Introduction to Stacks and Moduli Notes

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These are my course notes for "Math 582C: Introduction to Stacks and Moduli" at the University of Washington. Each lecture will get its own "chapter." These notes are live-texed or whatever, so there will likely to be some (but hopefully not too much) content missing from me typing more slowly than one lectures. They also, of course, reflect my understanding (or lack thereof) of the material, so they are far from perfect.[1] Finally, they contain many typos, but ideally not enough to distract from the mathematics. With all that taken care of, enjoy and happy mathing.

The instructor for this class is Jarod Alper, and the course website can be found by clicking this link. At some point after this class, Jarod uploaded an expanded version of his course notes here.

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[1] In particular, if things seem confused/false at any point, this is me being confused, not the speaker

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1 Lecture 1 (1/4/21): Motivation and History

"A lot more people than I expected are at a topics graduate course." There are over 120 people and counting. If we hit 200, this might (might) be the largest a UW math course has been.

1.1 Logistics

There is a website and an email list.

The goal of the course is to introduce the theory of algebraic spaces/stacks with moduli in mind. In particular, with the moduli space of curves in mind. We would like to establish

Theorem 1.1. *The moduli space $\overline{\mathcal{M}}_g$ of stable curves of genus $g \geq 2$ is a smooth, proper, and irreducible Deligne-Mumford stack of dimension $3g - 3$, which admits a projective course moduli space.*

There is a lot to even define before saying something of the proof, and a lot of background material that goes into it, so we will not give a complete proof.

Around 20 lectures (quarter system), each 80 minutes. They will be recorded and posted on Youtube. Probably 60 minutes of real lecture and 20 minutes devoted to questions (though quotions in the middle are fine too). Maybe a discord/zulip chat.

There are lecture notes on the site. We'll do a little motivation today, and then really just dive into the deep end starting next time. There are many references besides these notes which you can find on the course website.

We're currently at 150 participants.

For asking questions, preferred if you unmute and just ask.

1.1.1 Tips and advice

Learning the theory of stacks and moduli is hard, and requires active work. You need to simultaneously absorb

- functorial approach in AG
- working with groupoids and stacks
- replacing the Zariski topology with étale topology
- systematic use of descent theory
- several advanced topics not usually covered in a first course in AG such as
  - properties of flat, étale, smooth maps
  - existence of Hilbert/Quot schemes
  - algebraic groups and actions
  - deformation theory
  - Artin approximation
  - birational geometry of surfaces

Here are some tips.

- Work through the material yourself. Work out your own examples, do exercises, and fill in details in proofs.
- *Don't* read the notes linearly. As a graduate student, one of the first books Jarod read was Hartshorne, and he did read it linearly. However, this subject is a little different. There are many independent parts, so you can black box certain results and different parts and move around a bit.
- Accept that there are topics you don't know and still won't in 10 weeks, but try to use such results anyways.

Try to find a balance between accepting/using advanced results and understanding why they hold. Only need to balance accepting details vs. checking details. Have faith that either you could work out the details or that at some point you will properly learn the material.

1.2 Moduli

What are moduli? Let $*$ be your favorite mathematical object.

Definition 1.2. A **"moduli space of $*$"** is a space M such that the points of M are in bijection with isomorphism classes of $*$.

Example. If you take $*$ = smooth, connected, projective curves of genus g, then you get M_g as your moduli space.

Example. Can let $*$ =plane curves up to **projective equivalence**, so $M = \left\{C \subset \mathbb{P}^2 \text{ deg } d\right\} / \sim$ where $C \sim C'$ if $\exists$ auto of $\mathbb{P}^2$ sending C to C'.

Note that to define a moduli space, in addition to the objects you want to parameterize, you also need to specify equivalences between them. In previous example could have considered plane curves up to abstract isomorphism, for example.

Example (Hurwitz moduli spaces)**.** Instead of studying C in $\mathbb{P}^N$, can study branched covers $C \to \mathbb{P}^1$ of degree d.

$$\mathrm{Hur}_{d,g} := \left\{C \xrightarrow{d} \mathbb{P}^1 : C \text{ smooth, connected, projective}, g(C) = g\right\}.$$

Say $[C \to \mathbb{P}^1] \sim [C' \to \mathbb{P}^1]$ if they are isomorphic as $\mathbb{P}^1$-schemes.

Example (Vector bundles on a curve)**.** Fix C smooth, connected, projective curve. Can consider

$$M_{C,r,d} = \left\{ \begin{array}{c} \text{vector bundles on } C \\ \text{of rank } r, \text{ deg } d \end{array} \right\} / \sim$$

So far, we've really only defined these as sets. To call them "spaces" we need more structure. In fact, these spaces are algebraic varieties. Jarod calls this the Moduli Miracle.

1.3 History

Theorem 1.3 (Riemann, 1857). *The "number of moduli" of smooth curves of genus g is $3g-3$.*

(maybe this was more assertion than theorem. Whatever).

Riemann gave several arguments for this. Here's one

$$\begin{tikzcd} \mathrm{Hur}_{d,g} \arrow[r] \arrow[d] & \mathrm{Sym}^{2d+2g-2}\mathbb{P}^1 \\ M_g & \end{tikzcd}$$

The vertical map sends $[C \to \mathbb{P}^1] \mapsto [C]$ while the horizontal map sends $[C \to \mathbb{P}^1] \mapsto$ (branched points). Riemann-Hurwitz gives

$$2g(C) - 2 = d(2g(\mathbb{P}^1) - 2) + R \implies R = 2d + 2g - 2$$

where R is the number of branched points (counted properly). Can show the horizontal map has dense image and finite fibers, so $\dim \mathrm{Hur}_{d,g} = \dim \mathrm{Sym}^{2d+2g-2}\mathbb{P}^1 = 2d + 2g - 2$. For a fixed curve C, let $\mathrm{Hur}_{d,C} \subset \mathrm{Hur}_{d,g}$ be the fiber of the map $\mathrm{Hur}_{d,g} \to M_g$ over $[C]$ (i.e. its the moduli of degree d maps $C \to \mathbb{P}^1$). Then, $\dim \mathrm{Hur}_{d,g} = \dim M_g + \dim \mathrm{Hur}_{d,C}$. A degree d map $C \to \mathbb{P}^1$ is defined by a line bundle L of degree d along with 2 sections (mod scalars), so $L \in \mathrm{Pic}_d(C) \leftarrow \mathrm{Sym}^d C$. The data of a point of $\mathrm{Sym}^d C$ is a line bundle + a section (up to scaling), and a section up to scaling has $h^0(L) - 1 = d - g$ degrees of freedom. The conclusion is[2] that $\dim \mathrm{Hur}_{d,C} = 2d - g + 1$. Hence, $\dim M_g = 3g - 3$. We'll revisit this calculation later.

Riemann called M_g "mannigfaltigkeiten" (spelling) meanning "manifold-like." Manifolds were not introduced until 1940s.

- Weil 1958: "As for M_g, there is virtually no doubt that it can be provided the structure of an algebraic variety."
- Grothendieck 1960: Aware that M_g was not representable, he showed representability of the moduli of smooth curves with level n structure (i.e. C + basis of $H_1(C, \mathbb{Z}/n\mathbb{Z})$). He struggled with projectivity though.
- Mumford 1965: M_g is a variety

Mumford used GIT (Geometric Invariant Theory). The basic idea is to add additional data, and then try to quotient out by it. There are other approaches, including analytic or topological ones. Our approach will be entirely algebraic, integrating ideas from GIT with stack theory.

- We'll show $\overline{M}_g$ is a proper Deligne-Mumford stack
- Use Keel-Mori theorem to show $\exists$ coarse moduli space $\overline{M}_g \to \overline{\mathcal{M}}_g$.
- Show a line bundle on $\overline{M}_g$ descends to an ample line bundle using a result of Kollár.

Note 1. His M's and $\mathcal{M}$'s are hard to distinguish. In particular, I'm pretty sure I have M and $\mathcal{M}$ backwards in the above 3 steps.

[2]He may have said something like $\dim \mathrm{Pic}_d(C) = g$ at some point. I'm not sure...

1.4 Functorial Worldview

Grothendieck: Spaces are functors

Let Sch be the category of schemes. We are interested in contravariant functors $F : \mathrm{Sch} \to \mathrm{Set}$.

Example. If X is a scheme, then $h_X : \mathrm{Sch} \to \mathrm{Set}$ defined, on objects, by $S \mapsto \mathrm{Mor}(S, X) =: X(S)$ is a functor.

Remark 1.4. The k points of X are $h_X(\operatorname{spec} k)$, so as sets

$$X = \bigsqcup_{k \text{ field}} h_X(\operatorname{spec} k)/\sim$$

for a suitable relation $\sim$. One can then imagine similarly recovering the topology on X as well as its sheaf of rings by cleverly probing the functor h_X.

Lemma 1.5 (**Yoneda's Lemma**, "h_X determines X")**.** *For any contravariant functor $G : \mathrm{Sch} \to \mathrm{Set}$, the natural map*

$$\mathrm{Mor}(h_X, G) \xrightarrow{\sim} G(X)$$

is bijective.

This result is completely formal and easy to prove.

Example. Projective space $\mathbb{P}^n$ represents the functor $F : \mathrm{Sch} \to \mathrm{Set}$ with

$$F(S) = \{(L, s_0, \ldots, s_n)\}$$

where L is a line bundle on S and $s_0, \ldots, s_n \in \Gamma(S, L)$ are global sections generating L. Equivalently, the map $(s_0, \ldots, s_n) : \mathscr{O}_S^{n+1} \twoheadrightarrow L$ is surjective, so $\mathbb{P}^n$ represents the functor representing rank one quotients of $\mathscr{O}_S^{n+1}$.

Example. More generally, the **Grassmannian** functor

$$\mathrm{Gr}(k, n) : \mathrm{Sch} \to \mathrm{Set}$$

sends S to $\left\{\mathscr{O}_S^{\oplus n} \twoheadrightarrow V : V \text{ locally free of rank } k\right\}$. This functor is represented by a scheme, which we also denote as $\mathrm{Gr}(k, n)$, that is projective over $\mathbb{Z}$.

Warning 1.6. We will always conflate X with h_X.

Example. Can also consider functors $\mathrm{AffSch} \to \mathrm{Set}$ on affine schemes.

Exercise. A scheme X can be equivalently defined as a functor $F : \mathrm{AffSch} \to \mathrm{Set}$ where $\exists$ open subfunctors $F_i \subset F$ such that

This is missing the assumption that F be a Zariski sheaf

- each F_i is represented by an affine scheme
- $\bigsqcup_i F_i \to F$ is surjective.

Definition 1.7. We say $F_i \hookrightarrow F$ is an **open subfunctor** if for all maps $S \to F$ (S a scheme), the basechange $F_i \times_F S \hookrightarrow S$ is an open immersions of schemes (in particular, $F_i \times_F S$ is a scheme). Here, the fiber product of functors is simply

$$(F_i \times_F S)(T) = F_i(T) \times_{F(T)} S(T).$$

We say $\bigsqcup_i F_i \twoheadrightarrow F$ is surjective iff for all maps $S \to F$,

$$\bigsqcup_i F_i \times_F S \twoheadrightarrow S$$

is surjective as a map of schemes.

Note 2. For the exercise, things might be a little circular. If you want to be careful, should really say affine scheme everywhere we've said scheme.[3] Alternatively, given that we already know what a scheme is, we can interpret this exercise as giving a criterion for knowing a functor is represenatble by a scheme.

Based off chat comments, it seems here is one solution. Running this plan with affine schemes everywhere should result in describing "schemes with affine-diagonal" (since you'll be requiring that the intersections of any open affines are affine). These are slightly more general than separated schemes. Running things once more starting with "schemes with affine diagonal" then results in all schemes, and running them a third time starting with all schemes still only results in all schemes, so you're done.

We will use the analogues of this exercise to define algebraic spaces/stacks.

Algebro-geometric space	type of object	obtained by gluing
Schemes	sheaf	affine schemes in the Zariski topology
Algebraic spaces	sheaf	affine schemes in the étale topology
Deligne-Mumford stack	stack	affine schemes in the étale topology
Algebraic stack	stack	affine schemes in the smooth topology

Table 1: Comparison between various kinds of spaces

How can we view, e.g. M_g, as a functor?

Example. Consider $F_{M_g} : \mathrm{Sch} \to \mathrm{Set}$ sending S to the set of smooth maps $\mathcal{C} \to S$ whose geometric fibers are smooth, connected, projective curves of genus g.

Fact. F_{M_g} is not representable!

Here's a general principle. Let $F : \mathrm{Sch}/\mathbb{C} \to \mathrm{Set}$ be a functor (working with $\mathbb{C}$ just for concreteness). Let $\mathcal{E} \in F(S)$ be an object over a $\mathbb{C}$-variety S. For $s \in S(\mathbb{C})$, let $\mathcal{E}_s \in F(\mathbb{C})$ be restriction $\operatorname{spec} \mathbb{C} \xrightarrow{s} S$. If

- all $\mathcal{E}_s$ are isomorphic
- $\mathcal{E}$ is not trivial

then F is not representable by a scheme. Suppose X represents F. Then, $\mathcal{E}_s \in F(\mathbb{C})$ corresponds to a point $x \in X(\mathbb{C})$. The two maps $\mathcal{E}, x : S \rightrightarrows X$ must be the same, so $\mathcal{E}$ would have to be the constant x map.

	No Autos	Finite Auts	Infinite Auts
Type of space	algebraic variety/space	Deligne-Mumford stack	algebraic stack
Defining property	Zariski/étale locally affine schemes	étale-locally an affine scheme	smooth-locally an affine scheme
Examples	$\mathbb{P}^n, \mathrm{Gr}(k,n), \mathrm{Hilb}, \mathrm{Quot}$	$\mathcal{M}_g$, $\overline{\mathcal{M}}_{g,n}$	$\mathcal{M}_{C,r,d}$
Quotient stacks $[X/G]$	action is free	finite stabilizers	any action
Existence of moduli varieties/spaces	already an algebraic variety/space	coarse moduli space	good moduli space

Table 2: Trichotomoy of Moduli

Automorphisms give non-trivial families where all fibers are isomorphic. He gave some other example, but I did not follow....

