios and 10 AM-FM radios can be sold each week. Formulate this problem as an LP model so as to determine the optimal production mix of AM-FM radios that will maximize profits. [*Delhi Univ., MBA. 1998, 2004*]

15. The production of a certain manufacturing firm involves a machining process that acquires raw materials and then converts them into (unassembled) parts. These parts are then sent to one of the two divisions for being assembly into the final product. Division 1 is used for product A, and Division 2 for product B. Product A requires 40 units of raw material and 10 hours of machine processing time. Product B requires 80 units of raw material and 4 hours of machine processing time. During the period, 800 units of raw material and 80 hours of machine processing time are available. The capabilities of the two assembly divisions during the period are 6 and 9 units, respectively. The profit contribution per unit to profit and overhead (fixed costs) is of Rs 200 for each unit of product A and of Rs 120 for each unit of product B. With this information, formulate this problem as a linear programming model and determine the optimal level of output for the two products using the graphic method.

16. A local travel agent is planning a charter trip to a famous sea resort. The eight-day seven-night package includes the round-trip fare, surface transportation, boarding and lodging and selected tour options. The charter trip is restricted to 200 persons and past experience indicates that there will not be any problem in arranging 200 passangers. The problem faced by the travel agent is to determine the number of Deluxe Standard and Economy tour packages to offer for this charter. All three of these plans each differ in terms of their seating and service for the flight, quality of accommodation, meal plans and tour options. The following table summarizes the estimated prices of the three packages and the corresponding expenses of the travel agent. The travel agent has hired an aircraft for a flat fee of Rs 2,00,000 for the entire trip. The per person price and costs for the three packages are as follows:

Tour Plan	*Price (Rs)*	*Hotel Costs (Rs)*	*Other Meals and Expenses (Rs)*
Deluxe	10,000	3,000	4,750
Standard	7,000	2,200	2,500
Economy	6,500	1,900	2,200

In planning the trip, the following considerations must be taken into account:

(i) At least 10 per cent of the packages must be of the deluxe type.
(ii) At least 35 per cent but not more than 70 per cent must be of the standard type.
(iii) At least 30 per cent must be of the economy type.
(iv) The maximum number of deluxe packages available in any aircraft is restricted to 60.
(v) The hotel desires that at least 120 of the tourists should be on the deluxe standard packages together.

The travel agent wishes to determine the number of packages to offer in each type of trip so as to maximize the total profit.

(a) Formulate this problem as a linear programming problem.
(b) Restate the above linear programming problem in terms of two decision variables, taking advantage of the fact that 200 packages will be sold.
(c) Find the optimum solution using graphical methods for the restated linear programming problem and interpret your results. [*Delhi Univ., MBA, 2004*]

17. A publisher of textbooks is in the process of bringing a new book in the market. The book may be bound by either cloth or hard paper. Each cloth-bound book sold contributes Rs 24 towards the profit and each paperbound book contributes Rs 23. It takes 10 minutes to bind a cloth cover, and 9 minutes to bind a paperback. The total available time for binding is 80 hours. After considering a number of market surveys, it is predicted that the cloth-cover sales will be anything more than 10,000 copies, but the paperback sales will not be more than 6,000 copies. Formulate this problem as a LP problem and solve it graphically.

18. PQR Feed Company markets two feed mixes for cattle. The feed mix, Fertilex, requires at least twice as much wheat as barley. The second mix, Multiplex, requires at least twice as much barley as wheat. Wheat costs Rs 1.50 per kg, and only 1,000 kg are available this month. Barley costs Rs 1.25 per kg and 1,200 kg is available: Fertilex sells for Rs 1.80 per kg up to 99 kg and each additional kg over 99 sells for Rs 1.65. Multiplex sells at Rs 1.70 per kg up to 99 kg and each additional kg over 99 sells for Rs 1.55 per kg. Bharat Farms is sure to buy any and all amounts of both mixes that PQR Feed Company will mix. Formulate this problem as a LP problem to determine the product mix so as to maximize the profits.

19. On October 1st, a company received a contract to supply 6,000 units of a specialized product. The terms of contract requires that 1,000 units be shipped in the north of October; 3,000 units in November and 2,000 units in December. The company can manufacture 1,500 units per month on regular time and 750 units per month on overtime. The manufacturing cost per item produced during regular time is Rs 3 and the cost per item produced during overtime is Rs 5. The monthly storage cost is Re 1. Formulate this problem as a linear programming problem so as to minimize the total cost.

20. A small-scale manufacturer has production facilities for producing two different products. Each of the products requires three different operations: grinding, assembly and testing. Product I requires 15, 20 and 10 minutes to grind, assemble and test, respectively, on the other hand product II requires 7.5, 40 and 45 minutes for grinding, assembling and testing, respectively. The production run calls for at least 7.5 hours

of grinding time, at least 20 hours of assembly and at least 15 hours of testing time. If manufacturing product I costs Rs 60 and product II costs Rs 90, determine the number of units of each product the firm should produce in order to minimize the cost of operation.

21. A firm is engaged in breeding pigs. The pigs are fed on various products grown in the farm. Because of the need to ensure certain nutrient constituents, it is necessary to additionaly buy one or two products (call them A and B).

The content of the various products (per unit) in the nutrient constituents (e.g. vitamins, proteins, etc.) is given in the following table:

Nutrient	*Nutrient Content in Product*		*Minimum Amount of Nutrient*
	A	*B*	
M_1	36	6	108
M_2	3	12	36
M_3	20	10	100

The last column of the above table gives the minimum quantity of nutrient constituents M_1, M_2 and M_3 that must be given to the pigs. If products A and B cost Rs 20 and Rs 40 per unit, respectively, how much of each of these two products should be bought so that the total cost is minimized? Solve this LP problem graphically.

22. A company manufacturing television sets and radios has four major departments: chassis, cabinet, assembly and final testing. The monthly capacities of these are as follows:

Department	*Television Capacity*		*Radio Capacity*
Chassis	1,500	or	4,500
Cabinet	1,000	or	8,000
Assembly	2,000	or	4,000
Testing	3,000	or	9,000

The contribution of television is Rs 150 each and the contribution of radio is Rs 250 each. Assume that the company can sell any quantity of either product. Formulate this problem as an LP problem and solve it to determine the optimal combination of output.

23. The ABC Clothing Stores is planning their annual shirt and pant sale. The owner, Mr Jain is planning to use two different forms of advertising, viz., radio and newspaper ads, in order to promote the sale. Based on past experience, Mr Jain feels confident that each newspaper ad will reach 40 shirt customers and 80 pant customers. Each radio ad, he believes, will reach 30 shirt customers and 20 pant customers. The cost of each newspaper ad is Rs 300 and the cost of each radio spot is Rs 450. The advertising agency will prepare the advertising and it will require 5 man-hours of preparation for each newspaper ad and 15 man-hours of preparation for each radio spot. Mr Jain's sales manager says that a minimum of 75 man-hours should be spent on preparation of advertising in order to fully utilize the services of advertising agency. Mr Jain feels that in order to have a successful sale, the advertising must reach at least 360 shirt customers and at least 400 pant customers.
   (i) Formulate this problem as an LP model so as to determine how much should the sale be advertised using each media in order to minimize costs and still attain the objective of Mr Jain.
   (ii) Solve the LP problem using the graphical method.

24. The commander of a small tank has been ordered to win and occupy a valley located in a river delta area. He has 4 heavy tanks and 10 light tanks. Each heavy tank requires 4 men to operate, whereas the light tank requires 2 men. The total number of men available is 29. The fire power of the heavy tank is three times that of the light tank. Yet the commander feels that he should use more light tanks than heavy ones since the light ones are more effective against guerillas.
   (i) Formulate this problem as an LP model so as to determine the number of tanks in each type to be sent into the combat, keeping in mind that the commander wishes to maximize the total fire power.
   (ii) If you could persuade the commander that the rule 'more light tanks than heavy tanks' may not be applicable due to the fact that guerilla warfare is absent in the area, how many tanks of each type could be used?

25. A timber company cuts raw timber – oak and pine logs – into wooden boards. Two steps are required to produce boards from logs. The first step involves removing the bark from the logs. Two hours are required to remove the bark from 1,000 feet of oak logs and three hours per 1,000 feet of pine logs. After the logs have been debarked, they must be cut into boards. It takes 2.4 hours per 1,000 feet of oak logs to be cut into boards and 1.2 hours per 1,000 feet of pine logs. The bark removing machines can operate for up to 60 hours per week, while for cutting machines this number is limited to 48 hours per week. The company can buy a maximum of 18,000 feet of raw oak logs and 12,000 feet of raw pine logs each week. The profit per 1,000 feet of processed logs is Rs 1,800 and Rs 1,200 for oak and pine logs, respectively. Formulate this problem as an LP model and solve it to determine how many feet of each type of log should be processed each week in order to maximize the profit.

26. Upon completing the construction of his house, Mr Sharma discovers that 100 square feet of plywood scrap and 80 square feet of white pine scrap are in unusable form, which can be used for the construction of tables and bookcases. It takes 16 square feet of plywood and 16 square feet of white pine to construct a book case. By selling the finished products to a local furniture store, Mr Sharma can realize a profit of Rs 25 on each table and Rs 20 on each bookcase. How can he most profitably use the leftover wood? Use the graphical method to solve this LP problem.

27. A manufacturer produces electric hand saws and electric drills, for which the demand exceeds his capacity. The production cost of a saw is Rs 6 and the production cost of a drill Rs 4. The shipping cost is 20 paise for a saw and 30 paise for a drill. A saw sells for Rs 9 and a drill sells for Rs 5.50. The budget allows a maximum of Rs 2,400 for production costs and Rs 120 for shipping costs. Formulate this problem as an LP model and solve it to determine the number of saws and drills that should be produced in order to maximize the excess of sales over production and shipping costs.

28. A company produces two types of pens, say A and B. Pen A is of a superior quality and pen B is of an inferior quality. Profits on pen A and B are Rs 5 and Rs 3 per pen, respectively. The raw material required for producing single pen A is twice as that of pen B. The supply of raw material is sufficient only for 1,000 pens of B per day. Pen A requires a special clip and only 400 such clips are available per day. For pen B, only 700 clips are available per day. Solve this LP problem graphically to find the product mix so that the company can make the maximum profit. [*Delhi Univ., MBA, Nov. 1998*]

29. Two products A and B are to be manufactured. One single unit of product A requires 2.4 minutes of punch press time and 5 minutes of assembly time. The profit for product A is Re 0.60 per unit. One single unit of product B requires 3

minutes of punch press time and 2.5 minutes of welding time. The profit for product B is Re 0.70 per unit. The capacity of the punch press department available for these products is 1,200 minutes/week. The welding department has an idle capacity of 600 minutes/week and the assembly department has the capacity of 1,500 minutes/week. Formulate this problem as an LP model to determine the quantities of products A and B that would yield the maximum.

30. A rubber company is engaged in producing three different kinds of tyres A, B and C. These three different tyres are produced at two different plants of the company, which have different production capacities. In a normal 8-hour working day, Plant 1 produces 50, 100 and 100 tyres of types A, B and C, respectively. Plant 2, produces 60, 60 and 200 tyres of types A, B and C, respectively. The monthly demand for types A, B and C is 2,500, 3,000 and 7,000 units, respectively. The daily cost of operation of Plants 1 and 2 is Rs 2,500 and Rs 3,500, respectively. Formulate this problem as an LP model and solve it to determine how the company can minimize the number of days on which it operations, per month, at the two plants, so that the total cost is also minimized, while the demand is also met.

31. Kishore Joshi mixes pet food in his basement on a small scale. He advertises two types of pet food: Diet-Sup and Gro-More. Contribution from Diet-Sup is Rs 1.50 a bag and from Gro-More Rs 1.10 a bag. Both are mixed from two basic ingredients – a protein source and a carbohydrate source. Diet-Sup and Gro-More require ingredients in these amounts:

	Protein	*Carbohydrate*
Diet-Sup (7 kg bag)	4 kg	3 kg
Gro-More (3 kg bag)	2 kg	1 kg

Kishore has the whole weekend ahead of him, but will not be able to procure more ingredients over the weekend. He checks his bins and finds he has 700 kg of protein source and 500 kg of carbohydrate in the house. How many bags of each food should he mix in order to maximize his profits? Formulate this problem as an LP model and solve it to determine the optimum food-mix that would maximize the profit.

32. An advertising agency wishes to reach two types of audiences: customers with monthly income greater than Rs 15,000 (target audience A) and customers with annual incomes of less than Rs 15,000 (target audience B). The total advertising budget is Rs 2,00,000. One programme of TV advertising costs Rs 50,000; one programme of radio advertising costs Rs 20,000. For contract reasons, at least 3 programmes ought to be on TV, and the number of radio programmes must be limited to 5. Surveys indicate that a single TV programme reaches 4,50,000 customers in target audience A and 50,000 in the target audience B. One radio programme reaches 20,000 in target audience A and 80,000 in the target audience B. Formulate this problem as an LP model and solve it to determine the media-mix that would maximize the total reach. *[Delhi Univ., MBA, 2005]*

33. A company has two grades of inspectors – 1 and 2 – who are to be assigned a quality control inspection. It is required at least 2,000 pieces be inspected per 8-hour day. A grade 1 inspector can check pieces at the rate of 40 per hour, with an accuracy of 97 per cent. A grade 2 inspector checks at the rate of 30 pieces per hour with an accuracy of 95 per cent.

The wage rate of grade 1 inspector is Rs 5 per hour while that of a grade 2 inspector is Rs 4 per hour. An error made by an inspector costs Rs 3 to the company. There are only 9 grade 1 inspectors and 11 grade 2 inspectors available in the company. The company wishes to assign work to the available inspectors so as to minimize the total cost of the inspection. Formulate this problem as a LP model and solve it by using the graphical method. *[Delhi Univ., MBA, 2003]*

HINTS AND ANSWERS

1. (i) Alternative solutions (ii) Infeasible solution
   (iii) Unbounded solution
   (iv) Unbounded solution (v) Redundant constraints are second and third (vi) Infeasible solution
   (vii) No feasible solution (viii) Unbounded solution
   (ix) $x_1 = 20/19$, $x_2 = 45/19$ and Max $Z = 535/19$
   (x) No feasible solution
2. (i) $x_1 = 8$, $x_2 = 6$ and Min $Z = 20$
   (ii) $x_1 = 3$, $x_2 = 2$ and Min $Z = 19$
   (iii) $x_1 = 6$, $x_2 = 12$ and Min $Z = 240$
3. Let x_1 and x_2 = number of pieces of P_1 and P_2 to be manufactured, respectively.

   Max $Z = 50x_1 + 100x_2$

   subject to (i) $10x_1 + 15x_2 \leq 2{,}500$
   (ii) $4x_1 + 10x_2 \leq 2{,}000$
   (iii) $x_1 + 15x_2 \leq 450$

   and $x_1, x_2 \geq 0$

   Ans. $x_1 = 187.5$, $x_2 = 125$ and Max Z = Rs 21,875.
4. Let x_1 and x_2 = number of chairs and tables produced, respectively.

   Max $Z = 2x_1 + 10x_2$

   subject to (i) $2x_1 + 5x_2 \leq 16$; (ii) $6x_1 \leq 30$

   and $x_1, x_2 \geq 0$

   Ans. $x_1 = 0$, $x_2 = 3.2$ and Max Z = Rs 32.
5. Let x_1 and x_2 = number of belts of types A and B produced, respectively.

   Max $Z = 0.40x_1 + 0.30x_2$

   subject to (i) $x_1 + x_2 \leq 800$; (ii) $2x_1 + x_2 \leq 1{,}000$
   (ii) $x_1 \leq 400$; (iv) $x_2 \leq 700$

   and $x_1, x_2 \geq 0$

   Ans. $x_1 = 200$, $x_2 = 600$ and Max = Rs 260.
6. Let x_1 and x_2 = number of days plant at G and J work in July, respectively.

   Min $Z = 600x_1 + 400x_2$

   subject to (i) $1{,}500x_1 + 1{,}500x_2 \geq 20{,}000$
   (ii) $3{,}000x_1 + 1{,}000x_2 \geq 40{,}000$
   (iii) $2{,}000x_1 + 5{,}000x_2 \geq 44{,}000$

   and $x_1, x_2 \geq 0$

   Ans. $x_1 = 12$, $x_2 = 4$ and Min $Z = 8{,}800$.
7. Let x_1 and x_2 = number of parts manufactured of automobile A and B, respectively.

   $$\text{Max } Z = \{5-2+20/25+14/28+17.5/35\}x_1 + \{6-3+20/40+14/35+17.5/25\}x_2 = 4.80x_1 + 4.60x_2$$

subject to

(i) $\frac{x_1}{25} + \frac{x_2}{40} \le 1$, (ii) $\frac{x_1}{28} + \frac{x_2}{35} \le 1$, (iii) $\frac{x_1}{35} + \frac{x_2}{25} \le 1$

and $x_1, x_2 \ge 0$

8. Let x_1 and x_2 = number of units of casting X and Y to be manufactured, respectively.

$$\text{Max } Z = (300x_1 + 360x_2) - (120x_1 + 120x_2) - 1{,}000$$
$$= 180x_1 + 240x_2 - 1{,}000$$

subject to (i) $4x_1 + 2x_2 \le 2 \times 40$

(ii) $2x_1 + 5x_2 \le 3 \times 40$

and $x_1, x_2 \ge 0$

Ans. $x_1 = 10$, $x_2 = 20$ and Max $Z = 5{,}600$.

9. Let x_1 and x_2 = number of production hours for plants A and B needed to complete the orders, respectively.

$$\text{Min } Z = 9x_1 + 10x_2$$

subject to (i) $2x_1 + 4x_2 \ge 50$; (ii) $4x_1 + 3x_2 \ge 24$

(iii) $3x_1 + 2x_2 \ge 60$

and $x_1, x_2 \ge 0$

Ans. $x_1 = 35/2$ and $x_2 = 15/4$ and Min Z = Rs 195.

10. Let x_1 and x_2 = number of units of products A and B, respectively.

$$\text{Max } Z = 30x_1 + 40x_2$$

subject to (i) $x_1 \le 20$; $x_2 \ge 10$, (ii) $4x_1 + 4x_2 \le 100$,

(iii) $4x_1 + 6x_2 \le 180$, (iv) $x_1 + x_2 \le 40$

and $x_1, x_2 \ge 0$

12. Let x_1 and x_2 = number of units of A and B to be produced, respectively.

$$\text{Max } Z = 40x_1 + 50x_2$$

subject to (i) $3x_1 + x_2 \le 9$; (ii) $x_1 + 2x_2 \le 8$

and $x_1, x_2 \ge 0$

Ans. $x_1 = 2$, $x_2 = 3$ and Max Z = Rs 230.

13. Let x_1 and x_2 = number of chairs and tables to be made, respectively.

$$\text{Max } Z = 20x_1 + 30x_2$$

subject to (i) $3x_1 + 3x_2 \le 36$; (ii) $5x_1 + 2x_2 \le 50$

(iii) $2x_1 + 6x_2 \le 60$

and $x_1, x_2 \ge 0$

Ans. $x_1 = 3$, $x_2 = 9$ and Max Z = Rs 330.

14. Let x_1 and x_2 = number of units of AM radios and AM-FM radios to be sold, respectively.

$$\text{Max } Z = 40x_1 + 80x_2$$

subject to (i) $2x_1 + 3x_2 \le 48$; (ii) $x_1 \le 15$; (iii) $x_2 \le 10$

and $x_1, x_2 \ge 0$

Ans. $x_1 = 9$, $x_2 = 10$ and Max Z = Rs 1,160.

15. Let x_1 and x_2 = number of units to be produced of products A and B, respectively.

$$\text{Max } Z = 200x_1 + 120x_2$$

subject to (i) $40x_1 + 80x_2 \le 800$, (ii) $10x_1 + 04x_2 \le 80$,

(iii) $x_1 \le 6$; (iv) $x_2 \le 9$

and $x_1, x_2 \ge 0$

Ans. $x_1 = 5$, $x_2 = 7.5$ and Max Z = Rs 1,900.

16. Let x_1, x_2 and x_3 = number of Deluxe, Standard and Economy packages restricted to 200 persons, respectively.

Max (net profit) = [Price – (Hotel costs + Meals)] – Flat fee for the chartered aircraft

$$= (10{,}000 - 3{,}000 - 4{,}750)x_1 + (7{,}000 - 2{,}200 - 2{,}500)x_2 + (6{,}500 - 1{,}900 - 2{,}200)x_3 - 2{,}00{,}000$$
$$= 2{,}250x_1 + 2{,}300x_2 + 2{,}400x_3 - 2{,}00{,}000$$

subject to (a) $20 \le x_1 \le 60$ [from (i) and (iii) condition]

(b) $x_3 \ge 60$ [from (iii) condition]

(c) $70 \le x_2 \le 140$;

(d) $x_1 + x_2 + x_3 = 200$; (e) $x_1 + x_2 \ge 120$

and $x_1, x_2, x_3 \ge 0$

Since $x_1 + x_2 + x_3 = 200$, $x_3 = 200 - x_1 - x_2$. Substituting value of x_3 both in the objective function and set of constraints, we set

$$\text{Max } Z = -150x_1 - 100x_2 + 2{,}80{,}000$$

subject to (i) $20 \le x_1 \le 60$; (ii) $70 \le x_2 \le 140$;

(iii) $120 \le x_1 + x_2 \le 140$

and $x_1, x_2 \ge 0$

Ans. $x_1 = 20$, $x_2 = 100$, $x_3 = 80$ and Z = Rs 2,67,000.

21. Let x_1 and x_2 = number of units of products A and B to be bought, respectively

$$\text{Min (total cost) } Z = 20x_1 + 40x_2$$

subject to (i) $36x_1 + 6x_2 \ge 108$; (ii) $3x_1 + 12x_2 \ge 36$

(iii) $20x_1 + 10x_2 \ge 100$

and $x_1, x_2 \ge 0$

Ans. $x_1 = 4$, $x_2 = 2$ and Min Z = Rs 160.

22. Let x_1 and x_2 = number of units of television and radio tubes produced, respectively

$$\text{Max (total profit) } Z = 150x_1 + 250x_2$$

subject to (i) $x_1/1{,}500 + x_2/4{,}500 \ge 1$

(ii) $x_1/1{,}000 + x_2/8{,}000 \ge 1$

(iii) $x_1/2{,}000 + x_2/2{,}000 \ge 1$

(iv) $x_1/3{,}000 + x_2/9{,}000 \ge 1$

and $x_1, x_2 \ge 0$

25. Let x_1 and x_2 = number of feet of raw oak logs and raw pine logs to be processed each week, respectively

$$\text{Max } Z = 1{,}800x_1 + 1{,}200x_2$$

subject to (i) $2x_1 + 3x_2 \le 60$, (ii) $2.4x_1 + 1.2x_2 \le 48$,

(iii) $x_1 \le 18$, (iv) $x_2 \le 12$

and $x_1, x_2 \ge 0$

CHAPTER SUMMARY

The graphical solution approaches (or methods) provide a conceptual basis for solving large and complex LP problems. A graphical method is used to reach an optimal solution to an LP problem that has a number of constraints binding the objective function.

Both extreme point methods and the iso-profit (or cost) function line method are used for graphically solving any LP problem that has only two decision variables.

CHAPTER CONCEPTS QUIZ

True or False

1. The graphical method of solving linear programming problem is useful because of its applicability to many real life situations.
2. The problem of infeasibility in linear programming can only be solved by making additional resources available which in turn changes the constraints of the problem.
3. A linear programming problem is unbounded because constraints are incorrectly formulated.
4. If a LP problem has more than one solution yielding the same objective functional value, then such optimal solutions are known as alternative optimal solutions.
5. Since the constraints to a linear programming are always linear, we can graph them by locating only two different points on a line.
6. An optimal solution does not necessarily use up all the limited resources available.
7. The intersection of any two constraints is an extreme point which is a corner of the feasible region.
8. When there are more than one optimal solution to the problem then the decision-maker will be unable to judge the best optimal solution among them.
9. If all the constraints are ≥ inequalities in a linear programming problem whose objective function is to be maximized, then the solution of the problem is unbounded.
10. The problem caused by redundant constraints is that two isoprofit lines may not be parallel to each other.

Fill in the Blanks

11. Constraints in a linear programming requiring all variables to be zero or positive are known as .... constraints and rest from limited resources are referred as .... constraints.
12. Instead of maximizing profit in linear programming problem, we ensure linearity of the objective function by maximizing ....
13. The .... points of the convex set give the basic feasible solution to the linear programming.
14. A basic feasible solution is said to be .... if the values of all .... variables are nonzero and positive.
15. If an optimal solution to a linear programming problem exists, it will lie at .... of the feasible solution.
16. A constraint in linear programming must be expressed as a linear .... or a linear ....
17. When more than one solution best meets the objective of the linear programming problem then it is said to have .... solution.
18. .... occurs when no value of the variable is able to satisfy all the constraints in linear programming problem simultaneously.
19. An existence of objective of the problem for any firm is one of the major .... of the linear programming problem.
20. Any two isoprofit or isocost lines for a given linear programming problem are .... to each other.

Multiple Choice

21. The graphical method of LP problem uses
    (a) objective function equation
    (b) constraint equations
    (c) linear equations
    (d) all of the above
22. A feasible solution to an LP problem
    (a) must satisfy all of the problem's constraints simultaneously
    (b) need not satisfy all of the constraints, only some of them
    (c) must be a corner point of the feasible region
    (d) must optimize the value of the objective function
23. Which of the following statements is true with respect to the optimal solution of an LP problem
    (a) every LP problem has an optimal solution
    (b) optimal solution of an LP problem always occurs at an extreme point
    (c) at optimal solution all resources are completely used
    (d) if an optimal solution exists, there will always be at least one at a corner
24. An iso-profit line represents
    (a) an infinite number of solutions all of which yield the same profit
    (b) an infinite number of solution all of which yield the same cost
    (c) an infinite number of optimal solutions
    (d) a boundary of the feasible region
25. If an iso-profit line yielding the optimal solution coincides with a constraint line, then
    (a) the solution is unbounded
    (b) the solution is infeasible
    (c) the constraint which coincides is redundant
    (d) none of the above
26. While plotting constraints on a graph paper, terminal points on both the axes are connected by a straight line because
    (a) the resources are limited in supply
    (b) the objective function is a linear function
    (c) the constraints are linear equations or inequalities
    (d) all of the above
27. A constraint in an LP model becomes redundant because
    (a) two iso-profit line may be parallel to each other
    (b) the solution is unbounded
    (c) this constraint is not satisfied by the solution values
    (d) none of the above
28. If two constraints do not intersect in the positive quadrant of the graph, then
    (a) the problem is infeasible
    (b) the solution is unbounded
    (c) one of the constraints is redundant
    (d) none of the above
29. Constraints in LP problem are called active if they
    (a) represent optimal solution
    (b) at optimality do not consume all the available resources

(c) both of (a) and (b)
(d) none of the above

30. The solution space (region) of an LP problem is unbounded due to
(a) an incorrect formulation of the LP model
(b) objective function is unbounded
(c) neither (a) nor (b)
(d) both (a) and (b)

31. The while solving a LP model graphically, the area bounded by the constraints is called
(a) feasible region (b) infeasible region
(c) unbounded solution (d) none of the above

32. Alternative solutions exist of an LP model when
(a) one of the constraints is redundant
(b) objective function equation is parallel to one of the constraints
(c) two constraints are parallel
(d) all of the above

33. While solving a LP problem, infeasibility may be removed by
(a) adding another constraint
(b) adding another variable
(c) removing a constraint
(d) removing a variable

34. If a non-redundant constraint is removed from an LP problem, then
(a) feasible region will become larger
(b) feasible region will become smaller
(c) solution will become infeasible
(d) none of the above

35. If one of the constraint of an equation in an LP problem has an unbounded solution, then
(a) solution to such LP problem must be degenerate
(b) feasible region should have a line segment
(c) alternative solutions exist
(d) none of the above

Answers to Quiz

1. T	2. F	3. F	4. T	5. T	6. T	7. T	8. F	9. T	10. T

11. non-negative, structural 12. Contribution 13. extreme 14. non-degenerative, basic,
15. an extreme point 16. equation, inequality 17. alternative 18. infeasibility
19. requirement 20. parallel.

21. (d)	22. (a)	23. (d)	24. (a)	25. (d)	26. (c)	27. (d)	28. (a)	29. (a)	30. (c)
31. (a)	32. (b)	33. (c)	34. (a)	35. (b)					

chapter 4

Linear Programming: The Simplex Method

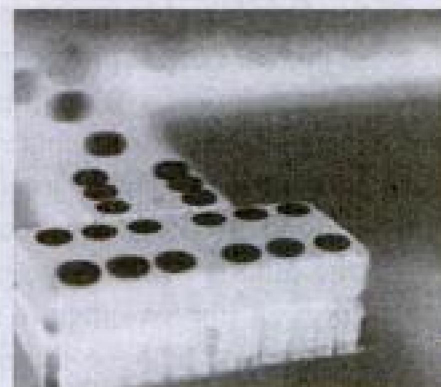
"Checking the results of a decision against its expectations shows executives what their strengths are, where they need to improve, and where they lack knowledge or information."

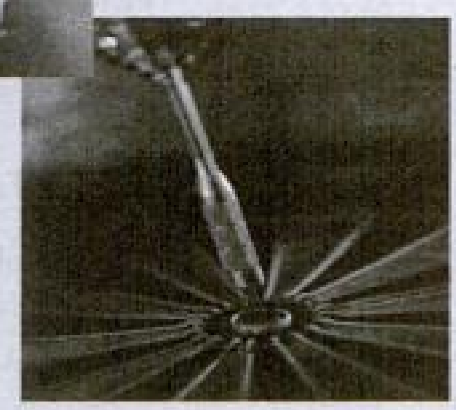
– Peter Drucker

Preview The purpose of this chapter is to help you gain an understanding of how the simplex method works. Understanding the underlying principles help to interpret and analyze solution of any LP problem

Learning Objectives After studying this chapter, you should be able to

- understand the meaning of the word 'simplex' and logic of using simplex method.
- convert an LP problem into its standard form by adding slack, surplus and artificial variables.
- set-up simplex tables and solve LP problems using the simplex algorithm.
- interpret the optimal solution of LP problems.
- recognize special cases such as degeneracy, multiple optimal solution, unbounded and infeasible solutions.





Chapter Outline

4.1 INTRODUCTION

Most real-life problems when formulated as an LP model have more than two variables and are too large to be interpret with the help of the graphical solution method. We therefore need a more efficient method to suggest an optimal solution for such problems. In this chapter, we shall discuss a procedure called the *simplex method*, which is used for solving an LP model of such problems. Thus method was developed by G B Dantzig in 1947.

The *simplex* is an important term in mathematics, one that represents an object in an n-dimensional space, connecting $n + 1$ points. In one dimension, a simplex is a line segment connecting two points; in two dimensions, it is a triangle formed by joining three points; in three dimensions, it is a four-sided pyramid, having four corners.

The concept of simplex method is similar to the graphical method. In the graphical method, extreme points of the feasible solution space are examined in order to search for the optimal solution that lies at one of these points. For LP problems with several variables, we may not be able to graph the feasible region, but the optimal solution will still lie at an extreme point of the many-sided, multidimensional figure (called an n-dimensional polyhedron) that represents the feasible solution space. The simplex method examines the extreme points in a systematic manner, repeating the same set of steps of the algorithm until an optimal solution is found. It is for this reason that it is also called the *iterative method.*

Since the number of extreme points (corners or vertices) of the feasible solution space are finite, the method assures an improvement in the value of objective function as we move from one iteration (extreme point) to another and achieve the optimal solution in a finite number of steps. The method also indicates when an unbounded solution is reached.

4.2 STANDARD FORM OF AN LP PROBLEM

Simplex method examines corner points of the feasible region, using matrix row operations, until an optimal solution is found

The use of the simplex method to solve an LP problem requires that the problem be converted into its standard form. The standard form of the LP problem should have the following characteristics:

(i) All the constraints should be expressed as equations by adding slack or surplus and/or artificial variables.

(ii) The right-hand side of each constraint should be made non-negative if it is not already, this should be done by multiplying both sides of the resulting constraint by – 1.

(iii) The objective function should be of the maximization type.

The standard form of the LP problem is expressed as:

Optimize (Max or Min) $Z = c_1 x_1 + c_2 x_2 + \ldots + c_n x_n + 0s_1 + 0s_2 + \ldots + 0s_m$

subject to the linear constraints

$$\begin{aligned} a_{11} x_1 + a_{12} x_2 + \ldots + a_{1n} x_n + s_1 &= b_1 \\ a_{21} x_1 + a_{22} x_2 + \ldots + a_{2n} x_n + s_2 &= b_2 \\ &\vdots \\ a_{m1} x_1 + a_{m2} x_2 + \ldots + a_{mn} x_n + s_m &= b_m \end{aligned}$$

and $\quad x_1, x_2, \ldots, x_n, s_1, s_2, \ldots, s_m \geq 0$

The standard form of the LP problem can also be expressed in the compact form as follows:

$$\text{Optimize (Max or Min) } Z = \sum_{j=1}^{n} c_j x_j + \sum_{i=1}^{m} 0 s_i \qquad \text{(Objective function)}$$

subject to the constraints

$$\sum_{j=1}^{n} a_{ij} x_j + s_i = b_i; \quad i = 1, 2, \ldots, m \qquad \text{(Constraints)}$$

and $\quad x_j, s_i \geq 0, \quad$ for all i and j $\qquad$ (Non-negativity conditions)

In matrix notations the standard form is expressed as:

Optimize (Max or Min) $Z = \mathbf{cx} + \mathbf{0s}$

subject to the constraints

$$\mathbf{Ax} + \mathbf{s} = \mathbf{b}, \text{ and } \mathbf{x}, \mathbf{s} \geq \mathbf{0}$$

where, $\mathbf{c} = (c_1, c_2, \ldots, c_n)$ is the row vector, $\mathbf{x} = (x_1, x_2, \ldots, x_n)^T$, $\mathbf{b} = (b_1, b_2, \ldots, b_m)^T$ and $\mathbf{s} = (s_1, s_2, \ldots, s_m)$ are column vectors, and $\mathbf{A}$ is the $m \times n$ matrix of coefficients of variables $x_1, x_2, \ldots, x_n$ in the constraints.

Remarks The constrained optimization (maximization or minimization) problem, may have

1. (a) no feasible solution, i.e. there may not exist values x_j ($j = 1, 2, \ldots, n$) that satisfy every constraint.
   (b) a unique optimum feasible solution.
   (c) more than one optimum feasible solution, i.e. alternative optimum feasible solution.
   (d) a feasible solution for which the objective function is *unbounded*, i.e. the value of the objective function can be made as large as possible in a maximization problem or as small as possible in a minimization problem by selecting an appropriate feasible solution.
2. Any minimization LP problem can be converted into an equivalent maximization problem by changing the sign of c_j's in the objective function. That is,

$$\text{Minimize } \sum_{j=1}^{n} c_j x_j = \text{Maximize } \sum_{j=1}^{n} (-c_j) x_j$$

3. Any constraint expressed by equality (=) sign may be replaced by two weak inequalities. For example, $a_{11}x_1 + a_{12}x_2 + \ldots + a_{1n}x_n = b_1$ is equivalent to following two simultaneous constraints,

$$a_{11}x_1 + a_{12}x_2 + \ldots + a_{1n}x_n \leq b_1 \quad \text{and} \quad a_{11}x_1 + a_{12}x_2 + \ldots + a_{1n}x_n \geq b_1$$

4. Three types of additional variables, namely (i) slack variables (s) (ii) surplus variables (– s), and (iii) artificial variables (A) are added in the given LP problem to convert it into the standard form for the following reasons:
   (a) These variables allow us to convert inequalities into equalities, thereby converting the given LP problem into a form that is amenable to algebraic solution.
   (b) These variables permit us to make a more comprehensive economic interpretation of a final solution.
   (c) Help us to get an initial feasible solution represented by the columns of the identity matrix.

Slack variable represents a quantity of unused resource; it is added to less-than or equal-to constraints in order to get an equality constraint

The summary of the extra variables to be added in the given LP problem in order to convert it into a standard form is given in Table 4.1.

Types of Constraint	*Extra Variable Needed*	*Coefficient of Extra Variables in the Objective Function*		*Presence of Extra Variables in the Initial Solution Mix*
		Max Z	*Min Z*	
• Less than or equal to (≤)	A slack variable is added	0	0	Yes
• Greater than or equal to (≥)	A surplus variable is subtracted, and	0	0	No
	an artificial variable is added	$-M$	$+M$	Yes
• Equal to (=)	Only an artificial variable is added	$-M$	$+M$	Yes

Table 4.1 Summary of Additional Variables Added in an LP Problem

Remark *A slack variable* represents an unused resource, either in the form of time on a machine, labour hours, money, warehouse space or any number of such resources in various business problems. Since these variables don't yield any profit, therefore such variables are added to the original objective function with zero coefficients.

A surplus variable represents the amount by which solution values exceed a resource. These variables are also called *negative slack variables*. Surplus variables, like slack variables carry a zero coefficient in the objective function.

Definitions

Basic solution Given a system of m simultaneous linear equations in n (> m) unknowns, $\mathbf{Ax} = \mathbf{B}$, where $\mathbf{A}$ is an $m \times n$ matrix and rank ($\mathbf{A}$) = m. Let $\mathbf{B}$ be any $m \times m$ non-singular submatrix of $\mathbf{A}$ obtained by reordering m

linearly independent columns of **A**. Then, a solution obtained by setting $n-m$ variables not associated with the columns of **B**, equal to zero, and solving the resulting system is called a *basic solution* to the given system of equations.

The m variables, which may be all different from zero, are called *basic variables*. The $m \times m$ non-singular sub-matrix **B** is called a *basis matrix* and the columns of **B** as *basis vectors*.

If **B** is the basis sub-matrix, then the basic solution to the system of equations will be $\mathbf{x_B} = \mathbf{B}^{-1}\mathbf{b}$.

Basic feasible solution A basic solution to the system $\mathbf{Ax} = \mathbf{b}$ is called *basic feasible* if $\mathbf{x_B} \geq 0$.

Degenerate solution A basic solution to the system $\mathbf{Ax} = \mathbf{b}$ is called *degenerate* if one or more of the basic variables vanish.

Associated cost vector Let $\mathbf{x_B}$ be a basic feasible solution to the LP problem.

Maximize $Z = \mathbf{cx}$

subject to the constraints

$$\mathbf{Ax} = \mathbf{b}, \quad \text{and} \quad \mathbf{x} \geq 0.$$

Then the vector $\mathbf{c_B} = (c_{B1}, c_{B2}, \ldots, c_{Bm})$, is called the *cost vector* associated with the basic feasible solution $\mathbf{x_B}$, and $\mathbf{c_B}$, is coefficient of basic variable x_i.

4.3 SIMPLEX ALGORITHM (MAXIMIZATION CASE)

The steps of the simplex algorithm for obtaining an optimal solution (if it exists) to a linear programming problem are as follows:

Step 1: Formulation of the mathematical model

Surplus variable represents the amount of resource usage above the minimum required and is added to greater-than or equal-to constraints in order to get equality constraint

(i) Formulate the mathematical model of the given linear programming problem.

(ii) If the objective function is of minimization, then convert it into one of maximization, by using the following relationship

$$\text{Minimize } Z = -\text{Maximize } Z^*, \text{ where } Z^* = -Z.$$

(iii) Check whether all the b_i ($i = 1, 2, \ldots, m$) values are positive. If any one of them is negative, then multiply the corresponding constraint by -1 in order to make $b_i > 0$. In doing so, remember to change a $\leq$ type constraint to a $\geq$ type constraint, and vice versa.

(iv) Express the mathematical model of the given LP problem in the standard form by adding additional variables to the left side of each constraint and assign a zero-cost coefficient to these in the objective function.

(v) Replace each *unrestricted* variable with the difference of the two non-negative variables; replace each non-positive variable with a new non-negative variable, whose value is the negative of the original variable.

Step 2: Set-up the initial solution Write down the coefficients of all the variables in the LP model in a tabular form, as shown in Table 4.2, in order to get an initial basic feasible solution $[\mathbf{x_B} = \mathbf{B}^{-1}\mathbf{b}]$.

		$c_j \rightarrow$	c_1	c_2	$\ldots$	c_n	0	0	$\ldots$	0
Coefficient of Basic Variables (c_B)	*Variables in Basis* B	*Value of Basic Variables* $b\,(= x_B)$	*Variables* x_1	x_2	$\ldots$	x_n	s_1	s_2	$\ldots$	s_m
c_{B1}	s_1	$x_{B1} = b_1$	a_{11}	a_{12}	$\ldots$	a_{1n}	1	0	$\ldots$	0
c_{B2}	s_2	$x_{B2} = b_2$	a_{21}	a_{22}	$\ldots$	a_{2n}	0	1	$\ldots$	0
$\vdots$	$\vdots$	$\vdots$	$\vdots$	$\vdots$		$\vdots$	$\vdots$	$\vdots$		$\vdots$
c_{Bm}	s_m	$x_{Bm} = b_m$	a_{m1}	a_{m2}	$\ldots$	a_{mn}	0	0	$\ldots$	1
$Z = \Sigma c_{Bi} x_{Bi}$ = (B.V. coefficients) × (Values of B.V.)	$z_j = \Sigma c_{Bi} x_j$ = Σ (B.V. coefficients) × (jth column of data matrix)		0	0	$\ldots$	0	0	0	$\ldots$	0
	$c_j - z_j$		$c_1 - z_1$	$c_2 - z_2$	$\ldots$	$c_n - z_n$	0	0	$\ldots$	0

Table 4.2 Initial Simplex Table

After having set up the initial simplex table, locate the identity matrix and column variables involved in it. This matrix contains all zeros except positive elements 1's on the diagonal. This identity matrix is always a square matrix and its size is determined by the number of constraints. The identity matrix so obtained is also called a *basis matrix* [because basic feasible solution is represented by $\mathbf{B} = \mathbf{I}$].

Assign the values of the constants (b_i's) to the column variables in the identity matrix [because $\mathbf{x_B} = \mathbf{B}^{-1}\mathbf{b} = \mathbf{I}\,\mathbf{b} = \mathbf{b}$].

The variables corresponding to the columns of the identity matrix are called *basic variables* and the remaining ones are *non-basic variables*. In general, if an LP model has n variables and m ($< n$) constraints, then m variables would be basic and $n - m$ variables non-basic. That is, simplex algorithm works with *basic feasible solution* (*bfs*), which is the algebraic version of extreme points. A *bfs* is a feasible solution obtained by choosing one basic variable for each constraint. The remaining ones are non-basic and have zero value. However, in certain cases some basic variables may also have zero values. This situation is called *degeneracy* and will discussed later.

Degeneracy arises when there is a tie in the minimum ratio value that determines the variable to enter into the next solution

The first row in Table 4.2 indicates the coefficients c_j of variables in the objective function that remain the same in successive simplex tables. These values represent the cost or the profit per unit, to the objective function of each of the variables and are used to determine the variable to be entered into the basis matrix **B**.

The second row provides the major column headings for the simplex table. Column '$\mathbf{c_B}$' lists the coefficients of the current basic variables in the objective function. These values are used to calculate the value of Z when one unit of any variable is brought into the solution. Column headed by $\mathbf{x_B}$ represents the current values of the corresponding variables in the basis.

The identity matrix (or basis matrix) represents the coefficients of slack variables that have been added to the constraints. Each column of the identity matrix also represents a basic variable to be listed in column **B**.

Numbers a_{ij} in the columns under each variable are also called *substitution rates* or *exchange coefficients* because these represent the rate at which resource i ($i = 1, 2, \ldots, m$) is consumed by each unit of an activity j ($j = 1, 2, \ldots, n$).

The values z_j represent the amount by which the value of objective function Z would be decreased (or increased) if one unit of the given variable is added to the new solution. Each of the values in the $c_j - z_j$ row represents the net amount of increase (or decrease) in the objective function that would occur when one unit of the variable represented by the column head is introduced into the solution. That is:

$$c_j - z_j \text{ (net effect)} = c_j \text{ (incoming unit profit/cost)} - z_j \text{ (outgoing total profit/cost)}$$

where z_j = Coefficient of basic variables column × Exchange coefficient column j

Basis is the set of variables present in the solution, have positive non-zero value and are listed in the $\mathbf{b}$ ($= \mathbf{x_B}$) column of a simplex table; these variables are also called **basic variables**

Step 3: Test for optimality Calculate the $c_j - z_j$ value for all non-basic variables. To obtain the value of z_j multiply each element under *'Variables' Column* (columns, $\mathbf{a_j}$ of the coefficient matrix) with the corresponding elements in $\mathbf{c_B}$-column. Examine the values of $c_j - z_j$. The following three cases may arise:

(i) If all $c_j - z_j \le 0$, then the basic feasible solution is optimal.

(ii) If at least one column of the coefficients matrix (i.e. $\mathbf{a}_k$) for which $c_k - z_k > 0$ and all other elements are negative (i.e. $a_{ik} < 0$), then there exists an unbounded solution to the given problem.

(iii) If at least one $c_j - z_j > 0$, and each of these columns have at least one positive element (i.e. a_{ij}) for some row, then this indicates that an improvement in the value of objective function Z is possible.

Step 4: Select the variable to enter the basis If Case (iii) of Step 3 holds, then select a variable that has the largest $c_j - z_j$ value to enter into the new solution. That is,

$$c_k - z_k = \text{Max } \{(c_j - z_j);\quad c_j - z_j > 0\}$$

The column to be entered is called the *key* or *pivot* column. Obviously, such a variable indicates the largest per unit improvement in the current solution. Such a variable therefore indicates the largest per unit improvement in the current solution.

Non-basic variables are those which are not in the 'basis' and have zero-value

Step 5: Test for feasibility (variable to leave the basis) After identifying the variable to become the basic variable, the variable to be removed from the existing set of basic variables is determined. For this, each number in $\mathbf{x_B}$-column (i.e. b_i values) is divided by the corresponding (but positive) number in the key column and a row is selected for which this ratio, [(*constant column*)/(*key column*)] is non-negative and minimum. This ratio is called the *replacement* (*exchange*) *ratio.* That is,

$$\frac{x_{Br}}{a_{rj}} = \text{Min}\left\{\frac{x_{Bi}}{a_{rj}};\ a_{rj} > 0\right\}$$

This ratio limits the number of units of the incoming variable that can be obtained from the exchange. *It may be noted here that division by a negative or by a zero element in key column is not permitted.*

Infeasibility is the situation in which no solution satisfies all constraints of an LP problem

The row selected in this manner is called the *key* or *pivot row* and it represents the variable which will leave the solution.

The element that lies at the intersection of the key row and key column of the simplex table is called *key* or *pivot* element.

Step 6: Finding the new solution

(i) If the key element is 1, then the row remains the same in the new simplex table.

(ii) If the key element is other than 1, then divide each element in the key row (including the elements in $\mathbf{x_B}$-column) by the key element, to find the new values for that row.

(iii) The new values of the elements in the remaining rows of the new simplex table can be obtained by performing elementary row operations on all rows so that all elements except the key element in the key column are zero.

In other words, for each row other than the key row, we use the formula:

$$\begin{matrix}\text{Number in}\\ \text{new row}\end{matrix} = \begin{pmatrix}\text{Number in}\\ \text{old row}\end{pmatrix} \pm \left[\begin{pmatrix}\text{Number above or below}\\ \text{key element}\end{pmatrix} \times \begin{pmatrix}\text{Corresponding number in}\\ \text{the new row, that is row}\\ \text{replaced in Step 6 (ii)}\end{pmatrix}\right]$$

The new entries in $\mathbf{c_B}$ (coefficient of basic variables) and $\mathbf{x_B}$ (value of basic variables) columns are updated in the new simplex table of the current solution.

$c_j - z_j$ row values represent net profit or loss resulting from introducing one unit of any variable into the 'basis' or solution mix

Step 7: Repeat the procedure Go to Step 3 and repeat the procedure until all entries in the $c_j - z_j$ row are either negative or zero.

Remark The flow chart of the simplex algorithm for both the maximization and the minimization LP problem is shown in Fig. 4.1.

Example 4.1 Use the simplex method to solve the following LP problem.

Maximize $Z = 3x_1 + 5x_2 + 4x_3$

subject to the constraints

(i) $2x_1 + 3x_2 \leq 8$, (ii) $2x_2 + 5x_3 \leq 10$, (iii) $3x_1 + 2x_2 + 4x_3 \leq 15$

and $x_1, x_2, x_3 \geq 0$

[*Rewa MSc (Maths), 1995; Meerut MSc (Stat.), 1996; MSc (Maths), 1998; Kurukshetra, MSc (Stat.), 1998*]

Solution Step 1: By introducing non-negative slack variables s_1, s_2 and s_3 in order to convert the inequality constraints to equality the LP problem becomes:

Maximize $Z = 3x_1 + 5x_2 + 4x_3 + 0s_1 + 0s_2 + 0s_3$

subject to the constraints

(i) $2x_1 + 3x_2 + s_1 = 8$, (ii) $2x_2 + 5x_3 + s_2 = 10$, (iii) $3x_1 + 2x_2 + 4x_3 + s_3 = 15$

and $x_1, x_2, x_3, s_1, s_2, s_3 \geq 0$

Step 2: Since all b_i (RHS values) > 0, (i = 1, 2, 3) we can choose the initial basic feasible solution as:

$$x_1 = x_2 = x_3 = 0; s_1 = 8, s_2 = 10, s_3 = 15 \text{ and Max } Z = 0$$

This solution can also be read from the initial simplex Table 4.3 by equating row-wise values in the basis (B) column and solution values ($\mathbf{x_B}$) column.

Step 3: To see whether the current solution given in Table 4.3 is optimal or not, calculate

$$c_j - z_j = c_j - \mathbf{c_B} \mathbf{B}^{-1} \mathbf{a}_j = c_j - \mathbf{c_B} \mathbf{y}_j$$

for non-basic variables x_1, x_2 and x_3 as follows.

$$z_j = (\text{Basic variable coefficients, } \mathbf{c_B}) \times (j\text{th column of data matrix})$$

That is,
$$z_1 = 0\,(2) + 0\,(0) + 0\,(3) = 0 \quad \text{for } x_1\text{-column}$$
$$z_2 = 0\,(3) + 0\,(2) + 0\,(2) = 0 \quad \text{for } x_2\text{-column}$$
$$z_3 = 0\,(0) + 0\,(5) + 0\,(4) = 0 \quad \text{for } x_3\text{-column}$$

These z_j values are now subtracted from c_j values in order to calculate the net profit. This is done by introducing one unit of each variable x_1, x_2 and x_3 into the new solution mix.

$$c_1 - z_1 = 3 - 0 = 3, \qquad c_2 - z_2 = 5 - 0 = 5, \qquad c_3 - z_3 = 4 - 0 = 4$$

The z_j and $c_j - z_j$ rows are added into the Table 4.3.

The values of basic variables, s_1, s_2 and s_3 are given in the *solution values* ($\mathbf{x_B}$) column of Table 4.3. The remaining variables that are non-basic at the current solution have zero value. The value of objective function at the current solution is given by

$$Z = (\text{Basic variable coefficients, } \mathbf{c_B}) \times (\text{Basic variable values, } \mathbf{x_B})$$
$$= 0\,(8) + 0\,(10) + 0\,(15) = 0$$

		$c_j \rightarrow$	3	5	4	0	0	0	
Profit per Unit c_B	*Variables in Basis* B	*Solution Values* $b\,(= x_B)$	x_1	x_2	x_3	s_1	s_2	s_3	*Min Ratio* x_B/x_2
0	s_1	8	2	3	0	1	0	0	8/3 →
0	s_2	10	0	2	5	0	1	0	10/2
0	s_3	15	3	2	4	0	0	1	15/2
$Z = 0$		z_j	0	0	0	0	0	0	
		$c_j - z_j$	3	5 ↑	4	0	0	0	

Table 4.3
Initial Solution

Since all $c_j - z_j \geq 0$ ($j = 1, 2, 3$), the current solution is not optimal. Variable x_2 is chosen to enter into the basis because $c_2 - z_2 = 5$ is the largest positive number in the x_2-column, where all elements are positive. This means that for every unit of variable x_2, the objective function will increase in value by 5. The x_2-column is the key column.

Step 4: The variable that is to leave the basis is determined by dividing the values in the $\mathbf{x_B}$-column by the corresponding elements in the key column as shown in Table 4.3. Since the exchange ratio, 8/3 is minimum in row 1, the basic variable s_1 is chosen to leave the solution (basis).

Step 5 (***Iteration 1***): Since the key element enclosed in the circle in Table 4.3 is not 1, divide all elements of the key row by 3 in order to obtain new values of the elements in this row. The new values of the elements in the remaining rows for the new Table 4.4 can be obtained by performing the following elementary row operations on all rows so that all elements except the key element 1 in the key column are zero.

$$R_1 \text{ (new)} \rightarrow R_1 \text{ (old)} \div 3 \textit{ (key element)}$$
$$\rightarrow (8/3,\ 2/3,\ 3/3,\ 0/3,\ 1/3,\ 0/3,\ 0/3) = (8/3,\ 2/3,\ 1,\ 0,\ 1/3,\ 0,\ 0)$$

R_2 (new) → R_2 (old) − $2R_1$ (new)
$10 - 2 \times 8/3 = 14/3$
$0 - 2 \times 2/3 = -4/3$
$2 - 2 \times 1 = 0$
$5 - 2 \times 0 = 5$
$0 - 2 \times 1/3 = -2/3$
$1 - 2 \times 0 = 1$
$0 - 2 \times 0 = 0$

R_3 (new) → R_3 (old) − $2R_1$ (new)
$15 - 2 \times 8/3 = 29/3$
$3 - 2 \times 2/3 = 5/3$
$2 - 2 \times 1 = 0$
$4 - 2 \times 0 = 4$
$0 - 2 \times 1/3 = -2/3$
$0 - 2 \times 0 = 0$
$1 - 2 \times 0 = 1$

		$c_j \rightarrow$	3	5	4	0	0	0	
Profit per Unit c_B	*Variables in Basis* B	*Solution Values* $b\ (= x_B)$	x_1	x_2	x_3	s_1	s_2	s_3	*Min Ratio* x_B/x_3
5	x_2	8/3	2/3	1	0	1/3	0	0	–
0	s_2	14/3	– 4/3	0	5	– 2/3	1	0	(14/3)/5 →
0	s_3	29/3	5/3	0	4	– 2/3	0	1	(29/3)/4
Z = 40/3		z_j	10/3	5	0	5/3	0	0	
		$c_j - z_j$	– 1/3	0	4	– 5/3	0	0	
					↑				

Table 4.4
Improved Solution

The improved basic feasible solution can be read from Table 4.4 as: $x_2 = 8/3$, $s_2 = 14/3$, $s_3 = 29/3$ and $x_1 = x_3 = s_1 = 0$. The improved value of the objective function is

$$Z = (\text{Basic variable coefficients, } \mathbf{c_B}) \times (\text{Basic variable values, } \mathbf{x_B})$$
$$= 5\,(8/3) + 0\,(14/3) + 0\,(29/3) = 40/3$$

Once again, calculate values of $c_j - z_j$ in the same manner as discussed earlier to check whether the solution shown in Table 4.4 is optimal or not. Since $c_3 - z_3 > 0$, the current solution is not optimal.

Iteration 2: Repeat Steps 3 to 5. Table 4.5 is obtained by performing following row operations to enter variable x_3 into the basis and to drive out s_2 from the basis.

$$R_2\,(\text{new}) = R_2\,(\text{old}) \div 5\ (\textit{key element}) = (14/15, -4/15, 0, 1, -2/15, 1/5, 0)$$

R_3 (new) → R_3 (old) – $4R_2$ (new)
29/3 – 4 × 14/15 = 89/15
5/3 – 4 × – 4/15 = 41/15
0 – 4 × 0 = 0
4 – 4 × 1 = 0
– 2/3 – 4 × – 2/15 = – 2/15
0 – 4 × 1/5 = – 4/5
1 – 4 × 0 = 0

Table 4.5 is completed by calculating the new z_j and $c_j - z_j$ values and the new value of objective function:

$z_1 = 5\,(2/3) + 4\,(-4/15) + 0\,(41/15) = 34/15$ $c_1 - z_1 = 3 - 34/15 = 11/15$ for x_1-column
$z_4 = 5\,(1/3) + 4\,(-2/15) + 0\,(2/15) = 17/15$ $c_4 - z_4 = 0 - 17/15 = -17/15$ for s_1-column
$z_5 = 5\,(0) + 4\,(1/5) + 0\,(-4/5) = 4/5$ $c_5 - z_5 = 0 - 4/5 = -4/5$ for s_2-column

The new objective function value is given by

$$Z = (\text{Basic variable coefficients, } \mathbf{c_B}) \times (\text{Basic variable values, } \mathbf{x_B})$$
$$= 5\,(8/3) + 4\,(14/15) + 0\,(89/15) = 256/15$$

The improved basic feasible solution is shown in Table 4.5.

		$c_j \rightarrow$	3	5	4	0	0	0	
Profit per Unit c_B	*Variables in Basis* B	*Solution Values* $b\ (= x_B)$	x_1	x_2	x_3	s_1	s_2	s_3	*Min Ratio* x_B/x_1
5	x_2	8/3	2/3	1	0	1/3	0	0	(8/3)/(2/3) = 4
4	x_3	14/15	– 4/15	0	1	– 2/15	1/5	0	–
0	s_3	89/15	41/15	0	0	2/15	– 4/5	1	(89/15)/(41/15) = 2.17 →
Z = 256/15		z_j	34/15	5	4	17/15	4/5	0	
		$c_j - z_j$	11/15	0	0	– 17/15	– 4/5	0	
			↑						

Table 4.5
Improved Solution

Iteration 3: In Table 4.5, since $c_1 - z_1$ is still a positive value, the current solution is not optimal. Thus, the variable x_1 enters the basis and s_3 leaves the basis. To get another improved solution as shown in Table 4.5 perform the following row operations in the same manner as discussed earlier.

$$R_3 \text{ (new)} \rightarrow R_3 \text{ (old)} \times 15/41 \textit{ (key element)}$$
$$\rightarrow (89/15 \times 15/41, 41/15 \times 15/41, 0 \times 15/41, 0 \times 15/41, -2/15 \times 15/41, -4/5 \times 15/41, 1 \times 15/41)$$
$$\rightarrow (89/41, 1, 0, 0, -2/41, -12/41, 15/41)$$

R_1 (new) $\rightarrow R_1$ (old) $-$ (2/3) R_3 (new)
$8/3 - 2/3 \times 89/3 = 50/41$
$2/3 - 2/3 \times 1 = 0$
$1 - 2/3 \times 0 = 1$
$0 - 2/3 \times 0 = 0$
$1/3 - 2/3 \times -2/41 = 15/41$
$0 - 2/3 \times -2/41 = 8/41$
$0 - 2/3 \times 15/41 = -10/41$

R_2 (new) $\rightarrow R_2$ (old)+ (4/15) R_3 (new)
$14/15 + 4/15 \times 89/41 = 62/41$
$-4/15 + 4/15 \times 1 = 0$
$0 + 4/15 \times 0 = 0$
$1 + 4/15 \times 0 = 1$
$-2/15 + 4/15 \times -2/41 = -6/41$
$1/5 + 4/15 \times -12/41 = 5/41$
$0 + 4/15 \times 15/41 = 4/41$

		$c_j \rightarrow$	3	5	4	0	0	0
Profit per Unit c_B	*Variables in Basis* B	*Solution Values* $b (= x_B)$	x_1	x_2	x_3	s_1	s_2	s_3
5	x_2	50/41	0	1	0	15/41	8/41	$-10/41$
4	x_3	62/41	0	0	1	$-6/41$	5/41	4/41
3	x_1	89/41	1	0	0	$-2/41$	$-12/41$	15/41
$Z = 765/41$		z_j	3	5	4	45/41	24/41	11/41
		$c_j - z_j$	0	0	0	$-45/41$	$-24/41$	$-11/41$

Table 4.6
Optimal Solution

In Table 4.6, all $c_j - z_j < 0$ for non-basic variables. Therefore, the optimal solution is reached with, $x_1 = 89/41$, $x_2 = 50/41$, $x_3 = 62/41$ and the optimal value of $Z = 765/41$.

Example 4.2 A company makes two kinds of leather belts N belt A and belt B. Belt A is a high quality belt and belt *B* is of lower quality. The respective profits are Rs 4 and Rs 3 per belt. The production of each of type *A* requires twice as much time as a belt of type B, and if all belts were of type B, the company could make 1,000 belts per day. The supply of leather is sufficient for only 800 belts per day (both A and B combined). Belt A requires a fancy buckle and only 400 of these are available per day. There are only 700 buckles a day available for belt B.

What should be the daily production of each type of belt? Formulate this problem as an LP model and solve it using the simplex method.

Solution Let x_1 and x_2 be the number of belts of type A and B, respectively manufactured each day. Then the mathematical LP model would be as follows:

Maximize (total profit) $Z = 4x_1 + 3x_2$

subject to the constraints

(i) $2x_1 + x_2 \le 1{,}000$ (Time availability), (ii) $x_1 + x_2 \le 800$ (Supply of leather)

(iii) $\left.\begin{array}{l} x_1 \le 400 \\ x_2 \le 700 \end{array}\right\}$ (Buckles availability)

and $x_1, x_2 \ge 0$

Standard form Introducing slack variables s_1, s_2, s_3 and s_4 to convert given LP model into its standard form as follows.

Maximize $Z = 4x_1 + 3x_2 + 0s_1 + 0s_2 + 0s_3 + 0s_4$

subject to the constraints

(i) $2x_1 + x_2 + s_1 = 1{,}000$, (ii) $x_1 + x_2 + s_2 = 800$

(iii) $x_1 + s_3 = 400$, (iv) $x_2 + s_4 = 700$

and $x_1, x_2, s_1, s_2, s_3, s_4 \geq 0$

Solution by simplex method An initial feasible solution is obtained by setting $x_1 = x_2 = 0$. Thus, the initial solution is: $s_1 = 1{,}000$, $s_2 = 800$, $s_3 = 400$, $s_4 = 700$ and Max $Z = 0$. This solution can also be read from the initial simplex Table 4.7.

Table 4.7
Initial Solution

		$c_j \rightarrow$	4	3	0	0	0	0	
Profit per Unit c_B	*Variables in Basis* B	*Solution Values* $b\ (= x_B)$	x_1	x_2	s_1	s_2	s_3	s_4	*Min Ratio* x_B/x_1
0	s_1	1,000	2	1	1	0	0	0	1,000/2 = 500
0	s_2	800	1	1	0	1	0	0	800/1 = 800
0	s_3	400	1	0	0	0	1	0	400/1 = 400 →
0	s_4	700	0	1	0	0	0	1	not defined
$Z = 0$		z_j	0	0	0	0	0	0	
		$c_j - z_j$	4	3	0	0	0	0	
			↑						

In Table 4.7, since $c_1 - z_1 = 4$ is the largest positive number, we apply the following row operations in the same manner as discussed earlier in order to get an improved basic feasible solution by entering variable x_1 into the basis and removing variable s_3 from the basis.

R_3 (new) $\rightarrow R_3$ (old) ÷ 1 (*key element*) $\qquad R_1$ (new) $\rightarrow R_1$ (old) − $2R_3$ (new)

R_2 (new) $\rightarrow R_2$ (old) − R_3 (new)

The new solution is shown in Table 4.8.

Table 4.8
An Improved Solution

		$c_j \rightarrow$	4	3	0	0	0	0	
Profit per Unit c_B	*Variables in Basis* B	*Solution Values* $b\ (= x_B)$	x_1	x_2	s_1	s_2	s_3	s_4	*Min Ratio* x_B/x_2
0	s_1	200	0	1	1	0	−2	0	200/1 = 200 →
0	s_2	400	0	1	0	1	−1	0	400/1 = 400
4	x_1	400	1	0	0	0	1	0	–
0	s_4	700	0	1	0	0	0	1	700/1 = 700
$Z = 1{,}600$		z_j	4	0	0	0	4	0	
		$c_j - z_j$	0	3	0	0	−4	0	
				↑					

The solution shown in Table 4.8 is not optimal because $c_2 - z_2 > 0$ in x_2-column. Thus, again applying the following row operations to get a new solution by entering variable x_2 into the basis and removing variable s_1 from the basis, we get

R_1 (new) $\rightarrow R_1$ (old) ÷ 1 (*key element*) $\qquad R_2$ (new) $\rightarrow R_2$ (old) − R_1 (new)

R_4 (new) $\rightarrow R_4$ (old) − R_1 (new)

The improved solution is shown in Table 4.9.

Profit per Unit c_B	Variables in Basis B	Solution Values $b(= x_B)$ $c_j \rightarrow$	x_1 4	x_2 3	s_1 0	s_2 0	s_3 0	s_4 0	Min Ratio x_B/s_3
3	x_2	200	0	1	1	0	−2	0	—
0	s_2	200	0	0	−1	1	1	0	200/1 = 200 →
4	x_1	400	1	0	0	0	1	0	400/1 = 400
0	s_4	500	0	0	−1	0	2	1	500/2 = 250
Z = 2,200		z_j	4	3	3	0	−2	0	
		$c_j - z_j$	0	0	−3	0	2 ↑	0	

Table 4.9
Improved Solution

The solution shown in Table 4.9 is not optimal because $c_5 - z_5 > 0$ in s_3-column. Thus, again applying the following row operations to get a new solution by entering variable, s_3 into the basis and removing variable s_2 from the basis, we get

R_2 (new) $\rightarrow R_2$ (old) $\div$ 1 (*key element*) $\quad$ R_1 (new) $\rightarrow R_1$ (old) + $2R_2$ (new)

R_3 (new) $\rightarrow R_3$ (old) − R_2 (new); $\quad$ R_4 (new) $\rightarrow R_4$ (old) − $2R_2$ (new)

The new improved solution is shown in Table 4.10.

Profit per Unit c_B	Variables in Basis B	Solution Values b $(= x_B)$ $c_j \rightarrow$	x_1 4	x_2 3	s_1 0	s_2 0	s_3 0	s_4 0
3	x_2	600	0	1	−1	2	0	0
0	s_3	200	0	0	−1	1	1	0
4	x_1	200	1	0	1	−1	0	0
0	s_4	100	0	0	1	−2	0	1
Z = 2,600		z_j	4	3	1	2	0	0
		$c_j - z_j$	0	0	−1	−2	0	0

Table 4.10
Optimal Solution

Since all $c_j - z_j < 0$ correspond to non-basic variables columns, the current solution cannot be improved further. This means that the current basic feasible solution is also the optimal solution. Thus, the company must manufacture, $x_1 = 200$ belts of type A and $x_2 = 600$ belts of type B in order to obtain the maximum profit of Rs 2,600.

Example 4.3 A pharmaceutical company has 100 kg of A, 180 kg of B and 120 kg of C ingredients available per month. The company can use these materials to make three basic pharmaceutical products namely 5-10-5, 5-5-10 and 20-5-10, where the numbers in each case represent the percentage of weight of A, B and C, respectively, in each of the products. The cost of these raw materials is as follows:

Ingredient	*Cost per kg (Rs)*
A	80
B	20
C	50
Inert ingredients	20

The selling prices of these products are Rs 40.5, Rs 43 and 45 per kg , respectively. There is a capacity restriction of the company for product 5-10-5, because of which the company cannot produce more than 30 kg per month. Determine how much of each of the products the company should produce in order to maximize its monthly profit. *[Delhi Univ., MBA, 2004, AMIE, 2005]*

Solution Let the P_1, P_2 and P_3 be the three products to be manufactured. The data of the problem can then be summarized as follows:

Product	Product Ingredients			Inert
	A	B	C	
P_1	5%	10%	5%	80%
P_2	5%	5%	10%	80%
P_3	20%	5%	10%	65%
Cost per kg (Rs)	80	20	50	20%

Cost of P_1 = 5% × 80 + 10% × 20 + 5% × 50 + 80% × 20 = 4 + 2 + 2.50 + 16 = Rs 24.50 per kg

Cost of P_2 = 5% × 80 + 5% × 20 + 10% × 50 + 80% × 20 = 4 + 1 + 5 + 16 = Rs 26 per kg

Cost of P_3 = 20% × 80 + 5% × 20 + 10% × 50 + 65% × 20 = 16 + 1 + 5 + 13 = Rs 35 per kg

Let x_1, x_2 and x_3 be the quantity (in kg) of P_1, P_2 and P_3, respectively to be manufactured. The LP problem can then be formulated as

Maximize (net profit) Z = (Selling price – Cost price) × (Quantity of product)

$$= (40.50 - 24.50)\,x_1 + (43 - 26)\,x_2 + (45 - 35)\,x_3 = 16x_1 + 17x_2 + 10x_3$$

subject to the constraints

$$\frac{1}{20}x_1 + \frac{1}{20}x_2 + \frac{1}{5}x_3 \le 100 \quad \text{or} \quad x_1 + x_2 + 4x_3 \le 2{,}000$$

$$\frac{1}{10}x_1 + \frac{1}{20}x_2 + \frac{1}{20}x_3 \le 180 \quad \text{or} \quad 2x_1 + x_2 + x_3 \le 3{,}600$$

$$\frac{1}{20}x_1 + \frac{1}{10}x_2 + \frac{1}{10}x_3 \le 120 \quad \text{or} \quad x_1 + 2x_2 + 2x_3 \le 2{,}400$$

$$x_1 \le 30$$

and $\quad x_1, x_2, x_3 \ge 0.$

Standard form Introducing slack variables s_1, s_2 and s_3 to convert the given LP model into its standard form as follows:

Maximize $Z = 16x_1 + 17x_2 + 10x_3 + 0s_1 + 0s_2 + 0s_3 + 0s_4$

subject to the constraints

(i) $x_1 + x_2 + 4x_3 + s_1 = 2{,}000$, (ii) $2x_1 + x_2 + x_3 + s_2 = 3{,}600$

(iii) $x_1 + 2x_2 + 2x_3 + s_3 = 2{,}400$, (iv) $x_1 + s_4 = 30$

and $\quad x_1, x_2, x_3, s_1, s_2, s_3, s_4 \ge 0$

Solution by simplex method An initial basic feasible solution is obtained by setting $x_1 = x_2 = x_3 = 0$. Thus, the initial solution shown in Table 4.11 is: $s_1 = 2{,}000$, $s_2 = 3{,}600$, $s_3 = 2{,}400$, $s_4 = 30$ and Max $Z = 0$.

Table 4.11 Initial Solution

		$c_j \rightarrow$	16	17	10	0	0	0	0	
Profit per Unit c_B	Variables in Basis B	Solution Values $b\ (= x_B)$	x_1	x_2	x_3	s_1	s_2	s_3	s_4	Min Ratio x_B/x_2
0	s_1	2,000	1	1	4	1	0	0	0	2,000/1 = 2,000
0	s_2	3,600	2	1	1	0	1	0	0	3,600/1 = 3,600
0	s_3	2,400	1	2	2	0	0	1	0	2,400/2 = 1,200 →
0	s_4	30	1	0	0	0	0	0	1	—
$Z = 0$		z_j	0	0	0	0	0	0	0	
		$c_j - z_j$	16	17 ↑	10	0	0	0	0	

Since $c_2 - z_2 = 17$ in x_2-column is the largest positive value, we apply the following row operations in order to get a new improved solution by entering variable x_2 into the basis and removing variable s_3 from the basis.

R_3 (new) → R_3 (old) ÷ 2 (*key element*) $\qquad$ R_1 (new) → R_1 (old) – R_3 (new)

R_2 (new) → R_2 (old) – R_3 (new)

The new solution is shown in Table 4.12.

Profit per Unit c_B	Variables in Basis B	Solution Values $b\,(= x_B)$	x_1	x_2	x_3	s_1	s_2	s_3	s_4	Min Ratio x_B/x_1
		$c_j \rightarrow$	16	17	10	0	0	0	0	
0	s_1	800	1/2	0	3	1	0	– 1/2	0	800/(1/2) = 1,600
0	s_2	2,400	3/2	0	0	0	1	– 1/2	0	2,400/(3/2) = 1,600
17	x_2	1,200	1/2	1	1	0	0	1/2	0	1,200/(1/2) = 2,400
0	s_4	30	1	0	0	0	0	0	1	30/1 = 30 →
Z = 20,400		z_j	17/2	17	17	0	0	17/2	0	
		$c_j - z_j$	15/2 ↑	0	– 7	0	0	– 17/2	0	

Table 4.12
Improve Solution

The solution shown in Table 4.12 is not optimal because $c_1 - z_1 > 0$ in x_1-column. Thus, again applying the following row operations to get a new improved solution by entering variable x_1 into the basis and removing the variable s_4 from the basis, we get

R_4 (new) → R_4 (old) ÷ 1 (*key element*); R_1 (new) → R_1 (old) – (1/2) R_4 (new)

R_2 (new) → R_2 (old) – (3/2) R_4 (new); R_3 (new) → R_3 (old) – (1/2) R_4 (new)

The new solution is shown in Table 4.13.

Profit per Unit c_B	Variables in Basis B	Solution Values $b\,(= x_B)$	x_1	x_2	x_3	s_1	s_2	s_3	s_4
		$c_j \rightarrow$	16	17	10	0	0	0	0
0	s_1	785	0	0	3	1	0	– 1/2	– 1/2
0	s_2	2,355	0	0	0	0	1	– 1/2	– 3/2
17	x_2	1,185	0	1	1	0	0	1/2	– 1/2
16	x_1	30	1	0	0	0	0	0	1
Z = 20,625		z_j	16	17	17	0	0	17/2	15/2
		$c_j - z_j$	0	0	–7	0	0	– 17/2	– 15/2

Table 4.13
Optimal Solution

Since all $c_j - z_j < 0$ corresponding to non-basic variables columns, the current solution cannot be improved further. This means that the current solution is also an optimal solution. Thus, the company must manufacture, $x_1 = 30$ kg of P_1, $x_2 = 1{,}185$ kg of P_2 and $x_3 = 0$ kg of P_3 in order to obtain the maximum net profit of Rs 20,625.

4.4 SIMPLEX ALGORITHM (MINIMIZATION CASE)

In certain cases as will be discussed below, it is difficult to obtain an initial basic feasible solution. Such cases arise

(i) when the constraints are of the ≤ type

$$\sum_{j=1}^{n} a_{ij}\, x_j \le b_i, \quad x_j \ge 0$$

but some right-hand side constants are negative [i.e. $b_i < 0$]. In this case after adding the non-negative slack variable s_i $(i = 1, 2, \ldots, m)$, the initial solution so obtained will be $s_i = -\,b_i$ for some i. This is not the feasible solution because it violates the non-negativity conditions of slack variables (i.e. $s_i \ge 0$).

(ii) when the constraints are of the ≥ type

$$\sum_{j=1}^{n} a_{ij}\, x_j \ge b_i, \quad x_j \ge 0$$

In this case to convert the inequalities into the equation form, adding surplus (negative slack) variables,

$$\sum_{j=1}^{n} a_{ij}\, x_j - s_i = b_i, \quad x_j \geq 0,\ s_i \geq 0$$

Letting $x_j = 0$ ($j = 1, 2, \ldots, n$), we get an initial solution $-s_i = b_i$ or $s_i = -b_i$. This is also not a feasible solution as it violates the non-negativity conditions of surplus variables (i.e. $s_i \geq 0$). In this case, we add artificial variables, A_i ($i = 1, 2, \ldots, m$) to get an initial basic feasible solution. The resulting system of equations then becomes:

$$\sum_{j=1}^{n} a_{ij}\, x_j - s_i + A_i = b_i$$

$$x_j,\ s_i,\ A_i \geq 0, \quad i = 1, 2, \ldots, m$$

and has m equations and $(n + m + m)$ variables (i.e. n decision variables, m artificial variables and m surplus variables). An initial basic feasible solution of the new system can be obtained by equating $(n + 2m - m) = (n + m)$ variables equal to zero. Thus the new solution to the given LP problem is: $A_i = b_i$ ($i = 1, 2, \ldots, m$), which does not constitute a solution to the original system of equations because the two systems of equations are not equivalent. Thus to get back to the original problem, artificial variables must be dropped out of the optimal solution. There are two methods for eliminating these variables from the solution.

- *Two-Phase Method*
- *Big-M Method or Method of Penalties*

The simplex method both for the minimization and the maximization LP problem may be summarized in the form of a flow chart. (Fig. 4.1)

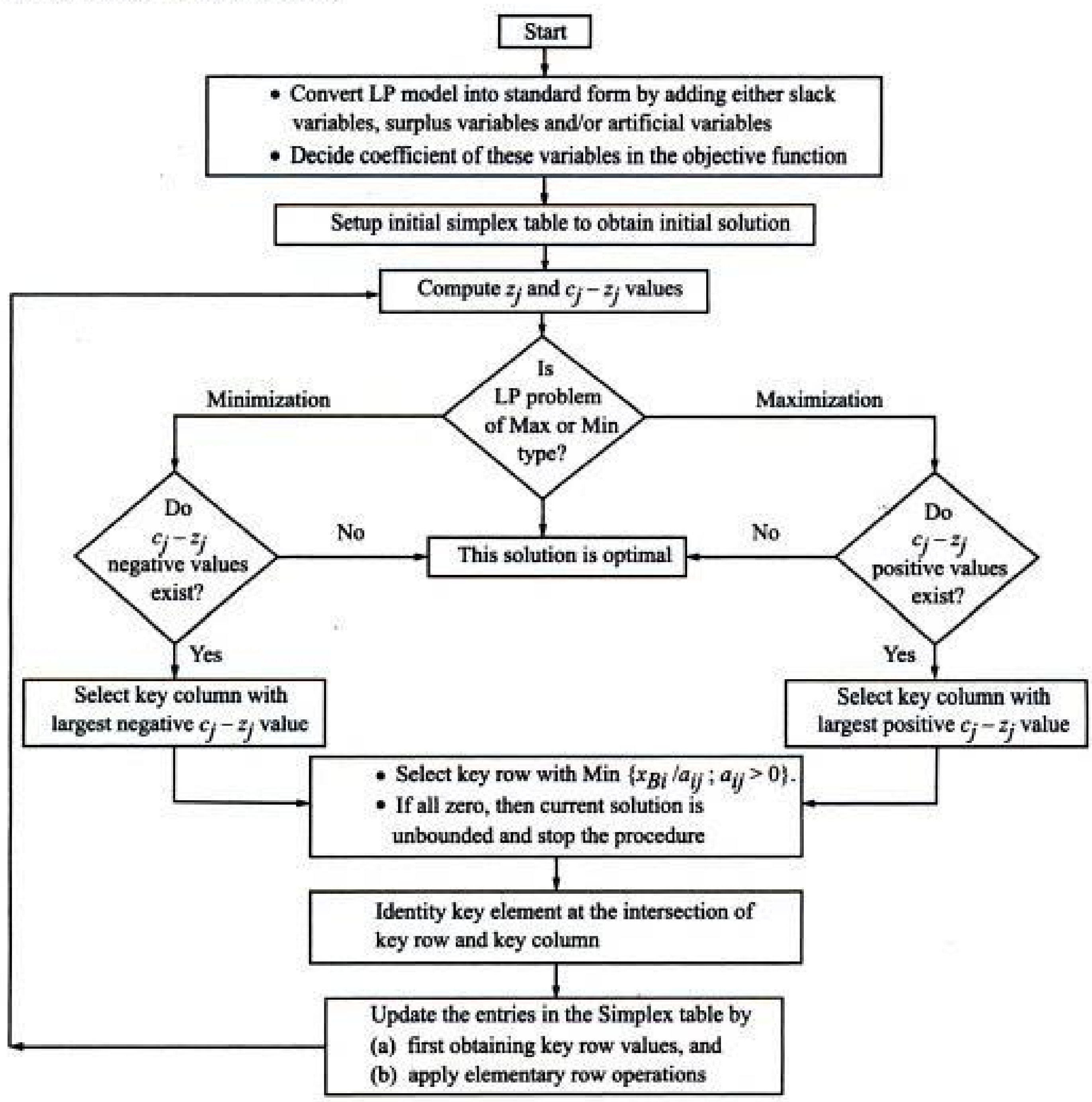


Fig. 4.1 Flow Chart of Simplex Algorithm

Remark Artificial variables have no meaning in a physical sense and are only used as a tool for generating an initial solution to an LP problem. Before the final simplex solution is reached, all artificial variables must be dropped out from the solution mix. This is done by assigning appropriate coefficients to these variables in the objective function. These variables are added to those constraints with equality (=) and greater than or equal to (≥) sign.

4.4.1 Two-Phase Method

In the first phase of this method the sum of the artificial variables is minimized subject to the given constraints in order to get a basic feasible solution of the LP problem. The second phase minimizes the original objective function starting with the basic feasible solution obtained at the end of the first phase. Since the solution of the LP problem is completed in two phases, this is called the *two-phase method.*

An artificial variable is added to the LP constraints to get an initial solution to an LP problem

Advantages of the method

1. No assumptions on the original system of constraints are made, i.e. the system may be redundant, inconsistent or not solvable in non-negative numbers.
2. It is easy to obtain an initial basic feasible solution for Phase I.
3. The basic feasible solution (if it exists) obtained at the end of phase I is used to start Phase II.

Steps of the Algorithm: ***Phase I***

Step 1 (a): If all the constraints in the given LP problem are of the (≤) type, then Phase II can be directly used to solve the problem. Otherwise, the necessary number of surplus and artificial variables are added to convert constraints into equality constraints.

(b) If the given LP problem is of minimization, then convert it to the maximization type by the usual method.

Step 2: Assign zero coefficient to each of the decision variables (x_j) and to the surplus variables; and assign – 1 coefficient to each of the artificial variables. This yields the following auxiliary LP problem.

$$\text{Maximize } Z^* = \sum_{i=1}^{m} (-1)\, A_i$$

subject to the constraints

$$\sum_{i=1}^{n} a_{ij}\, x_j + A_i = b_i, \qquad i = 1, 2, \ldots, m$$

and $\qquad x_j, A_j \geq 0$

Step 3: Apply the simplex algorithm to solve this auxiliary LP problem. The following three cases may arise at optimality.

(a) Max $Z^* = 0$ and at least one artificial variable is present in the basis with positive value. This means that no feasible solution exists for the original LP problem.

(b) Max $Z^* = 0$ and no artificial variable is present in the basis. This means that the basis consists of only decision variables (x_j's) and hence we may move to Phase II to obtain an optimal basic feasible solution on the original LP problem.

(c) Max $Z^* = 0$ and at least one artificial variable is present in the basis at zero value. This means that a feasible solution to the above LP problem is also a feasible solution to the original LP problem. Now in order to arrive at the basic feasible solution we may proceed directly to Phase II or else eliminate the artificial basic variable and then proceed to Phase II.

Once an artificial variable has left the basis, it has served its purpose and can, therefore, be removed from the simplex table. An artificial variable is never considered for re-entry into the basis.

Remark The LP problem defined above is also called an *auxiliary problem*. The value of the objective function in this problem is bounded from below, by zero, because the objective function represents the sum of artificial variables with negative unit coefficients. Thus, the solution to this problem can be obtained in a finite number of steps.

Phase II: Assign actual coefficients to the variables in the objective function and zero coefficient to the artificial variables which appear at zero value in the basis at the end of Phase I. The last simplex table of

Phase I can be used as the initial simplex table for Phase II. Then apply the usual simplex algorithm to the modified simplex table in order to get the optimal solution to the original problem. Artificial variables that do not appear in the basis may be removed.

Example 4.4 Use two-phase simplex method to solve the following LP problem.

Minimize $Z = x_1 + x_2$

subject to the constraints

(i) $2x_1 + x_2 \geq 4$, (ii) $x_1 + 7x_2 \geq 7$

and $x_1, x_2 \geq 0$

Solution Converting the given LP problem objective function into the maximization form and then adding surplus variables s_1 and s_2 and artificial variables A_1 and A_2, in the constraints. The problem becomes:

Maximize $Z^* = -x_1 - x_2$

subject to the constraints

(i) $2x_1 + x_2 - s_1 + A_1 = 4$, (ii) $x_1 + 7x_2 - s_2 + A_2 = 7$

and $x_1, x_2, s_1, s_2, A_1, A_2 \geq 0$

where $Z^* = -Z$

Phase I: This phase starts by considering the following auxiliary LP problem:

Maximize $Z^* = -A_1 - A_2$

subject to the constraints

(i) $2x_1 + x_2 - s_1 + A_1 = 4$, (ii) $x_1 + 7x_2 - s_2 + A_2 = 7$

and $x_1, x_2, s_1, s_2, A_1, A_2 \geq 0$

The initial solution is presented in Table 4.14.

		$c_j \rightarrow$	0	0	0	0	−1	−1
c_B	Variables in Basis B	Solution Values $b(= x_B)$	x_1	x_2	s_1	s_2	A_1	A_2
−1	A_1	4	2	1	−1	0	1	0
−1	A_2	7	1	7	0	−1	0	1 →
$Z^* = -11$		z_j	−3	−8	1	1	−1	−1
		$c_j - z_j$	3	8 ↑	−1	−1	0	0

Table 4.14 Initial Solution

Artificial variables A_1 and A_2 are now removed, one after the other, maintaining the feasibility of the solution.

Iteration 1: Applying the following row operations to get an improved solution by entering variable x_2 in the basis and first removing variable A_2 from the basis gives us the results shown in Table 4.15. Note that the variable x_1 cannot be entered into the basis as this would lead to an infeasible solution.

R_2 (new) $\rightarrow R_2$ (old) ÷ 7 (*key element*); R_1 (new) $\rightarrow R_1$ (old) − R_2 (new)

		$c_j \rightarrow$	0	0	0	0	−1	−1
c_B	Variables in Basis B	Solution Values $b(= x_B)$	x_1	x_2	s_1	s_2	A_1	A_2*
−1	A_1	3	13/7	0	−1	1/7	1	−1/7 →
0	x_2	1	1/7	1	0	−1/7	0	1/7
$Z^* = -3$		z_j	−13/7	0	1	−1/7	−1	1/7
		$c_j - z_j$	13/7	0	−1	1/7 ↑	0	−8/7

Table 4.15 Improved Solution

* This column can permanently be removed at this stage.

Iteration 2: To remove A_1 from the solution shown in Table 4.15, we enter variable s_2 in the basis by applying the row operations shown in Table 4.16. Here it may be noted that if variable x_1 is chosen as the value to be entered into the basis, it will lead to an infeasible solution

R_1 (new) $\rightarrow R_1$ (old) $\times$ 7; $\qquad R_2$ (new) $\rightarrow R_2$ (old) + (1/7) R_1 (new)

		$c_j \rightarrow$	*0*	*0*	*0*	*0*	*– 1*	*– 1*
c_B	*Variables in Basis* B	*Solution Values* $b(=x_B)$	x_1	x_2	s_1	s_2	A_1^*	A_2^*
0	s_2	21	13	0	– 7	1	7	– 1
0	x_2	4	2	1	– 1	0	1	0
$Z^* = 0$		z_j	0	0	0	0	0	0
		$c_j - z_j$	0	0	0	0	– 1	– 1

Table 4.16
Improved Solution

* Remove columns A_1 and A_2 from Table 4.16.

Since all $c_j - z_j \leq 0$ correspond to non-basic variables, the optimal solution: $x_1 = 0$, $x_2 = 4$, $s_1 = 0$, $s_2 = 21$, $A_1 = 0$, $A_2 = 0$ with $Z^* = 0$ is arrived at. However, this solution may or may not be the basic feasible solution to the original LP problem. Thus, we have to move to phase II to get an optimal solution to our original LP problem.

Phase II: The modified simplex table obtained from Table 4.16 is represented in Table 4.17.

		$c_j \rightarrow$	*– 1*	*– 1*	*0*	*0*	
c_B	*Variables in Basis* B	*Solution Values* $b(=x_B)$	x_1	x_2	s_1	s_2	*Min Ratio* x_B/x_1
0	s_2	21	13	0	– 7	1	21/13 $\rightarrow$
– 1	x_2	4	2	1	– 1	0	4/2
$Z^* = -4$		z_j	– 2	– 1	1	0	
		$c_j - z_j$	1	0	– 1	0	
			$\uparrow$				

Table 4.17

Iteration 1: Introducing variable x_1 into the basis and removing variable s_2 from the basis by applying the following row operations gives as

R_1 (new) $\rightarrow R_1$ (old) $\div$ 13 (*key element*); $\qquad R_2$ (new) $\rightarrow R_2$ (old) – $2R_1$ (new)

The improved basic feasible solution so obtained is given in Table 4.18. Since in Table 4.18, $c_j - z_j \leq 0$ for all non-basic variables, the current solution is optimal. Thus, the optimal basic feasible solution to the given LP problem is: $x_1 = 21/13$, $x_2 = 10/13$ and Max $Z^* = -31/13$ or Min $Z = 31/13$.

		$c_j \rightarrow$	*– 1*	*– 1*	*0*	*0*
c_B	*Variables in Basis* B	*Solution Values* $b(=x_B)$	x_1	x_2	s_1	s_2
– 1	x_1	21/13	1	0	– 7/13	1/13
– 1	x_2	10/13	0	1	1/13	– 2/13
$Z^* = -31/13$		z_j	– 1	– 1	6/13	1/13
		$c_j - z_j$	0	0	– 6/13	– 1/13

Table 4.18
Optimal Solution

Example 4.5 Solve the following LP problem by using the two-phase simplex method.

Minimize $Z = x_1 - 2x_2 - 3x_3$

subject to the constraints

(i) $-2x_1 + x_2 + 3x_3 = 2$, $\qquad$ (ii) $2x_1 + 3x_2 + 4x_3 = 1$

and $\qquad x_1, x_2, x_3 \geq 0$. $\qquad$ [*Meerut, MSc (Math.), 1996*]

Solution After converting the objective function into the maximization form and by adding artificial variables A_1 and A_2 in the constraints of the given LP problem, the problem becomes:

Maximize $Z^* = -x_1 + 2x_2 + 3x_3$

subject to the constraints

(i) $-2x_1 + x_2 + 3x_3 + A_1 = 2$, (ii) $2x_1 + 3x_2 + 4x_3 + A_2 = 1$

and $x_1, x_2, x_3, A_1, A_2 \geq 0$

where $Z^* = -Z$

Phase I: This phase starts by considering the following auxiliary LP problem:

Maximize $Z^* = -A_1 - A_2$

subject to the constraints

(i) $-2x_1 + x_2 + 3x_3 + A_1 = 2$ (ii) $2x_1 + 3x_2 + 4x_3 + A_2 = 1$

and $x_1, x_2, x_3, A_1, A_2 \geq 0$

The initial solution is presented in Table 4.19.

Table 4.19 Initial Solution

		$c_j \rightarrow$	0	0	0	−1	−1
c_B	*Variables in Basis* B	*Solution Values* $b (= x_B)$	x_1	x_2	x_3	A_1	A_2
−1	A_1	2	−2	1	3	1	0
−1	A_2	1	2	3	4	0	1 →
$Z^* = -3$		z_j	0	−4	−7	−1	−1
		$c_j - z_j$	0	4	7 ↑	0	0

To first remove the artificial variable A_2 from the solution shown in Table 4.19, introduce variable x_3 into the basis by applying the following row operations:

R_1 (new) $\rightarrow R_2$ (old) ÷ 4 (*key element*) ; R_1 (new) $\rightarrow R_1$ (old) − $3R_2$ (new)

The improved solution so obtained is given in Table 4.20. Since in Table 4.20, $c_j - z_j \leq 0$ corresponds to non-basic variables, the optimal solution is: $x_1 = 0$, $x_2 = 0$, $x_3 = 1/4$, $A_1 = 5/4$ and $A_2 = 0$ with Max $Z^* = -5/4$. But at the same time, the value of $Z^* < 0$ and the artificial variable A_1 appears in the basis with positive value 5/4. Hence, the given original LP problem does not possess any feasible solution.

Table 4.20 Optimal but not Feasible Solution

		$c_j \rightarrow$	0	0	0	−1
c_B	*Variables in Basis* B	*Solution Values* $b (= x_B)$	x_1	x_2	x_3	A_1
−1	A_1	5/4	−7/2	−5/4	0	1
0	x_3	1/4	1/2	3/4	1	0
$Z^* = -5/4$		z_j	7/2	5/4	0	−1
		$c_j - z_j$	−7/2	−5/4	0	0

Example 4.6 Use two-phase simplex method to solve following LP problem

Maximize $Z = 3x_1 + 2x_2 + 2x_3$

subject to the constraints

(i) $5x_1 + 7x_2 + 4x_3 \leq 7$, (ii) $-4x_1 + 7x_2 + 5x_3 \geq -2$, (iii) $3x_1 + 4x_2 - 6x_3 \geq 29/7$

and $x_1, x_2, x_3 \geq 0$. *[Punjab Univ., BE (E & C) 2006 ; BE (IT) 2004]*

Solution Since RHS of constraint 2 is negative, multiplying it by − 1 on both sides and expressed it as: $4x_1 - 7x_2 - 5x_3 \leq 2$.

Phase I: Introducing slack, surplus and artificial variables in the constraints, the standard form of LP problem becomes:

image
not
available

	$c_j \rightarrow$		3	2	2	0	0	0	
c_B	Variables in Basis B	Solution Values $b\,(= x_B)$	x_1	x_2	x_3	s_1	s_2	s_3	Min Ratio x_B/x_3
2	x_2	2/7	0	1	42	3	0	5	1/147
0	s_2	0	0	0	521	37	1	63	0 →
3	x_1	1	1	0	– 58	– 6	0	– 7	—
$Z = 25/7$		z_j	3	2	– 90	– 6	0	– 11	
		$c_j - z_j$	0	0	92	6	0	11	

Table 4.24 Initial Solution

In Table 4.24, $c_3 - z_3 = 92$ is the largest positive value, replacing basic variable s_2 with non-basic variable x_3 into the basis. For this apply necessary row operations as usual. The new solution is shown in Table 4.25.

	$c_j \rightarrow$		3	2	2	0	0	0
c_B	Variables in Basis B	Solution Values $b\,(= x_B)$	x_1	x_2	x_3	s_1	s_2	s_3
2	x_2	2/7	0	1	0	9/521	– 42/521	– 41/521
2	x_3	0	0	0	1	37/521	1/521	63/521
3	x_1	1	1	0	0	62/521	58/521	7/521
$Z = 25/7$		z_j	3	2	2	278/521	92/521	65/521
		$c_j - z_j$	0	0	0	–278/521	– 92/521	– 65/521

Table 4.25 Optimal Solution

Since all $c_j - z_j \le 0$ in Table 4.25, the current solution is the optimal basic feasible solution: $x_1 = 1, x_2 = 2/7$, $x_3 = 0$ and Max Z = 25/7.

4.4.2 The Big-M Method

The Big-M method is another method of removing artificial variables from the basis. In this method, we assign large undesirable (unacceptable penalty) coefficients to artificial variables from the point of view of the objective function. If the objective function Z is to be minimized, then a very large positive price (called *penalty*) is assigned to each artificial variable. Similarly, if Z is to be maximized, then a very large negative price (also called *penalty*) is assigned to each of these variables. The penalty is supposed to be designated by $-M$ for a maximization problem and $+M$ for a minimization problem, where $M > 0$. The Big-M method for solving an LP problem can be summarized in the following steps.

