NIVEDITHA V NAIR
22B2109

SOC (FEA using ML)

③ Given, $u'' + u = x^2, \quad 0 < x < 1$

$u(0) = 0, \quad u'(1) = 1$

(a) TWO-TERM :-

Let $u(x) \approx C_0 + C_1 x + C_2 x^2 + C_3 x^3$

$u(0) = 0 \Rightarrow C_0 = 0$

$u'(1) = 1 \Rightarrow C_1 + 2C_2 + 3C_3 = 1$

$\Rightarrow C_1 = 1 - 2C_2 - 3C_3$

$\therefore u(x) \approx (1 - 2C_2 - 3C_3)x + C_2 x^2 + C_3 x^3$ [Two const. C_2, C_3]

$u'(x) = 1 - 2C_2 - 3C_3 + 2C_2 x + 3C_3 x^2$

$u''(x) = 2C_2 + 6C_3 x$

(b) THREE-TERM :

Let $u(x) \approx C_0 + C_1 x + C_2 x^2 + C_3 x^3 + C_4 x^4$

$u(0) = 0 \Rightarrow C_0 = 0$

$u'(1) = 1 \Rightarrow C_1 + 2C_2 + 3C_3 + 4C_4 = 1$

$\therefore u(x) \approx (1 - 2C_2 - 3C_3 - 4C_4)x + C_2 x^2 + C_3 x^2 + C_4 x^4$

$u'(x) = 1 - 2C_2 - 3C_3 - 4C_4 + 2C_2 x + 3C_3 x^2 + 4C_4 x^3$

$u''(x) = 2C_2 + 6C_3 x + 12 C_4 x^2$

(b) COLLOCATION METHOD :

* Two-term ($x_i = 1/3, 2/3$).

$R(x) = 2C_2 + 6C_3 x + (1 - 2C_2 - 3C_3)x + C_2 x^2 + C_3 x^3 - x^2$

$R(1/3) = 2C_2 + 6C_3 \cdot (1/3) + (1 - 2C_2 - 3C_3) \cdot \frac{1}{3} + C_2 (1/3)^2 + C_3 (1/3)^3 - (1/3)^2$

$= 2C_2 + 2C_3 + 1/3 - \frac{2}{3}C_2 - C_3 + \frac{C_2}{9} + \frac{C_3}{27} - \frac{1}{9}$

$= \left(2 - \frac{2}{3} + \frac{1}{9}\right)C_2 + \left(2 - 1 + \frac{1}{27}\right)C_3 + \frac{1}{3} - \frac{1}{9}$

$= \frac{18 - 6 + 1}{9} C_2 + \left(\frac{27 + 1}{27}\right)C_3 + \frac{2}{9}$

$= \frac{13}{9} C_2 + \frac{28}{27} C_3 + \frac{2}{9}$

$$R(2/3) = 2c_2 + 6c_3 \cdot (2/3) + (1 - 2c_2 - 3c_3)(2/3) + c_2 (2/3)^2 + c_3 (2/3)^3 - (2/3)^2$$

$$= 2c_2 + 4c_3 + \frac{2}{3} - \frac{4}{3}c_2 - 2c_3 + \frac{4}{9}c_2 + \frac{8}{27}c_3 - \frac{4}{9}$$

$$= c_2\left(2 - \frac{4}{3} + \frac{4}{9}\right) + c_3\left(4 - 2 + \frac{8}{27}\right) + \frac{2}{3} - \frac{4}{9}$$

$$= c_2\left(\frac{18 - 12 + 4}{9}\right) + c_3\left(\frac{54 + 8}{27}\right) + \frac{6 - 4}{9}$$

$$= c_2(10/9) + c_3\left(\frac{62}{27}\right) + \frac{2}{9}$$

$R(1/3) = 0$, $R(2/3) = 0$

$$\Rightarrow \frac{13}{9}c_2 + \frac{28}{27}c_3 + \frac{2}{9} = 0$$

$$\frac{10}{9}c_2 + \frac{62}{27}c_3 + \frac{2}{9} = 0$$

solving,

$$c_2 = -\frac{34}{263}, \quad c_3 = \frac{-9}{263}$$

$$\therefore u(x) \approx \frac{358}{263}x - \frac{34}{263}x^2 - \frac{9}{263}x^3$$

