SRKR ENGINEERING COLLEGE (A): BHIMAVARAM
Department of Mathematics & Humanities

LINEAR ALGEBRA & CALCULAS

(SHORT ANSWER QUESTIONS WITH SOLUTIONS)

Linear Algebra and Calculus (LAC)
Unit-I
Short Answer Questions & Answers

1. Define the rank of a matrix

Ans: If A is a non-zero matrix, we say that 'r' is the rank of A if

(i) Everty $(r+1)^{th}$ order minor of A is zero.
(ii) There exists at least one r^{th} order minor of A which is not zero.

Rank of A is denoted by $\rho(A)$.

2. Define the Echelon form of a matrix

Ans: A matrix is said to be in Echelon form if it has the following properties:

(i) Zero rows, if any, are below any non-zero row.
(ii) The first non-zero entry in each non-zero row is equal to 1.
(iii) The number of zeros before the first non-zero element in a row is less than the number of such zeros in the next row.

3. Define the Normal form of a matrix

Ans: Every mxn matrix of rank 'r' can be reduced to the form I_r, $[I_r, 0]$, $\begin{bmatrix} I_r & 0 \\ 0 & 0 \end{bmatrix}$ by a finite chain of elementary row or column operations, where I_r is the r-rowed unit matrix. The above form is called "normal form" of a matrix.

4. How the rank of a given matrix is determined from its echelon form?

Ans: The number of non-zero rows in the echelon form is equal to the rank of the given matrix

5. Write Cauchy Binet formula.

Ans: Let A be an mxn matrix and B be an nxm matrix.

Then for $n > m, \det(AB) = \sum_{1 \le j_1 < j_2 < \cdots < j_m \le n} \det\left(A_{j_1, j_2, \dots, j_m}\right) . (B_{j_1, j_2, \dots, j_m})$.

Where $\boldsymbol{A}_{j_1, j_2, \dots, j_m}$ denotes the matrix formed from **A** using columns $j_1, j_2, \dots, j_m$ in **A** and $\boldsymbol{B}_{j_1, j_2, \dots, j_m}$ denotes the matrix formed from **B** using rows $j_1, j_2, \dots, j_m$ in **B**.

For $n < m, \det(AB) = 0$

For $n = m$, then $\det(AB) = \det A . \det B$.

6. Write the conditions for Consistency and solutions of system of homogeneous equations.

Ans: If the rank of coefficient matrix is equal to the number of unknowns of the system, then it has trivial solution and if the rank of coefficient matrix is less than the number of unknowns, then it has non-trivial solutions.

7. ***What is the necessary and sufficient condition for a homogenous system to possess non-trivial solution?***

Ans: The determinant of the coefficient matrix is zero.

8. ***Which method has faster convergence of the two methods Jacobi and Gauss seidal iterative methods?***

Ans: The Guass Seidal iterative method is faster convergent than Jacobi method.

9. ***What is the initial approximation solution in Jacobi and Gauss seidal iterative methods?***

Ans: Assuming all Unkonwns in the system of equations is equal to zero.

10. ***When do we say that a given system is diagonally dominant if it has 3 unknowns?***

Ans: If in the first equation, the coefficient of the first unknown is relatively large than others, in the second equation, the coefficient of the second unknown is relatively large than others and in the third equation, the coefficient of the third unknown is relatively large than other. Then the system is a diagonally dominant.

UNIT-II

Short Answer Questions.

1. Illustrate the nature of the quadratic form.

Ans: The quadratic form $X^T AX$ in n-variables ,where r = rank, s= *signature,* n =no. of. Unknowns Then the quadratic form is said to be

Positive definite : If $r = n$ and $s = n$ **or** If all the Eigen values of the symmetric matrix A are positive

Negative definite : If $r = n$ and $s = 0$ **or** If all the Eigen values of the symmetric matrix A are negative

Positive semi-definite: If $r < n$ and $s = r$ **or** If all the Eigen values of the symmetric matrix A are positive and at least one Eigen value is zero

Negative semi-definite: If $r < n$ and $s = 0$ **or** If all the Eigen values of the symmetric matrix A are negative and at least one Eigen value is zero

Indefinite: In all other cases (some Eigen values are positive and other negative)

2. Define Quadratic form of a matrix

Ans: A homogeneous expression of the second degree in any number of variables is called Quadratic form. The quadratic form generally denoted by $X'AX$

Ex: $(i) ax^2 + 2hxy + by^2$ (ii) $ax^2 + b\,y^2 + cz^2 + 2hxy + 2gzx + 2fyz$

3. Define canonical form of a quadratic form

Ans: Let $X^T AX$ be a quadratic form in n-variables, then there exists real non-singular linear transformation $X = PY$ which transforms $X^T AX$ to another quadratic form of type

$Y^T DY = \lambda_1 Y^2{}_1 + \lambda_2 Y^2{}_2 + .. . + \lambda_n Y^2{}_n$ then $Y^T DY$ is called canonical form of $X^T AX$.

4. Find the index and signature of quadratic form $x^2 + 2\,y^2 - 3\,z^2$

Ans: Index=2

Signature= 1

5. Define Diagonalization of a matrix and Similarity transformation

Ans: **Diagonalization of a matrix :** A matrix 'A' is diagonalizable if there exists an invertible matrix P such that $P^{-1}AP = D$, where 'D' is a diagonal matrix. Also, the matrix P is then said to diagonalize A to diagonal form.

Similarity transformation: If A and B are square matrices of order n then B is said to be similar to A. If there exists a non-singular matrix of P order n such that $D = P^{-1}AP$. The transformation $Y = PX$ called similarity transformation.