Step 1: Express the LP problem in the standard form by adding slack variables, surplus variables and artificial variables. Assign a zero coefficient to both slack and surplus variables. Then add a very large positive coefficient $+M$ (minimization case) and $-M$ (maximization case) to artificial variable in the objective function.

Step 2: The initial basic feasible solution is obtained by assigning zero value to original variables.

Step 3: Calculate the values of $c_j - z_j$ in last row of the simplex table and examine these values.

(i) If all $c_j - z_j \ge 0$, then the current basic feasible solution is optimal.

(ii) If for a column, k, $c_k - z_k$ is most negative and all entries in this column are negative, then the problem has an unbounded optimal solution.

(iii) If one or more $c_j - z_j < 0$ (minimization case), then select the variable to enter into the basis with the largest negative $c_j - z_j$ value (largest per unit reduction in the objective function value). This value also represents the opportunity cost of not having one unit of the variable in the solution. That is,

$$c_k - z_k = \text{Min } \{c_j - z_j : c_j - z_j < 0\}$$

Step 4: Determine the key row and key element in the same manner as discussed in the simplex algorithm of the maximization case.

Step 5: Continue with the procedure to update solution at each iteration till optimal solution is obtained.

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Iteration 2: Since the value of $c_2 - z_2$ in Table 4.28 is the largest negative value, variable x_2 is chosen to enter into the basis. For introducing variable x_2 into the basis and to remove s_1 from the basis we apply the following row operations.

R_1 (new) $\rightarrow R_1$ (old) × 5/16 (*key element*); R_2 (new) $\rightarrow R_2$ (old) – (6/5) R_1 (new).

R_3 (new) $\rightarrow R_3$ (old) – 2/5 R_1 (new).

The new solution is shown in Table 4.28.

Table 4.28 Improved Solution

		$c_j \rightarrow$	5	3	0	0	M	
Cost per Unit c_B	Variables in Basis B	Solution Values $b(= x_B)$	x_1	x_2	s_1	s_2	A_1	Min Ratio x_B/s_2
3	x_2	5/2	0	1	5/16	1/8	0	(5/2)/(1/8) = 40
M	A_1	3	0	0	– 3/8	1/4	1	3/(1/4) = 12 →
5	x_1	1	1	0	– 1/8	– 1/4	0	
$Z = 25/2 + 3M$		z_j	5	3	– 3M/8 + 5/16	$M/4 - 7/8$	M	
		$c_j - z_j$	0	0	3M/8 – 5/16	$-M/4 + 7/8$ ↑	0	

Iteration 3: As $c_4 - z_4 < 0$ in s_2-column, the current solution is not optimal. Thus, introduce s_2 into the basis and remove A_1 from the basis by applying the following row operations:

R_2 (new) $\rightarrow R_2$ (old) × 4 (*key element*); R_1 (new) $\rightarrow R_1$ (old) – (1/8) R_2 (new)

R_3 (new) $\rightarrow R_3$ (old) + (1/4) R_2 (new).

The new solution is shown in Table 4.29.

Table 4.29 Optimal Solution

		$c_j \rightarrow$	5	3	0	0
Cost per Unit c_B	Variables in Basis B	Solution Values $b(= x_B)$	x_1	x_2	s_1	s_2
3	x_2	1	0	1	1/2	0
0	s_2	12	0	0	– 3/2	1
5	x_1	4	1	0	– 1/2	0
$Z = 23$		z_j	5	3	– 1	0
		$c_j - z_j$	0	0	1	0

In Table 4.24, all $c_j - z_j \geq 0$. Thus an optimal solution is arrived at with the value of variables as: $x_1 = 4$, $x_2 = 1$, $s_1 = 0$, $s_2 = 12$ and Min $Z = 23$.

Example 4.8 Use penalty (Big-M) method to solve the following LP problem.

Maximize $Z = x_1 + 2x_2 + 3x_3 - x_4$

subject to the constraints

(i) $x_1 + 2x_2 + 3x_3 = 15$, (ii) $2x_1 + x_2 + 5x_3 = 20$, (iii) $x_1 + 2x_2 + x_3 + x_4 = 10$

and $x_1, x_2, x_3, x_4 \geq 0$ *[Calicut, BTech. (Engg), 1995; Bangalore BE (Mech.), 1995; AMIE, 2004]*

Solution Since all constraints of the given LP problem are equations, therefore we should add only artificial variables A_1 and A_2 in the constraints. The standard form of the problem is then stated as follows:

Maximize $Z = x_1 + 2x_2 + 3x_3 - x_4 - MA_1 - MA_2$

subject to the constraints

(i) $x_1 + 2x_2 + 3x_3 + A_1 = 15$, (ii) $2x_1 + x_2 + 5x_3 + A_2 = 20$

(iii) $x_1 + 2x_2 + x_3 + x_4 = 10$

and $x_1, x_2, x_3, x_4, A_1, A_2 \geq 0$

An initial basic feasible solution is given in Table 4.30.

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As $c_2 - z_2$ value in x_2-column of Table 4.34 is the largest negative, therefore variable x_2 should be entered to replace basic variable A_2 into the basis. For this, apply following row operations

$$R_2 \text{ (new)} = R_2 \text{ (old)} \div 2 \text{ (key element)}; \qquad R_1 \text{ (new)} \to R_1 \text{ (old)} - R_2 \text{ (new)}$$

to get the new solution as shown in Table 4.35.

	$c_j \to$		600	500	0	0	M	
Cost per Unit c_B	Variables in Basis B	Solution Values $b(= x_B)$	x_1	x_2	s_1	s_2	A_1	Min Ratio x_B/x_1
M	A_1	50	3/2	0	– 1	1/2	1	100/3 →
500	x_2	30	1/2	1	0	– 1/2	0	60
Z = 15,000 + 50M	z_j		3M/2 + 250	500	– M	M/2 – 250	M	
	$c_j - z_j$		350 – 3M/2	0	M	250 – M/2	0	
			↑					

Table 4.35
Improved Solution

As $c_1 - z_1$ value in x_1-column of Table 4.35 is the largest negative, enter variable x_1 to replace basic variable A_1 into the basis. For this, apply row operations

$$R_1 \text{ (new)} \to R_1 \text{ (old)} \times (2/3) \text{ (key element)}; \qquad R_2 \text{ (new)} \to R_2 \text{ (old)} - (1/2)\, R_1 \text{ (new)}$$

to get the new solution as shown in Table 4.36.

	$c_j \to$		600	500	0	0
Cost per Unit c_B	Variables in Basis B	Solution Values $b(= x_B)$	x_1	x_2	s_1	s_2
600	x_1	100/3	1	0	– 2/3	1/3
500	x_2	40/3	0	1	1/3	– 2/3
Z = 80,000/3	z_j		600	500	– 700/3	– 400/3
	$c_j - z_j$		0	0	700/3	400/3

Table 4.36
Optimal Solution

In Table 4.36, all $c_j - z_j \geq 0$ and also both artificial variables have been reduced to zero. An optimum solution has been arrived at with x_1 = 100/3 batches of hardcover books, x_2 = 40/3 batches of paperback books, at a total minimum cost, Z = Rs. 80,000/3.

Example 4.10 An advertising agency wishes to reach two types of audiences: Customers with annual income greater than Rs 15,000 (target audience A) and customers with annual income less than Rs 15,000 (target audience B). The total advertising budget is Rs 2,00,000. One programme of TV advertising costs Rs 50,000; one programme on radio advertising costs Rs 20,000. For contract reasons, at least three programmes ought to be on TV, and the number of radio programmes must be limited to five. Surveys indicate that a single TV programme reaches 4,50,000 customers in target audience A and 50,000 in target audience B. One radio programme reaches 20,000 in target audience A and 80,000 in target audience *B*. Determine the media mix to maximize the total reach. [*Delhi Univ., MBA, 1995*]

Solution Let x_1 and x_2 be the number of insertions in TV and radio, respectively. The LP problem can the be formulated as follows:

$$\begin{aligned} \text{Maximize (total reach) } Z &= (4{,}50{,}000 + 50{,}000)\, x_1 + (20{,}000 + 80{,}000)\, x_2 \\ &= 5{,}00{,}000\, x_1 + 1{,}00{,}000\, x_2 = 5x_1 + x_2 \end{aligned}$$

subject to the constraints

(i) $50{,}000\, x_1 + 20{,}000\, x_2 \leq 2{,}00{,}000$ or $5x_1 + 2x_2 \leq 20$ (Advt. budget)

(ii) $x_1 \geq 3$ (Advt. on TV) (iii) $x_2 \leq 5$ (Advt. on Radio)

and $x_1, x_2 \geq 0$.

Standard form After introducing slack/surplus and/or artificial variables in the inequalities of the constraints, the LP problem in standard form becomes:

$$\text{Maximize } Z = 5x_1 + x_2 + 0s_1 + 0s_2 + 0s_3 - MA_1$$

subject to the constraints

(i) $5x_1 + 2x_2 + s_1 = 20$, (ii) $x_1 - s_2 + A_1 = 3$, (iii) $x_2 + s_3 = 5$

and $x_1, x_2, s_1, s_2, s_3, A_1 \geq 0$

Solution by simplex method An initial basic feasible solution is obtained by setting $x_1 = x_2 = s_2 = 0$. Thus, $s_1 = 20$, $A_1 = 3$, $s_3 = 5$ and Max $Z = -3M$. This initial solution is shown in simplex Table 4.37.

Table 4.37 Initial Solution

		$c_j \rightarrow$	5	1	0	0	0	$-M$	
Profit per Unit c_B	Variables in Basis B	Solution Values $b(=x_B)$	x_1	x_2	s_1	s_2	s_3	A_1	Min Ratio x_B/x_1
0	s_1	20	5	2	1	0	0	0	20/5 = 4
$-M$	A_2	3	1	0	0	-1	0	1	3/1 = 3 →
0	s_3	5	0	1	0	0	1	0	—
$Z = -3M$		z_j	$-M$	0	0	M	0	$-M$	
		$c_j - z_j$	$M+5$ ↑	1	0	$-M$	0	0	

As $c_1 - z_1$ value in x_1-column of Table 4.37 is the largest positive value, enter variable x_1 to replace basic variable A_1 into the basis. For this apply following row operations

R_2 (new) → R_2 (old) ÷ 1 (*key element*); R_1 (new) → R_1 (old) – $5R_2$ (new)

to get the new solution as shown in Table 4.38.

Table 4.38 Improved Solution

		$c_j \rightarrow$	5	1	0	0	0	
Profit per Unit c_B	Variable in Basis B	Solution Values $b(=x_B)$	x_1	x_2	s_1	s_2	s_3	Min Ratio x_B/s_2
0	s_1	5	0	2	1	5	0	5/5 = 1 →
5	x_1	3	1	0	0	-1	0	—
0	s_3	5	0	1	0	0	1	—
$Z = 15$		z_j	5	0	0	-5	0	
		$c_j - z_j$	0	1	0	5 ↑	0	

The solution shown in Table 4.38 is not optimal as $c_4 - z_4$ value in s_2-column is the largest positive. Thus, enter variable s_2 to replace basic variable s_1 into the basis. For this apply the following row operations:

R_1 (new) → R_1 (old) ÷ 5 (*key element*); R_2 (new) → R_2 (old) + R_1 (new)

to get the new solution, shown in Table 4.39.

Table 4.39 Optimal Solution

		$c_j \rightarrow$	5	1	0	0	0
Profit per Unit c_B	Variables in Basis B	Solution Values $b(=x_B)$	x_1	x_2	s_1	s_2	s_3
0	s_2	1	0	2/5	1/5	1	0
5	x_1	4	1	2/5	1/5	0	0
0	s_3	5	0	1	0	0	1
$Z = 20$		z_j	5	2	1	0	0
		$c_j - z_j$	0	-1	-1	0	0

Since all $c_j - z_j \geq 0$ in Table 4.39, the total reach of target audience cannot be increased further. Hence, the optimal solution is: $x_1 = 4$ insertions in TV and $x_2 = 0$ in radio with Max (total audience) $Z = 20{,}00{,}000$.

Example 4.11 An Air Force is experimenting with three types of bombs P, Q and R in which three kinds of explosives, viz., A, B and C will be used. Taking the various factors into account, it has been decided to use the maximum 600 kg of explosive A, at least 480 kg of explosive B and exactly 540 kg of explosive C. Bomb P requires 3, 2, 2 kg, bomb Q requires 1, 4, 3 kg and bomb R requires 4, 2, 3 kg of explosives A, B and C respectively. Bomb P is estimated to give the equivalent of a 2 ton explosion, bomb Q, a 3 ton explosion and bomb R, a 4 ton explosion respectively. Under what production schedule can the Air Force make the biggest bang?

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Table 4.42 Optimal Solution

		$c_j \rightarrow$	2	3	4	0	0
Profit per Unit c_B	Variables in Basis B	Solution Values $b (= x_B)$	x_1	x_2	x_3	s_1	s_2
0	s_1	60	4/3	0	0	1	–3/2
3	x_2	60	1/3	1	0	0	–1/2
4	x_3	120	1/3	0	1	0	1/2
$Z = 660$		z_j	7/3	3	4	0	1/2
		$c_j - z_j$	–1/3	0	0	0	–1/2

In Table 4.42, all $c_j - z_j \le 0$ and artificial variables A_1 and A_2 have been reduced to zero. Thus, an optimal solution has been arrived at with $x_1 = 0$ bombs of type P, $x_2 = 60$ bombs of type Q, $x_3 = 120$ bombs of type R, at largest benefit of Z = 660.

SELF PRACTICE PROBLEMS A

1. A television company has three major departments for manufacturing two of its models – A and B. The monthly capacities of the departments are given as follows:

	Per Unit Time Requirement (hours)		Hours Available this Month
	Model A	Model B	
Department *I*	4.0	2.0	1,600
Department *II*	2.5	1.0	1,200
Department *III*	4.5	1.5	1,600

The marginal profit per unit from model A is Rs 400 and from model B is Rs 100. Assuming that the company can sell any quantity of either product due to favourable market conditions, determine the optimum output for both the models, the highest possible profit for this month and the slack time in the three departments.

2. A manufacturer of leather belts makes three types of belts A, B and C which are processed on three machines M_1, M_2 and M_3. Belt A requires 2 hours on machine M_1 and 3 hours on machine M_2 and 2 hours on machine M_3. Belt B requires 3 hours on machine M_1, 2 hours on machine M_2 and 2 hours on machine M_3 and Belt C requires 5 hours on machine M_2 and 4 hours on machine M_3. There are 8 hours of time per day available on machine M_1, 10 hours of time per day available on machine M_2 and 15 hours of time per day available on machine M_3. The profit gained from belt A is Rs 3.00 per unit, from Belt B is Rs 5.00 per unit, from belt C is Rs 4.00 per unit. What should be the daily production of each type of belt so that the products yield the maximum profit?

3. A company produces three products A, B and C. These products require three ores O_1, O_2 and O_3. The maximum quantities of the ores O_1, O_2 and O_3 available are 22 tonnes, 14 tonnes and 14 tonnes, respectively. For one tonne of each of these products, the ore requirements are:

	A	B	C
O_1	3	–	3
O_2	1	2	3
O_3	3	2	3
Profit per tonne (Rs in thousand)	1	4	5

The company makes a profit of Rs 1,000, 4,000 and 5,000 on each tonne of the products A, B and C, respectively. How many tonnes of each product should the company produce in order to maximize its profits.

4. A manufacturing firm has discontinued the production of a certain unprofitable product line. This has created considerable excess production capacity. Management is considering to devoting this excess capacity to one or more of three products; call them product 1, 2 and 3. The available capacity on the machines that might limit output is summarized in the following table:

Machine Type	Available Time (in Machine-hours per Week)
Milling Machine	250
Lathe	150
Grinder	50

The number of machine-hours required for each unit of the respective product is as follows:

Machine Type	Productivity (in Machine-hours per Unit)		
	Product 1	Product 2	Product 3
Milling Machine	8	2	3
Lathe	4	3	0
Grinder	2	–	1

The profit per unit would be Rs 20, Rs 6 and Rs 8, respectively for product 1, 2 and 3. Find how much of each product the firm should produce in order to maximize its profit.

[*Delhi Univ., MBA 2003*]

5. A farmer has 1,000 acres of land on which he can grow corn, wheat or soyabean. Each acre of corn costs Rs 100 for preparation, requires 7 men-days of work and yields a profit of Rs 30. An acre of wheat costs Rs 120 to prepare, requires 10 men-days of work and yields a profit of Rs 40. An acre of soyabean costs Rs 70 to prepare, requires 8 men-days of work and yields a profit of Rs 20. If the farmer has Rs 1,00,000 for preparation and can count on 8,000 men-days of work, determine how many acres should be allocated to each crop in order to maximize profits?

6. The annual handmade furniture show and sale is supposed to take place next month and the school of vocational studies is also planning to make furniture for this sale. There are three wood-working classes–I year, II year and III year, at the school and they have decided to make styles of chairs – *A*, *B* and *C*. Each chair must receive work in each class. The time in hours required for each chair in each class is:

Chair	*I Year*	*II Year*	*III Year*
A	2	4	3
B	3	3	2
C	2	1	4

During the next month there will be 120 hours available to the I year class, 160 hours to the II year class, and 100 hours to the III year class for producing the chairs. The teacher of the wood-working classes feels that a maximum of 40 chairs can be sold at the show. The teacher has determined that the profit from each type of chair will be: A, Rs 40; B, Rs 35 and C, Rs. 30. How many chairs of each type should be made in order to maximize profits at the show and sale?

7. Mr Jain, the marketing manager of ABC Typewriter Company is trying to decide how he should allocate his salesmen to the company's three primary markets. Market I is in the urban area and the salesman can sell, on the average, 40 typewriters a week. Salesmen in the other two markets, II and III can sell, on the average, 36 and 25 typewriters per week, respectively. For the coming week, three of the salesmen will be on vacation, leaving only 12 men available for duty. Also because of lack of company care, a maximum of 5 salesmen can be allocated to market area I. The selling expenses per week for salesmen in each area are Rs 800 per week for area I, Rs 700 per week for area II, and Rs 500 per week for area III. The budget for the next week is Rs 7,500. The profit margin per typewriter is Rs 150. Determine how many salesmen should be assigned to each area in order to maximize profits?

8. Three products – A, B and C – are produced in three machine centres X, Y and Z. All three products require a part of their manufacturing operation at each of the machine centres. The time required for each operation on various products is indicated in the following table. Only 100, 77 and 80 hours are available at machine centres X, Y and *Z*, respectively. The profit per unit from A, B and C is Rs 12, Rs 3 and Re 1, respectively.

Products	*Machine Centres*			*Profit per Unit (Rs)*
	X	*Y*	*Z*	
A	10	7	2	12
B	2	3	4	3
C	1	2	1	1
Available hours	100	77	80	

(a) Determine suitable product mix so as to maximize the profit. Comment on the queries (b) and (c) from the solution table obtained.

(b) Satisfy that full available hours of X and Y have been utilized and there is surplus hours of Z. Find out the surplus hours of Z.

(c) Your aim is to utilize surplus capacity of Z. Can you say from the table that the introduction of more units of Y is required?

9. A certain manufacturer of screw fastenings found that there is a market for packages of mixed screw sizes. His market research data indicated that two mixtures of three screw types (1, 2 and 3), properly priced, could be most acceptable to the public. The relevant data is:

Mixture	*Specifications*	*Selling Price (Rs/kg)*
A	≥ 50% type 1 ≤ 30% type 2 and quantity of type 3	5
B	≥ 35% type 1 ≤ 45% type 2 and quantity of type 3	4

For these screws, the plant capacity and manufacturing cost are as follows:

Screw Type	*Plant Capacity (kg/day × 100)*	*Manufacturing Cost (Rs/kg)*
1	10	4.50
2	10	3.50
3	6	2.70

What production shall this manufacturer schedule for greatest profit, assuming that he can sell all that he manufactures?

10. A blender of whisky imports three grades A, B and C. He mixes them according to the recipes that specify the maximum or minimum percentages of grades A and C in each blend. These are shown in the table below:

Blend	*Specification*	*Price per Unit (Rs)*
Blue Dot	Not less than 60% of A Not more than 20% of C	6.80
Highland Fling	Not more than 60% of C Not less than 15% of A	5.70
Old Frenzy	Not more than 50% of C	4.50

Following are the supplies of the three whiskies along with their cost.

Whisky	*Maximum Quantity Available per Day*	*Cost per Unit (Rs)*
A	2,000	7.00
B	2,500	5.00
C	1,200	4.00

Show how to obtain the first matrix in a simplex computation of a production policy that will maximize profits.

11. An animal feed company must produce on a daily basis 200 kg of a mixture that consists ingredients x_1 and x_2 ingredient. x_1 costs Rs 3 per kg and x_2 costs Rs 8 per kg. Not more than 80 kg of x_1 can be used and at least 60 kg of x_2 must be used. Find out how much of each ingredient should be used if the company wants to minimize costs.

12. A diet is to contain at least 20 ounces of protein and 15 ounces of carbohydrate. There are three foods A, B and C available in the market, costing Rs 2, Re 1 and Rs 3 per unit, respectively. Each unit of A contains 2 ounces of protein and 4 ounces of carbohydrate; each unit of B contains 3 ounces of protein and 2 ounces of carbohydrate; and each unit of C contains 4 ounces of protein and 2 ounces of carbohydrate. How many units of each food should the diet contain so that the cost per unit diet is minimum?

13. A person requires 10, 12 and 12 units of chemicals A, B and C, respectively for his garden. A typical liquid product contains 5, 2 and 1 unit of A, B and C, respectively per jar. On the other hand a typical dry product contains 1, 2 and 4 units of A, B and C per unit. If the liquid product sells for Rs 3 per jar and the dry product for Rs 2 per carton, how many of each should be purchased in order to minimize the cost and meet the requirement?

14. A scrap metal dealer has received an order from a customer for a minimum of 2,000 kg of scrap metal. The customer requires that at least 1,000 kg of the shipment of metal be of high quality copper that can be melted down and used to produce copper tubing. Furthermore, the customer will not accept delivery of the order if it contains more than 175 kg metal that he deems unfit for commercial use, i.e. metal that contains an excessive amount of impurities and cannot be melted down and refined profitably.

The dealer can purchase scrap metal from two different suppliers in unlimited quantities with following percentages (by weight) of high quality copper and unfit scrap.

	Supplier A	*Supplier B*
Copper	25%	75%
Unfit scrap	5%	10%

The cost per kg of metal purchased from supplier A and B are Re 1 and Rs 4, respectively. Determine the optimal quantities of metal to be purchased for the dealer from each of the two suppliers.

15. A marketing manager wishes to allocate his annual advertising budget of Rs 2,00,000 to two media vehicles – A and B. The unit cost of a message in media A is Rs 1,000 and that of B is Rs 1,500. Media A is a monthly magazine and not more than one insertion is desired in one issue, whereas at least five messages should appear in media B. The expected audience for unit messages in media A is 40,000 and that of media B is 55,000. Develop an LP model and solve it for maximizing the total effective audience.

16. An advertising agency wishes to reach two types of audiences: Customers with annual income of greater than Rs 15,000 (target audience A) and customers with annual income of less than Rs 15,000 (target audience B). The total advertising budget is Rs 2,00,000. One programme of TV advertising costs Rs 50,000; one programme of radio advertising costs Rs 20,000. For contract reasons, at least three programmes ought to be on TV and the number of radio programmes must be limited to five. A survey indicates that a single TV programme reaches 4,50,000 customers in target audience A and 50,000 in audience B; and that one radio programme reaches 20,000 in target audience A and 80,000 in target audience B. Determine the media mix to maximize the total reach.

17. A transistor radio company manufactures four models A, B, C and D. Models A, B and C, have profit contributions of Rs 8, Rs 15 and Rs 25 respectively and has model D a loss of Re 1. Each type of radio requires a certain amount of time for the manufacturing of components, for assembling and for packing. A dozen units of model A require one hour for manufacturing, two hours for assembling and one hour for packing. The corresponding figures for a dozen units of model B are 2, 1 and 2, and for a dozen units of C are 3, 5 and 1. A dozen units of model D however, only require 1 hour of packing. During the forthcoming week, the company will be able to make available 15 hours of manufacturing, 20 hours of assembling and 10 hours of packing time. Determine the optimal production schedule for the company.

18. A transport company is considering the purchase of new vehicles for providing transportation between the Delhi Airport and hotels in the city. There are three vehicles under consideration: Station wagons, minibuses and large buses. The purchase price would be Rs 1,45,000 for each station wagon, Rs 2,50,000 for each minibus and Rs 4,00,000 for each large bus. The board of directors has authorized a maximum amount of Rs 50,00,000 for these purchases. Because of the heavy air travel, the new vehicles would be utilized at maximum capacity, regardless of the type of vehicles purchased. The expected net annual profit would be Rs 15,000 for the station wagon, Rs 35,000 for the minibus and Rs 45,000 for the large bus. The company has hired 30 new drivers for the new vehicles. They are qualified drivers for all three types of vehicles. The maintenance department has the capacity to handle an additional 80 station wagons. A minibus is equivalent to 1.67 station wagons and each large bus is equivalent to 2 station wagons in terms of their use of the maintenance department. Determine the number of each type of vehicle that should be purchased in order to maximize profit.

19. Omega Data Processing Company performs three types of activities: Payroll, accounts receivables, and inventories. The profit and time requirement for keypunch, computation and office printing, for a standard job, are shown in the following table:

Job	*Profit/ Standard Job (Rs)*	*Time Requirement (Min)*		
		Keypunch	*Computation*	*Printing*
Payroll	275	1,200	20	100
A/c Receivable	125	1,400	15	60
Inventory	225	800	35	80

Omega guarantees overnight completion of the job. Any job scheduled during the day can be completed during the day or night. Any job scheduled during the night, however, must be completed during the night. The capacities for both day and night are shown in the following table:

Capacity (Min)	*Keypunch*	*Computation*	*Print*
Day	4,200	150	400
Night	9,200	250	650

Determine the mixture of standard jobs that should be accepted during the day and night.

20. A furniture company can produce four types of chairs. Each chair is first made in the carpentry shop and then furnished, waxed and polished in the finishing shop. The man-hours required in each are:

	Chair Type			
	1	*2*	*3*	*4*
Carpentry shop	4	9	7	10
Finishing shop	1	1	3	40
Contribution per chair (Rs)	12	20	18	40

The total number of man-hours available per month in carpentry and finishing shops are 6,000 and 4,000, respectively.

Assuming an abundant supply of raw material and an abundant demand for finished products, determine the number of each type of chairs that should be produced for profit maximization.

21. A farmer has a 100 acre farm. He can sell all the tomatoes, lettuce or radishes that he produces. The price he can obtain is Re 1 per kilogram for tomatoes, Re 0.75 a head for lettuce and Rs 2 per kilogram for radishes. The average yield per acre is 2,000 kilograms of tomatoes, 3,000 heads of lettuce and 1,000 kilograms of radishes. The fertilizer is available at Re 0.50 per kilogram and the required amount per acre is 100 kilograms each for tomatoes and lettuce and 50 kilograms for radishes. The labour required for sowing cultivating and harvesting per acre is 5 man-days for tomatoes and radishes and 6 man-days

for lettuce. A total of 400 man-days of labour are available at Rs 20 per man-day. Determine the crop mix so as to maximize the farmer's total profit. *[Delhi Univ., MBA, 2004]*

22. A metal products company produces waste cans, filing cabinets, file boxes for correspondence, and lunch boxes. Its inputs are sheet metal of two different thickness, called A and B, and manual labour. The input-output relationship for the company are shown in the table given below:

	Waste Cans	*Filing Cabinets*	*Correspondence Boxes*	*Lunch Boxes*
Sheet metal A	6	0	2	3
Sheet metal B	0	10	0	0
Manual labour	4	8	2	3

The sales revenue per unit of waste cans, filing cabinets, correspondence ooxes and lunch boxes are Rs 20, Rs 400, Rs 90 and Rs 20, respectively. There are 225 units of sheet metal A available in the company's inventory, 300 of sheet metal B, and a total of 190 units of manual labour. What is the company's optimal sales revenue? *[Delhi Univ., MBA, 2000, 2002]*

23. A bank is planning its operations for the next year. The bank makes five types of loans. These are listed below, together with the annual return (in per cent);

Type of Loan	*Annual Return (per cent)*
Education	15
Furniture	12
Automobile	9
Home Construction	10
Home repair	7

Legal requirements and bank policy place the following limits on the amounts of the various types of loans:

Education loans cannot exceed 10 per cent of the total amount of loans. The amount of education and furniture loans together cannot exceed 20 per cent of the total amount of loans. Home construction loan must be at least 40 per cent of the total amount of home loans and at least 20 per cent of the total amount of loans. Home construction loan must not exceed 25 per cent of the total amount of loans.

The bank wishes to maximize its revenue from loan interest, subject to the above restrictions. The bank can lend a maximum of Rs 75 crore. Formulate and solve this problem as a linear programming problem.

24. A company produces electric transformers for the electrical industry. The company has orders for transformer for the next six months. The cost of manufacturing a transformer is expected to somewhat vary over the next few months due to expected changes in materials costs and in labor rates. The company can produce up to 50 units per month in regular time and up to an additional 20 units per month during overtime. The costs for both regular and overtime production are shown in the table below:

	Month					
	Jan.	*Feb.*	*Mar.*	*April*	*May*	*June*
Orders (units)	58	36	34	69	72	43
Cost per unit at regular time (in '000 Rs)	180	170	170	185	190	190
Cost per unit at overtime (in '000 Rs)	200	190	190	210	220	220

The cost of carrying an unsold transformer in stock is Rs 25 000 per month. As on January 1 the company has 15 transformers in stock on and it wishes to have no less than 5 in stock on June 30. Formulate a linear programming problem to determine the optimal production schedule.

25. A company mined diamonds in three locations in the country. The three mines differed in terms of their capacities, number, weight of stones mined, and costs. These are shown in the table below:

Due to marketing considerations, a monthly production of exactly 1,48,000 stones was required. A similar requirement called for at least 1,30,000 carats (The average stone size was at least 130/148 = 0.88 carats). The capacity of each mine is measured in cubic meter. The mining costs are not included from the treatment costs and assume to be same at each mine. The problem for the company was to meet the marketing requirements at the least cost.

Mine	*Capacity (M^3 of earth processed)*	*Treatment Costs (Rs. per M^3)*	*Crude (Carats per M^3)*	*Stone Count (Number of stone per M^3)*
Plant 1	83,000	0.60	0.360	0.58
Plant 2	3,10,000	0.36	0.220	0.26
Plant 3	1,90,000	0.50	0.263	0.21

Formulate a linear programming model to determine how much should be mined at each location.

HINTS AND ANSWERS

1. Let x_1 and x_2 = units of models A and B to be manufactured, respectively.

$$\text{Max } Z = 400x_1 + 400x_2$$

subject to
$$4x_1 + 2x_2 \le 1,600$$
$$5x_1/2 + x_2 \le 1,200$$
$$9x_1/2 + 3x_2/2 \le 1,600$$
and
$$x_1, x_2 \ge 0$$

Ans. $x_1 = 355.5$, $x_2 = 0$ and Max $Z = 1,42,222.2$.

2. Let x_1, x_2 and x_3 = units of types A, B and C belt to be manufactured, respectively.

$$\text{Max } Z = 3x_1 + 5x_2 + 4x_3$$

subject to
$$2x_1 + 3x_2 \le 8$$
$$2x_2 + 5x_3 \le 10$$
$$3x_1 + 2x_2 + 4x_3 \le 15$$
and
$$x_1, x_2, x_3 \ge 0$$

Ans. $x_1 = 89/41$, $x_2 = 50/41$ $x_3 = 64/41$ and Max $Z = 775/41$.

3. Let x_1, x_2 and x_3 = quantity of products A, B and C to be produced, respectively.

$$\text{Max } Z = x_1 + 4x_2 + 5x_3$$

subject to
$$3x_1 + 3x_2 \le 22$$
$$x_1 + 2x_2 + 3x_3 \le 14$$
$$3x_1 + 2x_2 \le 14$$
and
$$x_1, x_2, x_3 \ge 0$$

Ans. $x_1 = 0$, $x_2 = 7$, $x_3 = 0$ and Max Z = Rs 28,000.

4. Let x_1, x_2 and x_3 = number of units of products 1, 2 and 3 to be produced per week, respectively.

$$\text{Max } Z = 20x_1 + 6x_2 + 8x_3$$

subject to
$$8x_1 + 2x_2 + 3x_3 \le 250$$
$$4x_1 + 3x_2 \le 150$$
$$2x_1 + x_3 \le 50$$
and $x_1, x_2, x_3 \ge 0$

Ans. $x_1 = 0$, $x_2 = 50$, $x_3 = 50$ and Max $Z = 700$.

5. Let x_1, x_2 and x_3 = acreage of corn, wheat and soyabean, respectively.

$$\text{Max } Z = 30x_1 + 40x_2 + 20x_3$$

subject to
$$10x_1 + 12x_2 + 7x_3 \le 10{,}000$$
$$7x_1 + 10x_2 + 8x_3 \le 8{,}000$$
$$x_1 + x_2 + x_3 \le 1{,}000$$
and $x_1, x_2, x_3 \ge 0$

Ans. $x_1 = 250$, $x_2 = 625$, $x_3 = 0$ and Max Z = Rs 32,500.

6. Let x_1, x_2 and x_3 = number of units of chair of styles A, B and C, respectively.

$$\text{Max } Z = 40x_1 + 35x_2 + 30x_3$$

subject to
$$2x_1 + 3x_2 + 2x_3 \le 120$$
$$4x_1 + 3x_2 + x_3 \le 160$$
$$3x_1 + 2x_2 + 4x_3 \le 100$$
$$x_1 + x_2 + x_3 \le 40$$
and $x_1, x_2, x_3 \ge 0$

Ans. $x_1 = 20$, $x_2 = 20$, $x_3 = 0$ and Max Z = Rs 1,500.

7. Let x_1, x_2 and x_3 = salesman assigned to area, 1, 2 and 3, respectively.

$$\text{Max. } Z = 40 \times 150x_1 + 36 \times 150x_2 + 25 \times 150x_3 - (800x_1 + 700x_2 + 500x_3)$$

subject to (i) $x_1 + x_2 + x_3 \le 12$; (ii) $x_1 \le 5$;
(iii) $800x_1 + 700x_2 + 500x_3 \le 7{,}500$
and $x_1, x_2, x_3 \ge 0$

11. Let x and y = number of kg of ingredients x_1, x_2, respectively.

Min (total cost) $Z = 3x + 8y$

subject to (i) $x + y = 200$; (ii) $x \le 80$; (iii) $y \ge 60$
and $x, y \ge 0$

Ans. $x = 80$, $y = 120$ and Min Z = Rs 1,200.

12. Let x_1, x_2 and x_3 = number of units of food A, B and C, respectively which a diet must contain.

Min (total cost) $Z = 2x_1 + x_2 + x_3$

subject to
$$2x_1 + 2x_2 + 4x_3 \ge 20$$
$$4x_1 + 2x_2 + 2x_3 \ge 15$$
and $x_1, x_2, x_3 \ge 0$

13. Let x_1 and x_2 = number of units of liquid and dry product produced, respectively.

Min (total cost) $Z = 3x_1 + 2x_2$

subject to (i) $5x_1 + x_2 \ge 10$; (ii) $2x_1 + 2x_2 \ge 12$;
(iii) $x_1 + 4x_2 \ge 12$
and $x_1, x_2 \ge 0$

Ans. $x_1 = 1$, $x_2 = 5$ and Min $Z = 13$.

14. Let x_1 and x_2 = number of scrap (in kg) purchased from suppliers A and B, respectively.

Min (total cost) $Z = x_1 + 4x_2$

subject to
$$2.25x_1 + 0.75x_2 \ge 1{,}000$$
$$0.05x_1 + 0.10x_2 \ge 175$$
$$x_1 + x_2 \ge 2{,}000$$
and $x_1, x_2 \ge 0$

Ans. $x_1 = 2{,}500$, $x_2 = 500$ and Min $Z = 4{,}500$.

15. Let x_1 and x_2 = number of insertions messages for media A and B, respectively.

Min (total effective audience) $Z = 40{,}000x_1 + 55{,}000x_2$

subject to (i) $1{,}000x_1 + 1500x_2 \le 2{,}00{,}000$;
(ii) $x_1 \le 12$; (iii) $x_2 \ge 5$
and $x_1, x_2 \ge 0$

Ans. $x_1 = 12$, $x_2 = 16/3$ and Max $Z = 77{,}3333.33$.

16. Let x_1 and x_2 = number of radio and television programmes, respectively.

Max (total audience) $Z = (4{,}50{,}000 + 50{,}000)\, x_1 + (20{,}000 + 80{,}000)\, x_2$

subject to (i) $50{,}000x_1 + 20{,}000x_2 \le 2{,}00{,}000$;
(ii) $x_1 \le 3$; (iii) $x_2 \ge 5$
and $x_1, x_2 \ge 0$

Ans. $x_1 = 4$, $x_2 = 0$ and Max $Z = 4 \times 5{,}00{,}000$.

17. Let x_1, x_2, x_3 and x_4 = unit of models A, B, C and D to be produce, respectively.

Max (total income) $Z = 8x_1 + 15x_2 + 25x_3 - x_4$

subject to (i) $x_1 + 2x_2 + 3x_3 = 15$; (ii) $2x_1 + x_2 + 5x_3 = 20$
(iii) $x_1 + 2x_2 + x_3 + x_4 = 10$
and $x_1, x_2, x_3, x_4 \ge 0$

Ans. $x_1 = 5/2$, $x_2 = 5/2$, $x_3 = 5/2$, $x_4 = 0$ and Max $Z = 120$.

18. Let x_1, x_2 and x_3 = number of station wagons, minibuses and large buses to be purchased, respectively.

$$\text{Max } Z = 15{,}000x_1 + 35{,}000x_2 + 45{,}000x_3$$

subject to
$$x_1 + x_2 + x_3 \le 30$$
$$1{,}45{,}000x_1 + 2{,}50{,}000x_2 + 4{,}00{,}000x_3 \le 50{,}00{,}000$$
$$x_1 + 1.67x_2 + 0.5x_3 \le 80$$
and $x_1, x_2, x_3 \ge 0$

19. Let x_{ij} represents ith job and jth activity

$$\text{Max } Z = 275\,(x_{11} + x_{12}) + 125(x_{21} + x_{22}) + 225\,(x_{31} + x_{32})$$

subject to
$$1{,}200\,(x_{11} + x_{12}) + 1{,}400\,(x_{21} + x_{22}) + 800\,(x_{31} + x_{32}) \le 13{,}400$$
$$20\,(x_{11} + x_{12}) + 15\,(x_{21} + x_{22}) + 35\,(x_{31} + x_{32}) \le 400$$
$$100\,(x_{11} + x_{12}) + 60\,(x_{21} + x_{22}) + 80\,(x_{31} + x_{32}) \le 1{,}050$$
$$1{,}200x_{12} + 1{,}400x_{22} + 800x_{32} \le 9{,}200$$
$$20x_{12} + 15x_{22} + 35x_{32} \le 250$$
$$100x_{12} + 60x_{22} + 80x_{32} \le 650$$
and $x_{ij} \ge 0$ for all i, j

20. Let x_1, x_2, x_3 and x_4 = chair types 1, 2, 3 and 4 to be produced, respectively.

$$\text{Max } Z = 12x_1 + 20x_2 + 18x_3 + 40x_4$$

subject to
$$4x_1 + 9x_2 + 7x_3 + 10x_4 \le 6{,}000$$
$$x_1 + x_2 + 3x_3 + 40x_4 \le 4{,}000$$
and $x_1, x_2, x_3, x_4 \ge 0$

Ans. $x_1 = 4,000/3$, $x_2 = x_3 = 0$, $x_4 = 200/3$ and Max $Z =$ Rs 56,000/3.

21. Let x_1, x_2 and x_3 = number of units of tomatoes, lettuce and radishes to be produced, respectively.

$$\begin{aligned}\text{Max (profit) } Z &= \text{Selling price} - \text{Fertilizer cost} - \text{Labour cost}\\ &= (1 \times 2{,}000 - 0.50 \times 100 - 20 \times 5)\,x_1\\ &\quad + (0.75 \times 3{,}000 - 0.50 \times 100 - 20 \times 6)\,x_2\\ &\quad + (2 \times 1{,}000 - 0.50 \times 50 - 20 \times 5)\,x_3\\ &= 1{,}850\,x_1 + 2{,}080\,x_2 + 1{,}875\,x_3\end{aligned}$$

subject to (i) $x_1 + x_2 + x_3 \le 100$; (ii) $5x_1 + 6x_2 + 5x_3 \le 400$

and $x_1, x_2, x_3 \ge 0$.

23. Let x_1, x_2, x_3, x_4 and x_5 = funds allocated to education, furniture, automobile, home construction and home repair, respectively.

Max: $Z = 0.15x_1 + 0.12x_2 + 0.09x_3 + 0.10x_4 + 0.07x_5$

subject to (i) $x_1 + x_2 + x_3 + x_4 + x_5 \le 1.5$ (funds available)

(ii) $x_1 \le 0.10(x_1 + x_2 + x_3 + x_4 + x_5)$

or $0.9x_1 - 0.1x_2 - 0.1x_3 - 0.1x_4 - 0.1x_5 \le 0$

(iii) $x_1 + x_2 \le 0.20\,(x_1 + x_2 + x_3 + x_4 + x_5)$

or $0.8x_1 + 0.8x_2 - 0.2x_3 - 0.2x_4 - 0.2x_5 \le 0$

(iv) $x_5 \ge 0.40(x_4 + x_5)$ or $-0.4\,x_4 + 0.6\,x_5 \ge 0$

(v) $x_5 \ge 0.20(x_1 + x_2 + x_3 + x_4 + x_5)$

or $-0.2x_1 - 0.2x_2 - 0.2x_3 - 0.2x_4 + 0.8x_5 \ge 0$

and $x_j \ge 0$ for all j

Ans. $x_1 = 0.15$; $x_2 = 0.15$; $x_3 = 0.525$; $x_4 = 0.375$; $x_5 = 0.30$; Max $Z = 0.146$

24. Let $x_1, x_2, \ldots, x_6$ = number of transformers produced in regular time each month

$y_1, y_2, \ldots, y_6$ = number of transformers produced on overtimes each month

$I_1, I_2, \ldots, I_6$ = number of transformers in stock at the end of each month

$$\begin{aligned}\text{Min Z} &= 18\,x_1 + 17x_2 + 17x_3 + 18.5x_4 + 19x_5\\ &\quad + 19x_6 + 20y_1 + 19y_2 + 19y_3\\ &\quad + 21y_4 + 22y_5 + 22y_6 + 0.5I_1 + 0.5I_2\\ &\quad + 0.5I_3 + 0.5I_4 + 0.5I_5 + 0.5I_6\end{aligned}$$

subject to $\left.\begin{array}{l} x_i \le 50 \text{ for } i = 1, 2, \ldots 6\\ y_i \le 20 \text{ for } i = 1, 2, \ldots 6\end{array}\right\}$ Capacity constraints

$I_1 = 15 + x_1 + y_1 - 58$ (Inventory for January)

$I_2 = I_{i-1} + x_i + y_i -$ Orders for $i = 2, 3, \ldots 6$

$I_6 \ge 5$ (Final inventory)

Ans.

Month	*Regular Production*	*Overtime Production*	*Inventory*
January	43	0	0
February	50	0	14
March	50	11	41
April	50	0	22
May	50	0	0
June	48	0	5

Total cost = Rs 2,75,550.

An alternative basic solution exists that involves regular production of 50 in January, and only 4 produced in overtime in March. Ending inventories in January = 7 and February = 21.

25. Let x_1, x_2 and x_3 = cubic metric of earth processed at plants 1, 2 and 3, respectively.

Min $Z = 0.60x_1 + 0.36x_2 + 0.50x_3$

subject to

$0.58x_1 + 0.26x_2 + 0.21x_3 = 1{,}48{,}000$ (Stone count requirement)

$0.36x_1 + 0.22x_2 + 0.263x_3 \le 1{,}30.000$ (Carat requirement)

$x_1 \le 83{,}000$; $x_2 \le 3{,}10{,}000$;

$x_3 \le 1{,}90{,}000$ (Capacity requirement)

Ans. $x_1 = 61{,}700$; $x_2 = 3{,}10{,}000$; $x_3 = 1{,}50{,}500$ and Min $Z =$ Rs 2,23,880.

4.5 SOME COMPLICATIONS AND THEIR RESOLUTION

We have discussed, through examples, how one can use the simplex method for solving both maximization and minimization problems. In this section some of the complications that may arise in applying the simplex method and the resolution of these problems are discussed.

4.5.1 Unrestricted Variables

An unrestricted variable in an LP model can have either positive, negative or zero value

Usually in an LP problem, it is assumed that all the variables x_j ($j = 1, 2, \ldots, n$) should have non-negative values. In many practical situations, however, one or more of the variables, can have either positive, negative or zero value. Variables that can assume positive, negative or zero value are called *unrestricted variables*. Since the use of the simplex method requires that all the decision variables must have non-negative value at each iteration, therefore, in order to convert an LP problem involving unrestricted variables into an equivalent problem having only restricted variables, we have to express each of unrestricted variables as the difference of two non-negative variables.

Let variable x_r be unrestricted in sign. We define two new variables say x_r' and x_r'' such that

$$x_r = x_r' - x_r''; \qquad x_r', x_r'' \ge 0$$

If $x_r' \ge x_r''$ of then $x_r \ge 0$ and if $x_r' \le x_r''$, then $x_r \le 0$. Also if $x_r' = x_r''$, then $x_r = 0$. Hence depending on the value of x_r' and x_r'', x_r can have any sign.

$$\text{Maximize } Z = \sum_{j=1}^{n} c_j x_j + c_r x_r$$

subject to the constraints

$$\sum_{j \neq r}^{n} a_{ij} x_j + a_{ir} x_r = b_i, \quad i = 1, 2, \ldots, m$$

and $\qquad x_j \geq 0; \; j = 1, 2, \ldots, n, \; j \neq r; \; x_r$ unrestricted in sign

it can then be converted into its equivalent standard form as follows:

$$\text{Maximize } Z = \sum_{j \neq r}^{n} c_j x_j + c_r (x_r' + x_r'')$$

subject to the constraints

$$\sum_{j \neq r}^{n} a_{ij} x_j + a_{ir} (x_r' - x_r'') = b_i, \quad i = 1, 2, \ldots, m$$

and $\qquad x_j, x_r', x_r'' \geq 0; \; j = 1, 2, \ldots, n, \; j \neq r.$

Variables x_r', x_r'' simultaneously cannot appear in the basis (since the column vectors corresponding to these variables are linearly dependent). Thus any of the following three cases may arise at the optimal solution:

(i) $x_r' = 0 \Rightarrow x_r = -x_r''$ (ii) $x_r'' = 0 \Rightarrow x_r = x_r'$ (iii) $x_r' = x_r'' = 0 \Rightarrow x_r = 0$

This indicates that the value of x_r' and x_r'' uniquely determines the values of the variable x_r.

Example 4.12 Use the simplex method to solve the following LP problem.

Maximize $Z = 3x_1 + 2x_2 + x_3$

subject to the constraints

(i) $2x_1 + 5x_2 + x_3 = 12$, (ii) $3x_1 + 4x_2 = 11$

and $\qquad x_2, x_3 \geq 0, \; x_1$ unrestricted *[Bombay, BSc (Maths), 1995]*

Solution Introducing an artificial variable A_1 in the second constraint of the given LP problem in order to obtain the basis matrix as shown in Table 4.47. Since x_1 is unrestricted in sign, introduce the non-negative variables x_1' and x_2'' so that $x_1 = x_1' - x_1''$; x_1', $x_1'' \geq 0$. The standard form of the LP problem non becomes:

Maximize $Z = 3(x_1' - x_1'') + 2x_2 + x_3 - MA_1$

subject to the constraints

(i) $2(x_1' - x_1'') + 5x_2 + x_3 = 12$, (ii) $3(x_1' - x_1'') + 4x_2 + A_1 = 11$

and $\qquad x_1', x_1'', x_2, x_3, A_1 \geq 0$

The initial solution is shown in Table 4.43.

Table 4.43
Initial Solution

		$c_j \rightarrow$	3	−3	2	1	−M	
c_B	Variables in Basis B	Solution Values $b (= x_B)$	x_1'	x_1''	x_2	x_3	A_1	Min Ratio x_B/x_2
1	x_3	12	2	−2	5	1	0	12/5 →
−M	A_1	11	3	−3	4	0	1	11/4
$Z = -11M + 12$		z_j	$-3M+2$	$3M-2$	$-4M+5$	1	−M	
		$c_j - z_j$	$3M+1$	$-3M-1$	$4M-3$ ↑	0	0	

In Table 4.43, $c_3 - z_3$ is the largest positive value, we need to enter variable x_2 into the basis and remove variable x_3 from the basis. For this, apply the following row operations:

R_1 (new) → R_1 (old) ÷ 5 (*key element*) $\quad R_2$ (new) → R_2 (old) − $4R_1$ (new).

The improved solution so obtained is given in Table 4.44.

		$c_j \rightarrow$	3	−3	2	1	−M	
c_B	Variables in Basis B	Solution Values $b(= x_B)$	x_1'	x_1''	x_2	x_3	A_1	Min Ratio x_B/x_1'
2	x_2	12/5	2/5	−2/5	1	1/5	0	$\frac{12}{5} \times \frac{5}{2} = 6$
−M	A_1	7/5	7/5	−7/5	0	−4/5	1	$\frac{7}{5} \times \frac{5}{7} = 1 \rightarrow$
$Z = -7M/5 + 24/5$		z_j	$-7M/5 + 4/5$	$7M/5 - 4/5$	2	$4M/5 + 2/5$	−M	
		$c_j - z_j$	$7M/5 + 11/5$ ↑	$-7M/5 - 11/5$	0	$-4M/5 + 3/5$	0	

Table 4.44 Improved Solution

In Table 4.44, as $c_1 - z_1$ in x'_1 column is still positive, introduce variable x'_1 into the basis and remove variable A_1 from the basis. For this apply the following row operations:

R_2 (new) $\rightarrow R_2$ (old) × (5/7) (*key element*); R_1 (new) $\rightarrow R_1$ (old) − (2/5) R_2 (new).

The improved solution so obtained is given in Table 4.45.

Further, in order to improve the solution given in Table 4.45, we need to introduce variable x_3 into the basis and remove variable x_2 from the basis. For this we must apply the following row operations:

R_1 (new) $\rightarrow R_1$ (old) × (7/3) (*key element*); R_2 (new) $\rightarrow R_2$ (old) + (4/7) R_1 (new)

		$c_j \rightarrow$	3	−3	2	1	
c_B	Variables in Basis B	Solution Values $b(= x_B)$	x_1'	x_1''	x_2	x_3	Min Ratio x_B/x_3
2	x_2	2	0	0	1	3/7	$2/(3/7) = 14/3 \rightarrow$
3	x_1'	1	1	−1	0	−4/7	—
$Z = 7$		z_j	3	−3	2	−6/7	
		$c_j - z_j$	0	0	0	13/7 ↑	

Table 4.45 Improved Solution

The improved solution so obtained is given in Table 4.46.

		$c_j \rightarrow$	3	−3	2	1
c_B	Variables in Basis B	Solution Values $b(= x_B)$	x_1'	x_1''	x_2	x_3
1	x_3	14/3	0	0	7/3	1
3	x_1'	11/3	1	−1	4/3	0
$Z = 47/3$		z_j	3	−3	19/3	1
		$c_j - z_j$	0	0	−13/3	0

Table 4.46 Optimal Solution

In Table 4.46, all $c_j - z_j \leq 0$, an optimal solution has been arrived at with the values of the variables as: $x_1' = 11/3$ or $x_1 = x_1' - x_1'' = 11/3 - 0 = 11/3$; $x_3 = 14/3$ and Max $Z = 47/3$.

4.5.2 Tie for Entering Basic Variable (Key Column)

A situation may arise at any iteration when two or more columns may have exactly the same $c_j - z_j$ value (positive or negative depending upon the type of LP problem). In order to break this tie, the selection for key column (entering variable) can be made arbitrary. However, the number of iterations required to arrive at the optimal solution can be minimized by adopting the following rules:

(i) If there is a tie between two decision variables, then the selection can be made arbitrarily.

(ii) If there is a tie between a decision variable and a slack (or surplus) variable, then select the decision variable to enter into basis first.

(iii) If there is a tie between two slack (or surplus) variables, then the selection can be made arbitrarily.

4.5.3 Tie for Leaving Basic Variable (Key Row) – Degeneracy

While solving an LP problem a situation may arise in which there is a tie between two or more basic variables for leaving the basis, i.e. the minimum ratio to identify the basic variable to leave the basis is not unique or in which values of one or more basic variables in the 'solution values' column ($\mathbf{x}_B$) become equal to zero. This causes the problem of degeneracy. However, if the minimum ratio is zero, then the iterations of simplex method are repeated (cycle) indefinitely without arriving at the optimal solution. Such a situation, of course, is very rare in practical problems.

The problem of degeneracy arises due to redundant constraint, i.e. in the LP problem one or more of the constraints makes another unnecessary. For example, constraints such as $x_1 \leq 5$, $x_2 \leq 5$ and $x_1 + x_2 \leq 5$ in the LP problem make constraint $x_1 + x_2 \leq 5$ unnecessary (redundant).

Degeneracy may occur at any iteration of the simplex method. In most of the cases when there is a tie in the minimum ratios, the selection is made arbitrarily. However, the number of iterations required to arrive at the optimal solution can be minimized by adopting the following rules.

(i) Divide the coefficients of slack variables in the simplex table where degeneracy is detected by the corresponding positive numbers of the key column in the row, starting from left to right.

(ii) The row that contains the smallest ratio comparing from left to right columnwise becomes the key row.

Remark When there is a tie between a slack and artificial variable to leave the basis, the preference should be given to the artificial variable for leaving the basis. There is no need to apply the procedure for resolving degeneracy under such cases.

Example 4.13 Solve the following LP problem

Maximize $Z = 3x_1 + 9x_2$

subject to the constraints

(i) $x_1 + 4x_2 \leq 8$, (ii) $x_1 + 2x_2 \leq 4$

and $x_1, x_2 \geq 0$

Solution Adding slack variables s_1 and s_2 to the constraints, the problem can be expressed as

Maximize $Z = 3x_1 + 9x_2 + 0s_2 + 0s_2$

subject to the constraints

(i) $x_1 + 4x_2 + s_1 = 8$, (ii) $x_1 + 2x_2 + s_2 = 4$

and $x_1, x_2, s_1, s_2 \geq 0$

The initial basic feasible solution is given in Table 4.47. As shown in Table 4.47, $c_2 - z_2 = 9$ is the largest positive value, therefore variable x_2 is selected to be entered into the basis. However, both variables s_1 and s_2 are eligible to leave the basis as the minimum ratio is same, i.e. 2, so there is a tie among the ratio in rows s_1 and s_2. This is an indication of the existence of degeneracy. To obtain the unique key row and for resolving degeneracy apply the following procedure:

(i) Write the coefficients of the slack variables as shown in Table 4.47.

Table 4.47 Initial Solution

		$c_j \rightarrow$	3	9	0	0	
c_B	Variables in Basis B	Solution Values $b(= x_B)$	x_1	x_2	s_1	s_2	Min Ratio x_B/x_2
0	s_1	8	1	4	1	0	8/4 = 2
0	s_2	4	1	2	0	1	4/2 = 2
$Z = 0$		z_j	0	0	0	0	
		$c_j - z_j$	3	9 ↑	0	0	

Row	Key Column	Column s_1	s_2
s_1	4	1	0
s_2	2	0	1

(ii) By dividing the coefficients by the corresponding element of the key column, we obtain the following ratios:

Row	Key Column	Column s_1	s_2
s_1	4	$1/4 = 1/4$	$0/4 = 0$
s_2	2	$0/2 = 0$	$1/2 = 1/2$

(iii) Comparing the ratios of Step (ii) from left to right columnwise, the minimum ratio occurs for the second row. Therefore, the variable s_2 is selected to leave the basis. The new solution is shown in Table 4.48.

		$c_j \rightarrow$	3	9	0	0
c_B	Variables in Basis B	Solution Values $b(= x_B)$	x_1	x_2	s_1	s_2
0	s_1	0	-1	0	1	-2
9	x_2	2	1/2	1	0	1/2
$Z = 18$		z_j	9/2	9	0	9/2
		$c_j - z_j$	$-3/2$	0	0	$-9/2$

Table 4.48
Optimal Solution

In Table 4.48, all $c_j - z_j \leq 0$. Therefore, an optimal solution has been arrived at. The optimal basic feasible solution is: $x_1 = 0$, $x_2 = 2$ and Max $Z = 18$.

4.6 TYPES OF LINEAR PROGRAMMING SOLUTIONS

In this section, we shall discuss three different types of solutions in terms of the termination of the simplex method.

4.6.1 Alternative (Multiple) Optimal Solutions

The alternative optimal solution can be obtained by considering the $c_j - z_j$ row of the simplex table. We know that an optimal solution to a maximization problem is reached if all $c_j - z_j \leq 0$. What will happen if $c_j - z_j = 0$ for some non-basic variable columns in the optimal simplex table? Each entry in the $c_j - z_j$ row indicates the contribution per unit of a particular variable in the objective function value if it is entered into the basis. Thus, if a non-basic variable corresponding to which $c_j - z_j = 0$ is entered into the basis, a new solution will be arrived at but in this case the value of the objective function will not change.

Alternative optimal solutions arise when $c_j - z_j = 0$ for non-basic variable columns in the simplex table

Example 4.14 Solve the following LP problem

Maximize $Z = 6x_1 + 4x_2$

subject to the constraints

(i) $2x_1 + 3x_2 \leq 30$, (ii) $3x_1 + 2x_2 \leq 24$, (iii) $x_1 + x_2 \geq 3$

and $x_1, x_2 \geq 0$.

Solution By adding slack variables s_1, s_2, surplus variable s_3 and artificial variable A_1 in the constraint set, the LP problem becomes:

Maximize $Z = 6x_1 + 4x_2 + 0s_1 + 0s_2 + 0s_3 - MA_1$

subject to the constraints

(i) $2x_1 + 3x_2 + s_1 = 30$, (ii) $3x_1 + 2x_2 + s_2 = 24$ (iii) $x_1 + x_2 - s_3 + A_1 = 3$

and $x_1, x_2, s_1, s_2, s_3, A_1 \geq 0$

The optimal solution for this LP problem is presented in Table 4.49.

		$c_j \rightarrow$	6	4	0	0	0	
c_B	Variables in Basis B	Solution Values $b(= x_B)$	x_1	x_2	s_1	s_2	s_3	Min Ratio x_B/x_2
0	s_1	14	0	5/3	1	– 2/3	0	14/(15/3) = 42/5 →
0	s_3	5	0	– 1/3	0	1/3	1	—
6	x_1	8	1	2/3	0	1/3	0	8/(2/3) = 12
$Z = 48$		z_j	6	4	0	2	0	
		$c_j - z_j$	0	0	0	– 2	0	
				↑				

Table 4.49
Optimal Solution

The optimal solution shown in Table 4.49 is: $x_1 = 8$, $x_2 = 0$ and Max $Z = 48$.

In Table 4.49, $c_2 - z_2 = 0$ corresponds to a non-basic variable, x_2 (i.e. $x_2 = 0$). Thus, an alternative optimal solution can also be obtained by entering variable x_2 into the basis and removing s_1 from the basis. The new solution is shown in Table 4.50.

		$c_j \rightarrow$	6	4	0	0	0
c_B	Variables in Basis B	Solution Values $b(= x_B)$	x_1	x_2	s_1	s_2	s_3
4	x_2	42/5	0	1	3/5	– 2/5	0
0	s_3	39/5	0	0	1/5	1/5	1
6	x_1	12/5	1	0	– 2/5	3/5	0
$Z = 48$		z_j	6	4	0	2	0
		$c_j - z_j$	0	0	0	–2	0

Table 4.50
Alternative Solution

The optimal solution shown in Table 4.50 is: $x_1 = 12/5$, $x_2 = 42/5$ and Max $Z = 48$.

Observe that in Table 4.50, $c_3 - z_3 = 0$ and variable s_1 is not in the basis. This again indicates that an alternative optimal solution exists. The infinite number of solutions that can be obtained for this LP problem are as follows:

Variables	Solution Values 1	Solution Values 2	General Solution
x_1	8	12/5	$x_1 = 8\lambda + (12/5)(1 - \lambda)$
x_2	0	42/5	$x_2 = 0\lambda + (42/5)(1 - \lambda)$
s_1	14	0	$s_1 = 14\lambda + (0)(1 - \lambda)$
s_3	5	39/5	$s_3 = 5\lambda + (39/5)(1 - \lambda)$

For each arbitrary value of λ, the value of objective function will remain same.

4.6.2 Unbounded Solution

Unboundedness describes an LP problem that does not have a finite solution. For example, in a maximization LP problem, if $c_j - z_j > 0$ ($c_j - z_j < 0$ for a minimization case) for a column variable not in the basis, and all entries in this column are negative, then for determining key row, we have to calculate the minimum ratio corresponding to each basic variable having negative or zero value in the denominator. Negative value in denominator cannot be considered, as it would indicate the entry of a non-basic variable in the basis with a negative value (an infeasible solution will occur). A zero value in the denominator would result in a ratio having a value of $+\infty$. This implies that the entering variable could be increased infinitely with any of the current basic variables being removed from the basis. In general, an unbounded solution occurs due to wrong formulation of the problem within the constraint set, and thus needs reformulation.

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Example 4.16 Solve the following LP problem

Maximize $Z = 6x_1 + 4x_2$

subject to the constraints

(i) $x_1 + x_2 \leq 5$, (ii) $x_2 \geq 8$

and $x_1, x_2 \geq 0$.

Solution By adding slack, surplus and artificial variables, the LP problem becomes

Maximize $Z = 6x_1 + 4x_2 + 0s_1 + 0s_2 - MA_1$

subject to the constraints

(i) $x_1 + x_2 + s_1 = 5$, (ii) $x_2 - s_2 + A_1 = 8$

and $x_1, x_2, s_1, s_2, A_1 \geq 0$

The initial solution to this LP problem is shown in Table 4.53.

Table 4.53 Initial Solution

		$c_j \rightarrow$	6	4	0	0	$-M$	
c_B	*Variables in Basis* B	*Solution Values* $b (= x_B)$	x_1	x_2	s_1	s_2	A_1	*Min Ratio* x_B/x_2
6	s_1	5	1	1	1	0	0	$5/1 \rightarrow$
$-M$	A_1	8	0	1	0	-1	1	8/1
$Z = 30 - 8M$		z_j	6	$6 - M$	0	M	$-M$	
		$c_j - z_j$	0	$-2 + M$	0	$-M$	0	
				$\uparrow$				

Iteration 1: Variable x_2 enters the basis and x_1 leaves the basis. The new solution is shown in Table 4.58.

Table 4.54 Optimal but Infeasible Solution

		$c_j \rightarrow$	6	4	0	0	$-M$
c_B	*Variables in Basis* B	*Solution Values* $b (= x_B)$	x_1	x_2	s_1	s_2	A_1
4	x_2	5	1	1	1	0	0
$-M$	A_1	3	-1	0	-1	-1	1
$Z = 20 - 3M$		z_j	$4 + M$	4	$4 + M$	M	$-M$
		$c_j - z_j$	$2 - M$	0	$-4 - M$	$-M$	0

Since all $c_j - z_j \leq 0$, the solution shown in Table 4.54 is optimal. But this solution is not feasible for the given problem since it has $x_1 = 0$ and $x_2 = 5$ (recall that in the second constraint $x_2 \geq 8$). The fact that artificial variable $A_1 = 3$ is in the solution also indicates the fact that the final solution violates the second constraint ($x_2 \geq 8$) by 3 units.

CONCEPTUAL QUESTIONS

1. Define slack and surplus variables in a linear programming problem.
2. Explain the various steps of the simplex method involved in the computation of an optimum solution to a linear programming problem.
3. (a) Give outlines of the simplex method in linear programming.
   (b) What is simplex? Describe the simplex method of solving linear programming problem.
   (c) What do you understand by the term two-phase method of solving linear programming problem?
   (d) Outline the simplex method in linear programming. Why is it called so?
   (e) Explain the purpose and procedure of the simplex method.
4. What do you mean by an optimal basic feasible solution to a linear programming problem?
5. Given a general linear programming problem, explain how you would test whether a basic feasible solution is an optimal solution or not. How would you proceed to change the basic feasible solution in case it is not optimal?
6. Explain the meaning of basic feasible solution and degenerate solution in a linear programming problem.
7. Explain what is meant by the terms degeneracy and cycling in linear programming? How can these problems be resolved?
8. Explain the term artificial variables and its use in linear programming.
9. What are artificial variables? Why do we need them? Describe the two-phase method of solving an LP problem with artificial variables.
10. What is the significance of $c_j - z_j$ numbers in the simplex table? Interpret their economic significance in terms of marginal worth.

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4. Solve the following LP problems and remove the complication (if any)

(i) Max $Z = 2x_1 + 3x_2 + 10x_3$
subject to $x_1 + 2x_3 = 2$
$x_2 + x_3 = 1$
and $x_1, x_2, x_3 \geq 0$

(ii) Max $Z = 5x_1 - 2x_2 + 3x_3$
subject to $2x_1 + 2x_2 - x_3 \geq 2$
$3x_1 - 4x_2 \leq 3$
$x_2 + 3x_3 \leq 5$
and $x_1, x_2, x_3 \geq 0$

(iii) Max $Z = 5x_1 + 3x_2$
subect to $x_1 + x_2 \leq 2$
$5x_1 + 2x_2 \leq 10$
$3x_1 + 8x_2 \leq 12$
and $x_1, x_2 \geq 0$

(iv) Max $Z = 22x_1 + 30x_2 + 25x_3$
subject to $2x_1 + 2x_2 + \quad \leq 100$
$2x_1 + x_2 + x_3 \geq 100$
$x_1 + 2x_2 + 2x_3 \leq 100$
and $x_1, x_2, x_3 \geq 0$

(v) Max $Z = 8x_2$
subject to $x_1 - x_2 \geq 0$
$2x_1 + 3x_2 \leq -6$
and x_1, x_2 unrestricted.

5. Solve the following LP problems to show that these have alternative optimal solutions.

(i) Max $Z = 6x_1 + 3x_2$
subject to $2x_1 + x_2 \leq 8$
$3x_1 + 3x_2 \leq 18$
$x_2 \leq 3$
and $x_1, x_2 \geq 0$

(ii) Min $Z = 2x_1 + 8x_2$
subject to $5x_1 + x_2 \geq 10$
$2x_1 + 2x_2 \geq 14$
$x_1 + 4x_2 \geq 12$
and $x_1, x_2 \geq 0$

(iii) Max $Z = x_1 + 2x_2 + 3x_3 - x_4$
subject to $x_1 + 2x_2 + 3x_3 = 15$
$2x_1 + x_2 + 5x_3 \geq 20$
$x_1 + x_2 + x_3 + x_4 \geq 10$
and $x_1, x_2, x_3, x_4 \geq 0$

6. Solve the following LP problems to show that these have an unbounded solution.

(i) Max $Z = -2x_1 + 3x_2$
subject to $x_1 \leq 5$
$2x_1 - 3x_2 \leq 6$
and $x_1, x_2 \geq 0$

(ii) Max $Z = 3x_1 + 6x_2$
subject to $3x_1 + 4x_2 \geq 12$
$-2x_1 + x_2 \leq 4$
and $x_1, x_2 \geq 0$

(iii) Max $Z = 107x_1 + x_2 + 2x_3$
subject to $14x_1 + x_2 - 6x_3 + 3x_4 = 7$
$16x_1 + 0.5x_2 - 6x_3 \leq 5$
$3x_1 - x_2 - x_3 \leq 0$
and $x_1, x_2 \geq 0$

(iv) Max $Z = 6x_1 - 2x_2$
subject to $2x_1 - x_2 \leq 2$
$x_1 \leq 4$
and $x_1, x_2, x_3, x_4 \geq 0$

7. Solve the following LP problems to show that these have no feasible solution.

(i) Max $Z = 2x_1 + 3x_2$
subject to $x_1 - x_2 \geq 4$
$x_1 + x_2 \leq 6$
$x_1 \leq 2$
and $x_1, x_2 \geq 0$

(ii) Max $Z = 4x_1 + x_2 + 4x_3 + 5x_4$
subject to $4x_1 + 6x_2 - 5x_3 + 4x_4 \geq -20$
$3x_1 - 2x_2 + 4x_3 + x_4 \leq 10$
$3x_1 - 2x_2 + 4x_3 + x_4 \leq 10$
$8x_1 - 3x_2 - 3x_3 + 2x_4 \leq 20$
and $x_1, x_2, x_3, x_4 \geq 0$

(iii) Max $Z = x_1 + 3x_2$
subject to $x_1 - x_2 \geq 1$
$3x_1 - x_2 \leq -3$
and $x_1, x_2 \geq 0$

(iv) Max $Z = 3x_1 + 2x_2$
subject to $2x_1 + x_2 \leq 2$
$3x_1 + 4x_2 \geq 12$
and $x_1, x_2 \geq 0$

[*Meerut, MSc (Maths), 1998*]

HINTS AND ANSWERS

1. (i) $x_1 = 3, x_2 = 1$ and Max $Z = 11$
(ii) $x_1 = 2, x_2 = 0$ and Max $Z = 10$
(iii) $x_1 = 0, x_2 = 100, x_3 = 230$ and Max $Z = 1{,}350$
(iv) Max $Z^* = -x_1 + 3x_2 - 2x_3$ where $Z^* = -Z$
$x_1 = 4, x_2 = 5, x_3 = 0$ and $Z^* = 11$.
(v) $x_1 = 50/7, x_2 = 0, x_3 = 55/7$ and Max $Z = 695/7$
(vi) $x_1 = 0, x_2 = 0, x_3 = 5$ and Max $Z = 5$
(vii) $x_1 = 0, x_2 = 0, x_3 = 1$ and Max $Z = 3$
(viii) $x_1 = 1, x_2 = 1, x_3 = 1/2$ and Max $Z = 13/2$
(ix) Divide the first equation by 3 (coefficient of x_4)
(x) $x_1 = 0, x_2 = 0$ and Max $Z = 200$
(xi) $x_1 = 0, x_2 = 6, x_3 = 4$ and Max $Z = 6$

2. (i) $x_1 = 1, x_2 = 0$, and Max $Z = 6$
(ii) All $c_j - z_j \leq 0$ but $Z = -5/4$ (< 0) and artificial variable $A_1 = 5/4$ appears in the basis with positive value. Thus the given LP problem has no feasible solution.

(iii) $x_1 = 5/4$, $x_2 = 0$, $x_3 = 0$ and Min $Z = 75/8$

(iv) $x_1 = 2$, $x_2 = 0$ and Max $Z = 6$

(v) $x_1 = 0$, $x_2 = 5$ and Max $Z = 40$

(vi) All $c_j - z_j \geq 0$, but $Z = -4$ (< 0) and artificial variable $A_1 = 4$ appears in the basis with a positive value. Thus the given LP problem has no feasible solution.

3. (i) $x_1 = 3$, $x_2 = 0$ and Max $Z = 9$

(ii) $x_1 = 3/5$, $x_2 = 6/5$ and Min $Z = 12/5$

(iii) All $c_j - z_j \geq 0$ artificial variable $A_1 = 0$ appears in the basis with zero value. Thus an optimal solution to the given LP problem exists.

(iv) Introduce artificial variable only in the third constraint.

$x_1 = 0, x_2 = 0, x_3 = 0, x_5 = 0$ and Max $Z = 4$.

(v) $x_1 = 4, x_2 = 5$ and Min $Z = -11$

(vi) $x_1 = 0, x_2 = 10, x_3 = 0$ and Min $Z = 20$

4. (i) $x_1 = 0$, $x_2 = 1$, $x_3 = 0$ and Max $Z = 3$

(ii) Degeneracy occurs at the initial stage. One of the variable eligible to leave the basis is artificial variable, therefore, there is no need of resolving degeneracy. Remove the artificial variable from the basis.

$x_1 = 23/3$, $x_2 = 5$, $x_3 = 0$ and Max $Z = 85/3$

(iii) $x_1 = 2$, $x_2 = 0$ and Max $Z = 10$

(iv) $x_1 = 100/3$, $x_2 = 50/3$, $x_3 = 50/3$ and Max $Z = 1,650$

(v) $x_1'' = 6/5$ or $x_1 = -6/5$ or $x_2'' = 6/5$ or $x_2 = -6/5$ and Max $Z = -48/5$.

5. (i) (a) $x_1 = 4$, $x_2 = 0$ and Max $Z = 24$

(b) $x_1 = 5/2$, $x_2 = 3$, and Max $Z = 24$

(ii) (a) $x_1 = 32/6$, $x_2 = 10/6$ and Min $Z = 24$

(b) $x_1 = 12$, $x_2 = 0$ and Min $Z = 24$

6. (i) At the current solution: $x_1 = 5$, $x_2 = 9$ and Max $Z = 15$, it may be observed that $c_2 - z_2 = 3/2$ but all elements in the second column are negative. Solution is unbounded.

(ii) At the current solution: $x_2 = 4$, $s_1 = 4$ and Max $Z = 24$, it may be observed that $c_2 - z_2 = 15$ but all elements in the second column are negative. Solution is unbounded.

(iii) At second best solution, $c_3 - z_3 = 113/3$ but all elements in the third column are negative. Solution is unbounded; $x_1 = 0$, $x_4 = 7/3$, $s_1 = 5$ and Max $Z = 0$.

(iv) Optimal solution: $x_1 = 4$, $x_2 = 6$ and Max $Z = 12$. Since in the initial simplex table all the elements are negative in the second column, the feasible solution is unbounded but the optimal solution is bounded.

7. (i) $x_1 = 2, x_2 = 0, A_1 = 2$ and M Max $Z = 9 - 4M$; Infeasible solution.

(iv) $x_1 = 0, x_2 = 2, A_1 = 2$ and Max $Z = 4 - 4M$; Infeasible solution.

CHAPTER SUMMARY

The simplex method is an interactive procedure for reaching the optimal solution to any LP problem. It consists of a series of rules that, in effect, algebraically examine corner (extreme) points of the solution space in a systematic way. Each step moves towards the optimal solution by increasing profit or decreasing cost, while maintaining feasibility. The simplex method consists of five steps: (i) identifying the pivot column, (ii) identifying the pivot row and key element, (iii) replacing the pivot row, (iv) computing new values for each remaining row; and (v) computing the z_j and $c_j - z_j$ values and examining for optimality. Each step of this iterative procedure is displayed and explained in a simplex table both for maximization and minimization LP problems.

CHAPTER CONCEPTS QUIZ

True or False

1. The major difference between slack and artificial variables is that an artificial can never be zero.
2. If an optimal solution is degenerate, then there are alternate optimal solutions of the LP problem.
3. An infeasible solution is characterized as one where one constraint is violated.
4. If the objective function coefficient in the c_j row above an artificial variable is $-$ M, then the problem is a minimization problem.
5. In the simplex method, the initial solution contains only slack variable in the product mix.
6. If there is a tie between decision variable and a slack (or surplus) variable, then select the decision variable to enter into the first basis.
7. An optimal solution to the maximized LP problem is reached if all $c_j - z_j \geq 0$.
8. Variables which can assume negative, positive or zero value are called unrestricted variables.
9. All the rules and procedures of the simplex method are identical whether solving a maximization or minimization LP problem.
10. Artificial variables are added to a linear programming problem to aid in the finding an optimal solution.

Fill in the Blanks

11. In the simplex method, the ———— column contains the variables which are currently in the solution; the values of these variables can be read from the ———— column.
12. A ———— variable represents amounts by which solution values exceed a resource.
13. The simplex method examines the ———— points in a systematic manner, repeating the same set of steps of the algorithm until an ———— solution is reached.
14. ———— occurs when there is no solution that satisfies all of the constraints in the linear programming problem.

15. The value for the replacing row must be ———— before computing the values for the ———— rows.
16. Entries in the $c_j - z_j$ rows are known as ———— costs.
17. Optimality is indicated for a maximization problem when all elements in the $c_j - z_j$ rows are ————, while for a minimization problem all elements must be ————.
18. In Big-M method, ———— basic feasible solution is obtained by assigning ———— value to the original value.
19. In the two-phase method, an ———— variable is never considered for re-entry into the basis.
20. ———— occurs when there is no finite solution in the LP problem.

Multiple Choice

21. The role of artificial variables in the simplex method is
    (a) to aid in finding an initial solution
    (b) to find optimal dual prices in the final simplex table
    (c) to start phases of simplex method
    (d) all of the above
22. For a maximization problem, the objective function coefficient for an artificial variable is
    (a) + *M* (b) – *M*
    (c) *Zero* (d) none of the above
23. If a negative value appears in the solution values ($\mathbf{x_B}$) column of the simplex table, then
    (a) the solution is optimal
    (b) the solution is infeasible
    (c) the solution is unbounded
    (d) all of the above
24. At every iteration of the simplex method, for minimization problem, a variable in the current basis is replaced with another variable that has
    (a) a positive $c_j - z_j$ value
    (b) a negative $c_j - z_j$ value
    (c) $c_j - z_j = 0$
    (d) none of the above
25. In the optimal simplex table, $c_j - z_j = 0$ value indicates
    (a) unbounded solution (b) cycling
    (c) alternative solution (d) infeasible solution
26. For maximization LP model, the simplex method is terminated when all values
    (a) $c_j - z_j \le 0$ (b) $c_j - z_j \ge 0$
    (c) $c_j - z_j = 0$ (d) $z_j \le 0$
27. A variable which does not appear in the basic variable (***B***) column of simplex table is
    (a) never equal to zero (b) always equal to zero
    (c) called a basic variable (d) none of the above
28. If for a given solution a slack variable is equal to zero then
    (a) the solution is optimal (b) the solution is infeasible
    (c) the entire amount of resource with the constraint in which the slack variable appears has been consumed.
    (d) all of the above
29. If an optimal solution is degenerate, then
    (a) there are alternative optimal solutions
    (b) the solution is infeasible
    (c) the solution is of no use to the decision-maker
    (d) none of the above
30. To formulate a problem for solution by the simplex method, we must add artificial variable to
    (a) only equality constraints
    (b) only 'greater than' constraints
    (c) both (a) and (b)
    (d) none of the above
31. If any value in x_B – Column of final simplex table is negative, then the solution is
    (a) unbounded (b) infeasible
    (c) optimal (d) none of the above
32. If all a_{ij} values in the incoming variable column of the simplex table are negative, then
    (a) solution is unbounded (b) there are multiple solutions
    (c) there exist no solution (c) the solution is degenerate
33. If an artificial variable is present in the 'basic variable' column of optimal simplex table, then the solution is
    (a) infeasible (b) unbounded
    (c) degenerate (d) none of the above
34. The per unit improvement in the solution of the minimization LP problem is indicated by the negative value of
    (a) $c_j - z_j$ in the decision-variable column
    (b) $c_j - z_j$ in the slack variable column
    (c) $c_j - z_j$ in the surplus variable column
    (d) none of the above
35. To convert ≥ inequality constraints into equality constraints, we must
    (a) add a surplus variable
    (b) subtract an artificial variable
    (c) substract a surplus variable and an artificial variable
    (d) add a surplus variable and substract an artificial variable

Answers to Quiz

1. F 2. F 3. T 4. F 5. F 6. T 7. T 8. T 9. F 10. T
11. product mix; quantity 12. Suplus 13. Extreme; optimal 14. Infeasibility 15. Computed; remaining
16. reduced 17. non positive, non negative 18. Initial; zero 19. Artificial 20. Unboundedness
21. (d) 22. (b) 23. (b) 24. (b) 25. (c) 26. (a) 27. (b) 28. (c) 29. (d) 30. (c)
31. (b) 32. (a) 33. (a) 34. (d) 35. (c)

CASE STUDY

Case 4.1: State Electricity Board*

The state electricity board is planning to construct a new plant for the next 10 years. It is possible to construct four types of electric power facilities-steam plants using coal for energy, hydroelectric plants with no reservoir, hydroelectric plants with small reservoirs (enough water storage capacity to meet daily fluctuations), and hydroelectric plants with large reservoirs (with enough water storage to meet seasonal fluctuations in power demands and waters flow).

Consumption of electricity is based on three characteristics: The first is the total annual usage – the requirement in the area is estimated to be 4,000 billion kilowatt-hours by the 10th year. The second characteristic is the peak usage of power – usually on a hot summer day at about 2 PM. Any plan should provide enough peaking capacity to meet a projected peak need of 3,000 million kilowatts in the 10th year. The third characteristic is guaranteed power output-measured as the averaged daylight output in midwinter when the consumption is high and water levels for hydroelectric power are low. The 10-year requirement is for 2,000 million kilowatts of guaranteed power.

The various possible power plants vary in terms of how they can satisfy characteristics. For example, hydroelectric plants with reservoirs are able to provide substantial peaking capacity, whereas steam plants and hydroelectric plants with no reservoirs are poor in this respect.

The characteristics of the various types of plants are shown in the table below. Each is measured in terms of a unit of capacity. The unit of capacity is defined to be the capacity to produce I billion kilowatt-hours per year. Note that the types of plants vary substantially in their investment costs. The annual operating costs of the various types of plants also vary considerably. For example, the cost of coal makes the annual costs of the steam plants quite high, whereas the annual costs of operating the hydroelectric plants are relatively less. The final column in the table shows the discounted total costs, including both the investments costs and the discounted annual operating costs.

Characteristics of Electric Plants per unit (1 billion kilowatt-hours) of Annual Output

Type	*Guaranteed Output (millions of kilowatts)*	*Peak Output (million of kilowatt)*	*Investment Cost (Rs '000)*	*Discounted Total Cost (Rs '000)*
Steam	0.15	0.20	1200	2600
Hydroelectric: no reservoir	0.10	0.10	1600	1680
Hydroelectric: small reservoir	0.10	0.40	2400	2560
Hydroelectric: large reservoir	0.80	0.90	4000	4400

Questions for Discussion

(a) Help the company in developing a 10-year plan that would detail the capacity of each type of plant that it should build.

(b) Develop an LP model and solve it to minimize the total discounted cost. However, there is a restriction that no more than Rs 14,000 million can be used for investment in plants over the 10 years.

Case 4.2: Nationwide Air lines**

Nationwide Airlines, faced with a sharply escalating cost of jet fuel, is interested in optimizing its purchase of jet fuel at its various locations around the country. Typically, there is some choice concerning the amount of fuel that can be placed on board any aircraft for any flight segment, as long as minimum and maximum limits are not violated. The flight schedule is considered as a chain of flight segments, or legs, that each aircraft follows. The schedule ultimately returns the aircraft to its starting point, resulting in a 'rotation'. Consider the following rotation: Delhi – Hyderabad – Cochin – Chennai – Delhi

* This case is based on, P. Masse and R. Gibrat, *Application of Linear Programming to Investments in Electric Power Industry*, Management Science, Jan 1957.

** This case is based on 'D. Wayne Darnell and Carln Loflin, *National Airlines Fuel Management and Allocation Model*, Interfaces' Feburary, 1977.

Fuel Requirements and Limits (1,000 gallons unless specified otherwise)

City	*Flight Sequence*	*Minimum Fuel Required*	*Maximum Fuel Allowed*	*Regular Fuel Consumption if Minimum Fuel Boarded*	*Additional Fuel Burned per Gallon of Tankered Fuel (i.e. fuel above minimum – in gallons)*	*Price per Gallons (Rs)*
1	Delhi to Hyderabad	23	33	12.1	0.040	8,200
2	Hyderabad to Cochin	8	19	2.0	0.005	7,500
3.	Cochin to Chennai	19	33	9.5	0.025	7,700
4.	Chennai to Delhi	25	33	13.0	0.045	8,900

The fuel for any one of these flight segments may be bought at its departure city, or it may be purchased at a previous city in the sequence and 'tankered' for the flight. Of course, it takes fuel to carry fuel, and thus an economic trade-off between purchasing fuel at the lowest-cost location and tankering it all around the country must be made.

In the table above the column 'Regular Fuel Consumption' takes into account fuel consumption if the minimum amount of fuel is on board, and the column 'Additional Fuel Burned' indicates the additional fuel burned in each flight segment per gallon of 'tankered' fuel carried; tankered fuel refers to fuel above the minimum amount.

The fuel originally carried into Delhi should equal fuel carried into Delhi on the next rotation, in order for the system to be in equilibrium.

If l_1= Leftover fuel inventory coming into city i (1,000 gallons) and x_i = Amount of fuel purchased at city i(1,000 gallons) then, $(l_i + x_i)$ is the amount of fuel on board the aircraft when it departs city i.

Questions for Discussion

Develop an LP model and solve it to suggest the optimal fuel quantity to be purchased by the Airlines.

chapter 5

Duality in Linear Programming

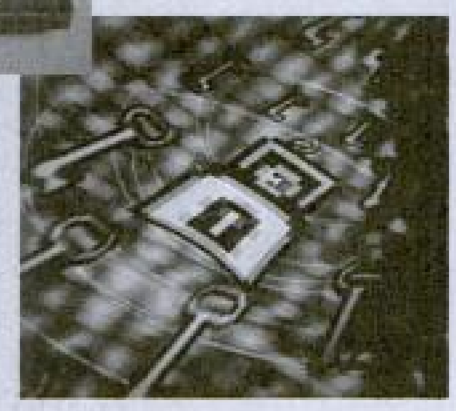
"A boat can't have two captains."

– Akira Mori

Preview The chapter deals with how to find the marginal value (also known as shadow price) of each resource. This in turn helps the resource to be best utilized.

Learning Objectives After studying this chapter, you should be able to

- appreciate the significance of the duality concept.
- formulate the dual LP problem and understand the relationship between primal and dual LP problems.
- understand the concept of shadow prices.



Chapter Outline

5.1 INTRODUCTION

The term 'dual' in a general sense implies two or double. The concept of duality is very useful in mathematics, physics, statistics, engineering and managerial decision-making. For example, in a two-person game theory, one competitor's problem is the dual of the opponent's problem.

In the context of linear programming, duality implies that each linear programming problem can be analyzed in two different ways but would have equivalent solutions. Any LP problem (either maximization and minimization) can be stated in another equivalent form based on the same data. The new LP problem is called *dual linear programming problem or in short dual*. In general, it is immaterial which of the two problems is called primal or dual, since the dual of the dual is primal.

Dual LP problem provides useful economic information about worth of resources to be used

For example, consider the problem of production planning. By using primal LP problem, the production manager attempts to optimize resource allocation by determining quantities for each product to be produced that will maximize profit. But through a dual LP problem approach, he attempts to achieve a production plan that optimizes resource allocation in a way that each product is produced at that quantity so that its *marginal opportunity cost equals its marginal return*. Thus, the *main focus of a dual problem is to find for each resource its best marginal value* (*also called shadow price*). This value reflects the scarcity of the resources, i.e. the maximum additional prices to be paid to obtain one additional unit of the resources in order to maximize profit under the resource constraints. If a resource is not completely used, i.e. there is slack, then its marginal profit is zero.

The shadow price is also defined as the rate of change in the optimal objective function value with respect to the unit change in the availability of a resource. To be more precise for any constraint, we have

$$\text{Shadow price} = \frac{\text{Change in optimal objective function value}}{\text{Unit change in the availability of resource}}$$

The interpretation of rate of change (increase or decrease) in the value of objective function depends on whether we are solving a maximization or minimization LP problem. The shadow price for a less than or equal to ($\leq$) type constraint will always be greater than or equal to zero. This is because increasing the right-hand side resource value cannot make the value of objective function worse. Similarly, the shadow price for a greater than or equal to ($\geq$) type constraint will always be less than or equal to zero because increasing the right-hand side resonance value cannot improve the value of the objective function.

The format of the simplex method is such that solving one type of problem is equivalent to solving the other simultaneously. Thus, if the optimal solution to one is known, the optimal solution of the other can also be read from the $c_j - z_j$ row of the final simplex table. In some cases, considerable computing time can be reduced by solving the dual.

5.2 FORMULATION OF DUAL LINEAR PROGRAMMING PROBLEM

There are two important forms of primal and dual problems, namely the *symmetrical (canonical) form* and the *standard form*.

Shadow price represents increase in the objective function value due to one-unit increase in the right hand side (resource) of any constraint

5.2.1 Symmetrical Form

Suppose the *primal LP problem* is given in the form

Maximize $Z_x = c_1 x_1 + c_2 x_2 + \ldots + c_n x_n$

subject to the constraints

$$\begin{aligned} a_{11} x_1 + a_{12} x_2 + \ldots + a_{1n} x_n &\leq b_1 \\ a_{21} x_1 + a_{22} x_2 + \ldots + a_{2n} x_n &\leq b_2 \\ \vdots \qquad\qquad & \quad \vdots \\ a_{m1} x_1 + a_{m2} x_2 + \ldots + a_{mn} x_n &\leq b_m \end{aligned}$$

and $\qquad x_1, x_2 \ldots, x_n \geq 0$

Then the corresponding *dual LP problem* is defined as:

image
not
available

or $\sum_{j=1}^{n}$ (Units of resource i, consumed per unit of variable, x_j) (Units of variable x_j) $\leq$ Units of resource, i available

and $x_j \geq 0$, for all j

Dual LP Problem

Minimize (cost) $Z_y = \sum_{i=1}^{m} b_i y_i = \sum_{i=1}^{m}$ (Units of resource, i) (Cost per unit of resource, i)

subject to the constraints

$$\sum_{i=1}^{m} a_{ji}\, y_i \geq c_j$$

or $\sum_{i=1}^{m}$ (Units of a resource, j consumed per unit of variable y_i) (Cost per unit of resource, i) $\geq$ Profit per unit for each activity (variable) x_j

and $y_i \geq 0$, for all i

From these expressions of parameters of both primal and dual problems, it is clear that for the unit of measurement to be consistent, *the dual variable (y_i) must be expressed in terms of return (or worth) per unit of resource i.* This is called *dual price* (*simplex multiplier* or *shadow price*) of resource i. In other words, optimal value of a dual variable associated with a particular primal constraint indicates the *marginal change (increase, if positive or decrease, if negative) in the optimal value of the primal objective function from increasing (or decreasing) in the right-hand side value (resource) of a constraint.* For example, if $y_2 = 5$, then this indicates that for every additional unit (up to a certain limit) of resource 2 (resource associated with constraint 2 in the primal), the objective function value will increase by 5 units. The value $y_2 = 5$ is called the marginal (or shadow or implicit) value or price of resource 2.

Similarly, for any two *feasible* primal and dual solutions, a finite value of objective function satisfy the inequality $Z_x \leq Z_y$. This inequality is interpreted as: *Profit* $\leq$ *Worth of resources*. Thus, so long as the total profit (return) from all activities is less than the worth of the resources, the solution of both primal and dual are not optimal. The optimality (maximum profit or return) is reached only when the resources have been completely utilized. This is only possible if the worth of the resources (i.e. input) is equal to profit (i.e. output).

5.2.3 Economic Interpretation of Dual Constraints

As stated earlier, the dual constraints are expressed as:

$$\sum_{i=1}^{m} a_{ji} y_i - c_j \geq 0$$

Since c_j represents profit (in Rs) per unit of activity x_j, therefore the LHS quantity $\Sigma\, a_{ji} y_i$ should also be in rupees per unit but it represents cost (due to profit with negative sign). Since coefficients a_{ji} represents the amount of resource b_i consumed by per unit of activity x_j, and the dual variable y_i represents shadow price per unit of resource b_i, the quantity $\Sigma\, a_{ji} y_i$ $(= z_j)$ should be total shadow price of all resources required to produce one unit of activity x_j.

As we know that for a maximization LP problem, if $c_j - z_j > 0$ corresponds to any non-basic (unused) activity (or variable), then the value of objective function can be increased. This implies that a variable, x_j value can be increased from zero to a positive level provided its unit profit (c_j) is more than its shadow price, i.e.

$$\sum_{i=1}^{m} a_{ji} y_i \leq c_j$$

(Shadow price of resources used per unit of activity, x_j) $\leq$ Profit per unit of activity x_j

5.2.4 Rules for Constructing the Dual from Primal

The rules for constructing the dual from the primal or primal from the dual, when using the symmetrical form are:

1. A dual variable is defined for each constraint in the primal LP problem and vice versa. Thus, in a given a primal LP problem with m constraints and n variables, there exists a dual LP problem with m variables and n constraints and vice-versa.

2. The right-hand side constants $b_1, b_2, \ldots, b_m$ of the primal LP problem becomes the coefficients of the dual variables $y_1, y_2, \ldots, y_m$ in the dual objective function Z_y. Also the coefficients $c_1, c_2, \ldots, c_n$ of the primal variables $x_1, x_2, \ldots, x_n$ in the objective function become the right-hand side constants in the dual LP problem.
3. For a maximization primal LP problem with all ≤ (less than or equal to) type constraints, there exists a minimization dual LP problem with all ≥ (greater than or equal to) type constraints and vice versa. Thus, the inequality sign is reversed in all the constraints except the non-negativity conditions.
4. The matrix of the coefficients of variables in the constraints of dual is the transpose of the matrix of coefficients of variables in the constraints of primal and vice versa, i.e. coefficients of the primal variables $x_1, x_2, \ldots, x_n$ in the constraints of a primal LP problem are the coefficients of dual variables in first, second, ..., nth, constraints for the dual problem, respectively.
5. If the objective function of a primal LP problem is to be maximized, the objective function of the dual is to be minimized and vice versa.
6. If the ith primal constraint is = (equality) type, then the ith dual variables is unrestricted in sign and vice versa.

The primal-dual relationships may also be remembered conveniently by using the following table:

Dual Variables	*Primal Variables*						*Maximize* Z_x
	x_1	x_2	...	x_j	...	x_n	
y_1	a_{11}	a_{12}	...	a_{1j}	...	a_{1n}	$\le b_1$
y_2	a_{21}	a_{22}	...	a_{2j}	...	a_{2n}	$\le b_2$
$\vdots$						$\vdots$	$\vdots$
y_m	a_{m1}	a_{m2}	...	a_{mj}	...	a_{mn}	$\le b_m$
Minimize Z_y	$\ge c_1$	$\ge c_2$	...	c_j	...	$\ge c_n$	Dual objective function coefficients

jth dual constraint

The primal constraints should be read across the rows, and the dual constraints should be read across the columns.