Get more into details on Wednesday. Start seeing Grothendieck topologies, sites, and sheaves.

2 Lecture 2 (1/6): Sites and sheaves

Began by displaying Figure 1 in his notes. He then spoke a bit about various references for learning about stacks. I did not write down any of what he said...

2.1 Motviation: étale topology

Definition 2.1. For a morphism $f : X \to Y$ of schemes of finite type over $\mathbb{C}$, the following are equivalent:

- f is **étale**
- f is smooth of relative dimension 0 (flat and all fibers X_y smooth of dimension 0).
- f is flat and unramified (fibers $X_y = f^{-1}(y) = \bigsqcup \operatorname{spec} \mathbb{C}$)
- f is flat and $\Omega_{X/Y} = 0$
- For all $x \in X(\mathbb{C})$, the map $\widehat{\mathscr{O}}_{Y,f(x)} \xrightarrow{\sim} \widehat{\mathscr{O}}_{X,x}$ is an iso;
- for any $A \twoheadrightarrow A_0$ of artinian $\mathbb{C}$-algebras, any commutative diagram

$$\begin{tikzcd} \operatorname{spec} A_0 \arrow[r] \arrow[d] & X \arrow[d, "f"] \\ \operatorname{spec} A \arrow[r] \arrow[ur, dashed] & Y \end{tikzcd}$$

  of solid arrows can be uniquely filled in.

- (assuming in addition that X, Y are smooth) for all $x \in X(\mathbb{C})$, the map $T_{X,x} \to T_{Y,f(x)}$ is an iso.

Example. For the lifting criterion above, can take $A_0 = \mathbb{C}$ and $A = \mathbb{C}[\varepsilon]/\varepsilon^2$, so $Y(A)$ consist of a point + a tangent vector. The unique lifting is saying that there is a unique lift of each tangent vector.

Example. The map $\mathbb{A}^1 \to \mathbb{A}^1, x \mapsto x^2$ is étale except at 0.

[3]Maybe actually, say open subscheme of an affine scheme. Working out the kinks is part of the exercise, I guess

Example. For a field extension K/L, $\operatorname{spec} L \to \operatorname{spec} K$ is étale $\iff$ $K \to L$ is finite and separable.

Question 2.2. *Why do we care about the étale topology?*

One reason is that it allows you to zoom in. "The étale topology is like putting on a new set of magnifying lenses for you algebraic geometry glasses that allows you to see what you could already see with your differential geometry glasses" (paraphrase).

Example (Nodes)**.** Let $C = V(y^2 - x^2(x-1)) \subset \mathbb{A}^2$ be a plane nodal cubic. This is irreducible in the Zariski topology. Any Zariski open around the nodal point is still irreducible. However, in the analytic topology, there are two branches near this point, so it is has reducible neighborhoods in the analytic topology. What about the étale topology? We can adjoin $t = \sqrt{x-1}$ so $y^2 - x^2(x-1) = (y-xt)(y+xt)$. The map

$$C' = \operatorname{spec} k[x,y,t]_t/(y^2 - x^2(x-1), t^2 - x + 1) \longrightarrow C$$

is étale with C' reducible.

Observe that the completion

$$\widehat{\mathscr{O}}_{C,c} = k\left[\!\left[x,y\right]\!\right]/(y^2 - x^2(x+1)) = k\left[\!\left[x,y\right]\!\right]/(y-xt)(y+xt)$$

where $t = \sqrt{x+1} = 1 + \frac{1}{2}x - \frac{1}{4}x^2 + \dots$ exists as a power series. Hence, $\operatorname{spec} \widehat{\mathscr{O}}_{C,c}$ is also irreducible.

Slogan (Artin approximation)**.** Algebraic properties that hold for the completion $\widehat{\mathscr{O}}_{X,x}$ also hold in an étale neighborhood of $x \in X$.

More formally...

Theorem 2.3 (**Artin approximation**)**.**

- *Let S be an excellent scheme (e.g. finite type / k or $\mathbb{Z}$).*
- *Let $F : \mathrm{Sch}/S \to \mathrm{Set}$ be a limit preserving contravariant functor.*[4]
- *Let $\widehat{\xi} \in F(\operatorname{spec} \widehat{\mathscr{O}}_{S,s})$ where $s \in S$ is a point.*

Liu talks about excellent schemes in his book

For any integer $N \geq 0$, there exist a residually-trivial étale morphism

$$(S', s') \to (S, s) \text{ and } \xi' \in F(S')$$

such that the restrictions of $\widehat{\xi}$ and ξ' to $\operatorname{spec}(\mathscr{O}_{S,s}/\mathfrak{m}_s^{N+1})$ are equal.

Question: How does this theorem justify the slogan?

Exercise. If $X \to S$ is an S-scheme, then $\operatorname{Mor}_S(-, X) : \mathrm{Sch}/S \to \mathrm{Set}$ is limit preserving iff $X \to S$ is locally of finite presentation.

Example. In the nodal example from before, take $F : \mathrm{Sch}/C \to \mathrm{Set}$ to be the functor

$$(C' \to C) \mapsto \left\{ \begin{matrix} \text{decomps} \\ C' = C_1' \cup C_2' \end{matrix} \right\} \stackrel{?}{=} \{\text{idempotents in } \Gamma(C')\}.$$

The completion being reducible gives an element $\xi \in F(\widehat{\mathscr{O}}_{C,c})$. Hence Artin approximation (with $N = 1$) gives an étale cover.

Answer: The choice of $\widehat{\xi} \in F(\operatorname{spec} \widehat{\mathscr{O}}_{S,s})$ is like a witness to an algebraic property at the completion. This is then saying

[4] $\operatorname{colim} F(\operatorname{spec} B_\lambda) = F(\operatorname{spec} \operatorname{colim} B_\lambda)$. Recall, $\operatorname{colim} = \varinjlim$

Example (Étale cohomology)**.** When C is a smooth, connected, projective curve, one has $\mathrm{H}^1(C, \mathbb{Z}/n\mathbb{Z}) = 0$. However, $\mathrm{H}^1(C_{\text{ét}}, \mathbb{Z}/n\mathbb{Z}) = (\mathbb{Z}/n\mathbb{Z})^{2g}$.

Example (Descent)**.** Recall that any good property P of morphisms of schemes satisfies

- stable under composition
- stable under base change
- Zariski-local on target, i.e. if $\{Y_i\}$ is an open cover

$$\begin{tikzcd} f^{-1}(Y_i) \arrow[r] \arrow[d, "f_i"'] & X \arrow[d, "f"] \\ Y_i \arrow[r] & Y \end{tikzcd}$$

  f has $P \iff$ each f_i has P.

The miracle is that they also tend to be étale-local on the target.

Remark 2.4. Can learn about descent theory from Néron Models (book), FGA explained, stacks project, etc.

2.2 Sites

Definition 2.5. A **Grothendieck topology** on a category $\mathcal{S}$ consists of the following data: for each object $U \in \mathcal{S}$ there is a set $\mathrm{Cov}(U)$ consisting of *coverings* of U, i.e. collections $\{U_i \to U\}_{i \in I}$ of morphisms in $\mathcal{S}$. We require

- $U' \xrightarrow{\sim} U$ iso $\implies \{U' \to U\} \in \mathrm{Cov}(U)$.
- For a map $V \to U$,

$$\{U_i \to U\}_{i \in I} \in \mathrm{Cov}(U) \implies \{U_i \times_U V \to V\}_{i \in I} \in \mathrm{Cov}(V)$$

  (in particular, the necessary fiber products exist in $\mathcal{S}$)

- $\{U_i \to U\} \in \mathrm{Cov}(U)$ and $\{U_{ij} \to U_i\}_{j \in J_i} \in \mathrm{Cov}(U_i)$ imply

$$\{U_{ij} \to U_i \to U\}_{i \in I, j \in J_i} \in \mathrm{Cov}(U).$$

A **site** is a category $\mathcal{S}$ with a Grothendieck topology.

Example (**small étale site**)**.** Let X be a scheme, and let $X_{\text{ét}}$ be the category of schemes étale over X (so objects are étale maps $U \to X$). We set

$$\mathrm{Cov}(U \to X) := \left\{ \{U_i \to U\} : \bigsqcup U_i \twoheadrightarrow U \right\}.$$

Example (**big étale site**)**.** The big étale site $\mathrm{Sch}_{\text{Ét}}$ is the category of all schemes Sch with coverings

$$\mathrm{Cov}(X) = \left\{ \left\{ X_i \xrightarrow{\text{ét}} X \right\} : \bigsqcup X_i \twoheadrightarrow X \right\}$$

Could also define big Zariski site, big étale site relative to S (i.e. $(\mathrm{Sch}/S)_{\text{Ét}}$), and other sites one cares above (smooth, fppf, fpqc, ...).

2.3 Sheaves

Definition 2.6. A **presheaf** on a category $\mathcal{S}$ is a contravariant functor $\mathcal{S} \to \mathrm{Set}$.

As a rule of thumb, assume all functors in this class are contravariant

Definition 2.7. A **sheaf** on a site $\mathcal{S}$ is a presheaf $F : \mathcal{S} \to \mathrm{Set}$ such that for any covering $\{S_i \to S\} \in \mathrm{Cov}(S)$ of an object $S \in \mathcal{S}$, the sequence

$$F(S) \longrightarrow \prod_i F(S_i) \rightrightarrows \prod_{i,j} F(S_i \times_S S_j)$$

is exact.

Remark 2.8. The two maps on the right are induced by the two projections $S_i \times_S S_j \rightrightarrows S_i, S_j$. There are two parts of this. One is that sections glue, and the other is that this gluing is unique.

Warning 2.9. I believe the sheaf condition is nontrivial even in the case that your covering $\{T \to S\}$ only consists of a single object!

Remark 2.10. If X is a topological space and $\mathrm{Open}(X)$ is the site of its open sets, then this recovers usual notion of sheaf.

Example. Let X be a scheme. Then $h_X : \mathrm{Sch} \to \mathrm{Set}$ given by $S \mapsto \mathrm{Mor}(S, X)$ is a sheaf on $\mathrm{Sch}_{\text{Ét}}$.

This comes from Descent theory. If $\{S_i \to S\}$ is an étale covering, consider

$$\begin{tikzcd}
S_i \times_S S_j \arrow[d, bend right] \arrow[d, bend left] & \\
S_i \arrow[d] \arrow[dr, "g_i"] & \\
S \arrow[r, dashed, "\exists!"] & X
\end{tikzcd}$$

If each $S_i \hookrightarrow S$ is open, then this would be the usual fact that morphisms glue uniquely in the Zariski topology.

Consider $G' \to G \leftarrow F$ maps of presheaves on a category $\mathcal{S}$.

Exercise. Show $F \times_G G'$ defined as

$$S \mapsto F(S) \times_{G(S)} G'(S)$$

is a fiber product in $\mathrm{Pre}(\mathcal{S})$, the category of presheaves.

Exercise. If F, G, G' are sheaves (on some site), then $F \times_G G'$ is also a sheaf.

Theorem 2.11. *Let $\mathcal{S}$ be a site. The forgetful functor* $\mathrm{Sh}(\mathcal{S}) \to \mathrm{Pre}(\mathcal{S})$ *admits a left adjoint $F \mapsto F^{sh}$, called the* ***sheafification****.*

Proof.

- Call a presheaf F **separated** if for every covering $\{S_i \to S\}$, the map $F(S) \to \bigcup_i F(S_i)$ is injective. Let $\mathrm{Pre}^{sep}(\mathcal{S}) \subset \mathrm{Pre}(\mathcal{S})$ be the full subcat of separated presheaves.
- We will construct left adjoints

$$\begin{tikzcd} \mathrm{Sh}(\mathcal{S}) \arrow[r, hook] & \mathrm{Pre}^{sep}(\mathcal{S}) \arrow[l, bend right, "sh_2"'] \arrow[r, hook] & \mathrm{Pre}(\mathcal{S}) \arrow[l, bend right, "sh_1"'] \end{tikzcd}$$

- Define $sh_1(F)$ by $S \mapsto F(S)/\sim$ where $a \sim b$ if there's a covering S_i s.t. $a|_{S_i} = b|_{S_i}$ for all i.
- Define $sh_2(F)$ by
$$S \mapsto \left\{(\{S_i \to S\}, \{a_i\}) : a_i|_{S_{ij}} = a_j|_{S_{ij}}\ \forall i,j\right\} / \sim$$
where $(\{S_i \to S\}, \{a_i\}) \sim (\{S'_j \to S\}, \{a'_j\})$ if $a_i|_{S_i \times_S S'_j} = a'_j|_{S_i|_S S'_j}$ for all i, j.

■

3 Lecture 3 (1/11): Groupoids and prestacks

3.1 Logistics

Zulip chat set up, so use that instead of Zoom chat from now on.

Course notes have been updated.

Something asked a question about "points" for general sites. Apparently, Olsson talks about this in section 2.5 (I think) of his book.

3.2 Last time

We defined sites as well as (pre)sheaves on a site.

Recall 3.1. A site is a category $\mathcal{S}$ and for each object $U \in \mathcal{S}$, we have a set $\mathrm{Cov}(U)$ of coverings, i.e. collections $\{U_i \to U\}_{i \in I}$. We require that isomorphisms are covers, pullbacks of covers are covers, and that covers can be refined.

Example. The big étale site $\mathrm{Sch}_{\text{Ét}}$ is the category of Sch where covers are collections $\{U_i \xrightarrow{\text{ét}} U\}$ of étale maps such that $\bigsqcup U_i \twoheadrightarrow U$ is surjective. A **special covering** is one consisting of a single surjective, étale map $\{U_i \overset{\text{ét}}{\twoheadrightarrow} U\}$.

