(R23) Differential Equations and Vector Calculus (B23BS1201)

Short Answer questions (Internal-I syllabus)

UNIT- I
(Differential equations of first order and first degree)

1) Write the standard form of Linear equation and also write its solution.

Ans: The standard form of Linear equation is $\frac{dy}{dx} + P(x)y = Q(x)$

and its solution is $y(I.F) = \int Q(I.F)dx \ + C$

2) Calculate the Integrating factor of $\frac{dy}{dx} + 2ytax = sinx$

Ans: It is of the form $\frac{dy}{dx} + P(x)y = Q(x)$, a linear equation in y.

Here $P(x) = 2tanx \ and \ Q(x) = sinx$

$$I.F = e^{\int P(x)dx} = e^{\int 2tanxdx} = e^{2\int \frac{sinx}{cosx}dx} = e^{2lnsecx} = \sec^2 x$$

3) What is the condition for exact equation

Ans: M(x, y) dx + N(x, y) dy = 0 to be *Exact if* $\frac{\partial M}{\partial y} = \frac{\partial N}{\partial x}$

4) Write the general solution for an Exact differential equation

Ans: Solution of exact equation is

$$\int_{\substack{\text{treating} \\ \text{y as constant}}} \text{M dx} + \int \begin{pmatrix}\text{terms of N not} \\ \text{containing x}\end{pmatrix} \text{dy} = \text{C}$$

5) Find the orthogonal trajectories of $x^{2/3} + y^{2/3} = a^{2/3}$, Where a is the parameter

Ans: Differentiating the given equation wilth respect to 'x'
We get $\frac{2}{3}x^{-1/3} + \frac{2}{3}y^{-1/3}\frac{dy}{dx} = 0$

$$x^{-1/3} + y^{-1/3}\frac{dy}{dx} = 0$$

$y^{-1/3}\frac{dy}{dx} = -x^{-1/3}$, which is the differential equation of family of cuves.

Replacing $\frac{dy}{dx}$ by $\frac{-dx}{dy}$, we get $-y^{-1/3}\frac{dx}{dy} = -x^{-1/3}$

Separating the variables, we get $x^{1/3}\,dx = y^{1/3}\,dy$
Integrating on both sides, we get

$$\int x^{1/3}\,dx = \int y^{1/3}\,dy$$

$\frac{x^{4/3}}{4/3} = \frac{y^{4/3}}{4/3}$+c $\rightarrow\rightarrow\rightarrow$ $x^{4/3} - y^{4/3} = C_1$ where $c_1 = \frac{3}{4}c$

6) what is the condition for the equation $M(x,y)dx + N(x,y)dy = 0$ to be non-exact ?

Ans: $\frac{\partial M}{\partial y} \neq \frac{\partial N}{\partial x}$

7) If the equation $M(x,y)dx + N(x,y)dy = 0$ is non-exact and it is homogeneous equation in $x\ and\ y$ then what is I.F.?

Ans: $\frac{1}{Mx+Ny}\ provided\ Mx + Ny \neq 0$

8) Define the orthogonal trajectories of the family of curves.

Ans: Orthogonal trajectories: Two families of curves are said to be orthogonal if every member of either family cuts each member of the other family at right angles.

9) Define the Self orthogonal trajectories of the family of curves.

Ans: Self orthogonal trajectories: If each member of a given family of curves cuts every member of the same family at right angles then the given family of curves is said to be self –orthogonal.

10) Write the statement of Newton's Law of cooling.

Ans: The rate of change of temperature of a body is proportional to the difference of the temperature of the body and that of surrounding medium.

11) Write the suitable differential equation when a resistance 'R', inductance 'L' is connected in series with battery 'E' volts.

Ans: The suitable equation is $\frac{di}{dt} + \frac{R}{L}i = \frac{E}{L}$

12) Write the suitable differential equation when a resistance 'R', Capacitance 'C' is connected in series with battery 'E' volts.

Ans: The suitable equation is $\frac{di}{dt} + \frac{1}{RC}i = \frac{E}{R}$

13) What is the voltage drop across resistance R

Ans: Ri

14) What is the voltage drop across inductance L

Ans: $L\frac{di}{dt}$

15) What is the voltage drop across capacitance C

Ans: $\frac{q}{C}$

UNIT-II
(Linear differential equations of higher order with constant coefficients)

1. Define complementary function.

Ans: The solution of a homogeneous linear differential equations

2. Define Auxiliary equation.

Ans: If the given linear differential equations is $F(D)y = X$, then $F(m) = 0$ is called auxiliary equation, where 'm' is a constant.

3. Define particular integral

Ans: If the given linear differential equations is $F(D)y = X$, then $y = \frac{1}{F(D)}X$ is called its particular integral.