Example 5.1 Write the dual to the following LP problem

Maximize $Z = x_1 - x_2 + 3x_3$

subject to the constraints

(i) $x_1 + x_2 + x_3 \le 10$, (ii) $2x_1 - x_2 - x_3 \le 2$, (iii) $2x_1 - 2x_2 - 3x_3 \le 6$

and $x_1, x_2, x_3 \ge 0$

Solution In the given LP problem there are $m = 3$ constraints and $n = 3$ variables. Thus, there must be $m = 3$ dual variables and $n = 3$ constraints. Further, the coefficients of the primal variables, $c_1 = 1$, $c_2 = -1$, $c_3 = 3$ become right-hand side constants of the dual. The right-hand side constants $b_1 = 10$, $b_2 = 2$, $b_3 = 6$ become the coefficients in the dual objective function. Finally, the dual must have a minimizing objective function with all ≥ type constraints. If y_1, y_2 and y_3 are dual variables corresponding to three primal constraints in the given order, the resultant dual is

Minimize $Z_y = 10y_1 + 2y_2 + 6y_3$

subject to the constraints

(i) $y_1 + 2y_2 + 2y_3 \ge 1$, (iii) $y_1 - y_2 - 2y_3 \ge -1$, (iii) $y_1 - y_2 - 3y_3 \ge 3$

and $y_1, y_2, y_3 \ge 0$

Example 5.2 Write the dual of the following LP problem

Minimize $Z_x = 3x_1 - 2x_2 + 4x_3$

subject to the constraints

(i) $3x_1 + 5x_2 + 4x_3 \ge 7$, (ii) $6x_1 + x_2 + 3x_3 \ge 4$, (iii) $7x_1 - 2x_2 - x_3 \le 10$

(iv) $x_1 - 2x_2 + 5x_3 \ge 3$, (v) $4x_1 + 7x_2 - 2x_3 \ge 2$

and $x_1, x_2, x_3 \ge 0$

Solution Since the objective function of the given LP problem is of minimization, the direction of each inequality has to be changed to ≥ type by multiplying both sides by – 1. The standard primal LP problem so obtained is:

Minimize $Z_x = 3x_1 - 2x_2 + 4x_3$

subject to the constraints

(i) $3x_1 + 5x_2 + 4x_3 \geq 7$, (ii) $6x_1 + x_2 + 3x_3 \geq 4$, (iii) $-7x_1 + 2x_2 + x_3 \geq -10$

(iv) $x_1 - 2x_2 + 5x_3 \geq 3$, (v) $4x_1 + 7x_2 - 2x_3 \geq 2$

and $x_1, x_2, x_3 \geq 0$

If y_1, y_2, y_3, y_4 and y_5 are dual variables corresponding to the five primal constraints in the given order, the dual of this primal LP problem is stated as:

Maximize $Z_y = 7y_1 + 4y_2 - 10y_3 + 3y_4 + 2y_5$

subject to the constraints

(i) $3y_1 + 6y_2 - 7y_3 + y_4 + 4y_5 \leq 3$, (ii) $5y_1 + y_2 + 2y_3 - 2y_4 + 7y_5 \leq -2$

(iii) $4y_1 + 3y_2 + y_3 + 5y_4 - 2y_5 \leq 4$

and $y_1, y_2, y_3, y_4, y_5 \geq 0$

Example 5.3 Obtain the dual problem of the following primal LP problem:

Minimize $Z = x_1 + 2x_2$

subject to the constraints

(i) $2x_1 + 4x_2 \leq 160$, (ii) $x_1 - x_2 = 30$, (iii) $x_1 \geq 10$

and $x_1, x_2 \geq 0$

Solution Since the objective function of the primal LP problem is of minimization, change all ≤ type constraints to ≥ type constraints by multiplying the constraint on both sides by –1. Also write = type constraint equivalent to two constraints of the type ≥ and ≤. Then the given primal problem can be written as:

Minimize $Z_x = x_1 + 2x_2$

subject to the constraint

(i) $-2x_1 - 4x_2 \geq -160$, (ii) $x_1 - x_2 \geq 30$

(iii) $x_1 - x_2 \leq -30$ or $-x_1 + x_2 \geq -30$, (iv) $x_1 \geq 10$

and $x_1, x_2 \geq 0$

Let y_1, y_2, y_3 and y_4 be the dual variables corresponding to the four constraints in the given order. The dual of the given primal problem can then be formulated as follows:

Maximize $Z_y = -160y_1 + 30y_2 - 30y_3 + 10y_4$

subject to the constraints

(i) $-2y_1 + y_2 - y_3 + y_4 \leq 1$, (ii) $-4y_1 - y_2 + y_3 \leq 2$

and $y_1, y_2, y_3, y_4 \geq 0$

Let $y = y_2 - y_3$ ($y_2, y_3 \geq 0$). The above dual problem then reduces to the form

Maximize $Z_y = -160y_1 + 30y + 10y_4$

subject to the constraints

(i) $-2y_1 + y + y_4 \leq 1$, (ii) $-4y_1 - y \leq 2$

and $y_1, y_4 \geq 0$; y being unrestricted in sign

Remark To apply rule 6, note that the second constraint in the primal is equality, therefore the corresponding second dual variable y (= $y_2 - y_3$) should be unrestricted in sign.

Example 5.4 Obtain the dual problem of the following primal LP problem:

Minimize $Z_x = x_1 - 3x_2 - 2x_3$

subject to the constraints

(i) $3x_1 - x_2 + 2x_3 \leq 7$, (ii) $2x_1 - 4x_2 \geq 12$, (iii) $-4x_1 + 3x_2 + 8x_3 = 10$

and $x_1, x_2 \geq 0$; x_3 unrestricted in sign.

Solution Since x_3 is an unrestricted variable, therefore, it can be expressed as the difference of two non- negative variables, i.e. $x_3 = x_3' - x_3''$, $x_3', x_3'' \geq 0$. The given LP problem can then be written as:

Minimize $Z_x = x_1 - 3x_2 - 2(x_3' - x_3'')$

subject to the constraints

(i) $3x_1 - x_2 + 2(x_3' - x_3'') \leq 7$ or $-3x_1 + x_2 - 2(x_3' - x_3'') \geq -7$

(ii) $2x_1 - 4x_2 \geq 12$, (iii) $-4x_1 + 3x_2 + 8(x_3' - x_3'') = 10$

and $x_1, x_2, x_3', x_3'' \geq 0$

Let y_1, y_2 and y_3 be the dual variables corresponding to three primal constraints in the given order. As the given problem is of minimization, all constraints can be converted to $\geq$ type by multiplying both sides by -1. Since the third constraint of the primal is an equation, the third dual variable y_3 will be unrestricted in sign. Now the dual of the given primal can be formulated as follows:

Maximize $Z_y = -7y_1 + 12y_2 + 10y_3$

subject to the constraints

(i) $-3y_1 + 2y_2 - 4y_3 \leq 1$, (ii) $y_1 - 4y_2 + 3y_3 \leq -3$, (iii) $-2y_1 + 8y_3 \leq -2$

and $y_1, y_2 \geq 0$; y_3 unrestricted in sign.

Example 5.5 Obtain the dual of the following primal LP problem

Maximize $Z_x = x_1 - 2x_2 + 3x_3$

subject to the constraints

(i) $-2x_1 + x_2 + 3x_3 = 2$, (ii) $2x_1 + 3x_2 + 4x_3 = 1$

and $x_1, x_2, x_3 \geq 0$

Solution Since both the primal constraints are of the equality type, the corresponding dual variables y_1 and y_2, will be unrestricted in sign. Following the rules of duality formulation, the dual of the given primal LP problem is

Minimize $Z_y = 2y_1 + y_2$

subject to the constraints

(i) $-2y_1 + 2y_2 \geq 1$, (ii) $y_1 + 3y_2 \geq -2$, (iii) $3y_1 + 4y_2 \geq 3$

and y_1, y_2 unrestricted in sign.

Example 5.6 Write the dual of the following primal LP problem

Maximize $Z = 3x_1 + x_2 + 2x_3 - x_4$

subject to the constraints

(i) $2x_1 - x_2 + 3x_3 + x_4 = 1$, (ii) $x_1 + x_2 - x_3 + x_4 = 3$

and $x_1, x_2 \geq 0$ and x_3, x_4 unrestricted in sign.

Solution Here we may apply the following rules of forming a dual of the given primal LP problem.

(i) The x_3 and x_4 variables in the primal are unrestricted in sign therefore, the third and fourth constraints in the dual shall be equalities.

(ii) The given primal problem is of maximization; the first two constraints in the dual will therefore be $\geq$ type constraints.

(iii) Since both the constraints in the primal are equalities, the corresponding dual variables y_1 and y_2 will be unrestricted in sign.

If y_1 and y_2 are dual variables corresponding to the two primal constraints in the given order, the dual of the given primal can be written as:

Minimize $Z_y = y_1 + 3y_2$
subject to the constraints

(i) $2y_1 + y_2 \geq 3$, (ii) $-y_1 + y_2 \geq 1$, (iii) $3y_1 - y_2 = 2$
(iv) $y_1 + y_2 = -1$

and y_1, y_2 unrestricted in sign.

SELF PRACTICE PROBLEMS A

Write the dual of the following primal LP problems

1. Max $Z_x = 2x_1 + 5x_2 + 6x_3$
   subject to (i) $5x_1 + 6x_2 - x_3 \leq 3$
   (ii) $-2x_1 + x_2 + 4x_3 \leq 4$
   (iii) $x_1 - 5x_2 + 3x_3 \leq 1$
   (iv) $-3x_1 - 3x_2 + 7x_3 \leq 6$
   and $x_1, x_2, x_3 \geq 0$
   [*Sambalpur MSc (Maths), 1996*]
2. Min $Z_x = 7x_1 + 3x_2 + 8x_3$
   subject to (i) $8x_1 + 2x_2 + x_3 \geq 3$
   (ii) $3x_1 + 6x_2 + 4x_3 \geq 4$
   (iii) $4x_1 + x_2 + 5x_3 \geq 1$
   (iv) $x_1 + 5x_2 + 2x_3 \geq 7$
   and $x_1, x_2, x_3 \geq 0$
3. Max $Z_x = 2x_1 + 3x_2 + x_3$
   subject to (i) $4x_1 + 3x_2 + x_3 = 6$, (ii) $x_1 + 2x_2 + 5x_3 = 4$
   and $x_1, x_2, x_3 \geq 0$
4. Max $Z_x = 3x_1 + x_2 + 3x_3 - x_4$
   subject to (i) $2x_1 - x_2 + 3x_3 + x_4 = 1$
   (ii) $x_1 + x_2 - x_3 + x_4 = 3$
   and $x_1, x_2, x_3, x_4 \geq 0$
5. Min $Z_x = 2x_1 + 3x_2 + 4x_3$
   subject to (i) $2x_1 + 3x_2 + 5x_3 \geq 2$
   (ii) $3x_1 + x_2 + 7x_3 = 3$
   (iii) $x_1 + 4x_2 + 6x_3 \leq 5$
   and $x_1, x_2, \geq 0, x_3$ is unrestricted
6. Min $Z_x = x_1 + x_2 + x_3$
   subject to (i) $x_1 - 3x_2 + 4x_3 = 5$, (ii) $x_1 - 2x_2 \leq 3$
   (ii) $2x_2 - x_3 \geq 4$
   and $x_1, x_2 \geq 0$, x_3 is unrestricted.
   [*Meerut, MSc (Maths), 1997*]
7. Max $Z_x = 8x_1 + 3x_2$
   subject to (i) $x_1 - 6x_2 \geq 2$, (ii) $5x_1 + 7x_2 = -4$
   and $x_1, x_2 \geq 0$.
8. Max $Z_x = 8x_1 + 8x_2 + 8x_3 + 12x_4$
   subject to (i) $30x_1 + 20x_2 + 25x_3 + 40x_4 \leq 800$
   (ii) $25x_1 + 10x_2 + 7x_3 + 15x_4 \leq 250$
   (iii) $4x_1 - x_2 = 0$
   $x_3 \geq 5$
   and $x_1, x_2, x_3, x_4 \geq 0$
9. Min $Z_x = 18x_1 + 10x_2 + 11x_3$
   subject to (i) $4x_1 + 6x_2 + 5x_3 \geq 480$
   (ii) $12x_1 + 10x_2 + 10x_3 \geq 1{,}200$
   (iii) $10x_1 + 15x_2 + 7x_3 \leq 1{,}500$
   (iv) $x_3 \geq 50$
   (v) $x_1 - x_2 \leq 0$
   and $x_1, x_2, x_3 \geq 0$
10. Min $Z_x = 2x_1 - x_2 + 3x_3$
    subject to (i) $x_1 + 2x_2 + x_3 \geq 12$
    (ii) $x_2 - 2x_3 \geq -6$
    (iii) $6 \leq x_1 + 2x_2 + 4x_3 \geq 24$
    and $x_1, x_2 \geq 0$, x_3 unrestricted.

HINTS AND ANSWERS

1. Min $Z_y = 3y_1 + 4y_2 + y_3 + 6y_4$
   subject to (i) $5y_1 - 2y_2 + 5y_3 - 3y_4 \geq 2$
   (ii) $6y_1 + y_2 - 5y_3 - 3y_4 \geq 5$
   (iii) $-y_1 + 4y_2 + 3y_3 + 7y_4 \geq 6$
   and $y_1, y_2, y_3, y_4 \geq 0$
2. Max $Z_y = 3y_1 + 4y_2 + y_3 + 7y_4$
   subject to (i) $8y_1 + 3y_2 + 4y_3 + y_4 \leq 7$
   (ii) $2y_1 + 6y_2 + y_3 + 5y_4 \leq 3$
   (iii) $y_1 + 4y_2 + 5y_3 + 2y_4 \leq 8$
   and $y_1, y_2, y_3, y_4 \geq 0$
3. Min $Z_y = 6y_1 + 4y_2$
   subject to (i) $4y_1 + y_2 \geq 2$, (ii) $3y_1 + 2y_2 \geq 3$
   (iii) $y_1 + 5y_2 \geq 1$
   and y_1, y_2 unrestricted in sign.
4. Min $Z_y = y_1 + 3y_2$
   subject to (i) $2y_1 + y_2 \geq 3$
   (ii) $-y_1 + y_2 \geq 1$
   (iii) $3y_1 - y_2 \geq 3$
   (iv) $y_1 + y_2 \geq -1$
   and y_1, y_2 unrestricted in sign.
5. Max $Z_y = 2y_1 + 3y_2 - 5y_3$
   subject to $2y_1 + 3y_2 - y_3 \leq 2$
   $3y_1 + y_2 - 4y_3 \leq 3$
   $5y_1 + 7y_2 - 6y_3 = 4$
   and $y_1, y_3 \geq 0$ and y_2 unrestricted.
6. Max $Z_y = -5y_1 - 3y_2 + 4y_3$
   subject to (i) $-y_1 - y_2 \leq 1$, (ii) $3y_1 + 2y_2 + 2y_3 \leq 1$
   $-4y_1 - y_3 \leq 1$
   and $y_2, y_3 \geq 0$ and y_1 is unrestricted.

7. Max $Z_y = 2y_1 - 4y_2$
subject to (i) $y_1 + 5y_2 \leq 8$, (ii) $-6y_1 + 7y_2 \leq 3$
and $y_1 \geq 0$ and y_2 is unrestricted.

8. Min $Z_y = 800y_1 + 250y_2 + y_3 + 5y_4$
subject to
$$30y_1 + 2y_2 + 4y_3 \geq 8$$
$$20y_1 + 10y_2 - y_3 \geq 8$$
$$25y_1 + 7y_2 + y_4 \geq 8$$
$$40y_1 + 15y_2 \geq 12$$
$y_1, y_2 \geq 0$ and y_3, y_4 are unrestricted.

5.3 STANDARD RESULTS ON DUALITY

See appendix for detail proof of the following standard results:

1. The dual of the dual LP problem is again the primal problem.
2. If either the primal or the dual problem has an unbounded objective function value, the other problem has no feasible solution.
3. If either the primal or dual problem has a finite optimal solution, the other one also possesses the same, and the optimal value of the objective functions of the two problems are equal, i.e. Max Z_x = Min Z_y. This analytical result is known as the *fundamental primal-dual relationship.* These results are summarized as follows.

Dual Problem (Max)	*Primal Problem (Min)*	
	Feasible	*Infeasible*
Feasible	Max Z_y = Min Z_x	Max $Z_y \to +\infty$
Infeasible	Min $Z_x \to -\infty$	Unbounded or infeasible

4. *Complementary slackness* property of primal-dual relationship states that for a positive basic variable in the primal, the corresponding dual variable will be equal to zero. Alternatively, for a non-basic variable in the primal (which is zero), the corresponding dual variable will be basic and positive.

5.3.1 Principle of Complementary Slackness

The principle of complementary slackness establishes the relationship between the optimal value of the main variables in one problem with their counterpart slack or surplus variables in other problem. Thus, this principle can also be helpful in obtaining the primal LP problem solution when only the dual solution is known. To illustrate this concept, let us consider the example of production planning.

1. If at the optimal solution of a primal problem, a primal constraint has a positive value of a slack variable, the corresponding resource is not completely used and must have zero opportunity cost (shadow price). This means that having more of this resource will not improve the value of the objective function. But if the value of slack variable is zero in that constraint, the entire resource is being used and must have a positive opportunity cost, i.e. additional resource will improve the value of objective function by allowing more production.

   Since resources are represented by slack variables in the primal and by main cost variables in the dual, therefore, the principle of complementary slackness states that for every resource, the following condition must hold:

   Primal slack variable × Dual main variable = 0

2. In cases where the resources are not completely used for producing any unit of a product, the opportunity cost of such resources exceeds the unit profit from that product. But, if few units of the product are made, the opportunity cost of resources used must equal the profit from each unit of the product, so that the surplus product cost is zero. A primal variable in primal LP problem represents the quantity of product made, and the surplus variables in dual represent surplus product costs. In this case the principle of complementary slackness states that for every unit of the product, the following condition must hold.

   Primal main variable × Dual surplus variable = 0

5.4 MANAGERIAL SIGNIFICANCE OF DUALITY

The importance of the dual LP problem is in terms of the information that it provides about the value of the resources. The economic analysis in this chapter is concerned about deciding whether or not one should secure more resources and how much should one pay for these additional resources.

The significance of the study of dual is as follows:

(i) The right-hand side of the constraint represents the amount of a resource available and the associated dual variable value is interpretated as the maximum amount likely to be paid for, for an additional unit of the resource.

However, such an interpretation is not always correct because of the two types of costs involved with each resource: (i) *sunk cost* that is not affected by the decision made, this will be incurred no matter what values the dual variables assume, and (ii) *relevant cost*; which depends on the decision made. This will vary with the values of decision variables.

(ii) The maximum amount that should be paid for one additional unit of a resource is called its *shadow price* (also called *simplex multiplier*).

(iii) The total marginal value of the resources equals the optimal objective function value. The dual variables equal the marginal value of resources (shadow prices).

(iv) The value of the *i*th dual variable represents the rate at which the primal objective function value will increase by increasing the right-hand side (resource value) of constraint i, assuming that all other data remain unchanged.

Remark When the cost of a resource is *sunk*, the shadow price represents the amount of money that is to be paid for the additional unit of a resource. When the cost of a resource is *relevant*, the shadow price represents the amount of money by which the value of the resource exceeds its cost. This implies that when the resource cost is relevant, the shadow price represents the extra money over the normal cost that is likely to be paid for one unit of the resource.

Example 5.7 The optimal solution simplex table for the primal LP problem

Maximize $Z = 3x_1 + 4x_2 + x_3$

subject to the constraints

(i) $x_1 + 2x_2 + 3x_3 \leq 90$ (time for operation 1), (ii) $2x_1 + x_2 + x_3 \leq 60$ (time for operation 2)

(iii) $3x_1 + x_2 + 2x_3 \leq 80$ (time for operation 3)

and $x_1, x_2, x_3 \geq 0$

is given below:

		$c_j \rightarrow$	3	4	1	0	0	0
Profit per Unit c_B	*Variables in Basis* B	*Solution Values* x_B (= b)	x_1	x_2	x_3	s_1	s_2	s_3
4	x_2	40	0	1	10/6	4/6	– 1/3	0
3	x_1	10	1	0	– 1/3	– 1/3	2/3	0
0	s_3	10	0	0	8/6	8/6	– 10/6	1
Z = 190		$c_j - z_j$	0	0	– 28/6	– 10/6	– 2/3	0

(a) Find the solution, maximum profit, idle capacity and the loss of the total contribution of every one unit reduced from the right-hand side of the constraints.

(b) Write the dual of the given problem and give the initial simplex table.

Solution To illustrate the interpretation of the dual variables let us consider this example in a profit maximization production context with constraints on time input resource.

The primal LP problem in this example is concerned with maximizing the profit contribution from three products say A, B and C, while the dual will be concerned with evaluating the time used in the three operations for producing the three products.

The productive capacity of the three operations is a valuable resource to the firm; the production manager wonders whether it would be possible to place a monetary value on its worth. If yes, then how much? His problem can be solved along the following lines.

(a) The optimal solution of the primal problem can be read from the given table as:

$$x_1 = 10, x_2 = 40, x_3 = 0 \text{ and Max (profit) } Z = 3 \times 10 + 4 \times 40 + 1 \times 0 = \text{Rs } 190$$

Since the value of slack variable, s_3, in the optimal solution is 10, the idle capacity in operation 3 is 10 units.

Interpretation of dual variables The absolute values of the numbers in the $c_j - z_j$ row, of final simplex table, under the slack variables columns represent the values of dual variables, i.e. marginal value (shadow price) of resources, 1 and 2. The optimal solution of dual LP problem is:

$$y_1 = 10/6, y_2 = 2/3, y_3 = 0 \text{ and Mix } Z_y = 190$$

The value of y_1, y_2 and y_3 shows the worth of one hour in each operation. That is, the optimal value of the dual variable y_1 represents the per unit price (worth or marginal price) of the first resource (i.e. time available in operation 1), viz., Rs 10/6 or Rs 1.66. This means that marginal contribution of operation 1 to the total profit is Rs 1.66. In other words, if one productive unit is removed from the right-hand side of the first constraint, the total contribution would reduce by Rs 10/6, or Rs 1.66.

However, if one productive unit is removed up to the extent of idle capacity from the right-hand side of the third constraint (time available in operation 1), the total contribution would not be affected.

(b) Let y_1, y_2 and y_3 be the dual variables representing the marginal value of one unit of the resource that corresponds to constraint 1, 2 and 3, respectively of the primal. That is to say that, the firm pays per unit price of y_1, y_2 and y_3 for the time available in the three operations. The firm, however, wants to make sure that it makes as much profit as it would if it remained in business for itself. Then the dual of the given problem is stated below:

The shadow price is represented by the $c_j - z_j$ value under slack variable columns in the optimal simplex table

Minimize $Z = 90y_1 + 60y_2 + 80y_3$

subject to the constraints

(i) $y_1 + 2y_2 + 3y_3 \geq 3$, (ii) $2y_1 + y_2 + y_3 \geq 4$, (iii) $3y_1 + y_2 + 2y_3 \geq 1$

and $y_1, y_2, y_3 \geq 0$

Interpretation of dual constraints Suppose the competition dictates that the company should also produce product *C*. To see how can this be achieved, examine $c_3 - z_3$ value under column-x_3. Producing *C* will be economical only if $c_3 > z_3$. This is possible by either increasing the profit per unit c_3 or decreasing the imputed cost of the used resources z_3 $(= 3y_1 + y_2 + 2y_3)$. An increase in the unit profit may not be possible because the company wants to remain competitive in the market. A decrease in z_3 is possible as this requires making improvement in operation 3 by reducing the unit usage of operation times. Let t_1, t_2 and t_3 represent the proportions by which the unit times of three operations are reduced. The company then needs to determine t_1, t_2 and t_3 such that the new imputed cost, z_3 of three operations falls below the unit profit, i.e.

$$3\,(1-t_1)\,y_1 + (1-t_2)y_2 + 2\,(1-t_3)\,y_3 < 1$$

For given values of $y_1 = 5/3$, $y_2 = 2/3$ and $y_3 = 0$, we get

$$3\,(1-t_1)\,(5/3) + (1-t_2)\,(2/3) < 1 \quad \text{or} \quad 5t_1 + 2t_2 > 16$$

Thus, any values of t_1 and t_2 between 3 and 4 that satisfy $5t_1 + 2t_2 > 16$ should make the product profitable. However, this may not possible because this requires a reduction in the times of operation 1 and 2, which is impractical.

Example 5.8 XYZ Company has three departments – Assembly, Painting and Packing. The company can make three types of almirahs. An almirah of type I requires one hour of assembly, 40 minutes of painting and 20 minutes of packing time, respectively. Similarly, an almirah of type II needs 80 minutes, 20 minutes and one hour, respectively. The almirah of type III requires 40 minutes each of assembly, painting and packing time. The total time available at assembly, painting and packing departments is 600 hours, 400 hours and 800 hours, respectively. Determine the number of each type of almirahs that should be produced in order to maximize the profit. The unit profit for types I, II and III is Rs 40, 80 and 60, respectively.

Suppose that the manager of this XYZ Company is thinking of renting the production capacities of the three departments to another almirah manufacturer – ABC Company. ABC Company is interested in minimizing the rental charges. On the other hand, the XYZ Company would like to know the worth of production hours to them, in each of the departments, in order to determine the rental rates. (a) Formulate this problem as an LP problem and solve it to determine the number of each type of almirahs that should be produced by the XYZ Company in order to maximize its profit. (b) Formulate the dual of the primal LP problem and interpret your results. [*CA, May 1994; Delhi Univ., MBA, 1999, 2002*]

Mathematical formulation The production data given in the problem may be summarized as below:

Types of Almirah	*Number of Hours Required per Unit*			*Profit per Unit*
Assembly	*Painting*	*Packing*	*of Almirah (Rs)*	
I	1	2/3	1/3	40
II	4/3	1/3	1	80
III	2/3	2/3	2/3	60
Total availability (hrs)	600	400	800	

Let x_1, x_2 and x_3 be the number of units of the three types of almirahs, respectively, to be produced. The given problem can then be represented as an LP model as:

Maximize $Z = 40x_1 + 80x_2 + 60x_3$

subject to the constraints

(i) $x_1 + (4/3)x_2 + (2/3)x_3 \leq 600$, (ii) $(2/3)\,x_1 + (1/3)\,x_2 + (2/3)\,x_3 \leq 400$

(iii) $(1/3)\,x_1 + x_2 + (2/3)\,x_3 \leq 800$

and $x_1, x_2, x_3 \geq 0$

If ABC Company pays the per hour rental price of y_1, y_2 and y_3 for assembly, painting and packing departments, respectively, the company XYZ must get a total rent equal to: $600y_1 + 400y_2 + 800y_3$. Since company ABC wants to earn as much profit as possible, its objective is to

$$\text{Min (total rental charges) } Z_y = 600y_1 + 400y_2 + 800y_3$$

After knowing this rental value, the objective of ABC Company is to know what minimum offer it should give to XYZ. The given data show that an almirah of type I requires 1 assembly hour, 2/3 painting hour and 1/3 packing hour. The imputed cost (i.e. rent) of time used for making almirah I is: $y_1 + (2/3)\,y_2 + (1/3)\,y_3$. If ABC Company used that time to make almirah I, it would earn Rs $c_1 = 40$ in contribution to profit, and so it will not rent out the time unless: $y_1 + (2/3)\,y_2 + (1/3)\,y_3 \geq 40$. Similarly, the company will work for almirahs II and III.

Since the total rent of all the departments should be greater than or equal to the profit from one unit of the almirah, the dual objective function along with the constraints that determines for ABC, the value of the productive resources, can be written as:

Minimize (total rent) $Z_y = 600y_1 + 400y_2 + 800y_3$

subject to the constraints

(i) $y_1 + (2/3)\,y_2 + (1/3)\,y_3 \geq 40$, (ii) $(4/3)\,y_1 + (1/3)\,y_2 + y_3 \geq 80$

(iii) $(2/3)\,y_1 + (2/3)\,y_2 + (2/3)\,y_3 \geq 60$

and $y_1, y_2, y_3 \geq 0$

Here the dual objective function, i.e. the minimum total rent acceptable by the ABC Company is equal to the primal objective function, i.e. maximum profit that could be earned by XYZ Company from its resources. The dual variables y_1, y_2 and y_3 represent the rental rate of the various departments corresponding to the slack or unused capacity (called slack variables s_1, s_2, and s_3, respectively) of these departments and are called the *shadow prices or marginal profitability* of various departments.

Example 5.9 A firm manufactures two products A and B on machines I and II as shown below:

Machine	Product A	Product B	Available Hours
I	30	20	300
II	5	10	110
Profit per unit (Rs)	6	8	

The total time available is 300 hours and 110 hours on machines I and II, respectively. Products A and B contribute a profit of Rs 6 and Rs 8 per unit, respectively. Determine the optimum product mix. Write the dual of this LP problem and give its economic interpretation.

Mathematical formulation The primal and the dual LP problems of the given problem are

Primal problem

x_1 and x_2 = number of units of A and B to be produced, respectively

$$\text{Max } Z_x = 6x_1 + 8x_2$$

subject to the constraints

$$30x_1 + 20x_2 \le 300$$
$$5x_1 + 10x_2 \le 110$$

and $x_1, x_2 \ge 0$

Dual problem

y_1 and y_2 = cost of one hour on machines I and II, respectively

$$\text{Min } Z_y = 300y_1 + 110y_2$$

subject to the constraints

$$30y_1 + 5y_2 \ge 6$$
$$20y_1 + 10y_2 \ge 8$$

and $y_1, y_2 \ge 0$

Solution of the primal problem The optimal solution of the primal problem is given in Table 5.2.

		$c_j \rightarrow$	6	8	0	0
c_B	Variables in Basis B	Solution Values x_B (= b)	x_1	x_2	s_{1p}	s_{2p}
6	x_1	4	1	0	1/20	– 1/10
8	x_2	9	0	1	– 1/10	3/20
Z = 96		$c_j - z_j$	0	0	– 1/10	– 6/10

Table 5.2 Optimal Solution of Primal Problem

Table 5.2 indicates that the optimal solution is to produce: x_1 = 4 units of product A; x_2 = 9 units of product B and Z_x = total maximum profit, Rs 96.

Solution of the dual problem The optimal solution of the dual problem can be obtained by applying the Big-M method. The optimal solution is shown in Table 5.3.

		$b_i \rightarrow$	300	110	0	0
b_B	Variables in Basis B	Solution Values y_B	y_1	y_2	s_{1d}	s_{2d}
300	y_1	1/10	1	0	– 1/20	1/40
110	y_2	6/10	0	1	1/10	– 3/20
Z = 96		$b_i - z_j$	0	0	4	9

Table 5.3 Optimal Solution of Dual Problem

The optimal solution as given in Table 5.3 is: y_1 = Rs 1/10 per hour on machine I; y_2 = Rs 6/10 per hour on machine II and Z_y = total minimum cost, Rs 96.

The values of y_1 = Rs 1/10 and y_2 = Rs 3/5, indicate the worth of one hour of machine time of machines I and II, respectively. However, since the machines hours can be increased beyond a specific limit, this result holds true only for a specific range of hours available on machines I and II.

Comparison of the solutions For interpreting the optimal solution of the primal (or dual), its solution values can be read directly from the optimal simplex table of the dual (or primal). The method can be summarized in the following steps.

1. The slack variables in the primal, correspond to the dual basic variables in the optimal solution and vice versa. For example, s_{1p} is the slack variable of the first primal constraint. It corresponds to the first dual variable y_1. Likewise s_{2p} corresponds to the second dual variable y_2. Moreover, s_{1d} and s_{2d} are the dual surplus variables and they correspond to x_1 and x_2, respectively of the primal problem. The correspondence between primal and dual variables is summarized in Table 5.4.

Table 5.4

Primal	*Dual*
Main variables $\begin{cases} x_1 \\ x_2 \end{cases}$	$\left. \begin{matrix} s_{1d} \\ s_{2d} \end{matrix} \right\}$ Surplus variables
Slack variables $\begin{cases} s_{1p} \\ s_{2p} \end{cases}$	$\left. \begin{matrix} y_1 \\ y_2 \end{matrix} \right\}$ Main variables

2. The value in the $c_j - z_j$ row under columns of the slack/surplus variables, ignoring the negative sign, directly gives the optimal values of the dual/primal basic variables. For example, $c_3 - z_3 = 1/10$ and $c_4 - z_4 = 6/10$ values in Table 5.2 under primal slack s_{1p} and s_{2p} correspond to solution values of dual variables y_1 and y_2 in Table 5.3 and vice versa. The primal-dual relationship for these problems which are feasible can be summarized in Table 5.5.

Primal	*Dual*
• Values of the basic variables	• $(c_j - z_j)$ values in the non-basic surplus variable columns
• $(c_j - z_j)$ values in the non-basic slack variable columns	• Value of the basic variables

3. The optimal value of the objective function is the same for primal and dual LP problems.
4. The dual variable y_i represents the worth (dual price or shadow price) of one unit of resource i. For example, for y_1 the worth of time on machine I, equals Rs 1/10 and for y_2, the worth of time on machine II, equals Rs 6/10, as shown in Table 5.3.

Example 5.10 A company wishes to get at least 160 million 'audience exposures' the number of times one of the advertisements is seen or heard by a person. Because of the nature of the product the company wants at least 60 million month and at least 80 million of the exposures to involve persons between 18 and 40 years of age. The relevant information pertaining to the two advertising media under consideration—magazine and television is given below:

	Magazine	*Television*
• Cost per advertisement (Rs. thousand)	40	200
• Audience per advertisement (million)	4	40
• Audience per advertisement with monthly income over Rs. 10,000 (million)	3	10
• Audience (per advertisement) in the age group 18–40 (million)	8	10

The company wishes to determine the number of advertisements to be released each in magazine and television so as to keep the advertisement expenditure to the minimum. Formulate this problem as a LP problem. What will be the minimum expenditure and its allocation among the two media? Write 'dual' of this problem. Solve the 'dual' problem to find answer to the problem.

Solution The primal and dual LP problems of the given problem are:

Primal Problem

x_1, x_2 = number of advertisements in magazine and television, respectively.

$$\text{Minimize } Z_x = 40x_1 + 200x_2$$

subject to $4x_1 + 40x_2 \geq 160$

$3x_1 + 10x_2 \geq 60$

$8x_1 + 10x_2 \geq 80$

and $x_1, x_2 \geq 0.$

Dual Problem

y_1, y_2, y_3 = shadow price (or worth) of one unit of advertisement over audience characteristics, respectively.

$$\text{Maximize } Z_y = 160y_1 + 60y_2 + 80y_3$$

subject to $4y_1 + 3y_2 + 8y_3 \leq 40$

$40y_1 + 10y_2 + 10y_3 \leq 200$

and $y_1, y_2, y_3 \geq 0.$

Example 5.11 XYZ manufacturing company operates a three-shift system at one of its plants. In a certain section of the plant, the number of operators required on each of the three shifts is as follows:

Shift	*Number of Operators*
Day (6 a.m. to 2 p.m.)	50
Afternoon (2 p.m. to 10 p.m.)	24
Night (10 p.m. to 6 p.m.)	10

The company pays its operators at the basic rate of Rs. 10 per hour for those working on the day shift. For the afternoon and night shifts, the rates are one and a half times the basic rate and twice the basic rate, respectively. In agreement with each operator at the commencement of his employment, he is allocated to one of three schemes *A*, *B* or *C*. These are as follows:

A : Work (on average) one night shift, one afternoon shift, and two day shifts in every four shifts.

B : Work (one average) equal number of day and afternoon shifts.

C : Work day shifts only.

In schemes A and B, it is necessary to work strictly alternating sequences of specified shifts, as long as the correct proportion of shifts is worked in the long run.

(a) Formulate a linear programming model to obtain the required number of operators at minimum cost.

(b) By solving the dual of the problem, determine how many operators must be employed under each of the three schemes. Does this result in over-provision of operators on any one of the three shifts?

Solution The primal and dual LP problems of the given problem are:

Primal Problem

x_1, x_2 and x_3 = number of operators employed under scheme *A*, *B* and *C* respectively.

$$\text{Minimize } Z_x = 20 \times \frac{1}{4}x_1 + 15\left(\frac{1}{4}x_1 + \frac{1}{2}x_2\right) + 10\left(\frac{1}{2}x_1 + \frac{1}{2}x_2 + x_3\right)$$

$$= \frac{55}{4}x_1 + \frac{25}{2}x_2 + 10x_3$$

subject to the constraints

$(1/4)x_1 \geq 10,$

$(1/4)x_1 + (1/2)x_2 \geq 24$

$(1/2)x_1 + (1/2)x_2 + x_3 \geq 50$

and $x_1, x_2, x_3 \geq 0.$

Dual Problem

y_1, y_2 and y_3 = shadow price (or worth) per unit of resources—operators in three shifts respectively.

$$\text{Maximize } Z_y = 10y_1 + 24x_2 + 50y_3$$

subject to the constraints

$(1/4)y_1 + (1/4)y_2 + (1/2)y_3 \leq 55/4$

$(1/2)y_2 + (1/2)y_3 \leq 25/2$

$y_3 \leq 10$

and $y_1, y_2, y_3 \geq 0.$

5.5 ADVANTAGES OF DUALITY

1. It is advantageous to solve the dual of a primal that has a less number of constraints because the number of constraints usually equals the number of iterations required to solve the problem.

2. This avoids the necessity for adding surplus or artificial variables and solves the problem quickly (the technique is known as the *primal-dual method*). In economics, duality is useful in the formulation of the input and output systems. It is also useful in physics, engineering, mathematics, etc.
3. The dual variables provide an important economic interpretation of the final solution of an LP problem.
4. It is quite useful when investigating changes in the parameters of an LP problem (the technique is known as the *sensitivity analysis*).
5. Duality is used to solve an LP problem by the simplex method in which the initial solution is infeasible (the technique is known as the *dual simplex method*).

CONCEPTUAL QUESTIONS

1. Define the dual of a linear programming problem. State the functional properties of duality.
2. Explain the primal-dual relationship.
3. Briefly discuss 'duality' in linear programming.
4. What is the principle of duality in linear programming? Explain its advantages.
5. What is duality? What is the significance of dual variables in an LP model?
6. State the general rules for formulating a dual LP problem from its primal.
7. What is a shadow price? How does the concept relate to the dual of an LP problem?
8. How can the concept of duality be useful in managerial decision-making?
9. State and prove the relationship between the feasible solutions of an LP problem and its dual.
10. Prove that the necessary and sufficient condition for any LP problem and its dual, in order to have optimal solutions is that both have feasible solutions.

SELF PRACTICE PROBLEMS B

Write the dual of the following primal LP problems

1. Max $Z_x = 2x_1 + 5x_2 + 6x_3$
   subject to (i) $5x_1 + 6x_2 - x_3 \le 3$ (ii) $-2x_1 + x_2 + 4x_3 \le 4$
   (iii) $x_1 - 5x_2 + 3x_3 \le 1$ (iv) $-3x_1 - 3x_2 + 7x_3 \le 6$
   and $x_1, x_2, x_3 \ge 0$.
2. Min $Z_x = 7x_1 + 3x_2 + 8x_3$
   subject to (i) $8x_1 + 2x_2 + x_3 \ge 3$ (ii) $3x_1 + 6x_2 + 4x_3 \ge 4$
   (iii) $4x_1 + x_2 + 5x_3 \ge 1$ (iv) $x_1 + 5x_2 + 2x_3 \ge 7$
   and $x_1, x_2, x_3 \ge 0$.
3. Max $Z_x = 2x_1 + 3x_2 + x_3$
   subject to (i) $4x_1 + 3x_2 + x_3 = 6$ (ii) $x_1 + 2x_2 + 5x_3 = 4$
   and $x_1, x_2, x_3 \ge 0$.
4. Max $Z_x = 3x_1 + x_2 + 3x_3 - x_4$
   subject to (i) $2x_1 - x_2 + 3x_3 + x_4 = 1$ (ii) $x_1 + x_2 - x_3 + x_4 = 3$
   and $x_1, x_2, x_3, x_4 \ge 0$.
5. Min $Z_x = 2x_1 + 3x_2 + 4x_3$
   subject to (i) $2x_1 + 3x_2 + 5x_3 \ge 2$ (ii) $3x_1 + x_2 + 7x_3 = 3$
   (iii) $x_1 + 4x_2 + 6x_3 \le 5$
   and $x_1, x_2 \ge 0$, x_3 is unrestricted.
6. Min $Z_x = x_1 + x_2 + x_3$
   subject to (i) $x_1 - 3x_2 + 4x_3 = 5$ (ii) $x_1 - 2x_2 \le 3$
   (iii) $2x_2 - x_3 \ge 4$
   and $x_1, x_2 \ge 0$, x_3 is unrestricted.
   [*Meerut Univ., M Sc (Maths), 1994*]
7. One unit of product A contributes Rs. 7 and requires 3 units of raw material and 2 hours of labour. One unit of product B contributes Rs. 5 and requires one unit of raw material and one hour of labour. Availability of raw material at present is 48 units and there are 40 hours of labour.
   (a) Formulate this problem as a linear programming problem.
   (b) Write its dual.
   (c) Solve the dual by the simplex method and find the optimal product mix and the shadow prices of the raw material and labour.
8. A company makes three products: X, Y and Z out of three raw materials A, B and C. The raw material requirements are given below:

Raw Materials	*Number of Units of Raw Material Required to Produce One Unit of Product*		
	X	*Y*	*Z*
A	1	2	1
B	2	1	4
C	2	5	1

   The unit profit contribution of the products: X, Y and Z is Rs. 40, 25 and 50, respectively. The number of units of raw material available are 36, 60 and 45, respectively.
   (a) Determine the product mix that will maximize the total profit.
   (b) Using the final simplex table, write the solution to the dual problem and give its economic interpretation.
9. Three food products are available at costs of Rs. 10, Rs. 36 and Rs. 24 per unit, respectively. They contain 1,000, 4,000 and 2,000 calories per unit, respectively and 200, 900 and 500 protein units per unit, respectively. It is required to find the minimum-cost diet containing at least 20,000 calories and 3,000 units of protein. Formulate and solve the given problem as an LP problem. Write the dual and use it to check the optimal solution of the given problem.
10. A company produces three products: P, Q and R from three raw materials A, B and C. One unit of product P requires 2 units of A and 3 units of B. A unit of product Q requires 2 units of B and 5 units of C and one unit of product R requires 3 units of A, 2 units of B and 4 units of C. The company has 8 units of material A, 10 units of material B and 15 units of material C available to it. Profits per unit of products P, Q and R are Rs. 3, Rs. 5 and Rs. 4, respectively.
   (a) Formulate this problem as an LP problem.
   (b) How many units of each product should be produced to maximize profit?
   (c) Write the dual of this problem.

11. A diet conscious housewife wishes to ensure certain minimum intake of vitamins A, B and C for the family. The minimum daily (quantity) needs of the vitamins A, B and C for the family are respectively 30, 20 and 16 units. For the supply of these minimum vitamin requirements, the housewife relies on two fresh food I and II. The first one provides 7, 5 and 2 units of the three vitamins per gram, respectively and the second one provides 2, 4 and 8 units of the same three vitamins per gram of the foodstuff, respectively. The first foodstuff costs Rs. 3 per gram and the second Rs. 2 per gram. The problem is how many grams of each foodstuff should the housewife buy everyday to keep her food bill as low as possible?
    (a) Formulate this problem as an LP model.
    (b) Write and then solve the dual problem.
12. A manufacturing firm has discontinued production of a certain unprofitable product line and this has created considerable excess production capacity. The management is considering to devote this excess capacity to produce one or more of three products 1, 2 and 3. The available excess capacity on the machines which might limit output, is summarized in the following table:

Machine type	*Available Excess Capacity (Machine Hours per Week)*
Milling machine	250
Lathe	150
Grinder	50

The number of machine-hours for each unit of the respective product is given below.

	Capacity Requirement (Machine-hours per Unit)		
Machine Type	*Product 1*	*Product 2*	*Product 3*
Milling machine	8	2	3
Lathe	4	3	0
Grinder	2	0	1

The per unit contribution would be Rs. 20, Rs. 6 and Rs. 8, respectively for products 1, 2 and 3.
    (a) Formulate this problem as an LP problem.
    (b) How much of each of the three products should the firm produce and sell in order to maximize the contribution? Determine the maximum contribution.
    (c) Write the dual of the above problem and give its economic interpretation.
13. A large distributing company buys coffee seeds from four different plantations. On these plantations, the seeds are available only in a blend of two types A and B. The company wants to market a blend consisting of 30 per cent of type A and 70 per cent of type B. The percentage of each type used by each plantation and the selling prices per 10 kg of the blends of each plantation are as follows:

Type	*Plantation*				*Desired*
	1	*2*	*3*	*4*	
A	40%	20%	60%	80%	30%
B	60%	80%	40%	20%	70%
Selling price per 5 kg	Rs. 3	Rs. 2	Rs. 1.20	Rs. 1.50	

What quantity of coffee seeds should the company buy from each plantation so that the total mixture will contain the desired percentages of A and B and at the same time keep the purchasing cost at a minimum? Also write the dual of the problem.
14. The XYZ Plastic Company has just received a government contract to produce three different plastic valves. These valves must be highly heat and pressure resistant and the company has developed a three-stage production process that will provide the valves with the necessary properties involving work in three different chambers. Chamber 1 provides the necessary pressure resistance and can process valves for 1,200 minutes each week. Chamber 2 provides heat resistance and can process valves for 900 minutes per week. Chamber 3 tests the valves and can work 1,300 minutes per week. The three valve types and the time in minutes required in each chamber are:

Valve Type	*Time Required in*		
	Chamber 1	*Chamber 2*	*Chamber 3*
A	5	7	4
B	3	2	10
C	2	4	5

The government will buy all the valves that can be produced and the company will receive the following profit margins on each valve: A, Rs. 15; B, Rs. 13.50; and C, Rs. 10.

How many valves of each type should the company produce each week in order to maximize profits? Write the dual of the given LP problem and give its economic interpretation?
15. The procurement manager of a company that manufactures special gasoline additives must determine the proper amounts of each raw material to purchase for the production of one of its products. Three raw materials are available. Each litre of the finished product must have at least a combustion point of 222°F. In addition, the gamma content (which cause hydrocarbon pollution) cannot exceed 6 per cent of volume. The zeta content (which is goods for cleaning the internal moving parts of engines) must be at least 12 per cent by volume. Each raw material contains three elements in varying amounts, as shown in the given table.

    Raw material A costs Rs. 9.00 per litre, whereas raw materials B and C cost Rs. 6.00 and Rs. 7.50 per litre, respectively.

	Raw Material		
	A	*B*	*C*
Combustion point, °F	200	180	280
Gamma content, %	4	3	10
Zeta content, %	20	10	8

The procurement manager wishes to minimize the cost of raw materials per litre of product. Use linear programming to find the optimal proportions of each raw material to use in a litre of the finished product. Also write the dual of the given problem and give its economic interpretation.
16. A medical scientist claims to have found a cure for the common cold that consists of three drugs called K, S and H. His results indicate that the minimum daily adult dosage for effective treatment is 10 mg of drug K, 6 mg of drug S and 8 mg of drug H. Two substances are readily available for preparing pills for distribution to cold sufferers. Both substances contain all three of the required drugs. Each unit of substance A contains 6 mg, 1 mg and 2 mg of drugs K, S and H, respectively and each unit of substance B contains 2 mg, 3 mg and 2 mg of the same drugs. Substance A costs Rs. 3 per unit and substance B costs Rs. 5 per unit.
    (a) Find the least-cost combination of the two substances that will yield a pill designed to contain the minimum daily recommended adult dosage.
    (b) Suppose that the costs of the two substances are interchanged so that substance A costs Rs. 5 per unit and substance B costs Rs. 3 per unit. Find the new optimal solution.

    [Delhi Univ., MBA (HCA), 2002]
17. The XYZ company has the option of producing two products during the period of slack activity. For the next period, production has been scheduled so that the milling machine is free for 10 hours and skilled labour will have 8 hours of time available.

Product	Machine Time per Unit	Skilled Labour Time per Unit	Profit Contribution per Unit (Rs.)
A	4	2	5
B	2	2	3

Solve the primal and dual LP problems and bring out the fact that the optimum solution of one can be obtained from the other. Also explain in the context of the example, what you understand by shadow prices (or dual prices or marginal value) of resource.
[Jammu Univ. MBA 1996]

18. A company produces three products *P*, *Q* and *R* whose prices per unit are 3, 5 and 4 respectively. On unit of product *P* requires 2 units of m_1 and 3 units of m_2. A unit of product *Q* requires 2 units of m_2 and 5 units of m_3 and one unit of product *R* requires 3 units of m_1, 2 units of m_2 and 4 units of m_3. The company has 8 units of material m_1, 10 units of material m_2 and 15 units of material m_3 available to it.
    (a) Formulate the problem as on LP model.
    (b) How many units of each product should be produced to maximize revenue?
    (c) Write the dual problem.

19. A person consumes two types of food A and B everyday to obtain 8 units of proteins, 12 units of carbohydrates and 9 units of fats which is his daily minimum requirements. 1 kg of food A contains 2, 6 and 1 units of protein, carbohydrates and fats, respectively. 1 kg of food B contains 1, 1 and 3 units of proteins, carbohydrates and fats respectively. Food A costs Rs. 8.50 per kg, while B costs Rs. 4 per kg. Determine how many kg of each food should he buy daily to minimize his cost of food and still meet the minimum requirements.

    Formulate on LP problem mathematically. Write its dual and solve the dual by the simplex method. *[Gujarat Univ. MBA, 1996]*

20. A firm produces three articles *X*, *Y*, *Z* at a total cost of Rs. 4 , Rs. 3, and Rs. 6 per item respectively. Total number of *X* and *Z* item produced should be at least 2 and number of *Y* and *Z* together be at least 5. The firm wants to minimize the cost. Formulate this problem as an LP problem. Write its dual. Solve the dual by the simplex method. Can you point out the solution of the primal problem? If yes, what is it?

21. A firm produces three types of biscuits *A*, *B* and *C*. It packs them in assortments of two sizes I and II. The size I contains 20 biscuits of type *A*, 50 of type *B* and 10 of type *C*. The size II contains 10 biscuits of type *A*, 80 of type *B* and 60 of type *C*. A buyer intends to buy at least 120 biscuits of type *A*, 740 of type *B* and 240 of type *C*. Determine the least number of packets he should buy. Write the dual LP problem and interpret your answer.

22. Consider the following product mix problem: Let x_1 denote number of units of Product 1 to be produced daily and x_2 the number of units of Product 2 to be produced daily. The production of Product 1 requires one hour of processing time in department D_1. Production of 1 unit of Product 2 requires 2 hours of processing time in department D_1 and one hour in department D_2. The number of hours available in department D_1 are 32 hours and in department D_2, 8 hours. The contribution of one unit of Product 1 is Rs. 200 and of Product 2 is Rs. 300.

    The solution to this LP model is given below:

		$c_j \rightarrow$	200	300	0	0
c_B	Basic Variables	Solution Values x_B	x_1	x_2	s_1	s_2
200	x_1	32	1	2	1	0
0	s_2	8	0	1	0	1
Z = 6400		$c_j - z_j$	–100	–200	0	6400

Given the dual of the primal model. Obtain the optimum solution to the dual LP model from the above table. Interpret the dual variables.

HINTS AND ANSWERS

1. Min $Z_y = 3y_1 + 4y_2 + y_3 + 6y_4$
   subject to (i) $5y_1 - 2y_2 + 5y_3 - 3y_4 \geq 2$
   (ii) $6y_1 + y_2 - 5y_3 - 3y_4 \geq 5$
   (iii) $-y_1 + 4y_2 + 3y_3 + 7y_4 \geq 6$
   and $y_1, y_2, y_3, y_4 \geq 0.$

2. Max $Z_y = 3y_1 + 4y_2 + y_3 + 7y_4$
   subject to (i) $8y_1 + 3y_2 + 4y_3 + y_4 \leq 7$
   (ii) $2y_1 + 6y_2 + y_3 + 5y_4 \leq 3$
   (iii) $y_1 + 4y_2 + 5y_3 + 2y_4 \leq 8$
   and $y_1, y_2, y_3, y_4 \geq 0.$

3. Min $Z_y = 6y_1 + 4y_2$
   subject to (i) $4y_1 + y_2 \geq 2$ (ii) $3y_1 + 2y_2 \geq 3$
   (iii) $y_1 + 5y_2 \geq 1$
   and y_1, y_2 unrestricted in sign.

4. Min $Z_y = y_1 + 3y_2$
   subject to (i) $2y_1 + y_2 \geq 3$ (ii) $-y_1 + y_2 \geq 1$
   (iii) $3y_1 - y_2 \geq 3$ (iv) $y_1 + y_2 \geq -1$
   and y_1, y_2, unrestricted in sign.

5. Max $Z_y = 2y_1 + 3y_2 - 5y_3$
   subject to (i) $2y_1 + 3y_2 - y_3 \leq 2$ (ii) $3y_1 + y_2 - 4y_3 \leq 3$
   (iii) $5y_1 + 7y_2 - 6y_3 = 4$
   and $y_1, y_3 \geq 0$ and y_2 unrestricted.

6. Max $Z_y = -5y_1 - 3y_2 + 4y_3$
   subject to (i) $-y_1 - y_2 \leq 1$ (ii) $3y_1 + 2y_2 + 2y_3 \leq 1$
   (iii) $-4y_1 - y_3 \leq 1$
   and $y_2, y_3 \geq 0$ and y_1 is unrestricted.

7. *Primal*
   x_1 and x_2 = number of units of products A and B, respectively to be produced.
   Max $Z_x = 7x_1 + 5x_2$
   subject to (i) $3x_1 + x_2 \leq 48$ (ii) $2x_1 + x_2 \leq 40$
   and $x_1, x_2 \geq 0$
   Ans. $x_1 = 0, x_2 = 40$, Max Z_x = Rs. 200
   Dual
   y_1 and y_2 = worth of one unit of raw material and labour, respectively.
   Min $Z_y = 48y_1 + 40y_2$
   subject to (i) $3y_1 + 2y_2 \geq 7$ (ii) $y_1 + y_2 \geq 5$
   and $y_1, y_2 \geq 0$
   Ans. $y_1 = 0, y_2 = 5$ and Min Z_y = Rs. 200

8. *Primal*

x_1, x_2 and x_3 = units of the products X, Y and Z, respectively to be produced.

Max $Z_x = 40x_1 + 25x_2 + 50x_3$

subject to (i) $x_1 + x_2 + x_3 \le 36$ (ii) $2x_1 + x_2 + 4x_3 \le 60$

(iii) $2x_1 + 5x_2 + x_3 \le 45$

and $x_1, x_2, x_3 \ge 0$.

Ans. $x_1 = 20, x_2 = 0, x_3 = 5$; Max $Z_x = 1{,}050$

Dual

y_1, y_2 and y_3 = worth (or shadow price) per unit of raw materials A, B and C, respectively.

Min $Z_x = 36y_1 + 60y_2 + 45y_3$

subject to (i) $y_1 + 2y_2 + 2y_3 \ge 40$ (ii) $y_1 + y_2 + 5y_3 \ge 25$

(iii) $y_1 + 4y_2 + y_3 \ge 50$

and $x_1, x_2, x_3 \ge 0$.

Ans. $y_1 = 0, y_2 = 0, y_3 = 10$; Min $Z_y = 1{,}050$

10. *Primal*

x_1, x_2 and x_3 = units of the products P, Q and R to be produced, respectively.

Max $Z = 3x_1 + 5x_2 + 4x_3$

subject to (i) $2x_1 + 3x_3 \le 8$ (ii) $5x_1 + 2x_2 + 2x_3 \le 10$

(iii) $5x_2 + 4x_3 \le 15$

and $x_1, x_2, x_3 \ge 0$.

Dual

Min $Z = 8y_1 + 10y_2 + 15y_3$

subject to (i) $2y_1 + 5y_2 \ge 3$ (ii) $2y_2 + 5y_3 \ge 5$

(iii) $3y_1 + 2y_2 + 4y_3 \ge 4$

and $y_1, y_2, y_3 \ge 0$.

11. *Primal*

x_1 and x_2 = units of fresh food I and II, to be bought, respectively

Min $Z = 3x_1 + 2x_2$

subject to (i) $7x_1 + 2x_2 \ge 30$ (ii) $5x_1 + 4x_2 \ge 20$

(iii) $2x_1 + 8x_2 \ge 16$

and $x_1, x_2 \ge 0$.

Dual

y_1, y_2 and y_3 = worth per unit of vitamins A, B and C, respectively to the body.

Max $Z = 30y_1 + 20y_2 + 16y_3$

subject to (i) $7y_1 + 5y_2 + 2y_3 \le 3$ (ii) $2y_1 + 4y_2 + 8y_3 \le 2$

and $y_1, y_2, y_3 \ge 0$.

(b) The optimal solution to the dual problem is: $y_1 = 20/52$, $y_2 = 0, y_3 = 8/52$ and Max Z = Rs. 14.

12. Product 1 = 0 unit; Product 2 = 50 units; Product 3 = 50 units and Max profit = Rs. 700.

13. *Primal*

x_1, x_2, x_3 and x_4 = quantities of coffee seeds, respectively which the company buys from each plantation.

Min = $3x_1 + 2x_2 + 1.2x_3 + 1.5x_4$

subject to (i) $0.4x_1 + 0.2x_2 + 0.6x_3 + 0.8x_4 \ge 0.3$

(ii) $0.6x_1 + 0.8x_2 + 0.4x_3 + 0.2x_4 \ge 0.7$

and $x_1, x_2, x_3, x_4 \ge 0$.

Dual

y_1 and y_2 = worth of coffee seeds A and B, respectively

Max $Z_y = 0.3y_1 + 0.7y_2$

subject to (i) $0.4y_1 + 0.6y_2 \le 3$ (ii) $0.2y_1 + 0.8y_2 \le 2$

(iii) $0.6y_1 + 0.4y_2 \le 1.2$ (iv) $0.8y_1 + 0.2y_2 \le 1.5$

and $y_1, y_2 \ge 0$

14. *Primal*

x_1, x_2 and x_3 = number of valves of the types A, B and C, respectively to be produced.

Max $Z_x = 15x_1 + 13.5x_2 + 10x_3$

subject to (i) $5x_1 + 3x_2 + 2x_3 \le 1{,}200$

(ii) $7x_1 + 2x_2 + 4x_3 \le 900$

(iii) $4x_1 + 10x_2 + 5x_3 \le 1{,}300$

and $x_1, x_2, x_3 \ge 0$.

Ans. $x_1 = 22{,}400/217, x_2 = 5{,}500/62, x_3 = 0$ and

Max Z_x = Rs. 58,125/31

Dual

y_1, y_2 and y_3 = work of Chambers 1, 2 and 3 production capacity, respectively.

Min $Z_y = 1{,}200y_1 + 900y_2 + 1{,}300y_3$

subject to (i) $5y_1 + 7y_2 + 4y_3 \ge 15$

(ii) $3y_1 + 2y_2 + 10y_3 \ge 13.5$

(iii) $2y_1 + 4y_2 + 5y_3 \ge 10$

and $y_1, y_2, y_3 \ge 0$.

Ans. $y_1 = 0, y_2 = -41/31, y_3 = -645/62$.

17. *Primal*

x_1, x_2 = number of units of product A and B respectively to be produced

Max $Z_x = 5x_1 + 3x_2$

subject to (i) $4x_1 + 2x_2 \le 10$ (ii) $2x_1 + 2x_2 \le 8$

and $x_1, x_2 \ge 0$

Ans. $x_1 = 1, x_2 = 3$ and Max $Z_x = 14$

Dual

y_1, y_2, y_3 = worth (or shadow price) per unit of resource operators in three shifts respectively.

Min $Z_y = 10y_1 + 8y_2$

subject to (i) $4y_1 + 2y_2 \ge 5$ (ii) $2y_1 + 2y_2 \ge 3$

and $y_1, y_2 \ge 0$.

Ans. $y_1 = 1, y_2 = 1/2$ and Min $Z_y = 14$.

18. *Primal*

x_1, x_2, x_3 = number of units of product P, Q and R to be produced, respectively.

Max $Z_x = 3x_1 + 5x_2 + 4x_3$

subject to (i) $2x_1 + 3x_3 \le 8$ (ii) $5x_1 + 2x_2 + 2x_3 \le 10$

(iii) $5x_2 + 4x_3 \le 15$

and $x_1, x_2, x_3 \ge 0$.

Ans. $x_1 = 44/41, x_2 = 59/41, x_3 = 80/41$

Dual

y_1, y_2, y_3 = worth per unit of material m_1, m_2 and m_3 respectively.

Min $Z_y = 8y_1 + 10y_2 + 15y_3$

subject to (i) $2y_1 + 3y_2 \geq 3$ (ii) $2y_2 + 5y_3 \geq 5$

(iii) $3y_1 + 2y_2 + 4y_3 \geq 4$

and $y_1, y_2, y_3 \geq 0$.

19. *Primal*

x_1, x_2 = number of units of food A and B to be consumed, respectively.

Max $Z_x = 8.50x_1 + 4x_2$

subject to (i) $2x_1 + x_2 \geq 8$ (ii) $6x_1 + x_2 \geq 12$

(iii) $x_1 + 3x_2 \geq 9$

and $x_1, x_2 \geq 0$.

Ans. $x_1 = 1, x_2 = 6$ and Min $Z_x = 65/2$;

Dual

y_1, y_2, y_3 = worth per unit of proteins, carbohydrates and fats respectively.

Min $Z_y = 8y_1 + 12y_2 + 9y_3$

subject to (i) $2y_1 + 6y_2 + y_3 \leq 8.50$ (ii) $y_1 + y_2 + 3y_3 \leq 4$

and $y_1, y_2, y_3 \geq 0$

Ans. $y_1 = 31/8, y_2 = 1/8, y_3 = 0$ and Max $Z_y = 65/2$;

20. *Primal*

x_1, x_2, x_3 = number of units of articles X, Y and Z respectively.

Max $Z_x = 4x_1 + 3x_2 + 6x_3$

subject to (i) $x_1 + x_3 \geq 2$ (ii) $x_2 + x_3 \geq 5$

and $x_1, x_2, x_3 \geq 0$.

Ans. $x_1 = 0, x_2 = 3, x_3 = 2$ with Min $Z = 21$

Dual

y_1, y_2 = worth per unit of resources to be used, respectively

Min $Z_y = 2y_1 + 5y_2$

subject to (i) $y_1 \leq 4$ (ii) $y_2 \leq 3$

(iii) $y_1 + y_2 \leq 6$

and $y_1, y_2 \leq 0$

Ans. $y_1 = 3, y_2 = 3$ with Max $Z^* = 21$

21. *Primal*

x_1, x_2 = number of assortments of size I and II, respectively.

Max $Z_x = x_1 + x_2$

subject to (i) $20x_1 + 10x_2 \geq 120$ (ii) $50x_1 + 80x_2 \geq 740$

(iii) $10x_1 + 60x_2 \geq 240$

and $x_1, x_2 \geq 0$

Ans. $x_1 = 2, x_2 = 8$ and Min $Z_x = 10$

Dual

y_1, y_2, y_3 = worth per unit of biscuits of type A, B and C, respectively.

Min $Z_y = 120y_1 + 740y_2 + 240$

subject to (i) $20y_1 + 50y_2 + 10y_3 \leq 1$

(ii) $10y_1 + 80y_2 + 60y_3 \leq 1$

and $y_1, y_2, y_3 \geq 0$

22. *Primal*

Max $Z_x = 200x_1 + 300x_2$

subject to (i) $x_1 + 2x_2 \leq 32$ (ii) $0x_1 + x_2 \leq 8$

and $x_1, x_2 \geq 0$.

Dual

Min $Z_y = 32y_1 + 8y_2$

subject to (i) $y_1 + 0y_2 \geq 200$ (ii) $2y_1 + y_2 \geq 300$

and $y_1, y_2 \geq 0$

The optimum solution to the dual LP problem is: $y_1 = 200, y_2 = 0s_1 = 0s_2 = 100, Z^* =$ Rs. 600

Note that, y_1 = marginal increase in profit for an addition 1 hour of capacity. Since y_1 = Rs. 200, profit becomes Rs. 6400 + 200 = Rs. 6600. Also y_2 = marginal increase in profit given an addition hour of capacity in D_2. Since $y_2 = 0$, there would be no change in the profit.

In primal LP problem solution, $s_2 = 8$ indicates that there is an excess capacity of 8 hours in D_2. Thus, in dual solution we have $y_2 = 0$. Dual constraint (i) is written as: $y_1 + 0y_2 \geq 200$ or

$$\begin{bmatrix}\text{total value of hours in department } D_1 \text{ and } D_2\\ \text{required to produce 1 unit of Product 1}\end{bmatrix} \geq \begin{bmatrix}\text{unit profit received from manufacturing}\\ \text{and selling 1 unit of Product 1}\end{bmatrix}$$

CHAPTER SUMMARY

In this chapter we discussed the relationship between an LP problem and its dual. The method of formulating a dual LP problem of the given LP problem was explained with several examples. The main focus, while solving a dual LP problem, was to find for each resource its best marginal value, also known as dual or shadow price. This value reflects the maximum additional price to be paid in order to obtain one additional unit of any resource in order to maximize profit (or minimize cost) under resource constraints.

CHAPTER CONCEPTS QUIZ

True or False

1. If the objective function of the primal LP problem is maximized, then the objective function of the dual is to be minimized.
2. A dual variable is not defined for each constraint in the primal LP problem.
3. The dual of dual is a primal problem.
4. The value in the $c_j - z_j$ row under columns of slack/surplus variables ignoring negative signs do not give the direct optimal values of the dual/primal basic variables.

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Answers to Quiz

1. T 2. F 3. T 4. F 5. F 6. T 7. T 8. T 9. T 10. T
11. Primal, dual 12. Dual 13. Simplex 14. Unbounded; infeasible 15. Equality; unrestricted
16. Complementary slackness; equal 17. Shadow price 18. Slack; basic
19. Decision variables; constraints; objective function 20. Dual; primal; minimization
21. (c) 22. (b) 23. (a) 24. (a) 25. (b) 26. (c) 27. (a) 28. (d) 29. (c) 30. (a) 31. (c)
32. (d) 33. (a) 34. (b) 35. (a)

APPENDIX: THEOREMS ON DUALITY

Theorem 5.1: The dual of the dual is the primal.

Proof: Consider the primal problem in canonical form·

Minimize $Z_x = \mathbf{cx}$

subject to $\mathbf{Ax} \geq \mathbf{b};\ \mathbf{x} \geq 0$ (1)

where $\mathbf{A}$ is an $m \times n$ matrix, $\mathbf{b}^T \in E^m$ and $\mathbf{c}, \mathbf{x}^T \in E^n$.

Applying the transformation rules, the dual of this problem is:

Maximize $Z_y = \mathbf{b}^T\mathbf{y}$

subject to $\mathbf{A}^T\mathbf{y} \leq \mathbf{c}^T;\ \mathbf{y} \geq 0$ (2)

This dual problem can also be written as:

Minimize $Z_y^* = (-\mathbf{b})^T\mathbf{y}$ (3)

subject to $(-\mathbf{A}^T)\mathbf{y} \geq (-\mathbf{c}^T);\ \mathbf{y} \geq 0$, where $Z_y^* = -Z_y$

If we consider LP problem (3) as primal, then its dual can be constructed by considering $\mathbf{x}$ as the dual variable. Thus, we have

Maximize $Z_x^* = (-\mathbf{c}^T)^T\mathbf{x} = (-\mathbf{c})\mathbf{x}$

subject to $(-\mathbf{A}^T)^T\mathbf{x} \leq (-\mathbf{b}^T)^T$ or $(-\mathbf{A})\mathbf{x} \leq (-\mathbf{b})$

and $\mathbf{x} \geq 0$ (4)

But LP problem (4) is identical to the given primal LP problem (1). This completes the proof of the theorem.

Theorem 5.2: Let $\mathbf{x}^*$ be any feasible solution to the primal LP problem

Maximize $Z_x = \mathbf{cx}$

subject to $\mathbf{Ax} \leq \mathbf{b};\ \mathbf{x} \geq 0$

and $\mathbf{y}^*$ be any feasible solution to the dual LP problem

Maximize $Z_y = \mathbf{b}^T\mathbf{y}$

subject to $\mathbf{A}^T\mathbf{y} \geq \mathbf{c}^T;\ \mathbf{y} \geq 0$

of the above primal problem. Then prove that $\mathbf{cx}^* \leq \mathbf{b}^T\mathbf{y}^*$, i.e. $Z_x \leq Z_y$

Proof: Since $\mathbf{x}^*$ and $\mathbf{y}^*$ are the feasible solutions to the primal and dual LP problems, respectively, therefore from the constraints in primal and dual, we have

$$\mathbf{Ax}^* \leq \mathbf{b};\quad \mathbf{x}^* \geq 0 \qquad (5)$$

$$\mathbf{A}^T\mathbf{y}^* \geq \mathbf{c}^T;\quad \mathbf{y}^* \geq 0 \qquad (6)$$

From inequality (6), we have $\mathbf{c}^T \leq \mathbf{A}^T\mathbf{y}^*$ or $\mathbf{c} \leq \mathbf{A}\,({}^*\mathbf{y})^T$

Multiplying both sides by $\mathbf{x}^*$, we get

$$\mathbf{cx}^* \leq (\mathbf{Ay}^{*T})\mathbf{x}^* = \mathbf{y}^{*T}(\mathbf{Ax}^*) = \mathbf{y}^{*T}\mathbf{b} = \mathbf{b}^T\mathbf{y}^* \qquad [\text{since } \mathbf{y}^{*T}\mathbf{b} = \mathbf{b}^T\mathbf{y}^*]$$

This completes the proof of the theorem.

Theorem 5.3: If $\mathbf{x}^*$ is the feasible solution to the primal LP problem,

Minimize $Z_x = \mathbf{cx}$

subject to $\mathbf{Ax} \leq \mathbf{b};\ \mathbf{x} \geq 0$

and y^* is the feasible solution to the dual problem

Minimize $Z_y = \mathbf{b}^T\mathbf{y}$

subject to $\mathbf{A}^T\mathbf{y} \geq \mathbf{c}^T;\ \mathbf{x} \geq 0$

of the above primal problem, such that, $\mathbf{cx}^* = \mathbf{b}^T\mathbf{y}$, then $\mathbf{x}^*$ and $\mathbf{y}^*$ are the optimal solutions to their respective problems.

Proof: Let $\mathbf{x}_0^*$ be any other feasible solution to the primal. Then from Theorem 5.2 we have $\mathbf{cx}_0^* \leq \mathbf{b}^T\mathbf{y}$ or $\mathbf{cx}_0^* \leq \mathbf{cx}^*$ [since $\mathbf{cx}^* = \mathbf{b}^T\mathbf{y}^*$]. Hence $\mathbf{x}^*$ is an optimal solution to the given maximization primal LP problem.

Similarly, for any other feasible solution $\mathbf{y}_0^*$ to the dual LP problem, we have $\mathbf{cx}^* \leq \mathbf{b}^T\mathbf{y}_0^*$ or $\mathbf{b}^T\mathbf{y}^* \leq \mathbf{b}^T\mathbf{y}_0^*$ [since $\mathbf{cx}^* = \mathbf{b}^T\mathbf{y}^*$]. Hence, $\mathbf{y}^*$ is also an optimal solution to the given minimization dual problem.

Theorem 5.4: If *i*th constraint in the primal is an equality, then the *i*th dual variable is unrestricted in sign.

Proof: Consider the primal LP problem in its standard form as

Minimize $Z_x = \mathbf{cx}$

subject to $\mathbf{Ax} = \mathbf{b};\ \mathbf{x} \geq 0$

This problem can also be written in equivalent form as:

Maximize $Z_x = \mathbf{cx}$

subject to $\mathbf{Ax} \leq \mathbf{b}$ and $(-\mathbf{A})\,\mathbf{x} \leq -\mathbf{b};\ \mathbf{x} \geq 0$

The dual of this problem can now be written as:

Minimize $Z_y = y_1\mathbf{b}^T + y_2(-\mathbf{b}^T) = (y_1 - y_2)\,\mathbf{b}^T$

subject to $\mathbf{A}^T y_1 + (-\mathbf{A}^T)\,y_2 \geq \mathbf{c};\ y_1, y_2 \geq 0$

Let $y = y_1 - y_2$. Then we can rewrite these equations as

Minimize $Z_y = \mathbf{yb}^T$

subject to $\mathbf{A}^T\mathbf{y} \geq \mathbf{c};$ $\mathbf{y}$ unrestricted in sign.

Theorem 5.5 (*Unboundedness Theorem*): If either the primal or the dual LP problem has an unbounded objective function value, then the other problem has no feasible solution.

Proof: Let the given primal LP problem have an unbounded solution. Then for any value of the objective function, say $+\infty$, there exists a feasible solution say $\mathbf{x}$ yielding this solution, i.e. $\mathbf{cx} \rightarrow \infty$.

From Theorem 5.2, for each feasible solution $\mathbf{y}^*$ of the dual, there exists a feasible solution $\mathbf{x}^*$ to the primal such that $\mathbf{cx}^* \leq \mathbf{b}^T\mathbf{y}^*$. That is, $\mathbf{b}^T\mathbf{y}^* \rightarrow +\infty$. As $\mathbf{b}$ is a constant number and $\mathbf{y}^*$ has to satisfy the constraint $\mathbf{A}^T\mathbf{y}^* \leq \mathbf{c}^T$. Hence the dual objective function $Z_y = \mathbf{b}^T\mathbf{y}^*$ must be finite. This contradicts the result $\mathbf{b}^T\mathbf{y}^* \rightarrow \infty$. Hence the dual LP problem has no feasible solution.

A similar argument can be used to show that when the dual LP problem has an unbounded solution, the primal LP problem has no solution.

Theorem 5.6 (*Duality Theorem*): If either the primal or the dual problem has a finite optimal solution, then the other one also possess the same, and the optimal values of the objective functions of the two problems are equal, Max Z_x = Min Z_y.

Proof: Let the following LP problem represent a primal problem:

Maximize $Z_x = \mathbf{cx}$

subject to $\mathbf{Ax} \leq \mathbf{b}$ or $\mathbf{Ax} + \mathbf{Is} = \mathbf{b};\ \mathbf{x} \geq 0$

where $\mathbf{I}$ is the identify matrix of order m and $\mathbf{s}$ is the slack variable.

Let $\mathbf{x}_B = \mathbf{B}^{-1}\mathbf{b}$ be an optimal basic feasible solution to this primal problem and $\mathbf{c}_B$ the cost vector of the basic variables. Then according to the optimality condition, we have, $c_j - z_j \leq 0$, for any vector $\mathbf{a}_j$ of $\mathbf{A}$ but not in $\mathbf{B}$. Therefore,

$$c_j - z_j = c_j - \mathbf{c}_B\mathbf{y}_j = c_j - \mathbf{c}_B(\mathbf{B}^{-1}\,\mathbf{a}_j) \leq 0, \quad \text{for all } j$$

This is equivalent to $\mathbf{c} \le \mathbf{c}_B\mathbf{B}^{-1}\mathbf{A}$, when written in matrix notation.

Let $\mathbf{y}^* = \mathbf{c}_B\mathbf{B}^{-1}$. Then $\mathbf{c} \le \mathbf{c}_B\mathbf{B}^{-1}\mathbf{A}$ becomes $\mathbf{c} \le \mathbf{y}^*\mathbf{A}$ or $\mathbf{c}^T \le \mathbf{y}^*\,\mathbf{A}^T$. Hence $\mathbf{y}^*$ is a feasible solution to the dual of the given primal because it satisfies the dual constraints. The corresponding dual objective function is given by

$$Z_y = \mathbf{y}^*\mathbf{b}^T = \mathbf{c}_B\mathbf{B}^{-1}\,\mathbf{b}^T = \mathbf{c}_B\mathbf{x}_B = Z_x$$

Hence, it has been shown that $\mathbf{x}_B$ and $\mathbf{c}_B\mathbf{B}^{-1}$ are the feasible solution to the primal and dual LP problems, respectively for which Max Z_x = Min Z_y.

Theorem 5.7 (*Complementary Slackness Theorem*): If $\mathbf{x}^*$ and $\mathbf{y}^*$ be feasible solutions to the primal and dual LP problems, respectively, then a necessary and sufficient condition for $\mathbf{x}^*$ and $\mathbf{y}^*$ to be optimal solutions to their respective problems is,

$$y_i \cdot x_{n+i} = 0, \qquad i = 1, 2, \ldots, m$$

and

$$x_j \cdot y_{m+j} = 0, \qquad j = 1, 2, \ldots, n$$

where x_{n+i} is the ith slack variable in the primal LP problem and y_{m+j} the jth surplus variable for the dual LP problem.

Proof: Let $\mathbf{x}^*$ be the feasible solution to the primal LP problem,

$$\text{Maximize } Z_x = \sum_{j=1}^{n} c_j x_j$$

$$\text{subject to} \quad \sum_{j=1}^{n} a_{ij}x_j + x_{n+i} = b_i;\; x_j \ge 0 \tag{7}$$

and $\mathbf{y}^*$ be the feasible solution to the dual of the above primal.

$$\text{Minimize } Z_y = \sum_{j=1}^{n} b_i y_i \tag{8}$$

$$\text{subject to} \quad \sum_{j=1}^{n} a_{ij}y_i - y_{m+j} = c_j;\; y_i \ge 0$$

Multiplying constraints in (7) by y_i, $i = 1, 2, \ldots, m$, and then adding we get

$$x_j \sum_{i=1}^{m} a_{ij}y_i + x_{n+i}\,y_i = \sum_{i=1}^{m} y_i\,b_i\,;\; j = 1, 2, \ldots, n \tag{9}$$

Subtracting (9) from the objective function of (7), we get

$$\left\{c_j - \sum_{i=1}^{m} a_{ij}y_i\right\} x_j - x_{n+i}\,y_i = Z_x - \sum_{i=1}^{m} y_i\,b_i \tag{10}$$

Substituting the value of Z_y and y_{m+j} from (8) to (10) we get,

$$-\sum_{j=1}^{n} x_j\,y_{m+j} - \sum_{i=1}^{m} y_i x_{n+i} = Z_x - Z_y \tag{11}$$

If $\mathbf{x}^*$ and $\mathbf{y}^*$ are optimal solutions to the primal and dual problems, then we have $\mathbf{cx}^* = \mathbf{b}^T\mathbf{y}^*$ (or $Z_x = Z_y$) Thus, from (11), we have

$$\sum_{j=1}^{n} x_j\,y_{m+j} = \sum_{i=1}^{m} y_i x_{n+i} = 0$$

Now it follows that, for each $x_j > 0$, j (= 1, 2, . . ., m) implies $y_{m+j} = 0$ and $\sum_{i=1}^{m} a_{ij}y_i = c_j$, i.e. jth constraint in the dual is an equation. Also $x_j = 0$ implies $y_{m+i} = 0$ and $\sum_{i=1}^{m} a_{ij}y_i > c_j$.

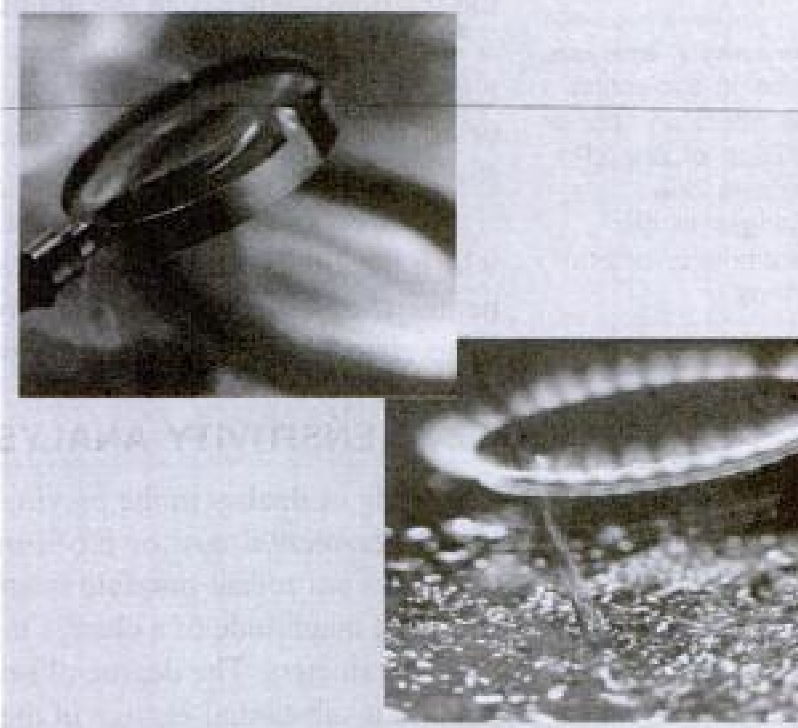
Similarly, for each $y_i > 0$, i (= 1, 2, . . ., m) implies $x_{n+i} = 0$ and $\sum_{j=1}^{n} a_{ij}x_i = b_i$, i.e. ith constraint in the primal is an equation. Also $y_i = 0$ implies $x_{n+i} > 0$ and $\sum_{j=1}^{n} a_{ij}x_j < b_i$. This completes the proof of the theorem.

chapter 6

Sensitivity Analysis in Linear Programming

"The most efficient way to produce anything is to bring together under one management as many as possible of the activities needed to turn out the product."

– Peter Drucker



Preview The purpase of sensitivity analysis is to know the effect on the optimal solution of an LP problem due to variations in the input coefficients (also called parameters), one at a time.

Learning Objectives After studying this chapter, you should be able to

- appreciate the significance of sensitivity analysis concept in managerial decision-making.
- perform sensitivity analysis on various parameters in an LP model without affecting the optimal solution.
- introduce a new variable and a constraint in the existing LP model with reformulation.

Chapter Outline

6.1 INTRODUCTION

In an LP model, the coefficients (also known as parameters) such as: (i) profit (cost) contribution (c_j) per unit of a decision variable, x_j (ii) availability of a resources (b_i), and (iii) consumption of resource per unit of decision variables (a_{ij}), are assumed to be constant and are known with certainty during a planning period. However, in real-world situations, these input parameters value may change over a period of time due to dynamic nature of the business environment. Such changes in any of these parameters may raise doubt on the validity of the optimal solution of the given LP model. Thus, a decision-maker, in such situations, would like to know how changes in these parameters may affect the optimal solution and the range within which the optimal solution will remain unchanged with changes in the original input data values.

Sensitivity analysis helps in evaluating the effect on optimal solution of any LP problem due changes in its parameters, one at a time

Sensitivity analysis and parametric linear programming are the two techniques that are used to evaluate the relationship between the optimal solution and changes in the LP model parameters. *Sensitivity analysis is the study of knowing the effect on optimal solution of the LP model due to variations in the input coefficients (also called parameters) one at a time. Parametric analysis on the other hand is the study of measuring the effect on the optimal solution of the LP model due to simultaneous changes in the input coefficients as a function of one parameter.*

However, while sensitivity analysis provides the sensitive ranges (both lower and upper limits) within which the LP model parameters can vary without changing the optimality of the current optimal solution; parametric linear programming provides information to such changes outside the sensitive range, as well as to changes in more than one parameter at a time.

6.2 SENSITIVITY ANALYSIS

The study of duality in the previous chapter allowed us to identify those resources whose adjustment could bring *incremental* cost or profit improvement in the value of objective function. However, dual variable values do not reveal possible *magnitude* of such changes. The study of sensitivity analysis, however, does show the magnitude of a change in the optimal solution of an LP model due to discrete variations (changes) in its parameters. The degree of sensitivity of the solution due to these variations can range from no change at all to a substantial change in the optimal solution of the given LP problem. Thus, in sensitivity analysis, we determine the range (or limit) over which the LP model parameters can change without affecting the current optimal solution. For this, instead of resolving the entire problem as a new problem with new parameters we may consider the optimal solution as an initial solution for the purpose of knowing the ranges, both lower and upper, within which a parameter may assume a value.

The sensitivity analysis is also referred to as *post-optimality analysis* because it does not begin until the optimal solution to the given LP model that has been obtained. Different parametric changes in the original LP model discussed in this chapter include:

(i) Profit (or cost) per unit (c_j) associated with both basic and non-basic decision variables (coefficients in the objective function).
(ii) Availability of resources (right-hand side of constants, b_i).
(iii) Consumption of resources per unit of decision variables x_j (coefficients of decision variables on the left-hand side of constraints, a_{ij}).
(iv) Addition of a new variable to the existing list of variables in LP problem.
(v) Addition of a new constraint to the original LP problem constraints.

Parametric analysis helps in evaluating the effect on optimal solution of any LP problem due to changes in its parameter, simultaneously as a function of one parameter

6.2.1 Change in Objective Function Coefficient (c_j)

The coefficient c_j in the objective function of an LP model represents either the profit or the cost per unit of an activity (variable) x_j. The question that may now arise is: *What happens to the optimal solution and the objective function value when this coefficient is changed?* For example, let Rs 10 be the per unit profit coefficient of a particular variable in the objective function. After obtaining an optimal solution to the LP problem with Rs 10 as one of the objective function coefficients, the decision-maker realizes that its true value might be any value between Rs 9 and Rs 11 per unit. The test of sensitivity of the objective function value, with respect to this coefficient (or on any other such coefficients), determines the range (both lower and upper) of values within which each c_j ($j = 1, 2, \ldots, n$) can lie, without changing the current optimal solution. Such an analysis can help the decision-maker in deciding whether resources from other activities (variables) should be diverted to (diverted away from) a more profitable (or less profitable) activity.

Change in the profit or cost coefficient (contribution) in the objective function can occur for any *basic variable* (the variable in the solution mix) or any *non-basic variable* (the variable not in the solution mix). The sensitivity range of these variables is determined differently. Thus, these two cases will be discussed separately.

Given an optimal basic feasible solution $[\mathbf{x_B} = \mathbf{B}^{-1}\mathbf{b}]$ with basis matrix $\mathbf{B}$, suppose that the coefficient c_k of a variable x_k is changed from c_k to $c_k + \Delta c_k$, where Δc_k represents the positive or negative amount of change in the value of c_k. Since the optimal basic feasible solution or solution values that appeared in the '$\mathbf{x_B}$' column of the simplex table do not involve cost (or profit) coefficients, c_j in their calculations, it will remain feasible even for any change in the coefficients in the objective function. However, such a change in coefficients (c_j's) only affect the optimality of the solution. Such a change thus requires recomputing z_j values in $c_j - z_j$ row of the optimal simplex table. In other words, as $c_k - z_k (= c_k - \mathbf{c_B}\,\mathbf{B}^{-1}\,\mathbf{a_k})$ involves coefficients c_k, the effect of this change will be seen in the $c_j - z_j$ row of the optimal simplex table.

While recomputing $c_j - z_j$ values the following two cases will arise

(i) The new $c_j - z_j$ values satisfy optimality condition and the solution remains unchanged. However, optimal value of objective function may change.

(ii) The optimality condition is not satisfied. In such a case usual simplex method is used to obtain optimality.

Case I: Change in the coefficient of a non-basic variable If $c_j - z_j \leq 0$ for all non-basic variables in a maximization LP problem, then the current optimal solution remains unchanged. Let c_k be the coefficient of a non-basic variable x_k in the objective function. Since c_k is the coefficient of non-basic variable x_k, therefore, it does not effect any of the c_j values listed in the '$\mathbf{c_B}$' column of optimal simplex table associated with basic variables. Since the calculation of $z_j = \mathbf{c_B}\,\mathbf{B}^{-1}\,\mathbf{a}_j$ values do not involve c_j, therefore changes in c_j do not change z_j values and hence $(c_j - z_j)$ values remain unchanged, except the $c_k - z_k$ value due to change in c_k. In other words, any change in this coefficient does not affect the feasibility of the optimal solution. This means that unit profit of x_k can be lowered to any level without causing the optimal solution to change. But any increase in its unit profit, beyond a certain level, (i.e. upper limit) should make this variable eligible to be a basic variable in the new solution mix. Obviously, $c_k - z_k$ will no longer then be negative.

To retain optimality of the current optimal solution for a change Δc_k in c_k, we must have $(c_k + \Delta c_k) - z_k \leq 0$ or $c_k + \Delta c_k \leq z_k$. Hence, for an LP problem with an objective function of the maximization type, the value of c_k may be increased up to the value of z_k, and decreased to negative infinity ($-\infty$) without affecting the optimal solution.

Case II: Change in the coefficient of a basic variable In the maximization LP problem the change in the coefficient, say c_k, of a basic variable x_k affects the $c_j - z_j$ values corresponding to all non-basic variables in the optimal simplex table. This is because the coefficient c_k is listed in the $\mathbf{c_B}$ column of the simplex table and affects the calculation of the z_j values.

Range of optimality is where the coefficient of any basic variable can change without causing change in the optimal solution mix

The sensitivity limits for the contribution per unit of a basic variable are calculated as under:

Lower limit = Original value, c_k − {Lowest absolute value of improvement ratio or $-\infty$ (if no ratio is negative)}

Upper limit = Original value, c_k + {Lowest positive value of improvement ratio or ∞ (if no ratio is positive)}

$$\text{where} \quad \text{Improvement ratio} = \frac{\text{Per unit improvement value}}{\text{Input} - \text{Output coefficient in the variable row}} = \frac{c_j - z_j}{a_{kj}}$$

Remark While performing sensitivity analysis, the *artificial variable columns in the simplex table are ignored.* Any exchange of quantities between basic variables and an artificial variable makes no sense, because an artificial variable has no economic interpretation. Thus, improvement ratios using coefficients corresponding to artificial variables should not be considered.

Case III: Change in coefficient of a non-basic variable in a cost minimization problem The procedure for calculating sensitivity limits of a cost minimization LP problem, where the objective function coefficients are unit costs, is identical to the Case I discussed above. In this case, the unit cost coefficient can be increased to any arbitrary level but it cannot be decreased by more than the per unit improvement value, without making it eligible so that a non-basic variable can be entered into the new solution mix. The sensitivity limits can be calculated as:

Lower limit = Original value c_k – Absolute value of per unit improvement value $(c_j - z_j)$
Upper limit = Infinity $(+\infty)$

Alternative method If we add Δc_k to the objective function coefficient c_k of the basic variable x_k, then its coefficient c_{Bk} will become $c^*_{Bk} = c_{Bk} + \Delta c_{Bk}$. The new $c_j - z^*_j$ can be calculated as follows:

$$c_j - z_j^* = c_j - \sum_{i=1}^{m} c_{Bi} y_{ij} = c_j - \left\{ \sum_{i \neq k}^{m} c_{Bi} y_{ij} + (c_{Bk} + \Delta c_{Bk}) y_{kj} \right\}$$

$$= c_j - \left\{ \sum_{i=k}^{m} c_{Bi} y_{ij} + \Delta c_{Bk} y_{kj} \right\}$$

$$= c_j - \{z_j + \Delta c_{Bk} y_{kj}\} = (c_j - z_j) - \Delta c_{Bk} y_{kj}$$

For a current basic feasible solution to remain optimal for a maximization LP problem, we must have $c_j - z_j^* \le 0$. That is,

$$c_j - z_j^* = (c_j - z_j) - \Delta c_{Bk} y_{kj} \le 0 \quad \text{or} \quad c_j - z_j \le \Delta c_{Bk} y_{kj}$$

or

$$(c_j - z_j)/y_{kj} \le \Delta c_{Bk} \quad \text{when } y_{kj} > 0$$
$$(c_j - z_j)/y_{kj} \ge \Delta c_{Bk} \quad \text{when } y_{kj} < 0$$

Hence, the value of Δc_{Bk} that satisfies the optimality criterion can be determined by solving the following system of linear inequalities:

$$\text{Min}\left\{\frac{c_j - z_j}{y_{kj} < 0}\right\} \ge \Delta c_{Bk} \ge \text{Max}\left\{\frac{c_j - z_j}{y_{kj} > 0}\right\} \qquad (1)$$

Here it may be noted that y_{kj}s are entries in the non-basic variable columns (i.e. variables) of the optimal simplex table. The new value of the objective function becomes:

$$Z^* = \sum_{i \neq k}^{m} c_{Bi} x_{Bi} + (c_{Bk} + \Delta c_{Bk}) x_{Bk} = \sum_{i=1}^{m} c_{Bi} x_{Bi} + \Delta c_{Bk} x_{Bk} = Z + \Delta c_{Bk} x_{Bk}$$

Hence if Δc_{Bk} satisfies inequality (1), then the optimal solution will remain unchanged but the value of Z will improve by an amount $\Delta c_{Bk} x_{Bk}$.

Example 6.1 Solve the following LP problem:

Maximize $Z = 3x_1 + 5x_2$

subject to the constraints

(i) $3x_1 + 2x_2 \le 18$, (ii) $x_1 \le 4$, (iii) $x_2 \le 6$

and $x_1, x_2 \ge 0$.

(a) Determine an optimal solution to the LP problem.
(b) Discuss the change in c_j on the optimality of the optimal basic feasible solution.

Solution (a) The optimal solution of the given LP problem is shown in Table 6.1. From Table 6.1, the optimal solution can be read as: $x_1 = 2, x_2 = 6$ and Max Z = 36.

(b) Here $c = (c_1, c_2, c_3, c_4, c_5)$, and the cost coefficients associated with basic variables x_1, x_2, and s_2 are $\mathbf{c_B} = (c_1, c_2, c_4) = (3, 5, 0)$. The changes in c_j can be classified as under:

(i) ***Changes in the coefficients c_j (i.e. c_3 and c_5) of non-basic variables s_1 and s_3:*** Let us add Δc_3 and Δc_5 to the objective function coefficients c_3 and c_5. Then the new objective function coefficients will become $c_3' = c_3 + \Delta c_3$ and $c_5' = c_5 + \Delta c_5$. Since $c_3 = 0$ and $c_5 = 0$, $c_3' = \Delta c_3$ and $c_5' = \Delta c_5$. Thus, the new values of $c_3 - z_3$ and $c_5 - z_5$ will become $\Delta c_3 - 1$ and $\Delta c_5 - 3$, respectively. Now in order to maintain optimality, we must have

$$\Delta c_3 - 1 \le 0 \text{ and } \Delta c_5 - 3 \le 0 \text{ (Maximization case)} \quad \text{or} \quad \Delta c_3 \le 1 \text{ and } \Delta c_5 \le 3$$

(ii) ***Change in the coefficients c_j (i.e. c_1, c_2 and c_4) of basic variables x_1, x_2 and s_2:*** The value of additional increment Δc_k, $k = 1, 2, 4$ in the coefficients c_1, c_2 and c_4 which satisfy the optimality condition can be determined by solving the following system of linear inequalities [refer to Eqn. (1)].

$$\text{Min}\left\{\frac{c_j - z_j}{y_{kj} < 0}\right\} \geq \Delta c_{B_k} \geq \text{Max}\left\{\frac{c_j - z_j}{y_{kj} > 0}\right\}$$

	$c_j \rightarrow$		3	5	0	0	0
c_B	*Variables in Basis* B	*Solution Values* $b\,(= x_B)$	x_1	x_2	s_1	s_2	s_3
3	x_1	2	1	0	1/3	0	–2/3
0	s_2	0	0	0	–2/3	1	4/3
5	x_2	6	0	1	0	0	1
$Z = 36$		$c_j - z_j$	0	0	–1	0	–3

Table 6.1 Optimal Solution

For $k = 1$ (i.e. basic variable x_1 in row 1), we have

$$\text{Min}\left\{\frac{-3}{-2/3}\right\} \geq \Delta c_1 \geq \text{Max}\left\{\frac{-1}{1/3}\right\};\quad j = 3, 5 \quad \text{or} \quad 9/2 \geq \Delta c_1 \geq -3$$

Here it may be noted that $y_{kj} = (y_{13} = 1/3, y_{15} = -2/3)$ are only for those columns corresponding to which variables are not in the optimal basis (i.e. non-basic variables). The current optimal solution will not change as long as

$$\left(3 + \frac{9}{2}\right) \geq c_1 \geq (3 - 3) \quad \text{or} \quad \frac{15}{2} \geq c_1 \geq 0$$

For $k = 3$ (i.e. basic variable x_2 in row 3), we have

$$\text{Min}\left\{\frac{-1}{0}\right\} \geq \Delta c_2 \geq \left\{\frac{-3}{1}\right\}; j = 3, 5 \quad \text{or} \quad \infty \geq \Delta c_2 \geq -3$$

Hence, the current solution will not change as long as: $(5 + \infty) \geq c_2 \geq (5 - 3)$ or $\infty \geq c_2 \geq 2$.