* Three-term : $(x_i = 1/4, 1/2, 3/4)$

$$R(x) = 2c_2 + 6c_3 x + 12c_4 x^2 + (1 - 2c_2 - 3c_3 - 4c_4)x + c_2 x^2 + c_3 x^3 + c_4 x^4 - x^2$$

$$R(1/4) = 2c_2 + 6c_3(1/4) + 12c_4(1/4)^2 + (1 - 2c_2 - 3c_3 - 4c_4)(1/4) + c_2(1/4)^2 + c_3(1/4)^3 + c_4(1/4)^4 - (1/4)^2$$

$$= 2c_2 + \frac{3}{2}c_3 + \frac{3}{4}c_4 + \frac{1}{4} - \frac{c_2}{2} - \frac{3c_3}{4} - c_4 + \frac{c_2}{16} + \frac{c_3}{64} + \frac{c_4}{256} - \frac{1}{16}$$

$$= c_2\left(2 - \frac{1}{2} + \frac{1}{16}\right) + c_3\left(\frac{3}{2} - \frac{3}{4} + \frac{1}{64}\right) + c_4\left(\frac{3}{4} - 1 + \frac{1}{256}\right) + \frac{1}{4} - \frac{1}{16}$$

$$= c_2\left(\frac{25}{16}\right) + c_3\left(\frac{49}{64}\right) + c_4\left(-\frac{63}{256}\right) + \frac{3}{16}$$

$$R(1/2) = 2C_2 + 6C_3(1/2) + 12C_4(1/2)^2 + (1-2C_2-3C_3-4C_4)(1/2) + C_2(1/2)^2 + C_3(1/2)^3 + C_4(1/2)^4 - (1/2)^2$$

$$= 2C_2 + 3C_3 + 3C_4 + 1/2 - C_2 - \frac{3}{2}C_3 - 2C_4 + \frac{C_2}{4} + \frac{C_3}{8} + \frac{C_4}{16} - 1/4$$

$$= C_2(2-1+1/4) + C_3(3-\frac{3}{2}+1/8) + C_4(3-2+1/16) + 1/2 - 1/4$$

$$= C_2(5/4) + C_3(13/8) + C_4(17/16) + 1/4$$

$$R(3/4) = 2C_2 + 6C_3(3/4) + 12C_4(3/4)^2 + (1-2C_2-3C_3-4C_4)(3/4) + C_2(3/4)^2 + C_3(3/4)^3 + C_4(3/4)^4 - (3/4)^2$$

$$= 2C_2 + \frac{9}{2}C_3 + \frac{27}{4}C_4 + \frac{3}{4} - \frac{3}{2}C_2 - \frac{9}{4}C_3 - 3C_4 + \frac{9}{16}C_2 + \frac{27}{64}C_3 + \frac{81C_4}{256} - 9/16$$

$$= C_2\left(2-\frac{3}{2}+\frac{9}{16}\right) + C_3\left(\frac{9}{2}-\frac{9}{4}+\frac{27}{64}\right) + C_4\left(\frac{27}{4}-3+\frac{81}{256}\right) + \frac{3}{4} - \frac{9}{16}$$

$$= \frac{17}{16}C_2 + \frac{171}{64}C_3 + \frac{1041}{256}C_4 + \frac{3}{16}$$

Solving $R(1/4)=0$, $R(1/2)=0$ & $R(3/4)=0$ gives

$$C_2 = \frac{279}{80765}, \quad C_3 = -\frac{17808}{80765}, \quad C_3 = \frac{7904}{80765}$$

$$\therefore u(x) \simeq 1.26310902x + 0.0034544x^2 - 0.22049x^3 + 0.097864x^4.$$