6. Find the sum and product of the eigen values of $\begin{bmatrix} 3 & 1 & 4 \\ 0 & 2 & 6 \\ 0 & 0 & 5 \end{bmatrix}$

Ans: Let A= $\begin{bmatrix} 3 & 1 & 4 \\ 0 & 2 & 6 \\ 0 & 0 & 5 \end{bmatrix}$

Given matrix is triangular matrix so diagonal elements are eigen values 3,2,5

Sum of eigen values = 3+2+5=10

Product of the eigen values = 3*2*5= 30.

7. Find the eigen values of Adj.A Where $A=\begin{bmatrix}2&3&4\\0&4&2\\0&0&3\end{bmatrix}$

Ans : The characteristic equation of A is $|A-\lambda I|=0$

The eigen values are 2, 4, 3

Now $|A|=\begin{vmatrix}2&3&4\\0&4&2\\0&0&3\end{vmatrix}=24$

The eigen values os Adj. A are $\frac{|A|}{\lambda}=$ 12,6,8.

8. If $'\lambda'$ is an eigen value of a non- singular matrix A, Prove that $\frac{|A|}{\lambda}$ is eigen value of $\boldsymbol{Adja.A}$

Ans: Since $'\lambda'$ is an eigen value of a non-singular matrix, corresponding to the eigen vector $'X'$, We have $AX=\lambda X$

$(Adj.A)AX=(AdjA)\lambda X$

$(Adj.A\,A)X=\lambda(AdjA)X$

$|A|IX=\lambda(AdjA)X$

$\frac{|A|}{\lambda}X=(Adj.A)X$

$(AdjA)X=\frac{|A|}{\lambda}X$

$\frac{|A|}{\lambda}$ is an eigen value of the matrix $Adj.A$.

9. Find $\boldsymbol{A^8}$, $\boldsymbol{If\ A=\begin{bmatrix}1&2\\2&-1\end{bmatrix}}$

Ans: The characteristic equation of A is $|A-\lambda I|=0$

$$\begin{vmatrix}1-\lambda&2\\2&-1-\lambda\end{vmatrix}=0$$

$$\lambda^2-5=0$$

By Cayley Hamilton theorem $A^2-5I=0$

$A^2=5I$

$A^4=25I$

$A^6=125I$

$A^8=625I=\begin{bmatrix}625&0\\0&625\end{bmatrix}$

10 If $A=\begin{bmatrix}5&3\\1&3\end{bmatrix}$**, find** A^n **and** A^4

Ans: Given that $A=\begin{bmatrix}5&3\\1&3\end{bmatrix}$

The characteristic equation of A is $|A-\lambda I|=0$

i.e $\begin{vmatrix}5-\lambda I&3\\1&3-\lambda I\end{vmatrix}=0$

$(5-\lambda)(3-\lambda)-3=0$

$15-3\lambda-5\lambda+\lambda^2-3=0$

$$\lambda^2 - 8\lambda + 12 = 0$$
$$\lambda = 2,6$$

Case(i):- when $\lambda = 2$, the given system of equations become,

$$x_1 + x_2 = 0 \Rightarrow x_1 = -x_2 \Rightarrow X_1 = \begin{bmatrix} 1 \\ -1 \end{bmatrix}$$

Case(ii):- when $\lambda = 6$ the given system of equations become,

$$x_1 - 3x_2 = 0 \Rightarrow x_1 = 3x_2 \Rightarrow X_2 = \begin{bmatrix} 3 \\ 1 \end{bmatrix}$$

We have Model matrix $P = \begin{bmatrix} X_1 & X_2 \end{bmatrix} = \begin{bmatrix} 1 & 3 \\ -1 & 1 \end{bmatrix}$

We have $P^{-1} = \frac{1}{4}\begin{bmatrix} 1 & -3 \\ 1 & 1 \end{bmatrix}$

We have $D = P^{-1}AP = \frac{1}{4}\begin{bmatrix} 1 & -3 \\ 1 & 1 \end{bmatrix}\begin{bmatrix} 5 & 3 \\ 1 & 3 \end{bmatrix}\begin{bmatrix} 1 & 3 \\ -1 & 1 \end{bmatrix} = \frac{1}{4}\begin{bmatrix} 8 & 0 \\ 0 & 24 \end{bmatrix} = \begin{bmatrix} 2 & 0 \\ 0 & 6 \end{bmatrix}$

$$D = P^{-1}AP$$
$$\Rightarrow D^2 = (P^{-1}AP)(P^{-1}AP)$$
$$\Rightarrow P^{-1}A^2P$$

Similarly, $D^n = P^{-1}A^nP$ ……(1)

Pre multiplying eq(1) by P and post multiplying by P^{-1} we get,

$$PD^nP^{-1} = PP^{-1}A^nPP^{-1} = A^n$$
$$\therefore A^n = P\ D^nP^{-1}$$

Put $n = 4$ in the above equation, we get

$$\therefore A^4 = P\ D^4P^{-1} = \begin{bmatrix} 1 & 3 \\ -1 & 1 \end{bmatrix}\begin{bmatrix} 16 & 0 \\ 0 & 1296 \end{bmatrix}\frac{1}{4}\begin{bmatrix} 1 & -3 \\ 1 & 1 \end{bmatrix} = \frac{1}{4}\begin{bmatrix} 3904 & 3840 \\ 1280 & 1344 \end{bmatrix} = \begin{bmatrix} 976 & 960 \\ 320 & 336 \end{bmatrix}$$

11. Define the characteristic equation of a matrix

Ans: By definition, let $A = \lfloor a_{ij} \rfloor$ be $n \times n$ matrix. Let X be an Eigen vector of A corresponding to the Eigen value λ

$$AX = \lambda X$$
$$AX - \lambda I\ X = 0$$
$$(A - \lambda I)\ X = 0$$

$|A - \lambda I| = 0$ is called the characteristic equation of A.

12. Write two properties of Eigen values

Ans: (i) The sum of the Eigen values of a matrix is equal to sum of its principal diagonal elements.