Example 6.2 A company wants to produce three products: A, B and C. The per unit profit on these products is Rs 4, Rs 6 and Rs 2, respectively. These products require two types of resources, manpower and raw material. The LP model formulated for determining the optimal product mix is as follows:

Maximize $Z = 4x_1 + 6x_2 + 2x_3$

subject to the constraints

(i) $x_1 + x_2 + x_3 \leq 3$ (Manpower required), (ii) $x_1 + 4x_2 + 7x_3 \leq 9$ (Raw material available)

and $x_1, x_2, x_3 \geq 0$

where x_1, x_2 and x_3 = number of units of products A, B and C, respectively, to be produced.

(a) Find the optimal product mix and the corresponding profit of the company.
(b) Find the range of the profit contribution of product C (i.e. coefficient c_3 of variable x_3) in the objective function, such that the current optimal product mix remains unchanged.
(c) What shall be the new optimal product mix when per unit profit from product C is increased from Rs 2 to Rs 10?
(d) Find the range of the profit contribution of product A (i.e. coefficient c_1 of variable x_1) in the objective function such that the current optimal product mix remains unchanged.

Solution (a) Convert the given LP model into the standard form by introducing the slack variables s_1 and s_2.

Maximize $Z = 4x_1 + 6x_2 + 2x_3 + 0s_1 + 0s_2$

subject to the constraints

(i) $x_1 + x_2 + x_3 + s_1 = 3$, (ii) $x_1 + 4x_2 + 7x_3 + s_2 = 9$

and $x_1, x_2, x_3, s_1, s_2 \geq 0$

The optimal solution obtained by applying the simplex method is shown in Table 6.2.

Table 6.2 Optimal Solution

		$c_j \rightarrow$	4	6	2	0	0
c_B	Variables in Basis B	Solution Values $b\ (= x_B)$	x_1	x_2	x_3	s_1	s_2
4	x_1	1	1	0	−1	4/3	−1/3
6	x_2	2	0	1	2	−1/3	1/3
$Z = 16$		z_j	4	6	8	10/3	2/3
		$c_j - z_j$	0	0	−6	−10/3	−2/3

The optimal solution is: $x_1 = 1$, $x_2 = 2$ and Max Z = Rs 16.

(b) ***Effect of change in the coefficient c_3 of non-basic variable x_3 for product C*** In optimal simplex Table 6.2, the variable x_3 is non-basic and its coefficient $c_3 = 2$ is not listed in the $\mathbf{c_B}$ column of the table. This means a further decrease in its profit contribution c_3 (= Rs 2) per unit will have no effect on the current optimal product mix. But, if c_3 is increased beyond a certain value, the product may become profitable to produce. Hence, there is only an upper limit on c_3 for which the current optimal product mix will be effected.

Table 6.3 Table with a Δc_3 change in c_3

		$c_j \rightarrow$	4	6	$2 + \Delta c_3$	0	0
c_B	Variables in Basis B	Solution Values $b\ (= x_B)$	x_1	x_2	x_3	s_1	s_2
4	x_1	1	1	0	−1	4/3	−1/3
6	x_2	2	0	1	2	−1/3	1/3
$Z = 16$		z_j	4	6	8	10/3	2/3
		$c_j - z_j$	0	0	$\Delta c_3 - 6$	−10/3	−2/3

As we already know, a change Δc_3 in c_3 will cause a change in z_3 and $c_3 - z_3$ values in the x_3-column. The change in c_3 results in the modified simplex Table 6.3.

For an optimal solution shown in Table 6.3 to remain unchanged, we must have: $\Delta c_3 - 6 \leq 0$ or $\Delta c_3 \leq 6$.

Recalling that $c_3 = 2 + \Delta c_3$, or $\Delta c_3 = c_3 - 2$, after substituting this amount in the above inequality, we get $c_3 - 2 \leq 6$ or $c_3 \leq 8$. This implies that as long as the profit contribution per unit of product C is less than Rs 8 (i.e. change should not be more than Rs 8) it is not profitable to produce it and therefore the current optimal solution will remain unchanged.

(c) ***If the value of c_3 is increased from Rs 2 to Rs 10, the new value of $c_3 - z_3 = (c_3 - \mathbf{c_B}\,\mathbf{a}_3)$ will be***

$$c_3 - z_3 = 10 - [4 \quad 6]\begin{bmatrix} -1 \\ 2 \end{bmatrix} \text{ or } 10 - (-4 + 12) = 2\ (> 0)$$

Thus, if coefficient of variable x_3 is increased from Rs 2 to Rs 10, the value of $c_3 - z_3$ will become positive as shown in Table 6.4. The variable x_3 (corresponding to product C) becomes eligible to enter into the basis. Hence, the solution cannot remain optimal any more.

Table 6.4

		$c_j \rightarrow$	4	6	10	0	0	
c_B	Variables in Basis B	Solution Values $b\ (= x_B)$	x_1	x_2	x_3	s_1	s_2	Min Ratio x_B/x_3
4	x_1	1	1	0	−1	4/3	−1/3	–
6	x_2	2	0	1	2	−1/3	1/3	$2/2 = 2 \rightarrow$
$Z = 16$		z_j	4	6	8	10/3	2/3	
		$c_j - z_j$	0	0	2	−10/3	−2/3	
					↑			

Applying the following row operations to enter variable x_3 into the new solution mix

R_2 (new) $\rightarrow R_2$ (old)/2 (*key element*) ; R_1 (new) $\rightarrow R_1$ (old) + R_2 (new)

The new optimal solution after entering non-basic variable x_3 into the solution is shown in Table 6.5.

	$c_j \rightarrow$		4	6	10	0	0
c_B	*Variables in Basis* B	*Solution Values* $b\ (= x_B)$	x_1	x_2	x_3	s_1	s_2
4	x_1	2	1	1/2	0	7/6	– 1/6
10	x_3	1	0	1/2	1	– 1/6	1/6
$Z = 18$		z_j	4	7	10	3	1
		$c_j - z_j$	0	– 1	0	– 3	– 1

Table 6.5

Since all $c_j - z_j \leq 0$ in Table 6.5, the new optimal solution obtained is: $x_1 = 2$, $x_2 = 0$ and $x_3 = 1$ and Max Z = Rs 18.

(d) *Effect of change in the coefficient c_1 of basic variable x_1 for product A* In the optimal simplex Table 6.2, the variable x_1 appears in 'Basis B'. This means a further decrease in its profit contribution c_1 (= Rs 4) will make it less profitable to produce and therefore the current optimal product mix will be affected. Also, an increase in the value of c_1, beyond a certain limit, will make the product A much more profitable and may force the decision-maker to decide to only produce product A. Thus, in either case, the current optimal product mix will be affected and hence we need to know both the lower as well as the upper limit on the value of c_1, within which the optimal solution will not be affected.

Referring again to Table 6.2, the range of change in the value of c_1 (and/or also in c_2) that does not affect the current optimal product mix can be determined once again by calculating the values of $c_j - z_j$, values that correspond to non-basic variables x_3, s_1 and s_2 respectively. For this reproduce Table 6.1 once again with unknown value of c_1 as shown in Table 6.6.

	$c_j \rightarrow$		$4 + \Delta c_1$	6	2	0	0
c_B	*Variables in Basis* B	*Solution Values* $b\ (= x_B)$	x_1	x_2	x_3	s_1	s_2
$4 + \Delta c_1$	x_1	1	1	0	– 1	4/3	– 1/3
6	x_2	2	0	1	2	– 1/3	1/3
$Z = 12 + (4 + \Delta c_1)$		z_j	$4 + \Delta c_1$	6	$8 - \Delta c_1$	$10/3 + 4\Delta c_1/3$	$2/3 - \Delta c_1/3$
		$c_j - z_j$	0	0	$\Delta c_1 - 6$	$-10/3 - 4\Delta c_1/3$	$\Delta c_1/3 - 2/3$

Table 6.6
Table with a Δc_1 change in c_1

For the solution shown in Table 6.6 to remain optimal we must have all $c_j - z_j \leq 0$. That is

$$\Delta c_1 - 6 \leq 0 \quad \text{gives } \Delta c_1 \leq 6$$

$$-\frac{10}{3} - \frac{4\Delta c_1}{3} \leq 0 \quad \text{gives } \Delta c_1 \geq -5/2$$

$$\frac{\Delta c_1}{3} - \frac{2}{3} \leq 0 \quad \text{gives } \Delta c_1 \leq 2$$

Thus, the range of values within which c_1 may change without affecting the current optimal solution is:

$$-\frac{5}{2} \leq \Delta c_1 \leq 2 \text{ or } 4 - \frac{5}{2} \leq c_1 \leq 4 + 2, \text{ i.e. } \frac{3}{2} \leq c_1 \leq 6$$

From these calculations, it may be concluded that in the case of competitive pressure or in the absence of any competition, the decision-maker knows to what extent the prices can be adjusted without changing the optimal product mix.

Example 6.3 Given the following LP problem

Maximize $Z = -x_1 + 2x_2 - x_3$,

subject to the constraints

(i) $3x_1 + x_2 - x_3 \le 10$, (ii) $-x_1 + 4x_2 + x_3 \ge 6$, (iii) $x_2 + x_3 \le 4$

and $x_1, x_2, x_3 \ge 0$.

Determine the effect of discrete changes in c_j ($j = 1, 2, ..., 6$) on the optimal basic feasible solution shown in Table 6.7.

Table 6.7 Optimal Solution

		$c_j \rightarrow$	−1	2	−1	0	0	0	−M
Cost per Unit c_B	*Variables in Basis* B	*Solution Values* $b\,(=x_B)$	x_1	x_2	x_3	s_1	s_2	s_3	A_1
0	s_1	6	3	0	−2	1	0	−1	0
2	x_2	4	0	1	1	0	0	1	0
0	s_2	10	1	0	3	0	1	4	−1
$Z = 8$		z_j	0	2	2	0	0	2	0
		$c_j - z_j$	−1	0	−3	0	0	−2	−M

Solution In Table 6.7, variable x_1 is a non-basic with $c_1 = -1$. If c_1 further decreases, x_1 will not enter the basis and hence will not affect the optimality of the solution. Thus there is no lower limit on the value of c_1. However, if c_1 increases and exceeds a certain value, it may become profitable to have x_1 in the basis. Thus, there should be an upper limit on the value of c_1.

Solution shown in Table 6.7 will remain optimal provided largest value of $\bar{c}_1 \le 0$, i.e.

$$c_1 - (0, 2, 0)\begin{bmatrix}3\\0\\1\end{bmatrix} \le 0 \quad \text{or} \quad c_1 \le 0.$$

The variable x_2 in Table 6.7 is a basic variable, the solution will remain optimal so long as non-basic variables remain non-positive, i.e.

$$c_1 = -1\,(0, c_2, 0)\begin{bmatrix}3\\0\\1\end{bmatrix} \le 0 \quad \text{or} \quad c_1 < 0$$

$$c_3 = -1 - (0, c_2, 0)\begin{bmatrix}-2\\1\\3\end{bmatrix} \le 0 \quad \text{or} \quad c_2 \ge -1$$

$$c_6 = 0 - (0, c_2, 0)\begin{bmatrix}-1\\1\\4\end{bmatrix} \le 0 \quad \text{or} \quad c_2 \ge 0$$

$$c_1 = -M - (0, c_2, 0)\begin{bmatrix}0\\0\\-1\end{bmatrix} \le 0 \quad \text{or} \quad M \ge 0$$

The variable x_3 in Table 6.7 is a non-basic variable, and to find its upper limit, we must have $c_3 \le 0$, i.e.

$$c_3 - (0, 2, 0)\begin{bmatrix}-2\\1\\3\end{bmatrix} \le 0 \quad \text{or} \quad c_3 - 2 \le 0 \quad \text{or} \quad c_3 \le 2.$$

The variable s_1 in Table 6.7 is a basic variable and its limiting value is obtained from largest value $\bar{c}_1, \bar{c}_3, \bar{c}_6$ and $\bar{c}_7$ of c_1, c_3, c_6 and c_7 respectively as follows:

$$\bar{c}_1 = -1 - (c_4, 2, 0)\begin{bmatrix}3\\0\\1\end{bmatrix} \le 0 \quad \text{or} \quad -1 - 3c_4 \le 0 \quad \text{or} \quad c_4 \ge -\frac{1}{3}$$

$$\bar{c}_3 = -1 - (c_4, 2, 0)\begin{bmatrix}-2\\1\\3\end{bmatrix} \le 0 \text{ or } -1 + 2c_4 - 2 \le 0 \text{ or } c_4 \le \frac{3}{2}$$

$$\bar{c}_6 = 0 - (c_4, 2, 0)\begin{bmatrix}-2\\1\\3\end{bmatrix} \le 0 \text{ or } 0 + c_4 - 2 \le 0 \text{ or } c_4 \le 2$$

$$\bar{c}_7 = -M - (c_4, 2, 0)\begin{bmatrix}0\\0\\-1\end{bmatrix} \le 0 \text{ or } -M \le 0, \text{ which is true.}$$

Hence, limits for c_4 are: $-1/3 \le c_4 \le 3/2$

The variable s_2 in Table 6.7 is a basic variable and its limiting value is obtained from largest value $\bar{c}_1, \bar{c}_3, \bar{c}_6$ and $\bar{c}_7$ of c_1, c_3, c_6 and c_7 respectively as follows:

$$\bar{c}_1 = -1 - (0, 2, c_5)\begin{bmatrix}3\\0\\1\end{bmatrix} \le 0 \quad \text{or} \quad -1 - c_5 \le 0 \text{ or } c_5 \ge -1,$$

$$\bar{c}_3 = -1 - (0, 2, c_5)\begin{bmatrix}-2\\1\\3\end{bmatrix} \le 0 \quad \text{or} \quad -1 - (2 + 3c_5) \le 0 \text{ or } c_5 \ge -1,$$

$$\bar{c}_6 = 0 - (0, 2, c_5)\begin{bmatrix}-1\\1\\4\end{bmatrix} \le 0 \quad \text{or} \quad 0 -(2 + 4c_5) \le 0 \text{ or } c_5 \ge -\frac{1}{2},$$

$$\bar{c}_7 = -M - (0, 2, c_5)\begin{bmatrix}0\\0\\-1\end{bmatrix} \le 0 \quad \text{or} \quad -M + c_5 \le 0 \text{ or } c_5 \le M.$$

Hence, limits for c_5 are: $-1/2 \le c_5 \le M$

The variable s_3 in Table 6.7 is non-basic variable and to find its limiting value $\bar{c}_6 \le 0$, we have

$$c_6 - (0, 2, 0)\begin{bmatrix}-1\\1\\4\end{bmatrix} \le 0 \text{ or } c_6 - 2 \le 0 \text{ or } c_6 \le 2.$$

6.2.2 Change in the Availability of Resources (b_i)

Case I : When slack variable is not in the solution mix We know that in the optimal simplex table, $c_j - z_j$ numbers (ignoring negative sign) corresponding to the slack variable columns represent shadow prices (or dual prices) of the available resources. The shadow price provides information about the change in the value of objective function due to the per unit increase or decrease in the RHS (resource value) of a constraint.

The optimal simplex Table 6.2 is reproduced as Table 6.8. Recall that slack variable s_1 represents the availability of manpower resource and s_2 represents the unused raw material. We cannot add an unlimited number of units of the resource without violating the LP problem constraints. For example, the knowledge of shadow price for an additional hour of manpower time ($c_4 - z_4$ = Rs 10/3) helps to determine how many hours we can actually afford to increase or decrease from the manpower so as to increase the profit. That is, we need to determine the range within which the shadow prices will remain valid.

In Table 6.8, there is no slack variable in the solution mix column **B**. The procedure for finding the range for 'resource values' within which the current optimal solution remains unchanged is summarized below.

(a) Treat the slack variable corresponding to *resource value* as if it was an entering variable in the solution mix. For this, calculate exchange ratio (minimum ratio) for every row.

$$\text{Exchange ratio} = \frac{\text{Solution value, } \mathbf{x_B}}{\text{Exchange (Input - output) coefficients in a slack variable column}}$$

(b) Find both the lower and upper sensitivity limits.

Lower limit = Original value – Least (smallest) positive ratio or $-\infty$ (if no ratio is positive)

Upper limit = Original value + Smallest absolute negative ratio or ∞ (if no ratio is negative)

Shadow price is the value of one additional unit of a scarce resource

		$c_j \rightarrow$	4	6	2	0	0
c_B	*Variables in Basis* **B**	*Solution Values* $b\ (= x_B)$	x_1	x_2	x_3	s_1	s_2
4	x_1	1	1	0	−1	4/3	−1/3
6	x_2	2	0	1	2	−1/3	1/3
$Z = 16$		z_j	4	6	8	10/3	2/3
		$c_j - z_j$	0	0	−6	−10/3	−2/3

Table 6.8

To illustrate the method of finding the range of variation in the availability of resources, we repeat s_1 column and the *Solution Values* column from Table 6.8 and calculate ratios as shown below:

Variables in Basis **B**	*Solution Values* $b\ (= x_B)$	*Exchange Coefficients in s_1-Column*	*Exchange Ratio*	*Exchanges Coeffi-cients in s_2-Column*	*Exchange Ratio*
x_1	1	4/3	1/(4/3) = 3/4	−1/3	1/(−1/3) = −3
x_2	2	−1/3	2/(−1/3) = −6	1/3	2/(1/3) = 6

The smallest positive ratio (3/4) indicates the number of hours that can be decreased from the manpower time resource while the smallest absolute negative ratio indicates the number of hours that can be increased (added), without changing the current optimal product mix. Thus, the shadow price for manpower resource (Rs 10/3) and raw material resource (Rs 2/3) are valid over the range as follows.

Solution Mix	*Lower Limit*	*Upper Limit*
Manpower (x_1)	3 − (3/4) = 9/4	3 + 6 = 9
Raw material (x_2)	9 − 6 = 3	9 + 6 = 15

The $c_j - z_j$ values in the slack variable columns with negative sign are the shadow prices

Case II: When a slack variable is in the basis (column B) When a slack variable is present in solution mix column **B** of the optimal simplex table, the procedure for finding the range of variation for the corresponding right hand of the constraint is as follows:

Lower limit = Original value – Solution value of slack variable

Upper limit = Infinity (∞)

Case III: Changes in right-hand side when constraints are of the mixed type

- *When surplus variable is not in the basis* (*column* **B**)

  Lower limits = Original value – Smallest absolute value of negative exchange ratios or $-\infty$ (if no ratio is negative)

  Upper limit = Original value + Smallest positive minimum ratio or ∞ (if no ratio is positive)

- *When surplus variable is in the basis* (*column* **B**)

  Lower limit = Minus infinity ($-\infty$)

  Upper limit = Original value + Solution value of surplus variable.

Alternative methods

1. Since the b_i values are not associated with the calculation of $c_j - z_j$ values, therefore, any change in the right-hand side of the constraints does not affect the optimality condition [$c_j - z_j = c_j - \mathbf{c_B} \mathbf{B}^{-1} \mathbf{a_j} \le 0$ (maximization case)]. However, it does affect the values of basic variables and the value of the objective function Z (= $\mathbf{c_B} \mathbf{x_B}$). This is because in the determination of solution values, ($\mathbf{x_B} = \mathbf{B}^{-1} \mathbf{b}$), the value of resource (i.e. $\mathbf{b}$) is involved. Thus, if resource b_k is changed to $b_k + \Delta b_k$, then the new values of resources becomes

$$\begin{aligned}\mathbf{b}^* &= (b_1, b_2, \ldots, b_k + \Delta b_k, \ldots, b_m)\\ &= (b_1, b_2, \ldots, b_m) + (0, 0, \ldots, \Delta b_k, \ldots, 0) = \mathbf{b} + \Delta b_k\end{aligned}$$

The range of values within which Δb_k can vary without affecting the optimality of the current solution is determined as follows:

$$\begin{aligned}\mathbf{x}_B^* &= \mathbf{B}^{-1} (\mathbf{b} + \Delta \mathbf{b}) = \mathbf{B}^{-1}\mathbf{b} + \mathbf{B}^{-1} (\Delta \mathbf{b})\\ &= \mathbf{B}^{-1}\mathbf{b} + \mathbf{B}^{-1} (0, 0, \ldots, \Delta b_k, \ldots, 0)\\ &= \mathbf{x}_B + (\boldsymbol{\beta}_1, \boldsymbol{\beta}_2, \ldots, \boldsymbol{\beta}_k, \ldots, \boldsymbol{\beta}_m) (0, 0, \ldots, \Delta b_k, \ldots, 0) = \mathbf{x}_B + \boldsymbol{\beta}_k (\Delta b_k)\end{aligned}$$

$$x_{Bi}^* = x_{Bi} + \beta_{ik} (\Delta b_k); \quad \text{for } i\text{th basic variable}$$

where β_{ik} is the (i, k) element of $\mathbf{B}^{-1}$ and is the kth column vector of $\mathbf{B}^{-1}$. In order to maintain the feasibility of the solution at each iteration, the solution values, $\mathbf{x_B}$ must be non-negative. That is, we must have:

$$\mathbf{x}_B^* = x_{Bi} + \beta_{ik} (\Delta b_k) \ge 0; \quad i = 1, 2, \ldots, m$$

Hence, the range of variation in b_k can be obtained by solving the following system of inequalities

$$\text{Min} \left\{ \frac{-x_{Bi}}{\beta_{ik} < 0} \right\} \ge \Delta b_k \ge \text{Max} \left\{ \frac{-x_{Bi}}{\beta_{ik} > 0} \right\}$$

2. The range of variation in the availability of resources (b_i), can also be obtained by using condition of feasibility of the current optimal solution, i.e. $\mathbf{x_B} = \mathbf{B}^{-1} \mathbf{b} \ge 0$,

where $\mathbf{B}^{-1}$ = matrix of coefficients corresponding to slack variables in the optimal simplex table
Δb_k = amount of change in the resource k
$\mathbf{x_B}$ = basic variables appearing in **B**-column of simplex table

Remarks 1. If one or more entries in the $\mathbf{x_B}$-column of the simplex table are negative, the dual simplex method can be used to get an optimal solution to the new problem by maintaining feasibility.

2. A resource whose shadow price is higher in comparison to others, should be increased first to ensure the best marginal increase in the objective function value.

Example 6.4 Solve the following LP problem

Maximize $Z = 4x_1 + 6x_2 + 2x_3$

subject to the constraints

(i) $x_1 + x_2 + x_3 \le 3$, (ii) $x_1 + 4x_2 + 7x_3 \le 9$

and $x_1, x_2, x_3 \ge 0$

(a) Discuss the effect of discrete change in the availability of resources from $[3, 9]^T$ to $[9, 6]^T$.

(b) Which resource should be increased (or decreased) in order to get the best marginal increase in the value of the objective function? *[AMIE, 2004]*

Solution The given LP problem in its standard form can be stated as follows:

Maximize $Z = 4x_1 + 6x_2 + 2x_3 + 0.s_1 + 0.s_2$

subject to the constraints

(i) $x_1 + x_2 + x_3 + s_1 = 3$, (ii) $x_1 + 4x_2 + 7x_3 + s_2 = 9$

and $x_1, x_2, x_3, s_1, s_2 \ge 0$

Applying the simplex method, the optimal solution so obtained is shown in Table 6.9. The optimal solution is: $x_1 = 1$, $x_2 = 2$ and Max $Z = 16$.

(a) If the new values of the right-hand side constants in the constraints are $[9, 6]^T$, then the new values of the basic variables ($\mathbf{x_B} = \mathbf{B}^{-1}\,\mathbf{b}$) shown in Table 6.9 will now become:

$$\begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = \begin{bmatrix} 4/3 & -1/3 \\ -1/3 & 1/3 \end{bmatrix} \begin{bmatrix} 9 \\ 6 \end{bmatrix} = \begin{bmatrix} 10 \\ -1 \end{bmatrix} \quad \text{or} \quad x_1 = 10, \text{ and } x_2 = -1$$

Table 6.9 Optimal Solution

		$c_j \rightarrow$	4	6	2	0	0
c_B	Variables in Basis B	Solution Values $b\ (= x_B)$	x_1	x_2	x_3	s_1	s_2
4	x_1	1	1	0	–1	4/3	–1/3
6	x_2	2	0	1	2	–1/3	1/3
$Z = 16$		z_j	4	6	8	10/3	2/3
		$c_j - z_j$	0	0	–6	–10/3	–2/3

Since the value of x_2 is negative, the optimal solution shown in Table 6.9 will become infeasible. To remove infeasibility, we use the dual simplex method. By reproducing Table 6.9 with new values we get the values shown in Table 6.10.

Table 6.10

		$c_j \rightarrow$	4	6	2	0	0
c_B	Variables in Basis B	Solution Values $b\ (= x_B)$	x_1	x_2	x_3	s_1	s_2
4	x_1	10	1	0	–1	4/3	–1/3
6	x_2	–1	0	1	2	–1/3	1/3 →
$Z = 34$		z_j	4	6	8	10/3	2/3
		$c_j - z_j$	0	0	–6 ↑	–10/3	–2/3

Since $x_2 = -1$, the second row is the key row and x_2 is the outgoing variable. For identifying the key column find the following ratios:

$$\text{Min}\left\{\frac{c_j - z_j}{y_{rj}},\ y_{rj} < 0\right\} = \left\{\frac{-10/3}{-1/3}\right\} = 10 \text{ (column } s_1)$$

Hence column 's_1' is the key column and variable s_1 will enter into the basis. Now revise the simplex table using the following row operations:

$$R_2 \text{ (new)} \rightarrow R_2 \text{ (old)} \times -3 \qquad R_1 \text{ (new)} \rightarrow R_1 \text{ (old)} - (4/3)\, R_2 \text{ (new)}$$

The new solution is shown in Table 6.11.

As all $c_j - z_j \leq 0$ and all $b_i > 0$, the solution given in Table 6.11 is optimal. The optimal solution is: $x_1 = 6$, $x_2 = 0$, $x_3 = 0$ and Max $Z = 24$.

In Table 6.11, slack variable s_1, is present at positive level in the solution mix. Thus, the sensitivity limits for the corresponding right-hand side, i.e. first constraint, are determined as follows:

$$\text{Lower limit} = 9 - 3 = 6; \qquad \text{Upper limit} = \infty$$

		$c_j \rightarrow$	4	6	2	0	0
c_B	*Variables in Basis* *B*	*Solution Values* $b\ (= x_B)$	x_1	x_2	x_3	s_1	s_2
4	x_1	6	1	4	7	0	1
0	s_1	3	0	−3	−6	1	−1
$Z = 24$		z_j	4	16	28	0	4
		$c_j - z_j$	0	−10	−26	0	−4

Table 6.11 Optimal Solution

(b) In Table 6.10, $c_4 - z_4 = 10/3$ and $c_5 - z_5 = 2/3$ values (ignoring negative sign) corresponding to s_1 and s_2 columns represent shadow prices of resources 1 and 2 respectively. Thus increasing the amount of resources 1 and 2 will increase the value of objective function Z by Rs 10/3 and Rs 2/3 respectively. Now in order to know how much these resources may be increased so that each additional unit continues to increase the objective function by 10/3 and 2/3 respectively, till the optimal solution remains feasible, we proceed as follows.

Let Δb_1 be an increase in first resource (RHS of first constraint) so that

$$\mathbf{x_B} = \mathbf{B}^{-1}\,\mathbf{b} \geq 0$$

or
$$\begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = \begin{bmatrix} 4/3 & -1/3 \\ -1/3 & 1/3 \end{bmatrix} \begin{bmatrix} 3+\Delta b_1 \\ 9 \end{bmatrix} = \begin{bmatrix} 1+(4/3)\Delta b_1 \\ 2-(1/3)\Delta b_1 \end{bmatrix} \geq 0$$

i.e. $1 + (4/3)\Delta b_1 \geq 0$ or $\Delta b_1 \geq -3/4$ and $2 - (1/3)\Delta b_1 \geq 0$ or $\Delta b_1 \leq 3$

From these inequalities, we have: $-3/4 \leq \Delta b_1 \leq 3$ or $3 - (3/4) \leq b_1 \leq 3 + 6$ or $9/4 \leq b_1 \leq 9$

Hence, value of first resource can be increased from 3 units to a maximum limit of 9 units. Similarly, second resource can be increased from 9 units to a maximum limit of 20 units (approx.).

Example 6.5 Solve the following LP problem

Maximize $Z = 5x_1 + 12x_2 + 4x_3$,

subject to the constraints

(i) $x_1 + 2x_2 + x_3 \leq 5$, (ii) $2x_1 - x_2 + 3x_3 = 2$

and $x_1, x_2, x_3 \geq 0$.

(a) Discuss the effect of changing the requirement vector from $[5, 2]^T$ to $[7, 2]^T$ on the optimum solution.

(b) Discuss the effect of changing the requirement vector from $[5, 2]^T$ to $[3, 9]^T$ on the optimum solution.

(c) Which resource should be increased and how much to achieve the best marginal increase in the value of the objective function? *[Dayalbagh Edu. Inst., M Tech., 1998]*

Solution The given LP problem in its standard form can be stated as follows:

Maximize $Z = 5x_1 + 12x_2 + 4x_3 + 0s_1 - MA_1$

subject to constraints

(i) $x_1 + 2x_2 + x_3 + s_1 = 5$, (ii) $2x_1 - x_2 + 3x_3 + A_1 = 2$,

and $x_1, x_2, x_3, s_1, A_1 \geq 0$.

Putting $x_1 = x_2 = x_3 = 0$ in the constraint equations, an initial solution: $s_1 = 5$, $A_1 = 2$ and Max $Z = -2M$ so obtained is shown in Table 6.12.

		$c_j \rightarrow$	5	12	4	0	$-M$	
c_B	*Variables in Basis* *B*	*Solution Values* $b\ (= x_B)$	x_1	x_2	x_3	s_1	A_1	*Min Ratio* x_B/x_3
0	s_1	5	1	2	1	1	0	5/1
$-M$	A_1	2	2	−1	3	0	1	2/3
$Z = -2M$		z_j	$-2M$	M	$-3M$	0	$-M$	
		$c_j - z_j$	$5+2M$	$12-M$	$4+3M$	0	0	

Table 6.12 Initial Solution

Applying the Big-M simplex method, the optimal solution so obtained is shown in Table 6.13.

Table 6.13 Optimal Solution

	$c_j \rightarrow$		5	12	4	0	$-M$
c_B	Variables in Basis B	Solution Values $b\ (=x_B)$	x_1	x_2	x_3	s_1	A_1
12	x_2	8/5	0	1	$-1/5$	2/5	$-2/5$
5	x_1	9/5	1	0	7/5	1/5	2/5
$Z = 141/5$		z_j	5	12	23/5	29/5	$-2/5$
		$c_j - z_j$	0	0	$-3/5$	$-29/5$	$-M + 2/5$

The optimal solution shown in Table 6.13 is: $x_1 = 9/5, x_2 = 8/5, x_3 = 0$ and Max $Z = 141/5$.

(a) If new values of right hand side constants in the constraints are changed from $[5, 2]^T$ to $[7, 2]^T$, then the new values of the basic variables ($x_B = B^{-1}b$) shown in Table 6.13 will become

$$\begin{bmatrix} x_2 \\ x_1 \end{bmatrix} = B^{-1}b = \begin{bmatrix} 2/5 & -1/5 \\ 1/5 & 2/5 \end{bmatrix}\begin{bmatrix} 7 \\ 2 \end{bmatrix} = \begin{bmatrix} 14/5 - 2/5 \\ 7/5 + 4/5 \end{bmatrix} = \begin{bmatrix} 12/5 \\ 11/5 \end{bmatrix}$$

Since both x_1 and x_2 are non-negative, the current solution remains feasible and optimal with new values: $x_1 = 11/5, x_2 = 12/5, x_3 = 0$ and Max Z = 199/5.

(b) New values of the current basic variables are

$$\begin{bmatrix} x_2 \\ x_1 \end{bmatrix} = B^{-1}b = \begin{bmatrix} 2/5 & -1/5 \\ 1/5 & 1/5 \end{bmatrix}\begin{bmatrix} 3 \\ 9 \end{bmatrix} = \begin{bmatrix} 6/5 - 9/5 \\ 3/5 + 18/5 \end{bmatrix} = \begin{bmatrix} -3/5 \\ 21/5 \end{bmatrix}$$

Since value of x_2 becomes negative, the current optimal solution becomes infeasible. To remove infeasibility, apply dual simplex method to remove infeasibility of the problem. Rewriting Table 6.13 with new values as shown in Table 6.14.

Table 6.14 Optimal (but infeasible) solution

	$c_j \rightarrow$		5	12	4	0	$-M$
c_B	Variables in Basis B	Solution Values $b\ (=x_B)$	x_1	x_2	x_3	s_1	A_1
12	x_2	$-3/5$	0	1	$-1/5$	2/5	$-1/5$
5	x_1	21/5	1	0	7/5	1/5	2/5
$Z = 69/5$		z_j	5	12	23/5	29/5	$-2/5$
		$c_j - z_j$	0	0	$-3/5$	$-29/5$	$-M + 2/5$

Since $x_2 = -3/5$, the first row is key row and x_2 is the outgoing variable. For identifying the key column, find the following ratio

$$\text{Min}\left\{\frac{c_j - z_j}{y_{r_j}},\ y_{r_j} < 0\right\} = \left\{\frac{-3/5}{-1/5}\right\} = 3\ (x_3 - \text{column})$$

Hence, 'x_3–column' is the key column and variable x_3 will enter into the basis. Revise the solution shown in Table 6.14 using following row operations:

$$R_1\ (\text{new}) \rightarrow R_1\ (\text{old}) \times -5 \qquad R_2\ (\text{new}) \rightarrow R_2\ (\text{old}) - (7/5)\ R_1\ (\text{new})$$

The new solution is shown in Table 6.15.

Since $b_1 = -3/5$, the first row is the key row and x_2 is the outgoing variable. To identify the key column, find the following ratios: $\text{Min}\{(c_j - z_j)/y_{r_j}\ (< 0)\} = \left\{\frac{-3/5}{-2/5}\right\} = 3$. Hence '$x_3$'–column becomes the key column and variable x_3 will enter into the basis. Revising the simplex table using suitable row operations. The new solution is shown in Table 6.15.

		$c_j \rightarrow$	5	12	4	0	0
c_B	Variables in Basis B	Solution Values $b\ (= x_B)$	x_1	x_2	x_3	s_1	A_1
4	x_3	3	0	–5	1	–2	1
5	x_1	0	1	7	0	3	–1
$z = 36$		z_j	5	15	4	7	–1
		$c_j - z_j$	0	–3	0	–7	$-M+1$

Table 6.15
Optimal Solution

Since all $c_j - z_j \leq 0$ and $b_i \geq 0$ in Table 6.15, the solution is optimal. The optimal solution is: $x_1 = 0, x_2 = 0, x_3 = 3$, Max $Z = 12$.

(d) To find the resource that should be increased (or decreased), write the dual objective function with the help of given LP model as $Z_y = 5y_1 + 2y_2$, where, $y_1 = 29/5$ an $y_2 = 2/5$ are the optimal dual variables (Table 6.13). Since value of y_1 is higher than the value of y_2, the first resource should be increased as each additional unit of the first resource increases the objective function by 29/5. Further to find how much the first resource should be increased so that each additional unit continues to increase the objective function by 29/5. This requirement will be met so long as the primal problem remains feasible. Let Δb_1 be the increase in the first resource, so that

$$\begin{bmatrix} x_2 \\ x_1 \end{bmatrix} = B^{-1}b = \begin{bmatrix} 2/5 & -1/5 \\ 1/5 & 2/5 \end{bmatrix} \begin{bmatrix} 5 + \Delta b_1 \\ 2 \end{bmatrix} = \begin{bmatrix} 10/5 + 2\Delta b_1/5 - 2/5 \\ 5/5 + \Delta b_1/5 + 4/5 \end{bmatrix} = \begin{bmatrix} \{8 + 2\Delta b_1\}/5 \\ \{9 + \Delta b_1\}/5 \end{bmatrix} \geq 0$$

i.e. $8 + \Delta b_1 \geq 0$ or $\Delta b_1 \geq -8$ and $9 + \Delta b_1 \geq 0$ or $\Delta b_1 \geq -9$

Since x_1 and x_2 remain feasible (≥ 0) for all values of $\Delta b_1 \geq 0$, the first resource can be increased indefinitely while maintaining the condition that each additional unit will increase the objective function by 29/5.

The second resource should be decreased as each additional unit of the second resource decreases the objective function by 2/5. Let Δb_2 be the decrease in the second resource, so that

$$\begin{bmatrix} x_2 \\ x_1 \end{bmatrix} = B^{-1}b = \begin{bmatrix} 2/5 & -1/5 \\ 1/5 & 2/5 \end{bmatrix} \begin{bmatrix} 5 \\ 2 - \Delta b_2 \end{bmatrix} = \begin{bmatrix} 10/5 - 2/5 + \Delta b_2/5 \\ 5/5 + 4/5 - 2\Delta b_2/5 \end{bmatrix} \begin{bmatrix} \{8 + \Delta b_2\}/5 \\ \{9 - 2\Delta b_2\}/5 \end{bmatrix} \geq 0$$

Thus, x_1 remains positive only so long as $9 - 2\Delta b_2 \geq 0$ or $\Delta b_2 \leq 9/2$. If $\Delta b_2 > 9/2$, x_1 becomes negative and must leave the solution.

Example 6.6 A factory manufactures three products A, B and C for which the data is given in the table below. Find the optimal product mix if the profit/unit is Rs. 32, Rs. 30 and Rs. 40 for product A, B and C respectively.

	Product			Available Resources
	A	B	C	
Material required (kg/unit)	5	4	3	2,500 kg
Machine hours required/unit	2	3	1	1,275 hours
Labour hours require/unit	3	2	4	2,100 hours

(a) Find the optimal solution if machine hours available become 1,350 instead of 1,275.

(b) Find the optimal solution if labour hours become 2,000 instead of 2,100.

(c) Find the optimal solution if 10 units of product A are to be produced.

Solution Let us define the following decision variables:

x_1, x_2 and x_3 = Number of units of product A, B and C to be produced, respectively.

Then the LP model based on problem data is written is:

Maximize $Z = 32x_1 + 30x_2 + 40x_3$

subject to constraints

(i) $5x_1 + 4x_2 + 3x_3 \le 2{,}500$, (ii) $2x_1 + 3x_2 + x_3 \le 1{,}275$, (iii) $3x_1 + 2x_2 + 4x_3 \le 2{,}100$

and $x_1, x_2, x_3 \ge 0$.

Initial basic feasible solution: $x_1 = 0, x_2 = 0, x_3 = 0, s_1 = 2{,}500, s_2 = 1{,}275, s_3 = 2{,}100$ and Max $Z = 0$ is shown in Table 6.16

		$c_j \rightarrow$	32	30	40	0	0	0	
c_B	Variables in Basis B	Solution Values $b\,(= x_B)$	x_1	x_2	x_3	s_1	s_2	s_3	Min Ratio x_B/x_3
0	s_1	2500	5	4	3	1	0	0	2500/3
0	s_2	1275	2	3	1	0	1	0	1275/1
0	s_3	2100	3	2	4	0	0	1	2100/4 →
$Z = 0$		z_j	0	0	0	0	0	0	
		$c_j - z_j$	32	30	40 ↑	0	0	0	

Table 6.16
Initial Solution

The initial solution shown in Table 6.16 is updated to obtain the optimal solution shown in Table 6.17 using suitable row operations:

		$c_j \rightarrow$	32	30	40	0	0	0
c_B	Variables in Basis B	Solution Values $b\,(= x_B)$	x_1	x_2	x_3	s_1	s_2	s_3
0	s_1	125	3/2	0	0	1	–1	–1/2
30	x_2	300	1/2	1	0	0	2/5	–1/10
40	x_3	375	1/2	0	1	0	–1/5	3/10
$Z = 24{,}000$		z_j	35	30	40	0	4	9
		$c_j - z_j$	–3	0	0	0	–4	–9

Table 6.17
Optimal Solution

The optimal solution shown in Table 6.17 is: $x_1 = 0$, $x_2 = 300$ units, $x_3 = 375$ units and Max Z = Rs. 24,000. Since $s_2 = 0$ and $s_3 = 0$, this implies this while producing product B and C all the machine hours and labour hours available are consumed. However, $s_1 = 125$ kg implies this only 125 kg of material remains unutilized.

(a) In Table 6.17, s_2-column shown that if one machine hour is increased, 1 kg of material will remain unconsumed and product B will increase by 2/5 unit and C will reduce by –1/5 unit. Thus, if machine hours available are increased by 75, the product B (= variable x_2) will increase by (2/5) × = 30 units and product C (= variable x_3) will decrease by (1/5) × 75 = 15 units. Thus the optimal solution will be: $x_1 = 330$, $x_3 = 360$. Further, if one machine hour remains unused, there is a loss of Rs. 4. In other words, for every additional machine hour used, there is a gain of Rs. 4 and hence profit will increase by Rs. 75 × 4 = Rs. 300 to attain a value of Rs. 24,300, which is equivalent to the new solution value: Rs. (0 × 125 + 30 × 330 + 40 × 360) = Rs. 24,300.

(b) In Table 6.17 s_3-column shows that if one labour hour is decreased, it would change the value of x_2 by – 1 × (–1/10) = (1/10) units and of x_3 by – 1 × (3/10) = – (3/10) units. Therefore, a reduction of 100 labour hours will increase value of variable x_2 by (1/10) × 100 = 10 units and decrease value of variable, x_3 by (3/10) × 100 = 30 units and the new optimal solution would be: $x_1 = 0$, $x_2 = 310$ units, $x_3 = 345$ units and Max Z = Rs. 23,100.

(c) In Table 6.17 x_1-column shows that when one unit of product A is produced, it would reduce the production of B and C each by 1/2 unit each. Therefore, if $x_1 = 10$, $x_2 = 300 - (1/2) \times 10 = 295$, $x_3 = 375 - (1/2) \times 10 = 370$. The profit contribution of Rs. 3 by product A (coefficient of variable x_1) will also reduces and hence it will reduced to Rs. 30 and become Rs. 23,970, which is equivalent to the new solution: Rs. (10 × 32 + 30 × 295 + 40 × 370) = Rs. 23,970.

6.2.3 Changes in the Input-Out Coefficients (a_{ij}'s)

Suppose that the elements of coefficient matrix **A** are changed. Then two cases arise

(i) Change in a coefficient, when variable is a basic variable, and
(ii) Change in a coefficient, when variable is a non-basic variable.

Case I: When a non-basic column $\mathbf{a}_k \notin \mathbf{B}$ changed to $\mathbf{a}_k^*$, the only effect of such change will be on the optimality condition. Thus the solution will remain optimal, if

$$c_k - z_k^* = c_k - \mathbf{c_B}\mathbf{B}^{-1}\mathbf{a}_k^* \le 0$$

otherwise the simplex method is continued, after column k of the simplex table is updated, by introducing the non-base variable x_k into the basis.

However, the range for the discrete change Δa_{ij} in the coefficient of non-basic variable x_j in the constraint, i can be determined by solving following linear inequalities:

$$\text{Max}\left\{\frac{c_j - z_j}{\mathbf{c_B}\,\boldsymbol{\beta}_i > 0}\right\} \le \Delta a_{ij} \le \text{Min}\left\{\frac{c_j - z_j}{\mathbf{c_B}\,\boldsymbol{\beta}_i < 0}\right\}$$

Here $\boldsymbol{\beta}_i$ is the ith column $\mathbf{B}^{-1}$. If $\mathbf{c_B}\,\boldsymbol{\beta}_i = 0$, then Δa_{ij} is unrestricted in sign.

Alternative method The change in the coefficients (a_{ij}'s) values associated with non-basic variables in the optimal simplex table can be analysed by forming a corresponding dual constraint from the original set of constraints:

$$\sum_{i=1}^{m} a_{ji}\, y_i \ge c_j;\ \text{for } x_j \text{ non-basic variable.}$$

The value of dual variables y_i's can be obtained from the optimal simplex table. The reason behind this dual constraint formulation is that an activity is considered as fully undertaken provided shadow price (or imputed cost) of all resources needed to produce one unit of an activity become equal to its per unit contribution to total profit.

Case II : Suppose a basic variable column $\mathbf{a}_k \in \mathbf{B}$ is changed to $\mathbf{a}_k^*$. Then conditions to maintain both feasibility and optimality of the current optimal solution are:

(a) $$\underset{k \ne p}{\text{Max}}\left\{\frac{-x_{Bk}}{x_{Bk}\,\beta_{pi} - x_{Bp}\,\beta_{ki} > 0}\right\} \le \Delta a_{ij} \le \underset{k \ne p}{\text{Min}}\left\{\frac{-x_{Bk}}{x_{Bk}\,\beta_{pi} - x_{Bp}\,\beta_{ki} < 0}\right\}$$

(b) $$\text{Max}\left\{\frac{c_j - z_j}{(c_j - z_j)\beta_{pi} - y_{pj}\,\mathbf{c_B}\,\boldsymbol{\beta}_i > 0}\right\} \le \Delta a_{ij} \le \text{Min}\left\{\frac{c_j - z_j}{(c_j - z_j)\beta_{pi} - y_{pj}\,\mathbf{c_B}\,\boldsymbol{\beta}_i < 0}\right\}$$

Example 6.7 Solve the following LP problem

Maximize $Z = -x_1 + 3x_2 - 2x_3$

subject to the constraints

(i) $3x_1 - x_2 + 2x_3 \le 7$, (ii) $-2x_1 + 4x_2 \le 12$, (iii) $-4x_1 + 3x_2 + 8x_3 \le 10$

and $x_1, x_2, x_3 \ge 0$.

Discuss the effect of the following changes in the optimal solution.

(a) Determine the range for discrete changes in the coefficients a_{13} and a_{23} consistent with the optimal solution of the given LP problem.
(b) 'x_1'-column in the problem is changed from $[3, -2, -4]^T$ to $[3, 2, -4]^T$.
(c) 'x_3'-column in the problem is changed from $[2, 0, 8]^T$ to $[3, 1, 6]^T$. *[AMIE, 2005]*

Solution The given LP problem in its standard form can be stated as follows:

Maximize $Z = -x_1 + 3x_2 - 2x_3 + 0.s_1 + 0.s_2 + 0.s_3$

subject to the constraints

(i) $3x_1 - x_2 + 2x_3 + s_1 = 7$, (ii) $-2x_1 + 4x_2 + s_2 = 12$, (iii) $-4x_1 + 3x_2 + 8x_3 + s_3 = 10$

and $x_1, x_2, x_3, s_1, s_2, s_3 \ge 0$

Applying the simplex (Big-M) method, the optimal solution so obtained is shown in Table 6.18.

		$c_j \rightarrow$	-1	3	-2	0	0	0
c_B	Variables in Basis B	Solution Values $b\ (= x_B)$	x_1	x_2	x_3	s_1	s_2	s_3
-1	x_1	4	1	0	4/5	2/5	1/10	0
3	x_2	5	0	1	2/4	1/5	3/10	0
0	s_3	11	0	0	10	1	$-1/2$	1
$Z = 11$	z_j	-1	3	2/5	1/5	4/5	0	
		$c_j - z_j$	0	0	$-12/5$	$-1/5$	$-4/5$	0

Table 6.18 Optimal Solution

The optimal basic feasible solution shown in Table 6.18 is: $x_1 = 4$, $x_2 = 5$, $x_3 = 0$ and Max $Z = 11$.

In table 6.18, the inverse of basis matrix, $\mathbf{B}$ is

$$\mathbf{B}^{-1} = \begin{bmatrix} 2/5 & 1/10 & 0 \\ 1/5 & 3/10 & 0 \\ 1 & -1/2 & 1 \end{bmatrix} = [\boldsymbol{\beta}_1, \boldsymbol{\beta}_2, \boldsymbol{\beta}_3]$$

Thus, we have

$$\begin{aligned} \mathbf{c_B}\,\boldsymbol{\beta}_1 &= -1(2/5) + 3(1/5) + 0(1) = 1/5 \\ \mathbf{c_B}\,\boldsymbol{\beta}_2 &= -1(1/10) + 3(3/10) + 0(-1/2) = 8/10 \\ \mathbf{c_B}\,\boldsymbol{\beta}_3 &= -1(0) + 3(0) + 0(1) = 0 \end{aligned}$$

Since variables x_1, x_2 and s_3 are in the basis (column $\mathbf{B}$ of Table 6.18), therefore any discrete change in coefficients, belonging to any of these column vectors may affect both the feasibility as well as optimality of the original optimal basic feasible solution, whereas any discrete change in the non-basic variables (i.e. x_3, s_1 and s_2) column vectors may affect only the optimality condition.

(a) Ranges for discrete change in coefficients a_{13} and a_{23} in the x_3-column vector of Table 6.18 are computed as:

$$\text{Max}\left\{\frac{c_3 - z_3}{\mathbf{c_B}\,\boldsymbol{\beta}_1}\right\} = \text{Max}\left\{\frac{-12/5}{1/5}\right\} \le \Delta a_{13} \quad \text{or} \quad \Delta a_{13} \ge -12$$

and

$$\text{Max}\left\{\frac{c_3 - z_3}{\mathbf{c_B}\,\boldsymbol{\beta}_2}\right\} = \text{Max}\left\{\frac{-12/5}{8/10}\right\} \le \Delta a_{23} \quad \text{or} \quad \Delta a_{23} \ge -3$$

(b) To measure the change Δa_{21} in the coefficient a_{21} (= 2) in the first column of variable x_1 in the second constraint of the original set of constraints, we need to check both the feasibility as well as optimality conditions. This needs to be done because variable x_1 is the basic variable, as shown in Table 6.18.

(i) *Feasibility Condition.* For $i = 2$ (constraint), $p = 1$ (column), and $k = 2, 3$ (columns of $\mathbf{B}^{-1}$), we have

For $k = 2$ $\quad x_{B2}\,\beta_{12} - x_{B1}\,\beta_{22} = 5(1/10) - 4(3/10) = -7/10$

For $k = 3$ $\quad x_{B3}\,\beta_{12} - x_{B1}\,\beta_{32} = 11(1/10) - 4(-1/2) = 31/10$

Hence, the range to maintain feasibility of the existing optimal solution is

$$\frac{-5}{31/10} \le \Delta a_{21} \le \frac{-5}{-7/10}$$

$$2 - (50/31) \le a_{21} \le 2 + (50/7) \quad \text{or} \quad -12/31 \le a_{21} \le 64/7$$

(ii) *Optimality Condition*

$$(c_3 - z_3)\,\beta_{12} - y_{13}\,c_B\,\beta_2 = -\frac{12}{5}\left(\frac{1}{10}\right) - \frac{4}{5}\left(\frac{8}{10}\right) = -\frac{44}{50}$$

$$(c_4 - z_4)\,\beta_{12} - y_{14}\,c_B\,\beta_2 = -\frac{1}{5}\left(\frac{1}{10}\right) - \frac{2}{5}\left(\frac{8}{10}\right) = -\frac{17}{50}$$

$$(c_5 - z_5)\,\beta_{12} - y_{15}\,c_B\,\beta_2 = -\frac{4}{5}\left(\frac{1}{10}\right) - \frac{1}{10}\left(\frac{8}{10}\right) = -\frac{16}{100}$$

Hence, the range to maintain optimality of the existing optimal solution is

$$-\infty \le \Delta a_{21} \le \text{Min}\left\{\frac{-12/5}{-44/50};\ \frac{-1/5}{-17/50};\ \frac{-4/5}{-16/100}\right\}$$

$$-\infty \le \Delta a_{21} \le 10/17 \quad \text{or} \quad -\infty \le a_{21} \le 44/17$$

(c) Suppose column vector $\mathbf{a}_3$ (x_3-column in Table 6.18) of original LP model is changed from $[2, 0, 8]^T$ to $[3, 1, 6]^T$. Then new value of $c_3 - z_3^*$ for this column is:

$$\mathbf{B}^{-1}\,\mathbf{a}_3^* = \begin{bmatrix} 2/5 & 1/10 & 0 \\ 1/5 & 3/10 & 0 \\ 1 & -1/2 & 1 \end{bmatrix}\begin{bmatrix} 3 \\ 1 \\ 0 \end{bmatrix} = \begin{bmatrix} 13/10 \\ 9/10 \\ 17/2 \end{bmatrix}$$

$$c_3 - z_3^* = c_3 - \boldsymbol{c}_B\,\mathbf{B}^{-1}\,\mathbf{a}_3^* = -2 - [-1, 3, 0]\begin{bmatrix} 13/10 \\ 9/10 \\ 17/2 \end{bmatrix} = -\frac{34}{10}$$

Since all entries in the x_3-column are non-zero we must replace column x_3 with new entries in Table 6.18 and proceed to get new optimal solution.

Example 6.8 Find the effect of the following changes on the optimal solution (Table 6.19) of the following LP problem.

Maximize $Z = 45x_1 + 100x_2 + 30x_3 + 50x_4$

subject to the constraint

(i) $7x_1 + 10x_2 + 4x_3 + 9x_4 \ge 1200$, (ii) $3x_1 + 40x_2 + x_3 + x_4 \le 800$,

and $x_2, x_2, x_3, x_4 \ge 0$.

		$c_j \rightarrow$	45	100	30	50	0	0
Cost per Unit c_B	Variables in Basis B	Solution Values $b\,(= x_B)$	x_1	x_2	x_3	x_4	s_1	s_2
30	x_3	800/3	5/3	0	1	7/3	4/5	–1/15
100	x_2	40/3	1/30	1	0	–1/30	–1/150	2/75
$Z = 28{,}000/3$		z_j	1600/30	100	30	–11/3	110/15	50/75
		$c_j - z_j$	–25/3	0	0	–55/3	–22/3	–2/3

Table 6.19 Optimal Solution

(a) 'x_1'-column in the problem changes from $[7, 3]^T$ to $[7, 5]^T$.

(b) 'x_1'-column changes from $[7, 3]^T$ to $[5, 8]^T$.

Solution (a) The variable x_1 is a non-basic variable in the optimal solution shown in Table 6.19. The upper limit $\bar{c}_1$ the coefficient c_1 of x_1 is calculated as follows:

$$\bar{c}_1 = c_1 - \boldsymbol{c}_\mathbf{B}\,\bar{\boldsymbol{a}}_1 = c_1 - \boldsymbol{c}_B\boldsymbol{B}^{-1}\boldsymbol{a}_1 = c_1 - \hat{y}_1\,\boldsymbol{a}_1, \quad \text{where} \quad c_1 = 45,\ \boldsymbol{a}_1 = \begin{bmatrix} 7 \\ 5 \end{bmatrix},$$

and

$$\hat{y} = \boldsymbol{c}_B\boldsymbol{B}^{-1} = (30, 100)\begin{bmatrix} 4/15 & -1/15 \\ -1/150 & 2/7 \end{bmatrix} = \left[\frac{22}{3}, \frac{2}{3}\right].$$

Thus

$$\bar{c}_1 = 45 - \left[\frac{22}{3}, \frac{2}{3}\right]\begin{bmatrix} 7 \\ 5 \end{bmatrix} = 45 - \left(\frac{154}{3} + \frac{10}{3}\right) = 45 - \frac{164}{3} = -\frac{29}{3}.$$

Since value of $\bar{c}_1$ is negative, the current optimal solution remains optimal for the new problem also.

(b) $\bar{c}_1 = c_1 - \boldsymbol{c_B}\,\bar{\boldsymbol{a}}_1 = c_1 - \boldsymbol{c_B}\,\boldsymbol{B}^{-1}\boldsymbol{a}_1 = c_1 - \hat{y}_1\,\boldsymbol{a}_1 = 45 - \left[\dfrac{22}{3}, \dfrac{2}{3}\right]\begin{bmatrix}5\\8\end{bmatrix} = 45 - \left(\dfrac{110}{3} + \dfrac{16}{3}\right) = 3.$

Since value of $\bar{c}_1$ is positive, the current optimal solution can be improved. Also

$$\bar{\boldsymbol{a}}_1 = \boldsymbol{B}^{-1}\,\boldsymbol{a}_1 = \begin{bmatrix} 4/15 & -1/5 \\ -1/150 & 2/75 \end{bmatrix}\begin{bmatrix}5\\8\end{bmatrix} = \begin{bmatrix} 4/5 \\ 27/150 \end{bmatrix}.$$

		$c_j \rightarrow$	45	100	30	50	0	0	
c_B	*Variables in Basis* B	*Solution Values* $b\,(=x_B)$	x_1	x_2	x_3	x_4	s_1	s_2	*Min Ratio* x_B/x_1
30	x_3	800/3	4/5	0	1	7/3	4/15	– 1/15	1000/3
100	x_2	40/3	27/150	1	0	– 1/30	– 1/150	2/75	2000/27 →
Z = 28,000/3		z_j	126/3	100	30	11/3	110/15	50/75	
		$c_j - z_j$	3 ↑	0	0	– 55/3	– 22/3	– 2/3	

Table 6.20
Optimal Solution

Introducing non-basic variable x_1 into the basis to replace basic variable x_2 using suitable row operations: In Table 6.21, all $c_j - z_j \leq 0$, the solution is optimal with: $x_1 = 2000/27$, $x_2 = 0$, $x_3 = 5600/27$ and Max $Z = 86{,}000/9$.

		$c_j \rightarrow$	45	100	30	0	0	0
c_B	*Variables in Basis* B	*Solution Values* $b\,(=x_B)$	x_1	x_2	x_3	x_4	s_1	s_2
30	x_3	5600/27	0	–40/9	1	67/27	8/27	–5/27
45	x_1	2000/27	1	50/9	0	–5/7	–1/27	4/27
Z = 86, 000/9		z_j	45	350	30	595/9	65/9	10/9
		$c_j - z_j$	0	–50/3	0	–145/9	–65/9	–10/9

Table 6.21
Optimal Solution

6.2.4 Addition of a New Variable (Column)

Let an extra variable x_{n+1} with coefficient c_{n+1} be added in the system of original constraint $\mathbf{Ax} = \mathbf{B}$, $\mathbf{x} \geq 0$. This in turn creates an extra column $\boldsymbol{a}_{n+1}$ in the matrix $\mathbf{A}$ of coeffcients. To see the impact of this addition on the current optimal solution, we compute

$$\mathbf{y}_{n+1} = \mathbf{B}^{-1}\,\mathbf{a}_{n+1}$$

and
$$c_{n+1} - z_{n+1} = c_{n+1} - \mathbf{c_B}\,\mathbf{y}_{n+1}$$

Two cases of the maximization LP model may arise:

(a) If $c_{n+1} - z_{n+1} \leq 0$, then $\mathbf{x_B} = 0$, and hence current solution remains optimal.

(b) If $c_{n+1} - z_{n+1} > 0$, then the current optimal solution can be improved upon by the introduction of a new column $\mathbf{a}_{n+1}$ into the basis to find the new optimal solution.

Example 6.9 Discuss the effect on optimality by adding a new variable to the following LP problem with column coefficients $(3, 3, 3)^T$ and coefficient 5 in the objective function.

Minimize $Z = 3x_1 + 8x_2$

subject to the constraints

(i) $x_1 + x_2 = 200$, (ii) $x_1 \leq 80$, (iii) $x_2 \geq 60$

and $x_1, x_2 \geq 0$

Solution The given LP problem in its standard form can be expressed as:

Minimize $Z = 3x_1 + 8x_2 + 0s_1 + 0s_2 + MA_1 + MA_2$
subject to the constraints

(i) $x_1 + x_2 + A_1 = 200$, (ii) $x_1 + s_1 = 80$, (iii) $x_2 - s_2 + A_2 = 60$

and $x_1, x_2, s_1, s_2, A_1, A_2 \geq 0$

By applying the simplex method the optimal solution that we get is shown in Table 6.22. The optimal solution is: $x_1 = 80$, $x_2 = 120$ and Min $Z = 1{,}200$.

		$c_j \rightarrow$	3	8	0	0	M
c_B	Variables in Basis B	Solution Values $b\ (= x_B)$	x_1	x_2	s_1	s_2	A_1
0	s_2	60	0	0	−1	1	1
3	x_1	80	1	0	1	0	0
8	x_2	120	0	1	−1	0	1
$Z = 1{,}200$		z_j	3	8	−5	0	8
		$c_j - z_j$	0	0	5	0	$M - 8$

Table 6.22
Primal-Dual Relationship

To see the changes in the optimal solution, we are given that $c_7 = 5$ and the column, $\mathbf{a}_7 = (3, 3, 3)^T$. Thus, to see the change we calculate:

$$c_7 - z_7 = c_7 - \mathbf{c_B}\, \mathbf{B}^{-1}\, \mathbf{a}_7 = 5 - (0, 3, 8) \begin{bmatrix} -1 & 1 & 1 \\ 1 & 0 & 0 \\ -1 & 0 & 1 \end{bmatrix} \begin{bmatrix} 3 \\ 3 \\ 3 \end{bmatrix} = -4$$

$$\mathbf{a}_7^* = \mathbf{B}^{-1}\, \mathbf{a}_7 = \begin{bmatrix} -1 & 1 & 1 \\ 1 & 0 & 0 \\ -1 & 0 & 1 \end{bmatrix} \begin{bmatrix} 3 \\ 3 \\ 3 \end{bmatrix} = \begin{bmatrix} 3 \\ 3 \\ 0 \end{bmatrix}$$

Since $c_7 - z_7 = -4$, which is a negative number, the existing optimal solution can be improved upon. For this, we start with the optimal solution shown in Table 6.22 and by adding entries corresponding to variable x_7 we get values as shown in Table 6.23.

		$c_j \rightarrow$	3	8	0	0	M	5	
c_B	Variables in Basis B	Solution Values $b\ (= x_B)$	x_1	x_2	s_1	s_2	A_1	x_7	Min. Ratio x_B/x_7
0	s_2	60	0	0	−1	1	1	3	60/3 →
3	x_1	80	1	0	1	0	0	3	80/3
8	x_2	120	0	1	−1	0	1	0	—
$Z = 1{,}200$		z_j	3	8	−5	0	8	9	
		$c_j - z_j$	0	0	5	0	$M - 8$	−4	
								↑	

Table 6.23
New Column with Variable x_7 Added

The variable x_7 in Table 6.23 must enter into the solution and s_2 should leave it. The new solution is shown in Table 6.24.

		$c_j \rightarrow$	3	8	0	0	M	5
c_B	Variables in Basis B	Solution Values $b\ (= x_B)$	x_1	x_2	s_1	s_2	A_1	x_7
5	x_7	20	0	0	−1/3	1/3	1/3	1
3	x_1	20	1	0	2	−1	−1	0
8	x_2	120	0	1	−1	0	1	0
$Z = 1{,}120$		z_j	3	8	−11/3	−4/3	−4/3	5
		$c_j - z_j$	0	0	11/13	4/3	$M + 4/3$	0

Table 6.24

Since all $c_j - z_j \geq 0$ in Table 6.24, the solution is optimal, with $x_1 = 20$, $x_2 = 120$, $x_7 = 20$ and Min $Z = 1{,}120$.

6.2.5 Addition of a New Constraint (Row)

After solving an LP model, the decision-maker may recall that a particular resource constraint was overlooked during the model formulation or perhaps he may desire to know the effect of adding a new resource to enhance the objective function value. The addition of a constraint in the existing constraints will cause a simultaneous change in the objective function coefficients (c_j), as well as coefficients a_{ij} of a corresponding non-basic variable. Thus, it will affect only the optimality of the problem. This means that the new variable should enter into the basis only if it improves the value of the objective function.

Suppose that a new constraint

$$\mathbf{a}_{m+1,1}\, x_1 + \mathbf{a}_{m+1,2}\, x_2 + \cdots + \mathbf{a}_{m+1,n}\, \mathbf{x}_n \leq \mathbf{b}_{m+1}$$

is added to the system of original constraints $\mathbf{Ax} = \mathbf{b}$, $\mathbf{x} \geq 0$, where b_{m+1} is positive, zero or negative. Then the following two cases may arise.

1. The optimal solution ($\mathbf{x_B}$) of the original problem satisfies the new constraint. If this is the case, then the solution remains feasible as well as optimal. This is because the new constraint either reduces or leaves unchanged the feasible region of the given LP problem.
2. The optimal solution ($\mathbf{x_B}$) of the original problem does not satisfy the new constraint. In this case the optimal solution to the modified LP problem should be re-obtained. Let $\mathbf{B}$ be the basis matrix for the original problem and $\mathbf{B}_1$ be the basis matrix for the new problem with $m + 1$ constraints. That is, matrix $\mathbf{B}_1$ of order $(m + 1)$ is given by

$$\mathbf{B}_1 = \begin{bmatrix} \mathbf{B} & 0 \\ \boldsymbol{\alpha} & 1 \end{bmatrix}$$

where the second column of $\mathbf{B}_1$ corresponds to slack, surplus or artificial variables added to the new constraint and $\boldsymbol{\alpha} = (a_{m+1,1}, a_{m+1,2}, \ldots, a_{m+1,n})$ is a row vector containing the coefficients in the new constraint and corresponds to variables in the optimal basis. In order to prove that the new solution

$$\mathbf{x}_\mathbf{B}^* = \begin{bmatrix} \mathbf{x_B} \\ \mathbf{s} \end{bmatrix} \qquad \mathbf{s} = \text{slack variable}$$

is a basic feasible solution to the new LP problem, we shall compute the inverse of $\mathbf{B}_1$ using partitioned methods, as given below:

$$\mathbf{B}_1^{-1} = \begin{bmatrix} \mathbf{B}^{-1} & 0 \\ -\boldsymbol{\alpha}\mathbf{B}^{-1} & 1 \end{bmatrix}$$

Since each column vector in the new LP problem is given by $\mathbf{a}_j^* = (\mathbf{a}_j, \mathbf{a}_{m+1,j})$, the new columns y_j^* are given by

$$\mathbf{y}_j^* = \mathbf{B}^{-1}\mathbf{a}_j^* = \begin{bmatrix} \mathbf{B}^{-1} & 0 \\ -\boldsymbol{\alpha}\mathbf{B}^{-1} & 1 \end{bmatrix} \begin{bmatrix} \mathbf{a}_j \\ \mathbf{a}_{m+1,j} \end{bmatrix}$$

$$= \begin{bmatrix} \mathbf{B}^{-1}\mathbf{a}_j \\ -\boldsymbol{\alpha}\mathbf{B}^{-1}\mathbf{a}_j + \mathbf{a}_{m+1,j} \end{bmatrix} = \begin{bmatrix} \mathbf{y}_j \\ \mathbf{a}_{m+1,j} - \boldsymbol{\alpha}\mathbf{y}_j \end{bmatrix}$$

The entries in the $c_j - z_j^*$ row of the simplex table for any non-basic variable x_j in the new problem are computed as follows:

$$c_j - z_j^* = c_j - z_j^* \mathbf{y}_j^* = c_j - [\mathbf{c_B}, \mathbf{c}_{Bm+1}] \begin{bmatrix} \mathbf{y}_j \\ a_{m+1,j} - \boldsymbol{\alpha}\mathbf{y}_j \end{bmatrix}$$

$$= c_j - (\mathbf{c_B}\mathbf{y}_j + \mathbf{c}_{Bm+1}\, a_{m+1} - \mathbf{c}_{Bm+1}\, \boldsymbol{\alpha}\mathbf{y}_j)$$

where c_{Bm+1} is the coefficient associated with the new variable introduced in the basis of the new LP problem.

If the new variable introduced in the basis of the new LP problem is slack or surplus variable, then $c_{Bm+1} = 0$. Hence, we have

$$c_j - z_j^* = c_j - \mathbf{c_B}\mathbf{y}_j = c_j - z_j$$

That is, the entries in the $c_j - z_j$ row are the same for the initial as well as the new LP problem. The value of objective function is given by:

$$Z^* = [\mathbf{c_B},\ 0]\begin{bmatrix} \mathbf{x_B} \\ \mathbf{s} \end{bmatrix} = \mathbf{c_B}\,\mathbf{x_B} = Z$$

This shows that the optimal simplex table of the original problem remains unchanged even after adding the new constraint. However, the slack or surplus variable appears with a negative value because the optimal solution of the original problem does not satisfy the new constraints. Thus, the dual simplex method may be used to get an optimal solution.

Remarks If the new constraint added is an equation and an artificial variable appears in the basis of the new problem, then the following two cases may arise:

1. If an artificial variable appears in the basis at negative value, a zero cost may be assigned to it. Apply the dual simplex method in order to obtain an optimal solution.
2. If the artificial variable appears in the basis at a positive value, then $-M$ cost may be assigned to it. Apply the usual simplex method to obtain an optimal solution.

Example 6.10 Consider the following LP problem

Maximize $Z = 3x_1 + 5x_2$

subject to the constraints

(i) $3x_1 + 2x_2 \leq 18$, (ii) $x_1 + 2x_2 \leq 4$, (iii) $x_2 \leq 6$

and $x_1, x_2 \geq 0$

Obtain an optimal solution of the given LP problem.

(a) Suppose variable x_6 is added to the given LP problem. Then obtain an optimal solution to the resulting LP problem. It is given that the coefficients of x_6 in the constraint of the problem are 1, 1 and 1, and that its coefficient in the objective function is 2.

(b) Discuss the effect on the optimal basic feasible solution by adding a new constraint $2x_1 + x_2 \leq 8$ to the given set of constraints. *[AMIE, 2005]*

Solution The optimal solution: $x_1 = 2, x_2 = 6$ with Max $Z = 36$ to the given LP problem is shown in Table 6.25.

		$c_j \rightarrow$	3	5	0	0	0
c_B	Variables in Basis B	Solution Values $b\ (= x_B)$	x_1	x_2	s_1	s_2	s_3
3	x_1	2	1	0	1/3	0	−2/3
0	s_2	0	0	0	−2/3	1	4/3
5	x_2	6	0	1	0	0	1
$Z = 36$		$c_j - z_j$	0	0	−1	0	−3

Table 6.25 Optimal Solution

(a) After adding the new variable, say x_6 in the given problem, the new LP problem becomes:

Maximize $Z = 3x_1 + 5x_2 + 2x_6$

subject to the constraints

(i) $3x_1 + 2x_2 + x_6 \leq 18$, (ii) $x_1 + x_6 \leq 4$, (iii) $x_2 + x_6 \leq 6$

and $x_1, x_2, x_6 \geq 0$

Given that the column vector associated with variable, x_6 is $\mathbf{a}_6 = (1, 1, 1)$, with the help of Table 6.25, we compute

$$\mathbf{y}_6 = \mathbf{B}^{-1}\,\mathbf{a}_6 = \begin{bmatrix} 1/3 & 0 & -2/3 \\ -2/3 & 1 & 4/3 \\ 0 & 0 & 1 \end{bmatrix}\begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix} = \begin{bmatrix} -1/3 \\ 5/3 \\ 1 \end{bmatrix}$$

Since in the current optimal solution, $\mathbf{c_B} = (3, 0, 5)$, we have

$$c_6 - z_6 = c_6 - \mathbf{c_B}\,\mathbf{y}_6 = 2 - (3, 0, 5)\begin{bmatrix} -1/3 \\ 5/3 \\ 1 \end{bmatrix} = -2\ (\leq 0)$$

As $c_6 - z_6 \leq 0$, the optimality of the current solution remains unaffected with the addition of x_6.

(b) The optimal basic feasible solution given in Table 6.25 does not satisfy the additional constraint $2x_1 + x_2 \leq 8$. Thus, a new optimal solution is obtained by adding slack variable s_4 to this constraint. Then, it is written together with the entries in Table 6.25, as shown in Table 6.26.

Table 6.26

		$c_j \rightarrow$	3	5	0	0	0	0
c_B	*Variables in Basis* B	*Solution Values* $b\ (= x_B)$	x_1	x_2	s_1	s_2	s_3	s_4
3	x_1	2	1	0	1/3	0	–2/3	0
0	s_2	0	0	0	–2/3	1	4/3	0
5	x_2	6	0	1	0	0	1	0
0	s_4	8	2	1	0	0	0	1
$Z = 36$		$c_j - z_j$	0	0	–1	0	–3	0

In Table 6.26 it may be noted that the basis matrix **B** has been disturbed due to row 4. Thus, the coefficients in row 4 must become zero. This can be done by using the following row operations.

$$R_4\ (\text{new}) \rightarrow R_4\ (\text{old}) - 2R_1 - R_3$$

We get a new table as shown in Table 6.27.

Table 6.27

		$c_j \rightarrow$	3	5	0	0	0	0	
c_B	*Variables in Basis* B	*Solution Values* $b\ (= x_B)$	x_1	x_2	s_1	s_2	s_3	s_4	
3	x_1	2	1	0	1/3	0	–2/3	0	
0	s_2	0	0	0	–2/3	1	4/3	0	
5	x_2	6	0	1	0	0	1	0	
0	s_4	–2	0	0	–2/3	0	1/3	1	$\rightarrow$
$Z = 36$		$c_j - z_j$	0	0	–1	0	–3	0	
					$\uparrow$				

Since the solution given in Table 6.27 is optimal but not feasible, apply the dual simplex method to get an optimal basic feasible solution. Introduce s_1 into the basis and remove s_4 from the basis. The new solution is shown in Table 6.28.

Table 6.28
Optimal Solution

		$c_j \rightarrow$	3	5	0	0	0	0
c_B	*Variables in Basis* B	*Solution Values* $b\ (= x_B)$	x_1	x_2	s_1	s_2	s_3	s_4
3	x_1	1	1	0	0	0	–1/2	1/2
0	s_2	2	0	0	0	1	1	–1
5	x_2	6	0	1	0	0	1	0
0	s_1	3	0	0	1	0	–1/2	–3/2
$Z = 33$		$c_j - z_j$	0	0	0	0	–7/2	–3/2

Since all $c_j - z_j \leq 0$, the solution shown in Table 6.28 is the optimal basic feasible solution: $x_1 = 1$, $x_2 = 6$ and Max $Z = 33$. It may be noted here that the additional constraint has decreased the optimal value of the objective function from 36 to 33.

Example 6.11 Solve the following LP problem

Maximize $Z = 3x_1 + 4x_2 + x_3 + 7x_4$

subject to the constraints

(i) $8x_1 + 3x_2 + 4x_3 + x_4 \leq 7$, (ii) $2x_1 + 6x_2 + x_3 + 5x_4 \leq 3$

(iii) $x_1 + 4x_2 + 5x_3 + 2x_4 \leq 8$

and $x_1, x_2, x_3, x_4 \geq 0$

(a) Discuss the effect of discrete changes in (i) RHS of constraints. (ii) Coefficients of variables in the objective function, (iii) Structural coefficients in the constraints on the optimal basic feasible solution of the LP problem.

(b) Discuss the effect on the optimal solution of the LP problem of adding an additional constraint: $2x_1 + 3x_2 + x_3 + 5x_4 \leq 4$.

Solution The optimal solution of the LP problem is shown in Table 6.29.

		$c_j \rightarrow$	3	4	1	7	0	0	0
c_B	Variables in Basis B	Solution Values $b\,(=x_B)$	x_1	x_2	x_3	x_4	s_1	s_2	s_3
3	x_1	16/19	1	9/38	1/2	0	5/38	–1/38	0
7	x_4	5/19	0	21/19	0	1	–1/19	4/19	0
0	s_3	126/19	0	59/38	9/2	0	–1/38	–15/38	1
Z = 83/9		z_j	3	321/38	3/2	7	1/38	53/38	0
		$c_j - z_j$	0	–169/38	–1/2	0	–1/38	–53/38	0

Table 6.29 Optimal Solution

The optimal solution is : $x_1 = 16/19$, $x_2 = 0$, $x_3 = 0$ and $x_4 = 5/19$ and Max $Z = 83/9$.

(a) (i) The range of variation in RHS values $b_k (k = 1, 2, 3)$ can be obtained by solving the following system of inequalities:

$$\text{Max}\left\{\frac{-x_{Bi}}{\beta_{ik>0}}\right\} \leq \Delta b_k \leq \text{Min}\left\{\frac{-x_{Bi}}{\beta_{ik<0}}\right\}$$

For $k = 1$, i.e. $b_1 = 7$ we have

$$\text{Max}\left\{\frac{-16/19}{5/28}\right\} \leq \Delta b_1 \leq \text{Min}\left\{\frac{-5/29}{-1/19}, \frac{-126/19}{-1/38}\right\}$$

$$-32/5 \leq \Delta b_1 \leq 5$$

$$7 - (32/5) \leq b_1 \leq 7 + 5, \text{ or } 3/5 \leq b_1 \leq 12$$

For $k = 2$, i.e. $b_2 = 3$, we have

$$\text{Max}\left\{\frac{-5/19}{4/19}\right\} \leq \Delta b_2 \leq \text{Min}\left\{\frac{-16/19}{-1/38}, \frac{-126/19}{-15/38}\right\}$$

$$-5/4 \leq \Delta b_2 \leq 85/5$$

$$3 - (5/4) \leq b_2 \leq 3 + (85/5) \text{ or } 7/4 \leq b_2 \leq 20$$

(ii) The range of variation in the coefficients c_j ($j = 1, 2, \ldots, 7$) of variables in the objective function without disturbing the current optimal solution is obtained by solving the following system of inequalities:

$$\text{Max}\left\{\frac{c_j - z_j}{y_{kj} > 0}\right\} \leq \Delta c_k \leq \text{Min}\left\{\frac{c_j - z_j}{y_{kj} < 0}\right\}$$

The range of variation in the coefficients of basic variables x_1, x_4 and s_3 is given by

For $k = 1$, i.e. $c_1 = 3$

$$\text{Max}\left\{\frac{-169/38}{9/38}, \frac{-1/2}{1/2}, \frac{-1/38}{5/38}\right\} \leq \Delta c_1 \leq \text{Min}\left\{\frac{-53/38}{-1/38}\right\}$$

$$-1/5 \leq \Delta c_1 \leq 53$$

$$3 - (1/5) \leq c_1 \leq 3 + 53, \quad \text{or} \quad 14/5 \leq c_1 \leq 56$$

For $k = 4$, i.e. $c_4 = 7$

$$\text{Max}\left\{\frac{-169/38}{21/19}, \frac{-53/38}{4/19}\right\} \le \Delta c_4 \le \text{Min}\left\{\frac{-1/38}{-1/19}\right\}$$

$$-169/42 \le \Delta c_4 \le 1/2$$

or $\quad 7 - (169/42) \le c_4 \le 7 + (1/2)$, or $125/42 \le c_4 \le 15/2$

For $k = 7$, i.e. $c_7 = 0$

$$\text{Max}\left\{\frac{-169/38}{59/38}\right\} \le \Delta c_7 \le \text{Min}\left\{\frac{-1/38}{-1/38}, \frac{-53/38}{-15/38}\right\}$$

$$-1/9 \le \Delta c_7 \le 1$$

The range of variation in the coefficients of non-basic variables x_2, x_3, s_1 and s_2, must satisfy the upper limit $c_k + \Delta c_k \le z_k$ or $\Delta c_k \le z_k - c_k$ thus:

For $k = 2$, $\Delta c_2 \le 169/28$; For $k = 3$, $\Delta c_3 \le 1/2$

For $k = 5$, $\Delta c_5 \le 1/38$; For $k = 6$ $\Delta c_6 \le 53/38$

(iii) Effect of change in coefficients a_{ij} of non-basic variables x_2, x_3, s_1 and s_2 on the optimal solution is determined by solving the following system of inequalities

$$\text{Max}\left\{\frac{c_j - z_j}{c_B\beta_i > o}\right\} \le \Delta c_{ij} \le \text{Min}\left\{\frac{c_j - z_j}{c_B\beta_i < 0}\right\}$$

Since $\quad \boldsymbol{c_B}\beta_i = (3\ 7\ 0)\begin{bmatrix} 5/28 & -1/38 & 0 \\ -1/19 & 4/19 & 0 \\ -1/38 & -15/38 & 1 \end{bmatrix}$, therefore

$$\boldsymbol{c_B}\beta_1 = 3\,(3/28) + 7\,(-1/19) + 0\,(-1/38) = 1/38$$

$$\boldsymbol{c_B}\beta_2 = 3\,(-1/38) + 7(4/19) + 0(-15/38) = 53/38$$

$$\boldsymbol{c_B}\beta_3 = 3\,(0) + 7\,(0) + 0\,(1) = 0$$

For row $i = 1, 2, 3$ and column $j = 2$, we have

$$(-169/38)/(1/38) \le \Delta a_{12} \text{ or } \Delta a_{12} \ge -169$$

$$(-169/38)/(53/38) \le \Delta a_{22} \text{ or } \Delta a_{22} \ge -169/53$$

$$(-169/38)/(0) \le \Delta a_{32} \text{ or } -\infty \le \Delta a_{32} \le \infty$$

For row $i = 1, 2, 3$ and column $j = 3$

$$(-1/2)/(1/38) \le \Delta a_{13} \text{ or } \Delta a_{13} \ge -19$$

$$(-1/2)/(53/38) \le \Delta a_{23} \text{ or } \Delta a_{23} \ge -19/53$$

$$(-1/2)/(0) \le \Delta a_{23} \text{ or } -\infty \le \Delta a_{33} \le \infty$$

For row $i = 1, 2, 3$ and column $j = 5$

$$(-1/38)/(1/38) \le \Delta a_{15} \text{ or } \Delta a_{15} \ge -1$$

$$(-1/38)/(53/38) \le \Delta a_{25} \text{ or } \Delta a_{25} \ge -1/53$$

$$(-1/38)/0 \le \Delta a_{35} \text{ or } -\infty \le \Delta a_{35} \le \infty$$

For row $i = 1, 2, 3$ and column $j = 6$

$$(-53/38)/(1/38) \le \Delta a_{16} \text{ or } a_{16} \ge -53$$

$$(-53/38)/(53/38) \le \Delta a_{26} \text{ or } \Delta a_{26} \ge -1$$

$$(-53/38)/0 \le \Delta a_{36} \text{ or } -\infty \ge \Delta a_{36} \le \infty$$

The effect of change in coefficient a_{ij} of basic variables x_1, x_4 and s_3 on the optimal solution is determined by solving the followings system of inequalities. Any change in the coefficients of basic variables may affect both feasibility and optimality of the solution. Considering a change in the element belonging to basic variable $x_4 = \beta_2$, i.e. a_{24}.

Feasible condition: To maintain the feasibility of the solution we need to solve the following inequalities:

$$\underset{k \ne p}{\text{Max}}\left\{\frac{-x_{Bk}}{x_{Bk}\,\beta_{pi} - x_{Bp}\,\beta_{ki} > 0}\right\} \le \Delta\, a_{ij} \le \text{Min}\left\{\frac{-x_{Bk}}{x_{Bk}\,\beta_{pi} - x_{Bp}\,\beta_{ki} < 0}\right\}$$

where β_{ij} are the entries under column s_1, s_2 and s_3 in the optimal simplex table.

Since
$$x_{B1}\,\beta_{24} - x_{B2}\,\beta_{21} = x_{B1}\,\beta_{22} - x_{B2}\,\beta_{21}$$
$$= (16/19)\,(4/19) - (15/19)\,(-1/19) = 7/38$$
$$x_{B3}\,\beta_{24} - x_{B2}\,\beta_{22} = x_{B1}\beta_{22} - x_{B2}\,\beta_{21}$$
$$= (126/19)\,(4/19) - (15/19)\,(-15/38) = 3/2$$

Thus the range of variation in the element a_{24} maintaining feasibility is given by

$$\text{Max}\left\{\frac{-16/19}{7/38}, \frac{-126/19}{3/2}\right\} \le \Delta a_{24} \quad \text{or} \quad -84/19 \le \Delta a_{24}$$

Optimality condition: Further, consider the discrete changes in a_{ij} belonging to column vector. $x_4 = \beta_2$ in the basis matrix $\boldsymbol{B}$.

$$\beta_{22}\,(c_2 - z_2) - y_{22} C_B\,\beta_2 = \frac{8}{38}\left(-\frac{169}{38}\right) - \frac{21}{19}\left(\frac{53}{38}\right) = -2.47$$

$$\beta_{22}\,(c_3 - z_3) - y_{23}\,C_B\,\beta_2 = \frac{8}{38}\left(-\frac{1}{2}\right) - 0\left(\frac{53}{38}\right) = -0.105$$

$$\beta_{22}\,(c_5 - z_5) - y_{25}\,c_B\,\beta_2 = \frac{8}{38}\left(-\frac{1}{38}\right) - \left(-\frac{1}{19}\right)\left(\frac{53}{38}\right) = 0.078$$

$$\beta_{22}\,(c_6 - z_6) - y_{26}\,c_B\,\beta_2 = \frac{8}{38}\left(-\frac{53}{38}\right) - \frac{4}{19}\left(\frac{53}{38}\right) = 0.293$$

Thus, the range of variation in the element a_{24}, maintaining optimality, is given by:

$$\text{Max}\left\{\frac{-1/38}{0.078}, -\frac{53/38}{0.293}\right\} \le \Delta a_{24} \le \text{Min}\left\{\frac{-169/38}{-2.47}, \frac{-1/2}{-0.105}\right\}$$
$$\text{Max}\ \{-0.337, -4.760\} \le \Delta\, a_{24} \le \text{Min}\ \{1.80,\ 4.901\}$$
$$-0.337 \le \Delta a_{24} \le 1.80$$

Since both the feasibility and the optimality conditions need to be maintained for the element a_{24}, we have: $-0.337 \le \Delta\, a_{24} \le 1.80$.

Similarly, the ranges for discrete change in elements a_{14}, a_{23}, . . . etc., can also be determined.

The optimal basic feasible solution in Table 6.29 does not satisfy the additional constraint. $2x_1 + 3x_2 + x_3 + 5x_4 \le 2$. Thus a new optimal solution is obtained by adding slack variable s_4 to this constraint. It is written together with entries in Table 6.30 as shown in Table 6.30.

	$c_j \to$		3	4	1	7	0	0	0	0
c_B	Variables in Basis B	Solution Values $b\ (= x_B)$	x_1	x_2	x_3	x_4	s_1	s_2	s_3	s_4
3	x_1	16/19	1	9/38	1/2	0	5/38	−1/38	0	0
7	x_4	5/19	0	21/19	0	1	−1/38	−15/38	0	0
0	s_3	126/19	0	59/38	9/2	0	−1/38	−15/38	1	0
0	s_4	2	2	3	1	5	0	0	0	1
$Z = 83/9$		z_j	3	321/38	3/2	7	1/38	53/38	7	0
		$c_j - z_j$	0	−169/38	−1/2	0	−1/38	−53/38	−7	0

Table 6.30 Addition of New Constraint

In Table 6.30 the basis matrix B has been disturbed due to Row 4. Thus the coefficients in Row 4 under column x_1 and x_4 should become zero. This can be done by applying following row operations.

$$R_4(\text{new}) = R_4\ (\text{old}) - 2R_1 - 5R_2$$

The new solution so obtained is shown in Table 6.31.

	$c_j \rightarrow$		3	4	1	7	0	0	0	0
c_B	*Variables in Basis* B	*Solution Values* $\mathbf{b}\ (= \mathbf{x}_B)$	x_1	x_2	x_3	x_4	s_1	s_2	s_3	s_4
3	x_1	16/19	1	9/38	1/2	0	5/38	–1/38	0	0
7	x_4	5/19	0	21/19	0	1	–1/19	4/19	0	0
0	s_3	126/19	0	59/38	9/2	0	–1/38	–15/38	1	0
0	s_4	–1	0	–3	0	0	0	–1	0	1 →
Z = 83/19	$c_j - z_j$		0	–159/38	–1/2	0	–1/38	–53/38 ↑	0	0

Table 6.31 Infeasible Solution

The solution shown in Table 6.31 is not feasible because $s_4 = -1$. Thus the dual simplex method is applied to the obtained optimal basic feasible solution. As per rule of dual simplex method variable s_4 should obviously leave the basis and s_2 should enter into the basis. The new solution is shown in Table 6.32.

	$c_j \rightarrow$		3	4	1	7	0	0	0	0
c_B	*Variables in Basis* B	*Solution Values* $\mathbf{b}\ (= \mathbf{x}_B)$	x_1	x_2	x_3	x_4	s_1	s_2	s_3	s_4
3	x_1	33/38	1	6/19	1/2	0	5/38	0	0	–1/38
7	x_4	1/19	0	9/19	0	1	– 1/19	0	0	4/19
0	s_3	267/38	0	52/19	1/2	0	– 1/38	0	1	–15/38
0	s_4	1	0	3	0	0	0	1	0	–1
Z = 113/38	$c_j - z_j$		0	– 5/19	– 1/12	0	– 1/38	0	0	– 53/38

Table 6.32 Optimal and Feasible Solution

Since all $c_j - z_j \le 0$ in Table 6.32 and all $x_{Bi} \ge 0$, therefore the current solution is optimal. The new solution value are: $x_1 = 33/38$, $x_2 = 0$, $x_3 = 0$ $x_4 = 1/19$ and Max $Z = 113/38$.

CONCEPTUAL QUESTIONS

1. Write a short note on sensitivity analysis.
2. Discuss the role of sensitivity analysis in linear programming. Under what circumstances is it needed, and under what conditions do you think it is not necessary?
3. (a) Explain how a change in a input-output coefficient can affect a problem's optimal solution?
   (b) How can a change in resource availability affect a solution?
4. What do you understand by the term 'sensitivity analysis'? Discuss the effect of (i) variation of c_j, (ii) variation of b_i and (iii) addition of a new constraint.
5. Discuss the changes in the coefficients a_{ij} for the given LP problem: Max $Z = \mathbf{cx}$, subject to $\mathbf{Ax} = \mathbf{b}$, $\mathbf{x} \ge 0$.
6. Given that the problem: Max $Z = \mathbf{cx}$ such that $\mathbf{Ax} = \mathbf{b}$, $\mathbf{x} \ge 0$ has an optimal solution, can one obtain a linear programming problem which has an unbounded solution changing $\mathbf{b}$ alone?
7. Consider the LP problem: Max $Z = \mathbf{cx}$; subject to $\mathbf{Ax} = \mathbf{b}$ and $\mathbf{x} \ge 0$, where $\mathbf{c}$, $\mathbf{x}^T \in E^n$, $\mathbf{b}^T \in E^m$ and $\mathbf{A}$ is $m \times n$ coefficients matrix. Determine how much can components of the cost vector $\mathbf{c}$ be changed without affecting the optimal solution of the LP problem.
8. Find the limits of variation of element a_{ik} so that the optimal feasible solution of $\mathbf{Ax} = \mathbf{b}$, $\mathbf{x} \ge 0$, Max $Z = \mathbf{cx}$ remains the optimal feasible solution when (i) $\mathbf{a}_k \in \mathbf{B}$, (ii) $\mathbf{a}_k \notin \mathbf{B}$.

SELF PRACTICE PROBLEMS

1. In a LP problem, Max Z = $\boldsymbol{cx}$, subject to $\boldsymbol{Ax} = \boldsymbol{b}$, $\boldsymbol{x} \ge 0$, obtain the variation in coefficients c_j which are permitted without changing the optimal solution. Find this for the following LP problem

   Max $Z = 3x_1 + 5x_2$

   subject to (i) $x_1 + x_2 \le 1$; (ii) $2x_1 + x_2 \le 1$

   and $x_1, x_2 \ge 0$. *[Meerut Univ., BSc, 1990]*

2. Discuss the effect of changing the requirement vector from [6 4 24] to [6 2 12] on the optimal solution of the following LP problem:

   Max $Z = 3x_1 + 6x_2 + x_3$

   subject to (i) $x_1 + x_2 + x_3 \ge 6$; (ii) $x_1 + 5x_2 - x_3 \ge 4$;

   (iii) $x_1 + 5x_2 + x_3 \le 24$

   and $x_1, x_2, x_3 \ge 0$.

3. Discuss the effect of discrete changes in the parameter b_i (i = 1, 2, 3) for the LP problem.

Max $Z = 3x_1 + 4x_2 + x_3 + 7x_4$

subject to (i) $8x_1 + 3x_2 + 4x_3 + x_4 \le 7$;

(ii) $2x_1 + 6x_2 + x_3 + 5x_4 \le 3$;

(iii) $x_1 + 4x_2 + 5x_3 + 2x_4 \le 8$

and $x_1, x_2, x_3, x_4 \ge 0$ [*Meerut Univ., MSc (Maths), 1992*]

4. Given the LP problem

Max $Z = -x_1 + 2x_2 - x_3$

subject to (i) $3x_1 + x_2 - x_3 \le 10$; (ii) $-x_1 + 4x_2 + x_3 \ge 6$;

(iii) $x_2 + x_3 \le 4$

and $x_1, x_2, x_3 \ge 0$.

Determine the range for discrete changes in the resource values $b_2 = 10$ and $b_3 = 6$ of the LP model so as to maintain optimality of the current solution. [*Meerut Univ., MSc (Maths), 1998*]

5. Given the LP problem

Max $Z = 3x_1 + 5x_2$

subject to (i) $3x_1 + 2x_2 \le 18$; (ii) $x_1 \le 4$; (iii) $x_2 \le 6$

and $x_1, x_2 \ge 0$.

Discuss the effect on the optimality of the solution when the objective function is changed to $3x_1 + x_2$.

6. In an LP problem, Max $Z = \mathbf{cx}$, subject to $\mathbf{Ax = b}$; $\mathbf{x} \ge 0$, obtain the variation in coefficients c_j which are permitted without changing the optimal solution. Conduct sensitivity analysis on c_j's for the following LP problem.

Max $Z = 3x_1 + 5x_2$

subject to (i) $x_1 + x_2 \le 1$; (ii) $2x_1 + x_2 \le 1$

and $x_1, x_2 \ge 0$. [*Meerut Univ., BSc, 1990*]

7. Given the LP problem

Max $Z = x_1 + 2x_2 + x_3$

subject to (i) $3x_1 + x_2 - x_3 \le 10$; (ii) $-x_1 + x_2 + x_3 \le 6$;

(iii) $x_2 + x_3 \le 4$

and $x_1, x_2, x_3 \ge 0$.

(a) Determine optimal solution to the problem.