(C) LEAST SQUARE METHOD

* Two-term :

$$u(x) \simeq (1-2C_2-3C_3)x + C_2x^2 + C_3x^3$$

$$R(x) = 2C_2 + 6C_3x + (1-2C_2-3C_3)x + C_2x^2 + C_3x^3 - x^2$$

$$W_1 = \frac{\partial R}{\partial C_2} = 2 - 2x + x^2, \quad W_2 = \frac{\partial R}{\partial C_3} = 6x - 3x + x^3$$

$$\int_0^1 W_1 R\,dx = 0, \quad \int_0^1 W_2 . R\,dx = 0$$

$$\Rightarrow \int_0^1 (2-2x+x^2)\left[2C_2+6C_3x+(1-2C_2-3C_3)x + C_2x^2 + C_3x^3 - x^2\right] = 0$$

$$\Rightarrow -\frac{89C_3}{60} - \frac{48C_2}{60} + \frac{35}{60} = 0$$

(or) $121\,C_3 + 112\,C_2 + 13 = 0$ — ①

$$\int (3x+x^3)\left[2C_2+6C_3x+(1-2C_2-3C_3)x + C_2x^2 + C_3x^3\right] = 0$$

$$\Rightarrow 1824\,C_3 + 847\,C_2 + 119 = 0 \quad — ②$$

Solving ① & ②, ~~$C_2 = -\frac{2856}{2153}, \; C_3 = \frac{2387}{2153}$~~ ~~$C_2 = -\frac{408}{14543}, \; C_3 = -\frac{3829}{14543}$~~

$C_2 = -0.091482, \; C_3 = -0.022760$

∴ ~~$u(x) = \frac{6704}{2153}x - \frac{2856}{2153}x^2 + \frac{2387}{2153}x^3$~~ ~~$1.84597x - 0.028x^2 - 0.2633x^3$~~

∴ $u(x) = 1.251244x - 0.091482x^2 - 0.022760x^3$

* Three-term :--

$$R(x) = 2C_2 + 6C_3x + 12C_4x^2 + (1-2C_2-3C_3-4C_4)x + C_2x^2 + C_3x^3 + C_4x^4 - x^2$$

$$W_1 = \frac{\partial R}{\partial C_2} = 2 - 2x + x^2$$

$$W_2 = \frac{\partial R}{\partial C_3} = 6x - 3x + x^3$$

$$W_3 = \frac{\partial R}{\partial C_4} = 12x^2 - 4x + x^4$$

To solve $\int_0^1 W_1 R\,dx = 0, \; \int_0^1 W_2 R\,dx = 0, \; \int_0^1 W_3 R\,dx = 0$

$784C_2 + 847C_3 + 956C_4 + 91 = 0$

$1694C_2 + 3648C_3 + 5733C_4 + 238 = 0$

$5736C_2 + 17199C_3 + 31096C_4 + 732 = 0$

$$\underbrace{\qquad\qquad\qquad\qquad}_{C_2 = 0.0012940, \; C_3 = -0.21779, \; C_4 = 0.096631}$$

Solving :- ~~$C_2 \approx -0.46057, \; C_3 \approx 22.613, \; C_4 \approx -2.3667$~~

$$\dot{u}(x) = \cancel{56.48106x - 0.46057x^2 + 22.613x^3 - 2.3667x^4}.$$
$$= 1.264258x + 0.0012940x^2 - 0.21770x^3 + 0.096631x^4$$