Recall 3.2. A presheaf on $\mathcal{S}$ is just a contravariant functor $F : \mathcal{S} \to \mathrm{Set}$. A sheaf is a presheaf F such that, for all covers $\{U_i \to U\}_{i \in I}$, the sequence

$$F(U) \to \prod_i F(U) \rightrightarrows \prod_{(i,j)} F(U_{ij})$$

is exact, where as usual,

$$U_{ij} := U_i \times_U U_j.$$

Exercise. A presheaf F is a sheaf on $\mathrm{Sch}_{\text{Ét}}$ iff

- It is a sheaf on the big Zariski site $\mathrm{Sch}_{\mathrm{Zar}}$; and
- for all étale surjections $U' \twoheadrightarrow U$ of *affine* schemes, the sequence

$$F(U) \to F(U') \to F(U' \times_U U')$$

is exact.

Proposition 3.3. *If X is a scheme, then $h_X = \mathrm{Mor}(-, X) : \mathrm{Sch} \to \mathrm{Set}$ is a sheaf on $\mathrm{Sch}_{\acute{E}t}$.*

3.2.1 Moduli perspective

Let $F : \mathrm{Sch} \to \mathrm{Set}$ be a moduli functor.[5] Then, F is a sheaf $\iff$ families glue uniquely in the étale topology.

Example. $F_{M_g} : \mathrm{Sch} \to \mathrm{Set}$ sending S to the set of smooth families $\mathcal{C} \to S$ of genus g curves.

Recall 3.4. We saw previously that this is not representable. For a similar reason, it is not even a sheaf (which also implies it is not representable since representable $\implies$ sheaf).

I guess we never said why representability is a good thing.

Question 3.5. *Why do we care about representable functors?*

Suppose there was a scheme $\mathcal{M}$ representing $F_{\mathcal{M}_g}$, i.e. $\mathrm{Mor}(S, \mathcal{M}) \simeq F_{\mathcal{M}_g}(S)$. Taking $S = \mathcal{M}$, the identity map $\mathrm{id}_{\mathcal{M}}$ would now correspond to some *universal family* $\mathcal{U} \to \mathcal{M}$ of genus g curves. That is, for any family $\mathcal{C} \to S$ over S, one gets a Cartesian diagram

$$\begin{tikzcd} \mathcal{C} \arrow[r] \arrow[d] & \mathcal{U} \arrow[d] \\ S \arrow[r] & \mathcal{M} \end{tikzcd}.$$

The bottom map sends $s \in S$ to the isomorphic class of the fiber $[\mathcal{C}_s] \in \mathcal{M}$.

Unfortunately though, $F_{\mathcal{M}_g}$ is not representable. Getting around this is why we'll introduce groupoids and stacks and whatnot.

3.3 Groupoids

Definition 3.6. A **groupoid** is a category $\mathcal{C}$ where every morphism is an isomorphism.

Example. Let $\mathcal{M}_g(\mathbb{C})$ be the category whose objects are smooth, connected, projective genus g curves C over $\mathbb{C}$. The morphisms are

$$\mathrm{Mor}(C, C') := \mathrm{Isom}_{\mathrm{Sch}\,/\mathbb{C}}(C, C'),$$

so this is clearly a groupoid.

Example. Let Σ be a set. Define the category $\mathcal{C}_\Sigma$ to be the category whose objects are elements of Σ and where the only morphisms are identity maps.

[5] By "moduli functor," we just mean functor. The word moduli is purely psychological and meant to make us think of viewing $F(S)$ as some family of objects over S

Remark 3.7. A morphism of groupoids is a functor, and an equivalence of groupoids is an equivalence of categories.

Example. Say $\mathcal{C}$ has 2 objects x_1, x_2 with morphisms $\mathrm{Mor}(x_i, x_j) = \{\pm 1\}$ for all i, j. Then $\mathcal{C}$ is equivalent to the groupoid with one object x and $\mathrm{Mor}(x, x) = \{\pm 1\}$.

Example. Let G be a group acting on a set Σ. We define the **quotient groupoid** $[\Sigma/G]$ to be the category whose objects are elements $x \in \Sigma$ of the set, and whose morphisms $\mathrm{Mor}(x, x') = \{g \in G : x' = gx\}$ are group elements sending x to x'.

Exercise. Show $[\Sigma/G]$ is equivalent to a set $\iff$ $G \curvearrowright \Sigma$ freely.

Example. If $\Sigma = *$ in the above example, then we get the classifying groupoid $BG = [*/G]$. This has one object $*$ and $\mathrm{Mor}(*, *) = G$.

Example. Let FB be the category of finite sets where morphisms are bijections. If you think about it,

$$FB = \bigsqcup_{n \geq 0} BS_n.$$

3.3.1 Fiber products of groupoids

Consider $\mathcal{C} \xrightarrow{f} \mathcal{D} \xleftarrow{g} \mathcal{D}'$, a diagram of groupoids. The **fiber product** is the category $\mathcal{C} \times_{\mathcal{D}} \mathcal{D}'$ of triples (c, d', α) with $c \in \mathcal{C}$, $d' \in \mathcal{D}'$, and $\alpha : f(c) \xrightarrow{\sim} g(d')$ whose morphisms are

$$\mathrm{Mor}\left((c_1, d'_1, \alpha_1), (c_2, d'_2, \alpha_2)\right) = \left\{ \begin{array}{c|c} \begin{array}{c} c_1 \xrightarrow{\beta} c_2 \\ d'_1 \xrightarrow{\gamma} d'_2 \end{array} & \begin{tikzcd} f(c_1) \arrow[r, "f(\beta)"] \arrow[d, "\alpha_1"'] & f(c_2) \arrow[d, "\alpha_2"] \\ g(d'_1) \arrow[r, "g(\gamma)"] & g(d'_2) \end{tikzcd} \end{array} \right\}$$

(the diagram above is required to commute).

Question 3.8 (Exercise)**.** *What is the universal property?*

Example.

$$\begin{tikzcd} G \arrow[r] \arrow[d] & * \arrow[d] \\ * \arrow[r] & BG \end{tikzcd}$$

is Cartesian, i.e. G is the above fiber product.

Remember: G is the category whose objects are $g \in G$ and whose morphisms are only identities. BG is the category with one object $*$ and morphisms $\mathrm{Mor}(*, *) = G$.

Example. Say G acts on a set Σ, and choose some $x \in \Sigma$. We have the orbit Gx and the stabilizer G_x. There is a natural map

$$BG_x \to [\Sigma/G]$$

as long as the natural projection $\Sigma \to [\Sigma/G]$. What is the fiber product? Exercise: it is the orbit, i.e.

$$\begin{tikzcd} Gx \arrow[r] \arrow[d] & \Sigma \arrow[d] \\ BG_x \arrow[r] & {[\Sigma/G]} \end{tikzcd}$$

is Cartesian.

3.4 Prestacks

3.4.1 Motivation

Specifying a moduli ~~functor~~ prestack requires specifying

- families of objects
- ~~when~~ how two families of objects are isomorphic; and
- and how families pull back under morphisms.

Here's a first attempt at making this formal. Consider maps

$$\mathrm{Sch} \xrightarrow{F} \mathrm{Groupoid}.$$

Note that Groupoid is a category of categories, and is really a 2-category.

How can we write down such a map?

- For every scheme S, $F(S)$ is a groupoid
- For all $f : S \to T$, we need a pullback $f^* : F(T) \to F(S)$
- Given composable $S \xrightarrow{f} T \xrightarrow{g} U$, we want an isomorphism

  $$\psi_{f,g} : f^* \circ g^* \xrightarrow{\sim} (g \circ f)^*$$

  of functors.

Is this enough? No, because of categorical issues. We need a compatibility of $\psi_{f,g}$ for triple compositions $S \to T \to U \to V$. This leads to the concept of "pseudofunctors" or "lax functors" or something like that. We won't go into this too deeply; we just wan to impress that there are extra difficulties/subtleties one needs to take care of.

One can do this, but we won't. Instead, we will build a massive category $\mathcal{X}$ with all of this data. Our $\mathcal{X}$ will live over Sch and an object in $\mathcal{X}$ will be something like (S, a) with $S \in \mathrm{Sch}$ and $a \in F(S)$. Intuively, we set

$$\mathcal{X} = \bigsqcup_{S \in \mathrm{Sch}} F(S).$$

3.4.2 Precise Definition

Let $p : \mathcal{X} \to \mathcal{S}$ be a functor of categories.

$$\begin{tikzcd} \mathcal{X} \arrow[d, "p"] \\ \mathcal{S} \end{tikzcd} \qquad \begin{tikzcd} a \arrow[r, "\alpha"] \arrow[d, mapsto] & b \arrow[d, mapsto] \\ S \arrow[r, "f"] & T \end{tikzcd}$$

Above, a, b are objects of $\mathcal{X}$ and S, T are objects of $\mathcal{S}$. We say a is over S and α is over f.

Definition 3.9. A functor $p : \mathcal{X} \to \mathcal{S}$ is a **prestack over** $\mathcal{S}$ if (pullbacks exist) for any diagram

$$\begin{tikzcd} a \arrow[r, dashed] \arrow[d, dashed, mapsto] & b \arrow[d, mapsto] \\ S \arrow[r] & T \end{tikzcd}$$

of solid arrows, there eixsts $a \to b$ over $S \to T$; and

(universal property of pullbacks) for any diagram

The vertical $\mapsto$'s are not arrows/morphisms. They just denote e.g. that a lies over R, i.e. $p(a) = R$.

$$\begin{tikzcd} a \arrow[r, dashed] \arrow[d, mapsto] \arrow[rr, bend left] & b \arrow[r] \arrow[d, mapsto] & c \arrow[d, mapsto] \\ R \arrow[r] & S \arrow[r] & T. \end{tikzcd}$$

of solid arrows, there exists unique arrow $a \to b$ over $R \to S$ filling in the diagram.

Remark 3.10. The second property above is secretly saying that every arrow in $\mathcal{X}$ is a pullback.

Getting comfortable with prestacks is very important with keeping up with the course as we continue.

Warning 3.11. This is not standard terminology. These are usually called "categories fibered in groupoids." When other people say "prestack," they usually mean one of these + an additional axiom (one of the two stack axioms). Of note, the word "prestack" apparently does not appear in the stacks project.

Abuse of Notation 3.12.

- We often only write $\mathcal{X}$, not $\mathcal{X} \to \mathcal{S}$.
- We often do not spell out the composition law.

Notation 3.13.

- We write f^*b or $b|_S$ to denote a *choice* of pullback.
- For $s \in \mathcal{S}$, the **fiber category** $\mathcal{X}(S)$ is the category of objects $a \in \mathcal{X}$ over S with morphisms over id_S.

Exercise. If $\mathcal{X} \to \mathcal{S}$ is a prestack, then $\mathcal{X}(S)$ is always a groupoid.

In below examples, $\mathcal{S} = \mathrm{Sch}$.

Example. Let $F : \mathrm{Sch} \to \mathrm{Set}$ be a presheaf. We build $\mathcal{X}_F$ as follows:

- objects are pairs (S, a) with $a \in F(S)$ and $S \in \mathrm{Sch}$
- $$\mathrm{Mor}((S,a),(T,b)) := \{f : S \to T : a = f^*b\}.$$

In this way, we can view any presheaf as a prestack.

Example. Let X be a scheme. We can apply previous construction to $h_X = \mathrm{Mor}(-, X) : \mathrm{Sch} \to \mathrm{Set}$ to get a prestack $\mathcal{X}_X$. Spelled out

- objects are maps $S \to X$ for any $S \in \mathrm{Sch}$
- morphisms $(S \to X) \to (T \to X)$ are X-morphisms $\alpha : S \to T$.

Example. $\mathcal{M}_g$ is the prestack whose objects are families $\mathcal{C} \to S$ of smooth curves of genus g (here, $S \in \mathrm{Sch}$). A morphism $(\mathcal{C} \to S) \to (\mathcal{C}' \to S')$ is a Cartesian diagram

$$\begin{tikzcd} \mathcal{C} \arrow[r] \arrow[d] & \mathcal{C}' \arrow[d] \\ S \arrow[r] & S'. \end{tikzcd}$$

Require morphisms to be Cartesian is what makes this a prestack (it gives the universal property of pullbacks).

Example. Let C be a smooth, connected, projective curve over k. The prestack $\mathrm{Bun}(C)$ has

- objects: pairs $(S, \mathcal{F})$ where $\mathcal{F}$ is a vector bundle on $C \times S$
- $$\mathrm{Mor}((S, \mathcal{F}), (S', \mathcal{F}')) = \left\{ S \to S' \ \text{ and } \ (f \times \mathrm{id})^* \mathcal{F}' \xrightarrow{\sim} \mathcal{F} \right\}.$$

  Above requires a choice of pullback, so maybe less canonical than one would like. Alternatively, could ask for a morphism $\mathcal{F}' \to (f \times \mathrm{id})_* \mathcal{F}$ such that the adjoint map is an isomorphism.

Example. Let $G \to S$ be an S-group scheme acting on an S-scheme X. The **quotient prestack** $[X/G]^{\mathrm{pre}}$ (over Sch/S) has

- objects: maps $T \to X$ over S
- $$\mathrm{Mor}(T \to X, T' \to X) = \{ (T \to T', g \in G(T)) : (T \to T' \to X) = g(T \to X) \}.$$

Exercise. $[X/G]^{\mathrm{pre}}(T) = [X(T)/G(T)]$, i.e. the fiber categories are quotient groupoids.

Definition 3.14. A **morphisms of prestacks** $f : \mathcal{X} \to \mathcal{Y}$ is a functor such that the diagram

$$\begin{tikzcd} \mathcal{X} \arrow[rr, "f"] \arrow[dr, "p_{\mathcal{X}}"'] & & \mathcal{Y} \arrow[dl, "p_{\mathcal{Y}}"] \\ & \mathcal{S} & \end{tikzcd}$$

strictly commutes, i.e. $p_{\mathcal{X}}(a) = p_{\mathcal{Y}}(f(a))$ is an equality.