(b) Determine the effect of discrete changes in those components of the cost vector which corresponds to the basic variable.

8. Find the optimal solution to the LP problem

Max $Z = 15x_1 + 45x_2$

subject to (i) $x_1 + 16x_2 \le 250$, (ii) $5x_1 + 2x_2 \le 162$,

(iii) $x_2 \le 50$

and $x_1, x_2 \ge 0$.

If Max $Z = \Sigma\, c_j x_j$, j = 1, 2 and c_2 is kept fixed at 45, determine how much can c_1 be changed without affecting the optimal solution of the problem. [*Bombay Univ., BSc (Maths), 1991*]

9. Given the LP problem

Max $Z = -x_1 + 2x_2 - x_3$

subject to (i) $3x_1 + x_2 - x_3 \le 10$, (ii) $-x_1 + 4x_2 + x_3 \ge 6$

(iii) $x_2 + x_3 \le 4$

and $x_1, x_2, x_3 \ge 0$.

Determine the range for discrete changes in the resource values $b_1 = 10$ and $b_2 = 6$ of the LP model so as to maintain optimality of the current solution.

10. Consider the following LP problem

Max $Z = 4x_1 + 6x_2$

subject to (i) $x_1 + 2x_2 \le 8$, (ii) $6x_1 + 4x_2 \le 24$

and $x_1, x_2 \ge 0$.

(a) What is the optimal solution?

(b) If the first constraint is altered as: $x_1 + 3x_2 \le 8$, does the optimal solution change?

11. A stainless steel utensil manufacturer makes three types of items. The restrictions, profits and requirements are tabulated below:

Utensil Type	*I*	*II*	*III*
Raw material requirement (kg per unit)	6	3	5
Welding and finishing time (hours per unit)	3	4	5
Profit per unit (Rs.)	3	1	4

If stainless steel (raw material) availability is 25 kg and welding and finishing time available is 20 hours per day, then the optimum product mix problem is expressed as:

Max $Z = 3x_1 + x_2 + 4x_3$

subject to

$6x_1 + 3x_2 + 5x_3 \le 25$ (raw material restriction)

$3x_1 + 4x_2 + 5x_3 \le 20$ (time restriction)

and $x_1, x_2, x_3 \ge 0$

where x_j (j = 1, 2, 3) is the number of units of the jth type of the item to be produced. Get the optimal simplex table and encircle the appropriate answer to the following questions.

(a) The second type of utensil would change the current optimal basis if its profit per unit is: ≥ 1; ≥ 1.02; ≥ 1.06; ≥ 2; ≥ 3.

(b) The simplex multiplier associated with the machine time restriction of 20 hours is (– 3/5). Thus, the multiplier remains unchanged for the upper limit on the machine time availability of 25; 27.5; 35; 42.5; 32.5 hours.

(c) The increase in the objective function for each unit availability of machine time higher than the upper limit indicated in part (b) is ... (show calculations).

(d) The profit of the third type of utensil is Rs. 4 per unit. The lower unit on its profitability such that the current basis is still optimal is 4; 3; 2.5; 2; < 2.

12. The following LP model applies for Shastri & Sons Wood Furniture Company which makes tables (T), chairs (C) and book shelves (B), along with other items.

Max $Z = 200x_T + 150x_C + 150x_B$

subject to

$10x_T + 3x_C + 10x_B \le 100$ (Wood)

$5x_T + 5x_C + 5x_B \le 60$ (Labour)

and $x_T, x_C, x_B \ge 0$

The optimal simplex table is shown below:

		$c_j \rightarrow$	200	150	150	0	0
c_B	*Variables in Basis* B	*Solution Values* $b\ (= x_B)$	x_T	x_C	x_B	s_W	s_L
200	T	64/7	1	0	1	1/7	– 6/70
150	C	20/7	0	1	0	– 1/7	2/7
Z = 15,800/7		z_j	200	150	200	50/7	80/7
		$c_j - z_j$	0	0	– 50	– 50/7	– 80/7

where s_W and s_L are the slack variables for unused word and labour, respectively.

(a) Determine the sensitivity limits for the available wood and labour within which the present product mix will remain optimal.
(b) Find the new optimal solution when the available wood is 90 board feet and labour is 100 hours.

13. Refer to the data in Practice Problem 12.
(a) Determine the sensitivity limits for unit profits within which the current optimal solution will remain unchanged.
(b) What is the total profit when each table yields a profit of Rs. 180 and each chair yields a profit of Rs. 100.

14. A company makes two products A and B. The production of both products requires processing time in two departments I and II. The hourly capacity of I and II, unit profits for products: A and B and the processing time requirements in I and II are given in the following table:

Department	*Product*		*Capacity (Hours)*
	A	*B*	
I	1	2	32
II	0	1	8
Unit profit (Rs.)	200	300	

Now the company is considering the addition of a new product C to its line. Product C requires one hour each of departments I and II. What must be product C's unit profit in order to profitably add it to the firm's product line?

15. A company produces three products A, B and C. Each product requires two raw materials: steel and aluminium. The following LP model describes the company's product mix problem.

Max $Z = 30x_A + 10x_B + 50x_C$

subject to

$$6x_A + 3x_B + 5x_C \le 450 \text{ (Steel)}$$
$$3x_A + 4x_B + 5x_C \le 300 \text{ (Aluminium)}$$

and $x_A, x_B, x_C \ge 0$

The optimal production plan is given in the following table:

		$c_j \rightarrow$	30	10	50	0	0
Unit Profit c_B	*Variables in Basis* B	*Solution Values* $b (= x_B)$	x_A	x_B	x_C	s_S	s_A
0	s_S	150	3	−1	0	1	−1
50	x_C	60	3/5	4/5	1	0	1/5
$Z = 3{,}000$		$c_j - z_j$	0	−30	0	0	−10

where s_s and s_A are the slack variables for unused steel and aluminium quantity, respectively.
(a) Suppose an additional 300 tonnes of steel may be procured at a cost of Rs. 100 per tonne. Should the company procure the additional steel?
(b) Unit profit of product A is Rs. 30. How much should this price be increased so that A is produced by the company?

16. A pig farmer is attempting to analyse his feeding operation. The minimum daily requirement of the three nutritional elements for the pigs and the number of units of each of these nutritional elements in two feeds is given in the following table:

Required Nutritional Element	*Units of Nutritional Elements (in kg)*		*Minimum Requirement*
	Food 1	*Food 2*	
A	20	30	200
B	40	25	350
C	30	45	430
Cost per kg (Rs.)	5	3	

(a) Formulate and solve this problem as an LP model.
(b) Assume that the farmer can purchase a third feed at a cost of Rs. 2 per kg, which will provide 35 units of nutrient A, 30 units of nutrient B and 50 units of nutrient C. Would this change the optimal mix of feeds? If yes, how?

17. A company sells two different products A and B. The selling price and incremental cost information is as follows:

	Product A	*Product B*
Selling price (Rs.)	60	40
Incremental cost (Rs.)	30	10
Incremental profit (Rs.)	30	30

The two products are produced in a common production process and are sold in two different markets. The production process has a capacity of 30,000 labour hours. It takes three hours to produce a unit of A and one hour to produce a unit of B. The market has been surveyed and company officials feel that the maximum number of units of A that can be sold is 8,000; and that of B is 12,000 units.
(a) Find the optimal product mix.
(b) Suppose maximum number of units of A and B that can be sold is actually 9,000 units and 13,000 units, respectively instead of as given in the problem, what effect does this have on the solution? What is the effect on the profit? What is the shadow price for the constraint on the sales limit for both these products.
(c) Suppose there are 31,000 labour hours available instead of 30,000 as in the base case, what effect does this have on the solution? What is the effect on the profit? What is the shadow price for the constraint on the number of labour hours.

18. An organization can produce a particular component for passenger cars, jeeps and trucks. The production of the component requires utilization of sheet metal working and painting facilities, the details of which are given below.

Resource	*Consumption (in hrs) to Produce a Unit for*			*Availability (hrs)*
	Passenger Car	*Jeep*	*Truck*	
Sheet metal working	0.25	1	0.5	12
Painting	0.5	1	2	30

The profits that can be earned by the three categories of components, i.e. for passenger cars, jeeps and trucks are Rs. 600, Rs. 1,400, and Rs. 1,300, respectively.
(a) Find the optimal product mix for that organization.
(b) What additional profit would be earned by increasing the availability of (i) Sheet metal working shop by an hour only, and (ii) Painting shop by an hour only.
(c) What would be the effect on the profit earned if at least one component for jeep had to be produced?

19. A firm uses three machines in the manufacture of three products. Each unit of product I requires 3 hours on machine 1, 2 hours on machine 2 and 1 hour on machine 3. Each unit of product II requires 4 hours on machine 1, 1 hour on machine 2 and 3 hours on machine 3. Each unit of product III requires 2 hours on machine 1, 2 hours on machine 2, and 2 hours on machine 3. The contribution margin of the three products is Rs. 30, Rs. 40 and Rs. 35 per unit, respectively. Available for scheduling are 90 hours of machine 1 time, 54 hours of machine 2 time, and 93 hours of machine 3 time.
(a) What is the optimal production schedule for the firm?
(b) What is the marginal value of an additional hour of time on machine?
(c) What is the opportunity cost associated with product 1? What interpretation should be given to this opportunity cost?
(d) Suppose that the contribution margin for product I is increased from Rs. 30 to Rs. 43, would this change the optimal production plan? Give reasons.

HINTS AND ANSWERS

1. $-\infty \le c_1 \le 3 + (1/3)$; $5 - (1/2) \le c_2 \le 5 + \infty$ or $9/2 \le c_2 \le \infty$
2. The current basic feasible solution consisting of s_1, x_1 and x_3 remains feasible and optimal at the new values: $x_1 = 7, x_3 = 5$ and $x_2 = 0$ with Min $Z = 26$.
3. $5 \ge b_1 \ge -32/5$; $84/5 \ge b_2 \ge -5/4$; $\Delta b_3 \ge -126/19$
4. $b_2 < 10$; $-5/2 \le b_3 \ge 6$
5. Initial optimal solution: $x_1 = 2, x_2 = 6$ and Max $Z = 36$. Where $c_3 - z_3 = -1$ and $c_5 - z_5 = -3$. After having $c_1 = 3$ and $c_2 = 1$, calculate new values of $c_3 - z_3 = 1$ and $c_5 - z_5 = 2/3$. Thus, introduce s_3 in the basis to get the new solution: $x_1 = 2, x_2 = 6$ and Max $Z = 12$.
7. (a) $x_1 = 0, x_2 = 4, x_3 = 0$ and Max $Z = 8$ (b) $-(1/2) \le c_2$
8. $(15 - 195/16) \le c_1 \le (15 + 195/16)$
9. $b_2 < 10$; $-5/2 \le b_3 \ge 6$
12. (a) $36 \le x_w \le 120$; $50 \le x_L \le 500/3$
    (b) $x_T = 54/7$; $40/7$; $x_C = 30/7$; $100/7$; $Z = 15{,}300/7$, $23{,}000/7$.
13. $15 \le x_T \le 50$; $6 \le x_C \le 20$; $-\infty \le x_B \le 20$
17. (a) $x_A = 6{,}000$, $x_B = 12{,}000$, Max Z = Rs. 5,40,000
    (b) As $x_A \le 8{,}000$ is not binding because $x_A = 6{,}000$ in the solution. Thus, an increase in the limit on A will have no effect on the solution or profit. Shadow price is zero. For increase in the limit of B, the new solution is, $x_A = 5{,}670$ and $x_B = 13{,}000$ and Max Z = Rs. 5,60,000. The shadow price is (5,60,000 – 5,40,000) = Rs. 20,000.
    (c) New solution is $x_A = 6{,}330$ and $x_B = 12{,}000$ with Max Z = Rs. 5,50,000. The shadow price is (5,50,000 – 5,40,000) = Rs. 10,000.

CHAPTER SUMMARY

Sensitivity analysis provides the sensitive ranges (both lower and upper limits) within which LP model parameters can vary without changing the optimally of the current solution. Formulae to conduct sensitivity analysis followed by numerical examples have been solved to illustrate how changes in the parameters specially, (i) the change in the objective function coefficients; right hand-side value of a constraint and (ii) input-output coefficients in any constraint of a LP model, affect its optimal solution.

CHAPTER CONCEPTS QUIZ

True or False

1. The solution to the dual yields shadow prices.
2. The absolute values of the numbers in the $c_j - z_j$ row under the slack variables represent the solutions to the dual problem.
3. The transpose of the primal constraints coefficients become the dual constraint coefficients.
4. The range over which shadow prices remain valid is called right-hand-side ranging.
5. The shadow price is the value of additional profit margin from an activity to be conducted.
6. The solution in optima as long as all $c_j - z_j \ge 0$.
7. Non-basic variables are those that have a value of zero.
8. Testing basic variables does not require reworking the final simplex table during sensitivity analysis.
9. In a simplex table, if all of the substitution rates in the key column are negative, then it indicates an infeasible solution.
10. Making changes in the resources value result in changes in the feasible region and often the optima solution.

Fill in the Blanks

11. The ________ for a constrain is the value of an additional unit of the resource.
12. The range of significance are values over which a ________ coefficient can vary without causing a change in the optimal solution mix.
13. The range of optimality are values over which a ________ coefficient can change without causing a change in the optimal solution mix.
14. Right-hand-side ranging method is used to find the range over which ________ remain valid.
15. In the optimal simplex table the absolute value of cj – zj numbers corresponds to slack variable column represent ________ of the available resources.
16. The addition of a constraint in the existing constraints will cause a ________ change in the objective function coefficients.
17. The addition of new constraint ________ the feasible region of the given LP problem.
18. Sensitivity analysis provides the range within which a parameter may change without offering ________.
19. When an additional variable is added in the LP problem, the ________ solution can be improved if $c_j - z_j \le 0$.
20. A non-basic variable should be brought into new solution mix provided its ________ is $c_j < c_j + (z_j - c_j)$.

Multiple Choice

21. Sensitivity analysis
    (a) is also called post-optimality analysis as it is carried out after the optimal solution is obtained
    (b) allows the decision-maker to get more meaningful information about changes in the LP model parameters
    (c) provides the range within which a parameter may change without affecting optimality
    (d) all of the above
22. When an additional variable is added in the LP model, the existing optimal solution can further be improved if
    (a) $c_j - z_j \ge 0$ (b) $c_j - z_j \le 0$
    (c) both (a) and (b)   (d) none of the above
23. Addition of an additional constraint in the existing constraints will cause a

(a) change in objective function coefficients (c_j)
(b) change in coefficients a_{ij}
(c) both (a) and (b)
(d) one of the above

24. If the additional constraint added is an equation and an artificial variable appears in the basis of the new problem, the new optimal solution is obtained by
(a) assigning zero cost coefficient to the artificial variable if it appears in the basis at negative value
(b) assigning $-M$ cost coefficient to the artificial variable if it appears in the basis at positive value
(c) either (a) and (b)
(d) none of the above

25. To ensure the best marginal increase in the objective function value, a resource value may be increased whose shadow price is comparatively
(a) larger (b) smaller
(c) neither (a) nor (b) (d) both (a) and (b)

26. A non-basic variable should be brought into the new solution mix provided its contribution rate (c_j^*) is
(a) $c_j^* = c_j + (z_j - c_j)$ (b) $c_j^* > c_j + (z_j - c_j)$
(c) $c_j^* < c_j + (z_j - c_j)$ (d) none of the above

27. While performing sensitivity analysis, the upper bound infinity on the value of the right-hand side of a constraint means that
(a) there is slack in the constraint
(b) the constraint is redundant
(c) the shadow price for that constraint is zero
(d) none of the above

28. The entering variable in the sensitivity analysis of objective function coefficients is always a
(a) decision variable (b) non-basic variable
(c) basic variable (d) slack variable

29. To maintain optimality of current optimal solution for a change Δc_k in the coefficient c_k of non-basic variable x_k, we must have
(a) $\Delta c_k = c_k - z_k$ (b) $\Delta c_k = z_k$
(c) $c_k + \Delta c_k = z_k$ (d) $\Delta c_k \geq z_k$

30. In sensitivity analysis of the coefficient of the non-basic variable in a cost minimization LP problem, the upper sensitivity limit is
(a) original value + Lowest positive value of improvement ratio
(b) original value – Lowest absolute value of improvement ratio
(c) positive infinity
(d) negative infinity

31. If the RHS value of a constraint is changed to another value within specified limits, then the new optimal solution would have
(a) same optimal solution
(b) a lower objective function value
(c) a higher objective function value
(d) same set of variables in the solution mix

32. A change in the objective function for a non-basic variable can affect
(a) $c_j - z_j$ values of all non-basic variables
(b) $c_j - z_j$ values of all basic variables
(c) Only the $c_j - z_j$ value of that variable
(d) none of the above

33. The 100 per cent rule for RHS values says that if the sum of the percentage change is less than or equal to 100%, then
(a) dual prices do not change
(b) different dual prices may exist
(c) both (a) and (b) may occur
(d) none of the above

34. The 100 per cent rule for objective function coefficients says that if the sum of the percentage change is less than or equal to 100%, then
(a) optimal solution does not change
(b) a different optimal solution may exist
(c) both (a) and (b) may occur
(d) none of the above

35. The right-hand side range is often referred to as the range of
(a) improvement (b) feasibility
(c) infeasibility (d) optimality

Answers to Quiz

1. T 2. T 3. T 4. T 5. F 6. F 7. T 8. F 9. F 10. T
11. shadow price 12. non-basic variable 13. basic variable 14. shadow price 15. shadow price
16. simultaneous 17. leaves unchanged 18. optimality 19. existing optimal 20. contribution rate
21. (d) 22. (a) 23. (c) 24. (c) 25. (a) 26. (c) 27. (d) 28. (d) 29. (c) 30. (c) 31. (d)
32. (c) 33. (a) 34. (a) 35. (b)

chapter 7

Integer Linear Programming

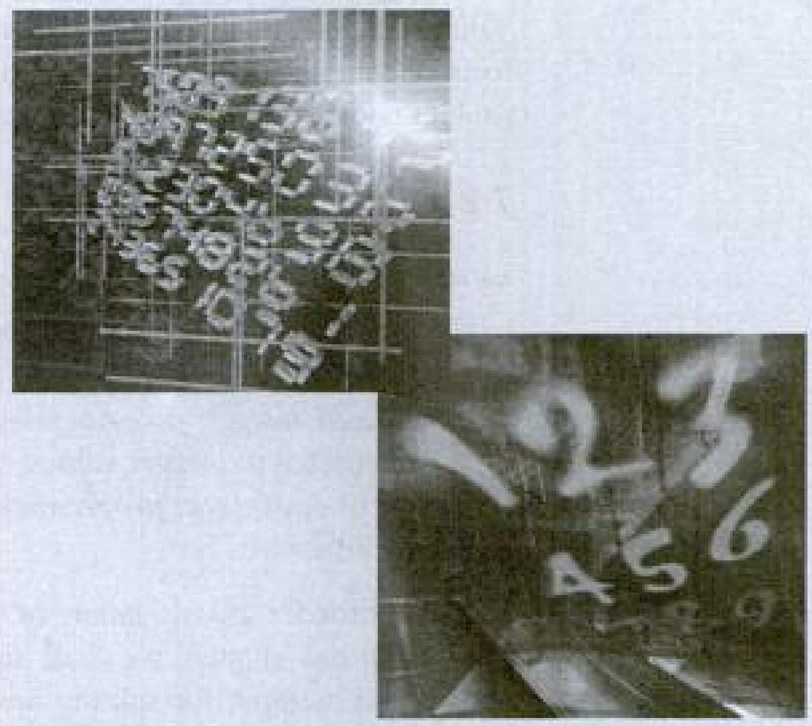
"The key is not to prioritize what's on your schedule, but to schedule your priorities."

– R. Covey

Preview In this chapter, Gomory's cutting plane method, and Branch and Bound method have been discussed for solving an extension of LP model called linear integer LP model. In a linear integer LP model one or more of the variables must be integer due to certain managerial considerations.

Learning Objectives After studying this chapter, you should be able to

- understand the limitations of simplex method in deriving integer solution to linear programming problems.
- apply cutting plane methods to obtain optimal integer solution value of variables in an LP problem.
- apply Branch and Bound method to solve integer LP problems.
- appreciate application of integer LP problem in several areas of managerial decision-making.



Chapter Outline

7.1 INTRODUCTION

In linear programming, each decision variable as well as slack and/or surplus variable is allowed to take any real or fractional value. However, there are certain real-life problems in which the fractional value of the decision variables has no significance. For example, it does not make sense to say that 1.5 men will be working on a project or 1.6 machines will be used in a workshop. The integer solution to a problem can, however, be obtained by rounding off the optimum value of the variables to the nearest integer value. This approach can be easy in terms of economy of effort, time, and the cost that might be required to derive an integer solution. This solution however may not satisfy all the given constraints. Secondly, the value of the objective function so obtained may not be the optimal value. All such difficulties can be avoided if the given problem, where an integer solution is required, is solved by integer programming techniques.

Integer LP problems are those in which some or all of the variables are restricted to integer (or discrete) values. An integer LP problem has important applications. Capital budgeting, construction scheduling, plant location and size, routing and shipping schedule, batch size, capacity expansion, fixed charge, etc., are few problems that demonstrate the areas of application of integer programming.

7.2 TYPES OF INTEGER PROGRAMMING PROBLEMS

Linear integer programming problems can be classified into three categories:

(i) *Pure (all) integer programming problems* in which all decision variables are restricted to integer values.

(ii) *Mixed integer programming problems* in which some, but not all, of the decision variables are restricted to integer values.

(iii) *Zero-one integer programming problems* in which all decision variables are restricted to integer values of either 0 or 1.

The broader classification of integer LP problems and their solution methods are summarized in Fig. 7.1 In this chapter, we shall discuss two methods: (i) Gomory's cutting plane method and (ii) Branch and Bound method, for solving integer programming problems.

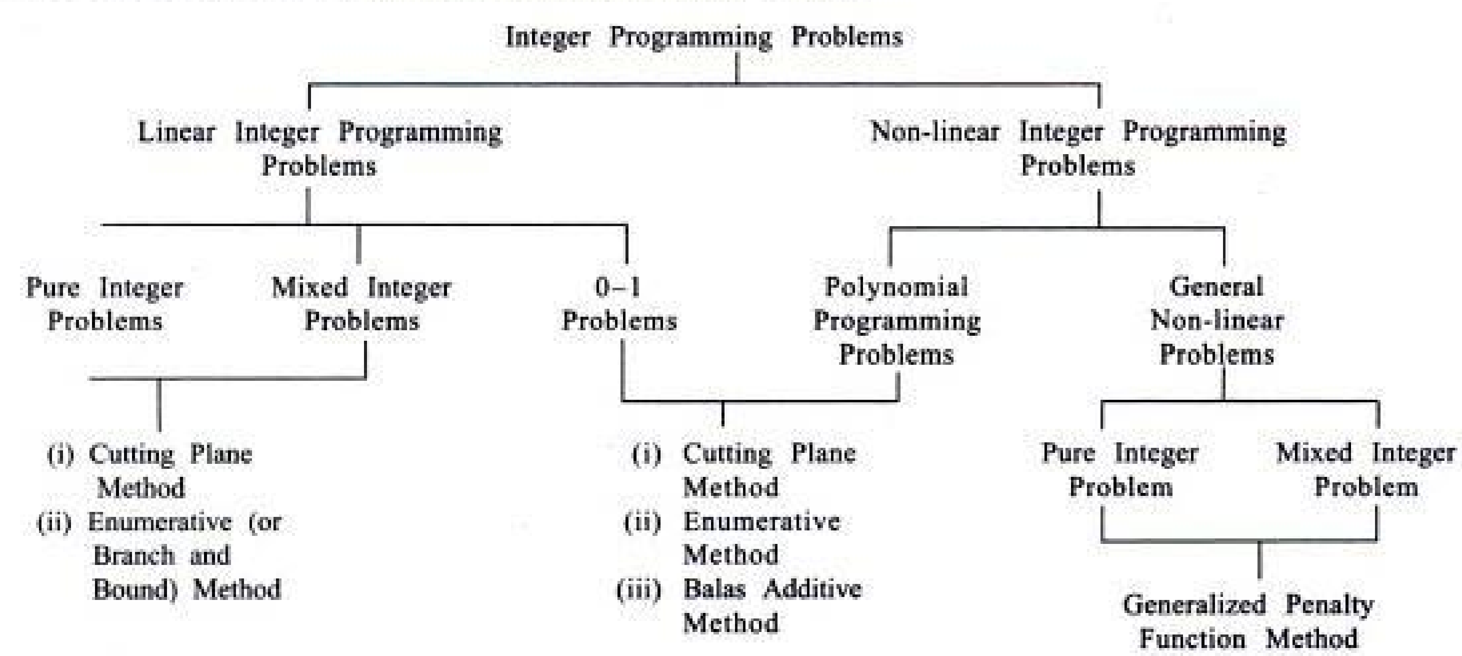


Fig. 7.1 Classification of Integer LP Problems and their Solution Methods

The pure integer linear programming problem in its standard form can be stated as follows:

Maximize $Z = c_1 x_1 + c_2 x_2 + \ldots + c_n x_n$

subject to the constraints

$$\begin{aligned} a_{11} x_1 + a_{12} x_2 + \ldots + a_{1n} x_n &= b_1 \\ a_{21} x_1 + a_{22} x_2 + \ldots + a_{2n} x_n &= b_2 \\ \vdots \qquad\qquad & \quad \vdots \\ a_{m1} x_1 + a_{m2} x_2 + \ldots + a_{mn} x_n &= b_m \end{aligned}$$

and $\quad x_1, x_2, \ldots, x_n \geq 0$ and are integers.

7.3 ENUMERATION AND CUTTING PLANE SOLUTION CONCEPT

The cutting-plane method to solve integer LP problems was developed by R.E. Gomory in 1956. He developed this by using the dual simplex method. This method is based on the generation of a *sequence of linear inequalities called a cut*. This *cut* cuts out only a part of the feasible region of the corresponding LP problem while leaving out the feasible region of the integer linear programming problem. The hyperplane boundary of a cut is called the *cutting plane.*

Illustration Consider the following linear integer programming (LIP) problem

Maximize $Z = 14x_1 + 16x_2$

subject to the constraints

(i) $4x_1 + 3x_2 \leq 12$, (ii) $6x_1 + 8x_2 \leq 24$

and $x_1, x_2 \geq 0$ and are integers.

A **cut** is the linear constraint added to the given LP problem constraints

Relaxing the integer requirements, the problem is solved graphically by using Fig. 7.2. We obtain the optimal solution to this LP problem as: $x_1 = 1.71$, $x_2 = 1.71$ and Max $Z = 51.42$

This solution does not satisfy the integer requirement of variables x_1 and x_2. The feasible region (solution space) formed by the constraints is marked by OABC in Fig. 7.1.

Rounding off this solution to $x_1 = 2$, $x_2 = 2$ does not satisfy both the constraints and therefore, the solution is infeasible. The dots in Fig. 7.1, also referred to as *lattice points,* represent all of the integer solutions that lie within the feasible solution space of the LP problem. However, it is difficult to evaluate every such point in order to determine the value of the objective function.

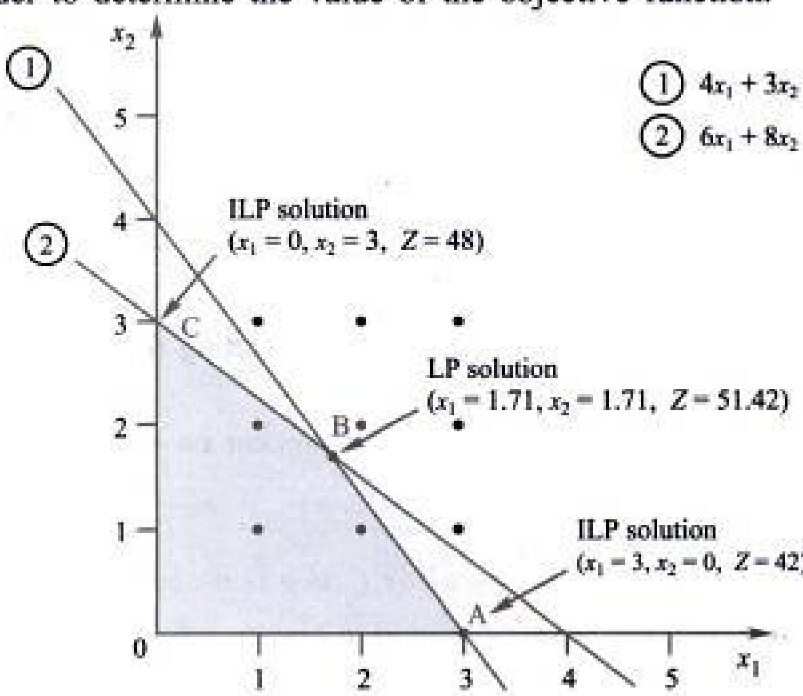


Fig. 7.2 Concept of Cutting Plane

Figure 7.2 suggests that we can find a solution to the problem when the problem is formulated as an LP problem (which as a matter of chance could contain integers). It may be noted that the optimal lattice point, point C, lies at the corner of the solution space OABC, obtained by cutting away the small portion above the dotted line. This suggests a solution procedure that successively cuts down (reduces) the feasible solution space until an integer-valued corner is found.

The optimal integer solution to the given LP problem is: $x_1 = 0$, $x_2 = 3$ and Max $Z = 48$. Notice that its lattice point is not even adjacent to the most desirable LP problem solution corner.

Hyperplane boundary of a cut is called the **cutting plane**

Remark Reducing the feasible region by adding extra constraints (cut) can never give an improved objective function value. Usually it makes it worse and if Z_{IP} represents the minimum value of objective function in an ILP problem and Z_{LP} the minimum value of objective function in an LP problem, then $Z_{IP} \geq Z_{LP}$.

7.4 GOMORY'S ALL INTEGER CUTTING PLANE METHOD

A systematic procedure called *Gomory's all-integer algorithm* will be discussed here for generating 'cuts' (additional linear constraints) so as to ensure an integer solution to the given LP problem in a finite number of steps. Gomory's algorithm has the following properties.

(i) Additional linear constraints never cutoff that portion of the original feasible solution space that contains a feasible integer solution to the original problem.
(ii) Each new additional constraint (or hyperplane) cuts off the current non-integer optimal solution to the linear programming problem.

7.4.1 Method for Constructing Additional Constraint (Cut)

An integer LP problem is usually worse in terms of higher cost or lower profit

Gomory's method begins by solving an LP problem without taking into consideration the integer value requirement of the decision variables. If the solution so obtained is an integer, i.e. all variables in the '$\mathbf{x_B}$' *column* (also called basis) of the simplex table assume non-negative integer values, the current solution is the optimal solution to the given ILP problem. However, if some of the basic variables do not have non-negative integer value, an additional linear constraint called the *Gomory constraint* (or cut) is generated. After having generated a linear constraint (or cutting plane), it is added to the bottom of the optimal simplex table so that the solution no longer remains feasible. The new problem is then solved by using the dual simplex method. If the optimal solution, so obtained, is again a non-integer, another cutting plane is generated. The procedure is repeated until all basic variables assume non-negative integer values.

The procedure of developing a cut is discussed in what follows. In the optimal simplex table, we select one of the rows, called *source row* for which basic variable is non-integer. The desired cut is developed by considering only fractional parts of the coefficients in source row. For this reason, such a cut is also referred to as *fractional cut.*

Suppose the basic variable x_r has the largest fractional value among all basic variables, restricted to be integers. Then the rth constraint equation (row) from the simplex table can be rewritten as:

$$x_{Br}\,(= b_r) = 1 \cdot x_r + (a_{r1}\,x_1 + a_{r2}\,x_2 + \ldots) = x_r + \sum_{j \neq r} a_{rj}\,x_j \tag{1}$$

where x_j ($j = 1, 2, 3, \ldots$) represents all the non-basic variables in the rth constraint (row), except the variables x_r and b_r ($= x_{Br}$) are the non-integer value of variable x_r. Let us decompose the coefficients of x_j, x_r variables and x_{Br} into integer and non-negative fractional parts in Eq. (1), as shown below:

$$[x_{Br}] + f_r = (1 + 0)\,x_r + \sum_{j \neq r} \{[a_{rj}] + f_{rj}\}\,x_j \tag{2}$$

where $[x_{Br}]$ and $[a_{rj}]$ denote the largest integer obtained by truncating the fractional part from x_{Br} and a_{rj} respectively.

Rearranging Eq. (2) so that all the integer coefficients appear on the left-hand side, we get

$$f_r + \{[x_{Br}] - x_r - \sum_{j \neq r} [a_{rj}]x_j\} = \sum_{j \neq r} f_{rj}\,x_j \tag{3}$$

where f_r is strictly a positive fraction ($0 < f_r < 1$) while f_{rj} is a non-negative fraction ($0 \leq f_{rj} \leq 1$).

Since all the variables (including slacks) are required to assume integer values, the terms in the bracket on the left-hand side as well as on the right-hand side must be non-negative numbers. Since the left-hand side in Eq. (3) is f_r plus a non-negative number, we may write it in the form of the following inequalities:

$$f_r \leq \sum_{j \neq r} f_{rj}\,x_j \tag{4}$$

or

$$\sum_{j \neq r} f_{rj}\,x_j = f_r + s_g \quad \text{or} \quad -f_r = s_g - \sum_{j \neq r} f_{rj}\,x_j \tag{5}$$

Constraints called **Gomory cuts** lead to a smaller feasible region that includes all feasible integer values

where s_g is a non-negative slack variable and is called the *Gomory slack variable.*

Equation (5) represents *Gomory's cutting plane constraint*. When this new constraint is added to the bottom of optimal simplex table, it would create an additional row in the table, along with a column for the new variable s_g.

7.4.2 Steps of Gomory's All Integer Programming Algorithm

An iterative procedure for the solution of an all integer programming problem by Gomory's cutting plane method can be summarized in the following steps.

Step 1: Initialization Formulate the standard integer LP problem. If there are any non-integer coefficients in the constraint equations, convert them into integer coefficients. Solve the problem by the simplex method, ignoring the integer requirement of variables.

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Iteration 3: Remove the variable s_{g_3} from the basis and enter variable s_{g_2} into the basis by applying the dual simplex method. The new solution is shown in Table 7.10.

Table 7.10 Optimal Solution

	$c_j \rightarrow$		2	20	−10	0	0	0	0
c_B	Variables in Basis B	Solution Values $b\ (= x_B)$	x_1	x_2	x_3	s_1	s_{g_1}	s_{g_2}	s_{g_3}
20	x_2	0	0	1	0	0	1	0	0
2	x_1	2	1	0	0	0	−4	0	2
0	s_1	3	0	0	0	1	−16	0	8
−10	x_3	2	0	0	1	0	1	0	−3
0	s_{g_2}	1	0	0	0	0	0	1	−2
$Z = -16$		$c_j - z_j$	0	0	0	0	−2	0	−34

Since all variables in Table 7.10 have assumed integer values and all $c_j - z_j \leq 0$, the solution is integer optimal solution: $x_1 = 2$, $x_2 = 0$, $x_3 = 2$ and Max $Z = -16$.

Example 7.3 The owner of a readymade garments store sells two types of shirts – Zee-shirts and Button-down shirts. He makes a profit of Rs 3 and Rs 12 per shirt on Zee-shirts and Button-down shirts, respectively. He has two tailors, A and B, at his disposal, for stitching the shirts. Tailors A and B can devote at the most 7 hours and 15 hours per day, respectively. Both these shirts are to be stitched by both the tailors. Tailors A and B spend 2 hours and 5 hours, respectively in stitching one Zee-shirt, and 4 hours and 3 hours, respectively on stitching a Button-down shirt. How many shirts of both types should be stitched in order to maximize the daily profit?

(a) Formulate and solve this problem as an LP problem.

(b) If the optimal solution is not integer-valued, use Gomory technique to derive the optimal integer solution. *[Delhi Univ., MBA, 1995]*

Mathematical formulation Let, x_1 and x_2 = number of Zee-shirts and Button-down shirts to be stitched daily, respectively.

Then the mathematical model of the LP problem is stated as:

Maximize $Z = 3x_1 + 12x_2$

subject to the constraints

(i) Availability of time with tailor A : $2x_1 + 4x_2 \leq 7$

(ii) Availability of time with tailor B : $5x_1 + 3x_2 \leq 15$

and $x_1, x_2 \geq 0$ and are integers.

Solution (a) Adding slack variables s_1 and s_2, the given LP problem is stated into its standard form as:

Maximize $Z = 3x_1 + 12x_2 + 0s_1 + 0s_2$

subject to the constraints

(i) $2x_1 + 4x_2 + s_1 = 7$, (ii) $5x_1 + 3x_2 + s_2 = 15$

and $x_1, x_2, s_1, s_2 \geq 0$ and are integers

The optimal solution of the LP problem, obtained by using the simplex method is given in Table 7.11.

Table 7.11 Optimal Non-integer Solution

	$c_j \rightarrow$		3	12	0	0
c_B	Variables in Basis B	Solution Values $b\ (= x_B)$	x_1	x_2	s_1	s_2
12	x_2	7/4	1/2	1	1/4	0
0	s_2	39/4	7/2	0	−3/4	1
$Z = 21$		$c_j - z_j$	−3	0	−3	0

The non-integer optimal solution shown in Table 7.11 is: $x_1 = 0$, $x_2 = 7/4$ and Max $Z = 21$.

(b) To obtain the integer-valued solution, we proceed to construct Gomory's fractional cut, with the help of x_2-row (because it has largest fraction value) as follows:

$$\frac{7}{4} = \frac{1}{2}x_1 + x_2 + \frac{1}{4}s_1 \qquad (x_2\text{-source row})$$

or

$$\left(1+\frac{3}{4}\right) = \left(0+\frac{1}{2}\right)x_1 + (1+0)\,x_2 + \left(0+\frac{1}{4}\right)s_1$$

$$\frac{3}{4} + (1 - x_2) = \frac{1}{2}x_1 + \frac{1}{4}s_1 \text{ or } \frac{3}{4} \le \frac{1}{2}x_1 + \frac{1}{4}s_1$$

On adding Gomory slack variable s_{g_1}, the required Gomory's fractional cut becomes:

$$-\frac{3}{4} = s_{g_1} - \frac{1}{2}x_1 - \frac{1}{4}s_1 \qquad \text{(Cut I)}$$

Adding this additional constraint to the bottom of the optimal simplex Table 7.11, the new table so obtained is shown in Table 7.12.

		$c_j \rightarrow$	3	12	0	0	0
c_B	Variables in Basis B	Solution Values $b\ (= x_B)$	x_1	x_2	s_1	s_2	s_{g_1}
12	x_2	7/4	1/2	1	1/4	0	0
0	s_2	39/4	7/2	0	– 3/4	1	0
0	s_{g_1}	– 3/4	– 1/2	0	– 1/4	0	1 →
$Z = 21$		$c_j - z_j$	– 3	0	– 3	0	0
		Ratio: Min $(c_j - z_j)/y_{3j}$ (< 0)	6 ↑	–	12	–	–

Table 7.12 Optimal but Infeasible Solution

Iteration 1: Remove variable s_{g_1} from the basis and enter variables x_1 into the basis by applying the dual simplex method. The new solution is shown in Table 7.13.

		$c_j \rightarrow$	3	12	0	0	0
c_B	Variables in Basis B	Solution Values $b\ (= x_B)$	x_1	x_2	s_1	s_2	s_{g_1}
12	x_2	1	0	1	0	0	1
0	s_2	9/2	0	0	– 5/2	1	7
3	x_1	3/2	1	0	1/2	0	2
$Z = 33/2$		$c_j - z_j$	0	0	– 3/2	0	– 6

Table 7.13

The optimal solution shown in Table 7.13 is still non-integer. Therefore, by adding one more fractional cut, with the help of the x_1-row, we get:

$$\frac{3}{2} = x_1 + \frac{1}{2}s_1 - 2\,s_{g_1} \quad (x_1\text{-source row})$$

$$\left(1+\frac{1}{2}\right) = (1+0)\,x_1 + \left(0+\frac{1}{2}\right)s_1 + (-2+0)\,s_{g_1}$$

$$\frac{1}{2} + (1 - x_1 + 2s_{g_1}) = \frac{1}{2}s_1 \text{ or } \frac{1}{2} \le \frac{1}{2}s_1$$

2. An airline owns an ageing fleet of Boeing 737 jet airplanes. It is considering a major purchase of up to 17 new Boeing models 757 and 767 jets. The decision must take into account several cost and capacity factors including the following: (i) the airline can finance up to Rs 4,000 million in purchases; (ii) each Boeing 757 will cost Rs 350 million, while each Boeing 767 will cost Rs 220 million; (iii) at least one-third of the planes purchased should be the longer-ranged 757; (iv) the annual maintenance budget is to be no more than Rs 80 million; (v) the annual maintenance cost per 757 is estimated to be Rs 8,00,000 and it is Rs 5,00,000 for each 767 purchased; (vi) each 757 can carry 1,25,000 passengers per year, while each 767 can fly 81,000 passengers annually. Formulate this problem as an integer programming problem in order in order to maximize the annual passenger carrying capacity and solve the problem by using the cutting plane method.
3. An air conditioning and refrigeration company has been awarded a contract for the air conditioning of a new computer installation. The company has to make a choice between two alternatives: (a) Hire one or more refrigeration technicians for 6 hours a day, or (b) hire one or more part-time refrigeration apprentice technicians for 4 hours a day. The wage rate of refrigeration technicians is Rs 400 per day, while the corresponding rate for apprentice technicians is Rs 160 per day. The company does not want to engage the technicians on work for more than 25 man hours per day. It also wants to limit the charges of technicians to Rs 4,800. The company estimates that the productivity of a refrigeration technician is 8 units and that of part time apprentice technician is 3 units. Formulate and solve this problem as an integer LP problem to enable the company to select the optimal number of technicians and apprentices.
[*Delhi Univ., MBA, 1997*]
4. A manufacturer of toys makes two types of toys, A and B. Processing of these two toys is done on two machines X and Y. The toy A requires two hours on machine X and six hours on machine Y. Toy B requires four hours on machine X and five hours on machine Y. There are sixteen hours of time per day available on machine X and thirty hours on machine Y. The profit obtained on both the toys is the same, i.e. Rs 5 per toy. Formulate and solve this problem as an integer LP problem to determine the daily production of each of the two toys?
[*Delhi Univ., MBA, 1998*]
5. The ABC Electric Appliances Company produces two products: Refrigerators and ranges. The production of these takes place in two separate departments. Refrigerators are produced in department I and ranges in department II. Both these are sold on a weekly basis. Due to the limited facilities in the departments the weekly production cannot exceed 25 refrigerators in department I and 35 ranges in department II. The company regularly employs a total of 50 workers in the departments. The production of one refrigerator requires two man-weeks of labour and of one range one man-week. A refrigerator contributes a profit of Rs 300 and a range of Rs 200. Formulate and solve this problem as an integer LP problem to determine the units of refrigerators and ranges that the company should produce to realize the maximum profit? [*Delhi Univ, MBA, 2003*]
6. XYZ Company produces two types of tape recorders: Reel-to-reel model and cassette model, on two assembly lines. The company must process each tape recorder on each assembly line. It has found that the this whole process requires the following amount of time:

Assembly Line	*Reel-to-reel*	*Cassette*
1	6 hours	2 hours
2	3 hours	2 hours

The production manager says that line 1 will be available for 40 hours per week and line 2 for only 30 hours per week. After the mentioned hours of operation each line must be checked for repairs. The company realizes a profit of Rs 300 on each reel-to-reel tape recorder and Rs 120 on each cassette recorder. Formulate and solve this problem as an integer LP problem in order to determine the number of recorders of each type to be produced each week so as to maximize profit.
[*Delhi Univ., MBA, 1996*]
7. A company that manufactures metal products is planning to buy any of the following three types of lathe machines – manual, semi-automatic and fully-automatic. The manual lathe machine costs Rs 1,500, while the semi-automatic and fully-automatic lathe machines cost Rs 4,000 and Rs 6,000 respectively. The company's budget for buying new machines is Rs 1,20,000. The estimated contribution towards profit from the manual, semi-automatic and fully-automatic lathe machines are: Rs 16, Rs 20 and Rs 22, respectively. The available floor space allows for the installation of only 36 new lathe machines. The maintenance required on fully automatic machines is low and the maintenance department can maintain 50 fully-automatic machines in a year. The maintenance of a semi-automatic machine takes 20 per cent more time than that of a fully automatic machine and a manually-operated machine takes 50 per cent more time for maintenance of a fully-automatic machine. Formulate and solve this problem as an integer LP problem in order to determine the optimal number of machines to be bought.
8. A stereo equipment manufacturer can produce two models – A and B – of 40 and 80 watts of total power, each. Each model passes through three manufacturing divisions 5 namely 1, 2 and 3, where model A takes 4, 2.5 and 4.5 hours each and model B takes 2, 1 and 1.5 hours each. The three divisions have a maximum of 1,600, 1,200 and 1,600 hours every month, respectively. Model A gives a profit contribution of Rs 400 each and B of Rs 100 each. Assuming abundant product demand, formulate and solve this problem as an integer LP problem, to determine the optimal product mix and the maximum contribution.
[*Delhi Univ., MBA, 1999*]
9. A manufacturing company produces two types of screws – metal and wooden. Each screw has to pass through the slotting and threading machines. The maximum time that each machine can be run is 150 hours per month. A batch of 50 wooden screws requires 2 minutes on the threading machine and 3 minutes on the slotting machine. Metal screws of the same batch size require 8 minutes on the threading machine and 2 minutes on the slotting machine. The profit contribution for each batch of wooden and metal screws is Re 1 and Rs 2.50, respectively. Formulate and solve this problem as an integer LP problem in order to determine the optimal product mix for maximum profit contribution.
10. A dietician for a hospital is considering a new breakfast menu that includes oranges and cereal. This breakfast must meet the minimum requirements for the Vitamins A and B. The number of milligrams of each of these vitamins contained in a purchasing unit, for each of these foods, is as follows:

Vitamin	*Milligrams per Purchasing Unit of Food*		*Minimum Requirement (mg)*
	Oranges (doz)	*Cereal (box)*	
A	1	2	20
B	3	2	50

The cost of the food ingredients is Rs 15 per dozen for oranges and Rs 12.50 per box for cereal. For dietary reasons, at least one unit of each food type must be used in the menu plan. Formulate and solve this problem as an integer programming problem.
11. The dietician at a local hospital is planning the breakfast menu for the maternity ward patients. She is primarily concerned with

Vitamin E and iron requirements, for planning the breakfast. According to the State Medical Association (SMA) new mothers must get at least 12 milligrams of Vitamin E and 24 milligrams of iron from breakfast. The SMA handbook reports that a scoop of scrambled egg contains 2 milligrams of Vitamin E and 8 milligrams of iron. The handbook also recommends that new mothers should eat at least two scoops of cottage cheese for their breakfast. The dietician considers this as one of the model constraints. The hospital's accounting department estimates that one scoop of cottage cheese costs Rs 2 and one scoop of scrambled egg also costs Rs 2. The dietician is attempting to determine the optimum breakfast menu that satisfies all the requirements and minimizes the total cost. The cook insists that he can serve foods by only full scoop, thus necessitating an integer solution. Determine the optimum integer solution to the problem. *[Delhi Univ., MBA (HCA), 2002, 2003]*

12. A building contractor has just won a contract to build a municipal library building. His present labour workforce is inadequate to immediately take up this work as the force is already involved with other jobs on hand. The convertor must therefore immediately decide whether to hire one or more labourers on a full-time basis (eight hours a day each) or to allow overtime to one or more of the existing labour force (five hours a day each). Extra labourers can be hired for Rs 40 per day (for eight hours) while overtime costs Rs 43 per day (for five hours per day). The contractor wants to limit his extra payment to Rs 400 per day and to use no more than twenty labourers (both full-time and overtime) because of limited supervision. He estimates that the new labour employed on a full-time basis will generate Rs 15 a day in profits, while overtime labour Rs 20 a day. Formulate and solve this problem as an integer LP problem to help the building contractor to decide the optimum labour force.

13. The ABC company requires an output of at least 200 units of a particular product per day. To accomplish this target it can buy machines A or B or both. Machine A costs Rs 20,000 and B Rs 15,000. The company has a budget of Rs 2,00,000 for the same. Machines A and B will be able to produce 24 and 20 units, respectively of this product per day. However, machine A will require a floor space of 12 square feet while machine B will require 18 square feet. The company only has a total floor space of 180 square feet. Formulate and solve this problem as an integer LP problem to determine the minimum number of machines that should be purchased. *[Delhi Univ., MBA, 1999, 2001]*

14. A company produces two products A and B. Each unit of product A requires one hour of engineering services and five hours of machine time. To produce one unit of product B, two hours of engineering and 8 hours of machine time are needed. A total of 100 hours of engineering and 400 hours of machine time is available. The cost of production is a non-linear function of the quantity produced as given in the following table:

Product A		*Product B*	
Production (units)	*Unit Cost (Rs)*	*Production (units)*	*Unit Cost (Rs)*
0– 50	10	0– 40	7
50–100	8	40–100	3

The unit selling price of product A is Rs 12 and of product B is Rs 14. The company would like a production plan that gives the number of units of A and the number of units of B to be produced that would maximize profit. Formulate and solve this problem as an integer linear programming problem to help the company maximize its total revenue. *[Delhi Univ., MBA, 1995, 1999, 2003]*

15. XYZ Corporation manufactures an electric device, final assembly of which is accomplished by a small group of trained workers operating simultaneously on different devices. Due to space limitations, the working group may not exceed ten in number. The firm's operating budget allows Rs 5,400 per month as salary for the group. A certain amount of discrimination is evidenced by the fact that the firm pays men in the group Rs 700 per month, while women doing the same work receive Rs 400. However, previous experience has indicated that a man will produce about Rs 1,000 in value added per month, while a woman worker adds Rs 900. If the firm wishes to maximize the value added by the group, how many men and women should be included? (A non-integer solution for this problem will not be accepted).

16. A manufacturer of baby dolls makes two types of dolls. One is sold under the brand name 'Molina' and the other under 'Suzie'. These two dolls are processed on two machines – A and B. The processing time for each 'Molina' is 2 hours and 6 hours on machines A and B, respectively and that for each 'Suzie' is 5 hour and 5 hours on machines A and B, respectively. There is 16 hours of time available per day on machine A and 30 hours on machine B. The profit contribution from a 'Molina' is Rs 6 and that from a 'Suzie' is Rs 18. Formulate and solve this problem as an integer LP problem to determine the optimal weekly production schedule of the two dolls. *[Delhi Univ, MBA (PSM), 2004]*

17. The dietician at the local hospital is planning the breakfast menu for the maternity ward patients. She is planning a special non-fattening diet, and has chosen cottage cheese and scrambled eggs for breakfast. She is primarily concerned with Vitamin E and Iron requirements in planning the breakfast.

According to the State Medical Association (SMA) new mothers must get at least 12 milligrams of Vitamin E and 24 milligrams of iron from breakfast. The SMA handbook reports that a scoop of cottage cheese contains 3 milligrams of Vitamin E and 3 milligrams of iron. An average scoop of scrambled egg contains 2 milligrams of Vitamin E and 8 milligrams of iron. The SMA handbook recommends that new mothers should eat at least two scoops of cottage cheese for their breakfast. The dietician considers this as one of the model constraints.

The hospital accounting department estimates that a scoop of cottage cheese costs Re 1, and a scoop of scrambled egg also costs Re 1. The dietician is attempting to determine the optimum breakfast menu that satisfies all the requirements and minimize total cost. The cook insists that he can serve foods by only full scoop, thus necessitating an integer solution. Formulate and solve this problem as an integer LP problem to determine the optimum-integer solution to the problem.

18. A firm makes two products: X and Y, and has total production capacity of 9 tonnes per day, X and Y requiring the same production capacity. The firm has a permanent contract to supply at least 2 tonnes of X and at least 3 tonnes of Y, per day, to another company. Each tonne of X requires 20 machine-hours production time and each tonne of Y requires 50 machine hours. The daily maximum possible number of machine-hours is 350. All the firm's output can be sold and the profit made is Rs 80 per tonne of X and Rs 120 per tonne of Y. It is required to determine the production schedule for attaining the maximum profit and to calculate this profit. (A non-integer solution for this problem will not be accepted). *[Delhi Univ., MBA, 1999, 2004]*

19. An Airline corporation is considering the purchase of three types of jet planes. The purchase price would be Rs 45 crore for each A type plane; Rs 40 crore for each B type plane and Rs 25 crore for each C type plane. The corporation has resources worth Rs 500 crore for these purchases. The three types of planes, if purchased, would be utilized essentially at maximum capacity. It is estimated that the net annual profit would be Rs 3 million for A type, Rs 2.25 million for B type and Rs 1.5 million for C type planes. Each plane requires one pilot and it is estimated that 25 trained pilots would be available. If only C type planes

image
not
available

Thus, the new cut can be expressed as

$$s_g = -f_r + \sum_{j \in R} f_{rj}^{*} x_j \tag{14}$$

where
$$f_{rj}^{*} = \begin{cases} a_{rj} & , \ a_{rj} \geq 0 \text{ and } x_j \text{ non-integer} \\ \left(\dfrac{f_r}{f_r - 1}\right) a_{rj} & , \ a_{rj} < 0 \text{ and } x_j \text{ non-integer} \\ f_{rj} & , \ f_{rj} \leq f_r \text{ and } x_j \text{ integer} \\ \left(\dfrac{f_r}{f_r - 1}\right)(1 - f_{rj}) & , \ f_{rj} > f_r \text{ and } x_j \text{ integer} \end{cases} \tag{15}$$

7.5.2 Steps of Gomory's Mixed-Integer Programming Algorithm

Gomory's mixed-integer cutting plane method can be summarized in the following steps:

Step 1: Initialization Formulate the standard integer LP problem. Solve it by simplex method, ignoring integer requirement of variables.

Step 2: Test of optimality

(a) Examine the optimal solution. If all integer restricted basic variables have integer values, then terminate the procedure. The current optimal solution, obtained, in Step 1 is the optimal basic feasible solution to the integer LP problem.

(b) If all integer restricted basic variables are not integers, then go to Step 3.

Step 3: Generate cutting plane Choose a row r corresponding to a basic variable x_r that has the highest fractional value f_r and generate a cutting plane as explained earlier in the form [Ref. Eq. 13]:

$$s_g = -f_r + \sum_{j \in R_+} a_{rj} x_j + \left(\frac{f_r}{f_r - 1}\right) \sum_{j \in R_-} a_{rj} x_j, \text{ where } 0 < f_r < 1.$$

Step 4: Obtain the new solution Add the cutting plane generated in Step 3 to the bottom of the optimal simplex table as obtained in Step 1. Find a new optimal solution by using the dual simplex method and return to Step 2. The process is repeated until all restricted basic variables are integers.

Example 7.4 Solve the following mixed-integer programming problem:

Maximize $Z = -3x_1 + x_2 + 3x_3$

subject to the constraints

(i) $-x_1 + 2x_2 + x_3 \leq 4$, (ii) $2x_2 - (3/2)x_3 \leq 1$, (iii) $x_1 - 3x_2 + 2x_3 \leq 3$

and $x_1, x_2 \geq 0$, x_3 non-negative integer.

Solution Given integer LP problem can be expressed in its standard form as:

Maximize $Z = -3x_1 + x_2 + 3x_3 + 0s_1 + 0s_2 + 0s_3$

subject to the constraints

(i) $-x_1 + 2x_2 + x_3 + s_1 = 4$, (ii) $2x_2 - (3/2)x_3 + s_2 = 1$, (iii) $x_1 - 3x_2 + 2x_3 + s_3 = 3$

and $x_1, x_2, s_1, s_2, s_3 \geq 0$, x_3 non-negative integer.

Step 1: Ignoring the integer requirement, the optimal solution of the problem using the simplex method is obtained, and is given in Table 7.16.

Table 7.16 Optimal Non-integer Solution

		$c_j \rightarrow$	−3	1	3	0	0	0
c_B	Variables in Basis B	Solution Values $b\ (= x_B)$	x_1	x_2	x_3	s_1	s_2	s_3
1	x_2	5/7	−3/7	1	0	2/7	0	−1/7
0	s_2	48/7	9/7	0	0	1/7	1	10/7
3	x_3	13/7	5/14	0	1	3/7	0	2/7
$Z = 44/7$		$c_j - z_j$	−51/14	0	0	−11/7	0	−5/7

The non-integer optimal solution as shown in Table 7.16 is: $x_1 = 0$, $x_2 = 5/7$, $x_3 = 13/7$ and Max $Z = 44/7$.

Step 2: Since, basic variable x_3 is required to be integer, constructing Gomory's mixed-integer cut with the help of x_3-row we get

$$\frac{13}{7} = \frac{5}{14}x_1 + x_3 + \frac{3}{7}s_1 + \frac{2}{7}s_3 \quad (x_3\text{-source row})$$

Since the coefficients of non-basic variables x_1, s_1 and s_3 are positive, therefore factorizing the coefficients using rule (15), we get:

$$\left(1+\frac{6}{7}\right) = \frac{1}{2}x_1 + x_2 \le \frac{11}{3}\left(0+\frac{5}{14}\right)x_1 + (1+0)x_3 + \left(0+\frac{3}{7}\right)s_1 + \left(0+\frac{2}{7}\right)s_3$$

$$\frac{6}{7} + (1 - x_3) = \frac{5}{14}x_1 + \frac{3}{7}s_1 + \frac{2}{7}s_3, \quad \text{i.e.} \quad \frac{6}{7} \le \frac{5}{14}x_1 + \frac{3}{7}s_1 + \frac{2}{7}s_3$$

On adding slack variable s_{g_1}, we obtain Gomory's mixed integer cut as follows:

$$-\frac{5}{14}x_1 - \frac{3}{7}s_1 - \frac{2}{7}s_3 + s_{g1} = -\frac{6}{7} \quad \text{(Mixed integer cut I)}$$

Adding this constraint to the bottom of the Table 7.16. The new values so obtained are shown in Table 7.17.

	$c_j \rightarrow$		−3	1	3	0	0	0	0
c_B	*Variables in Basis* B	*Solution Values* $b\ (= x_B)$	x_1	x_2	x_3	s_1	s_2	s_3	s_{g_1}
1	x_2	5/7	−3/7	1	0	2/7	0	−1/7	0
0	s_2	48/7	9/7	0	0	1/7	1	10/7	0
3	x_3	13/7	5/14	0	1	3/7	0	2/7	0
0	s_{g_1}	−6/7	−5/14	0	0	−3/7	0	−2/7	1 →
Z = 44/7		$c_j - z_j$	−51/14	0	0	−11/7	0	−5/7	0
		Ratio: Min $(c_j - z_j)/y_{4j}\ (< 0)$	51/5	—	—	11/3	—	5/2 ↑	—

Table 7.17 Optimal but Infeasible Solution

Iteration 1: Remove the variable s_{g_1} from the basis and enter the variable s_3 into the basis by applying the dual simplex method. The new solution so obtained is shown in Table 7.18.

	$c_j \rightarrow$		−3	1	3	0	0	0	0
c_B	*Variables in Basis* B	*Solution Values* $b\ (= x_B)$	x_1	x_2	x_3	s_1	s_2	s_3	s_{g_1}
1	x_2	8/7	−1/4	1	0	1/2	0	0	−1/2
0	s_2	18/7	−1/2	0	0	−2	1	0	5
3	x_3	1	0	0	1	0	0	0	1
0	s_3	3	5/4	0	0	3/2	0	1	−7/2
Z = 29/7		$c_j - z_j$	−11/4	0	0	−1/2	0	0	−5/2

Table 7.18 Optimal Solution

Since variable x_3 has assumed integer value and all $c_j - z_j \le 0$, the optimal mixed integer solution is: $x_1 = 0$, $x_2 = 8/7$, $x_3 = 1$ and Max $Z = 29/7$.

Example 7.5 Solve the following mixed-integer programming problem

Maximize $Z = x_1 + x_2$

subject to the constraints

(i) $3x_1 + 2x_2 \le 5$, (ii) $x_2 \le 2$

and $x_1, x_2 \ge 0$, x_1 non-negative integer.

Solution After converting the given LP problem into its standard form, obtain an optimal solution, ignoring the integer restriction on x_1 by the simplex method. The optimal solution is given in Table 7.19.

Table 7.19 Optimal Solution

	$c_j \rightarrow$		1	1	0	0
c_B	Variables in Basis B	Solution Values $b\ (= x_B)$	x_1	x_2	s_1	s_2
1	x_1	1/3	1	0	1/3	−2/3
1	x_2	2	0	1	0	1
$Z = 7/3$		$c_j - z_j$	0	0	−1/3	−1/3

Since all $c_j - z_j \le 0$, the optimal non-integer solution is: $x_1 = 1/3$, $x_2 = 2$ and Max $Z = 7/2$.

In the current optimal solution the variable x_1, which is restricted to take integer value, is not an integer, therefore, we generate Gomory cut considering x_1-row as follows:

$$\frac{1}{3} = x_1 + \frac{1}{3}s_1 - \frac{2}{3}s_2 \quad (x_1\text{-source row})$$

Since the coefficient of s_1 is positive, by applying rule (15) we get:

$$f_{13}^{*} = \frac{1}{3}\left[f_{rj}^{*} = a_{rj}\ ;\ a_{rj} \ge 0\right]$$

The coefficient of s_2 is also negative, and so by applying rule (15), we get:

$$f_{14}^{*} = \left(\frac{f_r}{f_r - 1}\right) f_r = \left\{\frac{1/3}{(1/3)-1}\right\}\left(-\frac{2}{3}\right) = \frac{1}{3}$$

Thus, Gomory's mixed integer cut becomes

$$-\frac{1}{3}s_1 - \frac{1}{3}s_2 + s_{g_1} = -\frac{1}{3} \quad \text{(Mixed integer cut I)}$$

where s_{g_1} is Gomory's slack variable.

By introducing this Gomory cut at the bottom of Table 7.19, we get a new table, as shown in Table 7.20.

Table 7.20 Optimal but Infeasible Solution

	$c_j \rightarrow$		1	1	0	0	0
c_B	Variables in Basis B	Solution Values $b\ (= x_B)$	x_1	x_2	s_1	s_2	s_{g_1}
1	x_1	1/3	1	0	1/3	−2/3	0
1	x_2	2	0	1	0	1	0
0	s_{g_1}	−1/3	0	0	−1/3	−1/3	1 →
$Z = 7/3$		$c_j - z_j$	0	0	−1/3	−1/3	0
	Ratio: Min $(c_j - z_j)/y_{3j}$ (< 0)		—	—	1 ↑	1	—

Applying the dual simplex method, we obtain the revised solution as shown in Table 7.21.

Table 7.21 Optimal Solution

	$c_j \rightarrow$		1	1	0	0	0
c_B	Variables in Basis B	Solution Values $b\ (= x_B)$	x_1	x_2	s_1	s_2	s_{g_1}
1	x_1	0	1	0	0	−1	1
1	x_2	2	0	1	0	1	0
0	s_1	1	0	0	1	1	−3
$Z = 2$		$c_j - z_j$	0	0	0	0	−1

Since all $c_j - z_j \le 0$, therefore the required mixed integer optimal solution, given in Table 7.21, is: $x_1 = 0$, $x_2 = 2$, $s_1 = 1$ and Max $Z = 2$.

7.6 BRANCH AND BOUND METHOD

The branch and bound method was first developed by AH Land and AG Doig, and it was further studied by JDC Little et al., and other researchers. This method can be used to solve all-integer, mixed-integer and zero-one linear programming problems.

The concept behind this method is to divide the entire feasible solution space of an LP problem into smaller parts called *subproblems* and then evaluate corner (extreme) points of each subproblem for an optimal solution. This approach is useful for solving LP problems where there is a large number of feasible solutions and so the enumeration becomes economically impractical.

The branch and bound method starts by imposing bounds on the value of objective function that help to determine the subproblem to be eliminated from consideration when the optimal solution has been found. If the solution to a subproblem does not yield an optimal integer solution, a new subproblem is selected for branching. At a point where no more subproblem can be created, an optimal solution is arrived at.

Branch and bound method breaks the feasible solution region into smaller regions until an optimal solution is obtained

The branch and bound method for the profit-maximization integer LP problem involves the following steps:

The Steps of the Algorithm

Step 1: Initialization Consider the following all integer programming problem

Maximize $Z = c_1 x_1 + c_2 x_2 + \ldots + c_n x_n$

subject to the constraints

$$\begin{aligned} a_{11} x_1 + a_{12} x_2 + \ldots + a_{1n} x_n &= b_1 \\ a_{21} x_1 + a_{22} x_2 + \ldots + a_{2n} x_n &= b_2 \\ \vdots \qquad\qquad & \quad \vdots \\ a_{m1} x_1 + a_{m2} x_2 + \ldots + a_{mn} x_n &= b_m \end{aligned} \qquad \text{(LP-A)}$$

and $x_j \geq 0$ and non-negative integers.

Obtain the optimal solution of the given problem ignoring integer restriction on the variables.

(i) If the solution to this LP problem (say LP-A) is infeasible or unbounded, the solution to the given all-integer programming problem is also infeasible or unbounded, as the case may be.

(ii) If the solution satisfies the integer restrictions, the optimal integer solution has been obtained. If one or more basic variables do not satisfy integer requirement, then go to Step 2. Let the optimal value of objective function of LP-A be Z_1. This value provides an initial upper bound on objective function value and is denoted by Z_U.

(iii) Find a feasible solution by rounding off each variable value. The value of objective function so obtained is used as a lower bound and is denoted by Z_L.

Step 2: Branching step

(i) Let x_k be one basic variable which does not have an integer value and also has the largest fractional value.

(ii) Branch (or partition) the LP-A into two new LP subproblems (also called *nodes*) based on integer values of x_k that are immediately above and below its non-integer value. That is, it is partitioned by adding two mutually exclusive constraints:

$$x_k \leq [x_k] \quad \text{and} \quad x_k \geq [x_k] + 1$$

to the original LP problem. Here $[x_k]$ is the integer portion of the current non-integer value of the variable x_k. This is obviously is done to exclude the non-integer value of the variable x_k. The two new LP subproblems are as follows:

LP Subproblem B	*LP Subproblem C*
Max $Z = \sum_{j=1}^{n} c_j x_j$	Max $Z = \sum_{j=1}^{n} c_j x_j$
subject to $\sum_{j=1}^{n} a_{ij} x_j = b_i$	subject to $\sum_{j=1}^{n} a_{ij} x_j = b_i$
$x_k \leq [x_k]$	$x_k \geq [x_k] + 1$
and $x_j \geq 0$.	and $x_j \geq 0$.

Step 3: Bound step Obtain the optimal solution of subproblems B and C. Let the optimal value of the objective function of LP-B be Z_2 and that of LP-C be Z_3. The best integer solution value becomes the lower bound on the integer LP problem objective function value (Initially this is the rounded off value). Let the lower bound be denoted by Z_L.

Step 4: Fathoming step Examine the solution of both LP-B and LP-C

(i) If a subproblem yields an infeasible solution then terminate the branch.
(ii) If a subproblem yields a feasible solution but not an integer solution then return to Step 2.
(iii) If a subproblem yields a feasible integer solution, examine the value of the objective function. If this value is equal to the upper bound, an optimal solution has been reached. But if it is not equal to the upper bound but exceeds the lower bound, this value is considered as new upper bound and return to Step 2. Finally, if it is less than the lower bound, terminate this branch.

Step 5: Termination The procedure of branching and bounding continues until no further sub-problem remains to be examined. At this stage, the integer solution corresponding to the current lower bound is the optimal all-integer programming problem solution.

Remark The above algorithm can be represented by an enumeration tree. Each node in the tree represents a subproblem to be evaluated. Each branch of the tree creates a new constraint that is added to the original problem.

Example 7.6 Solve the following all integer programming problem using the branch and bound method.

Maximize $Z = 2x_1 + 3x_2$

subject to the constraints

(i) $6x_1 + 5x_2 \leq 25$, (ii) $x_1 + 3x_2 \leq 10$

and $x_1, x_2 \geq 0$ and integers. [*Jammu Univ., BE (Mach.) 2006*]

Solution Relaxing the integer conditions, the optimal non-integer solution to the given integer LP problem obtained by graphical method as shown in Fig. 7.4 is: $x_1 = 1.92$, $x_2 = 2.69$ and max $Z_1 = 11.91$.

The value of Z_1 represents initial lower bound as: $Z_L = 11$. Selecting variable x_2 for branching. Then to divide the given problem into two sub-problems and to eliminate the fractional part of $x_2 = 2.69$, two new constraints $x_2 \leq 2$ and $x_2 \geq 3$ are created and added to the constraints of original LP problem as follows:

LP Sub-problem B

Max $Z = 2x_1 + 3x_2$,

subject to (i) $6x_1 + 5x_2 \leq 25$, (ii) $x_1 + 3x_2 \leq 10$,

(iii) $x_2 \leq 2$,

and $x_1, x_2 \geq 0$ integers.

LP Sub-problem C

Max $Z = 2x_1 + 3x_2$

subject to (i) $6x_1 + 5x_2 \leq 25$, (ii) $x_1 + 3x_2 \leq 10$

(iii) $x_2 \geq 3$,

and $x_1, x_2 \geq 0$ and integers.

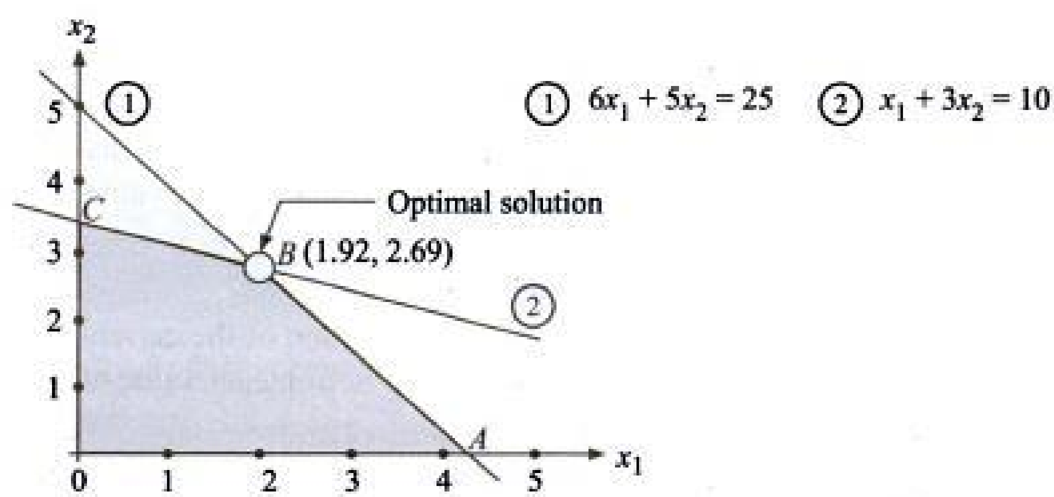

Fig. 7.4

Sub-problem B and C are solved graphically. The solutions are:

Sub-problem $B : x_1 = 2.5,\ x_2 = 2$ and Max $Z_2 = 11$

Sub-problem $C : x_1 = 1,\ x_2 = 3$ and Max $Z_3 = 11$

Graphical solutions of sub-problem B and C are shown in Fig. 7.5.

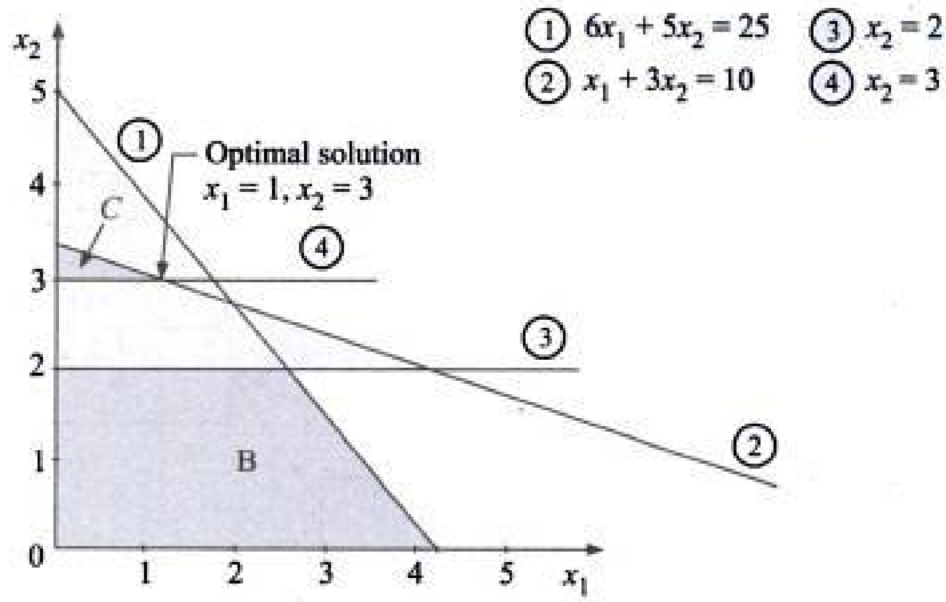

Fig. 7.5

In the solution space of LP sub-problem C, variables x_1 and x_2 are integers, so there is no need to branch this sub-problem further. The value Max $Z_L = 11$ is a lower bound on the maximum value of Z for future solutions.

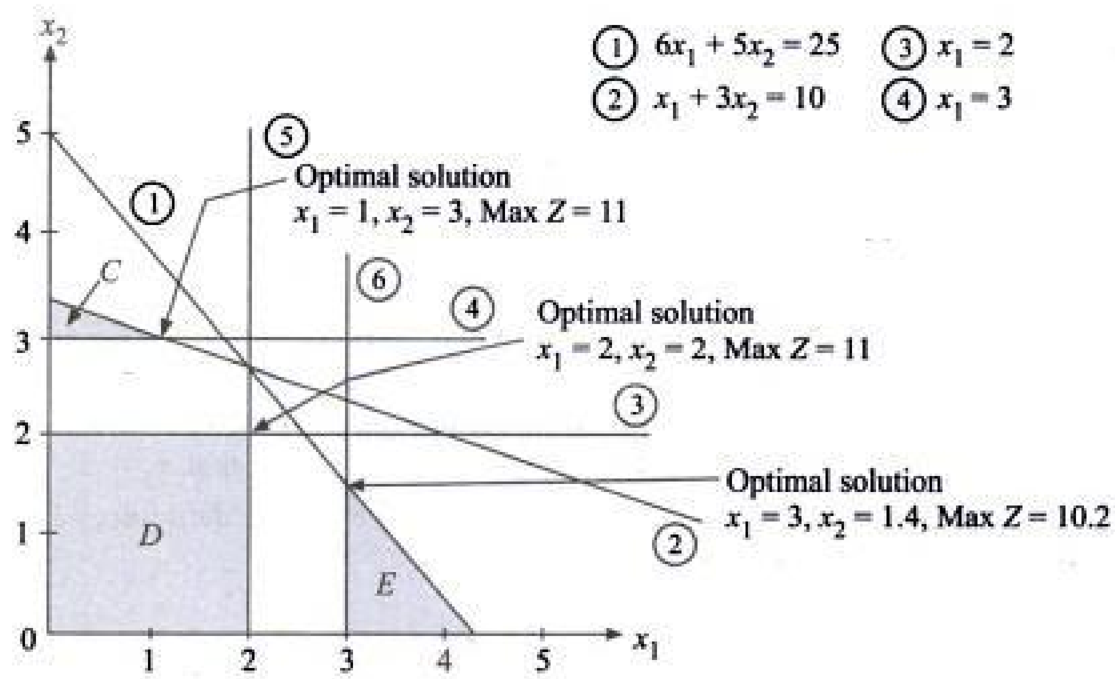

Fig. 7.6

LP sub-problem B is further subdivided into two LP sub-problem D and E (shown in Fig. 7.6) by taking variable $x_1 = 2.5$. Adding two new constraints $x_1 \leq 2$ and $x_1 \geq 3$. Also Max $Z = 11$ is also not inferior to the $Z_L = 11$.

LP Sub-problem D

Max $Z = 2x_1 + 3x_2$,

subject to (i) $6x_1 + 5x_2 \leq 25$, (ii) $x_1 + 3x_2 \leq 10$

(iii) $x_2 \leq 2$, (iv) $x_1 \leq 2$

and $x_1, x_2 \geq 0$ and integers.

LP Sub-problem E

Max $Z = 2x_1 + 3x_2$

subject to (i) $6x_1 + 5z_2 \leq 25$, (ii) $x_1 + 3x_2 \leq 10$,

(iii) $x_2 \geq 2$, (iv) $x_1 \geq 3$

and $x_1, x_2 \geq 0$ and integers.

Sub-problems D and E are solved graphically. The solutions as shown in Fig. 7.6 are

Sub-problem D : $x_1 = 2$, $x_2 = 2$ and max $Z_4 = 10$

Sub-problem E : $x_1 = 3$, $x_2 = 1.4$ and max $Z_5 = 10.2$

The solution of LP sub-problem D is integer feasible but is inferior to the best available solution of LP sub-problem E. Hence the value of lower bound $Z_L = 11$ remains unchanged and sub-problem D is not considered for further division.

Since the solution of sub-problem E is non-integer, it can be further branched with x_2 as the branching variable. But the value of its objective function (= 10.2) is inferior to the lower bound and hence this does not give a solution better than the one already obtained. The sub-problem E is also not considered for further branching. Hence, the best available solution corresponding to sub-problem C is the integer optimal solution: $x_1 = 1$, $x_2 = 3$ and Max $Z = 11$ of the given integer LP problem. The entire branch and bound procedure for the given problem is shown in Fig. 7.7.

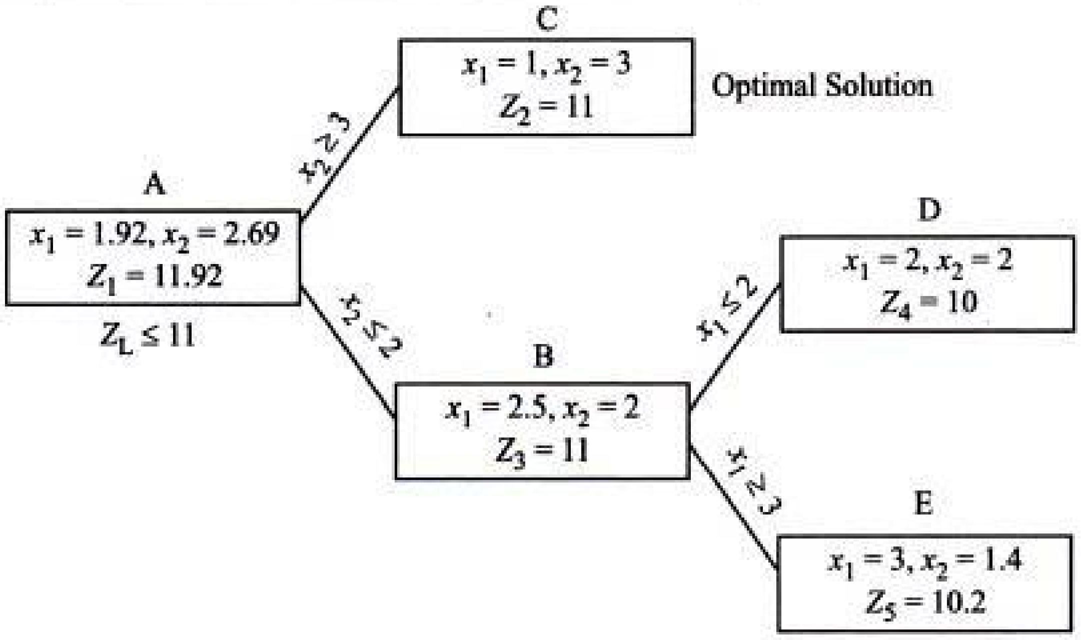

Fig. 7.7 Complete Branch and Bound Solution

Example 7.7 Solve the following all-integer programming problem using the branch and bound method.

Maximize $Z = 3x_1 + 5x_2$

subject to the constraints

(i) $2x_1 + 4x_2 \leq 25$, (ii) $x_1 \leq 8$, (iii) $2x_2 \leq 10$

and $x_1, x_2 \geq 0$ and integers.

Solution Relaxing the integer requirements, the optimal non-integer solution of the given integer LP problem obtained by the graphical method, as shown in Fig. 7.8, is: $x_1 = 8$, $x_2 = 2.25$ and $Z_1 = 35.25$. The value of Z_1 represents the *initial upper bound*, $Z_U = 35.25$ on the value of the objective function. This means that the value of the objective function in the subsequent steps should not exceed 35.25. The *lower bound* $Z_L = 34$ is obtained by the rounded off solution values to $x_1 = 8$ and $x_2 = 2$.

The variable x_2 (= 2.25) is the only non-integer solution value and is, therefore, selected for dividing the given LP-A problem into two subproblems LP-B and LP-C.

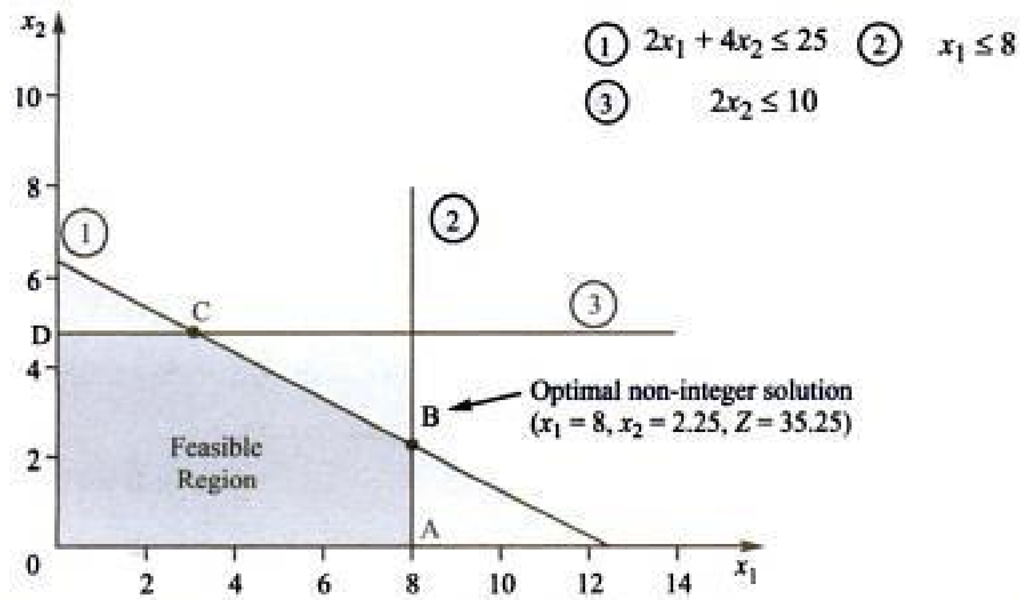

Fig. 7.8 Graphical Solution of LP–A

In order to eliminate the fractional part of 2.25 (the value of x_2), two new constraints $x_2 \leq 2$ and $x_2 \geq 3$ are created. These two constraints are then added to the LP-A problem as shown below:

LP Subproblem B

Max $Z = 3x_1 + 5x_2$

subject to (i) $2x_1 + 4x_2 \leq 25$, (ii) $x_1 \leq 8$

(iii) $2x_2 \leq 10$ (redundant), (iv) $x_2 \leq 2$

and $x_1, x_2 \geq 0$ and integers.

LP Subproblem C

Max $Z = 3x_1 + 5x_2$

subject to (i) $2x_1 + 4x_2 \leq 25$, (ii) $x_1 \leq 8$

(iii) $2x_2 \leq 10$, (iv) $x_2 \geq 3$

and $x_1, x_2 \geq 0$ and integers.

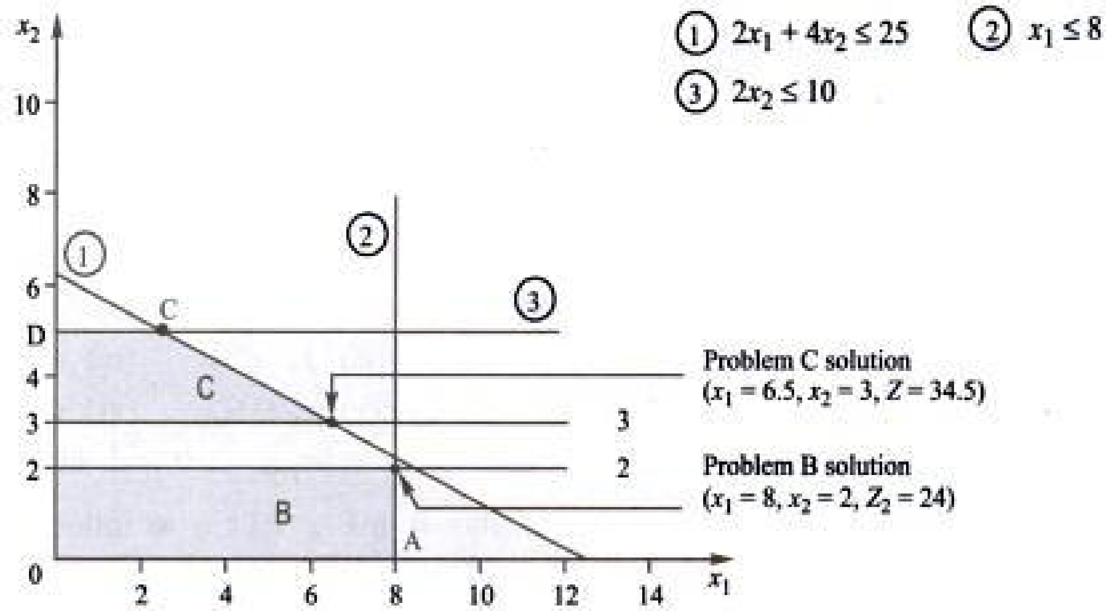

Fig. 7.9 Graphical Solution of Problems B and C

Subproblems B and C are solved graphically using Fig. 7.9. The solutions are:

Subproblem B : $x_1 = 8$, $x_2 = 2$, and Max $Z_2 = 34$

Subproblem C : $x_1 = 6.5$, $x_2 = 3$, and Max $Z_3 = 34.5$

Notice that both the solutions yield a value of Z that is lower than that of the original LP problem. The value of Z_1 establishes an upper bound on the values of the objective functions Z_2 and Z_3.

Since the solution of the subproblem B is an all-integer, we stop the search of this subproblem, i.e. no further branching is required from node B. The value of $Z_2 = 34$ becomes the *new lower bound* on the integer LP problem's optimal solution. A non-integer solution of subproblem C and also $Z_3 > Z_2$, both indicate that further branching is necessary from node C. However, if Z_3 would have been $\le Z_2$, then no further branching would have been possible, even from node C. The second upper bound takes on the value 34.5 instead of 35.25 at node A.

The subproblem C is now branched into two new subproblems: D and E, the value of which are obtained by adding constraints $x_1 \le 6$ and $x_1 \ge 7$ (x_1 is the only non-integer-valued variable in the subproblem C solution). The two subproblems are stated as follows:

LP Subproblem D

Max $Z = 3x_1 + 5x_2$

subject to (i) $2x_1 + 4x_2 \le 25$, (ii) $x_1 \le 8$ (redundant)

(iii) $2x_2 \le 10$, (iv) $x_2 \ge 3$, (v) $x_1 \le 6$

and $x_1, x_2 \ge 0$ and integers.

LP Subproblem E

Max $Z = 3x_1 + 5x_2$

subject to (i) $2x_1 + 4x_2 \le 25$, (ii) $x_1 \le 8$

(iii) $2x_2 \le 10$, (iv) $x_2 \ge 3$,

(v) $x_1 \ge 7$

and $x_1, x_2 \ge 0$ and integers.

Subproblems D and E are solved graphically using Fig. 7.10. The solutions are:

Subproblem D : $x_1 = 6$, $x_2 = 3.25$ and Max $Z_4 = 34.25$.

Subproblem E : No feasible solution exists because constraints $x_1 \ge 7$ and $x_2 \ge 3$ do not satisfy the first constraint. So this branch is terminated.

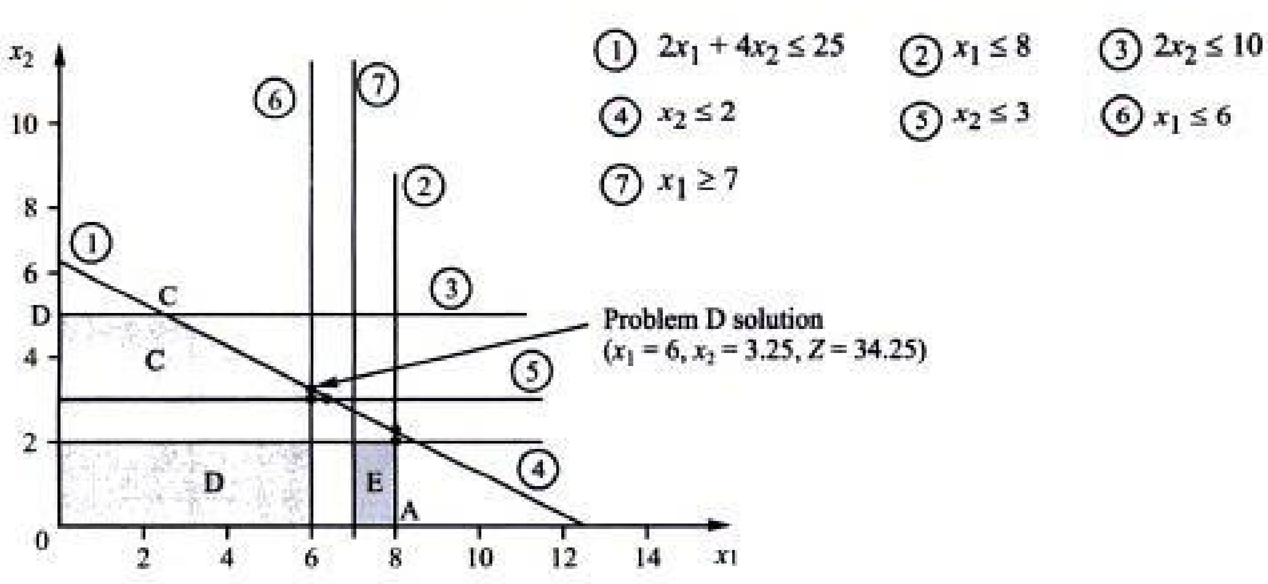

Fig. 7.10 Graphical Solution of Problems D and E

The non-integer solution obtained at sub-problem D yields an upper bound of 34.25 instead of 34.50 and also greater than Z_2 (an upper bound for sub-problem B).

Once again we create sub-problems F and G from sub-problem D with two new constraints $x_2 \le 3$ and $x_2 \ge 4$, as shown in Fig. 7.6.

LP Subproblem F	*LP Subproblem G*
Max $Z = 3x_1 + 5x_2$	Max $Z = 3x_1 + 5x_2$
subject to (i) $2x_1 + 4x_2 \le 25$	subject to (i) $2x_1 + 4x_2 \le 25$, (ii) $x_1 \le 8$
(ii) $x_1 \le 8$; (iii) $2x_2 \le 10$ (redundant)	(iii) $2x_2 \le 10$, (iv) $x_2 \ge 3$ (redundant)
(iv) $x_2 \ge 3$, (v) $x_1 \le 6$ (vi) $x_2 \le 3$	(v) $x_1 \le 6$, (vi) $x_2 \ge 4$
and $x_1, x_2 \ge 0$ and integers.	and $x_1, x_2 \ge 0$ and integers.

The graphical solution to subproblems F and G as shown in Fig. 7.11 is as follows:

Subproblem F : $x_1 = 6$, $x_2 = 3$ and Max $Z_5 = 33$.

Subproblem G : $x_1 = 4.25$, $x_2 = 4$ and Max $Z_6 = 33.5$.

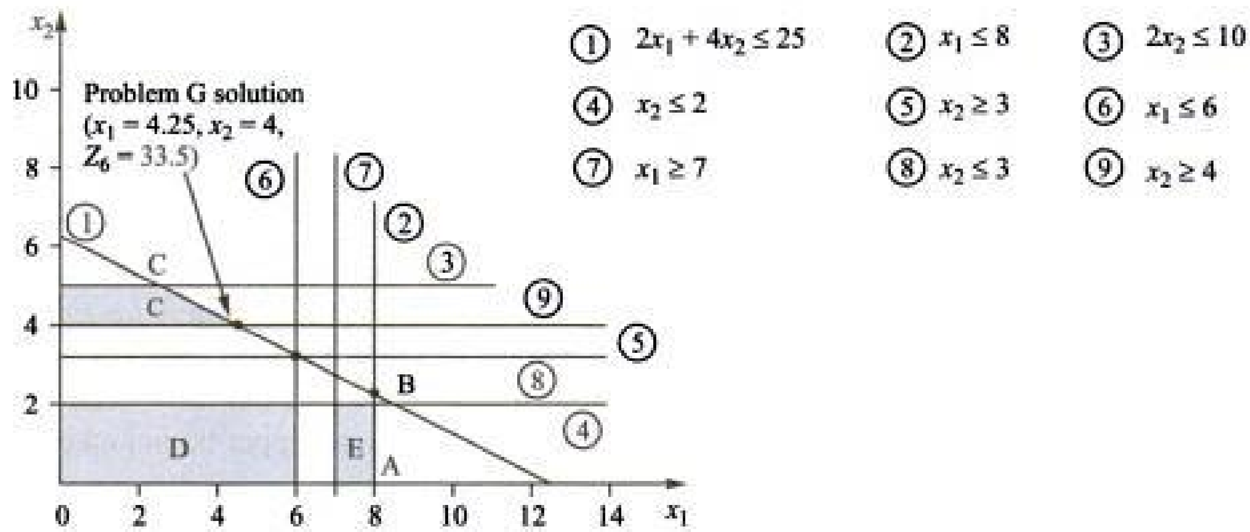

Fig. 7.11 Graphical Solution of Problems F and G

The branching process is terminated when new upper bound is less than or equal to the lower bounds of previous solutions or when no further branching is possible. Although the solution at node G is non-integer, no additional branching is required from this node because $Z_6 < Z_4$. The branch and bound algorithm thus terminated and the optimal integer solution is: $x_1 = 8$, $x_2 = 2$ and $Z = 34$ yielded at node B.

The entire branch and bound procedure for the given problem is shown in Fig. 7.12.