(ii) The product of the Eigen values of a matrix is equal to its determinant.

13. Prove that the sum of Eigen values of a matrix is equal to sum of its diagonal elements.

i.e, If A is an $n \times n$ matrix and $\lambda_1, \lambda_2, \lambda_3, ...\lambda_n$ are its n Eigen values, then

$$\lambda_1 + \lambda_2 + \lambda_3 + ... + \lambda_n = Tr(A) \quad \textbf{and} \quad \lambda_1.\lambda_2\lambda_3......\lambda_n = \det(A)$$

Ans: Characteristic equation of A is $|A - \lambda I| = 0$

$$\begin{vmatrix} a_{11}-\lambda & a_{12} & ...... & a_{1n} \\ a_{21} & a_{22}-\lambda & ...... & a_{2n} \\ .... & ...... & .... & .... \\ a_{n1} & a_{n2} & ..... & a_{nn}-\lambda \end{vmatrix} = 0$$

expanding this, we get

$$(a_{11}-\lambda)(a_{22}-\lambda)..........(a_{nn}-\lambda) - a_{12}(a polynomial of \deg ree \ \ n-2)$$
$$+ a_{13}(a \ \ polynomial \ \ \deg ree \ \ n-2) + ... = 0$$

$$(-1)^n (\lambda - a_{11})(\lambda - a_{22}).........(\lambda - a_{nn}) + a \ \ polynomial of \ \deg ree \ \ (n-2) = 0$$

$$(-1)^n [\lambda^n - (a_{11} + a_{22} + ... + a_{nn}) + a \ \ polynomial \ of \ \deg ree \ \ (n-2) = 0$$

$$(-1)^n \lambda^n + (-1)^{n+1}(a_{11} + a_{22} + ... + a_{nn})\lambda^{n-1} + a \ polynomial \ of \ \deg ree(n-2) \ in \ \lambda = 0$$

$$(-1)^n \lambda^n + (-1)^{n+1}(Trace)\lambda^{n-1} + a \ polynomial \ of \ \deg ree(n-2) \ in \ \lambda = 0$$

if $\lambda_1, \lambda_2, \lambda_3.....\lambda_n$ are the roots of the equation,

Sum of the roots $= \dfrac{-b}{a} = \dfrac{-(-1)^{n+1}Tr(A)}{(-1)^n} = Tr(A)$

On expanding the determinant eqn (1) takes the form

$$(-1)^n \lambda^n + a_{n-1}\lambda^{n-1} + a_{n-2}\lambda^{n-2} + ... + a_0 = 0 \quad ....(2)$$

Put $\lambda = 0$ in eq (2), we get $|A| = a_0$

Product of the roots $= \dfrac{c}{a} = \dfrac{(-1)^n a_0}{(-1)^n} = |A| = \det A$

14. If λ is the Eigen value of a matrix A, then prove that $\dfrac{1}{\lambda}$ is an Eigen value of A^{-1}

Ans: Since A is non-singular and product of the Eigen values is equal to $\det A$, it follows that none of the Eigen values of A is 0.

If λ is the Eigen value of a non - singular matrix A and X is the corresponding Eigen vector,

$\lambda \neq 0$ and $AX = IX$, pre multiplying this with A^{-1} we get,

$$A^{-1}(AX) = A^{-1}(IX) \Rightarrow (A^{-1}A)X \Rightarrow IX = \lambda A^{-1}X$$

$$X = \lambda A^{-1}X \Rightarrow A^{-1}X = \lambda^{-1}X (\because \lambda \neq 0)$$

Hence, by definition, it follows that $\dfrac{1}{\lambda}$ is an Eigen value of A^{-1} and X is the corresponding Eigen vector.

15. If λ is the Eigen value of an orthogonal matrix A, then prove that $\dfrac{1}{\lambda}$ is also its Eigen value

Ans: we know that If λ is the Eigen value of a matrix A, then prove that $\dfrac{1}{\lambda}$ is an Eigen value of

A^{-1}. Since A is an orthogonal matrix, we have $A^{-1} = A'$

$\therefore \frac{1}{\lambda}$ is an Eigen value of A'

But the matrices A and A' have the same Eigen values, since the determinants $|A-\lambda I|$ and

$|A'-\lambda I|$ are same.

Hence, $\frac{1}{\lambda}$ is also an Eigen value of A.

16. Prove that If $\lambda_1, \lambda_2, \lambda_3 .....\lambda_n$ are the Eigen values of A then are the Eigen values of A^k

Ans: Since λ is an Eigen value of A corresponding to the Eigen vector X, we have $AX = \lambda X$ ……(1)

Pre multiplying eq (1) by A, we get

$$A(AX) = A(\lambda X) \Rightarrow (AA)X = \lambda(AX) \Rightarrow A^2 X = \lambda.\lambda X = \lambda^2 X$$

Hence λ^2 is Eigen value of A^2 with X itself as the corresponding Eigen vector. Thus the result be true for n=2.

Let the result be true for n=k. Then

$$A^k X = \lambda^k X$$

pre multiplying this by A and using $AX = \lambda X$, we get $A^{k+1} X = \lambda^{k+1} X$

$\Rightarrow \lambda^{k+1}$ is Eigen value of A^{k+1} with X itself as the corresponding Eigen vector. Hence by the principle of mathematical induction, the theorem is true for all positive integer n.