(d) GALERKIN METHOD

* Two-term

$$u(x) = (1 - 2C_2 - 3C_3)x + C_2x^2 + C_3x^3$$

$$R(x) = 2C_2 + 6C_3x + (1 - 2C_2 - 3C_3)x + C_2x^2 + C_3x^3 - x^2$$

$$W_1 = (-2x + x^2), \quad W_2 = (-3x + x^3)$$

$$\int_0^1 W_1 R\,dx = 0, \qquad \int_0^1 W_2 R\,dx = 0$$

$$\cancel{12}\,623C_2 + 1200C_3 + 91 = 0$$

$$48C_2 + 89C_3 + 7 = 0$$

Solving, $C_2 = -0.1398$, $C_3 = -0.0032513$

$$u(x) = 1.289354x - 0.1398x^2 - 0.0032513x^3$$

* Three-term :-

$$u(x) = (1 - 2C_2 - 3C_3 - 4C_4)x + C_2x^2 + C_3x^3 + C_4x^4$$

$$R(x) = 2C_2 + 6C_3x + 12C_4x^2 + (1 - 2C_2 - 3C_3 - 4C_4)x + C_2x^2 + C_3x^3 + C_4x^4 - x^2$$

$$W_1 = (-2x + x^2), \quad W_2 = (-3x + x^3), \quad W_3 = (-4x + x^4)$$

$$\int_0^1 W_1 R\,dx = 0, \qquad \int_0^1 W_2 R\,dx = 0, \qquad \int_0^1 W_3 R\,dx = 0$$

$$336C_2 + 623C_3 + 892C_4 + 49 = 0$$

$$1246C_2 + 2400C_3 - 3507C_4 + 182 = 0$$

$$5352C_2 + 10521C_3 - 15560C_4 + 780 = 0$$

Solving, $C_2 = -0.16225$, $C_3 = 0.0086404$, $C_4 = 0.00016209$

$$u(x) = 1.29793x - 0.16225x^2 + 0.0086404x^3 + 0.00016209x^4$$

④ $u'' = -\cos(\pi x), \quad 0 < x < 1$

$u(0) = 0, \; u(1) = 0$

(a) Polynomial :- $u(x) = c_0 + c_1 x + c_2 x^2 + c_3 x^3 + c_4 x^4$

$u(0) = 0 \Rightarrow c_0 = 0$

$u(1) = 0 \Rightarrow c_1 + c_2 + c_3 + c_4 = 0$

$c_1 = -c_2 - c_3 - c_4$

$u(x) = (-c_2 - c_3 - c_4)x + c_2 x^2 + c_3 x^3 + c_4 x^4$

$u'(x) = -c_2 - c_3 - c_4 + 2c_2 x + 3c_3 x^2 + 4c_4 x^3$

$u''(x) = 2c_2 + 6c_3 x + 12 c_4 x^2$

Trigonometric :- $u(x) = c_0 + c_1 \sin(\pi x) + c_2 \sin(2\pi x) + c_3 \sin(3\pi x)$

$u(0) = 0 \Rightarrow c_0 = 0$

$u(1) = 0$ ✓

$u(x) = c_1 \sin(\pi x) + c_2 \sin(2\pi x) + c_3 \sin(3\pi x)$

$u'(x) = \pi c_1 \cos(\pi x) + 2\pi c_2 \cos(2\pi x) + 3\pi c_3 \cos(3\pi x)$

$u''(x) = -\pi^2 c_1 \sin(\pi x) - 4\pi^2 c_2 \sin(2\pi x) - 9\pi^2 c_3 \sin(3\pi x)$

(b) COLLOCATION ($x_i = 1/4, 1/2, 3/4$)

Polynomial:

$R(x) = 2c_2 + 6c_3 x + 12 c_4 x^2 + \cos(\pi x)$

$$\left.\begin{aligned} R(1/4) &= 2c_2 + \tfrac{3}{2} c_3 + \tfrac{3}{4} c_4 + 1/\sqrt{2} \\ R(1/2) &= 2c_2 + 3c_3 + 3c_4 \\ R(3/4) &= 2c_2 + \tfrac{9}{2} c_3 + \tfrac{27}{4} c_4 - 1/\sqrt{2} \end{aligned}\right\} \text{Solving,}$$