Definition 3.15. If $f, g : \mathcal{X} \to \mathcal{Y}$ are morphisms of prestacks, a **2-morphism** $\alpha : f \to g$ is a natural transformation such that for every object $a \in \mathcal{X}$, the morphism $\alpha_a : f(a) \to g(a)$ in $\mathcal{Y}$ is over the identity in $\mathcal{S}$.

Definition 3.16. The category $\mathrm{MOR}(\mathcal{X}, \mathcal{Y})$ is the category whose objects are morphisms of prestacks and whose morphisms are 2-morphisms.

Exercise. $\mathrm{MOR}(\mathcal{X}, \mathcal{Y})$ is a groupoid.

Note 3. Jarod also defined the notion of a "2-commutative diagram," but I didn't type this since I don't know how to draw 2-morphisms in commutative diagrams... Basically a diagram 2-commutes if you specify a 2-morphism between every composition of arrows starting and ending in the same place.

Definition 3.17. An **isomorphism** is a map $f : \mathcal{X} \to \mathcal{Y}$ with an inverse, i.e. $\exists g : \mathcal{Y} \to \mathcal{X}$ and 2-isomorphisms $g \circ f \xrightarrow{\sim} \mathrm{id}_{\mathcal{X}}$ and $f \circ g \xrightarrow{\sim} \mathrm{id}_{\mathcal{Y}}$.

Exercise. $\mathcal{X} \xrightarrow{f} \mathcal{Y}$ is an iso $\iff$ $\mathcal{X}(S) \to \mathcal{Y}(S)$ is an iso (i.e. equiv of categories/groupoids) for all S.

Lemma 3.18 (The 2-Yoneda Lemma). *Let $\mathcal{X}$ be a prestack over a category $\mathcal{S}$, and choose $S \in \mathcal{S}$. The functor*

Note S is a prestack in the same way schemes are prestacks (described earlier)

$$\mathrm{Mor}(S, \mathcal{X}) \to \mathcal{X}(S)$$

sending $f \mapsto f_S(\mathrm{id}_S)$ is an equivalence of categories.

Exercise. Work through details of a proof of this.

4 Lecture 4 (1/13): Stacks

4.1 Audience Questions Before We Start

Question 4.1. *If X, Y are schemes, are there more prestack morphisms $X \to Y$ than scheme morphisms $X \to Y$?*

Answer. No, there are not. This is a consequence of the 2-Yoneda lemma.

Question 4.2. *Is I have a Deligne-Mumford stack and a map to an affine scheme which induces an equivalence on the étale topologies (in a universal way), then is the source also affine?*

Answer. Answer not obvious off the top of the head. Maybe we'll address this later after we talk about Deligne-Mumford stacks and whatnot.

4.2 Recap of Last Time

Let $\mathcal{S}$ be a category.

Recall 4.3. A *prestack* over $\mathcal{S}$ is a functor $p : \mathcal{X} \to \mathcal{S}$ such that

- for all diagrams

$$\begin{tikzcd} a \arrow[r, dashed] \arrow[d, dashed, mapsto] & b \arrow[d, mapsto] \\ S \arrow[r] & T \end{tikzcd}$$

  there exists a with a morphism $a \to b$ over $s \to T$.

- for all diagrams

$$\begin{tikzcd} a \arrow[r, dashed] \arrow[rr, bend left] \arrow[d, mapsto] & b \arrow[r] \arrow[d, mapsto] & c \arrow[d, mapsto] \\ R \arrow[r] & S \arrow[r] & T. \end{tikzcd}$$

  there exists a morphism $a \to b$ over $R \to S$.

The *fiber category* $\mathcal{X}(S)$ has objects $a \in \mathcal{X}$ over S and morphisms $a \to b$ over id_S.

Example (Schemes are prestacks)**.** If X is a scheme, we have $\mathrm{Sch}/X \to \mathrm{Sch}$ where $(S \to X) \mapsto S$.

Example. The moduli space of smooth curves $\mathcal{M}_g$ is a prestack. It's objects are smooth families $\mathcal{C} \to S$ of genus g curves. Morphisms in this category are Cartesian diagrams.

Note 4. There are 100 "participants" on the dot right now.[6]

Recall 4.4. A *morphism* $f : \mathcal{X} \to \mathcal{Y}$ of prestacks is a functor such that $p_{\mathcal{Y}} \circ f = p_{\mathcal{X}}$ are strictly equal. A 2-*morphism* $\alpha : f \to g$ is a natural transformation such $\alpha_x : f(x) \to g(x)$ is over the identity.

Recall 4.5 (2-Yoneda)**.** Let $\mathcal{X}$ be a prestack over a category $\mathcal{S}$ and $S \in \mathcal{S}$. The functor

$$\mathrm{MOR}(S, \mathcal{X}) \to \mathcal{X}(S) \qquad f \mapsto f_S(\mathrm{id}_S)$$

is an equivalence of categories.

4.3 Fiber products of prestacks

Consider morphisms over $\mathcal{S}$

$$\begin{tikzcd} & \mathcal{Y}' \arrow[d, "g"] \\ \mathcal{X} \arrow[r, "f"] & \mathcal{Y} \end{tikzcd}$$

The **fiber product** $\mathcal{X} \times_{\mathcal{Y}} \mathcal{Y}'$ is the prestack with

- objects = triples (x, y', γ) with $x \in \mathcal{X}$, $y' \in \mathcal{Y}'$, and $\gamma : f(x) \xrightarrow{\sim} g(y')$ over id_S (in particular, $p_{\mathcal{X}}(x) = S = p_{\mathcal{Y}'}(y')$)
- A map $(x_1, y_1', \gamma_1) \to (x_2, y_2', \gamma_2)$ is a triple (q, χ, γ') with $\chi : x_1 \to x_2$, $\gamma' : y_1' \to y_2'$, and $q : S_1 \to S_2$ such that
$$\begin{tikzcd} f(x_1) \arrow[r, "f(\chi)"] \arrow[d, "\gamma_1"'] & f(x_2) \arrow[d, "\gamma_2"] \\ g(y_1') \arrow[r, "g(\gamma')"] & g(y_2') \end{tikzcd}$$
commutes.

Remark 4.6. The fiber categories above are fiber products of groupoids.

Remark 4.7. There is a 2-commutative diagram

$$\begin{tikzcd} \mathcal{X} \times_{\mathcal{Y}} \mathcal{Y}' \arrow[r, "p_2"] \arrow[d, "p_1"'] \arrow[dr, phantom, "\swarrow\alpha"] & \mathcal{Y}' \arrow[d, "g"] \\ \mathcal{X} \arrow[r, "f"] & \mathcal{Y} \end{tikzcd}.$$

Here, $\alpha : g \circ p_2 \to f \circ p_1$ is the 2-morphism associating to $(x, y', \gamma) \in \mathcal{X} \times_{\mathcal{Y}} \mathcal{Y}'$ the map

$$\alpha_{(x,y',\gamma)} f(x) \xrightarrow{\gamma} g(y').$$

[6] Air quotes e.g. since Jarod and his ipad are 2 "participants"

Theorem 4.8. *We have the following universal property: for any 2-commutative diagram*

$$\begin{tikzcd}
\mathcal{T} \arrow[drr, bend left, "q_2"] \arrow[ddr, bend right, "q_1"'] \arrow[dr, phantom, "\stackrel{\tau}{\Nearrow}"] & & \\
& \mathcal{X} \times_{\mathcal{Y}} \mathcal{Y}' \arrow[r, "p_2"] \arrow[d, "p_1"] \arrow[dr, phantom, "\stackrel{\alpha}{\Nearrow}"] & \mathcal{Y}' \arrow[d, "g"] \\
& \mathcal{X} \arrow[r, "f"] & \mathcal{Y}
\end{tikzcd}$$

there exists a morphism $h : \mathcal{T} \to \mathcal{X} \times_{\mathcal{Y}} \mathcal{Y}'$ and 2-isomorphisms $\beta : q_1 \to p_1 \circ h$ and $\gamma : q_2 \to p_2 \circ h$ such that

$$\begin{tikzcd}
f \circ q_1 \arrow[r, "f(\beta)"] \arrow[d, "\tau"'] & f \circ p_1 \circ h \arrow[d, "\alpha \circ h"] \\
g \circ q_2 \arrow[r, "g(\gamma)"] & g \circ p_2 \circ h
\end{tikzcd}$$

commutes.

Exercise. Let $\mathcal{M}_{g,1}$ be the prestack whose objects are

$$\begin{tikzcd}
\mathcal{C} \arrow[d, shift right] \\
S \arrow[u, bend right, shift right, "\sigma"']
\end{tikzcd}$$

smooth families of genus g curves with a section σ. The morphisms are again cartesian diagrams, but now must be compatible with section. There's a natural map $\mathcal{M}_{g,1} \to \mathcal{M}_g$ forgetting the section. Consider a map $S \to \mathcal{M}_g$ from some scheme (i.e. by 2-Yoneda, a family $\mathcal{C} \to S$ of curves). Then, the fiber product is this family, i.e.

$$\begin{tikzcd}
\mathcal{C} \arrow[r] \arrow[d] & \mathcal{M}_{g,1} \arrow[d] \\
S \arrow[r] & \mathcal{M}_g
\end{tikzcd}$$

is Cartesian. Hence, $\mathcal{M}_{g,1} \to \mathcal{M}_g$ is the universal family.

Exercise. Let $\mathcal{X}$ be a prestack. Let $a, b : T \to \mathcal{X}$ be morphisms from a scheme T. Define the presheaf

$$\underline{\mathrm{Isom}}_T(a,b) : \mathrm{Sch}/T \to \mathrm{Set}$$

sending $(S \xrightarrow{f} T) \mapsto \mathrm{Isom}_{\mathcal{X}(S)}(f^*a, f^*b)$. Show

$$\begin{tikzcd}
\underline{\mathrm{Isom}}_T(a,b) \arrow[r] \arrow[d] & T \arrow[d, "(a,b)"] \\
\mathcal{X} \arrow[r, "\Delta"] & \mathcal{X} \times \mathcal{X}
\end{tikzcd}$$

is Cartesian.

4.4 Stacks

Definition 4.9. A **stack over a site** $\mathcal{S}$ is a prestack $\mathcal{X} \to \mathcal{S}$ such that for all coverings $\{S_i \to S\}$ of $S \in \mathcal{S}$:

(1) (morphisms glue) For $a, b \in \mathcal{X}$ over S and maps $\varphi_i : a|_{S_i} \to b$ such that $\varphi_i|_{S_{ij}} = \varphi_j|_{S_{ij}}$

$$\begin{tikzcd}
& a|_{S_i} \arrow[dr] \arrow[drr, bend left, "\varphi_i"] & & & & & S_i \arrow[dr] & \\
a|_{S_{ij}} \arrow[ur] \arrow[dr] & & a \arrow[r, dashed] & b & \text{over} & S_{ij} \arrow[ur] \arrow[dr] & & S \\
& a|_{S_j} \arrow[ur] \arrow[urr, bend right, "\varphi_j"'] & & & & & S_j \arrow[ur] &
\end{tikzcd}$$

there exists a unique map $\varphi : a \to b$ with $\varphi|_{S_i} = \varphi_i$.

(2) (objects glue) For $a_i \in \mathcal{X}$ over S_i and isos $\alpha_{ij} : a_i|_{S_{ij}} \xrightarrow{\sim} a_j|_{S_{ij}}$ with $\alpha_{ij}|_{S_{ijk}} \circ \alpha_{jk}|_{S_{ijk}} = \alpha_{ik}|_{S_{ijk}}$ (cocycle condition), as displayed below

$$\begin{tikzcd}
& & a_i \arrow[dr, dashed] & & & & S_i \arrow[dr] & \\
a_i|_{S_{ij}} \arrow[r, "\alpha_{ij}"] \arrow[urr] \arrow[drr] & a_j|_{S_{ij}} & & a & \text{over} & S_{ij} \arrow[ur] \arrow[dr] & & S \\
& & a_j \arrow[ur, dashed] & & & & S_j \arrow[ur] &
\end{tikzcd}$$

then there exists $a \in \mathcal{X}$ over S and isos $\varphi_i : a|_{S_i} \to a_i$ such that $\alpha_{ij} \circ \varphi_i|_{S_{ij}} = \varphi_j|_{S_{ij}}$.

Remark 4.10. Axiom 1 above should be equivalent to $\underline{\mathrm{Isom}}_T(a, b)$ always being a sheaf[7], not just a presheaf. Further, being a stack means

$$\mathcal{X}(S) \longrightarrow \prod \mathcal{X}(S_i) \rightrightarrows \prod \mathcal{X}(S_{ij}) \substack{\longrightarrow\\\longrightarrow\\\longrightarrow} \prod \mathcal{X}(S_{ijk})$$

is "exact" (spelling out what this means will give above axioms[8]).

Example (sheaves are stacks)**.** Let $F : \mathcal{S} \to \mathrm{Set}$ be a presheaf. Recall this gives rise to a prestack $\mathcal{X}_F$ with fiber categories $\mathcal{X}_F(S) = F(S)$. Exercise: F sheaf $\iff$ $\mathcal{X}_F$ is a stack.

Example (Stack of qcoh sheaves over $\mathrm{Sch}_{\text{Ét}}$)**.** Let QCoh have

- objects: (S, F) with S a scheme and $F \in \mathrm{QCoh}(S)$ a quasi-coherent sheaf on S.
- morphisms:
$$\mathrm{Mor}((S, F), (S', F')) = \left\{ S \xrightarrow{f} S' \text{ and } F' \to f_* F \mid f^* F' \xrightarrow{\sim} F \right\}.$$

We know that QCoh is a stack over $\mathrm{Sch}_{\mathrm{Zar}}$. This follows already from Hartshorne exercise II.1.{15,22}. Étale descent for qcoh sheaves implies that QCoh is also a stack in $\mathrm{Sch}_{\text{Ét}}$.

[7]Someone made a comment to this effect in chat
[8]I think both of them

Example (stack of families over $\mathrm{Sch}_{\mathrm{Zar}}$)**.** Define Fam $\to$ Sch to be the prestack whose objects are maps $X \to S$ of schemes, and whose morphisms are Cartesian diagrams.[9]

Proposition 4.11. Fam *is a stack over the big Zariski site* Sch_{Zar}.