17. The Eigen values of an idempotent matrix are either zero (or) unity.

Ans: Let A be an idempotent matrix i.e., $A^2 = A.$

If λ be a Eigen value of A, then there exists a non-zero vector X such that

$$AX = \lambda X .....(1) \Rightarrow A(AX) = A(\lambda X) \Rightarrow A^2 X = \lambda(AX) \Rightarrow AX = \lambda^2 X .....(2)$$

From eq (1) and eq (2), we get

$$\lambda^2 X = \lambda x \ (or) \ (\lambda^2 - \lambda) X = 0 \Rightarrow \lambda^2 - \lambda = 0 \Rightarrow \lambda = 0 \ (or) \ 1$$

18. Find e^A and 4^A, if $A = \begin{bmatrix} 3/2 & 1/2 \\ 1/2 & 3/2 \end{bmatrix}$

The characteristic equation of A is $|A - \lambda I| = 0$

i.e, $\begin{vmatrix} \frac{3}{2} - \lambda & \frac{1}{2} \\ \frac{1}{2} & \frac{3}{2} - \lambda \end{vmatrix} = 0 \Rightarrow \lambda^2 - 3\lambda + 2 = 0 \Rightarrow \lambda = 1,2$

Case(i) :- when $\lambda = 1$, we have $(A - \lambda I) X_1 = 0 \Rightarrow (A - I) X_1 = 0$

$$\begin{bmatrix} 1/2 & 1/2 \\ 1/2 & 1/2 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$$

$$X_1 = \begin{bmatrix} 1 \\ -1 \end{bmatrix}$$

Case (ii);- when $\lambda = 2$, we have $(A - \lambda I) X_2 = 0 \Rightarrow (A - 2I) X_2 = 0$

$$\begin{bmatrix} -1 & 1 \\ 1 & -1 \end{bmatrix}\begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$$

$$X_2 = \begin{bmatrix} 1 \\ 1 \end{bmatrix}$$

Therefore the model matrix $P = \begin{bmatrix} 1 & 1 \\ -1 & 1 \end{bmatrix}$, $P^{-1} = \frac{1}{2}\begin{bmatrix} 1 & -1 \\ 1 & 1 \end{bmatrix}$

The Diagonal matrix $D = P^{-1}AP = \begin{bmatrix} 1 & 0 \\ 0 & 2 \end{bmatrix}$

$$f(A) = e^A \ , \ f(D) = e^D = \begin{bmatrix} e & 0 \\ 0 & e^2 \end{bmatrix}$$

$$e^A = Pf(D)P^{-1} = \frac{1}{2}\begin{bmatrix} 1 & 1 \\ -1 & 1 \end{bmatrix}\begin{bmatrix} e & 0 \\ 0 & e^2 \end{bmatrix}\begin{bmatrix} 1 & -1 \\ 1 & 1 \end{bmatrix} = \frac{1}{2}\begin{bmatrix} e+e^2 & -e+e^2 \\ -e+e^2 & e+e^2 \end{bmatrix}$$

Replacing e by 4, we get $4^A = \frac{1}{2}\begin{bmatrix} 20 & 12 \\ 12 & 20 \end{bmatrix} = \begin{bmatrix} 10 & 6 \\ 6 & 10 \end{bmatrix}$

3. CALCULUS

1. State Roll's theorem.

Sol.: Let $f(x)$ be a function, such that

(i) $f(x)$ is continuous in [a, b]
(ii) $f(x)$ is differentiable in (a, b)
(iii) $f(a)=f(b)$, then there exits at least one point c in (a, b) such that $f^1(c)=0$.

2. State Lagrange's mean value theorem.

Sol.: Let $f(x)$ be a function, such that

(i) $f(x)$ is continuous in [a, b]
(ii) $f(x)$ is differentiable in (a, b)
(iii) $f(a)=f(b)$, then there exits at least one point c in (a, b) such that $f^1(c)=\frac{f(b)-f(a)}{b-a}$.

3. State Cauchy's mean value theorem.

Sol.: Let $f(x)$, $g(x)$ be two functions such that

(i) $f(x)$, $g(x)$ is continuous in [a, b]
(ii) $f(x)$, $g(x)$ is differentiable in (a, b)
(iii) $g^1(x)\neq 0, \forall x\in(a,b)$, then there exits at least one point c in (a, b) such that

$$\frac{f(b)-f(a)}{g(b)-g(a)}=\frac{f^1(c)}{g^1(c)}.$$

4. State Second form of Lagrange's mean value theorem.

Sol.: Let $f(x)$ be a function, such that

(i) $f(x)$ is continuous in [a, b]
(ii) $f(x)$ is differentiable in (a, b)

then there is at least one number $\theta,(0<\theta<1)$ such that $f(a+h)=f(a)+f^1(a+\theta h)$.

5. If $f(x)=x+\frac{1}{x}$ in $\left[\frac{1}{2},1\right]$ satisfies Rolle's Theorem then find the value of 'c'.

Sol.: Given $f(x)=x+\frac{1}{x}\Rightarrow f^1(x)=1-\frac{1}{x^2}$

Form Rolle 's Theorem, $f^1(c)=0 \Rightarrow 1-\frac{1}{c_2}=0\Rightarrow c^2=1\Rightarrow c=\pm 1$. *So,* $c=1\in\left(\frac{1}{2},1\right)$

6. If $f(x)=x^2-3x+2$ in $[-2,\,3]$ satisfies Lagrange's mean value theorem then find the value of 'c'.

Sol.: Given $f(x)=x^2-3x+2 \Rightarrow f^1(x)=2x-3$

From Lagrange's mean value theorem, $f^1(c)=\dfrac{f(b)-f(a)}{b-a}$

$$\Rightarrow f^1(c)=\frac{f(3)-f(2)}{3-2}$$

$$\Rightarrow 2c-3=-2$$

$$c=\frac{1}{2}\in(-2,3).$$

7. If $f(x)=\sqrt{x}$ and $g(x)=\dfrac{1}{\sqrt{x}}$ satisfies Cauchy's mean value theorem in $[a,b]$ then find 'c'.

Sol.: Given $f(x)=\sqrt{x},\ g(x)=\dfrac{1}{\sqrt{x}}$

$$f^1(x)=\frac{1}{2\sqrt{x}},\quad g^1(x)=\frac{-1}{2}x^{-3/2}$$

From Cauchy's mean value theorem, $\dfrac{f^1(c)}{g^1(c)}=\dfrac{f(b)-f(a)}{g(b)-g(c)}$