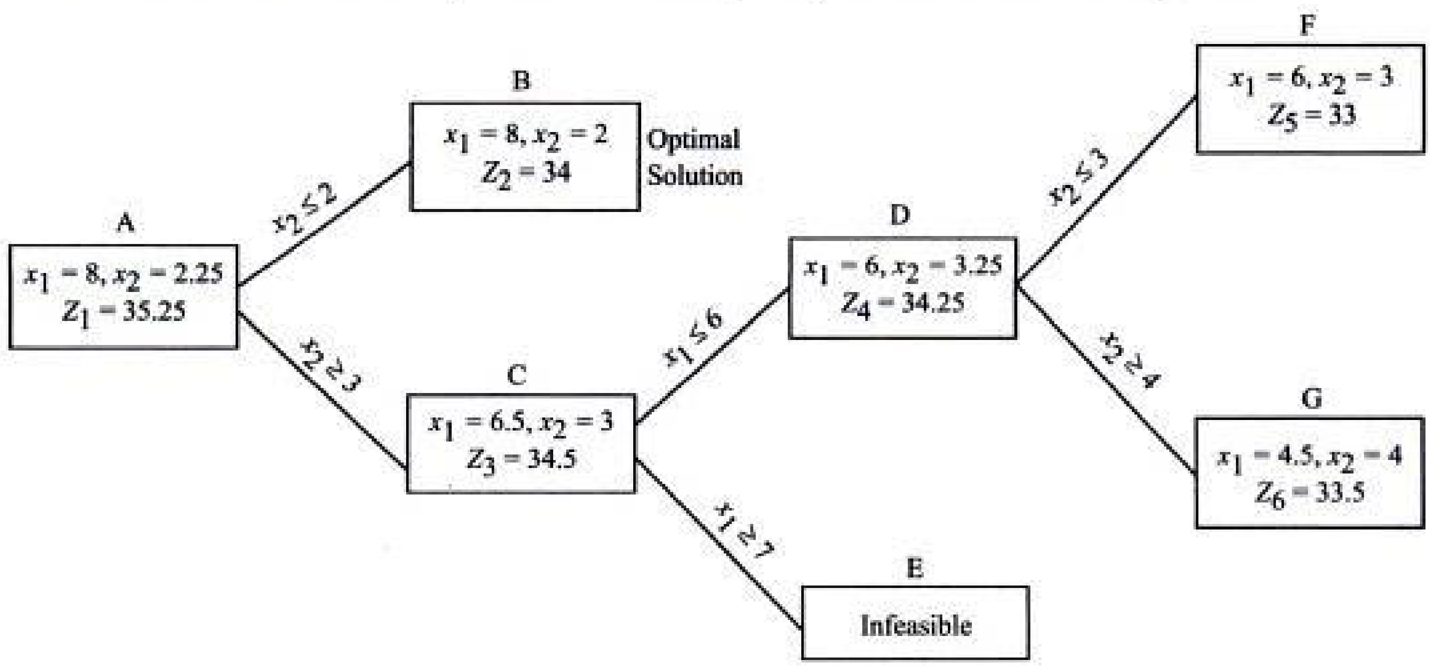

Fig. 7.12 Complete Branch and Bound Solution

Example 7.8 Solve the following all-integer programming problem using the branch and bound method

Minimize $Z = 3x_1 + 2.5x_2$

subject to the constraints

(i) $x_1 + 2x_2 \ge 20$, (ii) $3x_1 + 2x_2 \ge 50$

and $x_1, x_2 \ge 0$ and integers.

Solution Relaxing the integer requirements, the optimal non-integer solution of the given integer LP problem, obtained by the graphical method, is: $x_1 = 15, x_2 = 2.5$ and $Z_1 = 51.25$. This value of Z_1 represents the initial lower bound, $Z_L = 51.25$ on the value of the objective function, i.e. the value of the objective function in the subsequent steps cannot be less than 51.25.

The variable x_2 (= 2.5) is the only non-integer solution value and is therefore is selected for dividing the given problem into two subproblems – B and C. In order to eliminate the fractional part of 2.5, two new constraints $x_2 \leq 2$ and $x_2 \geq 3$ are created. These two constraints represent the two new parts of the problem as shown below:

LP Subproblem B

Max $Z = 3x_1 + 2.5x_2$

subject to (i) $x_1 + 2x_2 \geq 20$, (ii) $3x_1 + 2x_2 \geq 50$
(iii) $x_2 \leq 2$

and $x_1, x_2 \geq 0$ and integers.

LP Subproblem C

Max $Z = 3x_1 + 2.5x_2$

subject to (i) $x_1 + 2x_2 \geq 20$, (ii) $3x_1 + 2x_2 \geq 50$
(iii) $x_2 \geq 3$

and $x_1, x_2 \geq 0$ and integers.

Subproblems B and C are solved graphically. The solutions are:

Subproblem B : $x_1 = 16$, $x_2 = 2$ and Min $Z_2 = 53$.
Subproblem C : $x_1 = 14.66$, $x_2 = 3$ and Min $Z_3 = 51.5$.

Since the solution of subproblem B is all-integer, we stop the search of this subproblem, i.e. no further branching is required from the node B. The value of $Z_2 = 53$ becomes the new lower bound. A non-integer solution of sub-problem C and also $Z_3 < Z_2$ indicates that further branching is necessary from node C. However, if Z_3 would have been $\geq Z_2$, then no further branching would have been possible from node C. The second lower bound takes on the value $Z_L = 51.5$ instead of $Z_L = 51.25$ at node A.

We divide subproblem C into two new subproblems: D and E which are obtained by adding constraints $x_1 \leq 14$ and $x_1 \geq 15$. The two subproblems are stated as follows:

LP Subproblem D

Max $Z = 3x_1 + 2.5x_2$

subject to (i) $x_1 + 2x_2 \geq 20$, (ii) $3x_1 + 2x_2 \geq 50$
(iii) $x_2 \geq 3$, (iv) $x_1 \leq 14$

and $x_1, x_2 \geq 0$ and integers.

LP Subproblem E

Max $Z = 3x_1 + 2.5x_2$

subject to (i) $x_1 + 2x_2 \geq 20$, (ii) $3x_1 + 2x_2 \geq 50$
(iv) $x_2 \geq 3$, (v) $x_1 \geq 15$

and $x_1, x_2 \geq 0$ and integers.

Subproblems D and E are solved graphically. The solutions are:

Subproblem D : $x_1 = 14$, $x_2 = 4$ and Min $Z_4 = 52$.
Subproblem E : $x_1 = 15$, $x_2 = 3$ and Min $Z_5 = 52.5$.

The solutions obtained at node D and E are both all-integer and therefore branch and bound algorithm is terminated. The optimal integer solution to the given LP problem is at node D, where the value of the objective function is the lowest amongt the values at nodes B, D and E.

The entire branch and bound procedure for the given problem is shown in Fig. 7.13.

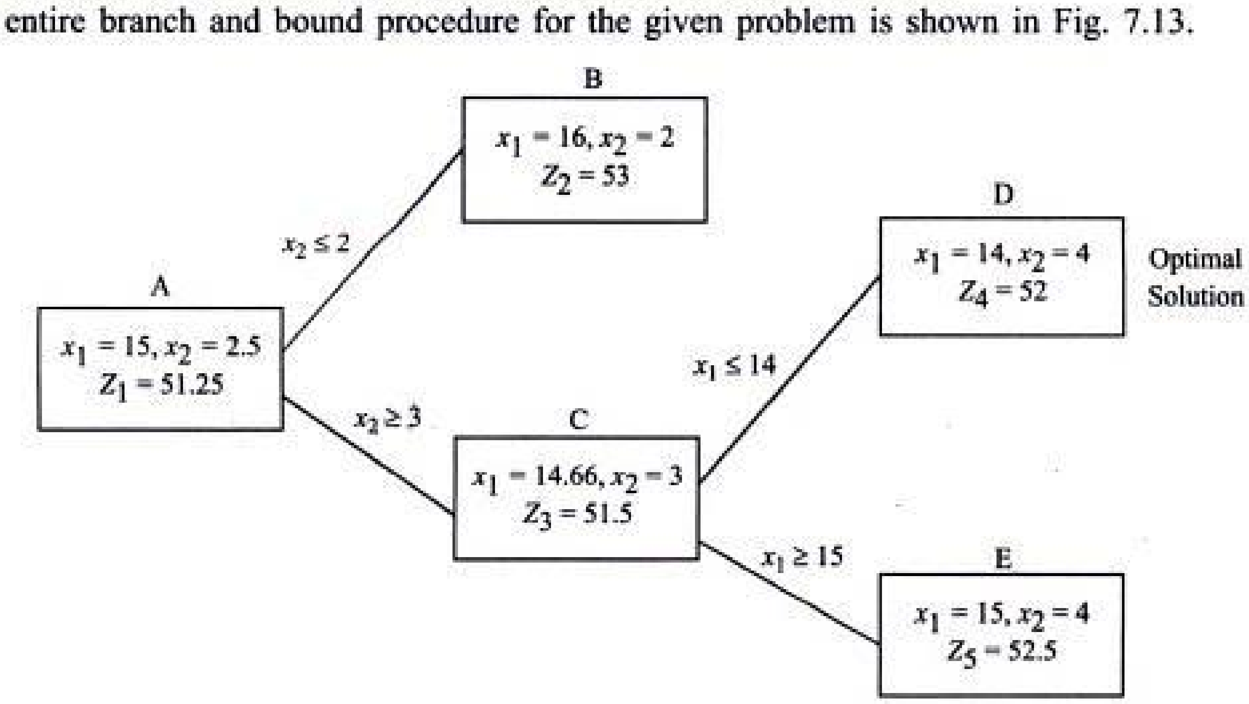

Fig. 7.13 Complete Branch and Bound Solution

7.7 APPLICATIONS OF ZERO-ONE INTEGER PROGRAMMING

A large number of real-world problems such as capital budgeting problem, matching problem, sequencing problem, scheduling problem, location problem, travelling salesman problem, etc., require all or some of the decision variables to assume the value of either zero or one. A few such problems are discussed here.

The zero-one integer programming problem can be stated as:

$$\text{Minimize } Z = \sum_{j=1}^{n} c_j x_j$$

subject to the constraints

$$\sum_{j=1}^{n} a_{ij} x_j \ge b_i; \quad i = 1, 2, \ldots, m$$

and $\quad x_j = 0 \text{ or } 1.$

7.7.1 Capital Budgeting Problem

Such problems cover the problem of allocating limited funds to various investment projects in order to maximize the discounted net return.

Example 7.9 A corporation is considering four possible investment opportunities. The following table presents information about the investment (in Rs thousand) profits:

Project	*Present Value of Expected Return*	*Capital Required Year-wise by Projects*		
		Year 1	*Year 2*	*Year 3*
1	6,500	700	550	400
2	7,000	850	550	350
3	2,250	300	150	100
4	2,500	350	200	–
Capital available for investment		1,200	700	400

In addition, projects 1 and 2 are mutually exclusive and project 4 is contingent on the prior acceptance of project 3. Formulate an integer programming model to determine which projects should be accepted and which should be rejected in order to maximize the present value from the accepted projects.

[Delhi Univ., MBA, 1998, 2004]

Model formulation Let

$$x_j = \begin{cases} 1 & \text{if project } j \text{ is accepted} \\ 0 & \text{if project } j \text{ is rejected} \end{cases}$$

Integer LP model

Maximize (Total present value) $Z = 6{,}500x_1 + 7{,}000x_2 + 2{,}250x_3 + 2{,}500x_4$

subject to the constraints

(i) Expenditure in years 1, 2 and 3

(i) $700x_1 + 850x_2 + 300x_3 + 350x_4 \le 1{,}200$

(ii) $550x_1 + 550x_2 + 150x_3 + 200x_4 \le 700$

(iii) $400x_1 + 350x_2 + 100x_3 \le 400$

(iv) $x_1 + x_2 \ge 1, \quad$ (v) $x_4 - x_3 \le 1$

and $\quad x_j = 0 \text{ or } 1.$

7.7.2 Fixed Cost (or Charge) Problem

In certain cases, while undertaking a particular set of activities, the fixed costs (fixed charge or setup costs) are incurred. In such cases, the objective is to minimize the total cost (sum of fixed and variable costs) associated with an activity:

Let us define the following decision variables

x_j = level of activity j

F_j = fixed cost associated with activity $x_j > 0$

c_j = variable cost associated with activity $x_j > 0$

Then the general fixed cost problem can be stated as:

$$\text{Minimize } Z = \sum_{j=1}^{n} (c_j x_j + F_j y_j)$$

subject to the constraints

$$\sum_{j=1}^{n} a_{ij} x_j \le b_i \,;\; i = 1, 2, \ldots, m$$

$$x_j \le M y_j \quad \text{or} \quad x_j - M y_j \le 0 \,;\; j = 1, 2, \ldots, n$$

and $\quad x_j \ge 0$ for all j ; $\quad y_j = 0$ or 1 for all j

where the symbol M denotes a large number so that $x_j \le M$.

Example 7.10 Consider the following production data:

Product	*Profit per Unit (Rs)*	*Direct Labour Requirement (hours)*
1	8	15
2	10	14
3	7	17

Fixed Cost (Rs)	*Direct Labour Requirement*
10,000	up to 20,000 hours
20,000	20,000–40,000 hours
30,000	40,000–70,000 hours

Formulate an integer programming problem to determine the production schedule so as to maximize the total net profit. [*Delhi Univ., MBA, 1999, 2000, 2001*]

Model formulation Let

x_1, x_2 and x_3 = number of units of products 1, 2 and 3, respectively to be produced

y_j = fixed cost (in Rs); $j = 1, 2, 3$.

Integer LP model

$$\text{Maximize } Z = 8x_1 + 10x_2 + 7x_3 - 10{,}000y_1 - 20{,}000y_2 - 30{,}000y_3$$

subject to the constraints

(i) $15x_1 + 14x_2 + 17x_3 \le 20{,}000y_1 + 40{,}000y_2 + 70{,}000y_3$

(ii) $y_1 + y_2 + y_3 = 1$

and $\quad x_j \ge 0$; $y_j = 0$ or 1, for $j = 1, 2, 3$.

7.7.3 Plant Location Problem

Suppose there are m possible sites (locations) at which the plants could be located. Each of these plants produces a single commodity for n customers (markets or demand points), each with a minimum demand required for b_j units ($j = 1, 2, \ldots, n$). The fixed setup cost (expenses associated with constructing and operating a plant) of a plant, in the ith location, is f_i ($i = 1, 2, \ldots, m$). The production capacity for each plant is limited to a_i units. The unit transportation cost from plant i to customer j is c_{ij}. The problem is to locate the plants in such a way that the sum of the fixed setup costs and transportation cost is minimized.

Let x_{ij} be the amount shipped from plant i to customer j, and y_i be the new variable associated with each of the possible plant locations, such that

$$y_i = \begin{cases} 1, \text{if plant is located at the } i\text{th location} \\ 0, \text{otherwise} \end{cases}$$

The value of f_i is assumed to be fixed and independent of the amount of x_{ij} shipped so long as $x_{ij} > 0$, i.e., for $x_{ij} = 0$, the value $f_i = 0$. The objective function in the following zero-one integer programming problem is to minimize the total cost (variable + fixed) of settings up and operating the network of transportation routes.

$$\text{Minimize } Z = \sum_{i=1}^{m} \sum_{j=1}^{n} c_{ij} x_{ij} + \sum_{i=1}^{m} f_i y_i$$

subject to the constraints

$$\sum_{i=1}^{m} x_{ij} = b_j; \qquad j = 1, 2, \ldots, n \tag{16}$$

$$\sum_{j=1}^{n} x_{ij} \le y_i u_i; \qquad i = 1, 2, \ldots, m \tag{17}$$

$$\sum_{j=1}^{n} x_{ij} \le a_i; \qquad i = 1, 2, \ldots, m \tag{18}$$

and $\qquad x_{ij} \ge 0$, for all i and j.

Here u_i denotes the annual capacity of plant i.

Constraints (16) guarantee that each customer's demand is met. If all the shipping costs are positive, i.e. there are no subsidized routes, then it never pays to send more than the needed amount, or we replace the inequality sign by an equality. Inequality (17) ensures that we do not ship from a plant which is not operating. u_i is the upper bound on the amount shipped that may be shipped from plant i. Inequality (18) restricts production from exceeding the limited capacity.

CONCEPTUAL QUESTIONS

1. What is integer linear programming? How does the optimal solution of an integer programming problem compare with that of the linear programming problem?
2. What is integer linear programming? Explain the merits and demerits of 'rounding-off' a continuous optimal solution to an LP problem in order to obtain an integer solution. [*Delhi Univ., MBA, 1997*]
3. What is the effect of the 'integer' restriction of all the variables on the feasible space of integer programming problem?
4. Explain how Gomory's cutting plane algorithm works.
5. Addition of a cut makes the previous non-integer optimal solution infeasible. Explain.
6. What is the meaning and the role of the lower bound and upper bound in the branch and bound method?
7. Describe any one method of solving a mixed-integer programming problem.
8. Sketch the branch and bound method in integer programming.
9. Discuss the advantages of the branch and bound method.
10. Discuss the advantages and disadvantages of solving integer programming problems by (a) the cutting plane method and (b) the branch and bound method.

SELF PRACTICE PROBLEMS B

Solve the following integer programming problems using Gomory's cutting plane algorithm or by branch or bound method:

1. Max $Z = 1.5x_1 + 3x_2 + 4x_3$
   subject to (i) $2.5x_1 + 2x_2 + 4x_3 \le 12$
   (ii) $2x_1 + 4x_2 - x_3 \le 7$
   and $x_1, x_2, x_3 \ge 0$
2. Max $Z = 2x_1 + 3x_2$
   subject to (i) $x_1 + 3x_2 \le 9$, (ii) $3x_1 + x_2 \le 7$
   (iii) $x_1 - x_2 \le 1$
   and $x_1, x_2 \ge 0$ and integers.
3. Max $Z = 7x_1 + 6x_2$
   subject to (i) $2x_1 + 3x_2 \le 12$, (ii) $6x_1 + 5x_2 \le 30$
   and $x_1, x_2 \ge 0$ and integers.
4. Max $Z = 5x_1 + 4x_2$
   subject to (i) $x_1 + x_2 \ge 2$, (ii) $5x_1 + 3x_2 \le 15$
   (iii) $3x_1 + 5x_2 \le 15$
   and $x_1, x_2 \ge 0$ and integers.
5. Max $Z = -3x_1 + x_2 + 3x_3$
   subject to (i) $-x_1 + 2x_2 + x_3 \le 4$, (ii) $2x_2 - 1.5x_3 \le 1$
   (iii) $x_1 - 3x_2 + 2x_3 \le 3$
   and $x_1, x_2 \ge 0$; x_3 non-negative integer.
6. Max $Z = x_1 + x_2$
   subject to (i) $2x_1 + 5x_2 \ge 16$, (ii) $6x_1 + 5x_2 \le 30$
   and $x_2 \ge 0$
   x_1 non-negative integer.
7. Min $Z = 4x_1 + 3x_2 + 5x_3$
   subject to (i) $2x_1 - 2x_2 + 4x_3 \ge 7$,
   (ii) $2x_1 + 6x_2 - 2x_3 \ge 5$
   and $x_2 \ge 0$; x_1, x_3 non-negative integers.
8. Max $Z = 110x_1 + 100x_2$
   subject to (i) $6x_1 + 5x_2 \le 29$
   (ii) $4x_1 + 14x_2 \le 48$
   and $x_1, x_2 \ge 0$ and integers.

9. A firm is considering investing in plant modernization and plant expansion. All of these proposed projects would be completed within 2 years, with varying requirements of money and plant engineering. The management is willing to use the following data in selecting the best set of proposals. Three resource limitations are:

First year expenditure	:	Rs 4,20,000
Second year expenditure	:	Rs 4,40,000
Engineering hours	:	15,000 hours

Project Description	Expenditure ('000s Rs)		Net Present Value ('000s Rs)	Engineering Hours (00)
	1st year	2nd year		
• Modernize shop floor	220	0	70	50
• Build new shop floor	95	270	95	80
• Equipment for new production line	0	170	60	43
• Modernize maintenance shop	60	100	80	90
• Processing sub-contract raw material	60	220	145	45
• Install new raw material processing plant	195	0	70	30
• Buy trucks and containers	80	32	45	0

The situation requires that a new or modernized shop floor be provided. The equipment for production line is applicable only to the new shop floor. The company may not want to buy or build raw material processing facilities. Formulate the given problem as an integer linear programming problem in order to maximize the net present value of the money.

10. ABC Manufacturing company faces the following problem: Should they make or buy each of their several products? The company policies specify that they will either make or buy the whole lot of each product in a complete lot. The company has four products to make or buy with six machines involved in making these products, if they are made in the shop. The time per unit (in hours) required are as follows:

Forty hours are available on each machine. One hundred ten units of each product are needed. The costs of making the products are listed below:

Products	:	1	2	3	4
Cost/unit (in Rs)	:	2.25	2.22	4.50	1.90

The cost to buy the products are :

Product	:	1	2	3	4
Cost/unit (in Rs)	:	3.10	2.60	4.75	2.25

Formulate this problem as a zero-one integer programming problem. [*Delhi, Univ., MBA, 1991, 2000*]

Product	Machine					
	A	B	C	D	E	F
1	0.04	0.02	0.02	0	0.03	0.06
2	0	0.01	0.05	0.15	0.09	0.06
3	0.02	0.06	0	0.06	0.02	0.02
4	0.06	0.04	0.15	0	0	0.05

11. ABC Construction Company is facing the problem of determining which projects it should undertake over the next 4 years. The following table gives information about each project:

Project	Estimated Present Value	Capital Requirements			
		Year 1	Year 2	Year 3	Year 4
A	18,00,000	3,00,000	4,00,000	4,00,000	3,00,000
B	2,00,000	1,20,000	80,000	0	40,000
C	7,20,000	3,00,000	2,00,000	2,00,000	2,00,000
D	8,00,000	2,00,000	4,00,000	4,00,000	1,00,000
	Fund available	6,50,000	8,00,000	8,00,000	5,00,000

Formulate a zero-one programming model to maximize estimated value. [*Delhi Univ., MBA, 2000*]

12. Write constraints to satisfy each of the following conditions in a project selection model. The projects are numbered 1, 2, 3, . . ., 10.

(i) Exactly one project from the set (1, 2, 3) must be selected.
(ii) Project 2 can be selected only if number 10 is selected. However, 10 can be selected without 2 being selected.
(iii) No more than one project from the set (1, 3, 5, 7, 9) can be selected.
(iv) If number 4 is selected, then number 8 cannot be selected.
(v) Projects 4 and 10 must both be selected or both be rejected. [*Delhi Univ., MBA, 1999*]

HINTS AND ANSWERS

1. $x_1 = 0, x_2 = 2, x_3 = 2$ and Max $Z = 14$
2. $x_1 = 0, x_2 = 3$ and Max $Z = 9$
3. $x_1 = 5, x_2 = 0$ and Max $Z = 35$
4. $x_1 = 3, x_2 = 0$ and Max $Z = 15$
5. $x_1 = 0, x_2 = 8/7, x_3 = 1$ and Max $Z = 29/7$
6. $x_1 = 4, x_2 = 6/5$ and Max $Z = 26/5$
7. $x_1 = 3, x_2 = 1.25, x_3 = 1$ and Min $Z = 20.75$
8. $x_1 = 4, x_2 = 1$, Max $Z = 540$

CHAPTER SUMMARY

In this chapter, important extension of linear programming, referred to as integer linear programming, was introduced where one or more of the variables must be an integer. If all variables of a problem are integers, then such problems are referred to as all-integer linear programming problems. If some, but not necessarily all, variables are integers, then such problems are referred to as mixed integer linear programming problems. Most integer programming applications involve 0-1 or binary variables.

The number of applications of integer linear programming continues to grow rapidly due to the availability of integer linear programming software packages.

The study of integer linear programming is important for two major reasons: (i) integer linear programming may be helpful when fractional values for the variables are not permitted and rounding off their values may not provide an optimal integer

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not
available

Bharat Chand & Co. also provides 1,000 units of steel free of cost if at least 1,500 units of steel and 1,400 units of aluminium are purchased from this vendor.

Seth & Co. has a minimum quality standard of 98 per cent for steel and 95 per cent for aluminium. In addition, it has a maximum acceptable delivery standard (in days) of steel as 7 and aluminium as 7.5. The total requirements for the coming month is 3,000 units of steel and 3,000 units of aluminium.

Questions for discussion Develop a mathematical model and specify the company's production schedule. Also advice the company to choose which seller it should purchase its two raw materials from and the quantity of the raw materials that should be purchased

Case 7.2: Apparel Industry

Zodiac is well known brand in the fashion industry. It manufactures different types of shirts in different sizes, as well as ties and other fashion accessories for upwardly mobile urban professionals. Recently, it decided to make use of some latest techniques in order to help itself in the cloth cutting operation.

The cutting operation for shirts is extremely time-consuming and gives rise to high set-up costs. The problem that the company was mainly facing was that a large amount of cloth were being wasted in the cutting operation. The process of cutting involved putting several layers of cloth of standard width on a table and putting stencils, i.e. the templates in order to cut the cloth. However, the choice of the templates was based on judgement, which was leading to a lot of wastage. To tackle the problem it was given to the R&D department of the company. The objective facing them was to minimize the setup cost and excess, subject to certain constraints.

The following data was available regarding the cutting operation of a shirt whose demand for different sizes is given as follows:

Size	:	38	40	42	44	46
Demand	:	54	84	91	60	29

The numbers of layers that can be cut is limited by the length of the knives and the thickness of the fabric. A second restriction of the cutting process is introduced by the length of the cutting table, which limits the number of stencils that can be cut in one operation. As the length of the stencils for the different sizes is almost equal, the maximum number of stencils on the cutting table is actually independent of the combination of the stencils used. Thus the R&D Department assumed that all stencils have equal length without loss of generality.

The spreading of the fabric on the cutting table, the fixing of the layers and stencils, and the cutting itself are extremely delicate and time-consuming operations. Therefore, the number of these operations should be minimized. The problem then consists of finding the optimal combination of the number of layers of cloth on the cutting table and the associated set of stencils that will result in the minimum number of setups, while satisfying the demand, with little or no excess.

There was an upper bound on the number of layers at 35 and the cutting table length could hold at most four stencils. Three cutting patterns were to be used.

Questions for discussion Develop an appropriate mathematical model and solve it so that it helps the management achieve its objective.

chapter 8

Goal Programming

"In the end, all business operations can be reduced to three words: people, product and profits. Unless you've got a good team, you can't do much with the other two."

– Lee Iacocca

Preview Goal programming is an approach used for solving any multi-objective optimization problem that balances trade-off in multiple and often conflicting incommensurable goals at different priority levels.

Learning Objectives After studying this chapter, you should be able to

- appreciate the need of a goal programming approach for solving multi-objective decision problems.
- distinguish between LP and GP approaches for solving a business decision problem.
- formulate GP model of the given multi-objective decision problem.
- understand the method of assigning different ranks and weights to unequal multiple goals.
- use both graphical and simplex method for solving a GP model.

Chapter Outline

8.1 INTRODUCTION

In previous chapters, linear programming models were formulated and solved in order to optimize an objective function value (a single measure of effectiveness) under a set of constraints. However, the optimization of such a single objective function is often not representative of the reality due to divergent and conflicting interests and objectives (economic as well as non-economic) of any business, service or commercial organization. Consequently there arises a need to attain a 'satisfactory' level of achievement amongst multiple and conflicting interests or goals of an organization or a decision-maker.

Goal programming (GP) is an approach used for solving a multi-objective optimization problem that balances trade-off in conflicting objectives. In other words, it is an approach of deriving a best possible 'satisfactory' level of goal attainment. A problem is modelled into a GP model in a manner similar to that of an LP model. However, the GP model accommodates multiple, and often conflicting, incommensurable (dimension of goals and unit of measurement may not be same) goals, in a particular priority order (hierarchy). A particular priority structure is established by ranking and weighing various goals and their subgoals, in accordance with their importance. The priority structure helps to deal with all goals (objectives) that cannot be completely and/or simultaneously achieved, in such a manner that more important goals are achieved first, at the expense of the less important ones.

8.2 DIFFERENCE BETWEEN LP AND GP APPROACH

Linear programming has two major limitations from its application point of view: (i) single objective function, and (ii) same unit of measurement of various resources.

The LP model has a single objective function to be optimized such as profit maximization, cost minimization, etc. However, in actual practice, the decision-maker may not be satisfied with a single objective. That is, he may desire to get simultaneous solution to a complex system of competing objectives.

GP approach establishes a specific numeric goal for each of the objective and then attempts to achieve each goal sequentially upto a satisfactory level rather than an optimal level

The solution of any LP problem is based on the cardinal value (the number that expresses exact amount such as, 1, 2, 3, . . .) such as profit or cost, whereas that of a GP allows ordinal ranking of goals in terms of their contribution or importance to the organization. Since it may not be possible to obtain information about the value or cost of a goal (the specific numerical target value desired to achieve) or a sub-goal, therefore their upper and lower limits are determined. Usually, desired goals are assigned priorities and then these priorities are ranked in an ordinal sequence.

Whenever there are multiple incommensurable (different units of measurement) goals, an LP incorporates only one of these goals in the objective function and treats the remaining goals as constraints. Since the optimal solution must satisfy all the constraints, this implies that (a) the several goals within the constraining equations are of equal importance, and (b) these goals have absolute priority over the goal incorporated into the objective function.

8.3 CONCEPT OF GOAL PROGRAMMING

The concept of GP was introduced by Channes and Cooper (1961). They suggested a method for solving an infeasible LP problem arising from various conflicting resource constraints (goals). A few examples of multiple conflicting goals are: (i) maximize profit and increase wages paid to employees, (ii) upgrade product quality and reduce product cost, (iii) reduce credit losses and increase sales.

Ijiri (1965) developed the concept of pre-emptive priority factors, assigning different priority levels to incommensurable goals and different weights to the goals at the same priority level. Lee (1972) and Ignizio (1976) have written textbooks on the subject of goal programming. Goal programming has been applied to a wide range of planning, resource allocation, policy analysis and functional management problems.

An important feature of GP is that the goals (*a specific numerical target values that the decision-maker would ideally like to achieve*) are satisfied in ordinal sequence. That is, the solution of the GP problem involves achieving some higher order (or priority) goals first, before the lower order goals are considered. Since it is not possible to achieve every goal (objective), to the extent desired by the decision-maker, attempts are made to achieve each goal *sequentially* rather than *simultaneously*, up to a *satisfactory* level rather than an optimal level.

In GP, instead of trying to minimize or maximize the objective function directly, as in the case of an LP, the deviations from established goals within the given set of constraints are minimized. In the simplex algorithm of linear programming such deviational variables are called *slack variables* and they are used only as dummy variables. In GP, these slack variables take on a new significance. The deviational variables are represented in two dimensions – both positive and negative deviations from each goal and subgoal. These deviational variables represent the extent to which the target goals are not achieved. The objective function then becomes the minimization of a sum of these deviations, based on the relative importance within the pre-emptive priority structure assigned to each deviation.

8.3.1 Distinction among Objectives, Goals and Constraints

The knowledge of the difference among the terms: *objectives*, *goals*, and *constraints* is important because these terms play a key role in the formulation of a GP model. A 'goal' describes a minimum acceptable or target value for its level of performance, whereas an 'objective', simply states optimization of the measure of performance, such as maximize profit, minimize cost, etc. As a result, the right-hand side of an objective is left unspecified.

The difference between a goal and a constraint can be more important. A goal and constraint look the same in terms of their mathematical formulation. However, the concept of a goal implies more flexibility and less rigidity than that of a 'constraint' because for a 'goal' the right-hand side value is the target level to be achieved. However, for a 'constraint', it is desirable to achieve the right-hand side value, otherwise it is considered violated, leading to an infeasible solution of the LP problem.

A single-objective LP problem ignores the concept of a goal. One objective is optimized subject to the satisfaction of a set of rigid constraints. However, multiple-objective models relax this rigidity and are based on the belief that in the real-life problems not all constraints are inflexible or as binding as implied by their strictly mathematical interpretation.

8.4 GOAL PROGRAMMING MODEL FORMULATION

8.4.1 Single Goal with Multiple Subgoals

An objective (goal) is the result desired by a decision-maker. The goal may be underachieved, fully achieved, or overachieved within the given decision environment. The degree of goal achievement depends upon the relative managerial effort applied to an activity. Mathematically, one unit of effort applied to activity x_j might contribute an amount a_{ij} toward the ith goal.

If the target level for the ith goal is fully achieved, then the ith constraint is written as:

Deviational variables represent the amount by which the objective is below or above the target

$$\sum_{j=1}^{n} a_{ij} x_j = b_i$$

To allow underachievement or overachievement in the target value (goal), let

d_i^- = negative deviation from ith goal (underachievement or amount below the target value)

d_i^+ = positive deviation from ith goal (overachievement or amount above the target value)

Using these notations, the above stated ith goal can be rewritten as:

$$\sum_{j=1}^{n} a_{ij} x_j + d_i^- - d_i^+ = b_i; \quad i = 1, 2, \ldots, m$$

$$\begin{pmatrix}\text{Value of the}\\ \text{objective}\end{pmatrix} + \begin{pmatrix}\text{Amount below}\\ \text{the goal}\end{pmatrix} - \begin{pmatrix}\text{Amount above}\\ \text{the goal}\end{pmatrix} = \text{Goal}$$

Since both underachievement and overachievement of a goal cannot be achieved simultaneously, one or both of these deviational variables (d_i^- or d_i^+) may be zero in the solution, i.e. $d_i^- \times d_i^+ = 0$. In other words, at optimality, if one variable assumes a positive value in the solution, the other must be zero and vice versa. The goal deviational variables must be *non-negative*.

Remark The deviational variables in goal programming model are equivalent to slack and surplus variables (the amount by which the objective is below or above the target) in linear programming model.

The deviational variable d_i^+ (called *surplus variable in LP*) is removed from the objective function of GP when overachievement is acceptable. Similarly, if underachievement is acceptable, d_i^- (called *slack variable in LP*) is removed from the objective function of the GP. But if the exact attainment of the goal is desired, then both d_i^- and d_i^+ are included in the objective function and ranked according to their pre-emptive priority factor, from the most important to the least important (See Section 8.4.3).

Example 8.1 A manufacturing firm produces two types of products: A and B. The unit profit from product A is Rs 100 and that of product B is Rs 50. The goal of the firm is to earn a total profit of exactly Rs 700 in the next week.

Model formulation To interpret the profit goal in terms of subgoals, which are sales volume of products, let

x_1 and x_2 = number of units of products A and B produced, respectively

The single goal of profit maximization is stated as:

$$\text{Maximize (profit) } Z = 100x_1 + 50x_2$$

Since the goal of the firm is to earn a target profit of Rs 700 per week, the profit goal can be restated to allow for underachievement or overachievement as:

$$100x_1 + 50x_2 + d_1^- - d_1^+ = 700$$

Now the goal programming model can be formulated as follows:

$$\text{Minimize } Z = d_1^- + d_1^+$$

subject to the constraints

$$100x_1 + 50x_2 + d_1^- - d_1^+ = 700$$

and $\quad x_1, x_2, d_1^-, d_1^+ \geq 0$

where $\quad d_1^-$ = underachievement of the profit goal of Rs 700

d_1^+ = overachievement of the profit goal of Rs 700

If the profit goal is not completely achieved, the slack in the profit goal will be expressed by a negative deviational (underachievement) variable, d_1^-, from the goal. But if the solution shows a profit in excess of Rs 700, the surplus in the profit will be expressed by positive deviational (overachievement) variable d_1^+ from the goal. If the profit goal of exactly Rs 700 is achieved, both d_1^- and d_1^+ will be zero.

In the given example, there are an infinite number of combinations of x_1 and x_2 that will achieve the profit goal. The required solution will be any linear combination of x_1 and x_2 between the two points: $x_1 = 7, x_2 = 0$ and $x_1 = 0, x_2 = 14$. This straight line is exactly the iso-profit function line when the total profit is Rs 700.

8.4.2 Equally Ranked Multiple Goals

In Example 8.1, we did not have any model constraints. Let us suppose that in addition to the profit goal constraint considered in this example, two more constraints are imposed as stated in Example 8.2.

Example 8.2 In Example 8.1, let us suppose that the manager in addition to the profit goal of Rs 700 also wants to achieve a sales volume for products A and B close to 5 and 4, respectively. Formulate this problem as a goal programming model.

Model formulation The constraints of the problem can be stated as:

$$100x_1 + 50x_2 = 700 \text{ (profit target goal)}$$

$$\left.\begin{array}{l} x_1 \leq 5 \\ x_2 \leq 4 \end{array}\right\} \text{(sales target goals)}$$

For this problem, the profit goal and the sales goals are expressed as:

$$100x_1 + 50x_2 + d_1^- - d_1^+ = 700$$
$$x_1 + d_2^- = 5$$
$$x_2 + d_3^- = 4$$

The problem can now be formulated as GP model as follows:

$$\text{Minimize Z} = d_1^- + d_1^+ + d_2^- + d_3^-$$

subject to the constraints

$$100x_1 + 50x_2 + d_1^- - d_1^+ = 700$$
$$x_1 + d_2^- = 5$$
$$x_2 + d_3^- = 4$$

and $\quad x_1, x_2, d_1^-, d_1^+, d_2^-, d_3^- \geq 0$

where d_2^- and d_3^- represent the underachievement of sales volume for products A and B, respectively. Since the sales target goals are given as the maximum possible sales volume, therefore, d_2^+ and d_3^+ are not included in the sales target constraints.

The solution to this problem can be found by a simple examination of the problem. If $x_1 = 5$ and $x_2 = 4$, then all targets will be completely achieved. Thus $d_1^- = d_2^- = d_3^- = d_1^+ = 0$.

Example 8.3 An office equipment manufacturer produces two types of products: chairs and lamps. The production of either a chair or a lamp, requires one hour of production capacity in the plant. The plant has a maximum production capacity of 50 hours per week. Because of the limited sales capacity, the maximum number of chairs and lamps that can be sold are 6 and 8 per week, respectively. The gross margin from the sale of a chair is Rs 90 and from the sale of a lamp is Rs 60.

The plant manager desires to determine the number of units of each product that should be produced per week in consideration of the following set of goals:

Goal 1: Available production capacity should be utilized as much as possible but should not exceed 50 hours per week.

Goal 2: Sales of two products should be as much as possible.

Goal 3: Overtime should not exceed 20 per cent of available production time.

Formulate and solve this problem as a GP model so that the plant manager may achieve his goals as closely as possible. *[Delhi Univ., MBA, 1999]*

Model formulation Let x_1 and x_2 = number of units of chair and lamp produced, respectively.

The first goal pertains to the production capacity attainment, with a target established at 50 hours per week. This constraint can be stated as:

$$x_1 + x_2 + d_1^- + d_1^+ = 50$$

where d_1^- = underutilization (idle time) of production capacity

d_1^+ = overutilization (overtime) of production capacity.

If this goal is not achieved, then d_1^- would take on a positive value and d_1^+ would be zero.

The second goal pertains to maximization of sales volume with a target of 6 units of chairs and 8 units of lamps per week. The sales constraints can be expressed as:

$$x_1 + d_2^- = 6 \quad \text{and} \quad x_2 + d_3^- = 8$$

Since the sales goals are the maximum possible sales volume, d_2^+ and d_3^+, will not appear in these constraints. Thus, the possibility of overachievement of sales goals is ruled out.

The third goal pertains to the minimization of overtime as much as possible. The constraint is stated as follows:

$$d_1^+ + d_4^- - d_4^+ = 0.2(50) = 10$$

where d_4^- = overtime less than 20 per cent of goal constraint

d_4^+ = overtime more than 20 per cent of goal constraint

d_1^+ = overtime beyond 50 hours.

Now, the given problem can be stated as a goal programming model as:

Minimize (total deviation) $Z = d_1^+ + d_2^- + d_3^- + d_4^+$

subject to the constraints

(i) Production capacity : $x_1 + x_2 + d_1^- - d_1^+ = 50$

(ii) Sales volume : $x_1 + d_2^- = 6$ and $x_2 + d_3^- = 8$

(iii) Overtime : $d_1^+ + d_4^- - d_4^+ = 10$

and $x_1, x_2, d_1^-, d_1^+, d_2^-, d_3^-, d_4^-, d_4^+ \geq 0.$

8.4.3 Ranking and Weighting of Unequal Multiple Goals

Since multiple and conflicting goals are usually not of equal rank (importance), deviations (negative and/ or positive) from these goals are not additive. Hence to achieve these goals according to their importance a *pre-emptive priority factor* $P_1, P_2, \ldots$ and so on is assigned to deviational variables in the formulation of the objective function to be minimized. The *P*s do not assume numerical value, they are simply a

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The Algorithm

1. Graph all system constraints (those not involving deviational variables) and identify the feasible solutions space. If no system constraints exists, then the feasible solutions space (or region) is the first quadrant. If no feasible solutions space exists, there is no solution to the problem.
2. Graph the straight lines corresponding to the goal constraints, labelling the deviational variables.
3. Within the feasible solutions space identified in Step 1, determine the point or points that best satisfy the highest priority goal.
4. Sequentially consider the remaining goals and the point that satisfy them to the greatest extent possible. Make sure that a lower priority goal is not achieved by reducing the degree of achievement of higher priority goals.

Example 8.4 A firm produces two products A and B. Each product must be processed through two departments namely 1 and 2. Department 1 has 30 hours of production capacity per day, and department 2 has 60 hours. Each unit of product A requires 2 hours in department 1 and 6 hours in department 2. Each unit of product B requires 3 hours in department 1 and 4 hours in department 2. Management has rank ordered the following goals it would like to achieve in determining the daily product mix:

P_1 : Minimize the underachievement of joint total production of 10 units.
P_2 : Minimize the underachievement of producing 7 units of product B.
P_3 : Minimize the underachievement of producing 8 units of product A.

Deviational variables are included in the objective function of the GP and are ranked according to their preemptive priority factor, from the most important to the least important

Formulate this problem as a GP model and then solve it by using the graphical method.

Model formulation Let

x_1 and x_2 = number of units of products A and B produced, respectively
d_i^- and d_i^+ = underachievement and overachievement associated with goal i, respectively

Then the GP model is stated as follows:

$$\text{Minimize } Z = P_1 d_1^- + P_2 d_3^- + P_3 d_2^-$$

subject to the constraints

(i) $2x_1 + 3x_2 \leq 30$, (ii) $6x_1 + 4x_2 \leq 60$

(iii) $x_1 + x_2 + d_1^- - d_1^+ = 10$, (iv) $x_1 + d_2^- - d_2^+ = 8$

(v) $x_1 + d_3^- - d_3^+ = 7$

and $x_1, x_2, d_i^-, d_i^+ \geq 0$, for all i.

Graphical solution The first two constraints in the GP model are system constraints. Constraints 3 to 5 are goal constraints. Figure 8.1 (a) illustrates the solutions space associated with the two system constraints and the lines associated with the goal constraints with deviational variables labelled. The farther a point is from a goal constraint line, the larger the value of the corresponding deviational variable. The closer a point is to a goal constraint line, the smaller the value of deviational variable associated with the goal.

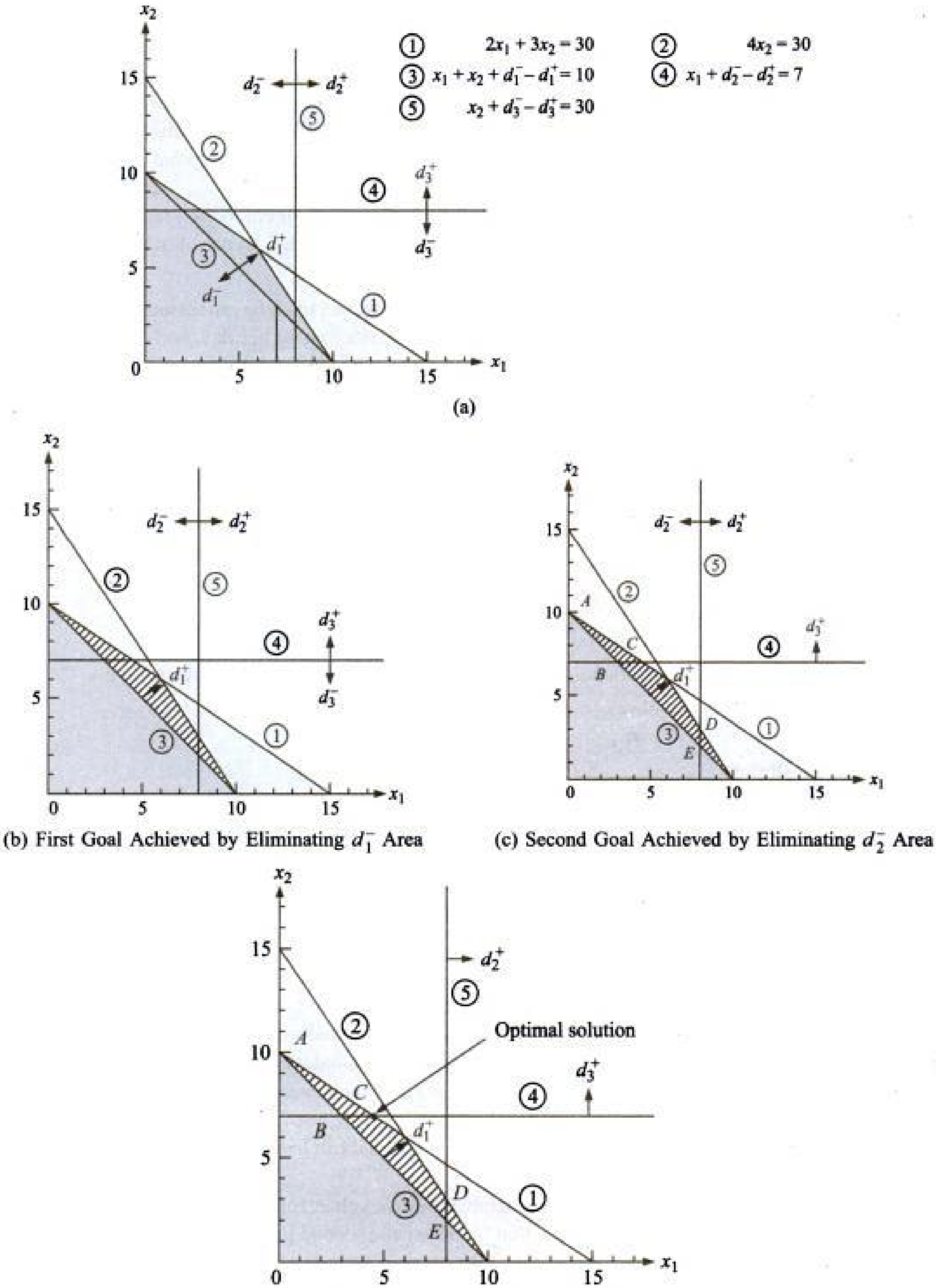
The highest priority is to minimize d_1^-, which represents underachievement of joint total production goal of 10 units. Thus all possible points that have positive values of d_1^- have been eliminated, as shown in Fig. 8.1(b). The lined area represents all combinations of product A and B that can be produced and that satisfy or exceed the production goal of 10 units.

To achieve second priority goal, all points that have positive values for d_3^- have been eliminated, as shown in Fig. 8.1(c). The points eliminated are those that lie below the second goal constraint line. These represent combinations of products A and B that fall short of the production goal of 7 units for product B.

The third priority goal is to minimize d_2^-, which represents underachievement of the production goal of 8 units of product A. In the lined area ABC each point involves the underachievement of the third goal. The optimal solution occurs at corner point C where d_3^- is made as small as possible. At points D or E in Fig. 8.1(d), d_2^- would become zero, and it would give positive values for d_3^-, sacrificing a higher priority goal, which is not allowed.

Since point C occurs at the intersection of constraints 1 and 4, by solving these equations, we get $x_1 = 4.5$ and $x_2 = 7$. Thus, the firm should produce 4.5 units of product A and 7 units of product B. Substituting $x_1 = 4.5$ and $x_2 = 7$ in the given constraints, we find that

- Department 1 has utilized its maximum capacity of 30 hours.
- Department 2 has unused time (slack) of 5 hours



(d) Third Goal Achieved by Eliminating d_3^- Area

Fig. 8.1(a) to (d) System and Goal Constraints

There is an overachievement of the joint production goal equal to 1.5 (4.5 + 17 – 10) units, while production goal of 7 units for product B has been fully achieved. There is an underachievement of the production goal for product A of 3.5 (8 – 4.5) units.

Example 8.5 A company produces motorcycle seats. The company has two production lines. The production rate for line 1 is 50 seats per hour and for line 2 it is 60 seats per hour. The company has entered into a contract to daily supply 1,200 seats daily to another company. Currently, the normal operation period for each line is 8 hours. The production manager of the company is trying to determine the best daily operation hours for the two lines. He has set the priorities to achieve his goals, as given below:

- P_1 : Produce and deliver 1,200 seats daily
- P_2 : Limit the daily overtime operation hours of line 2 to 3 hours
- P_3 : Minimize the underutilization of the regular daily operation hours of each line. Assign differential weights based on the relative productivity rate.
- P_4 : Minimize the daily overtime operation hours of each line as much as possible. Assign differential weights based on the relative cost of overtime. It is assumed that the cost of operation is identical for the two production lines.

Formulate this problem as a GP model and then solve it by using the graphical method.

Model formulation Let x_1 and x_2 = daily operation hours for the lines 1 and 2, respectively.

It may be noted in this problem that the first criterion for determining the differential weights in the third priority goal is the relative cost of overtime. The production rates ratio for the lines is 50 to 60. Therefore, the relative cost resulting from an hour of overtime is greater for line 1 than for line 2. The relative cost of overtime ratio for line 1 to line 2 will be line 6 to line 5. Second, the criterion for determining the differential weights in the fourth priority goal is relative productivity rate. Since the productivity of line 1 is 50 seats/hour and of line 2 is 60 seats/hour, the productivity goals are weighted for line 1 to line 2 as line 5 to line 6. The goal programming model for this problem can now be formulated as follows:

Minimize (total deviation) $Z = p_1\, d_1^- + P_2\, d_4^+ + P_3\,(5\,d_2^- + 6\,d_3^-) + P_4(6\,d_2^+ + 5\,d_3^+)$

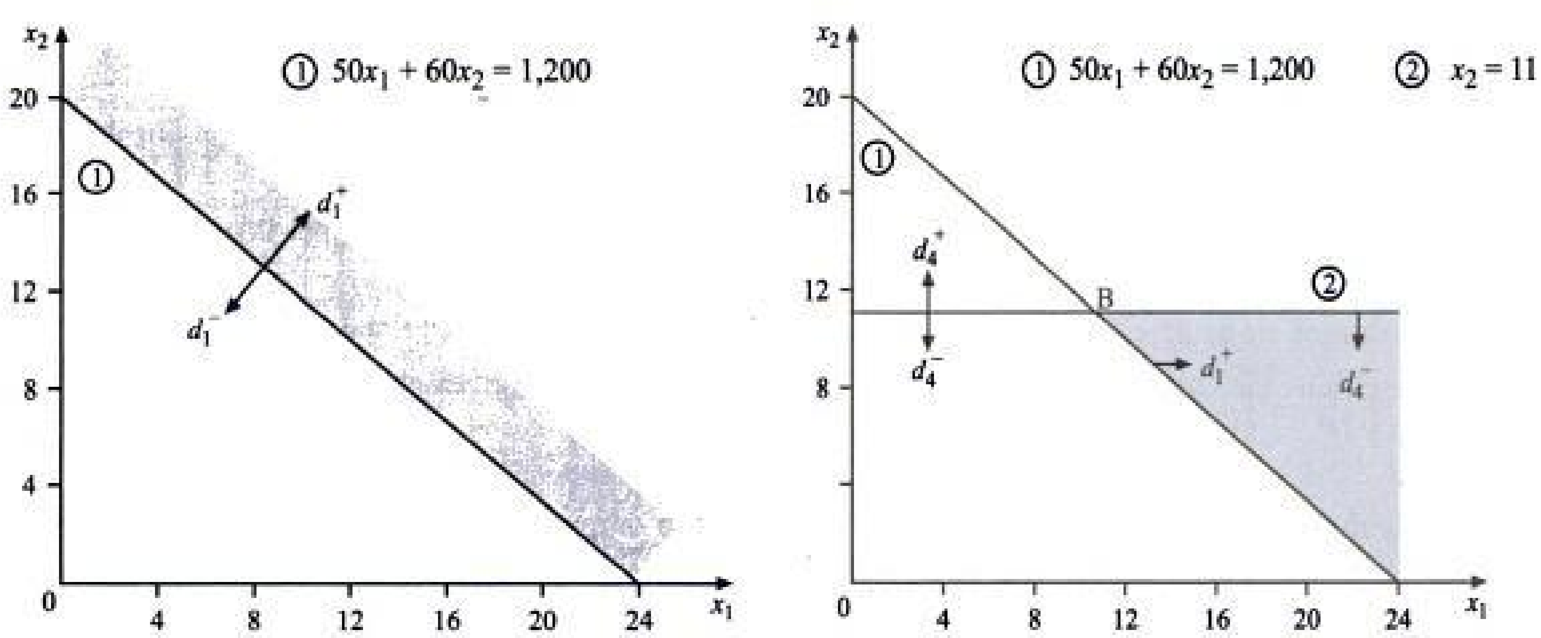
subject to the constraints

(i) $50x_1 + 60x_2 + d_1^- - d_1^+ = 1{,}200$, (ii) $x_1 + d_2^- - d_2^+ = 8$

(iii) $x_2 + d_3^- - d_3^+ = 8$, (iv) $x_2 + d_4^- - d_4^+ = 11$

and $x_1, x_2, d_i^+, d_i^- \geq 0$, for all i = 1, 2, 3, 4.

Graphical solution To solve this problem, each goal constraint is graphed at a time, starting with the one that has the highest-priority deviational variable. Since the profit goal constraint deviation d_1^- has priority P_1 in the objective function, this constraint is graphed, ignoring the deviational variables d_1^- and d_1^+, as shown in Fig. 8.2 (a).



(a) Analysis of Goal 1 (b) Analysis of Goal 1 and 2

Fig. 8.2 Analysis of First and Second Goals

To minimize d_1^- (underachievement of delivering 1,200 seats), the feasible area is the shaded region. Any point in the shaded region satisfies the first goal because the production exceeds 1,200 seats.

Figure 8.2 (b) includes the second priority goal of minimizing d_4^+. The region above the constraint line x_2 = 11 represents the value d_4^+, while the region below the line stands for d_4^-. To avoid the overachievement of the second goal, the area above the line is eliminated. But this must be attained within the feasible area that has already been defined by satisfying the first goal, as shown in Fig. 8.2 (b).

The third goal is to avoid underutilization of regular daily operation hours of each line. This means that both d_2^- and d_3^- should be as close to zero as possible.

At point A in Fig. 8.2 (c), both third and fourth goal are achieved. However, at this point, the first and second priority goals are not achieved. At point B, however, the solution is: x_1 = 10.8 and

$x_2 = 11$, $d_1^- = d_4^+ = 0$, $d_2^+ = 3.8$, $d_3^+ = 3$, $d_2^- = d_3^- = 0$. Hence, at this solution point, the first, second and third priority goals are completely achieved, but the fourth priority goal is not completely achieved. This is because the production line 1 has 3.8 hours of overtime ($d_2^+ = 3.8$) and production line 2 has 3 hours of overtime ($d_3^+ = 3$).

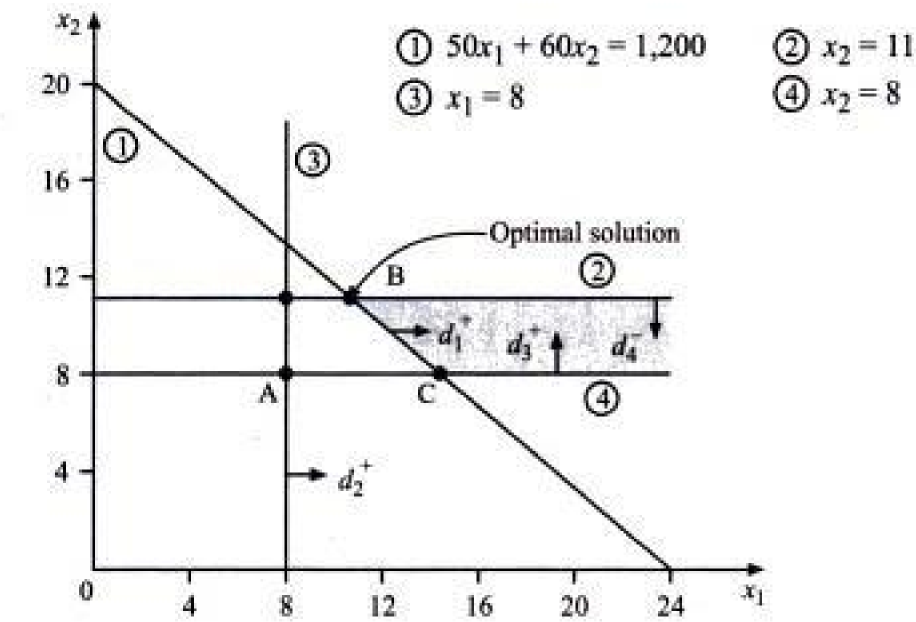


Fig. 8.2(c) Analysis of all Four Priority Goals

8.6 MODIFIED SIMPLEX METHOD OF GOAL PROGRAMMING

The simplex method for solving a GP problem is similar to that of an LP problem. The features of the simplex method for the GP problem are:

1. The z_j and $c_j - z_j$ values are computed separately for each of the ranked goals, P_1, P_2, ... This is because different goals are measured in different units. These are shown from bottom to top, i.e. first priority goal (P_1) is shown at the bottom and least priority goal at the top.

   The optimality criterion z_j or $c_j - z_j$ becomes a matrix of $k \times n$ size, where k represents the number of pre-emptive priority levels and n is the number of variables including both decision and deviational variables.

2. First examine $c_j - z_j$ values in the P_1-row. If all $c_j - z_j \leq 0$ at the highest priority levels in the same column, then the optimal solution been obtained.

   If $c_j - z_j > 0$, at a certain priority level, and there is no negative entry at higher unachieved priority levels, in the same column, the current solution is not optimal.

3. If the target value of each goal in $\mathbf{x_B}$-column is zero, the solution is optimal.

4. To determine the variable to be entered into the new solution mix, start examining $(c_j - z_j)$ row of highest priority (P_1) and select the largest negative value. Otherwise, move to the next higher priority (P_2) and select the largest negative value.

5. Apply the usual procedure for calculating the 'minimum ratio' to choose a variable that needs to leave the current solution mix (basis).

6. Any negative value in the $(c_j - z_j)$ row that has positive $(c_j - z_j)$ value under any lower priority rows are ignored. This is because that deviations from the highest priority goal would be increased with the entry of this variable in the solution mix.

Example 8.6 Use modified simplex method to solve the following GP problem.

Minimize $Z = P_1 d_1^- + P_2(2d_2^- + d_3^-) + P_3 d_1^+$

subject to the constraints

(i) $x_1 + x_2 + d_1^- - d_1^+ = 400$, (ii) $x_1 + d_2^- = 240$, (iii) $x_1 + d_3^- = 300$

and $x_1, x_2, d_1^-, d_1^+, d_2^-, d_3^- \geq 0$

Solution The initial simplex table for this problem is presented in Table 8.1. The basic assumption in formulating the initial table of the GP problem is the same as that of the LP problem. In goal programming, the pre-emptive priority factors and differential weights correspond to the c_j values in linear programming.

c_B	Variables in Basis B	$c_j \rightarrow$ Solution Values $b\ (= x_B)$	0 x_1	0 x_2	P_1 d_1^-	$2P_2$ d_2^-	P_2 d_3^-	P_3 d_1^+	Min Ratio x_B/x_1
P_1	d_1^-	400	1	1	1	0	0	−1	400/1
$2P_2$	d_2^-	240	1	0	0	1	0	0	240/1 →
P_2	d_3^-	300	0	1	0	0	1	0	—
	P_3	0	0	0	—	—	—	1	
$c_j - z_j$	P_2	780	−2	−1	—	—	—	0	
	P_1	400	−1	−1	—	—	—	1	
			↑						

Table 8.1
Initial Solution

In Table 8.1, the optimality criterion $(c_j - z_j)$ is 3×6 matrix because we have three priority levels and six variables (2 decision, 4 deviational) in the model.

By using the standard simplex method for the calculation of Z-value, we would obtain Z-value in GP as:

$$Z = P_1 \times 400 + 2P_2 \times 240 + P_2 \times 300 = 400P_1 + 780P_2$$

The values, $P_1 = 400$, $P_2 = 780$ and $P_3 = 0$ in the $\mathbf{x_B}$-column below the line represent the unachieved portion of each goal.

Now let us calculate $c_j - z_j$ values in Table 8.1. We have already said that c_j values represent the priority factors assigned to deviational variables and that the z_j values represent the sum of the product of entries in $\mathbf{c_B}$-column with columns of coefficient matrix. Thus, the $c_j - z_j$ value for each column is calculated as follows:

$$c_1 - z_1 = 0 \quad - (P_1 \times 1 + 2P_2 \times 1 + P_2 \times 0) = - P_1 - 2P_2$$
$$c_2 - z_2 = 0 \quad - (P_1 \times 1 + 2P_2 \times 0 + P_2 \times 1) = - P_1 - \ P_2$$
$$c_6 - z_6 = P_3 - (P_1 \times -1) = P_3 + P_1$$

Iteration 1: The selection of key column is based on the per unit contribution rate of each variable in achieving the most important goal (P_1). The pre-emptive priority factors are listed from the lowest to the highest so that the key column can be easily identified at the bottom of the table.

The column with the largest negative $c_j - z_j$ value at the P_1 level is selected as the key column. In Table 8.1, there are negative values (i.e. − 1) in the x_1 and x_2 columns. Remove this tie, as always, and choose variable x_1 to enter into the new solution mix.

The key row is the row with the minimum non-negative value, which is obtained by dividing the $\mathbf{x_B}$-values by the corresponding positive coefficients in the key column. The coefficient 1 is circled in Table 8.1 to indicate the fact that it is the key element at the intersection of the key column and key row.

By using the standard simplex method, the solution in Table 8.1 is revised to obtain the second improved solution, as shown in Table 8.2.

c_B	Variables in Basis (B)	$c_j \rightarrow$ Solution Values $b\ (= x_B)$	0 x_1	0 x_2	P_1 d_1^-	$2P_2$ d_2^-	P_2 d_3^-	P_3 d_1^+	Min Ratio x_B/x_2
P_1	d_1^-	160	0	1	1	−1	0	−1	160/1→
0	x_1	240	1	0	0	1	0	0	—
P_2	d_3^-	300	0	1	0	0	1	0	300/1
	P_3	0	–	0	–	–	–	1	
$c_j - z_j$	P_2	300	–	−1	–	2	–	–	
	P_1	160	–	−1	–	1	–	1	
				↑					

Table 8.2

As per Table 8.2, the value of objective function: $160 \times P_1 + 300 \times P_2$ indicates that the unachieved portion of the first and second goals has decreased. The revised solution is shown in Table 8.3.

Since the solution in Table 8.3 indicates that $c_j - z_j$ values in P_1-row are either positive or zero as well as the value of Z in terms of P_1 is completely minimized to zero, we turn our attention to the second priority level (P_2). An additional rule must be followed at this stage: *A column cannot be chosen as the key column with a positive value at a higher priority level.*

Table 8.3

		$c_j \rightarrow$	0	0	P_1	$2P_2$	P_2	P_3	
c_B	*Variables in Basis (B)*	*Solution Values $b\ (= x_B)$*	x_1	x_2	d_1^-	d_2^-	d_3^-	d_1^+	*Min Ratio x_B/d_1^+*
0	x_2	400	0	1	1	−1	0	−1	–
0	x_1	240	1	0	0	1	0	0	–
P_2	d_3^-	300	0	0	−1	1	1	1	300/1 →
	P_3	0	–	–	0	0	–	1	
$c_j - z_j$	P_2	140	–	–	1	1	–	−1	
	P_1	0	–	–	1	0	–	0 ↑	

The largest negative value in P_2-row is selected in order to determine the key column. The revised solution is shown in Table 8.4.

In Table 8.4, all $c_j - z_j$ values in the P_2-row are either positive or zero. Thus, the second goal (P_2) is fully achieved. It may be noted in Table 8.4 that there are two negative values in the P_3-row. However, we could not choose d_2^- or d_3^- as the key column because there is already a positive value at a higher priority level (P_2). Hence, the solution shown in Table 8.4 cannot be improved further. The optimal solution therefore is: $x_1 = 240, x_2 = 300,\ d_1^- = d_2^- = d_3^- = 0,\ d_1^+ = 140$.

Table 8.4

		$c_j \rightarrow$	0	0	P_1	$2P_2$	P_2	P_3
c_B	*Variables in Basis (B)*	*Solution Values $b\ (= x_B)$*	x_1	x_2	d_1^-	d_2^-	d_3^-	d_1^+
0	x_2	300	0	1	1	0	1	0
0	x_1	240	1	0	0	1	0	0
P_3	d_1^+	140	0	0	−1	1	1	1
	P_3	140	–	–	1	−1	−1	–
$c_j - z_j$	P_2	0	–	–	0	2	1	–
	P_1	0	–	–	1	0	0	–

8.7 ALTERNATIVE SIMPLEX METHOD FOR GOAL PROGRAMMING*

This proposed procedure is based on Baumol's simplex method for solving GP problems but is with minor modifications. It appears to be more efficient than the one suggested by Lee. The steps of the procedure are summarized below:

Step 1: Formulation of initial solution table The generalized formulation of a GP problem into an initial solution table begins in a similar fashion to the LP problem table. First, the goal constraints are reformulated in terms of their d_i^+ variables (i.e. basic variables):

$$d_i^+ = -b_i + \sum_{j=1}^{n} a_{ij}\, x_j + d_i^-;\quad i = 1, 2, \ldots, m$$

If a goal constraint does not possess a d_i^+ variable, it is artificially assigned with a zero priority for the sake of the initial table formulation. In Table 8.5 an initial table is presented. Row 1 in this table labels the decision variables x_j and negative deviational variable d_i^-. These variables are zero (non-basic) variables. The right-hand-values, b_i, are placed in column 3; the decision-variable coefficients a_{ij} are placed in column 4; and an identity matrix is placed in column 5, representing the inclusion of negative deviational variables d_i^-. In problems where a goal constraint does not have a d_i^- variable, the value of 'zero' would be placed in that column instead of 'one'. Mathematically, this requireds that all deviational variables, not included in the problem formulation, are equal to zero.

Column 1 in the table lists the appropriate priority factors P_i and weights w_i for each positive deviational variable (i.e. basic variables), including artificial deviational variables as shown in column 2. Row 2, column 3, in the table contains a value called 'total absolute deviation'. It represents the amount of total deviations from all goals for each table as the iterative process proceeds. Row 2, column 4, is simply a row vector of zero, representing the inclusion of all decision variables in the computational process. Row 2, column 5, lists the appropriate weights w_i for each negative deviational variable included in the objective function.

	(1)	(2)	(3)	(4)	(5)
(1)				$x_1, x_2, \ldots, x_n$	$d_1^-, d_2^-, \ldots, d_m^-$
(2)	*Weighted Priority*	Z	$\sum_{i=1}^{m} \lvert w_i \cdot b_i \rvert$	$0, 0, \ldots, 0$	$w_1, w_2, \ldots, w_m$
(3)	$w_1 P_1$	d_1^+	$-b_1$	$a_{11}, a_{12}, \ldots, a_{1n}$	$1 \; 0 \ldots 0$
	$w_2 P_2$	d_2^+	$-b_2$	$a_{21}, a_{22}, \ldots, a_{2n}$	$0 \; 1 \ldots 0$
	$\vdots$	$\vdots$	$\vdots$	$\vdots$	$\vdots$
	$w_m P_m$	d_m^+	$-b_m$	$a_{m1}, a_{m2}, \ldots, a_{mn}$	$0 \; 0 \ldots 1$

Table 8.5
Initial Table for a Generalized GP Model

Step 2: Modify the initial solution

(a) Determine which variable is to exit the solution basis. This is accomplished by selecting the variable with the highest-ranked priority. When two or more variables have the same priority ranking, the variable with the greatest mathematical weight determines which variable is to be selected first. When two or more variables have the same weighted priority level, the selection of the variable that has the greatest negative right-hand side value is used as the selection criterion. Its selection will eliminate the most negative values from the solution basis, thereby completing, quickly, the required computations for an optimal solution. This variable labels the *pivot row*.

(b) Determine which variable is to enter the solution basis. This is accomplished by selecting the column that has the smallest resulting ratio when the positive coefficients in the pivot row are divided into their respective positive elements in row 2. This variable labels the *pivot column*. The element at the intersection of the pivot row and pivot column is called the *pivot element*. If there is a tie in the column ratio, select the column that has the smaller ratio in the next priority row. If the tie cannot be broken by examining the next priority, then the selection of the pivot column would be based on the largest coefficient in the pivot row for the tied columns. (Note that artificial variables are not allowed to re-enter the solution basis once they have been removed.)

(c) Set a framework for the new table by exchanging the variables in the pivot row and pivot column. Also, the priority attached to the variable brought into the solution basis should be placed in column 1 of the new table. All other variables resume their places in the new table.

(d) In the new table, the new element that corresponds to the pivot element is found by taking the reciprocal of the pivot element. All other elements in the row are found by dividing the pivot-row elements by the pivot element and changing the resulting sign.

(e) Determine the new elements that correspond to the elements in the pivot column. These elements are found by dividing the pivot-column elements by the pivot element.

(f) Except for the element representing total absolute deviation (i.e. column 3, row 2), all other elements are found by the following formula:

$$\text{New element} = \text{Old Element} - \frac{\text{Product of Two Corner Elements}}{\text{Pivot Element}}$$

* Based on, Schniederjans, Marc J. and N.K. Kwak, *An Alternative Solution Method for Goal Programming Problems*, J. Opns. Res., Vol. 33, 1982.

The product of two corner elements is found by selecting elements out of the pivot row and pivot column.

(g) Determine the new total absolute deviation by the following formula: $Z = \sum_{i=1}^{m} |w_i \cdot b_i|$

(h) Check to see if the solution is optimal. The solution is optimal if all the basic variables are positive (i.e. positive b_i) and if the pre-emptive priority rule is satisfied. If one or more of the basic variables are negative, repeat Steps 1 to 8. If all the basic variables are positive but the pre-emptive priority rule is not satisfied, the solution is not optimal. Continue to Step (i).

(i) Determine which variable is to exit the solution basis. This is accomplished by selecting the largest positive element in column 3, with the highest priority level. The pivot element comes from this row.

(j) Determine which variable is to enter the solution basis. This is accomplished by selecting the column that has the smallest resulting ratio when the negative coefficients in the pivot row are divided into their respective positive elements in row 2, changing the resulting sign. This variable labels the pivot column. Repeat Steps (c) to (h).

(k) The solution is optimal if the basic variables are all positive and one or more of the objective function rows (i.e. row 2) have a negative sign.

Example 8.7 Solve the following GP problem

Minimize $Z = P_1 d_1^- + P_2 d_4^+ + 5P_3 d_2^- + 3P_3 d_3^- + P_4 d_1^+$

subject to the constraints

(i) $x_1 + x_2 + d_1^- - d_1^+ = 80,$ (ii) $x_1 + d_2^- = 70$

(iii) $x_2 + d_3^- = 45,$ (iv) $x_1 + x_2 + d_4^- - d_4^+ = 90$

and $x_j,\ d_i^-,\ d_i^+ \geq 0;\quad i = 1, 2, 3, 4;\ j = 1, 2.$

Using the steps of the method that have already been discussed, the given GP problem data is displayed in the initial Table 8.6.

Table 8.6 Initial Solution

			x_1	x_2	d_1^-	d_2^-	d_3^-	d_4^-
Weighted Priority	Z	*170*	*0*	*0*	*1*	*5*	*3*	*0*
P_4	d_1^+	−80	1	1	1	0	0	0
0 P_0	d_2^+	−70	1	0	0	1	0	0
0 P_0	d_3^+	−45	0	1	0	0	1	0
P_2	d_4^+	−90	1	1	0	0	0	1

In the simplex Table 8.6, zero elements in the Z row, (x_1 and x_2 columns) are treated as very small numbers for a pivot-column selection.

Table 8.7

			d_2^+	x_2	d_1^-	d_2^-	d_3^-	d_4^-
Weighted Priority	Z	*30*	*0*	*0*	*1*	*5*	*3*	*0*
P_4	d_1^+	−10	1	1	1	−1	0	0
	x_1	70	1	0	0	−1	0	0
0 P_0	d_3^+	−45	0	1	0	0	1	0
P_2	d_4^+	−20	1	1	0	−1	0	1

Table 8.8

			d_2^+	d_3^+	d_1^-	d_2^-	d_3^-	d_4^-
Weighted Priority	Z	*60*	*0*	*0*	*1*	*5*	*3*	*0*
P_4	d_1^+	35	1	1	1	−1	−1	0
	x_1	70	1	0	0	−1	0	0
	x_2	45	0	1	0	0	−1	0
P_2	d_4^+	25	1	1	0	−1	−1	1

			d_2^+	d_3^+	d_1^-	d_2^-	d_4^+	d_4^-
Weighted Priority	*Z*	*85*	*3*	*3*	*1*	*2*	*−3*	*3*
P_4	d_1^+	10	0	0	1	0	1	−1
	x_1	70	1	0	0	−1	0	0
	x_2	20	−1	0	0	1	1	−1
3 P_3	d_3^-	25	1	1	0	−1	−1	1

Table 8.9
Optimal Solution

It can be seen that the Table 8.9 presents an optimal solution, as column 3 contains all positive values (i.e. positive basic variables) and row 2 has a negative value in column d_4^+.

The solution can be read from the solution basis: $x_1 = 70$, $x_2 = 20$, $d_1^+ = 10$ and $d_3^- = 25$. This results in a total absolute deviation from desired goals of $Z = 85$. Examining the weighted priority column, it can be seen that the higher level priorities P_1 and P_2 are now fully satisfied as they do not appear in this column.

CONCEPTUAL QUESTIONS

1. What is goal programming? Clearly state its assumptions.
2. (a) Compare the differences/similarities between linear programming and goal programming.
   (b) Why are all goal linear programming problems minimization problems? Can a goal programming problem be infeasible? Discuss. *[Delhi Univ., MBA, 1999, 2003]*
3. Explain the following terms
   (i) Deviational variables
   (ii) Pre-emptive priority factors
4. Explain the difference between cardinal value and ordinal value.
5. Under what circumstances can cardinal weights be used in the objective function of a goal programming model? What happens if the cardinal weights are attached to all priorities in the objective function of a goal programming model?
6. (a) Explain the differences between solving a linear programming as against a goal programming problem by the simplex method.
   (b) What conditions require that a GP model rather than an LP model be used to solve a decision problem.
7. State some problem areas in management where goal programming might be applicable.
8. (a) Why are the derivational variables associated with a particular goal complementary?
   (b) What is the difference between a positive and negative deviational variable?
9. 'Goal programming appears to be the most appropriate, flexible and powerful technique for complex decision problems involving multiple conflicting objectives.' Discuss. *[Delhi Univ., MBA, 1999, 2000]*
10. What is goal programming? Why are all goal programming problems minimization problems? Why does altering the goal priorities result in a different solution to a problem? Explain.
11. What is meant by the terms 'satisfying' and why is the term often used in conjunction with goal programming?
12. What are deviational variables? How do they differ from decision variables in traditional linear programming problems?
13. What does 'to rank goals' mean in goal programming? How does this affect the problem's solution?

SELF PRACTICE PROBLEMS

1. Solve the following goal programming problems using both the graphical method and the simplex method to obtain the solution
   (a) Minimize $Z = P_1d_1^+ + P_2d_2^- + P_3d_3^-$
   subject to
   $$x_1 + x_2 + d_1^- - d_1^+ = 40$$
   $$x_1 + d_2^- - d_3^+ = 20$$
   $$x_1, x_2, d_i^-, d_i^+ \geq 0, \quad \text{for all } i$$
   (b) Minimize $Z = P_1d_1^- + P_2(8d_2^- + 6d_3^-) + P_3d_1^+$
   subject to
   $$2x_1 + x_2 + d_1^- - d_1^+ = 16$$
   $$x_1 + d_2^- - d_2^+ = 7$$
   $$x_2 + d_3^- - d_3^+ = 10$$
   and
   $$x_1, x_2, d_i^-, d_i^+ \geq 0, \quad \text{for all } i$$
   (c) Minimize $Z = P_1d_1^- + P_2d_2^- + P_3d_3^-$
   subject to
   $$2x_1 + 3x_2 \leq 30$$
   $$6x_1 + 4x_2 \leq 60$$
   $$x_1 + x_2 + d_2^- - d_2^+ = 8$$
   $$x_2 + d_3^- - d_3^+ = 7$$
   and
   $$x_1, x_2, d_i^-, d_i^+ \geq 0, \quad \text{for all } i$$
2. An office equipment manufacturer produces two kinds of products, chairs and lamps. The production of either, a chair or a lamp, requires 1 hour of production capacity in the plant. The plant has a maximum production capacity of 10 hours per week. Because of the limited sales capacity, the maximum number of chairs and lamps that can be sold are 6 and 8 per week, respectively. The gross margin from the sale of a chair is Rs 80 and from the sale of a lamp is Rs 40.

   The plant manager has set the following goals, arranged in order of importance:

(i) He wants to avoid any underutilization of production capacity.
(ii) He wants to sell as many chairs and lamps as possible. Since the gross margin from the sale of a chair is set at twice the amount of profit from a lamp, he has twice as much desire to achieve the sales goal for chairs as for lamps.
(iii) He wants to minimize the overtime operation of the plant as much as possible.

Formulate and solve this problem as a GP problem, so that the plant manager makes a decision that will help him achieve his goals as closely as possible. [*Delhi Univ., MBA, 1999*]

3. A production manager faces the problem of job allocation among three of his teams. The processing rates of the three teams are 5, 6 and 8 units per hour, respectively. The normal working hours for each team is 8 hours per day. The production manager has the following goals for the next day in order of priority:
   (i) The manager wants to avoid any underachievement of production level, which is set at 180 units of product.
   (ii) Any overtime operation of team 2 beyond two hours and team 3 beyond three hours should be avoided.
   (iii) Minimize the sum of overtime.

   Formulate and solve this problem as a goal programming problem.

4. XYZ company produces two products – record players and tape-recorders. Both the products are produced in two separate machine centres within the plant. Each record player requires two hours in machine centre A and one hour in machine centre B. Each tape-recorder, on the other hand, requires one hour in the machine centre A and three hours in machine centre B. In addition, each product requires some in-process inventory. The per-unit. The in-process inventory required is Rs 50 for the record player and Rs 30 for the tape-recorder. The firm has normal monthly operation hours of 120 for both machine centres A and B. The estimated profit per unit is Rs 100 for the record player and Rs 75 for the tape-recorder. According to the marketing department, the forecast of sales for the record player and the tape-recorder are 50 and 80, respectively for the coming month.

   The president of the firm has established the following goals for production in the next month, in ordinal rank of importance:
   (i) Limit the amount tied up in in-process inventory for the month to Rs 4,600.
   (ii) Achieve the sales goal of 80 tape-recorders for the month.
   (iii) Limit the overtime operation of machine centre A to 20 hours.
   (iv) Achieve the sales goal of 50 record players for the month.
   (v) Limit the sum of overtime operation for both machine centres.
   (vi) Avoid any underutilization of regular operation hours of both machine centres.

   Formulate and solve this problem as a goal programming problem.

5. ABC Furnitures produce three products – tables, desks and chairs. The furniture is produced in the central plant. The production of the desk requires 3 hours in the plant, the table 2 hours and the chair only 1 hour. The regular plant capacity is 40 hours a week. According to the marketing department, the maximum number of desks, tables and chairs that can be sold weekly are 10, 10 and 12, respectively. The president of the firm has established the following goals, according to their importance:
   (i) Avoid any underutilization of production capacity.
   (ii) Meet the order of XYZ Store for seven desks and five chairs.
   (iii) Avoid overtime operation of the plant beyond 10 hours.
   (iv) Achieve sales goal of 10 desks, 10 tables, and 12 chairs.
   (v) Minimize overtime operation as much as possible.

   Formulate and solve this problem as a goal programming problem. [*Delhi Univ., MBA, 1998*]

6. ABC Computer Company produces three different types of computers – Epic, Galaxie and Utopia. The production of all computers is conducted in a complex and modern assembly line. The production of an Epic requires 5 hours in the assembly line, a Galaxie requires 8 hours and a Utopia requires 12 hours. The normal operation hours of assembly line are 170 per month. The marketing and accounting departments have estimated that the profits, per unit for the three types of computers are Rs 1,00,000 for the Epic, Rs 1,44,000 for the Galaxie and Rs 2,52,000 for the Utopia. The marketing department further reports that the demand is such that the firm can expect to sell all the computers it produces in the month. The chairman of the company has established the following goals. These are listed below, according to their importance:
   (i) Avoid underutilization of capacity in terms of regular hours of operation of the assembly line.
   (ii) Meet the demand of the north-eastern sales district for five Epics, five Galaxies, and eight Utopias (differential weights should be assigned according to the profit ratios among the three types of computers).
   (iii) Limit overtime operation of the assembly line to 20 hours.
   (iv) Meet the sales goal for each type of computer: Epic – 10; Galaxie – 12; and Utopia – 10 (again assign weights according to the relative profit function for each computer).
   (v) Minimize the total overtime operation of the assembly line.

   Formulate and solve this problem as a goal programming model through two iterations (three tables) by the simplex method.

7. The manager of the only record shop in a town has a decision problem that involves multiple goals. The record shop employs five full-time and four part-time salesmen. The normal working hours per month for a full-time salesman are 160 hours and for a part-time salesman 80 hours. According to performance record of the salesmen, the average sales has been five records per hour for full-time salesmen and two records per hour for part-time salesmen. The average hourly wage rates are Rs 3 for full-time salesmen and Rs 2 for part-time salesmen. The average profit from the sales of a record is Rs 1.50. In view of past record of sales, the manager feels that the sales goal for the next month should be 5,500 records. Since the shop is open six days a week, salesmen are often required to work extra (not necessarily overtime but extra hours for the part-time salesmen). The manager believes that a good employer-employee relationship is an essential factor of business success. Therefore, he feels that a stable employment level with occasional overtime requirement is a better practice than an unstable employment level with no overtime. However, he feels that overtime of more than 100 hours among the full-time salesmen should be avoided because of the declining sales effectiveness caused by fatigue.

   The manager has set the following goals:
   (i) The first goal is to achieve a sales target of 5,500 records for the next month.
   (ii) The second goal is to limit the overtime of full-time salesmen to 100 hours.
   (iii) The third goal is to provide job security to salesmen. The manager feels that full utilization of employees' regular working hours (no layoffs) is an important factor for a good employer-employee relationship. However, he is twice as concerned with the full utilization of full-time salesmen as with the full utilization of part-time salesmen.
   (iv) The last goal is to minimize the number of hours of overtime for both full-time and part-time salesmen. The manager desires to assign differential weights to the minimization of overtime according to the net marginal profit ratio between the full-time and part-time salesmen.

   Formulate and solve the given problem as a goal programming problem.

8. A hospital administration is reviewing departmental requests prior to the design of a new emergency room. At issue is the number of beds for each department. The current plans call for a 15,000 square feet facility. The hospital board has established the following goals in order of importance:

Department	No. of Beds Requested	Cost per Bed (incl. equimpment) (Rs)	Area per bed (Sq. ft)	Peak Requirement (Max. no. of Patients at one time
A	5	12,600	474	3
B	20	5,400	542	18
C	20	8,600	438	15

(i) Avoid overspending of the budget Rs 3,00,000.
(ii) Avoid plan requiring more than 15,000 sq. ft.
(iii) Meet the peak requirement.
(iv) Meet the departmental requirements.

Formulate and solve (up to three iterations) the given problem as a goal programming problem. [*Delhi Univ., MBA, 2003*]

9. Mr X has inherited Rs 10,00,000 and seeks your advice concerning investing his money. You have determined that 10 per cent can be earned on a bank account and 14 per cent by investing in certificates of deposit. In real estate, you estimate an annual return of 15 per cent, while on the stock market you estimate that 20 per cent can be earned annually.

Mr X has established the following goals in order of importance:
(i) Minimize risk; therefore he wants to invest not more than 35 per cent in any one type of investment.
(ii) Must have Rs 1,00,000 in the bank account to meet any emergency.
(iii) Maximum annual cash return.

Formulate and solve (up to three iterations only) this problem as a goal programming model that determines the amount of money which X should invest in each investment option.

10. A company plans to schedule its annual advertising campaign. The total advertising budget is set at Rs 10,00,000. The firm can purchase local radio spots at Rs 2,000 per spot, local television spots at Rs 12,000 per spot or magazine advertising at Rs 4,000 per insertion. The payoff from each advertising medium is a function of its audience size and audience characteristics. Let this payoff be defined as audience points. Audience points for the three advertising vehicles are:

Radio	50 points per spot
Television	250 points per spot
Magazine	200 points per insertion.

The Advertising Manager of the firm has established the following goals for the advertising campaign, listed in the order of importance.
(i) The total budget should not exceed Rs 10,00,000.
(ii) The contract with the radio and television station requires that the firm spend at least Rs 3,00,000 for television and radio ads.
(iii) The company does not wish to spend more than Rs 2,00,000 for magazine ads.
(iv) Audience points from the advertising campaign should be maximized.

Formulate and solve (up to three iterations only) this problem as a goal programming problem by the simplex method. [*Delhi Univ., MBA, 1995, 99*]

11. A shoe manufacturer produces hiking boots and ski boots. The manufacturing process of the boots consist of sewing and stitching. The company has available 60 hours per week for the sewing process and 80 hours per week for the stitching process at normal capacity. The firm realizes profits of Rs 150 per pair on hiking boots and Rs 100 per pair on ski boots. It requires 2 hours of sewing and 5 hours of stitching to produce one pair of hiking boots and 3 hours of sewing and 2 hours of stitching to produce one pair of ski boots. The president of the company wishes to achieve the following goals, listed in the order of their importance:
(i) Achieve the profit goal of Rs 5,250 per week.
(ii) Limit the overtime operation of the sewing center to 30 hours.
(iii) Meet the sales goal for each type of boot – 25 hiking boots and 20 ski boots.
(iv) Avoid any underutilization of regular operation hours of the sewing center.

Formulate and solve this problem as a goal programming model. [*Delhi Univ., MBA, 2000*]

12. Delta Hospital is a medium-size many healthcare facilities (HCF) hospital, located in a small city of Bihar. HCF is specialized in performing four types of surgeries: T, A, H, and C. The performance of these surgeries is constrained by three resources: operating room hours, recovery room bed hours, and surgical service bed days. The director of HCF would like to achieve the following objectives, in order of their importance. Also given below is the information pertaining to the hospital.

Resources	Types of Surgical Patients				Capacity
	P_1	P_2	P_3	P_4	
Operating room	3	4	8	6	1,100 hours
Recovery room	8	2	4	2	1,400 bed-hours
Surgical service	4	6	2	4	400 bed days
Average contribution to profit (Rs)	2,100	2,600	2,800	3,000	

P_1: Achievement of at least Rs 5,00,000 in profit in a specified period of time given the available resources.
P_2: Minimization of idle capacity of available resources.

Formulate and solve this problem as a GP problem.

CHAPTER SUMMARY

Goal programming (GP) is an approach used for solving a multi-objective optimization problem that balances conflicting objectives to reach a 'satisfactory' level of goal attainment. A problem is modelled into a GP model in a manner similar to that of an LP model. However, the GP model accommodates multiple, and often conflicting, incommensurable goals, in a particular priority order (hierarchy). A particular priority structure is established by ranking and weighing various goals and their subgoals, in accordance with their importance. The priority structure helps to deal with all goals that cannot be completely and/or simultaneously achieved in such a manner that more important goals are achieved first, at the expense of the less important ones.

An important feature of a GP is that the goals (*a specific numerical target values that the decision-maker would ideally like to achieve*) are satisfied in ordinal sequence. That is, the solution of a GP problem involves achieving some higher order (or priority) goals first, before the lower order goals are considered. Since it is not possible to achieve every goal (objective), to the extent desired by the decision-maker, attempts are made to achieve each goal *sequentially* rather than *simultaneously*, up to a *satisfactory* level rather than an optimal level.

In GP, instead of trying to minimize or maximize the objective function directly, as in the case of an LP, the deviations from established goals within the given set of constraints are minimized. The deviational variables are represented in two dimensions – both positive and negative deviations from each goal and subgoal. These deviational variables represent the extent to which the target goals are not achieved. The objective function then becomes the minimization of a sum of these deviations, based on the relative importance within the pre-emptive priority structure assigned to each deviation.

CHAPTER CONCEPTS QUIZ

True or False

1. The main advantage of the goal programming over linear programing is its ability to solve problems comprising multiple constraints.
2. Goal programming appears to be the most appropriate, flexible and powerful technique for complex decision problem involving multiple conflicting objective.
3. In goal programming formulation each goal generates a new constraint and adds at least one new variable to the objective function.
4. In goal programming, a goal constraint having underachievement and overachievement variables is expressed as an equality constraint.
5. Managers, who make use of goal programming, have to specify the relative importance of the goals as well as how much more important one is expected to other.
6. The value of the objective function at the optimal solution to the GP problem reflects the overachievement of the goal in it.
7. In goal programming, the degree of goal achievement depends upon the relative managerial effort applied to an activity.
8. Unlike the simplex method of the linear programming, goal programming does not use the least-positive quotient rule when deciding upon the replaced variable.
9. In order to apply goal programming, management must make the assumption that linearity exists in the usage of resources to the attainment of the goals.
10. Assume that in goal programming, the variable Du and Do are defined as the amount we either fall short of or exceed the target for some goal. Thus, these variables can both be absent in the product mix if the goal is underachieved.

Fill in the Blanks

11. ________ is used for solving a multi-objective optimization problem that balances trade-off in conflicting objectives.
12. In a goal programming problem with prioritized goal, the coefficients of the under achievement variables in the objective function are called ________ .
13. In contrast to goal programming, the objective function of the linear programming is measured in only ________ .
14. ________ Programming is an extension of the ________ programming in which multiple goals ranked according to priorities may be included in the formulation.
15. Goal programming used for ________ technology management.
16. If goal ________ is stated in terms of ________ variable only, then they have to be restated in terms of ________ variable before foregoing with the graphical solution.
17. The solution of the GP problem is ________ if the targeted value of each goal in x_B column is ....
18. In goal programming, if there are two or more z_j and $c_j - z_j$ rows, then the problem has ________ goals.
19. The deviational variable with the identical ________ level are ________ otherwise out.
20. While solving a GP problem we enter the variable heading the column with the negative $c_j - z_j$ value provided corresponding $c_j - z_j$ value in the P_1 area is ________.

Multiple Choice

21. The use of GP model is preferred when
    (a) goals are satisfied in an ordinal sequence
    (b) goals are multiple incommensurable
    (c) more than one objective is set to achieve
    (d) all of the above
22. Deviational variables in GP model must satisfy following conditions
    (a) $d_i^+ \times d_i^+ = 0$ (b) $d_i^+ - d_i^- = 0$
    (c) $d_i^+ + d_i^- = 0$ (d) none of the above
23. In GP, at optimality, which of the following conditions indicate that a goal has been exactly satisfied
    (a) positive deviational variable is in the solution mix with a negative value
    (b) both positive and negative deviational variables are in the solution mix
    (c) both positive and negative deviational variables are not in the solution mix
    (d) none of the above
24. In GP problem, goals are assigned priorities such that
    (a) higher priority goals must be achieved before lower priority goals
    (b) goals may not have equal priority
    (c) goals of greatest importance are given lowest priority
    (d) all of the above
25. In a GP problem, a constraint that has an unachieved variable is expressed as:
    (a) an equality constraint
    (b) a less than or equal to type constraint
    (c) a greater than or equal to type constraint
    (d) all of the above
26. For applying a GP approach, the decision-maker must
    (a) set targets for each of the goals
    (b) assign pre-emptive priority to each goal
    (c) assume that linearity exists in the use of resources to achieve goals
    (d) all of the above
27. Consider a goal with constraint: $g_1(x_1, x_2, \ldots, x_n) + d_1^- - d_1^+ = b_1$ and the term $3d_1^- + 2d_1^+$ in the objective function, the decision-maker
    (a) prefers $g_1(x_1, x_2, \ldots, x_n) \geq b_1$, rather than $\leq b_1$
    (b) prefers $g_1(x_1, x_2, \ldots, x_n) \leq b_1$ rather than $\geq b_1$
    (c) not concerned with either $\leq$ or $\geq$
    (d) none of the above

28. Consider a goal with constraint: $g_1(x_1, x_2, \ldots, x_n) + d_1^- \geq b_1$ ($d_1^- \geq 0$) with d_1^- in the objective function. Then
    (a) the goal is to minimize underachievement
    (b) the constraint is active provided $d_1^- > 0$
    (c) both (a) and (b)
    (d) none of the above
29. Goal programming
    (a) requires only that decision-maker knows whether the goal is direct profit maximization or cost minimization
    (b) allows you to have multiple goals, with or without priorities
    (c) is an approach to achieve goal of a solution to all integer LP problems
    (d) none of the above
30. In simplex method of goal programming, the variable to enter the solution mix is selected with
    (a) lowest priority row and most negative $c_j - z_j$ value in it
    (b) lowest priority row and largest positive $c_j - z_j$ value in it
    (c) highest priority row and most negative $c_j - z_j$ value in it
    (d) highest priority row and most positive $c_j - z_j$ value in it
31. In optimal simplex table of GP problem, two or more $c_j - z_j$ rows indicate
    (a) unequal priority goals (b) equal priority goals
    (c) priority goals (d) unattainable goals
32. The GP approach attempts to achieve each objective
    (a) sequentially (b) simultaneously
    (c) both (a) and (b) (d) none of the above
33. The deviational variable in the basis of the initial simplex table of GP problem is
    (a) positive deviational variable
    (b) negative deviational variable
    (c) both (a) and (b)
    (d) artificial variable
34. If the largest value of each goal in the 'solution-value, x_B' column is zero, then it indicates
    (a) multiple solution (b) infeasible solution
    (c) optimal solution (d) none of the above
35. In GP problem, a goal constraint having over achievement variable is expressed as a
    (a) $\geq$ constraint (b) $\leq$ constraint
    (c) = constraint (d) all of the above

Answers to Quiz

1. T 2. T 3. T 4. T 5. F 6. F 7. T 8. F 9. T 10. F
11. Goal Programming 12. preemptive priority factors 13. One dimension 14. Goal; dimension
15. Ballistic missile defense 16. constraints; deviational; real 17. optimal; zero
18. prioritized 19. priority; commensurable 20. zero 21. (d), 22. (a), 23. (c), 24. (a), 25. (c),
26. (d), 27. (a), 28. (c), 29. (a), 30. (c), 31. (c), 32. (a), 33. (a), 34. (c), 35. (c)

CASE STUDY

Case 8.1 : Blended Gasoline Company

The Blended Gasoline Company purchase blended gasoline from a network of three vendors. This brand of gasoline is composed of one or more of three blending constituents, each with a different octane rating. There is a different vendor for each blending constituent. The blended gasoline is characterized by its overall octane blend. The vendors have the following characteristics

Vendor	*Unit Price (Rs/barrel)*	*Octane Rating*	*Lead Time (days)*	*Service Level (%)*	*Ratio Limit (barrels)*
1	680	102	27	94	5,000
2	560	99	28	93	6,000
3	720	110	29	96	12,000

Service is measured in terms of buyer satisfaction and ranges from zero to one, with zero being satisfactory and one being the most satisfactory. The ration limit of a vendor represents the maximum purchase that a buyer can make from that vendor. The limit is imposed by the government and the vendor must abide by it.