Axiom **(1)** is just that morphisms glue. Axiom **(2)** is that schemes glue (Hartshorne exercise II.2.12).

Note that Fam is *not* a stack over $\mathrm{Sch}_{\text{Ét}}$. We cannot glue schemes in the étale topology. Later, we'll see that we can glue algebraic spaces in the étale topology though.

Example ($\mathcal{M}_g$)**.** First recall that a **family of smooth curves of genus** g is a smooth proper morphism $\mathcal{C} \to S$ of schemes s.t. every geometric fiber $\mathcal{C}_{\overline{\kappa(s)}}$ is a connected curve of genus g.

Proposition 4.12. *Let* $\mathcal{C} \to S$ *be a family of smooth curves of genus* $g \geq 2$*. Then for* $k \geq 3$, $\Omega_{\mathcal{C}/S}^{\otimes k}$ *is relatively very ample and* $\pi_*(\Omega_{\mathcal{C}/S}^{\otimes k})$ *is a vector bundle of rank* $(2k-1)(g-1)$.

This is a consequence of cohomology and base change. This will allow us the prove that $\mathcal{M}_g$ is a stack over $\mathrm{Sch}_{\text{Ét}}$. Morphisms glue, so only need to worry about the second axiom. Say we have

$$\begin{tikzcd} \mathcal{C}_i|_{S_{ij}} \arrow[r] \arrow[d] & \mathcal{C}_i \arrow[d, "\pi_i"] & \\ S_{ij} \arrow[r] & S_i \arrow[r] & S \end{tikzcd}$$

Set $\mathcal{C}_{ij} := \mathcal{C}_i|_{S_{ij}}$ and $E_i := \pi_{i,*}\Omega_{\mathcal{C}_i/S_i}^{\otimes 3}$. We know that

$$E_i|_{S_{ij}} \simeq \pi_*\Omega_{\mathcal{C}_{ij}/S_{ij}} \simeq E_j|_{S_{ij}}.$$

Now Étale descent gives us a vector bundle E on S.

Proposition 4.13 (Étale descent for closed immersions)**.** *Let* $\{X_i \to X\}$ *be an étale cover. Closed subschemes* $Z_i \subset X_i$ *with*[10] $Z_i|_{X_{ij}} = Z_j|_{X_{ij}}$ *glue to a closed subscheme* $Z \subset X$.

e.g. Think of closed subscheme as an ideal sheaf, and apply (effective) descent for qcoh sheaves

We have $\mathcal{C}_i \hookrightarrow \mathbb{P}(E_i)$ and $\mathbb{P}(E_i) \to \mathbb{P}(E)$ an étale map (I think?) with $\mathbb{P}(E)$ over S. The above descent result gives now $\mathcal{C} \to S$. More descent arguments shows that it is a smooth family, so we can glue objects as desired.

Let $G \to S$ be a smooth, affine group scheme, let $X \to S$ be an S-scheme, and suppose $G \curvearrowright X$ (functorially, we have actions $G(T) \curvearrowright X(T)$ for all $T \to S$). Last time we defined $[X/G]^{pre}$ over Sch/S with fiber categories

$$[X/G]^{pre}(T) = [X(T)/G(T)]$$

given by the quotient groupoid.

Exercise. This is not a stack on $(\mathrm{Sch}/S)_{\text{Ét}}$.

An object over T_i is a map $T_i \to X$. However, we can also think of this as a trivial G-torsor

$$\begin{tikzcd} G \times T_i \arrow[r, "G\text{-equiv}"] \arrow[d] & X \\ T_i & \end{tikzcd}$$

[9] Note $\mathcal{M}_g$ is a substack of Fam

[10] Don't need a triple overlap condition for some reason?

So, intuitively at least, the issue with this being a stack is that these trivial G-bundles won't glue to a trivial G-bundle.

Definition 4.14. A **principle G-bundle over** T is a scheme P along with a G-action such that $P \to T$ is G-invariant and there is an étale cover $\{T_i \to T\}$ such that $T_i \times_T P \simeq G \times T_i$, G-equivariantly.

Example (Quotient stacks)**.** The **quotient stack** $[X/G]$ over Sch/S has

- objects:

$$\begin{tikzcd} P \arrow[r, "f"] \arrow[d] & X \\ T & \end{tikzcd}$$

  where f is G-equivariant.

- morphisms: Cartesian diagrams

$$\begin{tikzcd} P \arrow[r] \arrow[d] & P' \arrow[d] \\ T \arrow[r] & T' \end{tikzcd}$$

  such that $f : P \to X$ factors as $P \to P' \xrightarrow{f'} X$.

The map $P \to P'$ needs to be G-equivariant, i.e. the induced isomorphism $P \xrightarrow{\sim} P'_T$ is an iso of G-torsors over T

This is a Morphism "from P to P'"

Exercise: this is a stack over $(\mathrm{Sch}/S)_{\text{Ét}}$.

4.5 Fiber products of stacks

Exercise. If $\mathcal{X}, \mathcal{Y}, \mathcal{Y}'$ are stacks over a site $\mathcal{S}$, then $\mathcal{X} \times_{\mathcal{Y}} \mathcal{Y}'$ is also a stack over $\mathcal{S}$.

Exercise.

$$\begin{tikzcd} G \times X \arrow[r] \arrow[d] & X \arrow[d, "p"] \\ X \arrow[r] & {[X/G]} \end{tikzcd}$$

is Cartesian.

Exercise.

$$\begin{tikzcd} G \times X \arrow[r] \arrow[d] & X \times X \arrow[d] \\ {[X/G]} \arrow[r, "\Delta"] & {[X/G] \times [X/G]} \end{tikzcd}$$

is Cartesian.

Exercise. For any G-bundle $P \to T$ with G-equivariant map $P \xrightarrow{f} X$,

$$\begin{tikzcd} P \arrow[r] \arrow[d] & X \arrow[d] \\ T \arrow[r] & {[X/G]} \end{tikzcd}$$

is Cartesian. This says that $X \to [X/G]$ is the universal G-bundle, i.e. it classifies principle G-bundles along with the data of an equivariant map to X.

If you take $X = S$, the terminal object, then get $BG = [S/G]$ which really is the universal G-bundle, it classifies principle G-bundles full stop.

This is different from BG the groupoid from before.

4.6 Stackification

Proposition 4.15 (**Stackification**). *For a prestack $\mathcal{X}$ over a site $\mathcal{S}$, there exists a morphism $\mathcal{X} \to \mathcal{X}^{st}$ to a stack such that for any stack $\mathcal{Y}$ over $\mathcal{S}$, the functor*

$$\mathrm{MOR}(\mathcal{X}^{st}, \mathcal{Y}) \to \mathrm{MOR}(\mathcal{X}, \mathcal{Y})$$

is an equivalence of categories.

Proof Sketch. A 2-step process[11]

- Construct $\mathcal{X} \to \mathcal{X}_1$ with $\mathcal{X}_1$ a prestack where morphisms glue *uniquely*.[12] We say $\mathrm{ob}\,\mathcal{X}_1 = \mathrm{ob}\,\mathcal{X}$ and a map $a \to b$ in $\mathcal{X}_1$ over $S \xrightarrow{f} T$ is by definition an element of[13]

  $$\left(\underline{\mathrm{Isom}}_S(a, f^*b)\right)^{sh}(S).$$

  (since this involves sheafification, this is secretly 2 steps).

- Construct $\mathcal{X}_1 \to \mathcal{X}^{st}$. Formally define an object of $\mathcal{X}^{st}$ as a triple

  $$\left(\{S_i \to S\}, a_i \mapsto S_i, \alpha_{ij} : a_i|_{S_{ij}} \to a_j|_{S_{ij}}\right)$$

  with the α_{ij}'s satisfying the cocycle condition.

∎

Exercise.

$$\left([X/G]^{\mathrm{pre}}\right)^{\mathrm{st}} \cong [X/G].$$

Exercise. $\mathcal{M}_0 \simeq B\mathrm{PGL}(2)$

5 Lecture 5 (1/20): Algebraic spaces and stacks

Admistrative stuff

- Zoom chat disabled. Use Zulip instead.
- No class next Monday either

"I've been talking over monumental moments in American history. In the last three Wednesdays we went from an insurrection to an impeachment and now to an inauguration, and that sort of mirrors the development in this class. We've went from prestacks to stacks, and now to algebraic spaces/stacks" (paraphrase)

[11] Ravi says 3-step
[12] I think Ravi doesn't want to get uniqueness in the first step
[13] A morphism $a \to b$ in $\mathcal{X}$ is, by definition, an element $\mathrm{Isom}_S(a, f^*b)$

5.1 Recap

Let $\mathcal{S}$ be a site.

Recall 5.1. A prestack over $\mathcal{S}$ is a functor $p : \mathcal{X} \to \mathcal{S}$ such that pullbacks exist and any map in $\mathcal{X}$ satisfies a universal property.

Recall 5.2. A prestack $\mathcal{X}$ over $\mathcal{S}$ is a stack if morphisms glue uniquely w.r.t to covers $\{S_i \to S\}$, and if objects glue with respect to covers.

Example. If X is a scheme, we get a stack $\mathcal{X}_X$ (whose objects are maps $S \to X$). In fact any sheaf $\mathscr{F}$ gives rise to a stack $\mathcal{X}_{\mathscr{F}}$.

Example. We saw the stack QCoh $\to$ Sch of quasi-coherent sheaves where objects are $(S, \mathscr{F})$ with S a scheme and $\mathscr{F}$ qcoh.

Example. We say $\mathcal{M}_g$ whose objects are smooth families $\mathcal{C} \to S$ of curves, and whose morphisms are Cartesian diagrams.

These are all stacks over $\mathrm{Sch}_{\mathrm{Zar}}$. Étale descent implies that they are even stacks over $\mathrm{Sch}_{\text{Ét}}$.

5.2 Summary of descent

Everything relies on one key fact

Fact. If $\varphi : A \to B$ is a faithfully flat ring map, and M is an A-module, then the sequence

$$M \to M \otimes_A B \rightrightarrows M \otimes_A B \otimes_A B$$

is exact.

Let's schemify this. Say $S = \operatorname{spec} A$, $S' = \operatorname{spec} B$, and $\mathscr{F} = \widetilde{M}$. We have the sequence

$$S' \times_S S' \rightrightarrows S' \to S.$$

Let $\mathscr{F}' := \mathscr{F}_{S'} = \widetilde{M \otimes_A B}$. This fact is saying that $\mathscr{F}$ is recoverable from $\mathscr{F}'$.

Fix an étale cover $\{S_i \to S\}$. Descent on quasi-coh sheaves

- Morphisms: given $\mathscr{F}, \mathscr{G} \in \mathrm{QCoh}(S)$,

$$\left\{\begin{matrix} \text{maps } \mathscr{F}|_{S_i} \xrightarrow{\varphi_i} \mathscr{G}|_{S_i} \\ \text{s.t. } \varphi_i|_{S_{ij}} = \varphi_j|_{S_{ij}} \end{matrix}\right\} \longleftrightarrow \left\{\mathscr{F} \xrightarrow{\varphi} \mathscr{G}\right\}.$$

- Objects:

$$\left\{\begin{matrix} \mathscr{F}_i \in \mathrm{QCoh}(S_i) \text{ and } \mathscr{F}_i|_{S_{ij}} \xrightarrow[\sim]{\alpha_{ij}} \mathscr{F}_j|_{S_{ij}} \\ \text{s.t. } \alpha_{jk} \circ \alpha_{ij} = \alpha_{ik} \\ \end{matrix}\right\} \longleftrightarrow \{\mathscr{F} \in \mathrm{QCoh}(S)\}.$$

These two say QCoh a stack over $\mathrm{Sch}_{\text{Ét}}$.

Descending affine morphisms:

$$\left\{\begin{array}{c} X_i \xrightarrow{\text{aff}} S_i \text{ and } X_i|_{S_{ij}} \overset{\alpha_{ij}}{\xrightarrow{\sim}} X_j|_{S_{ij}} \\ \text{s.t. } \alpha_{jk}\circ\alpha_{ij} = \alpha_{ik} \end{array}\right\} \longleftrightarrow \left\{X \xrightarrow{\text{aff}} S\right\}.$$

As a special cases of this, we see

- (closed immersions) We saw/used this when showing $\mathcal{M}_g$ is a stack (for $g \geq 2$).
- (principal G-bundle) Fix $G \to T$ a smooth and affine group scheme. A **G-bundle over** T is a map $P \to T$ and a G-action on P over T so that there exists an étale cover $\{T_i \to T\}$ of T so that $G \times T_i \simeq P \times_T T_i$ equivariantly. The corresponding descent statement is

$$\left\{\begin{array}{c} X_i \xrightarrow{G\text{-bundle}} S_i \text{ and } X_i|_{S_{ij}} \overset{\alpha_{ij}}{\xrightarrow{\sim}} X_j|_{S_{ij}} \\ \text{s.t. } \alpha_{jk}\circ\alpha_{ij} = \alpha_{ik} \end{array}\right\} \longleftrightarrow \left\{X \xrightarrow{G\text{-bdl}} S\right\}.$$

The key point is that $P \to T$ is necessarily affine since $G \to T$ is.

More generally, we have descent for quasi-affine morphisms (closed immersion gives open immersion, and then use Zariski main theorem?). Even more generally, can descend morphisms which are only separated and **locally quasi-finite** (i.e. locally of finite type with discrete fibers).