$$\Rightarrow \frac{\dfrac{1}{2\sqrt{c}}}{\dfrac{-1}{2}c^{-3/2}}=\frac{\sqrt{b}-\sqrt{a}}{\dfrac{1}{\sqrt{b}}-\dfrac{1}{\sqrt{a}}}$$

$$\Rightarrow c=\sqrt{ab}\in(a,b).$$

8. State Taylor's theorem.

Sol.: If $f:[a,b]\to R$ is a function such that

(i) $f^{(n-1)}$ is continuous on $[a,b]$

(ii) $f^{(n-1)}$ is differentiable on (a,b), then for $p\in Z^+$ there exists a point $c\in(a,b)$ such that

$$f(b)=f(a)+\frac{b-a}{1!}f^1(a)+\frac{(b-a)^2}{2!}f^{11}(a)+------\frac{(b-a)^{n-1}}{(n-1)!}f^{(n-1)}(a)+R_n,\ \text{where}$$

$$R_n=\frac{(b-a)^p(b-c)^{n-p}}{(n-1)!\,p}f^{(n)}(c).$$

9. State Maclaurin's theorem.

Sol.: If $f:[o,x]\to R$ is a function such that

(i) $f^{(n-1)}$ is continuous on $[0,x]$

(ii) $f^{(n-1)}$ is differentiable on $(0,x)$, then there exists a real number $\theta\in(0,1)$ such that

$$f(x)=f(0)+x\,f^{1}(0)+\frac{x^{2}}{2!}f^{11}(0)+------\frac{x^{n-1}}{(n-1)!}f^{(n-1)}(0)+\frac{x^{n}(1-\theta)^{n-p}}{p(n-1!)}f^{(n)}(\theta x).$$

10. State another form of Taylor's theorem with Lagrange's form of remainder.

Sol.: If $f:[a,a+h]\to R$ is a function such that

(i) $f^{(n-1)}$ is continuous on $[a,a+h]$

(ii) $f^{(n-1)}$ is differentiable on $(a,a+h)$, and $p\in Z^{+}$ then there exists a real number $0<\theta<1$ such that

$$f(a+h)=f(a)+h\,f^{1}(a)+\frac{h^{2}}{2!}f^{11}(a)+------\frac{h^{n-1}}{(n-1)!}f^{(n-1)}(a)+R_{n}, \text{ where}$$

$$R_{n}=\frac{h^{n}\,f^{(n)}(a+\theta h)}{n!}.$$

UNIT-IV
PARTIAL DIFFERENTIATION

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1. *Find the first & second order partial derivatives of* $ax^2+2hxy+by^2$ *and verify*

$$\frac{\partial^2 f}{\partial x\partial y}=\frac{\partial^2 f}{\partial y\partial x}$$

Sol. Given $f(x,y)=ax^2+2hxy+by^2$ then

$$\frac{\partial f}{\partial x}=2ax+2hy,\frac{\partial f}{\partial y}=2by+2hx,\frac{\partial^2 f}{\partial x^2}=2a,\frac{\partial^2 f}{\partial y^2}=2b$$

$$\frac{\partial^2 f}{\partial x\partial y}=\frac{\partial}{\partial x}\left(\frac{\partial f}{\partial y}\right)=2h,\ \frac{\partial^2 f}{\partial y\partial x}=\frac{\partial}{\partial y}\left(\frac{\partial f}{\partial x}\right)=2h$$

Therefore $\dfrac{\partial^2 f}{\partial x\partial y}=\dfrac{\partial^2 f}{\partial y\partial x}$

2. *Define Homogeneous function.*

Sol. A function $f(x,y)$ is said to a homogeneous function of degree n in variables x,y if $f(kx,ky)=k^n f(x,y)$, where n is a real number

3. ***State Euler's theorem***.

Sol. If $z=f(x,y)$ is a homogeneous function of degree n then $x\dfrac{\partial z}{\partial x}+y\dfrac{\partial z}{\partial y}=nz,\forall x,y$ in the domain of the function.

4. ***Define Jacobians***.

Sol. Let u=u(x,y) and v=v(x,y) then these two simultaneous relations constitute a transformation from (x,y) to (u,v), the determinant $\begin{vmatrix} u_x & u_y \\ v_x & v_y \end{vmatrix}$ is called the Jacobian of u,v w.r.t. x and y.

5. ***Write any two properties of Jacobians***.

Sol. i) If $J=\dfrac{\partial(u,v)}{\partial(x,y)}$ & $J'=\dfrac{\partial(x,y)}{\partial(u,v)}$ then $JJ'=1$

ii) if u,v are functions of r,s and r,s functions of x,y then

$$\frac{\partial(u,v)}{\partial(x,y)}=\frac{\partial(u,v)}{\partial(r,s)}.\frac{\partial(r,s)}{\partial(x,y)}$$

6. Write the Chain rule and functionally dependence of Jacobians.

Sol. ***Chain Rule***: If u,v are functions of r,s and r,s functions of x,y then

$$\frac{\partial(u,v)}{\partial(x,y)}=\frac{\partial(u,v)}{\partial(r,s)}.\frac{\partial(r,s)}{\partial(x,y)}.$$

Functionally dependence: If u_1, u_2, u_3 be the functions of x_1, x_2, x_3 then the necessary and sufficient condition for the existence of a function relationship of the form $f(u_1, u_2, u_3) = 0$ is $J\left(\frac{u_1, u_2, u_3}{x_1, x_2, x_3}\right) = 0$

7. *State Taylor's theorem.*

Sol. If $f(x, y)$ possess continuous partial derivatives of n^{th} order in any neighborhood of a point (x,y) and if (x+h,y+k) is any point of this neighborhood, then

$$f(x+h, y+k) = f(x, y) + \left(h\frac{\partial}{\partial x} + k\frac{\partial}{\partial y}\right) f(x, y) + -----$$

8. Write the formula for Taylor's theorem in powers of (x-a) and (y-b).

Sol. If $f(x, y)$ possess continuous partial derivatives of n^{th} order in any neighborhood of a point (a,b) then $f(x, y) = f(a, b) + \lfloor (x-a) f_x(a, b) + (y-b) f_y(a, b) \rfloor + -----$

9. ***Write the formula for maclaurin's series***.