The company wishes to purchase 10,000 barrels of blended gasoline, while meeting the following goals, listed in order of priority to the company:

(i) *Quality goal* : The aggregate octane level of the blended gasoline purchased should be at least 100.
(ii) *Lead time goal* : All the gasoline purchased should arrive within 28 days.
(iii) *Service goal* : The service level of each vendor supplying gasoline should be at least 0.95
(iv) *Price goal* : The aggregate cost of the gasoline purchased should be no more than Rs 600 per barred on an average.

Suggest an appropriate quantitative model to help the management of the company to trade-off conflicting goals for the replenishment of gasoline.

Case 8.2 : Paper Manufacturing Company

Green Wood Paper Manufacturing Company is in the business of manufacturing different types of paper used in the printing of newspapers and magazines. Hardwood and bamboo chips are the inputs required to produce paper. These chips mixed in certain proportion are cooked in a kamyr digester with the help of chemicals to produce pulp, the intermediary to paper. This pulp is further processed to finished paper. There are three quality characteristics of pulp which are to be maintained in order to get right quality of paper. These are:

(i) K-number
(ii) Burst factor, and
(iii) Breaking length

K-number is an indirect index of digestion or cooking that has been done by the chemical called *cooking liquor*, at the kamyr digestor. If the chips fed into the digestor are overcooked, resulting in excess disintegration of chips, the K-number will accordingly be low. This disintegration of raw material is a loss to the company. On the other hand, if chips are undercooked, indicated by a high K-number, then at a later stage consumption of the bleaching agent will be high. Thus, K-number is the most important characteristic that should be strictly maintained within limits. The other two characteristics determine the strength of finished paper.

The proportion of hardwood and bamboo can easily be adjusted. All the process variables are controllable and can be measured directly. The details of the input, process variables, and output are given as follows:

*Specification/Permissible Limits**	
Input: (x_1) Hardwood (%)	20 – 40
Process variables	
R_1 : Upper cooking zone temperature (°C)	140 – 175
R_2 : Lower cooking zone temperature (°C)	140 – 173
R_3 : LP steam pressure (kg/cm^2)	2.0 – 4.4
R_4 : HP steam pressure (kg/cm^2)	8.0 – 20.5
R_5 : Active alkali as NaOH (%)	20 – 35
R_6 : Sulphidity of white liquor (%)	13 – 25
R_7 : Alkaldi index (no.)	12.5 – 18.7
Output characteristics	
Y_1 : K-number	16 – 18
Y_2 : Burst factor	close to 35
Y_3 : Breaking length	close to 5,000 m

*In some cases specification did not exist and hence permissible limits were considered instead.

The company was unable to maintain the desired characteristics of pulp. The problem was to fix the levels of the input and the process variables so that the specification was met.

Pre-emptive priority factors The K-number is the most important characteristic to be fulfilled and gets the top priority. Priorities for others were fixed by the management after giving due consideration to the quality aspect as well as the ease of adjusting/modifying the levels of these variables.

The company hired the services of a consultant. A follow-up study was undertaken by the consultants, which linked the input with output via the process variable. Exactly 46 sets of such data were collected over a period of 13 days. Multiple linear regression analysis was undertaken and the following relationships were obtained.

$$Y_1 = 22.840 + 0.06x_1 - 0.05R_1 + 0.004R_2 - 0.67R_3 + 0.24R_4 - 0.13R_5 + 0.19R_6 - 0.18R_7$$

(Multiple correlation coefficient = 0.74)

$$Y_2 = 38.94 + 0.05x_1 - 0.02R_1 + 0.002R_2 + 1.67R_3 + 0.21R_4 + 0.06R_5 + 0.02R_6 - 0.69R_7$$

(Multiple correlation coefficient = 0.72)

$$Y_3 = 3272.40 - 24.37x_1 + 9.997R_1 + 8.48R_2 - 268.68R_3 + 120.92R_4 + 67.27R_5 + 27.89R_6 - 138.46R_7$$

(Multiple correlation coefficient = 0.66)

It should be noted that the relationship of these output characteristics with the variables are conflicting in the sense that for increase (decrease) in the level of some of the variables, the values of some of these characteristics increase (decrease) whereas the values of other decrease (increase).

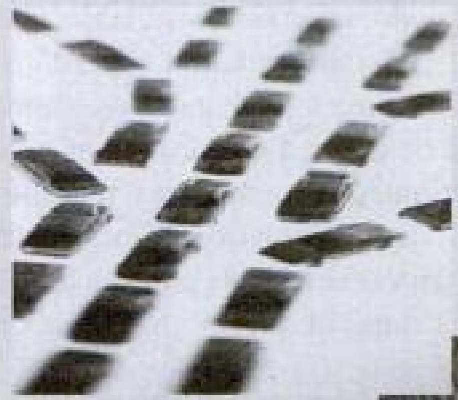
Suppose, your services are hired by the company, then suggest optimum values of input and process variables while satisfying conflicting goals.

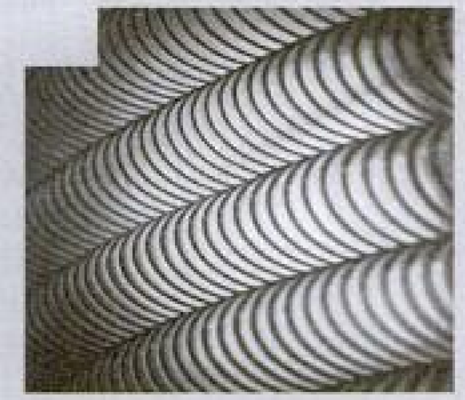
chapter 9

Transportation Problem

"We want these assets to be productive. We buy them. We own them. To say we care only about the short term is wrong. What I care about is seeing these assets in the best hands."

– Carl Icahn





Preview The structure of transportation problem involves a large number of shipping routes from several supply origins to several demand destinations. The objective is to determine the number of units of an item (commodity or product) that should be shipped from an origin to a destination in order to satisfy the required quantity of goods or services at each destination centre.

Learning Objectives After studying this chapter, you should be able to

- recognize and formulate a transportation problem involving a large number of shipping routes.
- drive initial feasible solution using several methods.
- drive optimal solution by using Modified Distribution Method.
- handle the problem of degenerate and unbalanced transportation problem.
- examine multiple optimal solutions, and prohibited routes in the transportation problem.
- construct the initial transportation table for a trans-shipment problem.
- solve a profit maximization transportation problem using suitable changes in the transportation algorithm.

Chapter Outline

9.1 INTRODUCTION

One important application of linear programming is in the area of physical distribution (transportation) of goods and services from several supply centres to several demand centres. It is easy to mathematically express a transportation problem in terms of an LP model, which can be solved by the simplex method. But because it involves a large number of variables and constraints, it takes a long time to solve it. However, transportation algorithms, namely the *Stepping Stone Method* and the *MODI* (modified distribution) *Method,* have been developed for this purpose.

The structure of transportation problem involves a large number of shipping routes from several supply origins to several demand destinations. The objective is to determine the number of units of an item (commodity or product) that should be shipped from an origin to a destination in order to satisfy the required quantity of goods or services at each destination centre. This should be done within the limited quantity of goods or services available at each supply centre, at the minimum transportation cost and/or time.

The study of **transportation problem** helps to identify optimal transportation routes along with units of commodity to be shipped in order to minimize total transportation cost

The transportation algorithm discussed in this chapter is applied to minimize the total cost of transporting a homogeneous commodity (product) from supply centres to demand centres. However, it can also be applied to the maximization of some total value or utility. For example, financial resources are distributed in such a way that the profitable return is maximized.

There are various types of transportation models and the simplest of them was first presented by F L Hitchcock (1941). It was further developed by T C Koopmans (1949) and G B Dantzig (1951). Several extensions of transportation models and methods have been subsequently developed.

9.2 MATHEMATICAL MODEL OF TRANSPORTATION PROBLEM

Let us consider Example 9.1 to illustrate the mathematical model formulation of transportation problem of transporting a single commodity from three sources of supply to four demand destinations. The *sources of supply* are production facilities, warehouses, or supply points, characterized by available capacities. The *destinations* are consumption facilities, warehouses or demand points, characterized by required levels of demand.

Example 9.1 A company has three production facilities S_1, S_2 and S_3 with production capacity of 7, 9 and 18 units (in 100s) per week of a product, respectively. These units are to be shipped to four warehouses D_1, D_2, D_3 and D_4 with requirement of 5, 6, 7 and 14 units (in 100s) per week, respectively. The transportation costs (in rupees) per unit between factories to warehouses are given in the table below:

	D_1	D_2	D_3	D_4	*Capacity*
S_1	19	30	50	10	7
S_2	70	30	40	60	9
S_3	40	8	70	20	18
Demand	5	8	7	14	34

Formulate this transportation problem as an LP model to minimize the total transportation cost.

Model formulation Let x_{ij} = number of units of the product to be transported from factory i (i = 1, 2, 3) to warehouse j (j = 1, 2, 3, 4)

The transportation problem is stated as an LP model as follows:

$$\text{Minimize (total transportation cost) } Z = 19x_{11} + 30x_{12} + 50x_{13} + 10x_{14} + 70x_{21} + 30x_{22} + 40x_{23} + 60x_{24} + 40x_{31} + 8x_{32} + 70x_{33} + 20x_{34}$$

subject to the constraints

$$\left.\begin{aligned} x_{11} + x_{12} + x_{13} + x_{14} &= 7 \\ x_{21} + x_{22} + x_{23} + x_{24} &= 9 \\ x_{31} + x_{32} + x_{33} + x_{34} &= 18 \end{aligned}\right\} \text{(Capacity available)}$$

In this problem, there are $(m + n)$ constraints, one for each source of supply, and distinction and $m \times n$ variables. Since all $(m + n)$ constraints are equations, and since the transportation model is always balanced (total supply = total demand), one of these equations is extra (redundant). The extra constraint equation can be derived from the other constraint equations, without affecting the feasible solution. It follows that any feasible solution for a transportation problem must have exactly $(m + n - 1)$ non-negative basic variables (or allocations) x_{ij} satisfying the rim conditions.

Remarks 1. When the total supply's equal to the total demand, the problem is called a *balanced transportation problem,* otherwise it is called an *unbalanced transportation problem*. The unbalanced transportation problem can be made balanced by adding a dummy supply centre (row) or a dummy demand centre (column) as the need arises.

When total demand equals total supply, the transportation problem is said to be balanced

2. When the number of positive allocations (values of decision variables) at any stage of the feasible solution is less than the required number (rows + columns – 1), i.e. number of independent constraint equations, the solution is said to be *degenerate,* otherwise *non-degenerate*. For proof, see Appendix at the end of this chapter.
3. Cells in the transportation table that have positive allocation are called *occupied cells,* otherwise they are known as *empty* or *non-occupied cells.*

9.3 THE TRANSPORTATION ALGORITHM

The algorithm for solving to a transportation problem may be summarized into the following steps:

Step 1: Formulate the problem and arrange the data in the matrix form The formulation of the transportation problem is similar to the LP problem formulation. Here the objective function is the total transportation cost and the constraints are the supply and demand available at each source and destination, respectively.

Step 2: Obtain an initial basic feasible solution In this chapter, following three different methods are discussed to obtain an initial solution:

- *North-West Corner Method,*
- *Least Cost Method,* and
- *Vogel's Approximation (or Penalty) Method.*

The initial solution obtained by any of the three methods must satisfy the following conditions:

(i) The solution must be feasible, i.e. it must satisfy all the supply and demand constraints (also called *rim conditions*).
(ii) The number of positive allocations must be equal to $m + n - 1$, when m is the number of rows and n is the number of columns.

Any solution that satisfies the above conditions is called *non-degenerate basic feasible solution,* otherwise, *degenerate solution.*

Step 3: Test the initial solution for optimality In this chapter, the Modified Distribution (MODI) method is discussed to test the optimality of the solution obtained in Step 2. If the current solution is optimal, then stop. Otherwise, determine a new improved solution.

Step 4: Updating the solution Repeat Step 3 until an optimal solution is reached.

9.4 METHODS OF FINDING INITIAL SOLUTION

There are several methods available to obtain an initial basic feasible solution. Here we shall discuss only three different methods:

9.4.1 North-West Corner Method (NWCM)

It is a simple and efficient method to obtain an initial solution. This method does not take into account the cost of transportation on any route of transportation. The method can be summarized as follows:

Step 1: Start with the cell at the upper left (north-west) corner of the transportation matrix and allocate commodity equal to the minimum of the rim values for the first row and first column, i.e. min (a_1, b_1).

Step 2: (a) If allocation made in Step 1 is equal to the supply available at first source (a_1, in first row), then move vertically down to the cell (2, 1) in the second row and first column. Apply Step 1 again, for next allocation.

(b) If allocation made in Step 1 is equal to the demand of the first destination (b_1 in first column), then move horizontally to the cell (1, 2) in the first row and second column. Apply Step 1 again for next allocation.

(c) If $a_1 = b_1$, allocate $x_{11} = a_1$ or b_1 and move diagonally to the cell (2, 2).

Step 3: Continue the procedure step by step till an allocation is made in the south-east corner cell of the transportation table.

Remark If during the process of allocating commodity at a particular cell, the supply equals demand, then the next allocation of magnitude zero can be made in a cell either in the next row or column. This condition is known as degeneracy.

Example 9.2 Use North-West Corner Method (NWCM) to Example 9.1 to find an initial basic feasible solution to the transportation problem.

Solution The cell (S_1, D_1) is the north-west corner cell in the given transportation table. The rim values for row S_1 and column D_1 are compared. The smaller of the two, i.e. 5, is assigned as the first allocation; otherwise it will violate the feasibility condition. This means that 5 units of a commodity are to be transported from source S_1 to destination D_1. However, this allocation leaves a supply of 7 – 5 = 2 units of commodity at S_1.

Move horizontally and allocate as much as possible to cell (S_1, D_2). The rim value for row S_1 is 2 and for column D_2 is 8. The smaller of the two, i.e. 2, is placed in the cell. Proceeding to row S_2, since the demand of D_1 has been met, nothing further can be allocated to D_1. The unfulfilled demand of D_2 is now 8 – 2 = 6 units. This can be fulfilled by S_2 with capacity of 9 units. So 6 units are allocated to cell (S_2, D_2). The demand of D_2 is now satisfied and a balance of 9 – 6 = 3 units remains with S_2.

	D_1	D_2	D_3	D_4	*Supply*
S_1	19 (5)	30 (2)	50	10	7
S_2	70	30 (6)	40 (3)	60	9
S_3	40	8	70 (4)	20 (14)	18
Demand	5	8	7	14	34

Table 9.2 Initial Solution using NWCM

We now move horizontally and vertically in the same manner. This should be done because successive demand and supply are met. Ensure that the solution is feasible, that is, the number of positive allocations (occupied cells) is equal to $m + n - 1 = 3 + 4 - 1 = 6$.

The total transportation cost of the initial solution derived by the NWCM is obtained by multiplying the quantity x_{ij} in the occupied cells with the corresponding unit cost c_{ij} and adding all the values together. Thus, the total transportation cost of this solution is

Total cost = 5 × 19 + 2 × 30 + 6 × 30 + 3 × 40 + 4 × 70 + 14 × 20 = Rs 1,015

9.4.2 Least Cost Method (LCM)

Since the main objective is to minimize the total transportation cost, we must try to transport as much as possible through those routes (cells) where the unit transportation cost is lowest. This method takes into account the minimum unit cost of transportation for obtaining the initial solution and can be summarized as follows:

Step 1: Select the cell with the lowest unit cost in the entire transportation table and allocate as much as possible to this cell. Then eliminate (line out) that row or column in which either the supply or demand is exhausted. If a row and a column are both satisfied simultaneously, then only one may be crossed out.

In case the smallest unit cost cell is not unique, then select the cell where the maximum allocation can be made.

Step 2: After adjusting the supply and demand for all uncrossed-out rows and columns repeat the procedure with the next lowest unit cost among the remaining rows and columns of the transportation table and allocate as much as possible to this cell. Then eliminate (line out) that row and column in which either supply or demand is exhausted.

Step 3: Repeat the procedure until the entire available supply at various sources and demand at various destinations is satisfied. The solution so obtained need not be non-degenerate.

Example 9.3 Use Least Cost Method (LCM) to Example 9.1 in order to find initial basic feasible solution to the transportation problem.

Solution The cell with lowest unit cost (i.e., 8) is (S_3, D_2). The maximum units which we can allocate to this cell is 8. This meets the complete demand of D_2 and leave 10 units with S_3, as shown in Table 9.3.

In the reduced table without column D_2, the next smallest unit transportation cost, is 10 in cell (S_1, D_4). The maximum which can be allocated to this cell is 7. This exhausts the capacity of S_1 and leaves 7 units with D_4 as unsatisfied demand. This is shown in Table 9.3.

Table 9.3

	D_1	D_2	D_3	D_4	Supply
S_1	19	30	50	10 (7)	7
S_2	70	30	40	60	9
S_3	40	8 (8)	70	20	18
Demand	5	8	7	14	34

In Table 9.3, the next smallest cost is 20 in cell (S_3, D_4). The maximum that can be allocated to this cell is 7 units. This satisfies the entire demand of D_4 and leaves 3 units with S_3, as the remaining supply, shown in Table 9.4.

In Table 9.4, the next smallest unit cost cell is not unique. That is, there are two cells – (S_2, D_3) and (S_3, D_1) – that have the same unit transportation cost of 40. Allocate 7 units in cell (S_2, D_3) first because it can accommodate more units as compared to cell (S_3, D_1). Then allocate 3 units (only supply left with S_3) to cell (S_3, D_1). The remaining demand of 2 units of D_1 is fulfilled from S_2. Since supply and demand at each origin and destination is exhausted, the initial solution is arrived at, and is shown in Table 9.4.

Table 9.4

	D_1	D_2	D_3	D_4	Supply
S_1	19	30	50	10 (7)	7
S_2	70 (2)	30	40 (7)	60	9
S_3	40 (3)	8 (8)	70	20 (7)	18
Demand	5	8	7	14	34

The total transportation cost of the initial solution by LCM is calculated as given below:

$$\text{Total cost} = 7 \times 10 + 2 \times 70 + 7 \times 40 + 3 \times 40 + 8 \times 8 + 7 \times 20 = \text{Rs } 814$$

The total transportation cost obtained by LCM is less than the cost obtained by NWCM. The optimal solution can also be obtained faster by using LCM.

9.4.3 Vogel's Approximation Method (VAM)

Vogel's approximation (penalty or regret) method is a heuristic method and is preferred more than the other two methods described above. In this method, each allocation is made on the basis of the opportunity (or penalty or extra) cost that would have been incurred if the allocations in certain cells with minimum unit transportation cost were missed. In this method allocations are made so that the penalty cost is minimized. The advantage of this method is that it gives an initial solution which is nearer to an optimal solution or is the optimal solution itself. The steps of VAM are as follows:

Step 1: Calculate the penalties for each row (column) by taking the difference between the smallest and next smallest unit transportation cost in the same row (column). This difference indicates the penalty or extra cost that has to be paid if one fails to allocate to the cell with the minimum unit transportation cost.

Step 2: Select the row or column with the largest penalty and allocate as much as possible in the cell that has the least cost in the selected row or column and satisfies the rim conditions. If there is a tie in the values of penalties, it can be broken by selecting the cell where the maximum allocation can be made.

Step 3: Adjust the supply and demand and cross out the satisfied row or column. If a row and a column are satisfied simultaneously, only one of them is crossed out and the remaining row (column) is assigned a zero supply (demand). Any row or column with zero supply or demand should not be used in computing future penalties.

Step 4: Repeat Steps 1 to 3 until the entire available supply at various sources and demand at various destinations are satisfied.

Example 9.4 Use Vagel's Approximation Method (VAM) to Example 9.1 in order to find the initial basic feasible solution to the transportation problem.

Solution The differences (penalty costs) for each row and column have been calculated as shown in Table 9.5. In the first round, the maximum penalty, 22 occurs in column D_2. Thus the cell (S_3, D_2) having the least transportation cost 8 is chosen for allocation. The maximum possible allocation in this cell is 8 and it satisfies demand in column D_2. Adjust the supply of S_3 from 18 to 10 (18 – 8 = 10).

	D_1	D_2	D_3	D_4	*Supply*	*Row differences*
S_1	19 (5)	30	50	10 (2)	7	9 9 40 40
S_2	70	30	40 (7)	60 (2)	9	10 20 20 20
S_3	40	8 (8)	70	20 (10)	18	12 20 50 –
Demand	5	8	7	14	34	
Column differences	21	22	10	10		
	21	–	10	10		
	–	–	10	10		
	–	–	10	50		

Table 9.5 Initial Solution Using VAM

The new row and column penalties are calculated except column D_2 because D_2,s demand has been satisfied. The second round allocation is made in column D_1 with target penalty 21 in the same way as in the first round as shown in cell (S_1, D_1) of Table 9.5.

In the third round, the maximum penalty 50 occurs at S_3. The maximum possible allocation of 10 units is made in cell (S_3, D_4) that has the least transportation cost of 20, as shown in Table 9.5.

The process is continued with new allocations till a complete solution is obtained. The initial solution using VAM is shown in Table 9.5. The total transportation cost associated with this method is calculated as:

$$\text{Total cost} = 5 \times 19 + 2 \times 10 + 7 \times 40 + 2 \times 60 + 8 \times 8 + 10 \times 20 = \text{Rs } 779$$

Example 9.5 A dairy firm has three plants located in a state. The daily milk production at each plant is as follows:

Plant 1 : 6 million litres, Plant 2 : 1 million litres, and Plant 3 : 10 million litres

Each day, the firm must fulfil the needs of its four distribution centres. The minimum requirement of each centre is as follows:

Distribution centre 1 : 7 million litres, Distribution centre 2 : 5 million litres,
Distribution centre 3 : 3 million litres, and Distribution centre 4 : 2 million litres

Cost (in hundreds of rupees) of shipping one million litre from each plant to each distribution centre is given in the following table:

		Distribution Centre			
		D_1	D_2	D_3	D_4
Plant	P_1	2	3	11	7
	P_2	1	0	6	1
	P_3	5	8	15	9

Find the initial basic feasible solution for given problem by using

(a) North-west corner rule
(b) Least cost method
(c) Vogel's approximation method

if the objective is to minimize the total transportation cost.

Solution (a) ***North-West Corner Rule***

Table 9.6

		Distribution Centre				
		D_1	D_2	D_3	D_4	*Supply*
	P_1	2 (6)	3	11	7	$6 = a_1$
Plant	P_2	1 (1)	0	6	1	$1 = a_2$
	P_3	5	8 (5)	15 (3)	9 (2)	$10 = a_3$
Demand		$7 = b_1$	$5 = b_2$	$3 = b_3$	$2 = b_4$	

(i) Comparing a_1 and b_1, since $a_1 < b_1$; allocate $x_{11} = 6$. This exhausts the supply at P_1 and leaves 1 unit as unsatisfied demand at D_1.

(ii) Move to cell (P_2, D_1). Compare a_2 and b_1 (i.e. 1 and 1). Since $a_2 = b_1$, allocate $x_{21} = 1$.

(iii) Move to cell (P_3, D_2). Since supply at P_3, is equal to the demand at D_2, D_3 and D_4, therefore, allocate $x_{32} = 5$, $x_{33} = 3$ and $x_{34} = 2$.

It may be noted that the number of allocated cells (also called *basic cells*) are 5 which is one less than the required number $m + n - 1$ $(3 + 4 - 1 = 6)$. Thus, this solution is the degenerate solution. The transportation cost associated with this solution is:

Total cost = Rs $(2 \times 6 + 1 \times 1 + 8 \times 5 + 15 \times 3 + 9 \times 2) \times 100$ = Rs 11,600

(b) ***Least Cost Method***

Table 9.7

		Distribution Centre				
		D_1	D_2	D_3	D_4	*Supply*
	P_1	2 (6)	3	11	7	6
Plant	P_2	1	0 (1)	6	1	1
	P_3	5 (1)	8 (4)	15 (3)	9 (2)	10
Demand		7	5	3	2	

(i) The lowest unit cost in Table 9.7 is 0 in cell (P_2, D_2), therefore the maximum possible allocation that which can be made here is 1. This exhausts the supply at plant P_2, therefore, row 2 is crossed out.
(ii) The next lowest unit cost is 2 in cell (P_1, D_1). The maximum possible allocation that can be made here is 6. This exhausts the supply at plant P_1, therefore, row P_1 is crossed out.
(iii) Since the total supply at plant P_3 is now equal to the unsatisfied demand at all the four distribution centres, therefore, the maximum possible allocations satisfying the supply and demand conditions, are made in cells (P_3, D_1), (P_3, D_2), (P_3, D_3) and (P_3, D_4).

The number of allocated cells in this case are six, which is equal to the required number $m + n - 1$ $(3 + 4 - 1 = 6)$. Thus, this solution is non-degenerate. The transportation cost associated with this solution is

Total cost = Rs $(2 \times 6 + 5 \times 1 + 8 \times 4 + 15 \times 3 + 9 \times 2) \times 100$ = Rs 11,200

(c) ***Vogel's Approximation Method:*** The method is self-explanatory as shown in Table 9.8.

Table 9.8

		D_1	D_2	D_3	D_4	*Supply*	*Row penalty*
	P_1	2 (1)	3 (5)	11	7	6	1 1 5
Plant	P_2	1	0	6	1 (1)	1	0 – –
	P_3	5 (6)	8	15 (3)	9 (1)	10	3 3 4
Demand		7	5	3	2		
Column penalty		1	3	5	6		
		3	5	4	2		
		3	–	4	2		
		Distribution Centre					

The number of allocated cells in Table 9.8 are six, which is equal to the required number $m + n - 1$ $(3 + 4 - 1 = 6)$, therefore, this solution is non-degenerate. The transportation cost associated with this solution is

Total cost = Rs $(2 \times 1 + 3 \times 5 + 1 \times 1 + 5 \times 6 + 15 \times 3 + 9 \times 1) \times 100$ = Rs 10,200

It can be seen that the total transportation cost found by VAM is lower than the costs of transportation determined by the other two methods. Therefore, it is of advantage to use this method in order to reduce computational time required to obtain optimum solution.

CONCEPTUAL QUESTIONS A

1. Show that all the bases for a transportation problem are triangular.
2. With reference to a transportation problem define the following terms:
   (i) Feasible solution (ii) Basic feasible solution
   (iii) Optimal solution
   (iv) Non-degenerate basic feasible solution
3. Given a mathematical formulation of the transportation problem and the simplex methods, what are the differences in the nature of problems that can be solved by using these methods?
4. Prove that there are only $m + n - 1$ independent equations in a transportation problem, m and n being the number of origins and destination, and only one equation can be dropped as being redundant. (For proof see Appendix).
5. Describe the transportation problem with its general mathematical formulation.
6. Show that a transportation problem is a special type of LP problem. In what areas of management can the transportation model be effectively used? Discuss.
7. What are the characteristics of transportation problem of linear programming?
8. What is meant by the triangular form of a system of linear equations? When does a system of linear equations have a triangular basis? (See Appendix for proof.)
9. What is meant by non-degenerate basic feasible solution of a transportation problem?
10. Explain in brief three, methods of initial feasible solution for transportation problem.
11. Explain the various steps involved in solving transportation problem using (i) Least cost method, and (ii) Vogel's approximation method.
12. Explain the (i) North-West Corner method, (ii) Least-Cost method, and (iii) Vogel's Approximation method, for obtaining an initial basic feasible solution of a transportation problem.
13. State the transportation problem. Describe clearly the steps involved in solving it.
14. Is the transportation model an example of decision-making under certainty or under uncertainty? Why?
15. Why does Vogel's approximation method provide a good initial feasible solution? Can the North-West Corner method ever be able to provide an initial solution with a cost as low as this?

SELF PRACTICE PROBLEMS A

1. Determine an initial basic feasible solution to the following transportation problem by using (a) NWCR, (b) LCM and (c) VAM.

Destination

		D_1	D_2	D_3	D_4	*Supply*
	S_1	21	16	15	3	11
Source	S_2	17	18	14	23	13
	S_3	32	27	18	41	19
	Demand	6	10	12	15	

2. Determine an initial basic feasible solution to the following transportation problem by using (a) the least cost method, and (b) Vogel's approximation method.

Destination

		D_1	D_2	D_3	D_4	*Supply*
	S_1	1	2	1	4	30
Source	S_2	3	3	2	1	50
	S_3	4	2	5	9	20
	Demand	20	40	30	10	

3. Determine an initial basic feasible solution to the following transportation problem by using (a) NWCM, (b) LCM, and (c) VAM.

Destination

		D_1	D_2	D_3	D_4	*Supply*
	A	11	13	17	14	250
Source	*B*	16	18	14	10	300
	C	21	24	13	10	400
	Demand	200	225	275	250	

4. Determine an initial basic feasible solution to the following transportation problem by using the North-West corner rule, where O_i and D_j represent ith origin and jth destination, respectively.

Destination

		D_1	D_2	D_3	D_4	*Supply*
	O_1	6	4	1	5	14
Source	O_2	8	9	2	7	16
	O_3	4	3	6	2	5
	Demand	6	10	15	4	

HINTS AND ANSWERS

1. $x_{14} = 11$, $x_{21} = 6$, $x_{22} = 3$, $x_{24} = 4$, $x_{32} = 3$, $x_{33} = 4$, $x_{43} = 14$; Total cost = 686.
2. (a) and (b): $x_{11} = 20$, $x_{13} = 10$, $x_{22} = 20$, $x_{33} = 20$, $x_{24} = 10$, $x_{32} = 20$; Total cost = 180.
3. (a) $x_{11} = 200$, $x_{12} = 50$, $x_{22} = 175$, $x_{23} = 125$, $x_{33} = 150$, $x_{34} = 250$; Total cost = 12,200.
   (b) $x_{11} = 200$, $x_{12} = 50$, $x_{22} = 175$, $x_{23} = 125$, $x_{33} = 150$, $x_{34} = 250$; Total cost = 12,200.
   (c) $x_{11} = 200$, $x_{12} = 50$, $x_{22} = 175$, $x_{24} = 125$, $x_{33} = 275$, $x_{34} = 125$; Total cost = 12,075.
4. $x_{11} = 6$; $x_{12} = 8$; $x_{22} = 2$; $x_{23} = 14$; $x_{33} = 1$; $x_{34} = 4$; Total cost = Rs 128.

9.5 TEST FOR OPTIMALITY

Once an initial solution is obtained, the next step is to check for its optimality. An optimal solution is one where there is no other set of transportation routes (allocations) that will further reduce the total transportation cost. Thus, we have to evaluate each unoccupied cell (represents unused route) in the transportation table in terms of an opportunity of reducing total transportation cost.

An unoccupied cell with the largest negative opportunity cost is selected to include in the new set of transportation routes (allocations). *This value indicates the per unit cost reduction that can be achieved by raising the shipment allocation in the unoccupied cell from its present level of zero.* This is also known as an *incoming cell (or variable)*. The outgoing cell (or variable) in the current solution is the occupied cell (basic variable) in the unique closed path (loop) whose allocation will first become zero as more units are allocated to the unoccupied cell with the largest negative opportunity cost. Such an exchange reduces the total transportation cost. The process is continued until there is no negative opportunity cost. That is, the current solution cannot be improved further. This would be the optimal solution.

The negative opportunity cost indicates the per unit cost reduction that can be achieved by raising the shipment allocation in the unoccupied cell from its present level of zero

An efficient technique called the *modified-distribution* (MODI) method (also called *u-v method* or *method of multipliers*), which helps in comparing the relative advantage of alternative allocations for all the unoccupied cells simultaneously, is discussed below. The MODI method is based on the concept of duality.

9.5.1 Dual of Transportation Model

For a given basic feasible solution if we associate numbers (also called *dual variables* or *multipliers*) u_i and v_j with row i ($i = 1, 2, \ldots, m$) and column j ($j = 1, 2, \ldots, n$) of the transportation table, respectively, then u_i and v_j must satisfy the equation

$$u_i + v_j = c_{ij}, \quad \text{for each occupied cell } (i, j)$$

These equations yield $m + n - 1$ equations in $m + n$ unknown dual variables. The values of these variables can be determined by arbitrarily assigning a zero value to any one of these variables. The value of the remaining $m + n - 2$ variables can then be obtained algebraically by using the above relationship for occupied the cells. Once the values of u_i and v_j have been determined, evaluation in terms of opportunity cost of each unoccupied cell (called *non-basic variable* or *unused route*) is done by using the equation:

$$d_{rs} = c_{rs} - (u_r + v_s), \quad \text{for each unoccupied cell } (r, s)$$

For proving these two results, let us consider the general transportation model using sigma notation.

$$\text{Minimize } Z = \sum_{i=1}^{m} \sum_{j=1}^{n} c_{ij}\, x_{ij}$$

subject to the constraints

$$\sum_{j=1}^{n} x_{ij} = a_i, \quad i = 1, 2, \ldots, m \quad \text{(Supply)}$$

$$\sum_{i=1}^{m} x_{ij} = b_j, \quad j = 1, 2, \ldots, n \quad \text{(Demand)}$$

and $\quad x_{ij} \geq 0 \quad$ for all i and j

Modi method helps in comparing the relative advantage of alternative allocations for all the unoccupied cells simultaneously

Since all of the constraints are equalities, write each equality constraint equivalent to two inequalities as follows:

$$\left.\begin{aligned} &\sum_{j=1}^{n} x_{ij} \geq a_i, \quad i = 1, 2, \ldots, m \\ &\sum_{j=1}^{n} - (x_{ij}) \geq - a_i \end{aligned}\right\} \text{(Supply constraints)}$$

$$\left.\begin{aligned} &\sum_{i=1}^{m} x_{ij} \geq b_j, \quad j = 1, 2, \ldots, m \\ &\sum_{i=1}^{m} - (x_{ij}) \geq - b_j \end{aligned}\right\} \text{(Demand constraints)}$$

Let u_i^+ and u_i^- be the dual variables, one for each supply constraint i. Similarly v_j^+, v_j^- be the dual variables one for each demand constraint j. Now the dual of the transportation model is given by

$$\text{Maximize } Z^* = \sum_{i=1}^{m} (u_i^+ - u_i^-) a_i + \sum_{j=1}^{n} (v_j^+ - v_j^-) b_j$$

subject to the constraints

$$(u_i^+ - u_i^-) + (v_j^+ - v_j^-) \le c_{ij}$$

and $\quad u_i^+, u_i^-, v_j^+, v_j^- \ge 0, \quad$ for all i and j.

The variables u_i^+ and u_i^- that appear in the objective function, may take positive, negative or zero values. Thus, either of these will appear in the optimal basic feasible solution because one is the negative of the other. The same argument may be given for v_j^+ and v_j^-. Thus, let

$$u_i = u_i^+ - u_i^-, \quad i = 1, 2, \ldots, m$$
$$v_j = v_j^+ - v_j^-, \quad j = 1, 2, \ldots, n$$

The values of u_i and v_j will then be unrestricted in sign. Hence, the dual of the transportation model can now be written as

$$\text{Maximize } Z^* = \sum_{i=1}^{m} u_i a_i + \sum_{j=1}^{n} v_j b_j$$

subject to the constraints

$$u_i + v_j \le c_{ij}$$

and $\quad u_i, v_j$ unrestricted in sign for all i and j.

The variables x_{ij} form an optimal solution to the given transportation problem provided

(i) solution x_{ij} is feasible for all (i, j) with respect to original transportation model.
(ii) solution u_i and v_j is feasible for all (i, j) with respect to the dual of the original transportation model.
(iii) $(c_{ij} - u_i - v_j)\, x_{ij} = 0$ for all i and j.

The relationship $(c_{ij} - u_i - v_j)\, x_{ij} = 0$ is also known as *complementary slackness* for a transportation problem and indicates that

(a) if $x_{ij} > 0$ and is feasible, then $c_{ij} - u_i - v_j = 0$ or $c_{ij} = u_i + v_j$, for each occupied cell,
(b) if $x_{ij} = 0$ and $c_{ij} > u_i + v_j$, then it is not desirable to have $x_{ij} > 0$ in the solution mix because it would cost more to transport on a route (i, j),
(c) if $c_{ij} \le u_i + v_j$ for some $x_{ij} = 0$, then x_{ij} can be brought into the solution mix.

The per unit net reduction in the total cost of transportation for a route (i, j) is given by

$$d_{ij} = c_{ij} - (u_i + v_j), \text{ for all } i \text{ and } j.$$

Here it may be noted that $d_{ij} = 0$ for occupied cells (basic variables).

9.5.2 Economic Interpretation of u_i's and v_j's

The **dual variables** u_is and v_js represent the shadow price (value of the commodity) for the supply centres and demand centres, respectively

The value of each variable u_i measures the comparative advantage of either the location or the value of a unit of capacity at the supply centre i and, therefore, may be termed as *location rent*. Similarly, the value of each variable v_j measures the comparative advantage of an additional unit of commodity transported to demand centre j and, therefore, may be termed as *market price*.

Illustration The concept of duality in transportation problem is applied on Example 9.1 in the following manner:

Reproducing transportation data of Example 9.1 for ready reference in Table 9.9. In Table 9.9, there are $m = 3$ rows and $n = 4$ columns. Let u_1, u_2 and u_3 be dual variables corresponding to each of the supply constraint in that order. Similarly, v_1, v_2, v_3 and v_4 be dual variables corresponding to each of demand constraint in that order. The dual problem then becomes

Table 9.9

	D_1	D_2	D_3	D_4	*Supply*	u_i
S_1	19	30	50	10	7	u_1
S_2	70	30	40	60	9	u_2
S_3	40	8	70	20	18	u_3
Demand	5	8	7	14	34	
v_j	v_1	v_2	v_3	v_4		

Maximize $Z = (7u_1 + 9u_2 + 18u_3) + (5v_1 + 8v_2 + 7v_3 + 14v_4)$

subject to the constraints

(i) $u_1 + v_1 \le 19$, (ii) $u_1 + v_2 \le 30$, (iii) $u_1 + v_3 \le 50$, (iv) $u_1 + v_4 \le 10$,
(v) $u_2 + v_1 \le 70$, (vi) $u_2 + v_2 \le 30$, (vii) $u_2 + v_3 \le 40$, (viii) $u_2 + v_4 \le 60$,
(ix) $u_3 + v_1 \le 40$, (x) $u_3 + v_2 \le 8$, (xi) $u_3 + v_3 \le 70$, (xii) $u_3 + v_4 \le 20$,

and u_i, v_j unrestricted in sign for all i and j.

For interpreting the data let us consider the constraint $u_1 + v_1 \le 19$ or $v_1 \le 19 - u_1$. This represents the delivered market value of the commodity at destination D_1 which should be less than or equal to the unit cost of transportation from S_1 to D_1 minus the per unit value of commodity at D_1. A similar interpretation can also be given for other constraints.

Now, the optimal values of dual variables can be obtained either by solving this LP problem or by reading values of these variables from the transportation table that contains the optimal solution. Readers may verify that the total transportation cost at optimal solution obtained by the MODI method would be the same as obtained by putting values of u_i's and v_j's from optimal transportation table in the dual objective function:

$$\text{Maximize } Z = \sum_{i=1}^{3} a_i u_i + \sum_{j=1}^{4} b_j v_j$$

9.5.3 Steps of MODI Method (Transportation Algorithm)

The steps to evaluate unoccupied cells are as follows:

Step 1: For an initial basic feasible solution with $m + n - 1$ occupied cells, calculate u_i and v_j for rows and columns. The initial solution can be obtained by any of the three methods discussed earlier.

To start with, any one of u_is or v_js is assigned the value zero. It is better to assign zero to a particular u_i or v_j where there are maximum number of allocations in a row or column respectively, as this will reduce the considerably arithmetic work. The complete the calculation of u_is and v_js for other rows and columns by using the relation

$$c_{ij} = u_i + v_j, \quad \text{for all occupied cells } (i, j).$$

Step 2: For unoccupied cells, calculate the opportunity cost (the difference that indicates the per unit cost reduction that can be achieved by an allocation in the unoccupied cell). Do this by using the relationship

$$d_{ij} = c_{ij} - (u_i + v_j), \quad \text{for all } i \text{ and } j.$$

Step 3: Examine sign of each d_{ij}

(i) If $d_{ij} > 0$, then the current basic feasible solution is optimal.
(ii) If $d_{ij} = 0$, then the current basic feasible solution will remain unaffected but an alternative solution exists.
(iii) If one or more $d_{ij} < 0$, then an improved solution can be obtained by entering unoccupied cell (i, j) in the basis. An unoccupied cell having the largest negative value of d_{ij} is chosen for entering into the solution mix (new transportation schedule).

Changing the shipping route involves adding to cells on the closed path with plus signs and subtracting from cells with negative signs

Step 4: Construct a closed-path (or loop) for the unoccupied cell with largest negative opportunity cost. Start the closed path with the selected unoccupied cell and mark a plus sign (+) in this cell. Trace a path along the rows (or columns) to an occupied cell, mark the corner with a minus sign (–) and continue down the column (or row) to an occupied cell. Then mark the corner with plus sign (+) and minus sign (–) alternatively. Close the path back to the selected unoccupied cell.

Step 5: Select the smallest quantity amongst the cells marked with minus sign on the corners of closed loop. Allocate this value to the selected unoccupied cell and add it to other occupied cells marked with plus signs. Now subtract this from the occupied cells marked with minus signs.

Step 6: Obtain a new improved solution by allocating units to the unoccupied cell according to Step 5 and calculate the new total transportation cost.

Step 7: Further test the revised solution for optimality. The procedure terminates when all $d_{ij} \ge 0$ for unoccupied cells.

Remarks 1. The closed-loop (path) starts and ends at the selected unoccupied cell. It consists of successive horizontal and vertical (connected) lines whose end points must be occupied cells, except for a end point associated with entering unoccupied cell. This means that every corner element of the loop must be an occupied cell.

It is immaterial whether the loop is traced in a clockwise or anti-clockwise direction and whether it starts up, down, right or left (but never diagonally). However, for a given solution only one loop can be constructed for each unoccupied cell.

2. There can only be one plus (+) sign and only one minus (–) sign in any given row or column.
3. The closed path indicates changes involved in reallocating the shipments.

The steps of MODI method for solving a transportation problem can also be described by the flow chart shown in Fig. 9.1.

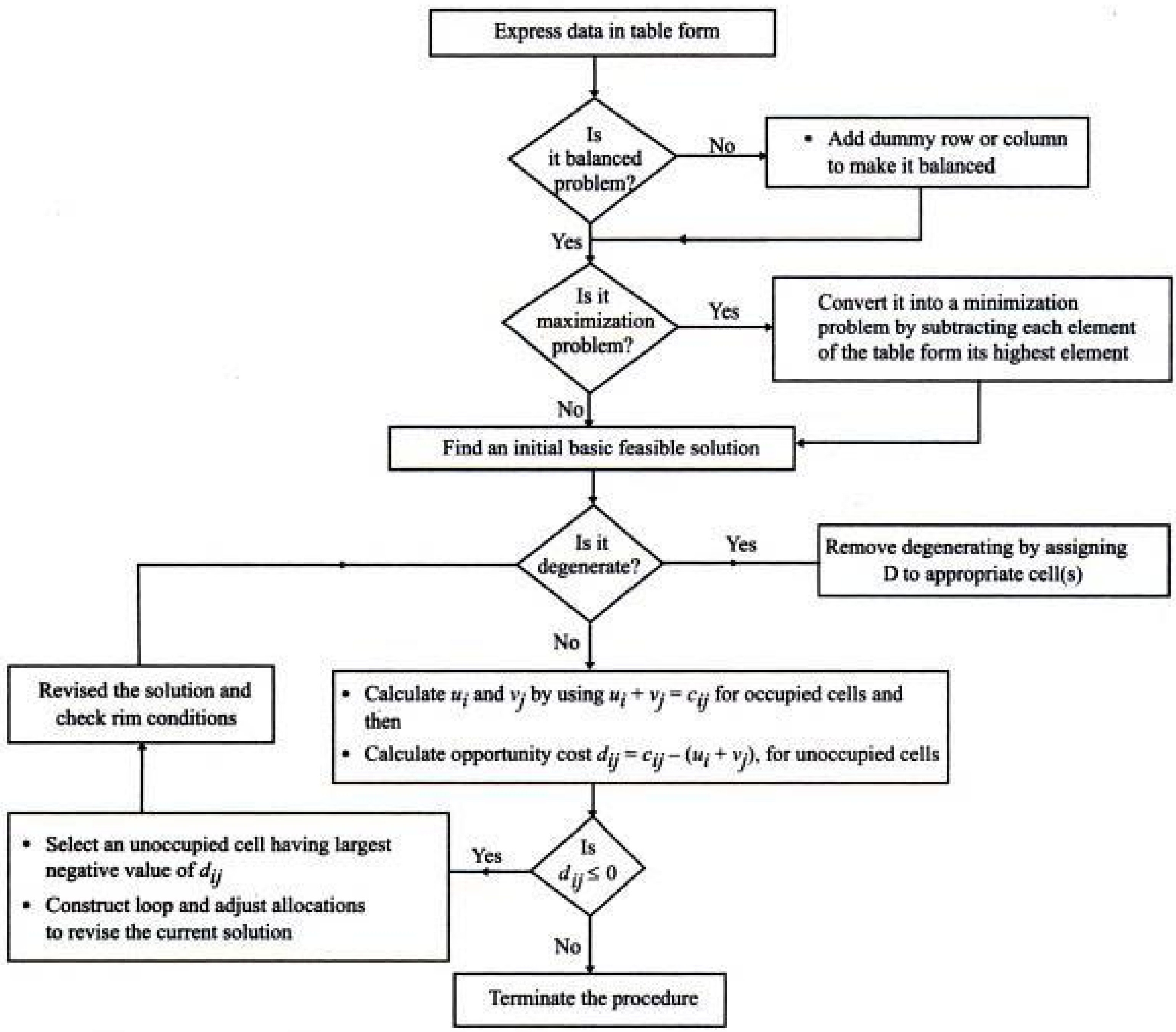


Fig. 9.1 Flow Chart of MODI Method

9.5.4 Close-loop in Transportation Table and Its Properties

An ordered set of at least four cells in a transportation table forms a **loop**

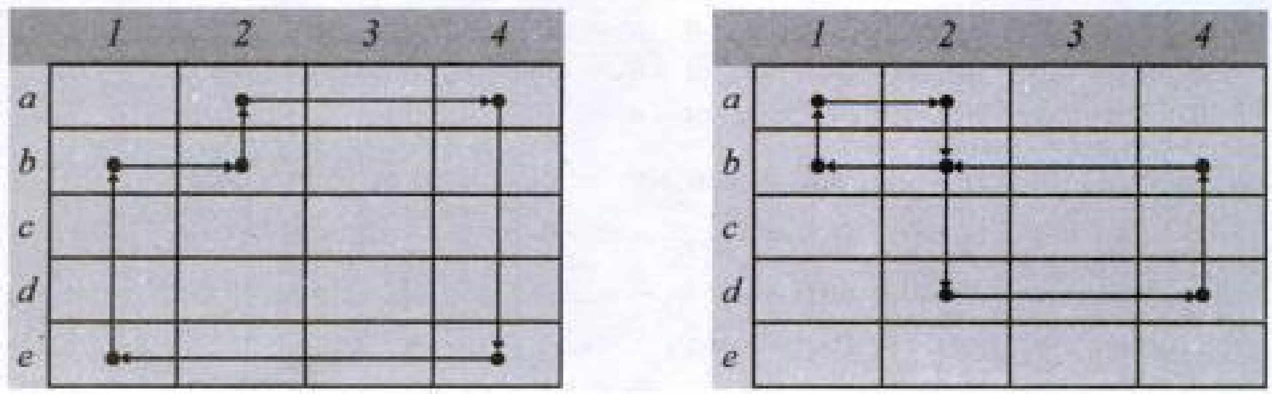
Any basic feasible solution must contain $m + n - 1$ independent non-zero allocations. The independent non-zero allocations imply that one cannot form a closed circuit (loop) only by joining the positive allocations made by horizontal and vertical lines. Mathematically, provided:

(i) any two adjacent cells of the ordered set lie either in the same row or in the same column, and
(ii) no three or more adjacent cells in the ordered set lie in the same row or column. The first cell of the set must follow the last in the set, i.e. each cell (except the last) must appear only once in the ordered set.

Consider the following two cases represented in Tables 9.10(a) and 9.10(b). In Table 9.10(a), if we join the positive allocations by horizontal and vertical lines, then a closed loop is obtained. The ordered set of cells forming a loop is:

$$L = \{(a, 2), (a, 4), (e, 4), (e, 1), (b, 1), (b, 2), (a, 2)\}$$

The loop in Table 9.10(b) is not allowed because it does not satisfy the conditions in the definition of a loop. That is, the cell $(b, 2)$ appears twice.



(a) (b)

Table 9.10

Remarks 1. Every loop has an even number of cells and has at least four cells.

2. The allocations are said to be in independent position if it is not possible to increase or decrease any independent individual allocation without changing the positions of these allocations, or if a closed loop cannot be formed through these allocations without violating the rim conditions.
3. Each row and column in the transportation table should have only one plus and minus sign. All cells that have a plus or a minus sign, except the starting unoccupied cell, must be occupied cells.
4. Closed loops may or may not be in the shape of a square.

Example 9.6 Use Example 9.1 data, to obtain an optimal solution using MODI method.

	D_1	D_2	D_3	D_4	*Supply*
S_1	19	30	50	10	7
S_2	70	30	40	60	9
S_3	40	8	70	20	18
Demand	5	8	7	14	34

Solution We apply Vogel's approximation method to obtain an initial basic feasible solution, as shown in Table 9.5. This solution is again reproduced in Table 9.11 for ready reference.

1. In Table 9.11, the number of occupied cells are $m + n - 1 = 3 + 4 - 1 = 6$, and initial solution is non-degenerate. Thus, an optimal solution can be obtained. The total transportation cost associated with this solution is 779.

2. In order to calculate the values of u_is ($i = 1, 2, 3$) and v_js ($j = 1, 2, 3, 4$) for each occupied cell, we arbitrarily assign $v_4 = 0$ in order to simplify calculations. Given $v_4 = 0$, u_1, u_2 and u_3 can be immediately computed by using the relation $c_{ij} = u_i + v_j$ for occupied cells, as shown as follows:

	D_1	D_2	D_3	D_4	*Supply*	u_i
S_1	19 (5)	30 +32	50 +60	10 (2)	7	$u_1 = 10$
S_2	70 +1	30 (+) −18	40 (7)	60 (2) (−)	9	$u_2 = 60$
S_3	40 +11	8 (−) (8)	70 +70	20 (10) (+)	18	$u_3 = 20$
Demand	5	8	7	14	34	
v_j	$v_1 = 9$	$v_2 = -12$	$v_3 = -20$	$v_4 = 0$		

Table 9.11
Initial Solution, VAM

$$c_{34} = u_3 + v_4 \quad \text{or} \quad 20 = u_3 + 0 \quad \text{or} \quad u_3 = 20$$
$$c_{24} = u_2 + v_4 \quad \text{or} \quad 60 = u_2 + 0 \quad \text{or} \quad u_2 = 60$$
$$c_{14} = u_1 + v_4 \quad \text{or} \quad 10 = u_1 + 0 \quad \text{or} \quad u_1 = 10$$

Given u_1, u_2, and u_3, value of v_1, v_2 and v_3 can also be calculated as shown below:

$$c_{11} = u_1 + v_1 \quad \text{or } 19 = 10 + v_1 \text{ or } v_1 = 9$$
$$c_{23} = u_2 + u_3 \quad \text{or } 40 = 60 + v_3 \text{ or } v_3 = -20$$
$$c_{32} = u_3 + v_2 \quad \text{or } 8 = 20 + v_2 \text{ or } v_2 = -12$$

3. The opportunity cost for each of the occupied cell is determined by using the relation $d_{ij} = c_{ij} - (u_i + v_j)$ and is shown below.

$$d_{12} = c_{12} - (u_1 + v_2) = 30 - (10 - 12) = 32$$
$$d_{13} = c_{13} - (u_1 + v_3) = 50 - (10 - 20) = 60$$
$$d_{21} = c_{21} - (u_2 + v_1) = 70 - (60 + 9) = 1$$
$$d_{22} = c_{22} - (u_2 + v_2) = 30 - (60 - 12) = -18$$
$$d_{31} = c_{31} - (u_3 + v_1) = 40 - (20 + 9) = 11$$
$$d_{33} = c_{33} - (u_3 + v_3) = 70 - (20 - 20) = 70$$

4. According to the optimality criterion for cost minimizing transportation problem, the current solution is not optimal, since the opportunity costs of the unoccupied cells are not all zero or positive. The value of $d_{22} = -18$ in cell (S_2, D_2) is indicating that the total transportation cost can be reduced in the multiple of 18 by shifting an allocation to this cell.

5. A closed-loop (path) is traced along row S_2 to an occupied cell (S_3, D_2). A plus sign is placed in cell (S_2, D_2) and minus sign in cell (S_3, D_2). Now take a right-angle turn and locate an occupied cell in column D_4. An occupied cell (S_3, D_4) exists at row S_3, and a plus sign is placed in this cell.
   Continue this process and complete the closed path. The occupied cell (S_2, D_3) must be bypassed otherwise they will violate the rules of constructing closed path.

6. In order to maintain feasibility, examine the occupied cells with minus sign at the corners of closed loop, and select the one that has the smallest allocation. This determines the maximum number of units that can be shifted along the closed path. The minus signs are in cells (S_3, D_2) and (S_2, D_4). The cell (S_2, D_4) is selected because it has the smaller allocation, i.e. 2. The value of this allocation is then added to cell (S_2, D_2) and (S_3, D_4), which carry plus signs. The same value is subtracted from cells (S_2, D_4) and (S_3, D_2) because they carry minus signs.

7. The revised solution is shown in Table 9.12. The total transportation cost associated with this solution is shown in Table 9.12.

   Total cost = 5 × 19 + 2 × 10 + 2 × 30 + 7 × 40 + 6 × 8 + 12 × 20 = Rs 743

	D_1	D_2	D_3	D_4	Supply	u_i
S_1	19 (5)	30 +32	50 +42	10 (2)	7	$u_1 = 0$
S_2	70 +19	30 (2)	40 (7)	60 +14	9	$u_2 = 32$
S_3	40 +11	8 (6)	70 +52	20 (12)	18	$u_3 = 10$
Demand	5	8	7	14	34	
v_j	$v_1 = 19$	$v_2 = -2$	$v_3 = 8$	$v_4 = 10$		

Table 9.12
Optimal Solution

8. Test the optimality of the revised solution once again in the same way as discussed in earlier steps. The values of u_is, v_js and d_{ij}s are shown in Table 9.12. Since each of d_{ij}s is positive, therefore, the current basic feasible solution is optimal with a minimum total transportation cost of Rs 743.

Example 9.7 A company has factories at F_1, F_2, and F_3 that supply products to warehouses at W_1, W_2 and W_3. The weekly capacities of the factories are 200, 160 and 90 units, respectively. The weekly warehouse requirements are 180, 120 and 150 units, respectively. The unit shipping costs (in rupees) are as follows:

Warehouse

Factory		W_1	W_2	W_3	*Supply*
	F_1	16	20	12	200
	F_2	14	8	18	160
	F_3	26	24	16	90
	Demand	180	120	150	450

Determine the optimal distribution for this company in order to minimize its total shipping cost.

Solution Initial basic feasible solution obtained by North-West Corner Rule is given in Table 9.13. The initial solution has $m + n - 1 = 3 + 3 - 1 = 5$ allocations. Therefore, it is a non-degenerate solution. The optimality test can, therefore, be performed. The total transportation cost associated with this solution is:

	W_1	W_2	W_3	*Supply*
F_1	16 (180)	20 (20)	12	200
F_2	14	8 (100)	18 (60)	160
F_3	26	24	16 (90)	90
Demand	180	120	150	450

Table 9.13
Initial Solution

$$\text{Total cost} = 16 \times 180 + 20 \times 20 + 8 \times 100 + 18 \times 60 + 16 \times 90 = \text{Rs } 6{,}800$$

We now determine the values of u_is and v_js as usual, by arbitrarily assigning $u_1 = 0$. Given $u_1 = 0$, the values of others variables obtained by using the equation $c_{ij} = u_i + v_j$ for occupied cells, are shown in Table 9.14.

	W_1	W_2	W_3	*Supply*	u_i
F_1	16 (180)	20 (−) (20)	12 (+) −18	200	$u_1 = 0$
F_2	14 +10	8 (+) (100)	18 (60) (−)	160	$u_2 = -12$
F_3	26 +24	24 +18	16 (90)	90	$u_3 = -14$
Demand	180	120	150	450	
v_j	$v_1 = 16$	$v_2 = 20$	$v_3 = 30$		

At each step a non-occupied cell with largest negative opportunity cost is selected to get maximum reduction in total transportation cost

Table 9.14

$$\begin{aligned}
c_{11} &= u_1 + v_1 \quad \text{or} \quad 16 = 0 + v_1 \quad \text{or} \quad v_1 = 16\\
c_{12} &= u_1 + v_2 \quad \text{or} \quad 20 = 0 + v_2 \quad \text{or} \quad v_2 = 20\\
c_{22} &= u_2 + v_2 \quad \text{or} \quad 8 = u_2 + 20 \quad \text{or} \quad u_2 = -12
\end{aligned}$$

$$c_{23} = u_2 + v_3 \quad \text{or} \quad 18 = -12 + v_3 \quad \text{or } v_3 = 30$$
$$c_{33} = u_3 + v_3 \quad \text{or} \quad 16 = u_3 + 30 \quad \text{or } u_3 = -14$$

The opportunity cost for each of the unoccupied cells is determined by using the equation, $d_{ij} = c_{ij} - (u_i + v_j)$ as follows:

$$d_{13} = c_{13} - (u_1 + v_3) = 12 - (0 + 30) = -18$$
$$d_{21} = c_{21} - (u_2 + v_1) = 14 - (-12 + 16) = 10$$
$$d_{31} = c_{31} - (u_3 + v_1) = 26 - (-14 + 16) = 24$$
$$d_{32} = c_{32} - (u_3 + v_2) = 24 - (-14 + 20) = 18$$

The value of $d_{13} = -18$ in the cell (F_1, W_3) indicates that the total transportation cost can be reduced in a multiple of 18 by introducing this cell in the new transportation schedule. To see how many units of the commodity could be allocated to this cell (route) we shall form a closed path, as shown in Table 9.14.

The largest number of units of the commodity that should be allocated to the cell (F_1, W_3) is 20 units because it does not violate the supply and demand restrictions (minimum allocation among the occupied cells bearing negative sign at the corners of the loop). The new transportation schedule (solution) so obtained is shown in Table 9.15.

Table 9.15

	W_1	W_2	W_3	*Supply*
F_1	16 (180)	20	12 (20)	200
F_2	14	8 (120)	18 (40)	160
F_3	26	24	16 (90)	90
Demand	180	120	150	450

The total transportation cost associated with this solution is

Total cost = 16 × 180 + 12 × 20 + 8 × 120 + 18 × 40 + 16 × 90 = Rs 6,240

To test the optimality of the new solution shown in Table 9.15, we need to again calculate the opportunity cost of each unoccupied cell in the same manner as discussed earlier. The calculations for u_is, v_js and d_{ij}s are shown in Table 9.16.

$$c_{13} = u_1 + v_3 \quad \text{or} \quad 12 = u_1 + 0 \quad \text{or} \quad u_1 = 12$$
$$c_{23} = u_2 + v_3 \quad \text{or} \quad 18 = u_2 + 0 \quad \text{or} \quad u_2 = 18$$
$$c_{33} = u_3 + v_3 \quad \text{or} \quad 16 = u_3 + 0 \quad \text{or} \quad u_3 = 16$$
$$c_{11} = u_1 + v_1 \quad \text{or} \quad 16 = 12 + v_1 \quad \text{or} \quad v_1 = 4$$
$$c_{22} = u_2 + v_2 \quad \text{or} \quad 8 = 18 + v_2 \quad \text{or} \quad v_2 = -10$$

$$d_{12} = c_{12} - (u_1 + v_2) \quad \text{or} \quad 20 - (12 - 10) = 18$$
$$d_{21} = c_{21} - (u_2 + v_1) \quad \text{or} \quad 14 - (18 + 4) = -8$$
$$d_{31} = c_{31} - (u_3 + v_1) \quad \text{or} \quad 26 - (16 + 4) = 6$$
$$d_{32} = c_{32} - (u_3 + v_2) \quad \text{or} \quad 24 - (16 - 10) = 18$$

The value of $d_{21} = -8$ in the cell (F_2, W_1) indicates that the total cost of transportation can further be reduced in a multiple of 8 by introducing this cell in the new transportation schedule. The new solution is obtained in the same manner by introducing 40 units of the commodity in the cell (F_2, W_1), as indicated in Table 9.16. The new solution is shown in Table 9.17.

	W_1	W_2	W_3	Supply	u_i
F_1	16 (–) 180	20 18	12 20 (+)	200	$u_1 = 12$
F_2	14 (+) –8	8 120	18 40 (–)	160	$u_2 = 18$
F_3	26 6	24	16 90	90	$u_3 = 16$
Demand	180	120	150		
v_j	$v_1 = 4$	$v_2 = -10$	$v_3 = 0$		

Table 9.16

The total transportation cost associated with this solution is

Total cost = 16 × 140 + 12 × 60 + 14 × 40 + 8 × 120 + 16 × 90 = Rs 5,920

	W_1	W_2	W_3	Supply
F_1	16 140	20	12 60	200
F_2	14 40	8 120	18	160
F_3	26	24	16 90	90
Demand	180	120	150	

Table 9.17

To test the optimality of the new solution shown in Table 9.17 we need to again calculate the opportunity cost of each unoccupied cell in the same manner as discussed earlier. The calculations are shown in Table 9.18.

	W_1	W_2	W_3	Supply	u_i
F_1	16 140	20 +10	12 60	200	$u_1 = 16$
F_2	14 40	8 120	18 +8	160	$u_2 = 14$
F_3	26 +6	24 +10	16 90	90	$u_3 = 20$
Demand	180	120	150		
v_j	$v_1 = 0$	$v_2 = -6$	$v_3 = -4$		

Table 9.18

$$d_{12} = c_{12} - (u_1 + v_2) \quad \text{or} \quad 20 - (16 - 6) = 10$$
$$d_{23} = c_{23} - (u_2 + v_3) \quad \text{or} \quad 18 - (14 - 4) = 8$$
$$d_{31} = c_{31} - (u_3 + v_1) \quad \text{or} \quad 26 - (20 + 0) = 6$$
$$d_{32} = c_{32} - (u_3 + v_2) \quad \text{or} \quad 24 - (20 - 6) = 10$$

Since none of the unoccupied cells in Table 9.18 has a negative opportunity cost value, therefore, the total transportation cost cannot be reduced further. Thus, the solution shown in Table 9.18 is the optimal solution, giving the optimal transportation schedule with a total cost of Rs 5,920.

Example 9.8 The following table provides all the necessary information on the availability of supply to each warehouse, the requirement of each market, and the unit transportation cost (in Rs) from each warehouse to each market.

Market

Warehouse		*P*	*Q*	*R*	*S*	*Supply*
	A	6	3	5	4	22
	B	5	9	2	7	15
	C	5	7	8	6	8
	Demand	7	12	17	9	45

The shipping clerk of the shipping agency has worked out the following schedule, based on his own experience: 12 units from A to Q, 1 unit from A to R, 8 units from A to S, 15 units from B to R, 7 units from C to P and 1 unit from C to R.

(a) Check and see if the clerk has the optimal schedule.
(b) Find the optimal schedule and minimum total transport cost.
(c) If the clerk is approached by a carrier of route C to Q, who offers to reduce his rate in the hope of getting some business, by how much should the rate be reduced before the clerk would offer him the business.

Solution (a) The shipping schedule determined by the clerk based on his experience is shown in Table 9.19. The total transportation cost associated with this solution is

$$\text{Total cost} = 3 \times 12 + 5 \times 1 + 4 \times 9 + 2 \times 15 + 5 \times 7 + 8 \times 1 = \text{Rs } 150$$

Since the number of occupied cells (i.e. 6) is equal to the required number of occupied cells (i.e. $m + n - 1$) in a feasible solution, therefore the solution is non-generate feasible solution. Now, to test the optimality of the solution given in Table 9.19 we evaluate each unoccupied cell in terms of the opportunity cost associated with it. This is done in the usual manner and is shown in Table 9.20.

Table 9.19
Initial Solution

	P	*Q*	*R*	*S*	*Supply*
A	6	3 (12)	5 (1)	4 (9)	22
B	5	9	2 (15)	7	15
C	5 (7)	7	8 (1)	6	8
Demand	7	12	17	9	45

In Table 9.20, cell (C, S) has a negative opportunity cost (i.e. – 1). Thus, this solution is not the optimal solution and, therefore, the schedule prepared by the shipping clerk is not optimal.

(b) By forming a closed-loop to introduce the cell (C, S) into the new transportation schedule as shown in Table 9.20, we get a new solution that is shown in Table 9.21.

	P	Q	R	S	Supply	u_i
A	6 +4	3 (12)	5 (+) (1)	4 (9) (–)	22	$u_1 = 0$
B	5	9 +9	2 (15)	7 +6	15	$u_2 = -3$
C	5 (7)	7 +1	8 (–) (1)	6 (+) –1	8	$u_3 = 3$
Demand	7	12	17	9	45	
v_j	$v_1 = 2$	$v_2 = 3$	$v_3 = 5$	$v_4 = 4$		

Table 9.20

While testing the optimality of the improved solution shown in Table 9.21, we found that the opportunity costs in all the unoccupied cells are positive. Thus the current solution is optimal and the optimal schedule is to transport 12 units from A to Q; 2 units from A to R; 8 units from A to S; 15 units from B to R; 7 units from C to P and 1 unit from *C* to *S*. The total minimum transportation cost associated with this solution is

Total cost = 3 × 12 + 5 × 2 + 4 × 8 + 2 × 15 + 5 × 7 + 6 × 1 = Rs 149

	P	Q	R	S	Supply	u_i
A	6 +3	3 (12)	5 (2)	4 (8)	22	$u_1 = 0$
B	5 +5	9 +9	2 (15)	7 +6	15	$u_2 = -3$
C	5 (7)	7 +2	8 +1	6 (1)	8	$u_3 = 2$
Demand	7	12	17	9	45	
v_j	$v_1 = 3$	$v_2 = 3$	$v_3 = 5$	$v_4 = 4$		

Table 9.21

(c) The total transportation cost will increase by Rs 2 (opportunity cost) if one unit of commodity is transported from C to Q. This means that the rate of the carrier on the route C to Q should be reduced by Rs 2, i.e. from Rs 7 to Rs 5 so as to get some business of one unit of commodity only.

In case all the 8 units available at C are shipped through the route (C, Q), then the solution presented in Table 9.21 may be read as shown in Table 9.22.

	P	Q	R	S	Supply
A	6 (7)	3 (4)	5 (2)	4 (9)	22
B	5	9	2 (15)	7	15
C	5	7 (8)	8	6	8
Demand	7	12	17	9	45

Table 9.22

The total cost of transportation associated with this solution is

$$\text{Total cost} = 6 \times 7 + 3 \times 4 + 5 \times 2 + 4 \times 9 + 2 \times 15 + 7 \times 8 = \text{Rs } 186.$$

Thus, the additional cost of Rs 37 (= 186 – 149) should be reduced from the transportation cost of 8 units from C to Q. Hence transportation cost per unit from C to Q should be at the most 7 – (37/8) = Rs 2.38.

Example 9.10 ABC Limited has three production shops that supply a product to five warehouses. The cost of production varies from shop to shop and cost of transportation from one shop to a warehouse also varies. Each shop has a specific production capacity and each warehouse has certain amount of requirement. The costs of transportation are given below:

Warehouse

		I	*II*	*III*	*IV*	*V*	*Supply*
	A	6	4	4	7	5	100
Shop	*B*	5	6	7	4	8	125
	C	3	4	6	3	4	175
	Demand	60	80	85	105	70	400

The cost of manufacturing the product at different production shops is

Shop	*Variable Cost*	*Fixed Cost*
A	14	7,000
B	16	4,000
C	15	5,000

Find the optimum quantity to be supplied from each shop to different warehouses at the minimum total cost. *[Delhi Univ., MBA, 1997]*

Solution In this case, the fixed cost data is of no use. The transportation cost matrix will include the given transportation cost plus the variable cost, as shown in Table 9.23.

Table 9.23

	I	*II*	*III*	*IV*	*V*	*Supply*
A	6 + 14 = 20	4 + 14 = 18	4 + 14 = 18	7 + 14 = 21	5 + 14 = 19	100
B	5 + 16 = 21	6 + 16 = 22	7 + 16 = 23	4 + 16 = 20	8 + 16 = 24	125
C	3 + 15 = 18	4 + 15 = 19	6 + 15 = 21	3 + 15 = 18	4 + 15 = 19	175
Demand	60	80	85	105	70	400

The optimal solution obtained by applying MODI method is shown in Table 9.24.

Table 9.24

	I	*II*	*III*	*IV*	*V*	*Supply*	u_i
A	20 +3	18 (15)	18 (85)	21 +5	19 +1	100	$u_1 = 18$
B	21 0	22 (20)	23 +1	20 (105)	24 +2	125	$u_2 = 22$
C	18 (60)	19 (45)	21 +2	18 +1	19 (70)	175	$u_3 = 19$
Demand	60	80	85	105	70		
v_j	$v_1 = -1$	$v_2 = 0$	$v_3 = 0$	$v_4 = -2$	$v_5 = 0$		

The transportation cost associated with the solution is

Total cost = 18 × 15 + 18 × 85 + 22 × 20 + 20 × 105 + 18 × 60 + 19 × 45 + 19 × 70 = Rs 7,605

CONCEPTUAL QUESTIONS B

1. Describe the computational procedure of the optimality test in a transportation problem.
2. Indicate how you will test for optimality of initial feasible solution of a transportation problem.
3. Let S_i and D_j ($i = 1, 2, \ldots, m$; $j = 1, 2, \ldots, n$) be the supply and demand available, respectively, for a commodity at m godowns and n markets. Let c_{ij} be the cost of transporting one unit of the commodity from godown i to market j. Assuming that

$$\sum_{i=1}^{m} S_i = \sum_{j=1}^{n} D_j$$

Symbolically state the transportation problem. Establish that the optimal solution is not altered when the c_{ij}s are replaced by c_{ij}^*s, where $c_{ij}^* = c_{ij} + u_i + v_j$, u_i ($i = 1, 2, \ldots, m$) and v_j ($j = 1, 2, \ldots, n$) are arbitrary real numbers.

SELF PRACTICE PROBLEMS B

1. Consider four bases of operation B_i and three targets T_j. The tons of bombs per aircraft from any base that can be delivered to any target are given in the following table:

Target (T_j)

Base (B_i)		T_1	T_2	T_3
	B_1	8	6	5
	B_2	6	6	6
	B_3	10	8	4
	B_4	8	6	4

The daily sortie capability of each of the four bases is 150 sorties per day. The daily requirement of sorties spread over each individual target is 200. Find the allocation of sorties from each base to each target which maximizes the total tonnage over all three targets. Explain each step in the process.

2. A company has four warehouses, a, b, c and d. It is required to deliver a product from these warehouses to three customers A, B and C. The warehouses have the following amounts in stock:

Warehouse	:	a	b	c	d
No. of units	:	15	16	12	13

and the customers' requirements are

Customer	:	A	B	C
No. of units	:	18	20	18

The table below shows the costs of transporting one unit from warehouse to the customer.

Warehouse

		a	b	c	d
Customer	A	8	9	6	3
	B	6	11	5	10
	C	3	8	7	9

Find the optimal transportation routes.

3. A firm manufacturing a single product has three plants I, II and III. They have produced 60, 35 and 40 units, respectively during this month. The firm had made a commitment to sell 22 units to customer A, 45 units to customer B, 20 units to customer C, 18 units to customer D and 30 units to customer E. Find the minimum possible transportation cost of shifting the manufactured product to the five customers. The net unit cost of transporting from the three plants to the five customers is given below:

Customers

		A	B	C	D	E
Plants	I	4	1	3	4	4
	II	2	3	2	2	3
	III	3	5	2	4	4

4. The following table gives the cost of transporting material from supply points A, B, C and D to demand points E, F, G, H and I.

To

		E	F	G	H	I
From	A	8	10	12	17	15
	B	15	13	18	11	9
	C	14	20	6	10	13
	D	13	19	7	5	12

The present allocation is as follows:
A to E 90; A to F 10; B to F 150; C to F 10; C to G 50; C to I 120; D to H 210; D to I 70.

(a) Check if this allocation is optimum. If not, find an optimum schedule.

(b) If in the above problem, the transportation cost from A to G is reduced to 10, what will be the new optimum schedule?

[*IAS (Main), 1992*]

5. A wholeselling company has three warehouses from which the supplies are drawn for four retail customers. The company deals in a single product, the supplies of which at each warehouse are:

Warehouse Number	*Supply (units)*	*Customer Number*	*Demand (units)*
1	20	1	15
2	28	2	19
3	17	3	13
		4	18

Total supply at the warehouses is equal to total demand from the customers. The table below gives the transportation costs, per unit, shipped from each warehouse to each customer.

Customer

		C_1	C_2	C_3	C_4
Warehouse	W_1	3	6	8	5
	W_2	6	1	2	5
	W_3	7	8	3	9

Determine what supplies should be despatched from each of the warehouses to each customer so as to minimize the overall transportation cost.

6. A manufacturer has distribution centres at Agra, Allahabad and Kolkata. These centres have availability of 40, 20 and 40 units of his product, respectively. His retail outlets at A, B, C, D and E require 25, 10, 20, 30 and 15 units of the products, respectively. The transport cost (in rupees) per unit between each centre outlet is given below:

Distribution Centre	*Retail Outlets*				
	A	*B*	*C*	*D*	*E*
Agra	55	30	40	50	40
Allahabad	35	30	100	45	60
Kolkata	40	60	95	35	30

Determine the optimal distribution so as to minimize the cost of transportation. [*Delhi Univ., MBA, 1998*]

7. A manufacturer has distribution centres located at Agra, Allahabad and Kolkata. These centres have available 40, 20 and 40 units of his product. His retail outlets at A, B, C, D and E requires 25, 10, 20, 30 and 15 units of the product, respectively. The shipping cost per unit (in rupees) between each centre and outlet is given in the following table.

Distribution Centre	*Retail Outlets*				
	A	*B*	*C*	*D*	*E*
Agra	55	30	40	50	40
Allahabad	35	30	100	45	60
Kolkata	40	60	95	35	30

Determine the optimal shipping cost.
[*Allahabad, MBA, 1990; Delhi Univ. MBA, 1998*]

8. A steel company is concerned with the problem of distributing imported ore from three ports to four steel mills. The supplies of ore arriving at the ports are:

Port	*Tonnes per week*
a	20,000
b	38,000
c	16,000

The demand at the steel mills is as follows:

Steel mills	: A	B	C	D
Tonnes per week	: 10,000	18,000	22,000	24,000

The transportation cost is Re 0.05 per tonne per km. The distance between the ports and the steel mills is as given below:

	A	*B*	*C*	*D*
a	50	60	100	50
b	80	40	70	50
c	90	70	30	50

Calculate a transportation plan that will minimize the distribution cost for the steel company. State the cost of this distribution plan.

9. A company has three factories at Amethi, Baghpat and Gwalior that have a production capacity of 5,000, 6,000, and 2,500 tonnes, respectively. Four distribution centres at Allahabad, Bombay, Kolkata and Delhi, require 6,000 tonnes, 4,000 tonnes, 2,000 tonnes and 1,500 tonnes, respectively, of the product. The transportation costs per tonne from different factories to different centres are given below:

Factories	*Distribution Centres*			
	Allahabad	*Bombay*	*Kolkata*	*Delhi*
Amethi	3	2	7	6
Baghpat	7	5	2	3
Gwalior	2	5	4	5

Suggest an optimum transportation schedule and find the minimum cost of transportation. [*Gujarat Univ., MBA, 1990*]

10. A company has three plants and four warehouses. The supply and demand in units and the corresponding transportation costs are given. The table below has been taken from the solution procedure of a transportation problem:

Warehouses

		I		*II*		*III*		*IV*		*Supply*
Plants	*A*	5		10		4	10	5		10
	B	6	20	8		7		2	5	25
	C	4	5	2	10	5	5	7		20
	Demand		25		10		15		5	55

Answer the following questions, giving brief reasons for the same:

(a) Is this solution feasible?
(b) Is this solution degenerate?
(c) Is this solution optimum?
(d) Does this problem have more than one optimum solution? If so, show all of them.
(e) If the cost for the route B-III is reduced from Rs 7 to Rs 6 per unit, what will be the optimum solution?

11. A baking firm can produce a speciality bread in either of its two plants, the details of which are as follows:

Plant	*Production Capacity Loaves*	*Production Cost Rs/Loaf*
A	2,500	2.30
B	2,100	2.50

Four restaurant chains are willing to purchase this bread; their demand and the prices they are willing to pay are as follows:

Chain	*Maximum Demand Loaves*	*Price Offered Rs/Loaf*
1	1,800	3.90
2	2,300	3.70
3	550	4.00
4	1,750	3.60

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In Table 9.26, all opportunity costs d_{ij}s are not positive, the current solution is not optimal. Thus, the unoccupied cell (X, C), where $d_{23} = -8$ must enter into the basis and cell (W, C) must leave the basis, as shown by the closed path. The new solution is shown in Table 9.27.

	A	*B*	*C*	D_{excess}	*Supply*	u_i
W	4 +4	8 76	8 +8	0 +16	76	$u_1 = -16$
X	16 0	24 21	16 41	0 20	82	$u_2 = 0$
Y	8 72	16 5	24 +16	0 +8	77	$u_3 = -8$
Demand	72	102	41	20	235	
v_j	$v_1 = 16$	$v_2 = 24$	$v_3 = 16$	$v_4 = 0$		

Table 9.27

Since all opportunity costs d_{ij}s are non-negative in Table 9.27, the current solution is optimal. The total minimum transportation cost associated with this solution is:

Total cost = 8 × 76 + 24 × 21 + 16 × 41 + 0 × 20 + 8 × 72 + 16 × 5 = Rs 2,424.

Example 9.11 A product is manufactured at four factories A, B, C and D. Their unit production costs are Rs 2, Rs 3, Re 1 and Rs 5, respectively. Their production capacities are 50, 70, 30 and 50 units, respectively. These factories supply the product to four stores, demands of which are 25, 35, 105 and 20 units respectively. Unit transportation cost in rupees from each factory to each store is given in the table below.

		Stores			
		I	*II*	*III*	*IV*
Factories	*A*	2	4	6	11
	B	10	8	7	5
	C	13	3	9	12
	D	4	6	8	3

Determine the extent of deliveries from each of the factories to each of the stores, so that the total production and transportation cost is the minimum.

Solution The new transportation costs that include both the production and the transportation costs is given in Table 9.28.

	I	*II*	*III*	*IV*	*Supply*
A	2 + 2 = 4	4 + 2 = 6	6 + 2 = 8	11 + 2 = 13	50
B	10 + 3 = 13	8 + 3 = 11	7 + 3 = 10	5 + 3 = 8	70
C	13 + 1 = 14	3 + 1 = 4	9 + 1 = 10	12 + 1 = 13	30
D	4 + 5 = 9	6 + 5 = 11	8 + 5 = 13	3 + 5 = 8	50
Demand	25	35	105	20	200 185

Table 9.28

Since the total supply of 200 units exceeds the total demand of 185 units by 15 units, a dummy destination (store) is added (or created) to absorb the excess capacity. The associated cost coefficients in dummy store are taken as zero. This may be due to the reason that the surplus quantity remains lying in the respective factories and is not shipped at all. The modified table is shown in Table 9.29.

	I	II	III	IV	Dummy	Supply
A	4 (25)	6 (5)	8 (20)	13	0	50
B	13	11	10 (70)	8	0	70
C	14	4 (30)	10	13	0	30
D	9	11	13 (15)	8 (20)	0 (15)	50
Demand	25	35	105	20	15	200

Table 9.29
Initial Solution

Using the VAM method the initial solution is shown in Table 9.29. It can be seen that 15 units are allocated to dummy store from factory D. This means that the company may cut down the production by 15 units at the factory that is proving to be uneconomical. Now to test the optimality of the solution shown in Table 9.29 we evaluate each unoccupied cell in terms of opportunity cost associated with it in the usual manner as shown in Table 9.30.

	I	II	III	IV	Dummy	Supply	u_i
A	4 (25)	6 (5)	8 (20)	13 +10	0 +5	50	$u_1 = -5$
B	13 +7	11 +3	10 (70)	8 +3	0 +3	70	$u_2 = -3$
C	14 +12	4 (30)	10 +4	13 +12	0 +7	30	$u_3 = -7$
D	9 0	11 0	13 (15)	8 (20)	0 (15)	50	$u_4 = 0$
Demand	25	35	105	20	15	200	
v_j	$v_1 = 9$	$v_2 = 11$	$v_3 = 13$	$v_4 = 8$	$v_5 = 0$		

Table 9.30

Since the opportunity cost in all the unoccupied cells is positive, the initial solution is an optimal solution. The total cost of transportation associated with this solution is

Total cost = 4 × 25 + 6 × 5 + 8 × 20 + 10 × 70 + 4 × 30 + 13 × 15 + 8 × 20 + 0 × 15 = Rs 1,465.

9.6.2 Degeneracy and Its Resolution

Degeneracy arises when the number of occupied cells are less than the number of rows + columns − 1.

A basic feasible solution for the general transportation problem must consist of exactly $m + n - 1$ (number of rows + number of columns − 1) positive allocations in independent positions in the transportation table. A solution will only be called degenerate if the number of occupied cells is less than the required number, $m + n - 1$. In such cases, the current solution cannot be improved upon because it is not possible to draw a closed path for every occupied cell. Also, the values of dual variables u_i and v_j that are used to test the optimality cannot be computed. Thus, we need to remove the degeneracy in order to improve the given solution. The degeneracy in the transportation problems may occur at two stages:

(a) When obtaining an initial basic feasible solution we may have less than $m + n - 1$ allocations.
(b) At any stage while moving towards optimal solution. This happens when two or more occupied cells with the same minimum allocation are simultaneously unoccupied.

Case 1: Degeneracy at the initial solution To resolve degeneracy at the initial solution, we proceed by allocating a very small quantity close to zero to one or more (if needed) unoccupied cells so as to get $m + n - 1$ number of occupied cells. This amount is denoted by a Greek letter ε (epsilon) or Δ (delta). This quantity would neither affect the total cost nor the supply and demand values. In a minimization transportation problem it is better to allocate Δ to unoccupied cells that have lowest transportation costs, whereas in maximization problems it should be allocated to a cell that has a high payoff value. In some cases, Δ must be added in one of those unoccupied cells that uniquely makes possible the determination of u_i and v_j.

The quantity Δ is considered to be so small that if it is transferred to an occupied cell it does not change the quantity of allocation. That is,

$$x_{ij} + \Delta = x_{ij} \qquad - \Delta = x_{ij}$$
$$\Delta - \Delta = 0; \qquad \Delta + \Delta = \Delta$$
$$0 + \Delta = \Delta; \qquad k \times \Delta = \Delta$$

It is also obvious then that Δ does not affect the total transportation cost of the allocation. Hence, the quantity Δ is used to evaluate unoccupied cells and to reduce the number of improvement cycles necessary to reach an optimal solution. Once the purpose is over, Δ can be removed from the transportation table.

Example 9.12 A manufacturer wants to ship 22 loads of his product as shown below. The matrix gives the kilometres from sources of supply to the destinations.

Destination

		D_1	D_2	D_3	D_4	D_5	*Supply*
	S_1	5	8	6	6	3	8
Source	S_2	4	7	7	6	5	5
	S_3	8	4	6	6	4	9
	Demand	4	4	5	4	8	25 \ 22

The shipping cost is Rs 10 per load per km. What shipping schedule should be used in order to minimize the total transportation cost? *[Delhi Univ., MBA, 2001]*

Solution Since the total destination requirement of 25 units exceeds the total resource capacity of 22 by 3 units, the problem is unbalanced. The excess requirement is handled by adding a dummy plant, S_{excess} with a capacity equal to 3 units. We use zero unit transportation cost to the dummy plant. The modified transportation table is shown in Table 9.31.

	D_1	D_2	D_3	D_4	D_5	*Supply*
S_1	5	8	6 (5)	6	3 (3)	8
S_2	4 (4)	7	7	6 (1)	5	5
S_3	8	4 (4)	6	6	4 (5)	9
S_{excess}	0	0	0	0 (3)	0	3
Demand	4	4	5	4	8	25

Table 9.31
Initial Solution

The initial solution is obtained by using Vogel's approximation method as shown in Table 9.31. Since the solution includes 7 occupied cells, therefore, the initial solution is degenerate. In order to remove degeneracy we assign Δ to unoccupied cell (S_2, D_5), which has the minimum cost amongst the unoccupied cells, as shown in Table 9.32.

	D_1	D_2	D_3	D_4	D_5	*Supply*	u_i
S_1	5 +3	8 +5	6 (–) 5	6 +2	3 3 (+)	8	$u_1 = 0$
S_2	4 4	7 +2	7 –1	6 1 (+)	5 Δ (–)	5	$u_2 = 2$
S_3	8 +5	4 4	6 –1	6 +1	4 5	9	$u_3 = 1$
S_{excess}	0 +2	0 +1	0 (+) –2	0 3 (–)	0 –7	3	$u_4 = -4$
Demand	4	4	5	4	8	25	
v_j	$v_1 = 2$	$v_2 = 3$	$v_3 = 6$	$v_4 = 4$	$v_5 = 3$		

Table 9.32

Determine u_i and v_j for occupied cells as shown in Table 9.32. Since the opportunity cost in the cell (S_{excess}, D_3) is largest negative, it must enter the basis and the cell (S_2, D_5) must leave the basis. The new solution is shown in Table 9.33.

	D_1	D_2	D_3	D_4	D_5	*Supply*	u_i
S_1	5 +1	8 +5	6 (–) 5	6 0	3 3 (+)	8	$u_1 = 0$
S_2	4 4	7 +4	7 +1	6 1	5 +2	5	$u_2 = 0$
S_3	8 +3	4 4	6 –1	6 (+) –1	4 5 (–)	9	$u_3 = 1$
S_{excess}	0 +2	0 +3	0 (+) Δ	0 3 (–)	0 +3	3	$u_4 = -6$
Demand	4	4	5	4	8	25	
v_j	$v_1 = 4$	$v_2 = 3$	$v_3 = 6$	$v_4 = 6$	$v_5 = 3$		

Table 9.33

Repeat the procedure of testing optimality of the solution given in Table 9.33. The optimal solution is shown in Table 9.34.

	D_1	D_2	D_3	D_4	D_5	*Supply*
S_1	5	8	6	6	3 8	8
S_2	4 4	7	7	6 1	5	5
S_3	8	4 4	6 2	6 3	4	9
S_{excess}	0	0	0 3	0	0	3
Demand	4	4	5	4	8	25

Table 9.34

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	D_1	D_2	D_3	Supply
S_1	8 (120)	5	6	120
S_2	15	10 (80)	12	80
S_3	3 (30)	9 (Δ)	10 (50)	80
Demand	150	80	50	280

Table 9.37

To remove degeneracy a quantity Δ is assigned to one of the cells that has become unoccupied so that there are $m + n - 1$ occupied cells. Assign Δ to either (S_2, D_1) or (S_3, D_2) and proceed with the usual solution procedure. The optimal solution is given in Table 9.38.

	D_1	D_2	D_3	Supply
S_1	8 (70)	5	6 (50)	120
S_2	15	10 (80)	12	80
S_3	3 (80)	9	10	80
Demand	150	80	50	280

Table 9.38

9.6.3 Alternative Optimal Solutions

The existence of alternative optimal solutions can be determined by an inspection of the opportunity costs, d_{ij} for the unoccupied cells. If an unoccupied cell in an optimal solution has an opportunity cost of zero, an alternative optimal solution can be formed with another set of allocations, without increasing the total transportation cost.

Illustration Consider the optimal solution of Example 9.7, given in Table 9.27. For ready reference Table 9.27 is reproduced as Table 9.39.

	A	B	C	D_{excess}	Supply	u_i
W	4 +4	8 (76)	8 +8	0 +16	76	$u_1 = -16$
X	16 0	24 (21)	16 (41)	0 (20)	82	$u_2 = 0$
Y	8 (72)	16 (5)	24 +16	0 +8	77	$u_3 = -8$
Demand	72	102	41	20	235	
v_j	$v_1 = 16$	$v_2 = 24$	$v_3 = 16$	$v_4 = 0$		

Table 9.39

The opportunity costs in all unoccupied cells are positive except for the cell (X, A) which has a zero opportunity cost. This means if (X, A) is entered into the basis, no change in the transportation cost would occur. To determine this alternative solution, form a closed path for cell (X, A) as shown in Table 9.40.

	A	*B*	*C*	D_{excess}	*Supply*	u_i
W	4 +4	8 (76)	8 +8	0 +16	76	$u_1 = -16$
X	16 (+) 0	24 (21) (−)	16 (41)	0 (20)	82	$u_2 = 0$
Y	8 (−)(72)	16 (5) (+)	24 +16	0 +8	77	$u_3 = -8$
Demand	72	102	41	20	235	
v_j	$v_1 = 16$	$v_2 = 24$	$v_3 = 16$	$v_4 = 0$		

Table 9.40
Optimal Solution

The maximum quantity that can be allocated to cell (*X*, *A*) is 21. After this change, the new solution is shown in Table 9.41.

Since all d_{ij} values are positive or zero, the solution given in Table 9.41 is optimal with a minimum total transportation cost of Rs 2,424, which is same as in the previous solution.

	A	*B*	*C*	D_{excess}	*Supply*	u_i
W	4 +4	8 (76)	8 +8	0 +16	76	$u_1 = -16$
X	16 (21)	24 0	16 (41)	0 (20)	82	$u_2 = -0$
Y	8 (51)	16 (26)	24 +16	0 +8	77	$u_3 = -8$
Demand	72	102	41	20	235	
v_j	$v_1 = 16$	$v_2 = 24$	$v_3 = 16$	$v_4 = 0$		

Table 9.41
Initial Solution, VAM

Example 9.14 XYZ tobacco company purchases tobacco and stores in warehouses located in the following four cities:

Warehouse Location (City)	:	A	B	C	D
Capacity (Tonnes)	:	90	50	80	60

The warehouses supply tobacco to cigarette companies in three cities that have the following demand:

Cigarette Company	*Demand (Tonnes)*
Bharat	120
Janata	100
Red Lamp	110

The following railroad shipping costs per tonne (in hundred rupees) have been determined:

Warehouse Location	*Bharat*	*Janata*	*Red Lamp*
A	7	10	5
B	12	9	4
C	7	3	11
D	9	5	7

Because of railroad construction, shipments are temporarily prohibited from warehouse at city A to Bharat Cigarette company.

(a) Find the optimum distribution for XYZ tobacco company.
(b) Are there multiple optimum solutions? If yes, identify them.
(c) Write the dual of the given transportation problem and use it for checking the optimum solution.

[*Delhi Univ., MBA, 97, 1999, 2002*]

Solution Since the total demand of 330 units exceeds the total capacity of 280 units by 50 units of the product, a dummy company is created to handle the excess demand. The associated cost coefficients for the dummy warehouse location are taken as zero. Further, the cost element (i.e. 7) on the route city A–Bharat company is replaced by M, since the route is prohibited. The modified table is shown in Table 9.42.

Table 9.42

	Bharat	*Janata*	*Red Lamp*	*Supply*
A	M	10	5	90
B	12	9	4	50
C	7	3	11	80
D	9	5	7	60
Dummy	0	0	0	50
Demand	120	100	110	330

Using the VAM method, the initial solution is shown in Table 9.43. Now to test the optimality of the solution as shown in Table 9.43, we evaluate each unoccupied cell in terms of opportunity cost associated with it in the usual manner. This is shown in Table 9.43.

Table 9.43
Optimal Solution

	Bharat	*Janata*	*Red Lamp*	*Supply*	u_i
A	M M − 13	10 +1	5 90	90	$u_1 = 13$
B	12 30	9	4 20	50	$u_2 = 12$
C	7 (+) 0	3 80 (−)	+12	11 80	$u_3 = 7$
D	9 (−) 40	5 20 (+)	7 +6	60	$u_4 = 9$
Dummy	0 50	0 +4	0 +8	50	$u_5 = 0$
Demand	120	100	110	330	
v_j	$v_1 = 0$	$v_2 = -4$	$v_3 = -8$		

Since the opportunity cost in all the unoccupied cells is positive, the initial solution shown in Table 9.43 is also an optimal solution. The total transport cost associated with this solution is

$$\text{Total cost} = 5 \times 90 + 12 \times 30 + 4 \times 20 + 3 \times 80 + 9 \times 40 + 5 \times 20 = \text{Rs } 1{,}59{,}000$$

(b) Since opportunity cost in cell (C, Bharat), $d_{31} = 0$, there exists an alternative optimal solution:

$$x_{13} = 90,\ x_{21} = 30,\ x_{23} = 20,\ x_{31} = 40,\ x_{32} = 40,\ x_{42} = 60 \text{ and } x_{51} = 50$$

and total cost = Rs 1,59,000

(c) The dual of the given problem is

Maximize $Z = (90\,u_1 + 50\,u_2 + 80\,u_3 + 60\,u_4 + 50\,u_5) + (120\,v_1 + 100\,v_2 + 110\,v_3)$

subject to the constraints

$$\begin{array}{llll} u_1 + v_1 \le M & u_2 + v_1 \le 12 & u_3 + v_1 \le 7 & u_4 + v_1 \le 9 \\ u_1 + v_2 \le 10 & u_2 + v_2 \le 9 & u_3 + v_2 \le 3 & u_4 + v_2 \le 5 \\ u_1 + v_3 \le 5 & u_2 + v_3 \le 4 & u_3 + v_3 \le 11 & u_4 + v_3 \le 7 \end{array}$$

and u_i, v_j unrestricted in sign, for all i and j.

Now by substituting the values of u_is and v_js from the optimal transportation in Table 9.43, we get

$$\text{Maximize } Z = 90 \times 13 + 50 \times 12 + 80 \times 7 + 60 \times 9 + 50 \times 0 + 120 \times 0 + 100 \times -4 + 110 \times -8 = \text{Rs } 1{,}59{,}000$$

which is the same value as obtained earlier.

9.6.4 Prohibited Transportation Routes

Situations like road hazards (snow, flood, etc.), traffic regulations, etc., may arise because of which it is usually not possible to transport goods from certain sources to certain destinations. Such type of problems can be handled but by assigning a very large cost, say M (or ∞) to that route (or cell).