Consequences of descent:

- Let $\mathcal{P}$ be a property of morphisms: open imm, closed imm, affine, quasi-affine, or sep and loc q. finite
- Let $S' \overset{\text{ét}}{\twoheadrightarrow} S$ be étale surjection of schemes
- Let $F \to S$ be a map of sheaves (maybe presheaves enough?)

$$\begin{tikzcd} F' \arrow[r] \arrow[d] & F \arrow[d] \\ S' \arrow[r] & S \end{tikzcd}$$

Then,

$$\begin{pmatrix} F' \text{ is a scheme and} \\ F' \to S' \text{ has } \mathcal{P} \end{pmatrix} \Longrightarrow \begin{pmatrix} F \text{ is a scheme and} \\ F \to S \text{ has } \mathcal{P} \end{pmatrix}.$$

5.3 Hilbert schemes

Theorem 5.3 (Grothendieck)**.** *Let $X \to T$ be a projective morphism of noetherian schemes, and let $\mathscr{O}_X(1)$ be a relatively ample line bundle on X. Let $P \in \mathbb{Q}[z]$ be a polynomial. The functor*

$$\begin{array}{ccc} \mathrm{Sch}/T & \longrightarrow & \mathrm{Set} \\ (S \to T) & \longmapsto & \left\{ \begin{tikzcd} Z \arrow[r, hook, "closed"] \arrow[dr, "flat of fin pres"'] & X \times_T S \arrow[d] \\ & S \end{tikzcd} \;\middle|\; \forall s \in S : \mathrm{Hilb\,poly}(Z_s) = P \right\} \end{array}$$

is represented by a scheme $\mathrm{Hilb}^P(X/T)$ *which is projective over* T.

5.4 Main definitions

Definition 5.4. A map $F \to G$ of presheaves/prestacks over Sch is **representable by schemes** if for all maps $S \to G$ from a scheme, $F \times_G S$ is a scheme. Let $\mathcal{P}$ be a property of maps of schemes (e.g. surjective or étale). A map $F \to G$ representable by schemes **has property** $\mathcal{P}$ if for all $S \to G$ from a scheme, $F \times_G S \to S$ has $\mathcal{P}$.

Definition 5.5. An **algebraic space** is a sheaf X on $\mathrm{Sch}_{\text{Ét}}$ such that there exists a scheme U and a morphism $U \to X$ which is representable by schemes, étale, and surjective. We call $U \to X$ an **étale presentation**.

We can generalize previous definitions now.

Definition 5.6. A map $F \to G$ of presheaves/prestacks over Sch is **representable** if for all maps $S \to G$ from a scheme (equivalently, all from an alg. spaces), $F \times_G S$ is an algebraic space. Let $\mathcal{P}$ be a property of maps of schemes *which is* ***étale local on the source***, i.e. if $X' \overset{\text{ét}}{\twoheadrightarrow} X$, then $X \to Y$ has $P \iff X' \to X \to Y$ has P. A map $F \to G$ which is representable **has property** $\mathcal{P}$ if for all $S \to G$ from a scheme, and all étale presentations $U \to F \times_S G$, the composition $U \to F \times_G S \to S$ has $\mathcal{P}$. The relevant diagram here is

$$\begin{tikzcd}
\text{scheme} \arrow[d, equal] & \text{algebraic space} \arrow[d, equal] & \text{scheme} \arrow[d, equal] \\
U \arrow[r] & F \times_G S \arrow[r] \arrow[d] & S \arrow[d] \\
 & F \arrow[r, "\text{repr}"] & G
\end{tikzcd}$$

Remark 5.7. Not always, but usually, you can check this properties by checking on a single étale presentation.

Definition 5.8. A **Deligne-Mumford stack** is a stack $\mathcal{X}$ over $\mathrm{Sch}_{\text{Ét}}$ such that there exists a scheme U, and a morphism $U \to \mathcal{X}$ which is representable, étale, and surjective. The map $U \to \mathcal{X}$ is called an **étale presentation**.

This is almost like taking the definition of an algebraic space, and replacing the word sheaf with stack. However, now we require our map to just be representable (by algebraic spaces), not necessarily representable by schemes.

Definition 5.9. An **algebraic stack** is a stack $\mathcal{X}$ over $\mathrm{Sch}_{\text{Ét}}$ such that there exists a scheme U and a **smooth presentation** $U \to \mathcal{X}$, which is a map that is representable, smooth, and surjective.

History.

- Algebraic spaces were introduced by Artin and Knutson ca 1969,71. They had a qc assumption on $\Delta_{\mathcal{X}}$.
- DM stacks were introduced by Deligne and Mumford '69 (but called them algebraic stacks). They assumed Δ was representable by schemes.[14]

[14] They pointed out this was not the 'right' general definition, but sufficed for their cases

- Algebraic stacks were introduced by Artin '74 (also called them algebraic stacks). He assumed everything was locally of finite type over an excellent Dedekind domain.

Warning 5.10. Our definitions are not standard.

- Usually one requires that the diagonal $\Delta : \mathcal{X} \to \mathcal{X} \times \mathcal{X}$ is representable. We will show this is automatic from our definition.

- Different authors have different hypotheses on Δ.

We follow Olsson and Stacks project.[15]
We never defined morphisms...

- Maps of algebraic spaces are maps of sheaves

- Maps of DM or algebraic stacks are maps of stacks over $\mathrm{Sch}_{\text{Ét}}$

Exercise. Show that fiber products exist for algebraic spaces, DM stacks, and algebraic stacks.

Exercise. Show that any stack over $(\mathrm{Sch}/S)_{\text{Ét}}$ can be viewed as a stack over $\mathrm{Sch}_{\text{Ét}}$.

We've just seen many definitions. Our immediate goals are algebraicity of quotient stacks and of $\mathcal{M}_g$.

5.5 Algebraicity of Quotient Stacks

Here's the setup

- $G \to T$ smooth and affine group scheme.

- U an algebraic space with a G-action.

- $[U/G]$ denotes the quotient stack over $(\mathrm{Sch}/T)_{\text{Ét}}$. Its objects are diagrams

$$\begin{tikzcd} P \arrow[r, "G\text{-equiv}"] \arrow[d, "G\text{-bundle}"] & U \\ S & \end{tikzcd}$$

Theorem 5.11. *$[U/G]$ is an algebraic stack over T. Moreover, $U \to [U/G]$ is a G-torsor, and in particular, it is representable, surjective, smooth, and affine.*

Proof. Set $\mathcal{X} = [U/G]$. Let $S \to \mathcal{X}$ be a map from a scheme. Consider

$$\begin{tikzcd} U_S \arrow[r] \arrow[d] & U \arrow[d] \\ S \arrow[r] & \mathcal{X} \end{tikzcd}$$

where $U \to \mathcal{X} = [U/G]$ is the natural map. We need to show that $U_S \to S$ is a G-torsor. In particular, we will need to show U_S is a scheme.[16] Since $[U/G] = ([U/G]^{pre})^{st}$ we know we have a 2-commutative

[15] We use $\mathrm{Sch}_{\text{Ét}}$, but stacks project uses fppf topology $\mathrm{Sch}_{\mathrm{fppf}}$

[16] In general, the relevant fiber product will only need to be an algebraic space, but in the current setting, we'll actually get schemes

diagram

$$\begin{tikzcd}
S' \arrow[r] \arrow[d, "\text{ét}"'] \arrow[dr, phantom, "\swArrow"] & U \arrow[d] \\
S \arrow[r] & \mathcal{X}
\end{tikzcd}$$

with $S' \overset{\text{ét}}{\twoheadrightarrow} S$ surjective. We now form a cube

$$\begin{tikzcd}
& U_{S'} \arrow[rr] \arrow[dl] \arrow[dd] & & S' \arrow[dl] \arrow[dd, two heads, "\text{ét}"] \\
G \times U \arrow[rr] \arrow[dd] & & U \arrow[dd] & \\
& U_S \arrow[rr] \arrow[dl] & & S \arrow[dl] \\
U \arrow[rr] & & \mathcal{X} &
\end{tikzcd}$$

whose top, bottom, back, and front are all Cartesian.

At the moment, we only know U_S is a sheaf. What else do we know? We know $G \times U \to U$ is a G-bundle, so it's pullback $U_{S'} \to S'$ is also a G-bundle. Furthermore, $S' \to S$ is an étale surjection *of schemes*! By étale descent for G-bundles, we get that $U_S \to S$ is a G-bundle (so U_S a scheme), which is exactly what we wanted to show. ■

Corollary 5.12. *If G is a finite group acting freely on an algebraic space X, then the quotient sheaf X/G is an algebraic space. Here, $X \to X/G$ is an étale presentation (since G finite?).*

The upshot is that quotients by free, finite group actions (on schemes) always exist as an algebraic space. Further, algebraic spaces are closed under taking quotients by free actions of finite groups.

Remember: Any finite set of points on a quasi-projective scheme is contained in an open affine (someone said this in chat once, I think). See also this stack-exchange

5.6 Algebraicity of $\mathcal{M}_g$

Theorem 5.13. *If $g \geq 2$, then $\mathcal{M}_g$ is algebraic.*

We know $\mathcal{M}_g$ is a stack over $\mathrm{Sch}_{\text{Ét}}$. The strategy will be to show that

$$\mathcal{M}_g \cong [H/G]$$

where $H \subset \mathrm{Hilb}$ is locally closed, parameterizing 3-canonical embedded smooth curves.

Proof. Consider

$$\begin{tikzcd}
\mathcal{C} \arrow[r, hook] \arrow[d, "\text{sm fam}"', "\pi"] & \mathbb{P}(\pi_* \Omega_{\mathcal{C}/S}^{\otimes 3}) \arrow[dl] \\
S &
\end{tikzcd}$$

Over a fiber $s \in S$, this becomes

$$\begin{tikzcd}
\mathcal{C}_s \arrow[r, hook] \arrow[d] & \mathbb{P}^{5g-6} \arrow[dl] \\
\operatorname{spec} k(s) &
\end{tikzcd}$$

By Riemann-Roch, the Hilbert polynomial is $P(n) = (6n-1)(g-1)$. Define $H := \mathrm{Hilb}_P(\mathbb{P}^{5g-6})$ which is a scheme projective over $\mathbb{Z}$. We have a universal family

$$\begin{tikzcd} \mathcal{C} \arrow[r, hook] \arrow[d, "\text{univ fam}"'] & \mathbb{P}^{5g-6} \times H \arrow[dl] \\ H. & \end{tikzcd}$$

The idea now is to look at the locus in H of smooth curves tri-canonically embedded.

Claim 5.14. *$\exists!$ locally closed $H' \hookrightarrow H = \mathrm{Hilb}_P(\mathbb{P}^{5g-6})$ containing the $h \in H$ s.t.*

(a) *$\mathcal{C}_h$ is smooth and geometrically connected*

(b) *Setting $\mathcal{C}' = \mathcal{C}|_{H'}$, $\Omega^{\otimes 3}_{\mathcal{C}'/H'}$ and $\mathscr{O}_{\mathcal{C}'}(1)$ differ by the pullback of a line bundle on H'. Over $h \in H$, we have $\Omega^{\otimes 3}_{\mathcal{C}_h} = \mathscr{O}_{\mathcal{C}_h}(1)$.*

(c) *$\mathcal{C}_h \hookrightarrow \mathbb{P}^{5g-6}$ is embedded via $\left|\Omega^{\otimes 3}_{\mathcal{C}_h}\right|$, i.e.*

$$\Gamma(\mathbb{P}^{5g-6}, \mathscr{O}(1)) \xrightarrow{\sim} \Gamma(\mathcal{C}_h, \mathscr{O}_{\mathcal{C}_h}(1)).$$

Proof of Claim. **(a)** For smoothness, just note that $\{h \in H : \mathcal{C}_h \text{ smooth}\} \subset H$ is open. Use that the map is proper (so sends closed non-smooth points to a closed set downstairs). For geom connected, have Stein factorization

$$\mathcal{C} \to \mathbf{Spec}\, \pi_* \mathscr{O}_{\mathcal{C}} \to H$$

with first map having geometrically connected fibers. Let $\lambda : \mathscr{O}_? \to \pi_* \mathscr{O}_{\mathcal{C}}$. Then,

$$\{h \in H : \mathcal{C}_h \text{ geom conn}\} = \{h \in H : \lambda \text{ iso at } h\} = H \setminus (\operatorname{supp} \ker \lambda \cup \operatorname{supp} \operatorname{coker} \lambda)$$

Note that λ is a map of coherent sheaves ($\mathcal{C} \to H$ proper) so these supports are closed.

(b) We will need the following result.

Theorem 5.15. *Let $f : X \to Y$ be a flat, proper map of noeth schemes with geometrically integral fibers. Let L be a line bundle on X, and assume $f_* \mathscr{O}_X = \mathscr{O}_Y$ holds after base change. Then, there exists a closed subscheme $Z \hookrightarrow Y$ s.t. $T \to Y$ factors through Z if and only if $L|_{X_T}$ is the pullback of a line bundle on T.*

We won't prove this[17], but it follows mostly from cohomology and basechange. This theorem gives us **(b)**.

Question: Is this (related to) Seesaw theorem?

(c) Want to consider

$$\mathrm{H}^0(\mathbb{P}^{5g-6}, \mathscr{O}(1)) \to \pi_* \Omega^{\otimes 3}_{\mathcal{C}/S}.$$

Look at locus in H where this is an isomorphism (throw out kernel and cokernel). ■

Given that claim, note that there exists a functorial action of $\mathrm{PGL}_{5g-5} = \underline{\mathrm{Aut}}(\mathbb{P}^{5g-6})$ on H, and $H' \subset H$ is invariant. Our goal is

Remember: 4 step strategy to showing a stack is isomorphic to a quotient stack

[17] Not necessary to know, but this theorem is equivalent to the separatedness of the Picard functor $\mathrm{Pic}_{X/Y} : \mathrm{Sch}/Y \to \mathrm{Set}$ sending $(S \to Y) \mapsto \mathrm{Pic}(X \times_Y X)/\operatorname{Pic} S$

$$\mathcal{M}_g \cong [H'/\mathrm{PGL}_{5g-5}].$$

Do it in a few steps

- First construct a map
$$[H'/\mathrm{PGL}_{5g-6}]^{pre} \to \mathcal{M}_g.$$
Since H' lives in the Hilbert scheme, we can send $(S \to H')$ to the family $\mathcal{C} \to S$ it naturally corresponds to.

- This map is fully faithful. This is because an automorphism of $\mathcal{C} \to S$ induces an automorphism of $\Omega_{\mathcal{C}/S}$ (since its canonical), and therefore induces an automorphism of projective space.