Sol. If $f(x, y)$ possess continuous partial derivatives of n^{th} order in any neighborhood of a point (0,0) then $f(x, y) = f(0,0) + \lfloor x f_x(0,0) + y f_y(0,0) \rfloor + -----$

10. ***Define maximum & minimum values***.

Sol. Let f(x,y) be a function of two variables x & y, at x=a, y=b then f(x,y) is said to be maximum or minimum value, if f(a,b)>f(a+h,b+k) or f(a,b)<f(a+h,b+k) respectively, where h, k are small values.

11. ***Define extreme value***.

Sol. If f(a,b) is said to be an extreme value of f, if it is maximum or minimum value.

12. ***Define Stationary value***.

Sol. If f(a,b) is said to be a stationary value of f(x,y) if $f_x(a, b) = 0$ & $f_y(a, b) = 0$. Thus every extreme value is a stationary value but the converse may not be true.

13. ***Discuss the maximum and minimum*** **of** $x^2 + y^2 + 6x + 12$

Sol. Given $f(x, y) = x^2 + y^2 + 6x + 12$

$$\frac{\partial f}{\partial x} = 2x + 6 = 0, \frac{\partial f}{\partial y} = 2y = 0, \frac{\partial^2 f}{\partial x^2} = 2, \frac{\partial^2 f}{\partial y^2} = 2, \frac{\partial^2 f}{\partial x \partial y} = 0$$

x=-3 & y=0; at the point (-3,0)

$\ln - m^2 = 4 > 0 \,\&\, l = 2 > 0$

Hence f(x,y) will be minimum, when x=-3 & y=0

Therefore minimum value is f(-3,0)=9+0-18+12=3

14. ***Write the formula for Leibnitz's rule***.

Sol. If $f(x, \alpha)$ & $\frac{\partial f(x, \alpha)}{\partial \alpha}$ be continuous functions of x & α then

$$\frac{d}{d\alpha}\left[\int_a^b f(x, \alpha) dx\right] = \int_a^b \frac{\partial}{\partial \alpha} f(x, \alpha) dx$$ where a,b are constants.

15. ***Write the formula for Leibnitz's for the variable limits of integration***.

Sol. If $f(x,\alpha)$ & $\frac{\partial f(x,\alpha)}{\partial \alpha}$ be continuous functions of x & α then

$\frac{d}{d\alpha}\left[\int_{\phi(\alpha)}^{\psi(\alpha)} f(x,\alpha)dx\right] = \int_{\phi(\alpha)}^{\psi(\alpha)} \frac{\partial}{\partial \alpha} f(x,\alpha)dx + \frac{d\psi}{d\alpha} f(\psi(\alpha),\alpha) - \frac{d\phi}{d\alpha} f(\phi(\alpha),\alpha)$, provided $\psi(\alpha)$ & $\phi(\alpha)$ possess continuous first order derivatives w.r.t. α

UNIT – 5

MULTIPLE INTEGRALS

1. Double Integrals

Generally, a Double Integral may be of the form $I = \int_{y_1}^{y_2} \int_{x_1}^{x_2} f(x,y)dx\,dy$.

2. Evaluation of Double Integrals in cartesian co-ordinates.

A Double Integral $\int_{y_1}^{y_2} \int_{x_1}^{x_2} f(x,y)dx\,dy$ can be evaluated in one of the following ways.

Case.1: When *x* limits are variables and *y* limits are constants

In this case, x_1, x_2 are functions of y and y_1, y_2 are constants *i.e.*, the double integral will be of the form $\int_a^b \left[\int_{\phi_1(y)}^{\phi_2(y)} f(x,y)\,dx\right] dy$.

- Here, the order of integration is $dx\,dy$.
- The region of integration will be bounded by the cross section of horizontal strips or by sliding a horizontal strip in the given domain (region).
- First, we have to integrate *w.r.t.* x between the limits $x = \phi_1(y)$ and $x = \phi_2(y)$ treating y as constant (so that the resultant will be entirely function of y) and then integrate *w.r.t.* y between the limits a and b.

Case.2: When *y* limits are variables and *x* limits are constants

In this case, y_1, y_2 are functions of x and x_1, x_2 are constants *i.e.*, the double integral will be of the form $\int_c^d \left[\int_{\varphi_1(x)}^{\varphi_2(x)} f(x,y)\,dy\right] dx$.

- Here, the order of integration is $dy\,dx$.
- The region of integration will be bounded by the cross section of vertical strips or by sliding a vertical strip.
- First, we have to integrate *w.r.t.* x treating y as constant (so that the resultant will be entirely function of y) and then integrate *w.r.t.* y.

Case.3: When both *x* and *y* limits are are constants

In this case, the double integral will be of the form $\int_c^d \int_a^b f(x,y)\;dx\,dy$.

- Here, the order of integration is immaterial. *i.e.*, either $dx\,dy$ (*i.e.*, we can integrate first *w.r.t.* x between the limits a and b and then *w.r.t.* y between c and d) or $dy\,dx$ (*i.e.*, first *w.r.t.* y between c and d and then *w.r.t.* x between a and b).
- The region of integration will be a rectangle.