$c_2 = -1/\sqrt{2}, \; c_3 = \frac{\sqrt{2}}{3}, \; c_4 = 0$

Trigonometric :-

$R(x) = -\pi^2 c_1 \sin(\pi x) - 4\pi^2 c_2 \sin(2\pi x) - 9\pi^2 c_3 \sin(3\pi x) + \cos(\pi x)$

$$\left.\begin{aligned} R(1/4) &= -\frac{\pi^2 c_1}{\sqrt{2}} - 4\pi^2 c_2 - \frac{9\pi^2 c_3}{\sqrt{2}} + 1/\sqrt{2} \\ R(1/2) &= -\pi^2 c_1 + 9\pi^2 c_3 \\ R(3/4) &= -\frac{\pi^2 c_1}{\sqrt{2}} + 4\pi^2 c_2 - \frac{9\pi^2 c_3}{\sqrt{2}} - 1/\sqrt{2} \end{aligned}\right\} \text{Solving}$$

$c_1 = 0, \; c_2 = \frac{1}{4\sqrt{2}\pi^2}, \; c_3 = 0$

④ (c) GALERKIN METHOD :-

Polynomial :-

$R(x) = 2C_2 + 6C_3x + 12C_4x^2 + \cos(\pi x)$

$W_1 = (-x + x^2),\ W_2 = (-x + x^3),\ W_3 = (-x + x^4)$

$\cancel{10C_2 + 15C_3 + 18C_4 = 0} \rightarrow ①$

$\int W_1 R\,dx = 0,\ \int W_2 R\,dx = 0,\ \int W_3 R\,dx = 0$

$10C_2 + 15C_3 + 18C_4 = 0 \qquad 5\pi^4C_2 + 8\pi^4C_3 + 10\pi^4C_4 + 10\pi^2 - 120 = 0$

$21\pi^4C_2 + 35\pi^4C_3 + 45\pi^4C_4 + 70\pi^2 - 840 = 0$

Solving, $C_2 = \dfrac{30(\pi^2 - 12)}{\pi^4},\quad C_3 = \dfrac{-20(\pi^2 - 12)}{\pi^4},\quad C_4 = 0$

Trigonometric :-

$R(x) = -\pi^2c_1\sin(\pi x) - 4\pi^2c_2\sin(2\pi x) - 9\pi^2c_3\sin(3\pi x) + \cos(\pi x)$

$W_1 = \sin(\pi x),\ W_2 = \sin(2\pi x),\ W_3 = \sin(3\pi x)$

$\cancel{-\frac{\pi^2c_1}{2} = 0}$

$\int W_1 R\,dx = 0,\ \int W_2 R\,dx = 0,\ \int W_3 R\,dx = 0$

$-\dfrac{\pi^2c_1}{2} = 0 \qquad -6\pi^3c_2 - 4 = 0 \qquad -\dfrac{9\pi^2c_3}{2} = 0$

Solving, $c_1 = 0,\ c_2 = \dfrac{2}{3\pi^3},\ c_3 = 0$

(d) Weak form :-

$\int w(u'' + \cos\pi x)\,dx = 0$

$= \int wu''\,dx + \int w\cos\pi x\,dx$

$= w\dfrac{du}{dx}\Big|_0^1 - \int_0^1 \dfrac{du}{dx}\dfrac{dw}{dx}\,dx + \int_0^1 w\cos\pi x\,dx$

Theoretically, both the strong form and weak form should give the same solⁿ for u(x) if the exact solⁿ can be represented by the chosen approxⁿ functions and the problem is solved correctly. But, in practice, differences arise since the weak form naturally incorporates boundary conditions while the strong form requires explicit enforcement.