- By univ property of stackification, there exists an induced map $[H'/\mathrm{PGL}_{5g-6}] \to \mathcal{M}_g$ which is also fully faithful.

- Now just need to show essential surjectivity. Suffices to show that for all families $\mathcal{C} \to S$ of smooth curves, there exists an étale cover $\{S_i \to S\}$ s.t. $\mathcal{C}_{S_i} \to S_i$ is in the image.[18] Have $\pi : \mathcal{C} \to S$. Always get an embedding $\mathcal{C} \hookrightarrow \mathbb{P}(\pi_*\Omega_{C/S}^{\otimes 3})$. To get map into honest projective space, choose a Zariski cover where $\pi_*\Omega_{\mathcal{C}/S}^{\otimes 3}$ trivializes.

■

6 Lecture 6 (1/27): First properties of algebraic spaces and stacks

6.1 Admistrative Stuff

Next week we'll meet both Monday and Wednesday. There will be another holiday in February (president's day), so there'll be another half-week later. Hoping to give 20 lectures total, so may end up going past week 10.

(Down to 75 participants)

6.2 Goal of Coarse

Remember that we are after the following theorem.

Theorem 6.1. *The moduli space $\overline{\mathcal{M}}_g$ of stable curves of genus $g \geq 2$ is a smooth, proper and irreducible Deligne-Mumford stack of dimension $3g-3$ which admits a projective coarse moduli space.*

We are still working on making sense of the terms in this theorem.

Where are we?

- Defined $\mathcal{M}_g$ (but *not* $\overline{\mathcal{M}}_g$)

- We've shown $\mathcal{M}_g$ is an algebraic stack.[19]

How do we get from here to the theorem? Let's take a second to go over the outline for this course.

[18] Know things locally lift and then can glue together using fully faithfulness

[19] It is Deligne-Mumford, but we have not shown that

(1) Sites, sheaves and stacks

(2) algebraic spaces and stacks (you are here)

(3) geometry of DM stacks (e.g. get to Keel-Mori theorem)

(4) Moduli of stable curves

Question 6.2 (Audience)**.** *Is $\mathcal{M}_g$ an algebraic space?*

Answer. No. But when we show Keel-Mori, we'll see that there is an algebraic space closely related to $\mathcal{M}_g$, but which no longer has a universal family.

6.3 Review

Definition 6.3. Let $F \to G$ be a map of presheaves/prestacks over $\mathrm{Sch}_{\text{Ét}}$.

- We say $F \to G$ is *represented by schemes* if for all $S \to G$ from a scheme S, $F \times_G S$ is a scheme.
- We say $F \to G$ is *representable* if for all $S \to G$ from a scheme S, $F \times_G S$ is an algebraic space

Recall 6.4. We can discuss properties of maps which are representable or representable by schemes.

Definition 6.5. Here are some of the key definitions of this class:

- An *algebraic space* is a sheaf X on $\mathrm{Sch}_{\text{Ét}}$ such that $\exists$ a scheme U and a map $U \to X$ which is representable by schemes, étale and surjective.
- A *DM stack* is a stack $\mathcal{X}$ on $\mathrm{Sch}_{\text{Ét}}$ s.t. there exists a scheme U, and a map $U \to \mathcal{X}$ which is representable, étale and surjective.
- An *algebraic stack* is a stack $\mathcal{X}$ on $\mathrm{Sch}_{\text{Ét}}$ s.t. there exists a scheme U, and a map $U \to \mathcal{X}$ which is representable, smooth and surjective.

6.4 Examples

Last time, we showed that quotient stacks $[X/G]$ as well as $\mathcal{M}_g$ (for $g \geq 2$) are algebraic.

Exercise. Show that the stack $\mathrm{Bun}_{r,d}(C)$ of vector bundles on C of rank r and degree d is algebraic (should be easier than the $\mathcal{M}_g$ case).

Remark 6.6. You'll want to get a smooth presentation using the Quot scheme. Secretly, $\mathrm{Bun}_{r,d}(C)$ is not quasi-compact, so you'll need infinitely many copies of Quot schemes.

Example. Let $G = \mathbb{Z}/2\mathbb{Z}$ acts on $\mathbb{A}^2$ via $-1 \cdot (x, y) = (-x, -y)$. Then, $[\mathbb{A}^2/G]$ is a DM stack (since G finite) and comes with a natural map

$$[\mathbb{A}^2/G] \to \mathbb{A}^2/G = \operatorname{spec} k[x^2, xy, y^2]$$

where $\mathbb{A}^2/G$ is a (singular) cone over a quadric in $\mathbb{P}^2$. We'll see later that $[\mathbb{A}^2/G]$ is a smooth DM stack, and $\mathbb{A}^2/G$ is its coarse moduli space. The map $[\mathbb{A}^2/G] \to \mathbb{A}^2/G$ is birational, so can think of this as a sort of stacky resolution of singularities.

Example. $\mathbb{G}_m \curvearrowright \mathbb{A}^1$, so can take $[\mathbb{A}^1/\mathbb{G}_m]$. This is an algebraic stack, but it is not DM (it has an infinite stabilizer group). In this case, $\mathbb{A}^1 \to [\mathbb{A}^1/\mathbb{G}_m]$ is a smooth presentation.

Example (Hironaka)**.** There exists a smooth, proper 3-fold X (say, over $\mathbb{C}$) with a free $\mathbb{Z}/2\mathbb{Z}$ action s.t. there exists an orbit not contained in any affine open (hence, X proper but not quasiprojective). Since this action is free, the quotient $X/(\mathbb{Z}/2\mathbb{Z})$ is an algebraic space. We have a finite map $X \to X/(\mathbb{Z}/2\mathbb{Z})$, so $X/(\mathbb{Z}/2\mathbb{Z})$ is an algebraic space, but not a scheme.[20]

Example. $G = \mathbb{Z}/2\mathbb{Z} \curvearrowright X$ where X is "the line with two origins" (or the "non-sepearated affine line"). Away from the origins, $-1 : x \mapsto -x$. Furthermore, -1 swaps the two origins. Now, $Y := X/G$ is an algebraic space, but not a scheme. Here are two reasons it is not a scheme

- The two origins are not contained in any affine.
- The diagonal $Y \to Y \times Y$ is not a locally closed immersion.

  Consider the Cartesian diagram
  $$\begin{tikzcd} G \times X \arrow[r] \arrow[d] & X \times X \arrow[d] \\ Y \arrow[r] & Y \times Y \end{tikzcd}.$$
  We now basechange further using the map $\mathbb{A}^1 \to X \times X$ sending $x \mapsto (x, -x)$. The base-change $\mathbb{A}^1 \times_{X \times X} (G \times X)$ is something like $\{(-1, x) \mid x \neq 0\} \sqcup \{(1, 0)\}$ so we do not have a locally closed immersion.

6.5 Summary of Important results

(all to be proven later)

Recall 6.7. Recall that for all $a, b : T \to \mathcal{X}$, there is a commutative diagram

$$\begin{tikzcd} \underline{\mathrm{Isom}}_T(a,b) \arrow[r] \arrow[d] & T \arrow[d, "(a,b)"] \\ \mathcal{X} \arrow[r, "\Delta"] & \mathcal{X} \times \mathcal{X} \end{tikzcd}$$

Slogan. Diagonal encodes "stackiness."

Definition 6.8. For a "point" $x : \operatorname{spec} k \to \mathcal{X}$ for k a field, we define the **stabilizer of** x via

$$\begin{tikzcd} G_x \arrow[r] \arrow[d] & \operatorname{spec} k \arrow[d, "(x,x)"] \\ \mathcal{X} \arrow[r] & \mathcal{X} \times \mathcal{X}, \end{tikzcd}$$

i.e. $G_x = \underline{\mathrm{Isom}}_{\operatorname{spec} k}(x, x)$.

If $x \in [X/G](k)$, then (I think) $G_x = \underline{\mathrm{Stab}}_G(X)$

Theorem 6.9. *If X is an algebraic space, then $X \to X \times X$ is represented by schemes.*

If $\mathcal{X}$ is an algebraic stack, then $\mathcal{X} \to \mathcal{X} \times \mathcal{X}$ is representable.

[20] If it were, any point would be contained in an affine open whose preimage (since finite maps are affine!) would be an affine open containing the corresponding orbit.

Hence, the stabilizer G_x is always an algebraic space. In fact, it is usually a scheme (a group scheme). Often, we will impose further conditions on $\mathcal{X} \to \mathcal{X} \times \mathcal{X}$, e.g. that it be affine or finite.

Type of Space	Propery of the diagonal	Property of stabilizers
algebraic space	monomorphism	trivial
DM stack	unramified	discrete and reduced groups
algebraic stack	arbitrary	arbitrary

Table 3: Characterization of algebraic spaces and DM stacks

For the rest of these properties, assume everything noetherian (don't wanna worry about technical quasi-compactess, quasi-separatedness, etc. hypotheses).

Properties of algebraic spaces

- $R \rightrightarrows X$ étale equivalence relation of schemes $\implies$ the quotient sheaf X/R is an algebraic space.
- X algebraic space $\implies \exists$ dense open scheme $U \subset X$
- $X \to Y$ separated and quasi-finite morphism of algebraic spaces $\implies X \to Y$ quasi-affine (*Zariski's Main Theorem*)

Properties of Deligne-Mumford stacks

- $R \rightrightarrows X$ is an étale groupoid of scheme $\implies$ the quotient stack $[X/R]$ is a DM stack
- $\mathcal{X}$ a DM stack $\implies \exists$ scheme U and finite morphism $U \to \mathcal{X}$ (*Global structure of DM stacks*)
- $\mathcal{X}$ a DM stack $+\ x \in \mathcal{X}(k) \implies \exists$ étale nbdh of x

  $$[\mathrm{spec}(A)/G_x] \to \mathcal{X}$$

  (*Local structure of DM stacks*)
- $\mathcal{X}$ separated DM stack $\implies \exists$ a coarse moduli space $\mathcal{X} \to X$ where X is a separated algebraic space (*Keel-Mori theorem*)

6.6 Properties of algebraic spaces/stacks

6.6.1 Properties of morphisms

Definition 6.10. Let $\mathcal{P}$ be a property of maps of schemes.

- $\mathcal{P}$ is **étale-local on the source** if for any $X' \overset{\text{ét}}{\twoheadrightarrow} X$, then $X \to Y$ has $\mathcal{P} \iff X' \to X \to Y$ has $\mathcal{P}$

  Example. étale, surjective

- $\mathcal{P}$ is **étale-local on the target** if for any $Y' \overset{\text{ét}}{\twoheadrightarrow} Y$, then $X \to Y$ has $\mathcal{P} \iff X \times_Y Y' \to Y'$ has $\mathcal{P}$.

  Example. almost everything (except projectivity).

Same definition works for **smooth-local**.

The following are all smooth-local on the source

- étale
- surjectivity
- smooth
- flat
- locally of finite type/presentation

Definition 6.11. Assume $\mathcal{P}$ is stable under composition and base change.

- If $\mathcal{P}$ is étale-local on the source and target, a map $\mathcal{X} \to \mathcal{Y}$ of DM stacks **has property** $\mathcal{P}$ if for all étale presentations $V \to \mathcal{Y}$ and $U \to \mathcal{X} \times_{\mathcal{Y}} V$, the composition $U \to \mathcal{X} \times_Y V \to V$ has $\mathcal{P}$.
- If $\mathcal{P}$ is smooth-local (on both source/target), can define property $\mathcal{P}$ for maps $\mathcal{X} \to \mathcal{Y}$ of algebraic stacks
- A map $\mathcal{X} \to \mathcal{Y}$ of algebraic stacks which is representable by schemes has property $\mathcal{P}$ if for every map $T \to Y$ from a scheme, the base change $\mathcal{X} \times_{\mathcal{Y}} T \to T$ has $\mathcal{P}$.

When $\mathcal{X} \to \mathcal{Y}$ is representable by schemes, we can define basically any property for it.

6.6.2 Properties of stacks

Definition 6.12. Let $\mathcal{P}$ be a property of schemes.

- $\mathcal{P}$ is **étale-local** if for any $X' \overset{\text{ét}}{\twoheadrightarrow} X$, then X has $\mathcal{P} \iff X'$ has $\mathcal{P}$.

Let $\mathcal{P}$ be étale-local. We say a DM stack $\mathcal{X}$ has $\mathcal{P} \iff \forall$ étale presentations $U \to \mathcal{X}$, U has $\mathcal{P}$ (equivalent to change $\forall$ to $\exists$).

Can make the same definition for **smooth-local** and properties of algebraic stacks.

Example. locally noetherian, regular, reduced are all three smooth-local.

6.6.3 Topolgical properties

Definition 6.13. The **topological space of an algebraic stack** $\mathcal{X}$ is

$$|\mathcal{X}| := \left\{ \operatorname{spec} K \xrightarrow{x} \mathcal{X} : K \text{ a field} \right\} / \sim,$$

where $(\operatorname{spec} K_1 \xrightarrow{x_1} \mathcal{X}) \sim (\operatorname{spec} K_2 \xrightarrow{x_2} \mathcal{X})$ if there exists $K_1 \to K_3$ and $K_2 \to K_3$ s.t. $x_1|_{\operatorname{spec} K_3} \xrightarrow{\sim} x_2|_{\operatorname{spec} K_3}$.

$$\begin{tikzcd}
\operatorname{spec} K_3 \arrow[r] \arrow[d] & \operatorname{spec} K_1 \arrow[d] \arrow[dl, phantom, "\swarrow"] \\
\operatorname{spec} K_2 \arrow[r] & \mathcal{X}
\end{tikzcd}$$

This gives a set. To get a topology, we say $U \subset |\mathcal{X}|$ is open if there exists an open immersion $\mathcal{U} \hookrightarrow \mathcal{X}$ such that U is the image of $|\mathcal{U}| \to |\mathcal{X}|$.