NOTE: To evaluate a double Integral, observe the variable limits carefully as we have to evaluate with variable limits first,

1. If the variable limits are functions of *x*, then proceed as in case 2.
2. If the variable limits are functions of *x*, then proceed as in case 1.
3. If the limits are not given and the region is specified, then select either case1 or case2. If the

region is a rectangle or square, then follow case.3

3. Evaluation of Double Integrals in polar co-ordinates.

A Double Integral in polar co-ordinates will be of the form $I = \int_{\theta_1}^{\theta_2} \int_{r_1}^{r_2} f(x,y) r dr\, d\theta$. Here, *r* limits will be variables (*i.e., r* will be expressed as function of θ) and θ limits will be constants. Hence, r_1 and r_2 will be expressed as functions of θ.

In the case of polar co-ordinates, we have to introduce radial strip in the given region.

and $I = \int_{\theta_1}^{\theta_2} \int_{f_1(\theta)}^{f_2(\theta)} f(x,y)\, r\, dr\, d\theta$.

4. Change of order of integration

Sometimes, evaluation of integrals become quite easier when we change the order of integration.

Procedure

Step.1 Trace out the region of integration *R*, from the given limits.
(From the variable limits, we can decide whether the region is bounded by vertical strip or horizontal strip. Suppose the variable limits are in terms of *x* (which represent *y* limits), then *R* will be bounded by vertical strip and vice-versa.

Step.2 Find the boundaries of *R* from the constant limits and by finding intersection points of the two curves, so that *R* will be specified completely.

Step.3 Introduce the alternate strip in *R*, to change the order of integration.

Step.4 Now, find the limits for integration.
(Generally, strip end points give variable limits and sliding of the strip gives constant Limits).

Step.5 Finally, evaluate the integral with new limits.

5. Area enclosed between two plane curves is given by

$A = \iint_R dxdy$ or $\iint_R dydx$ in case of cartesian co-ordinates

or $A = \iint_R r\, dr\, d\theta$ in case of polar co-ordinates.

Here *R* is the region enclosed between the two curves.

6. Triple Integrals

Generally, it will be of the form $I = \int_{z_1}^{z_2} \int_{y_1}^{y_2} \int_{x_1}^{x_2} f(x,y,z)\, dx\, dy\, dz$

If the order of integration is $dx\, dy\, dz$, then $x_1$and x_2 will be expressed as functions of *y* and *z*, $y_1$and y_2 will be expressed as functions of *z* and $z_1$and z_2 will be constants.

Here, we integrate first *w.r.t. x* treating *y* and *z* as constants, then *w.r.t. y* treating *z* as constant and finally *w.r.t. z.*

NOTE: In a similar manner, depending on the variable limits, the order of Integration will be decided.

7. Change of Variables

This concept is used when one type of co-ordinate system is transferred into another.

I. Two – dimension

(i) When cartesian co-ordinate system (X, Y) is transformed in to a new co-ordinate system (U,V), then the transformations are $x = \phi(u,v)$, $y = \psi(u,v)$ and

$$\iint_{R_{XY}} \boldsymbol{f(x,y)\, dx\, dy} = \iint_{R_{UV}} \boldsymbol{f(\phi,\psi)|\,J\,|\, du\, dv}$$

Here $\boldsymbol{R_{XY}}$ is the region in XY-plane.
$\boldsymbol{R_{UV}}$ is the region in UV-plane which is the image of $\boldsymbol{R_{XY}}$.

$|\,\boldsymbol{J}\,|$ is the jacobian of the transformation given by $|\,\boldsymbol{J}\,| = \begin{vmatrix} \frac{\partial x}{\partial u} & \frac{\partial x}{\partial v} \\ \frac{\partial y}{\partial u} & \frac{\partial y}{\partial v} \end{vmatrix}$

(ii) Polar Co-ordinates

When cartesian co-ordinate system (X, Y) is transformed in to polar co-ordinates (r, θ):

Transformations

$\underline{(x,y) \to (r,\theta)}$	$\underline{(r,\theta) \to (x,y)}$
$x = r\cos\theta$ $y = r\sin\theta$	$r = \sqrt{x^2+y^2}$ $\theta = \tan^{-1}\left(\frac{y}{x}\right)$

The **jacobian** of the transformation is given by

$$|\,\boldsymbol{J}\,| = \begin{vmatrix} \frac{\partial x}{\partial r} & \frac{\partial x}{\partial \theta} \\ \frac{\partial y}{\partial r} & \frac{\partial y}{\partial \theta} \end{vmatrix} = \begin{vmatrix} \cos\theta & -r\sin\theta \\ \sin\theta & r\cos\theta \end{vmatrix} = \boldsymbol{r}$$

$$\therefore\ dx\, dy = |J|\, dr\, d\theta = r\, dr\, d\theta$$

Hence, $\iint_R \boldsymbol{f(x,y)\, dx\, dy} = \iint_R \boldsymbol{f(r\cos\theta, r\sin\theta)\, r\, dr\, d\theta}$

where R is the region of integration.

<u>II. Three - dimension</u>

(i) When cartesian co-ordinate system (*X, Y, Z*) is transformed in to a new co-ordinate system (*U,V, W*),

Transformations: are $x = \phi(u,v,w)$, $y = \varphi(u,v,w)$ and $z = \xi(u,v,w)$

$|\,\boldsymbol{J}\,| =$ **Jacobian** of the transformation given by $|\,\boldsymbol{J}\,| = \begin{vmatrix} \frac{\partial x}{\partial u} & \frac{\partial x}{\partial v} & \frac{\partial x}{\partial w} \\ \frac{\partial y}{\partial u} & \frac{\partial y}{\partial v} & \frac{\partial y}{\partial w} \\ \frac{\partial z}{\partial u} & \frac{\partial z}{\partial v} & \frac{\partial z}{\partial w} \end{vmatrix}$

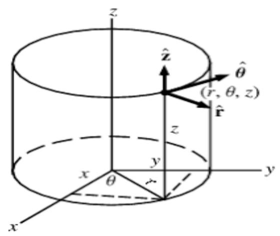
Let $\boldsymbol{R_{xyz}}$ be the given region in *XYZ*-plane and let image of $\boldsymbol{R_{xyz}}$ be $\boldsymbol{R_{uvw}}$ which is the region in *UVW*-plane.