Example 9.15 Consider the problem of scheduling the weekly production of certain items for the next four weeks. The production cost of the item is Rs 10 for the first two weeks and Rs 15 for the last two weeks. The weekly demands are 300, 700, 900 and 800, which must be met. The plant can produce a maximum of 700 units per week. In addition, the company can use overtime during the second and third week. This increases the weekly production by an additional 200 units, but the production cost also increases by Rs 5. Excess production can be stored at a unit cost of Rs 3 per week. How should the production be scheduled so as to minimize the total cost?

Solution The given information is presented as a transportation problem in Table 9.44. The cost elements in each cell are determined by adding the production cost, the overtime cost of Rs 5, and the storage cost of Rs 3. Thus, in the first row, the cost of Rs 3 is added during second week onward. Since the output of any period cannot be used in a period preceding it, the cost element is written in the appropriate cells. A dummy column has been added because the supply exceeds demand.

Week (Origin)	*Production Cost per Week (Destination)*					*Supply*
	I	*II*	*III*	*IV*	*Dummy*	
R_1	10	13	16	19	0	700
R_2	–	10	13	16	0	700
O_2	–	15	18	21	0	200
R_3	–	–	15	18	0	700
O_3	–	–	20	23	0	200
R_4	–	–	–	15	0	700
Demand	300	700	900	800	500	3,200

Table 9.44

Note: R: Regular, *O:* Overtime

The problem can be solved by using MODI method for the usual transportation problem. The solution is left as an exercise for the reader. Degeneracy occurs at the initial stage if initial basic feasible solution is obtained by Vogel's method. Degeneracy may be removed by adding Δ in the cell (R_2, Dummy). Table 9.45 provides the optimal solution.

Table 9.45
Optimal Solution

	I	II	III	IV	Dummy	Supply
R_1	10 (300)	13	16 (200)	19 (100)	0 (100)	700
R_2	–	10 (700)	13 (Δ)	16	0	700
O_2	–	15	18	21	0 (200)	200
R_3	–	–	15 (700)	18	0	700
O_3	–	–	20	23	0 (200)	200
R_4	–	–	–	15 (700)	0	700
Demand	300	700	900	800	500	3,200

The production schedule is given in Table 9.46.

Table 9.46
Production Schedule

Production in Week	*Units*		*For Use in Week*			
			I	II	III	IV
I		700	300	–	200	100
II	R_2	700	–	700	–	–
	O_2	Nil				
III	R_3	700	–	–	700	–
	O_3	Nil				
IV		700	–	–	–	700
Demand			300	700	900	800

The total minimum cost for the optimal production schedule given in Table 9.47 is

$$\text{Total cost} = 10 \times 300 + 16 \times 200 + 19 \times 100 + 10 \times 700 + 15 \times 700 + 15 \times 700 = \text{Rs } 36,100$$

Example 9.16 ABC company wishes to develop a monthly production schedule for the next three months. Depending upon the sales commitments, the company can either keep the production constant, allowing fluctuations in inventory, or inventories can be maintained at a constant level, with fluctuating production. Fluctuating production necessitates working overtime, the cost of which is estimated to be double the normal production cost of Rs 12 per unit. Fluctuating inventories result in inventory carrying cost of Rs 2 per unit. If the company fails to fulfil its sales commitment, it incurs a shortage cost of Rs 4 per unit per month. The production capacities for the next three months are shown in the following table:

Month	*Production Capacity*		*Sales*
	Regular	*Overtime*	
M_1	50	30	60
M_2	50	0	120
M_3	60	50	40

Determine the optimal production schedule. [*Delhi Univ, MBA, 1998, 2001, AMIE 2005*]

Solution The given information is presented as a transportation problem in Table 9.47. The cost elements in each cell are determined as follows:

(i) If items are produced in a month for sales during the same month, there will be no inventory carrying cost. Thus, the total cost will either be the normal production cost or the overtime production cost. Thus the cost elements for cells (R_1, 1), (R_2, 2) and (R_3, 3) are Rs 12 each and for cells (O_1, 1), (O_2, 3) are Rs 24 each.

(ii) If items are produced in a particular month for sales during the subsequent month, in addition to the production costs (normal or overtime) inventory carrying cost at the rate of Rs 2 per month will be incurred.

Cell	*Production Cost (Rs)*	*Inventory Carrying Cost (Rs)*	*Total Cost (Rs)*
$(R_1, 2)$	12	2	14
$(R_1, 3)$	12	4	16
$(O_1, 2)$	24	2	26
$(O_1, 3)$	24	4	28
$(R_2, 3)$	12	2	14

(iii) If the company fails to fulfil its sales commitment, it incurs a shortage cost of Rs 4 per unit per month, in addition to the production costs (normal or overtime), carrying (storage) cost at the rate of Rs 2 per month will be incurred.

Cell	*Production Cost (Rs)*	*Shortage Cost (Rs)*	*Total Cost (Rs)*
$(R_2, 1)$	12	4	16
$(R_3, 2)$	12	4	16
$(R_3, 1)$	12	8	20
$(O_3, 2)$	24	4	28
$(O_3, 1)$	24	8	32

The solution is left as an exercise for the reader. The initial basic feasible solution obtained by Vogel's method (shown in Table 9.47) is also the optimal solution.

	M_1	M_2	M_3	*Dummy*	*Product Supply*
R_1	12 (50)	14	16	0	50
O_1	24 (10)	26 (20)	28	0	30
R_2	16	12 (50)	14	0	50
R_3	20	16 (20)	12 (40)	0	60
O_3	32	28 (30)	24	0 (20)	50
Sales Demand	60	120	40	20	240

Table 9.47

The production schedule is given in Table 9.48.

Table 9.48
Production Schedule

Production in Month		Units	For Use in Month M_1	M_2	M_3
M_1	R_1	50	50	–	–
	O_1	30	10	20	–
M_2	R_2	50	–	50	–
M_3	R_3	60	–	20	40
	O_3	30	–	30	–
		Demand	60	120	40

The total minimum cost for the optimal production schedule given in Table 9.49 is

$$\text{Total cost} = 12 \times 50 + 24 \times 10 + 16 \times 20 + 12 \times 50 + 16 \times 20 + 12 \times 40 + 28 \times 30$$
$$= \text{Rs } 3{,}400.$$

Example 9.17 The following is the information that concerns the operations of the XYZ manufacturing company. The production cost of the company is estimated to be Rs 5 per unit.

	Month 1	Month 2
Units on order	800	1,400
Production Capacity		
Regular time	920	920
Overtime	250	250
Excess cost/unit (overtime)	1.25	1.25
Storage cost/unit	0.50	0.50

Formulate and solve the above problem as transportation problem. [*Delhi Univ., MBA, 2002; AMIE 2004*]

Solution The storage cost of Re 0.50 per unit per month is charged only if production during the month 1 is used for supplies during month 2.

Costs for supplies against first month's order from the previous month have been assumed at infinity as this is treated not only as prohibitive but also as undesirable. This is because the order quantity of first month is even less than a regular time production capacity for the same month.

This problem is unbalanced as net supply is of 2,340 units while the demand is only of 2,200 units. A dummy demand centre of 140 units with supply cost zero is added to solve the problem. The data are summarized in Table 9.49 along with initial solution.

The initial basic solution obtained by Vogel's method (shown in Table 9.49) is updated in order to obtain optimal solution, which is shown in Table 9.50.

The least cost production schedule to meet the sale demand is shown below:

Table 9.49
Initial Solution

	M_1	M_2	Dummy	Supply
M_1	5.0 (800)	5.5 (120)	0	920
$M_1(OT)$	6.25	6.75 (250)	0	250
M_2	∞	5.0 (920)	0	920
$M_2(OT)$	∞	6.25 (110)	0 140	250
Demand	800	1,400	140	

Table 9.50
Optimal Solution

	M_1	M_2	Dummy	Supply
M_1	5.0 (800)	5.5 (120)	0	920
$M_1(OT)$	6.25	6.75 (110)	0 140	250
M_2	∞	5.0 (920)	0	920
$M_2(OT)$	∞	6.25 (250)	0	250
Demand	800	1,400	140	

	Production		Supply	
	M_1	M_2	M_1	M_2
Regular time	920	110	800	230
Overtime	920	250	–	1,170
			800	1,400

9.7 MAXIMIZATION TRANSPORTATION PROBLEM

In general, the transportation model is used for cost minimization problems. However, it is also used to solve problems in which the objective is to maximize total value or benefit. That is, instead of unit cost c_{ij}, the unit profit or payoff p_{ij} associated with each route, (i, j) is given. The objective function in terms of total profit or payoff is then stated as follows:

$$\text{Maximize } Z = \sum_{i=1}^{m} \sum_{j=1}^{n} p_{ij}\, x_{ij}$$

The algorithm for solving this problem is same as that for the minimization problem. However, since we are given profits instead of costs, therefore, a few adjustments in Vogel's approximation method (VAM) for finding initial solution and in the MODI optimality test are required.

For finding the initial solution by VAM, the penalties are computed as difference between the largest and next largest payoff in each row or column. In this case, row and column differences represent payoffs. Allocations are made in those cells where the payoff is largest, corresponding to the highest row or column difference.

Since it is a maximization problem, the criterion of optimality is the converse of the rule for minimization. The rule is: *A solution is optimal if all opportunity costs d_{ij} for the unoccupied cells are zero or negative.*

Example 9.18 A company has four manufacturing plants and five warehouses. Each plant manufactures the same product, which is sold at different prices in each warehouse area. The cost of manufacturing and cost of raw materials are different in each plant due to various factors. The capacities of the plants are also different. The relevant data is given in the following table:

	Plant			
Item	*1*	*2*	*3*	*4*
Manufacturing cost (Rs) per unit	12	10	8	8
Raw material cost (Rs) per unit	8	7	7	5
Capacity per unit time	100	200	120	80

The company has five warehouses. The sale prices, transportation costs and demands are given in the following table:

Warehouse	*Transportation Cost (Rs) per Unit*				*Sale Price*	*Demand per Unit (Rs)*
	1	*2*	*3*	*4*		
A	4	7	4	3	30	80
B	8	9	7	8	32	120
C	2	7	6	10	28	150
D	10	7	5	8	34	70
E	2	5	8	9	30	90

(a) Formulate this problem as a transportation problem in order to maximize profit.
(b) Find the solution using VAM method.
(c) Test for optimality and find the optimal solution. *[ICWA, June 1990]*

Solution Based on the given data, the profit matrix can be derived by using following equation.

Profit = Sales price – Production cost – Raw material cost – Transportation cost

The matrix, so obtained, is shown in Table 9.49.

Table 9.49 representing profit can be converted to an equivalent minimization of loss by subtracting all the profit values in the table from the highest profit value. As the highest profit value is 15, by subtracting all cell values including itself from it, the new values that we obtain are shown in Table 9.50. The problem now becomes a usual cost minimizing transportation problem.

Table 9.49
Profit Matrix

	1	*2*	*3*	*4*	*Dummy*	*Demand*
A	6	6	11	15	0	80
B	4	6	10	12	0	120
C	6	4	7	6	0	150
D	4	10	14	14	0	70
E	8	8	7	9	0	90
Supply	100	200	120	80	10	510

Apply Vogel's method to find the initial basic feasible solution, as shown in Table 9.50.

Table 9.50
Initial Feasible Solution

	1	*2*	*3*	*4*	*Dummy*	*Demand*	u_i
A	9 +7	9 +5	4 +4	0 (80)	15 +12	80	$u_1 = 4$
B	11 +4	9 (70)	5 (–)(50)	3 (+) –2	15 +7	120	$u_2 = 9$
C	9 (100)	11 40	8 +1	11 +4	15 (10)	150	$u_3 = 11$
D	11 +8	5 0	1 (+)(70)	1 (Δ) (–)	15 +11	70	$u_4 = 5$
E	7 +2	7 (90)	8 +5	6 +3	15 +9	90	$u_5 = 7$
Supply	100	200	120	80	10	510	
v_j	$v_1 = -2$	$v_2 = 0$	$v_3 = -4$	$v_4 = -4$	$v_5 = -1$		

Since initially the number of occupied cells was 8, which is one less than the required number, $m + n - 1 = 9$, therefore, the solution is degenerate. However, after making an allocation of Δ to the cell (D, 4), the initial solution has now become eligible for optimality test.

Apply MODI method to evaluate each unoccupied cell in terms of opportunity cost associated with it in the usual manner. This is shown in Table 9.50.

The cell (B, 4) has a negative opportunity cost (i.e. – 2) as shown in Table 9.50. Introduce it into the new solution by constructing a loop shown in Table 9.50. The new solution is given in Table 9.51, where Δ has shifted from cell (D, 4) to cell (B, 4).

	1	2	3	4	Dummy	Demand	u_i
A	9 +5	9 +3	4 +2	0 (80)	15 +5	80	$u_1 = -5$
B	11 +4	9 (70)	5 (50)	3 (Δ)	15 +2	120	$u_2 = -2$
C	9 (100)	11 (40)	8 +1	11 +6	15 (10)	150	$u_3 = 0$
D	11 +8	5 0	1 (70)	1 +2	15 +6	70	$u_4 = -6$
E	7 +2	7 (90)	8 +5	6 +5	15 +4	90	$u_5 = -4$
Supply	100	200	120	80	10	510	
v_j	$v_1 = 9$	$v_2 = 11$	$v_3 = 7$	$v_4 = 5$	$v_5 = 15$		

Table 9.51

Since there is no negative opportunity cost in the unoccupied cells in Table 9.51, therefore, this solution is the optimal solution. However, the zero opportunity cost in cell (D, 2) indicates the existence of an alternative solution. The total maximization profit associated with the solution is

Total profit = 9 × 70 + 5 × 50 + 9 × 100 + 11 × 40 + 15 × 10 + 1 × 70 + 7 × 90 = Rs 4,580.

9.8 TRANS-SHIPMENT PROBLEM

In a transportation problem, the shipment of a commodity takes place among sources and destinations. But instead of direct shipments to destinations, the commodity can be transported to a particular destination through one or more intermediate or trans-shipment points. Each of these points, in turn supply to other points. Thus, when the shipments pass from destination to destination and from source to source, we have a trans-shipment problem. A problem dealing with four sources and three destinations is shown diagrammatically in Figs. 9.2(a) and (b).

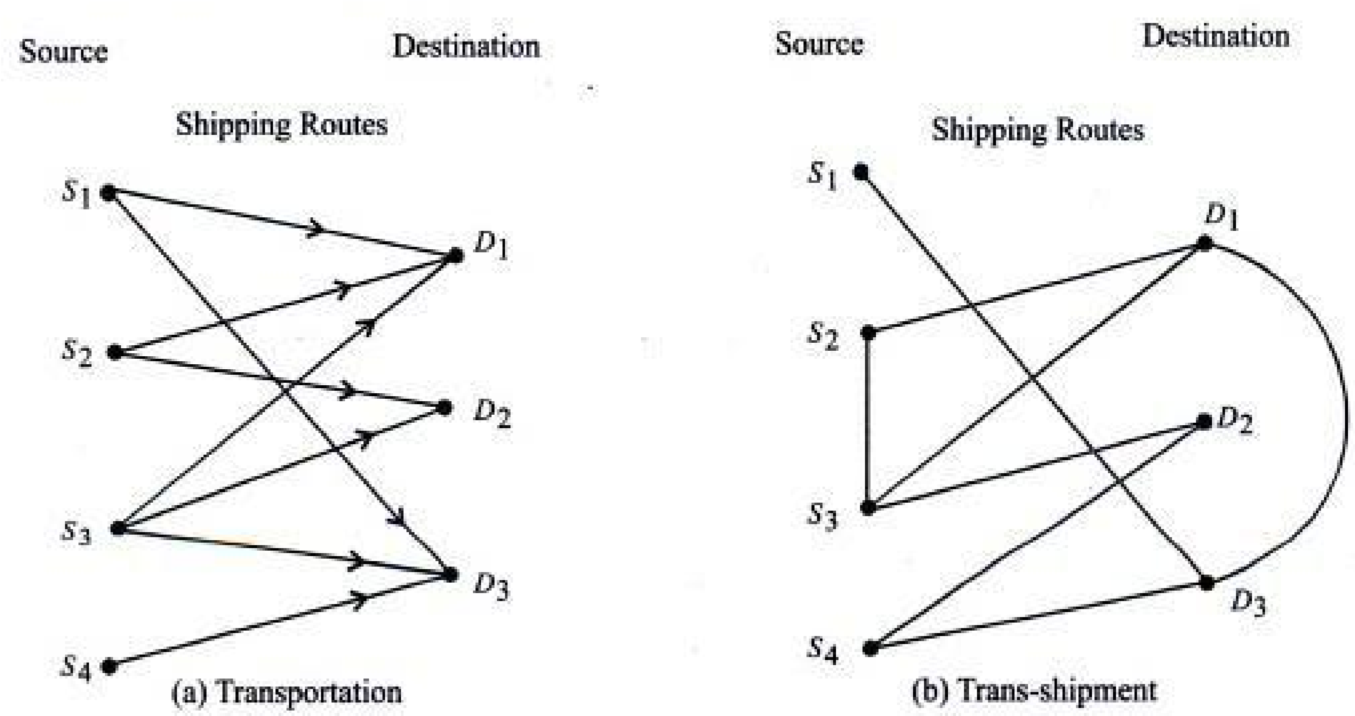


Fig. 9.2

Since the flow of commodity can be in both directions, arrows are not shown in Fig. 9.2(b). The solution to this problem can be obtained by using the transportation model. The solution procedure is as follows: If there are m sources and n destinations, we shall have a transportation table of size $(m + n) \times (m + n)$ instead of $m \times n$ as in the usual case. If the total number of units transported from all sources to all destinations is N, then the given supply at each source and demand at each destination are added to N. The demand at source and the supply at each destination are set to be equal to N. The problem can then be solved by the usual MODI method for transportation problems. In the final solution, ignore the units transported from a point to itself, i.e. diagonal cells, because they do not have any physical meaning (no transportation).

Example 9.19 Consider a firm having two factories. The firm is to ship its products from the factories to three retail stores. The number of units available at factories X and Y are 200 and 300, respectively, while those demanded at retail stores A, B and C are 100, 150 and 250, respectively. Rather than shipping the products directly from factories to retail stores, it is asked to investigate the possibility of trans-shipment. The transportation cost (in rupees) per unit is given in the table:

		Factory		*Retail Store*		
		X	*Y*	*A*	*B*	*C*
Factory	*X*	0	8	7	8	9
	Y	6	0	5	4	3
Retail Store	*A*	7	2	0	5	1
	B	1	5	1	0	4
	C	8	9	7	8	0

Find the optimal shipping schedule.

Solution The number of units available at X and Y are 200 and 300, respectively and the demand at A, B and C is 100, 150 and 250, respectively. The maximum amount which can be transported through a factory or retail store is the total supply and demand, i.e. N = 500 units. If all these 500 units are not transported through a factory or retail store then the remaining units will play the role of dummy. The trans-shipment table is shown in Table 9.52 where 500 units have been added to the supply and demand at factory and to a retail store.

Table 9.52

	X	*Y*	*A*	*B*	*C*	*Supply*
X	0	8	7	8	9	200 + 500
Y	6	0	5	4	3	300 + 500
A	7	2	0	5	1	500
B	1	5	1	0	4	500
C	8	9	7	8	0	500
Demand	500	500	100 + 500	150 + 500	250 + 500	

The initial solution to trans-shipment problem given in Table 9.53 can be obtained by putting 500 units to each route on the diagonal and making allocation in the matrix, by using Vogel's approximation method.

	A	*B*	*C*	*Supply*
X	7	8	9	700
Y	5	4	3	800
Demand	600	650	750	

Applying the MODI method to test the optimality of the solution given in Table 9.53. For this first determine u_i and v_j and then opportunity cost d_{ij} for each unoccupied cell as shown in Table 9.53. Since the opportunity cost corresponding to each unoccupied cell is positive, therefore, the solution given in Table 9.53 is also optimal. In order to interpret the optimal solution, allocations in the diagonal cells are ignored as these values show the extra dummies have been added in order to allow as much as flow possible.

	X	Y	A	B	C	Supply	u_i
X	0 (500)	8 +4	7 (100)	8 (100)	9 +2	700	$u_1 = 4$
Y	6 +10	0 (500)	5 +2	4 (50)	3 (250)	800	$u_2 = 0$
A	7 +14	2 +5	0 (500)	5 +4	1 +1	500	$u_3 = -3$
B	1 +9	5 +9	1 +2	0 (500)	4 +5	500	$u_4 = -4$
C	8 +15	9 +12	7 +7	8 +7	0 (500)	500	$u_5 = -3$
Demand	500	500	600	650	750	3,000	
v_j	$v_1 = -4$	$v_2 = 0$	$v_3 = 3$	$v_4 = 4$	$v_5 = 3$		

Table 9.53
Initial Solution

CONCEPTUAL QUESTIONS C

1. What is meant by unbalanced transportation problem? Explain the method for solving such a problem.
2. Explain how a profit maximization transportation problem can be converted to an equivalent cost minimization transportation problem.
3. What is degeneracy in transportation problems? How is a transportation problem solved when the demand and supply are not equal?
4. (a) Explain how to resolve degeneracy in a transportation problem.
   (b) How does the problem of degeneracy arise in a transportation problem? Explain how one can overcome it.
5. State a transportation problem in general terms and explain the problem of degeneracy. How does one overcome it? *[IAS (Maths), 1990]*
6. Explain a trans-shipment problem.
7. What are the main characteristics of a transshipment problem?
8. Explain how a trans-shipment problem can be solved as a transportation problem.
9. What is a trans-shipment problem? Explain how it can be formulated and solved as a transportation problem. *[Delhi Univ., MBA, 1995, 99, 2000]*
10. Explain the method for solving trans-shipment problem.

SELF PRACTICE PROBLEMS C

1. A steel company has three open hearth furnaces and five rolling mills. The transportation costs (rupees per quintal) for shipping steel from furnaces to rolling mills are given in the following table:

	M_1	M_2	M_3	M_4	M_5	Supply
F_1	4	2	3	2	6	8
F_2	5	4	5	2	1	12
F_3	6	5	4	7	7	14
Demand	4	4	6	8	8	

What is the optimal shipping schedule?

2. Consider the following unbalanced transportation problem.

		To			
		I	*II*	*III*	*Supply*
	A	5	1	7	10
From	B	6	4	6	80
	C	3	2	5	15
	Demand	75	20	50	

Since there is not enough supply, some of the demands at these destinations may not be satisfied. Suppose there are penalty costs for every unsatisfied demand unit which are given by 5, 3 and 2 for destinations I, II and III, respectively. Find the optimal solution.

3. A company produces a small component for an industrial product and distributes it to five wholesalers at a fixed delivered price of Rs 2.50 per unit. The sales forecasts indicate that the monthly deliveries will be 3,000, 3,000, 10,000, 5,000 and 4,000 units to wholesalers I, II, III, IV and V, respectively. The company's monthly production capacities are 5,000, 10,000 and 12,500 at plants 1, 2 and 3, respectively. Respective direct costs of production of each unit are Re 1.00, Re 0.90, and Re 0.80 at plants W, X and Y. Transportation costs of shipping a unit from a plant to a wholesaler are as follows.

Wholesaler

		I	*II*	*III*	*IV*	*V*
	W	0.05	0.07	0.10	0.25	0.15
Plant	*X*	0.08	0.06	0.09	0.12	0.14
	Y	0.10	0.09	0.08	0.10	0.15

Find how many components each plant should supply to each wholesaler in order to maximize its profit.

4. ABC Tool Company has a sales force of 25 men, who operate from three regional offices. The company produces four basic product lines of hand tools. Mr Jain, the sales manager, feels that 6 salesmen are needed to distribute product line *I*, 10 to distribute product line *II*, 4 for product line III and 5 salesmen for product line IV. The cost (in Rs) per day of assigning salesmen from each of the offices for selling each of the product lines are as follows:

Product Lines

		I	*II*	*III*	*IV*
	A	20	21	16	18
Regional Office	*B*	17	28	14	16
	C	29	23	19	20

At the present time, 10 salesmen are allocated to office *A*, 9 to office *B* and 7 salesmen to office *C*. How many salesmen should be assigned from each office to sell each product line in order to minimize costs? Identify alternate optimum solutions, if any. [*Delhi Univ., MBA, 1996*]

5. The Purchase Manager, Mr Shah, of the State Road Transport Corporation must decide on the amount of fuel that should be bought from three possible vendors. The corporation refuels its buses regularly at the four depots within the area of its operations.

The oil companies have said that they can furnish up to the following amounts of fuel during the coming month: 2,75,000 litres by oil company 1; 5,50,000 litres by oil company 2; and 6,60,000 litres by oil company 3. The required amount of the fuel is 1,10,000 litres by depot 1; 2,20,000 litres at depot 2; 3,30,000 litres at depot 3; and 4,40,000 litres at depot 4.

When the transportation costs are added to the bid price per litre supplied, the combined cost per litre for fuel from each vendor, servicing a specific depot, is as under:

	Company 1	*Company 2*	*Company 3*
Depot 1	25.00	24.75	24.25
Depot 2	25.00	25.50	26.75
Depot 3	24.50	26.00	25.00
Depot 4	25.50	26.00	24.50

Determine the optimal schedule. [*Delhi Univ., MBA, 1988, 2000*]

6. A departmental store wishes to purchase the following quantities of sarees:

Types of sarees	:	A	B	C	D	E
Quantity	:	150	100	75	250	200

Tenders are submitted by four different manufacturers who undertake to supply not more than the quantities mentioned below (all types of sarees combined);

Manufacturer	:	W	X	Y	Z
Total quantity	:	300	250	150	200

Sarees

		A	*B*	*C*	*D*	*E*
	W	275	350	425	225	150
Manufacture	*X*	300	325	450	175	100
	Y	250	350	475	200	125
	Z	325	275	400	250	175

How should the orders be placed?

7. A company has four factories F_1, F_2, F_3 and F_4 that manufacture the same product. Production and raw material costs differ from factory to factory and are given in the following table in the first two rows. The transportation costs from the factories to the sales depots, S_1, S_2 and S_3 are also given. The last two columns in the table give the sales price and the total requirement at each depot. The production capacity of each factory is given in the last row.

		F_1	F_2	F_3	F_4	*Sales Price per Unit*	*Requirement*
Production cost/unit		15	18	14	13		
Raw material cost/unit		10	9	12	9		
Transportation cost/unit	S_1	3	9	5	4	34	80
	S_2	1	7	4	5	32	120
	S_3	5	8	3	6	31	150
Supply		50	150	50	100		

Determine the most profitable production and distribution schedule and the corresponding profit. The deficit production should be taken to yield zero profit. Write the dual of this transportation problem and use it for checking the optimal solution. [*Delhi Univ., MBA, 1996*]

8. Link Manufacturing Company has several plants, three of which manufacture two principal products – a standard card table and deluxe card table. A new deluxe card table will be introduced, which must be considered in terms of selling price and costs. The selling prices are: Standard, Rs 14.95, deluxe Rs 18.85 and New Deluxe, Rs 21.95.

		Variable Costs (in Rs)				
Model	*Qty.*	*Plant A*	*Plant B*	*Plant C*	*Plant*	*Capacity*
Standard	450	8.00	7.95	8.10	A	800
Deluxe	1,050	8.50	8.60	8.45	B	600
New Deluxe	600	9.25	9.20	9.30	C	700

Solve this problem by the transportation technique for the maximum contribution.

9. City Super Market keeps five different patterns of a particular size of readymade garment for sale. There are four manufacturers

available to the manager of the market to whom be can order the required quantity. The demand for the coming season and the maximum quantity that can be produced by a manufacturer is as follows:

Pattern	1 U. Cut	2 P. Form	3 Maxi	4 Mini	5 Midi
Quantity required	200	150	200	150	300
Manufacturer	A	B	C	D	
Capacity	200	300	250	250	

The following quotations have been submitted by the manufacturers for different patterns:

	1 U. Cut	2 P. Form	3 Maxi	4 Mini	5 Midi
A	Rs 100	Rs 130	Rs 160	Rs 70	Rs 140
B	100	120	150	75	130
C	115	140	155	80	120
D	125	145	140	60	125

The super market is selling different patterns at the following prices:

Pattern	:	1	2	3	4	5
Selling Price (Rs)	:	120	150	170	80	140

How should the super market manager place the order? Write the dual of this transportation problem and give its economic interpretation.

10. A company wishes to determine an investment strategy for each of the next four years. Five investment types have been selected. The investment capital has been allocated for each of the coming four years, and maximum investment levels have been established for each investment type. There is an assumption that amounts invested in any year will remain invested until the end of the planning horizon of four years. The following table summarizes the data of this problem. The values in the body of the table represent the net return on investment of one rupee up to the end of the planning horizon. For example, a rupee invested in investment type B at the beginning of year will grow to Rs 1.90 by the end of the fourth year, yielding a net return of Re 0.90.

Investment Made at the Beginning of Year	Investment Type A	B	C	D	E	Rupees Available (in '000s)
1	0.80	0.90	0.60	0.75	1.00	500
2	0.55	0.65	0.40	0.60	0.50	600
3	0.30	0.25	0.30	0.50	0.20	750
4	0.15	0.12	0.25	0.35	0.10	800
Maximum rupees investment (in '000s)	750	600	500	800	1,000	

The objective in this problem is to determine the amount to be invested at the beginning of each year in an investment type so as to maximize the net rupee return for the four year period. Solve the above transportation problem and get an optimal solution. Also calculate the net return on investment for the planning horizon of four-year period. *[CA, May 1993]*

11. XYZ Company has provided the following data and seek your advice on the optimum investment strategy:

Investment Made at the Beginning of Year	Net Return Data (in paise) of Selected Investment P	Q	R	S	Amount Available (lacs)
1	95	80	70	60	70
2	75	65	60	50	40
3	70	45	50	40	90
4	60	40	40	30	30
Max. Investment (Rs lakh)	40	50	60	60	–

The following additional information is also provided:

(a) P, Q, R and S represent the selected investments.
(b) The company has decided to have a four years investment plan.
(c) The policy of the company is that the amount invested in any year will remain so until the end of fourth year.
(d) The values (paise) in the table represent net return on investment of one rupee till the end of the planning horizon (for example, a rupee in investment P at the beginning of year will grow to Rs 1.95 by the end of the fourth year, yielding a return of 95 paise).

Using the above, determine the optimum investment strategy. *[CA, Nov. 1996]*

12. A manufacturer must produce a certain product in sufficient quantity in order to meet contracted sales of the next four months. The production facilities available for this product are limited, and vary in different months. The unit cost of production also changes accordingly to the facilities and personnel available. The product may be produced in one month and then held for sale in a later month, at an estimated storage cost of Re 1 per unit per month. No storage cost is incurred for goods that are sold in the same month in which they are produced. Presently there is no inventory of this product and none is desired at the end of four months. Given the following table, show how much to produce in each of four months in order to minimize total cost.

Month	Contracted Sales (in units)	Maximum Production (in units)	Unit Cost of Production (Rs)	Unit Storage Cost per Month (Rs)
1	20	40	14	1
2	30	50	16	1
3	50	30	15	1
4	40	50	17	1

Formulate the problem as a transportation problem and solve it.

13. A company has factories at A, B and C, which supply its products to warehouses at D, E, F and G. The factory capacities are 230, 280 and 180, respectively for regular production. If overtime production is utilized, the capacities can be increased to 300, 360 and 190, respectively. Increment unit overtime costs are Rs 5, Rs 4 and Rs 6, respectively. The current warehouse requirements are 165, 175, 205 and 165, respectively. Unit shipping costs in rupees between the factories and the warehouses are:

		To D	E	F	G
	A	6	7	8	10
From	B	4	10	7	6
	C	3	22	2	11

Determine the optimum distribution for the company to minimize costs.

14. The products of three plants X, Y and Z are to be transported to four warehouses I, II, III and IV. The cost of transportation of each unit from plants to the warehouses along with the normal capacities of plants and warehouses are indicated below:

		Warehouse I	II	III	IV	Available
	X	25	17	25	14	300
Plant	Y	15	10	18	24	500
	Z	16	20	8	13	600
Required		300	300	500	500	

(a) Solve the problem for minimum cost of transportation. Are there any alternative solutions? If any, explain the methodology.
(b) Overtime can be used in each plant to raise the capacity by 50 per cent of the normal but the corresponding cost of trans-shipment will also increase by 10, 15 and 20 to the unit costs of production at each plant.

15. A firm manufacturing industrial chemicals has got 3 plants P_1, P_2 and P_3 each having capacities to produce 300 kg, 200 kg, and 500 kg, respectively of a particular chemical per day. The production costs per kg in plants P_1, P_2 and P_3, respectively are Re 0.70, Re 0.60 and Re 0.66. Four bulk consumers have placed orders for the products on the following basis:

Consumer	*kg Required per Day*	*Price Offered Rs/kg*
I	400	1.00
II	250	1.00
III	350	1.02
IV	150	1.03

Shipping costs (paise per kg) from plants to consumers are given in the table below:

		Consumer C_1	C_2	C_3	C_4
	P_1	3	5	4	6
Plant	P_2	8	11	9	12
	P_3	4	6	2	8

Work out an optimal schedule for the above situation. Under what conditions would you change schedule?

16. The personnel manager of a manufacturing company is in the process of filling 175 jobs in six different entry level skills due to the establishment of a third shift by the company. Union wage scale and requirement for the skills are shown in the following table:

Pay Scales and Skills Requirements

Entry Level Skills	A	B	C	D	E	F
Wage scale Rs/month	1,000	1,100	1,200	1,300	1,400	1,500
No. required	25	29	31	40	33	17

230 applicants for the jobs have been tested and their aptitudes and skills for the jobs in question have been matched against company standards and evaluated. The applicants have been grouped into four categories by their abilities; the grouping and values of each category to the company are shown in the table below:

Applicant	*Category Value (Rs per month)*						*Number of*
Category	*A*	*B*	*C*	*D*	*E*	*F*	*Applicants*
I	1,000	1,100	1,500	1,400	1,400	1,450	54
II	1,200	1,250	1,200	1,350	1,400	1,400	54
III	1,000	1,100	1,200	1,400	1,500	1,600	45
IV	1,500	1,500	1,600	1,400	1,400	1,500	74

How many applicants of each category should the personnel manager hire and for which jobs?

17. Debonair Private Ltd. is in the business of manufacturing and selling office shirts for men. It has four factories located in different parts of the country and the monthly capacities of the factories in thousand are as given below. The shirts are made in a few standard designs and colours, and each factory can make all types of shirts in any size subject to the overall capacity of the factory.

Factories	:	I	II	III	IV
Monthly capacity	:	3	4.5	2.5	5

From the factories, the shirts are transported to five warehouses located in five different regions in India. The warehouses in turn supply to the distributors and the retailers. The monthly demand of shirts (in thousand) from the warehouses is as follows:

Warehouses	:	A	B	C	D	E
Monthly capacity	:	3	5	1.5	2	2.5

The cost of transporting a shirt from a factory to a warehouse depends on the distance between them and the cost of transporting a shirt from each factory to each warehouse is given in the table below:

		Warehouse *A*	*B*	*C*	*D*	*E*
	I	6	3	4	2	5
	II	11	7	5	10	9
Factory	*III*	10	7	1	2	8
	IV	12	10	5	3	5

How many shirts are to be produced, in which factory, and how are these to be despatched to the warehouse so that the total cost involved in transportation is minimized.
(a) Use the North-West Corner Method to get an initial feasible solution.
(b) Check if the solution obtained in (a) above is an optimal allocation and if not, then find the optimal solution.

18. A manufacturer of jeans is interested in developing an advertising campaign that will reach four different age groups. Advertising campaigns can be conducted through TV, radio and magazines. The following table gives the estimated cost in paise per exposure for each age group according to the medium employed. In addition, maximum exposure levels possible in each of the media, namely TV, radio and magazines are 40, 30 and 20 millions, respectively. Also the minimum desired exposure within each age group, namely 13–18, 19–25, 26–35, 36 and olders are 30, 25, 15 and 10 millions. The objective is to minimize the cost of attaining the minimum exposure level in each age group.

Media	*Age Groups*			
	13–18	*19–25*	*26–35*	*36 and older*
TV	12	7	10	10
Radio	10	9	12	10
Magazine	14	12	9	12

(a) Formulate the above as a transportation problem, and find the optimal solution.

(b) Solve this problem if the policy is to provide at least 4 million exposures through TV in the 13–18 age group and at least 8 million exposures through TV in the age group 19–25. [*CA, May 1991*]

19. Two drug companies have inventories of 1.1 and 0.9 million doses of a particular flu vaccine, and an epidemic of the flu seems imminent in three states. Since the flu could be fatal to senior citizens, it is imperative that they be vaccinated first; others will be vaccinated on a first-come-first-served basis while the vaccine supply lasts. The amounts of vaccine (in millions of doses) each state estimates it could administer are as follows:

	State 1	*State 2*	*State 3*
Elders	0.325	0.260	0.195
Others	0.750	0.800	0.650

The shipping costs (paise per dose) between drug companies and states are as follows:

	State 1	*State 2*	*State 3*
Company *1*	30	30	60
Company *2*	10	40	70

Determine a minimum-cost shipping schedule which will provide each state with at least enough vaccine to care for its senior citizens. Write the dual of this transportation problem and use it for checking the optimal solution.

[*Delhi Univ., MBA (HCA), 1990, 95*]

20. A leading firm has three auditors. Each auditor can work up to 160 hours during the next month, during which time three projects must be completed. Project 1 will take 130 hours, project 2 will take 140 hours, and project 3 will take 160 hours. The amount per hour that can be billed for assigning each auditor to each project is given in the table.

Auditor	*Project*		
	1(Rs)	*2(Rs)*	*3(Rs)*
1	1,200	1,500	1,900
2	1,400	1,300	1,200
3	1,600	1,400	1,500

Formulate this as a transportation problem and find the optimal solution. Also find out the maximum total billing during the next month. [*CA, May 1995*]

Degeneracy

21. Obtain an optimum basic feasible solution to the following degenerate transportation problem.

		To			
		A	*B*	*C*	*Supply*
	X	7	3	4	2
From	*Y*	2	1	3	3
	Z	3	4	6	5
Demand		4	1	5	

22. A manufacturer wants to ship 8 loads of his product as shown in the table. The matrix gives the mileage from origin to destination. Shipping costs are Rs 10 per load per mile. What shipping schedule should be used?

	D_1	D_2	D_3	*Supply*
O_1	50	30	220	1
O_2	90	45	170	3
O_3	250	200	50	4
Demand	4	2	2	

Trans-shipment

23. Consider the following trans-shipment problem with two sources and three destinations, the cost for shipments (in rupees) is given below.

		Source		*Destination*			*Supply*
		S_1	S_2	D_1	D_2	D_3	
Source	S_1	0	80	10	20	30	100 + 300
	S_2	10	0	20	50	40	200 + 300
Destination	D_1	20	30	0	4	10	300
	D_2	40	20	10	0	20	300
	D_3	60	70	80	20	0	300
Demand		300	300	100 + 300	100 + 300	100 + 300	

Determine the optimal shipping schedule.

24. A firm having two sources, S_1 and S_2 wishes to ship its product to two destinations, D_1 and D_2. The number of units available at S_1 and S_2 are 10 and 30 and the product demanded at D_1 and D_2 are 25 and 15 units, respectively. The firm instead of shipping from sources to destinations, decides to investigate the possibility of transshipment. The unit transportation cost (in Rs) is given in the following table:

		Source		*Destination*		*Supply*
		S_1	S_2	D_1	D_2	
Source	S_1	0	3	4	5	10 + 40
	S_2	3	0	3	5	30 + 40
Destination	D_1	4	3	0	2	40
	D_2	5	5	2	0	40
Demand		40	40	25 + 40	15 + 40	

Determine the optimal shipping schedule.

HINTS AND ANSWERS

1. Total requirement (30) < Total capacity (34), add a dummy mill with requirement (34 – 30) = 4. Degeneracy occur at the initial solution (VAM).

Ans. $x_{12} = 4$, $x_{14} = 4$, $x_{24} = 2$, $x_{25} = 8$, $x_{31} = 4$, $x_{33} = 6$ and $x_{36} = 4$. Total cost = Rs 80.

2. Demand (145) > Supply (105), add dummy source with supply (145 – 105) = 40 and transportation costs 5, 3 and 2 for destination 1, 2 and 3, respectively.
Ans. $x_{12} = 0, x_{21} = 60, x_{22} = 10, x_{23} = 10, x_{31} = 5$ and $x_{43} = 40$. Transportation cost = Rs 515. Penalty for transportation of 40 units to destination 3 at the cost of Rs 2 per unit = Rs 80. Thus, total cost = 515 + 80 = Rs 595.
3. Maximization as well as unbalanced problem. Total capacity (27,500) > demand (25,000). Add a dummy wholesaler with demand 2,500 units. Since direct costs of production of each unit are Re 1, Re 0.90 and Re 0.80 at plants 1, 2 and 3, add this cost to each figure row wise to get exact data of the problem.
Ans. x_{11} = 2,500, x_{21} = 500, x_{22} = 3,000, x_{23} = 2,500, x_{25} = 4,000, x_{33} = 7,500 and x_{34} = 5,000. Total cost Rs 23,730.
Since in total 25,000 units are supplied to the wholesalers at a fixed price of Rs 2.50 per unit, therefore, the total cost is Rs 62,500.
The net maximum profit to the manufacturer = Rs 62,500 – Rs 23,730 = Rs 38,770
4. Supply (26) > Demand (25), add dummy column with demand 1.
Ans. $x_{12} = 4, x_{13} = 1, x_{14} = 5; x_{21} = 6, x_{23} = 3, x_{32} = 2, x_{35} = 1$ Total cost = Rs 472
Alternative solution exists because opportunity cost is zero in cell (B, 4) and (C, 4).
5. Supply (14,85,000) > Demand (11,00,000), add dummy column with demand of 3,85,000 litres of oil; $x_{13} = 110, x_{21} = 55, x_{22} = 165, x_{31} = 220, x_{33} = 110, x_{43} = 440, x_{52} = 385$. Total cost = Rs 51,700.
6. Profit maximization problem. First add a dummy column with demand of 125 sarees and then subtract all elements of the profit matrix from the highest element 475.

Manufacturer	*Variety*	*Quantity*
W	*B, D, E*	25, 50, 200
X	*A*	150
Y	*B, C*	75, 75
Z	*D*	200

7. Profit maximization problem. First add dummy row with supply of 40 units and then subtract all elements of the profit matrix from highest element 8.

Profit = Sales price – (Production cost + Raw material cost + Transportation cost)

Factory	*Sales Depot*	*Unit Profit*
F_1	S_2	6
F_2	S_2, S_3	– 2, – 4
F_3	S_3	2
F_4	S_1, S_2	8, 5

Total profit = Rs 2,000

12. Transportation cost table and optimal solution

From/To	*1*	*2*	*3*	*4*	*Dummy*	*Supply*
1	14 (20)	15 (20)	16	17	0	40
2	∞	16 (10)	17 (20)	18	0 (20)	50
3	∞	∞	15 (30)	16	0	30
4	∞	∞	∞	17 (40)	0 (10)	50
Demand	20	30	50	40	30	170

13. Transportation cost table and optimal solution

	D	*E*	*F*	*G*	*Dummy*	*Supply*
A	6 (40)	7 (175)	8 (15)	10	0	230
B	4 (115)	10	7	6 (165)	0	280
C	3	22	2 (180)	11	0	180
A_1	11	12	13	15	0 (70)	70
B_1	8 (10)	14	11	10	0 (70)	80
C_1	9	28	8 (10)	17	0	10
Demand	165	175	205	165	140	850

22. Using NWCM, initial solution is degenerate.
Ans. x_{13} = 2, x_{22} = 1, x_{23} = 2, x_{31} = 4 and x_{33} = 1, Total cost = Rs 33.

23. *Ans.* x_{12} = 10, x_{13} = 20, x_{15} = 10, x_{21} = 20, x_{31} = 5, x_{34} = 30, x_{35} = 5, Total cost = Rs 3,600.

CHAPTER SUMMARY

Transportation problems (and their variants) are special types of linear programming problems that have a variety of important applications.

A transportation problem is concerned with distributing a commodity from given sources to their respective destinations. Each source has a fixed supply of the commodity and each destination has a fixed demand. A basic assumption is that the cost of distribution from each source to each destination is directly proportional to the amount distributed. Formulating a transportation problem requires constructing a special purpose table that gives the unit costs of distribution, the supplies and the demands.

The objective of this chapter is to enable readers to recognize a problem that can be formulated and analyzed as a transportation problem or as a variant of one of these problem types.

CHAPTER CONCEPTS QUIZ

True or False

1. The test for degenecay of a solution is to check if there are unused cells with an improvement index equal to zero. If so, degeneccy exists.
2. The advantage of the most method over the stepping stone method of computing improvement indices for unused cells lies in the greater computational efficiency.
3. If you have to solve a transportation problem with *m* rows and n columns using the simplex method, you would have to formulate the problem with *m* variables and *n* constraints.
4. A solution to the cost minimization transportation problem is optimal when all unused cells have an improvement index which is non-negative.
5. A dummy row or column is introduced in the transportation method is order to handle an unbalanced problem. The dummy serve, the same purpose as a slack variable in the simplex method.
6. Each iteration of the transportation method involves the elimination of one occupied cell and the introduction of one unoccupied cell which is similar to a pivot in the simplex method.
7. In the transportation problem, extra constraint equation can not be derived from any other constraint equations as it affects the feasible solution of the problem.
8. In the north-west corner method, the cost of transportation on any route of transportation is taken into account.
9. The unbalanced transportation problem can be balanced by adding a dummy supply row or a demand column as per the need.
10. In the transportation problem, the rim requirement for a row is the capacity of a supplier, white a rim requirement for a column is the demand of a user.

Fill in the Blanks

11. The northwest corner rule provides a __________ for obtaining an __________ solution to the transportation problem.
12. The degnecacy may occurs when there are two or more cells with the same smallest __________ value in a closed __________ for an incoming cell.
13. We have alternative optimal solutions to a minimization transportation problem whenever we find a solution where the improvement indices are all __________ with at least __________ to zero.
14. An improvement index in the transportation method is analogous to a value in the quantity column in the __________ method.
15. The __________ serves the same purpose for the transportation method as all slack variables in the simplex method.
16. An unused cell in the transportation table is analogous to a variable not in the solution column in the simplex method.
17. The end result of an iteration in the transportation method is the same as the end result in the simplex method in that value of the solution is __________.
18. In Vogel's approximation method, each allocation is made on the basis of the __________ cost that would have been incurred if allocation in certain cells with __________ unit transportation cost were missed.
19. __________ method is based on the concept of duality.
20. If the total supply __________ total demand, then an additional column known as __________ added to the transportation table to absorb the same.

Multiple Choice

21. The initial solution of a transportation problem can be obtained by applying any known method. However, the only condition is that
    (a) the solution be optimal
    (b) the rim conditions are satisfied
    (c) the solution not be degenerate
    (d) all of the above
22. The dummy source or destination in a transportation problem is added to
    (a) satisfy rim conditions
    (b) prevent solution from becoming degenerate
    (c) ensure that total cost does not exceed a limit
    (d) none of the above
23. The occurrence of degeneracy while solving a transportation problem means that
    (a) total supply equals total demand
    (b) the solution so obtained is not feasible
    (c) the few allocations become negative
    (d) none of the above
24. An alternative optimal solution to a minimization transportation problem exists whenever opportunity cost corresponding to unused route of transportation is:

(a) positive and greater than zero
(b) positive with at least one equal to zero
(c) negative with at least one equal to zero
(d) none of the above

25. One disadvantage of using North-West Corner Rule to find initial solution to the transportation problem is that
(a) it is complicated to use
(b) it does not take into account cost of trans-portation
(c) it leads to a degenerate initial solution
(d) all of the above

26. The solution to a transportation problem with *m*-rows (supplies) and *n*-columns (destination) is feasible if number of positive allocations are
(a) $m + n$
(b) $m \times n$
(c) $m + n - 1$
(d) $m + n + 1$

27. The calculation of opportunity cost in the MODI method is analogous to a
(a) $c_j - z_j$ value for non-basic variable columns in the simplex method
(b) value of a variable in $\mathbf{x_B}$-column of the simplex method
(c) variable in the **B**-column in the simplex method
(d) none of the above

28. An unoccupied cell in the transportation method is analogous to a
(a) $c_j - z_j$ value in the simplex table
(b) variable in the **B**-column in the simplex table
(c) variable not in the **B**-column in the simplex table
(d) value in the $\mathbf{x_B}$-column in the simplex table

29. If an opportunity cost value is used for an unused cell to test optimality, it should be
(a) equal to zero
(b) most negative number
(c) most positive number
(d) any value

30. During an iteration while moving from one solution to the next, degeneracy may occur when
(a) the closed path indicates a diagonal move
(b) two or more occupied cells are on the closed path but neither of them represents a corner of the path.
(c) two or more occupied cells on the closed path with minus sign are tied for lowest circled value
(d) either of the above

31. The large negative opportunity cost value in an unused cell in a transportation table is chosen to improve the current solution because
(a) it represents per unit cost reduction
(b) it represents per unit cost improvement
(c) it ensure no rim requirement violation
(d) none of the above

32. The smallest quantity is chosen at the corners of the closed path with negative sign to be assigned at unused cell because
(a) it improve the total cost
(b) it does not disturb rim conditions
(c) it ensure feasible solution
(d) all of the above

33. When total supply is equal to total demand in a transportation problem, the problem is said to be
(a) balanced
(b) unbalanced
(c) degenerate
(d) none of the above

34. Which of the following methods is used to verify the optimality of the current solution of the transportation problem,
(a) Least cost method
(b) Vagel's approximation method
(c) Modified distribution method
(d) all of the above

35. The degeneracy in the transportation problem indicates that
(a) dummy allocation (s) needs to be added
(b) the problem has no feasible solution
(c) the multiple optimal solution exist
(d) (a) and (b) but not (c)

Answers to Quiz

1. F 2. T 3. F 4. T 5. T 6. T 7. F 8. F 9. T 10. T
11. mechanism; initial 12. negative; path 13. non-negative; equals 14. simplex
15. north west corner rule 16. product mix 17. improved 18. opportunity; minimum
19. modified, distribution 20. exceed; dummy demand centre
21. (b) 22. (a) 23. (b) 24. (b) 25. (b) 26. (c) 27. (a) 28. (c) 29. (b) 30. (c)
31. (a) 32. (c) 33. (a) 34. (c) 35. (d)

CASE STUDY

Case 9.1: Asian Games*

For maintaining law and order during the Asian Games, 1982 police force has been requisitioned from the various central police organizations like CRPF, BSF, ITBP and other states such as Haryana, Himachal Pradesh, Uttar Pradesh, Orissa and Gujarat. Besides this there are 10 batallions of Delhi Police available for deployment. The outside and the local force have been stationed at various places in the Union Territory of Delhi. The figures are in the number of sections (10 Men) available for deployment.

Place	*Abbreviation*	*Force (Section)*
New Police Lines	NPL	400
Jharoda Kalan	JK	160
Pitam Pura	PP	160
Model Town	MT	90
Police Training School	PTS	90
Old Police Lines	OPL	120
New Kotwali	NK	20
Kamala Market	KMT	120
Parliament Street	PST	240
Rajpura Lines	RPL	120
Shakarpur	SP	120
Moti Nagar	MN	120
Mehram Nagar	MRN	240
		Total 2,000

The force is to be deployed at various stadia on the day of events. The requirement of the force has been calculated keeping in view the factors like capacity of the stadium, traffic problems, security of VIPs and participants etc. The requirement of force is also mentioned in number of sections.

Stadium	*Abbreviation*	*Force (Section)*
Chhatrasal	CS	60
Delhi University	DU	40
Indraprastha	IP	400
Yamuna Velodrome	YV	40
Talkatora Indoor	TKI	80
Talkatora Swimming	TKS	80
National	NS	240
Hall of sports	HS	60
Harbaksh	HB	60
Shivaji	SV	60
Jawaharlal Nehru	JLN	500
Hauz Khas	HK	20
Games Village	GV	60
Ambedkar	AK	200
Tughlakabad Range	TR	40
Golf Club	GC	20
Karnail Singh	KS	40
		Total 2,000

The cost of transportation of one section, i.e. 10 Men from every location to each stadium has been calculated as shown in the matrix below.

It is assumed that the availability and requirement of force is equal, i.e. 2,000 sections. It will not be possible on any day during the games as all stadia are not having games on all the days. Hence, there will always be a surplus force available. Consequently dummy stadium will have to be planned everyday so as to absorb the unutilized force. The cost (in Rs 100s for 10 persons – one section) matrix for the movement of the police force is given below:

* Based on the class assignment prepared by the MBA(FMS) student Mr M.S Upadhya, IPS.

	CS	*DU*	*IP*	*YV*	*TKI*	*TKS*	*NS*	*HS*	*HB*	*JLN*	*SV*	*HK*	*GV*	*AK*	*TR*	*GC*	*KS*	*Force Availability*
NPL	2	1	13	12	15	15	16	14	25	20	14	22	24	11	35	16	10	400
JK	28	31	36	37	130	30	35	36	16	42	30	35	36	35	50	35	36	160
PP	15	14	26	25	28	28	28	17	38	33	17	35	37	24	48	29	23	160
MT	4	3	15	14	17	17	18	16	27	22	16	24	26	13	37	18	12	90
PTS	26	24	15	16	13	13	12	15	15	12	15	33	3	16	10	17	16	90
OPL	7	2	8	7	10	10	11	13	20	15	9	17	13	6	30	11	6	120
NK	11	6	4	3	6	6	7	9	16	11	5	13	15	2	26	7	4	20
KMT	13	8	3	4	4	4	5	5	21	16	2	25	25	1	24	10	2	120
PST	15	12	5	6	2	2	4	5	16	10	12	12	12	4	22	4	3	240
RPL	4	1	11	10	13	13	14	14	23	18	12	20	22	9	33	14	8	120
SP	19	16	3	4	7	7	6	5	21	12	8	18	18	5	30	8	8	120
MN	10	12	16	17	13	13	15	16	20	20	15	25	25	14	35	18	12	120
MRN	30	28	18	19	15	15	15	16	6	22	15	12	12	20	20	20	20	240
Force Requirement	60	40	400	40	80	80	180	240	60	60	500	60	20	200	40	20	40	2,000

Suggest an optimal transportation schedule of the police force so as to minimize the total transportation cost.

APPENDIX: THEOREMS AND RESULTS

Theorem 9.1 (*Existence of Feasible Solution*) A necessary and sufficient condition for the existence of a feasible solution to the transportation problem is

$$\sum_{i=1}^{m} a_i = \sum_{j=1}^{n} b_j \quad \text{(Rim condition)}$$

That is, the total capacity (or supply) must equal total requirement (or demand).

Proof (a) *Necessary Condition* Let there exist a feasible solution to the transportation problem. Then, we have

$$\sum_{i=1}^{m} \sum_{j=1}^{n} x_{ij} = \sum_{i=1}^{m} a_i \tag{1}$$

and

$$\sum_{j=1}^{n} \sum_{i=1}^{m} x_{ij} = \sum_{j=1}^{n} b_j \tag{2}$$

Since the left-hand side of (1) and (2) are the same, therefore $\sum_{i=1}^{m} a_i = \sum_{j=1}^{n} b_j$

(b) *Sufficient Condition* Suppose

$$\sum_{i=1}^{m} a_i = \sum_{j=1}^{n} b_j = k \text{ (say)}$$

If there exists a real number $\lambda_i \neq 0$ such $x_{ij} = \lambda_i b_j$ for all i and j, then value of λ_i is given by

$$\sum_{j=1}^{n} x_{ij} = \sum_{j=1}^{n} \lambda_i b_j = \lambda_i \sum_{j=1}^{n} b_j = k\lambda_i \quad \text{or} \quad \lambda_i = \frac{1}{k} \sum_{j=1}^{n} x_{ij} = \frac{a_i}{k}$$

Thus

$$x_{ij} = \lambda_i b_j = \frac{a_i b_j}{k} \text{ for all } i \text{ and } j.$$

Since $a_i > 0$ and $b_j > 0$ for all i and j, therefore $a_i b_j / k \geq 0$ and hence a feasible solution exists, i.e. $x_{ij} \geq 0$.

Theorem 9.2 (*Basic Feasible Solution*) The number of basic variables (positive allocations) in any basic feasible solution are $m + n - 1$ (the number of independent constraint equations) satisfying all the rim conditions.

Proof In the mathematical model of a transportation problem it can be seen that there are m rows (capacity or supply constraint equations) and n columns (requirement or demand constraint equations). Thus there are in total $m + n$ constraint equations. But because of Theorem 9.1 (total capacity be equal to the total requirement) out of $m + n$ constraint equations one of the equations is redundant and can also be eliminated. Thus there are $m + n - 1$ linearly independent equations. It can be verified by adding all the m rows equations and subtracting from the sum the first $n - 1$ column equations, thereby getting the last column equation. That is,

$$\sum_{i=1}^{m} \sum_{j=1}^{n} x_{ij} - \sum_{j=1}^{n-1} \sum_{i=1}^{m} x_{ij} = \sum_{i=1}^{m} a_i - \sum_{j=1}^{n-1} b_j$$

$$\sum_{i=1}^{m} \sum_{j=1}^{n} x_{ij} - \left[\sum_{j=1}^{n} \sum_{i=1}^{m} x_{ij} - \sum_{i=1}^{m} x_{in} \right] = \sum_{i=1}^{m} a_i - \left[\sum_{j=1}^{n} b_j - b_n \right]$$

$$\sum_{i=1}^{m} x_{in} = b_n; \qquad \text{since } \sum_{i=1}^{m} a_i = \sum_{j=1}^{m} b_j$$

Triangular Basis We know that the number of basic variables are equal to the number of constraints in linear programming. In the same way when capacity and requirement constraint equations are expressed in terms of basic variables and all non-basic variables are given zero value, the matrix of coefficients of variables in the system of equations is triangular, that is, there is an equation in which a single basic variable occurs; in the second equation one more basic variable is added but the total number of variables does not exceed two; similarly, in the third equation one more basic variable occurs, but total does not exceed three, and so on.

Theorem 9.3 The transportation problem has a triangular basis.

Proof To prove this theorem, consider capacity and requirement constraint equations represented in the tabular form (Table 9.1).

In the system of capacity and requirement constraint equations, every equation has a basic variable otherwise the equation cannot be satisfied for $a_i \neq 0$ or $b_j \neq 0$. Suppose, every row and column equation has at least two basic variables. Since there are m rows and n columns, therefore, the total number of basic variables in row equations and column equations will be at least $2m$ and $2n$, respectively. If N be the total number of basic variables, then obviously $N \geq 2m$, $N \geq 2n$. Now three cases may arise:

Case I : If $m > n$, then $m + m > m + n$, or $2m > m + n$. Thus $N \geq 2m \geq m + n$.

Case II : If $m < n$, then $m + n < n + n$ or $m + n < 2n$. Thus $N \geq 2n > m + n$.

Case III : If $m = n$, then $m + m = m + n$ or $2m = m + n$. Thus $N \geq 2m = m + n$.

In each of these cases, we observed that $N > m + n$. But the number of basic variables in the transportation problem are $N = m + n - 1$. This is a contradiction. Thus our assumption that every equation has at least two basic variables is wrong. Therefore, there is at least one equation, either row or column, having only one basic variable.

Let the rth equation have only one basic variable, and let x_{rt} be the only basic variable in row r and column t. Then $x_{rt} = a_r$. Eliminate rth row from the system of equations and substituting $x_{rt} = a_r$ in t-th column equation and replace b_t by $b_t' = b_t - a_r$.

After eliminating the r-th row, the system has $m - 1$ row equations and n column equations of which $m + n - 2$ are linearly independent. This implies that the number of basic variables are $m + n - 2$. Repeating the argument given earlier and conclude that in the reduced system of equation, there is an equation which has only one basic variable. But if this equation happens to be the t-th column equation in the original system, then it will have two basic variables. This indicates that in our original system of equations, there is an equation which has at least two basic variables.

Continue, repeating the argument to prove that the system has an equation which has at least three basic variables and so on.

Unbalanced Transportation Problem For a feasible solution to exist it is necessary that the total supply must equal total demand. That is,

$$\sum_{i=1}^{m} a_i = \sum_{j=1}^{n} b_j$$

But a situation may arise when the total available supply is not equal to the total requirement.

Case 1: When the supply exceeds demand, the constraints of the transportation problem will appear as,

$$\sum_{j=1}^{n} x_{ij} \leq a_i; \quad i = 1, 2, \ldots, m$$

$$\sum_{i=1}^{m} x_{ij} = b_j; \quad j = 1, 2, \ldots, n$$

$$x_{ij} \geq 0 \text{ for all } i, j$$

Adding slack variables $s_{i,\, n+1}$, $(i = 1, 2, \ldots, m)$ in the first m constraints, we get

$$\sum_{j=1}^{m} x_{ij} + s_{i,\, n+1} = a_i$$

$$\sum_{i=1}^{m} \left\{ \sum_{j=1}^{n} x_{ij} + s_{i,n+1} \right\} = \sum_{i=1}^{m} a_i$$

$$\sum_{i=1}^{m} s_{i,n+1} = \sum_{i=1}^{m} a_i - \sum_{j=1}^{n} b_j = \text{excess supply available.}$$

If b_{n+1} denotes the excess supply available, then the modified transportation model can be stated as

$$\text{Minimize } Z = \sum_{i=1}^{m} \sum_{j=1}^{n} \left(c_{ij}x_{ij} + c_{i,n+1}\, s_{i,n+1}\right)$$

subject to
$$\sum_{j=1}^{n} x_{ij} + s_{i,n+1} = a_i; \quad i = 1, 2, \ldots, m$$
$$\sum_{j=1}^{n} x_{ij} = b_j \quad ; \quad j = 1, 2, \ldots, n+1$$

and $x_{ij} \geq 0$ for all i, j

where $c_{i,n+1} = 0$ $(i = 1, 2, \ldots, m)$ and

$$\sum_{i=1}^{m} a_i = \sum_{j=1}^{n} b_j + b_{n+1} \quad \text{or} \quad b_{n+1} = \sum_{i=1}^{m} a_i - \sum_{j=1}^{n} b_j$$

It follows that, if $\sum_{i=1}^{m} a_i > \sum_{j=1}^{n} b_j$, then a dummy column (demand centre) can be added to the transportation table to absorb the excess supply. The unit transportation cost for the cells in this column is set equal to zero because these represent product items that are not being made and not being sent.

Case 2: When demand exceeds supply, the constraints of the transportation table will appear as

$$\sum_{j=1}^{n} x_{ij} = a_i; \quad i = 1, 2, \ldots, m$$
$$\sum_{i=1}^{m} x_{ij} \leq b_j; \quad j = 1, 2, \ldots, n$$

and $x_{ij} \geq 0$ for all i, j

Adding slack variables $s_{m+1,\,j}$ $(j = 1, 2, \ldots, n)$ in the last n constraints, we get

$$\sum_{j=1}^{n} x_{ij} = a_i \quad ; \quad i = 1, 2, \ldots, m$$
$$\sum_{i=1}^{m} x_{ij} + s_{m+1,j} = b_j; \quad j = 1, 2, \ldots, n$$
$$\sum_{j=1}^{n} s_{m+1,j} = \sum_{j=1}^{n} b_j - \sum_{i=1}^{m} a_i = \text{excess demand.}$$

If a_{m+1} denotes the excess demand, then the modified transportation model can be stated as:

$$\text{Minimize } Z = \sum_{i=1}^{m} \sum_{j=1}^{m} \left(c_{ij}x_{ij} + c_{m+1,j}\, s_{m+1,j}\right)$$

subject to
$$\sum_{j=1}^{n} x_{ij} = a_i \quad ; \quad i = 1, 2, \ldots, m+1$$
$$\sum_{i=1}^{m} x_{ij} + s_{m+1,j} = b_j; \quad j = 1, 2, \ldots, n$$

and $x_{ij} \geq 0$ for all i, j

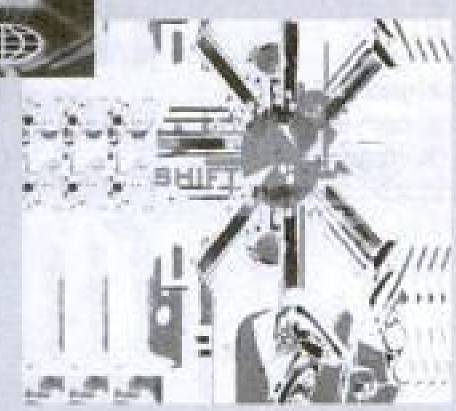
where $c_{m+1,j} = 0$, for all j and

$$\sum_{i=1}^{m} a_i + a_{m+1} = \sum_{j=1}^{n} b_j \quad \text{or} \quad a_{m+1} = \sum_{j=1}^{n} b_j - \sum_{i=1}^{m} a_i$$

It follows that if $\sum_{j=1}^{n} b_j > \sum_{i=1}^{m} a_i$, then a dummy row (supply centre) can be added to the transportation table to account for excess demand quantity. The unit transportation cost here also for the cells in the dummy row is set equal to zero.

chapter 10

Assignment Problem



"We don't have as many managers as we should, but we would rather have too few than too many."

– Larry Page

Preview An assignment problem is a particular case of a transportation problem where the given resources are allocated to an equal number of activities with an aim of either minimizing total cost, distance, time or maximizing profit.

Learning Objectives After studying this chapter, you should be able to

- understand the features of assignment problems and transportation problems.
- formulate an assignment problem as a square matrix.
- apply the Hungarian method to solve an assignment problem.
- make appropriate changes in the Hungarian method to solve an unbalanced assignment problem, profit maximization assignment problem, etc.
- solve a travelling salesman problem.

Chapter Outline

10.1 INTRODUCTION

An assignment problem is a particular case of a transportation problem where the sources are assignees and the destinations are tasks. Further more, every source has a supply of 1 (since each assignee is to be assigned to exactly one task) and every destination has a demand of 1 (since each task is to be performed by exactly one assignee). Also, the objective is to minimize the total cost or to maximize the total profit of allocation.

The problem of assignment arises because the resources that are available such as men, machines, etc., have varying degrees of efficiency for performing different activities. Therefore, the cost, profit or time of performing different activities is also different. Thus, the problem is: *How should the assignments be made so as to optimize the given objective.*

Some of the problems where the assignment technique may be useful are assignment of: workers to machines, salesmen to different sales areas, clerks to various checkout counters, classes to rooms, vehicles to routes, contracts to bidders, etc.

10.2 MATHEMATICAL MODEL OF ASSIGNMENT PROBLEM

Given n resources (or facilities) and n activities (or jobs), and effectiveness (in terms of cost, profit, time, etc.), of each resource (facility) for each activity (job), the problem lies in assigning each resource to one and only one activity (job) so that the given measure of effectiveness is optimized. The data matrix for this problem is shown in Table 10.1.

Assignment table is a convenient way to summarize available data

From Table 10.1, it may be noted that this data matrix is the same as the transportation cost matrix except that the supply (or availability) of each of the resources and the demand at each of the destinations is taken to be one. It is due to this fact that assignments are made on a one-to-one basis.

Table 10.1
Data Matrix

Resources (workers)	*Activities (jobs)* J_1	J_2	...	J_n	*Supply*
W_1	c_{11}	c_{12}	...	c_{1n}	1
W_2	c_{21}	c_{22}	...	c_{2n}	1
⋮	⋮	⋮		⋮	⋮
W_n	c_{n1}	c_{n2}	...	c_{nn}	1
Demand	1	1	...	1	n

Let x_{ij} denote the assignment of facility i to job j such that

$$x_{ij} = \begin{cases} 1 & \text{if facility } i \text{ is assigned to job } j \\ 0 & \text{otherwise} \end{cases}$$

Then, the mathematical model of the assignment problem can be stated as:

$$\text{Minimize } Z = \sum_{i=1}^{n} \sum_{j=1}^{n} c_{ij} x_{ij}$$

subject to the constraints

$$\sum_{j=1}^{n} x_{ij} = 1, \text{ for all } i \text{ (resource availability)}$$

$$\sum_{i=1}^{n} x_{ij} = 1, \text{ for all } j \text{ (activity requirement)}$$

and $\qquad x_{ij} = 0 \text{ or } 1, \text{ for all } i \text{ and } j$

where c_{ij} represents the cost of assignment of resource i to activity j.

From the above discussion, it is clear that the assignment problem is nothing but a variation of the transportation problem with two characteristics: (i) the cost matrix is a square matrix, and (ii) the optimal solution for the problem would always be such that there would be only one assignment in a given row or column of the cost matrix.

Remark In an assignment problem if a constant is added to or subtracted from every element of any row or column of the given cost matrix, then an assignment that minimizes the total cost in one matrix also minimizes the total cost in the other matrix. (See Appendix 10.A for proof.)

10.3 SOLUTION METHODS OF ASSIGNMENT PROBLEM

An assignment problem can be solved by the following four methods:

- Enumeration method
- Simplex method
- Transportation method
- Hungarian method

1. **Enumeration Method** In this method, a list of all possible assignments among the given resources (men, machines, etc.) and activities (jobs, sales areas, etc.) is prepared. Then an assignment that involves the minimum cost (or maximum profit), time or distance is selected. If two or more assignments have the same minimum cost (or maximum profit), time or distance, the problem has multiple optimal solutions.

   In general, if an assignment problem involves n workers/jobs, then in total there are $n!$ possible assignments. For example, for an $n = 5$ workers/jobs problem, we have to evaluate a total of 5! or 120 assignments. However, when n is large, the method is unsuitable for manual calculations. Hence, this method is suitable only when the value of n is small.

2. **Simplex Method** Since each assignment problem can be formulated as a 0 or 1 integer linear programming problem, such a problem can also be solved by the simplex method. As can be seen in the general mathematical formulation of the assignment problem, there are $n \times n$ decision variables and $n + n$ or $2n$ equalities. In particular, for a problem that involves 5 workers/jobs, there will be 25 decision variables and 10 equalities. This, again, is difficult to solve manually.

3. **Transportation Method** Since an assignment problem is a special case of the transportation problem, it can also be solved by transportation methods discussed in Chapter 9. However, every basic feasible solution of a general assignment problem that has a square payoff matrix of order n should have $m + n - 1 = n + n - 1 = 2n - 1$ assignments. But due to the special structure of this problem, none of the solutions can have more than n assignments. Thus, the assignment problem is inherently degenerate. In order to remove degeneracy, $(n - 1)$ number of dummy allocations (deltas or epsilons) will be required in order to proceed with the algorithm for solving a transportation problem. Thus, the problem of degeneracy at each solution makes the transportation method computationally inefficient for solving an assignment problem.

Hungarian method to solve an assignment problem is more efficient than using simplex method

4. **Hungarian Method** The Hungarian method (developed by Hungarian mathematician D. Konig) of assignment provides us with an efficient method of finding the optimal solution, without having to make a direct comparison of every solution. It works on the principle of reducing the given cost matrix to a matrix of opportunity costs. Opportunity costs show the relative penalties associated with assigning a resource to an activity as opposed to making the best or least cost assignment. If we can reduce the cost matrix to the extent of having at least one zero in each row and column, it will be possible to make optimal assignments (when the opportunity costs are all zero).

10.3.1 Hungarian Method for Solving Assignment Problem

The Hungarian method (minimization case) can be summarized in the following steps:

Step 1: Develop the cost table from the given problem If the number of rows are not equal to the number of columns, then as required a dummy row or dummy column must be added. The cost element in dummy cells are always zero.

Step 2: Find the opportunity cost table

(a) Identify the smallest element in each row of the given cost table and then subtract it from each element of that row, and

(b) In the reduced matrix obtained from 2(a), identify the smallest element in each column and then subtract it from each element of that column. Each row and column now have at least one zero element.

Step 3: Make assignments in the opportunity cost matrix The procedure of making assignments is as follows:

(a) First round for making assignments
- Identify rows successively from top to bottom until a row with exactly one zero element is found. Make an assignment to this single zero by making a square (□) around it. Then cross off (×) all other zeros in the corresponding column.
- Identify columns successively from left to right hand with exactly one zero element that has not been assigned. Make assignment to this single zero by making a square (□) around it and then cross off (×) all other zero elements in the corresponding row.

(b) Second round for making assignments
- If a row and/or column has two or more unmarked zeros and one cannot be chosen by inspection, then choose the zero cell arbitrarily for assignment.
- Repeat steps (a) and (b) successively until one of the following situations arise.

Step 4: Optimality criterion

(a) If all zero elements in the matrix are either marked with square (□) or are crossed off (×) and if there is exactly one assignment in each row and column, then it is an optimal solution. The total cost associated with this solution is obtained by adding the original cost figures in the occupied cells.

(b) If a zero element in a row or column was chosen arbitrarily for assignment in Step 4(a), there exists an alternative optimal solution.

(c) If there is no assignment in a row (or column), then this implies that the total number of assignments are less than the number of rows/columns in the square matrix. In such a situation proceed to Step 5.

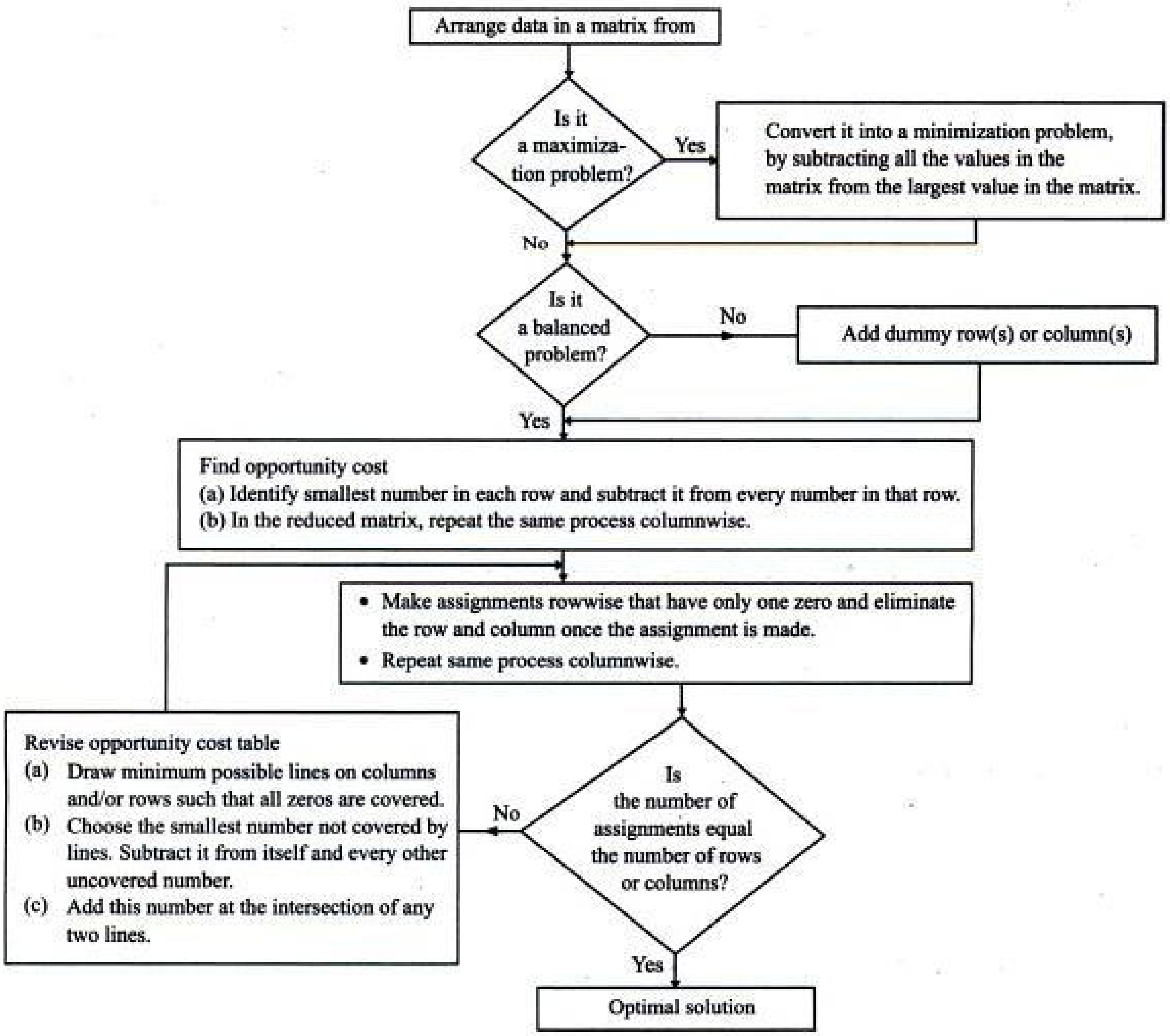


Fig. 10.1 Flow Chart of Steps in the Hungarian method

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Determine the allocations which minimize the total cost of transportation.

5. A departmental head has four subordinates and four tasks to be performed. The subordinates differ in efficiency and the tasks differ in their intrinsic difficulty. His estimates of the times that each man would take to perform each task is given in the matrix below:

		Tasks			
		I	II	III	IV
Subordinates	A	8	26	17	11
	B	13	28	4	26
	C	38	19	18	15
	D	19	26	24	10

How should the tasks be allocated to subordinates so as to minimize the total man-hours? *[Nagpur Univ., MBA, 1990]*

6. An automobile dealer wishes to put four repairmen to four different jobs. The repairmen have somewhat different kinds of skills and they exhibit different levels of efficiency from one job to another. The dealer has estimated the number of man-hours that would be required for each job-man combination. This is given in matrix form in the following table:

		Jobs			
		A	B	C	D
Men	1	5	3	2	8
	2	7	9	2	6
	3	6	4	5	7
	4	5	7	7	8

Find the optimal assignment that will result in minimum man-hours needed.

7. A lead draftsman has five drafting tasks to accomplish and five idle draftsmen. Each draftsman is estimated to require the following number of hours for each task.

		Tasks				
		A	B	C	D	E
Draftsmen	1	60	50	100	85	95
	2	65	45	100	75	90
	3	70	60	110	97	85
	4	70	55	105	90	93
	5	60	40	120	85	97

If each draftsman costs the company Rs 15.80 per hour, including overhead, find the assignment of draftsmen to tasks that will result in the minimum total cost. What would be the total cost?

8. A construction company has requested bids for subcontracts on five different projects. Five companies have responded. Their bids are represented below.

		Bid Amounts ('000s Rs)				
		I	II	III	IV	V
Bidders	1	41	72	39	52	25
	2	22	29	49	65	81
	3	27	39	60	51	40
	4	45	50	48	52	37
	5	29	40	45	26	30

Determine the minimum cost assignment of subcontracts to bidders, assuming that each bidder can receive only one contract. *[Delhi Univ., MBA, 1991]*

9. A shipbuilding company has been awarded a big contract for the construction of five cargo vessels. The contract stipulates that the company must subcontract a portion of the total work to at least five small ancillary companies. The company has invited bids from the small ancillary companies (A_1, A_2, A_3, A_4, and A_5) to take care of the subcontract work in five fields – materials testing, fabrication, assembly, scrap removal and painting. The bids received from the ancillary companies are given in the table.

Subcontract Bids (Rs)

Ancillary Companies	Materials Testing	Fabrication	Assembly	Scrap Removal	Painting
A_1	2,50,000	3,00,000	3,80,000	5,00,000	1,50,000
A_2	2,80,000	2,60,000	3,50,000	5,00,000	2,00,000
A_3	3,00,000	3,50,000	4,00,000	5,50,000	1,80,000
A_4	1,50,000	2,50,000	3,00,000	4,80,000	1,20,000
A_5	3,00,000	2,70,000	3,20,000	4,80,000	1,60,000

Which bids should the company accept in order to complete the contract at minimum cost? What is the total cost of the subcontracts? *[Delhi Univ., MBA, 1988]*

10. In a textile sales emporium, four salesmen A, B, C and D are available to four counters *W*, *X*, *Y* and *Z*. Each salesman can handle any counter. The service (in hours) of each counter when manned by each salesman is given below:

		Salesmen			
		A	B	C	D
Counters	W	41	72	39	52
	X	22	29	49	65
	Y	27	39	60	51
	Z	45	50	48	52

How should the salesmen be allocated to appropriate counters so that the service time is minimized? Each salesman must handle only one counter.

11. A hospital wants to purchase three different types of medical equipments and five manufacturers have come forward to supply one or all the three machines. However, the hospital's policy is not to accept more than one machine from any one of the manufacturers. The data relating to the price (in thousand of rupees) quoted by the different manufacturers is given below:

		Machines		
		1	2	3
Manufacturers	A	30	31	27
	B	28	29	26
	C	29	30	28
	D	28	31	27
	E	31	29	26

Determine how best the hospital can purchase the three machines. *[Delhi Univ., MBA (HCA), 2001]*

12. The secretary of a school is taking bids on the city's four school bus routes. Four companies have made the bids (in Rs), as detailed in the following table:

		Route 1	Route 2	Route 3	Route 4
Bus	1	4,000	5,000	–	–
	2	–	4,000	–	4,000
	3	3,000	–	2,000	–
	4	–	–	4,000	5,000

Suppose each bidder can be assigned only one route. Use the assignment model to minimize the school's cost of running the four bus routes. *[CA, Nov., 1995]*

13. A large oil company operating a number of drilling platforms in the North Sea is forming a high speed rescue unit in order to cope with emergency situations that may occur. The rescue unit comprises 6 personnel who, for reasons of flexibility, undergo the same comprehensive training programme. The six personnel are assessed as to their suitability for various specialist tasks and the marks they received in the training programme are given in the following table:

	Trainee Number					
Specialist Task	*I*	*II*	*III*	*IV*	*V*	*VI*
Unit Leader	21	5	21	15	15	28
Helicopter Pilot	30	11	16	8	16	4
First Aid	28	2	11	16	25	25
Drilling Technology	19	16	17	15	19	8
Firefighting	26	21	22	28	29	24
Communications	3	21	21	11	26	26

Based on the marks awarded, what role should each of the trainees be given in the rescue unit?

14. The personnel manager of ABC Company wants to assign Mr *X*, Mr *Y* and Mr *Z* to regional offices. But the firm also has an opening in its Chennai office and would send one of the three to that branch if it were more economical than a move to Delhi, Mumbai or Kolkata. It will cost Rs 2,000 to relocate Mr *X* to Chennai, Rs 1,600 to reallocate Mr *Y* there, and Rs 3,000 to move Mr *Z*. What is the optimal assignment of personnel to offices?

		Office		
		Delhi	*Mumbai*	*Kolkata*
Personnel	*Mr X*	1,600	2,200	2,400
	Mr Y	1,000	3,200	2,600
	Mr Z	1,000	2,000	4,600

HINTS AND ANSWERS

1. $A-III$, $B-V$, $C-I$, $D-IV$, $E-II$;
   Optimal value = 13 hours.
2. $A-e$, $B-c$, $C-b$, $D-a$, $E-d$;
   Minimum distance = 570 km.
4. $1-11$, $2-8$, $3-7$, $4-9$, $5-10$, $6-12$;
   Minimum distance = 125 km.
5. $A-I$, $B-III$, $C-II$, $D-IV$;
   Total man-hours = 41 hours.
6. $1-B$, $2-C$, $3-D$, $4-A$; $1-C$, $2-D$, $3-B$, $4-A$;
   Total man-hours = 17 hours.
7. $1-A$, $2-D$, $3-E$, $4-C$, $5-B$;
   Minimum cost = Rs 365 × 15.8
8. $1-V$, $2-II$, $3-I$, $4-III$, $5-IV$;
   Minimum cost = Rs 155.
9. A_1 – scrap, A_2 – Fabrication, A_3 – Painting, A_4 – Testing, A_5 – Assembly; Minimum cost = Rs 14,10,000
10. $W-C$, $X-B$, $Y-A$, $Z-D$; Optimal value = 147 hours.

10.4 VARIATIONS OF THE ASSIGNMENT PROBLEM

10.4.1 Multiple Optimal Solutions

While making an assignment in the reduced assignment matrix, it is possible to have two or more ways to strike off a certain number of zeros. Such a situation indicates that there are multiple optimal solutions with the same optimal value of objective function. In such cases the more suitable solution may be considered by the decision-maker.

10.4.2 Maximization Case in Assignment Problem

There may arise situations when the assignment problem calls for maximization of profit, revenue, etc., as the objective function. Such problems may be solved by converting the given maximization problem into a minimization problem in either of the following two ways:

(i) Put a negative sign before each of the payoff elements in the assignment table so as to convert the profit values into cost values.
(ii) Locate the largest payoff element in the assignment table and then subtract all the elements of the table from the largest element.

The transformed assignment problem, so obtained, can be solved by using the Hungarian method.

Example 10.4 A company operates in four territories, and four salesmen available for an assignment. The territories are not equally rich in their sales potential. It is estimated that a typical salesman operating in each territory would bring in the following annual sales:

Territory	:	*I*	*II*	*III*	*IV*
Annual sales (Rs)	:	1,26,000	1,05,000	84,000	63,000

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Cost of Repairs (Rs in lakh)

Contractors/Road		R_1	R_2	R_3	R_4
	C_1	9	14	19	15
	C_2	7	17	20	19
	C_3	9	18	21	18
	C_4	10	12	18	19
	C_5	10	15	21	16

(a) Find the best way of assigning the repair work to the contractors and the costs.
(b) If it is necessary to seek supplementary grants, what should be the amount sought?
(c) Which of the five contractors will be unsuccessful in his bid? *[ICWA, June 1987, AMIE 2005]*

Solution (a) The given cost matrix is not balanced; add one dummy column (road, R_5) with a zero cost in that column. The cost matrix so obtained is given in Table 10.22.

	R_1	R_2	R_3	R_4	R_5
C_1	9	14	19	15	0
C_2	7	17	20	19	0
C_3	9	18	21	18	0
C_4	10	12	18	19	0
C_5	10	15	21	16	0

Table 10.22
Cost Matrix

Apply the Hungarian method to solve this problem. This is left as an exercise for the reader. An optimal assignment is shown in Table 10.23.

	R_1	R_2	R_3	R_4	R_5
C_1	1	1	$\boxed{0}$	$\cancel{0}$	1
C_2	$\boxed{0}$	5	2	5	2
C_3	$\cancel{0}$	4	1	2	$\boxed{0}$
C_4	3	$\boxed{0}$	$\cancel{0}$	5	2
C_5	1	1	1	$\boxed{0}$	$\cancel{0}$

Table 10.23
Optimal Solution

The total minimum cost (in rupees) and optimal assignment made are as follows:

Road	*Contractor*	*Cost (Rs in lakh)*
R_1	C_2	7
R_2	C_4	12
R_3	C_1	19
R_4	C_5	16
R_5	C_3	0
	Total	54

(b) Since the total cost exceeds 50 lakh, the excess amount of Rs 4 lakh (= 54 – 50) is to be sought as supplementary grant.
(c) Contractor C_3 who has been assigned to dummy row, R_5 (roads) loses out in the bid.

CONCEPTUAL QUESTIONS B

1. Can there be multiple optimal solutions to an assignment problem? How would you identify the existence of multiple solutions, if any?
2. How would you deal with the assignment problems, where (a) the objective function is to be maximized? (b) some assignments are prohibited?
3. Explain how can one modify an effectiveness matrix in an assignment problem, if a particular assignment is prohibited.
4. What is an unbalanced assignment problem? How is the Hungarian method applied for obtaining a solution if the matrix is rectangular?

SELF PRACTICE PROBLEMS B

1. A project work consists of four major jobs for which an equal number of contractors have submitted tenders. The tender amount quoted (in lakh of rupees) is given in the matrix.

		Job			
		a	*b*	*c*	*d*
Contractor	*1*	10	24	30	15
	2	16	22	28	12
	3	12	20	32	10
	4	9	26	34	16

Find the assignment which minimizes the total cost of the project when each contractor has to be assigned at least one job.

2. Alpha Corporation has four plants, each of which can manufacture any one of four products *A, B, C* or *D*. Production costs differ from one plant to another and so do the sales revenue. The revenue and the cost data are given below. Determine which product should each plant produce in order to maximize profit.

Sales Revenue (in '000 Rs)

		Plant			
		1	*2*	*3*	*4*
Product	*A*	50	68	49	62
	B	60	70	51	74
	C	52	62	49	68
	D	55	64	48	66

Production Cost (in '000 Rs)

		Plant			
		1	*2*	*3*	*4*
Product	*A*	49	60	45	61
	B	55	63	45	49
	C	55	67	53	70
	D	58	65	54	68

3. A company has four machines that are to be used for three jobs. Each job can be assigned to one and only one machine. The cost of each job on each machine is given in the following table.

		Machines			
		W	*X*	*Y*	*Z*
Jobs	*A*	18	24	28	32
	B	8	13	17	18
	C	10	15	19	22

What are the job-assignment pairs that shall minimize the cost?
[*Gauhati, MCA, 1996*]

4. Five workers are available to work with the machines and the respective costs (in rupees) associated with each worker-machine assignment are given below. A sixth machine is available to replace one of the existing ones and the associated of that machine costs are also given below.

		Machines					
		M_1	M_2	M_3	M_4	M_5	M_6
Workers	W_1	12	3	6	–	5	9
	W_2	4	11	–	5	–	8
	W_3	8	2	10	9	7	5
	W_4	–	7	8	6	12	10
	W_5	5	8	9	4	6	1

(a) Determine whether the new machine can be accepted.
(b) Also determine the optimal assignment and the associated saving in cost.

5. A firm is contemplating the introduction of three products 1, 2 and 3, in its three plants *A, B* and *C*. Only a single product is decided to be introduced in each of the plants. The unit cost of producing one product in a plant, is given in the following matrix.

		Plant		
		A	*B*	*C*
Product	*1*	8	12	–
	2	10	6	4
	3	7	6	6

(a) How should the product be assigned so that the total unit cost is minimized?
(b) If the quantity of different products to be produced is as follows, then what assignment shall minimize the aggregate production cost?

Product	*Quantity (in units)*
1	2,000
2	2,000
3	10,000

(c) What would your answer be if the three products were to be produced in equal quantities?
(d) It is expected that the selling prices of the products produced by different plants would be different. The prices are shown in the following table:

		Plant		
		A	*B*	*C*
Product	*1*	15	18	–
	2	18	16	10
	3	12	10	8

Assuming the quantities mentioned in (b) above would be produced and sold, how should the products be assigned to the plants in order to obtain maximum profits?
[*Delhi Univ., MBA, 2000*]

6. A fast-food chain wants to build four stores. In the past, the chain has used six different construction companies, and having been satisfied with each, has invited each to bid on each job. The final bids (in lakh of rupees) are shown in the following table:

Construction Companies

	1	2	3	4	5	6
Store 1	85.3	88.0	87.5	82.4	89.1	86.7
Store 2	78.9	77.4	77.4	76.5	79.3	78.3
Store 3	82.0	81.3	82.4	80.6	83.5	81.7
Store 4	84.3	84.6	86.2	83.3	84.4	85.5

Since the fast-food chain wants to have each of the new stores ready as quickly as possible, it will award at the most one job to a construction company. What assignment would result in minimum total cost to the fast-food chain?

[Delhi Univ., MBA, 1998, 2001, 2003]

7. A methods engineer wants to assign four new methods to three work centres. The assignment of the new methods will increase production. The methods are given below.

Increase in Production (unit)

Work Centres

		A	B	C
Method	1	10	7	8
	2	8	9	7
	3	7	12	6
	4	10	10	8

If only one method can be assigned to a work centre, determine the optimum assignment.

8. Consider a problem of assigning four clerks to four tasks. The time (hours) required to complete the task is given below:

Tasks

		A	B	C	D
Clerks	1	4	7	5	6
	2	–	8	7	4
	3	3	–	5	3
	4	6	6	4	2

Clerk 2 cannot be assigned task A and clerk 3 cannot be assigned task B. Find all the optimum assignment schedules.

9. The marketing director of a multi-unit company is faced with a problem of assigning 5 senior managers to six zones. From past experience he knows that the efficiency percentage judged by sales, operating costs, etc., depends on the manager-zone combination. The efficiency of different managers is given below:

Zones

		I	II	III	IV	V	VI
Manager	A	73	91	87	82	78	80
	B	81	85	69	76	74	85
	C	75	72	83	84	78	91
	D	93	96	86	91	83	82
	E	90	91	79	89	69	76

Find out which zone should be managed by a junior manager due to the non-availability of a senior manager.

10. A head of department in a college has the problem of assigning courses to teachers with a view to maximize educational quality in his department. He has available to him one professor, two associate professors, and one teaching assistant (TA). Four courses must be offered. After appropriate evaluation, he has arrived at the following relative ratings (100 = best rating) regarding the ability of each instructor to teach each of the four courses.

	Course 1	Course 2	Course 3	Course 4
Prof. 1	60	40	60	70
Prof. 2	20	60	50	70
Prof. 3	20	30	40	60
TA	30	10	20	40

How should he assign his staff to the courses in order to realize his objective? *[Delhi Univ., MBA (HCA), 1999]*

11. At the end of a cycle of schedules, a transport company has a surplus of one truck in each of the cities 1, 2, 3, 4, 5 and a deficit of one truck in each of the cities A, B, C, D, E and F. The distance (in kilometres) between the cities with a surplus, and cities with a deficit, is given below:

To City

		A	B	C	D	E	F
From City	1	80	140	80	100	56	98
	2	48	64	94	126	170	100
	3	56	80	120	100	70	64
	4	99	100	1,100	104	80	90
	5	64	80	90	60	60	70

How should the trucks be despatched so as to minimize the total distance travelled? Which city will not receive a truck?

[Madras, MBA, Oct. 1994]

12. A company is considering expanding into five new sales territories. The company has recruited four new salesmen. Based on the salesmen's experience and personality traits, the sales manager has assigned ratings to each of the salesmen for each of the sales territories. The ratings are as follows:

Territory

		1	2	3	4	5
Salesmen	A	75	80	85	70	90
	B	91	71	82	75	85
	C	78	90	85	80	80
	D	65	75	88	85	90

Suggest optimal assignment of the salesmen. If for certain reasons, salesman D cannot be assigned to territory 3, will the optimal assignment be different? If so, what would be the new assignment schedule? *[Delhi Univ., MCom, 1990]*

13. The personnel manager of a medium-sized company has decided to recruit two employees D and E in a particular section of the organization. The section has five fairly defined tasks 1, 2, 3, 4 and 5; and three employees A, B and C are already employed in the section. Considering the specialized nature of task 3 and the special qualifications of the recruit D for task 3, the manager has decided to assign task 3 to employee D and then assign the remaining tasks to remaining employees so as to maximize the total effectiveness. The index of effectiveness of each employee of different tasks is as under.

Tasks

		1	2	3	4	5
Employees	A	25	55	60	45	30
	B	45	65	55	35	40
	C	10	35	45	55	65
	D	40	30	70	40	60
	E	55	45	40	55	10

Assign the tasks for maximizing total effectiveness. Critically examine whether the decision of the manager to assign task 3 to employee D was correct. *[Delhi Univ., MBA, 2000]*

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Index

Operations Research
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