4. What is the complete solution of $F(D)y = X$?
   Ans: The complete solution = Complementary function + Particular integral

5. Solve $y'' - 4y' - 5y = 0$

Sol. The given linear differential equation can be expressed in the operator form as

$$(D^2 - 4D - 5)y = 0$$

Its auxiliary equation is $m^2 - 4m - 5 = 0 \Rightarrow (m-5)(m+1) = 0 \Rightarrow m = -1,5$

$\therefore$ its complementary function is the complete solution $y = c_1e^{-x} + c_2e^{5x}$

6. Solve $\frac{d^3y}{dx^3} - 3\frac{d^2y}{dx^2} + 3\frac{dy}{dx} - y = 0$

Sol. The given linear differential equation can be expressed in the operator form as

$$(D^3 - 3D^2 + 3D - 1)y = 0$$

Its auxiliary equation is $m^3 - 3m^2 + 3m - 1 = 0 \Rightarrow (m-1)^3 = 0 \Rightarrow m = 1,1,1$

$\therefore$ its complementary function is the complete solution $y = (c_1 + c_2 x + c_3 x^2)e^x$

7. Solve $(D^2 - D + 1)y = 0$

Sol. Its auxiliary equation is $m^2 - m + 1 = 0 \Rightarrow m = \dfrac{1 \pm i\sqrt{3}}{2}$

$\therefore$ its complementary function is the complete solution

$$y = e^{\frac{x}{2}}(c_1 \cos\frac{\sqrt{3}}{2}x + c_2 \sin\frac{\sqrt{3}}{2}x)$$

8. Solve $(D^4 + 18D^2 + 81)y = 0$

Sol. Its auxiliary equation is $m^4 + 18m^2 + 81 = 0 \Rightarrow (m^2 + 9)^2 = 0 \Rightarrow m = \pm 3i, \pm 3i$

$\therefore$ its complementary function is the complete solution

$$y = [(c_1 + c_2 x)\cos 3x + (c_3 + c_4 x)\sin 3x]$$

9. Prove that $\dfrac{1}{D}X = \int X\,dx$

sol. Let $y = \dfrac{1}{D}X \Rightarrow Dy = X \Rightarrow \dfrac{dy}{dx} = X$

Integrating on both sides , we get $\dfrac{1}{D}X = \int X\,dx$

10. Prove that $\dfrac{1}{D-a}X = e^{ax}\int Xe^{-ax}dx$

Let $\dfrac{1}{D-a}X = y$

$$X = (D-a)y \Rightarrow \frac{dy}{dx} - ay = X \qquad ....(A)$$

It represents Leibnitz's linear equation with P=-a,Q=X

Therefore it's $I.F = e^{\int p dx} = e^{-ax}$

The solution of equation (A) is $y e^{-ax} = \int X e^{-ax} dx$

$y = e^{ax} \int X e^{-ax} dx$.

11. Find the particular integral of $(D^2 - 6D + 9)y = 6e^{3x}$

Sol. We have

$$P.I = \frac{1}{D^2 - 6D + 9} 6e^{3x}$$
$$= \frac{x}{2D - 6} 6e^{3x}$$
$$= \frac{x^2}{2} 6e^{3x}$$
$$= 3x^2 e^{3x}$$

12. Find the particular integral of $(D-2)^2 y = Sin2x$

Sol. We have

$$P.I = \frac{1}{D^2 - 4D + 4} \sin 2x$$
$$= \frac{1}{(-2^2) - 4D + 4} \sin 2x$$
$$= -\frac{1}{4} \int \sin 2x \, dx$$
$$= -\frac{1}{4}\left(\frac{-\cos 2x}{2}\right)$$
$$= \frac{1}{8} \cos 2x$$

UNIT-III
(Partial differential equations)

1. ***Form the partial differential equation by eliminating arbitrary constants*** $z = ax + by + ab$

Sol: The given partial differential equation is $z = ax + by + ab$ …… (1)

Differentiating equation (1) partially with respect to x on both sides

We get $\frac{\partial z}{\partial x} = a \Rightarrow p = a$

Differentiating equation (1) partially with respect to y on both sides

We get $\frac{\partial z}{\partial y} = b \Rightarrow q = b$

Substituting the values of a &b in equation (1) we get the required partial differential equation is $z = px + qy + pq$

2. *Form the partial differential equation by eliminating arbitrary constants*

$z = ax + by + \sqrt{a^2 + b^2}$

Sol: The given partial differential equation is $z = ax + by + \sqrt{a^2 + b^2}$ …….(1)

Differentiating equation (1) partially with respect to x on both sides

We get $\frac{\partial z}{\partial x} = a \Rightarrow p = a$

Differentiating equation (1) partially with respect to y on both sides

We get $\frac{\partial z}{\partial y} = b \Rightarrow q = b$

Substituting the values of a & b in equation (1) we get the required partial differential equation is $z = px + qy + \sqrt{p^2 + q^2}$

3. *Form the partial differential equation by eliminating arbitrary constants* $z = ax^3 + by^3$

Sol: The given partial differential equation is $z = ax^3 + by^3$ ……(1)

Differentiating equation (1) partially with respect to x on both sides

We get $\frac{\partial z}{\partial x} = 3ax^2 \Rightarrow p = 3ax^2 \Rightarrow a = \frac{p}{3x^2}$