Hence, $\iint_{R_{xyz}} \boldsymbol{f(x,y,z)\, dx\, dydz} = \iint_{R_{uvw}} \boldsymbol{f(\phi,\varphi,\xi)|\,J\,|\, du\, dv\, dw}$

(ii) Cylindrical Polar Co-ordinates: ρ, ϕ, z

The three coordinates (ρ, ϕ, z) at a point P are defined as radial co-ordinate, azimuthal co-ordinate and axial co-ordinate.

- ρ is the radius vector or radial distance from the z-axis to the point P.
- ϕ is the *azimuthal* co-ordinate which is the angle made by the radius vector ρ *w.r.t.* positive X-axis to get the base of the cylinder or chosen plane.
- z is the *axial coordinate* or height of the cylinder



NOTE: Sometimes, cylindrical polar co-ordinates may also be represented as r, θ, z

Transformations

$(x, y, z) \to (\rho, \phi, z)$	$(\rho, \phi, z) \to (x, y, z)$
$x = \rho\cos\phi$ $y = \rho\sin\phi$ $z = z$	$\rho = \sqrt{x^2 + y^2}$ $\phi = \tan^{-1}\left(\frac{y}{x}\right)$ $z = z$

The **jacobian** of the transformation is given by

$$|\boldsymbol{J}| = \begin{vmatrix} \frac{\partial x}{\partial \rho} & \frac{\partial x}{\partial \phi} & \frac{\partial x}{\partial z} \\ \frac{\partial y}{\partial \rho} & \frac{\partial y}{\partial \phi} & \frac{\partial y}{\partial z} \\ \frac{\partial z}{\partial \rho} & \frac{\partial z}{\partial \phi} & \frac{\partial z}{\partial z} \end{vmatrix} = \begin{vmatrix} \cos\phi & -\rho\sin\phi & 0 \\ \sin\phi & \rho\cos\phi & 0 \\ 0 & 0 & 1 \end{vmatrix} = \rho$$

$$\therefore\ dx\,dy\,dz = |J|\,d\rho\,d\phi\,dz = \rho d\rho\,d\phi\,dz$$

Hence a triple integral in cylindrical polar co-ordinates can be evaluated as

$$\iiint_R \boldsymbol{f(x, y, z)\,dx\,dy\,dz} = \iiint_R \boldsymbol{f(\rho\cos\phi, \rho\sin\phi, z)\,\rho\,d\rho\,d\phi\,dz}$$

where R is the region of integration.

(iii) Spherical polar co-ordinates (r, θ, ϕ)

Significance of the co-ordinates:

r indicates the length of the radius vector

θ is the polar angle, *i.e.,* angle made by the radius vector r (OP) with positive z-axis

Here O is the origin and P is any point on the sphere.

ϕ is the azimuthal angle which is the rotational orientation of the plane (which is formed by the rotation of radius vector) with Z-axis.

Transformations

$\underline{(x,y,z) \to (r,\theta,\phi)}$	$\underline{(r,\theta,\phi) \to (x,y,z)}$
$x = r\sin\theta\cos\phi$ $y = r\sin\theta\sin\phi$ $z = r\cos\theta$	$r = \sqrt{x^2+y^2+z^2}$ $\theta = \cos^{-1}\left(\frac{z}{\sqrt{x^2+y^2+z^2}}\right)$ $\phi = \tan^{-1}\left(\frac{y}{x}\right)$

The volume element in spherical polar coordinates can be determined from the Jacobian:

$$J = \frac{\partial(x,y,z)}{\partial(r,\theta,\phi)} = \begin{vmatrix} x_r & y_r & z_r \\ x_\theta & y_\theta & z_\theta \\ x_\phi & y_\phi & z_\phi \end{vmatrix} =$$

$$\begin{vmatrix} \sin\theta\cos\phi & \sin\theta\sin\phi & \cos\theta \\ r\cos\theta\cos\phi & r\cos\theta\sin\phi & -r\sin\theta \\ -r\sin\theta\sin\phi & r\sin\theta\cos\phi & 0 \end{vmatrix} = r^2\sin\theta.$$

Hence a triple integral in spherical polar co-ordinates can be evaluated as

$$\iiint_R f(x,y,z)\,dx\,dy\,dz = \iiint_R f(r\sin\theta\cos\phi, r\sin\theta\sin\phi, r\cos\theta)\,r^2\sin\theta\,dr\,d\theta\,d\phi$$

where R is the region of integration.

HINT for Limits:

Sphere is formed by rotating the semi-circle one revolution.

Semi-circle is formed by rotating the radius vector 180^0

$\therefore 0 \le r \le a,\ \ 0 \le \theta \le \pi,\ \ 0 \le \phi \le 2\pi$

8. Volume of a solid

1. Volume of a solid as a double integral = $\iint_S z\,dx\,dy = \iint_S f(x,y)\,dx\,dy$

   or $\iint_S z\,r dr\,d\theta$

   Here S is the projection of the given region on XY- plane.
2. Volume as triple integral = $\iiint_V dxdydz$
3. Volume formed by the revolution of the plane (of area A) around X-axis = $\iint_A 2\pi y\,dxdy$

   $= \iint_A 2\pi r^2\sin\theta\ dr d\theta$ (in polar co-ordinates)

   Volume formed by the revolution of the plane around Y-axis = $\iint_A 2\pi x\,dxdy$

   $= \iint_A 2\pi r^2\cos\theta\ dr d\theta$ (in polar co-ordinates)