⑤ $\frac{d}{dx}\left(u\frac{du}{dx}\right) - f(x) = 0 \quad , 0<x<L$

$$u\left.\frac{du}{dx}\right|_{x=0} = 0, \quad u(L) = u_0$$

$$\int w\left[\frac{d}{dx}\left(u\frac{du}{dx}\right) - f(x)\right]dx = \int w.\left(\frac{d}{dx}\left(u\frac{du}{dx}\right)\right)dx - \int w f(x) dx$$

$$= w\left(u.\frac{du}{dx}\right)_0^L - \int_0^L \frac{dw}{dx}\left(u.\frac{du}{dx}\right)dx - \int w f(x) dx$$

$$= w(L)\left(u\frac{du}{dx}\right)_{x=L} - w(0).\left(\cancel{u.\frac{du}{dx}}\right)_{x=0} - \int_0^L \frac{dw}{dx}\left(u.\frac{du}{dx}\right)dx$$

$$- \int w f(x) dx$$

$$= w(L).u_0.\left(\frac{du}{dx}\right)_{x=L} - \int_0^L \frac{dw}{dx}\left(u.\frac{du}{dx}\right)dx - \int w.f(x)dx$$

⑥ $2uu'' - (u')^2 + 4 = 0, \quad 0<x<1$

$u(0) = 1, \quad u(1) = 0$

(a) $\int_0^1 w\,(2uu'' - (u')^2 + 4) = 2\int_0^1 wuu''dx - \int_0^1 w(u')^2 dx + 4\int_0^1 w\,dx$

$$= 2\int_0^1 (wu)\,u''dx - \int_0^1 (wu')\,u'\,dx + 4\int_0^1 w\,dx$$

$$= 2\left[\cancel{wu}.u'\Big|_0^1 - \int_0^1 u'\frac{d}{dx}(wu)\,dx\right] - \int_0^1 wu'^2 dx + 4\int_0^1 w\,dx$$

$$= -2\int_0^1 u'(wu' + uw')\,dx - \int_0^1 w(u')^2 dx + 4\int_0^1 w\,dx$$

$$= -2\int_0^1 (wu'^2 + uu'w')\,dx - \int_0^1 w(u')^2 dx + 4\int_0^1 w\,dx$$

$$= -3\int_0^1 wu'^2 dx - 2\int_0^1 uu'w'\,dx + 4\int_0^1 w\,dx$$

$$= -3\left[\cancel{wu'}u\Big|_0^1 - \int_0^1 u.\frac{d}{dx}(wu')\,dx\right] - 2\int_0^1 uu'w'\,dx + 4\int_0^1 w\,dx$$

$$= +3\int_0^1 u.(wu'' + w'u')\,dx - 2\int_0^1 uu'w'dx + 4\int_0^1 w\,dx$$

$$= 3\int_0^1 wuu'' + 3\int_0^1 w'u'u\,dx - 2\int_0^1 w'u'u\,dx + 4\int_0^1 w\,dx$$

$$= 3\int_0^1 w u u'' dx + \int_0^1 w' u' u dx + 4\int_0^1 w dx. \rightarrow \text{WEAK FORM}$$

The weight functions should vanish at the boundaries, i.e,

$W(0) = 0, \quad W(1) = 0$

(b) Let $u(x) = a_1(x - x^3) + a_2(x^2 - x^3) + (1 - x^3)$

$W_1 = (x - x^3), \; W_2 = (x^2 - x^3), \; W_1' = (1 - 3x^2), \; W_2' = (2x - 3x^2)$

$u' = a_1(1 - 3x^2) + a_2(2x - 3x^2) - 3x^2$

$u'' = a_1(-6x) + a_2(2 - 6x) - 6x$

Plugging in these, we get,

- $29a_2^2 + (134a_1 + 162)a_2 + 175a_1^2 + 518a_1 + 133 = 0$

$35a_2^2 + (158a_1 + 214a_2) + 193a_1^2 + 582a_1 + 179 = 0$

Solving: $a_1 = 0, a_2 = -1$

(or) $a_1 = -0.391, a_2 = 0.358$.

(c) Trigonometric:-

$u(x) = 1 + b_1 \sin(\pi x) + b_2 \sin$

(unable to find trig sol' satisfying essential Boundary conditions).