Differentiating equation (1) partially with respect to y on both sides

We get $\frac{\partial z}{\partial y} = 3by^2 \Rightarrow q = 3by^2 \Rightarrow b = \frac{q}{3y^2}$

Substituting the values of a & b in equation (1) we get the required partial differential equation

is $z = \frac{px + qy}{3}$

4. *Form the partial differential equation by eliminating arbitrary constants*

$z = (x^2 + a)(y^2 + b)$

Sol: The given partial differential equation is $z = (x^2 + a)(y^2 + b)$ …… (1)

Differentiating equation (1) partially with respect to x on both sides

We get $\frac{\partial z}{\partial x} = 2x(y^2 + b) \Rightarrow p = 2x(y^2 + b) \Rightarrow y^2 + b = \frac{p}{2x}$

Differentiating equation (1) partially with respect to y on both sides

We get $\frac{\partial z}{\partial y}=(x^2+a)(2y) \Rightarrow q=(x^2+a)(2y) \Rightarrow x^2+a=\frac{q}{2y}$

Substituting the values of x^2+a & y^2+b in equation (1) we get the required partial differential equation is $z=\frac{pq}{4xy}$

5. Write the standard form of Lagrange's linear equation.

Sol: A linear partial differential equation of the first order, commonly known as Lagrange's Linear equation, is of the form $Pp+Qq=R$ .where P, Q and R are functions of x, y, z. This equation is called Quasi-linear equation.

6. Form the partial differential equation for the following equation $z=f(x^2+y^2)$

Sol: Given equation is $z=f(x^2+y^2)$

$$\frac{\partial z}{\partial x}=2x\, f'(x^2+y^2) \quad \ldots\ldots\ldots\ldots. \quad (1)$$

$$\frac{\partial z}{\partial y}=2y\ f'(x^2+y^2) \ldots\ldots\ldots\ldots\ldots. (2)$$

$$(1)\div(2) \Rightarrow \frac{p}{q}=\frac{x}{y}$$

So, the required partial differential equations is $yp-xq=0$ $where\ p=\frac{\partial z}{\partial x}\ and\ q=\frac{\partial z}{\partial y}$

7. Form the partial differential equation for the following equation $z=f_1(x)f_2(y)$

Sol: Given equation is $z=f_1(x)f_2(y)\ldots\ldots\ldots$ (1)

$$\frac{\partial z}{\partial x}=f_1'(x)\ f_2(y)\ldots\ldots\ldots\ldots. \quad (2)$$

$$\frac{\partial z}{\partial y}=f_1(x)\ f_2'(y)\ldots\ldots\ldots\ldots. \quad (3)$$

$$\frac{\partial^2 z}{\partial x\partial y}=f_1'(x)f_2'(y)\ldots\ldots\ldots.(4)$$

$$\frac{\partial z}{\partial x}\frac{\partial z}{\partial y}=f_1'(x)\ f_2(y)f_1(x)\ f_2'(y)$$

$$\frac{\partial z}{\partial x}\ \frac{\partial z}{\partial y}=z\ \frac{\partial^2 z}{\partial x\partial y}$$

The required partial differential equation is $pq=sz$

8. Form the partial differential equation for the following equation $z=x+y+f(x^2+y^2)$

Sol: Given equation is $z = x + y + f(x^2 + y^2)$

$$\frac{\partial z}{\partial x} = 1 + 2x\, f^{'}(x^2 + y^2)$$
$$\Rightarrow \frac{\partial z}{\partial x} - 1 = 2x\, f^{'}(x^2 + y^2) \ldots\ldots\ldots\ldots \quad (1)$$

$$\frac{\partial z}{\partial y} = 1 + 2y\ \ f^{'}(x^2 + y^2)$$
$$\Rightarrow \frac{\partial z}{\partial y} - 1 = 2y\ \ f^{'}(x^2 + y^2) \ldots\ldots\ldots\ldots\ldots (2)$$

$$(1) \div (2) \Rightarrow \frac{p-1}{q-1} = \frac{x}{y}$$

$$\Rightarrow\ yp - xq = y - x$$

The required partial differential equation is $yp - xq = y - x\ \ where\ p = \frac{\partial z}{\partial x}\ and\ q = \frac{\partial z}{\partial y}$.

9. *Solve the PDE* $p + q = 1$

Sol: Given equation is

The corresponding simultaneous equations are

$$\frac{dx}{1} = \frac{dy}{1} = \frac{dz}{1}$$

Solving these equations we get $x - y = c_1\ \ and\ y - z = c_2$

Hence, the solution is $f(x - y, y - z) = 0\ where\ f\ is\ an\ arbitray\ function$

10. *Solve the PDE* $xp + yq = z$

Sol: Given equation is $xp + yq = z$

The corresponding simultaneous equations are

$$\frac{dx}{x} = \frac{dy}{y} = \frac{dz}{z}$$

Solving these equations we get $x/y = c_1 and\ y/z = c_2$

Hence, the solution is $f(x/y, y/z) = 0\ where\ f\ is\ an\ arbitray\ function$