Example. Say $G = \mathbb{Z}/2\mathbb{Z} \curvearrowright \mathbb{A}^1$ via negation $-1 \cdot x = -x$. We have

$$\begin{tikzcd} \mathbb{A}^1 \arrow[rr] \arrow[dr] & & {[\mathbb{A}^1/G]} \arrow[dl] \\ & \mathbb{A}^1/G & \end{tikzcd}$$

where $\mathbb{A}^1/G = \operatorname{spec} k[x^2]$ and the map $\mathbb{A}^1 \to \mathbb{A}^1/G$ is $x \mapsto x^2$. In this case, $\big|[\mathbb{A}^1/G]\big| = \big|\mathbb{A}^1\big|$.

Example. $\mathbb{G}_m \curvearrowright \mathbb{A}^1$. Then, $\big|[\mathbb{A}^1/\mathbb{G}_m]\big|$ consists of two points, a closed point with stabilizer $\mathbb{G}_m$, and an open point.

Example. Take $\mathbb{G}_m \curvearrowright \mathbb{A}^2$ via $t \cdot (x, y) = (tx, ty)$. So the origin is fixed, and the other orbits are just lines through the origin. We have $\mathbb{P}^1 \subset [\mathbb{A}^2/\mathbb{G}_m]$, and $\big|[\mathbb{A}^2/\mathbb{G}_m]\big| = \big|\mathbb{P}^1\big| \sqcup \{0\}$.[21]

Definition 6.14. We use this to define topological properties.

- An algebraic stack is **quasi-compact**, **connected**, or **irreducible** if $|\mathcal{X}|$ is.
- A morphism $\mathcal{X} \to \mathcal{Y}$ is **quasi-compact** if $|\mathcal{X}| \to |\mathcal{Y}|$ is.
- A morphism $\mathcal{X} \to \mathcal{Y}$ is **finite type** if it is locally of finite type and quasi compact.

Exercise. Show that $\mathcal{X}$ is quasi-compact $\iff$ $\exists$ a smooth presentation $\operatorname{spec} A \twoheadrightarrow \mathcal{X}$.

Exercise (Harder). If $\mathcal{X} \to \mathcal{X} \times \mathcal{X}$ is quasi-compact, then $|\mathcal{X}|$ is a **sober topological space**, i.e. every irreducibe closed subset has a generic point.

6.7 Equivalence Relations and Groupoids

Definition 6.15. An **étale groupoid of schemes** if a pair of étale maps $s, t : R \rightrightarrows U$ of schemes call the *source* and *target*, along with a *composition* morphism $c : R \times_{t,U,s} R \to R$ satisfying

(associativity)

$$\begin{tikzcd} R \times_{t,U,s} R \times_{t,U,s} R \arrow[r, "c \times \mathrm{id}"] \arrow[d, "\mathrm{id} \times c"'] & R \times_{t,U,s} \times R \arrow[d, "c"] \\ R \times_{r,U,s} R \arrow[r, "c"] & R \end{tikzcd}$$

(identity) $\exists e : U \to R$ such that

$$\begin{tikzcd} & U \arrow[dl, "\mathrm{id}"'] \arrow[d, "e"] \arrow[dr, "\mathrm{id}"] & \\ U & R \arrow[l, "s"] \arrow[r, "t"'] & U \end{tikzcd} \qquad \begin{tikzcd} R \arrow[r, "e \circ s{,}\mathrm{id}"] \arrow[dr, "\mathrm{id}"'] & R \times_{t,U,s} R \arrow[d, "c"] & R \arrow[l, "e \circ t{,}\mathrm{id}"'] \arrow[dl, "\mathrm{id}"] \\ & R & \end{tikzcd}$$

(inverse) $\exists i : R \to R$ such that

$$\begin{tikzcd} R \arrow[r, "i"] \arrow[dr, "s"'] & R \arrow[r, "i"] \arrow[d, "t"] & R \arrow[dl, "s"] \\ & U & \end{tikzcd} \qquad \begin{tikzcd} R \arrow[r, "s"] \arrow[d, "(\mathrm{id}{,}i)"] & U \arrow[d, "e"] \\ R \times_{t,U,s} R \arrow[r, "c"] & R \end{tikzcd} \qquad \begin{tikzcd} R \arrow[r, "t"] \arrow[d, "(i{,}\mathrm{id})"] & U \arrow[d, "e"] \\ R \times_{t,U,s} R \arrow[r, "c"] & U \end{tikzcd}$$

[21] And I think he said 0 is in the closure of every point?

If $(s,t) : R \to U \times U$ is a monomorphism, then we say $s, t : R \rightrightarrows U$ is an **étale equivalence relation**.

We think of R as a "scheme of relations," i.e. $r \in R$ "determines a relation $s(r) \xrightarrow{r} t(r)$."

If $R \rightrightarrows U$ is an equivalence relation, then there exists at most one relation between any two points of U.

Example. Let G be a smooth algebraic group over a field k, and let U be a k-schme on which G acts. Define $R := G \times U$ with source $s = \sigma : G \times U \to U$ (multiplication) and target $t = p_2 : G \times U \to U$ (projection). So $g \in G$ gives a relation $u \xrightarrow{g} g \cdot u$ for any $u \in U$. This gives a smooth groupoid (étale if G finite, for example). This is an equivalence relation $\iff$ $G \curvearrowright U$ freely $\iff$ $G \times U \to U \times U$ is a monomorphism.

Example. Let $\mathcal{X}$ be a DM stack with étale presentation $U \to \mathcal{X}$. Let $R = U \times_{\mathcal{X}} U$ with source/target $(s,t) = (p_1, p_2) : U \times_{\mathcal{X}} U \rightrightarrows U$ the projections. This is an equivalence relation $\iff$ $\mathcal{X}$ is an algebraic space.

Definition 6.16. Let $s, t : R \rightrightarrows U$ be a smooth groupoid of schemes (abusing notation by leaving out composition law). We define the prestack $[U/R]^{pre}$ whose fiber categories are $[U/T]^{pre}(S) := [U(S)/R(S)]$ where this is the groupoid quotient of an equivalence relation on sets. One also needs to say what morphisms between objects over different schemes are, but this is easy to do.

We define $[U/R]$ to be the stackification of $[U/R]^{pre}$.

Exercise. There exists a Cartesian diagram

$$\begin{tikzcd} R \arrow[r, "s"] \arrow[d, "t"'] & U \arrow[d, "p"] \\ U \arrow[r, "p"] & {[U/R]} \end{tikzcd}$$

as well as one

$$\begin{tikzcd} R \arrow[r] \arrow[d] & U \times U \arrow[d, "p \times p"] \\ {[U/R]} \arrow[r, "\Delta"] & {[U/R] \times [U/R]} \end{tikzcd}$$

Theorem 6.17. *If $R \rightrightarrows U$ is an étale (resp. smooth) groupoid, then $[U/R]$ is a DM (resp. algebraic) stack.*

We'll prove this on Monday.

Remark 6.18. If you start with a presentation $U \to \mathcal{X}$ of a (DM or algebraic) stack, then you can form the (étale or smooth) groupoid $U \times_{\mathcal{X}} U \rightrightarrows U$ and the quotient $[U/(U \times_{\mathcal{X}} U)]$ is isomorphic to the stack $\mathcal{X}$ you started with. Hence, you can consider an (étale or smooth) groupoid as being the data of a (DM or algebraic) stack along with a presentation.

Someone asked a question about group actions on stacks. I was distracted typing above remark, so I did not quite get what all was said... seems like one can make sense of a group scheme G acting on a stack $\mathcal{X}$, and the quotient $[\mathcal{X}/G]$ will still be a stack. However, if you have a group stack $\mathcal{G}$ acting on a stack $\mathcal{X}$ (and you figure out how to make sense of this), then the quotient object is no longer a stack; it is now a higher (categorical) stack and Lurie probably makes sense of things in this case.

Question 6.19 (Audience)**.** *Mentioned earlier that projectivity is not étale-local on the source. I guess being representable by schemes is also not étale-local (Raynaud or someone has some complicated example). Is there a simple example of this phenomenon?*

Answer. We'll give an example next time. To be clear, we want a diagram like

$$\begin{tikzcd} X' \arrow[r] \arrow[d] & X \arrow[d, "\text{rep}"] \\ Y' \arrow[r, "\text{ét}"] & Y \end{tikzcd}$$

with X an algebraic space, and X', Y', Y all schemes (so $X' \to Y'$ certainly representable by schemes). The map $X \to Y$ won't be representable by schemes since the basechange along $Y = Y$ is not a scheme.

7 Lecture 7 (2/1): Representability of the diagonal

7.1 Review

Definition 7.1. An *algebraic space* is a sheaf X on $\mathrm{Sch}_{\text{Ét}}$ such that there exists a scheme U and a surjective, étale morphism $U \to X$ representable by schemes. A *Deligne-Mumford stack* is a stack $\mathcal{X}$ over $\mathrm{Sch}_{\text{Ét}}$ such that there exist a scheme U and a surjective, étale and representable morphism $U \to \mathcal{X}$. An *algebraic stack* is a stack $\mathcal{X}$ over $\mathrm{Sch}_{\text{Ét}}$ such that there exist a scheme U and a surjective, smooth and representable morphism $U \to \mathcal{X}$.

Remark 7.2. In the literature, one usually adds a hypothesis that the diagonal is representable. We will see today that this is automatic from the above definition.

Definition 7.3. An *étale groupoid of schemes* is a groupoid object $s, t : R \rightrightarrows U$ in the category of schemes where s, t are both étale maps. Can given analogous definition for *smooth groupoid of schemes.*

Example. If $G \curvearrowright U$ a scheme, then have $\sigma, p_2 : R = G \times U \rightrightarrows U$.

Definition 7.4. The quotient stack $[U/R]$ of a smooth groupoid $R \rightrightarrows U$ is the stackification of $[U/R]^{pre}$ where $[U/R]^{pre}(S) = [U(S)/R(S)]$ with the RHS denoting the groupoid quotient of set-theoretic groupoid $R(S) \rightrightarrows U(S)$.

Warning 7.5. It seems that if you want to make sense of these quotient stacks in the fppf topology, for example, then stackifying this naive prestack is not the "right" construction.

Fact. There exists a Cartesian diagram

$$\begin{tikzcd} R \arrow[r, "s"] \arrow[d, "t"'] & U \arrow[d, "p"] \\ U \arrow[r, "p"] & {[U/R]} \end{tikzcd}$$

as well as one

$$\begin{tikzcd} R \arrow[r] \arrow[d] & U \times U \arrow[d, "p \times p"] \\ {[U/R]} \arrow[r, "\Delta"] & {[U/R] \times [U/R]} \end{tikzcd}$$

We left off last time showing that the quotient stack is algebraic.

Definition 7.6. Let $R \rightrightarrows U$ be an étale (resp. smooth) groupoid. Then, $[U/R]$ is a DM (resp. algebraic) stack.

Proof. We claim that $U \to [U/R]$ is representable. Let $T \to [U/R]$ be a map from a scheme. We need to show that the fiber product $U_T := U \times_{[U/R]} T$ is an algebraic space, and that $U_T \to T$ is étale (resp. smooth) and surjective. We know that exists an étale cover $T' \twoheadrightarrow T$ fitting into a 2-commutative diagram

$$\begin{tikzcd} T' \arrow[r] \arrow[d] \arrow[dr, phantom, "\Swarrow"] & U \arrow[d] \\ T \arrow[r] & {[U/R]} \end{tikzcd}$$

Now consider the cube

$$\begin{tikzcd} & U_{T'} \arrow[rr] \arrow[dl] \arrow[dd] & & T' \arrow[dl] \arrow[dd, two heads, "\text{ét}"] \\ R \arrow[rr, crossing over] \arrow[dd] & & U & \\ & U_T \arrow[rr] \arrow[dl] & & T \arrow[dl] \\ U \arrow[rr] & & {} \arrow[from=uu, crossing over] & \end{tikzcd}$$

where every face is Cartesian except the left and right.

We know that T, T' and $U_{T'}$ are schemes, while U_T is so far only a sheaf. Since $T' \twoheadrightarrow T$ is representable by schemes and étale, the same is true for $U_{T'} \twoheadrightarrow U_T$. Since $U_{T'}$ is a scheme, this is exactly what we need to show that U_T is an algebraic space. ■

Remark 7.7. A similar argument shows that if $s, t : R \rightrightarrows U$ is an étale equivalence relation and s, t are quasi-compact and separated, then U/R is an algebraic space.[22]

7.2 Examples

We'll give three descriptions of the "**bug-eyed cover**".

- $\mathbb{Z}/2\mathbb{Z} \curvearrowright \mathbb{A}^1 \cup_{\mathbb{A}^1 \setminus 0} \mathbb{A}^1$, the non-separated affine line, by swapping the origins and by the involuztion $x \mapsto -z$. Let $X = U/(\mathbb{Z}/2\mathbb{Z})$. We saw this previous as an example of an algebraic space which is not a scheme.

- $\mathbb{Z}/2\mathbb{Z} \curvearrowright \mathbb{A}^1$ via $x \mapsto -x$. Consider the corresponding étale groupoid $\sigma, p_2 : \mathbb{Z}/2\mathbb{Z} \times \mathbb{A}^1 \rightrightarrows \mathbb{A}^1$. This is not an equiv relation since the origin is fixed by -1, so let $R \subset \mathbb{Z}/2\mathbb{Z} \times \mathbb{A}^1$ be the complement of $(-1, 0)$. Then, $R \rightrightarrows \mathbb{A}^1$ is an étale equivalence relation. Then, $X = \mathbb{A}^1/R$ is the same space as above.

  Remark 7.8. Let X be the space above. The map $\mathbb{A}^1 \to \mathbb{A}^1, x \mapsto x^2$ factors through X, and the map $X \to \mathbb{A}^1$ is an isomorphism away from the origin, but something fishy happens at 0. Recall X

[22] Something like use Zariski main to get $R \to U$ to be quasi-affine, and then effective descent for quasi-affine morphisms to show that U_T is a scheme

is not a scheme, e.g. by considering the Cartesian squares

$$\begin{tikzcd}
\{(-1,z) \mid z \neq 0\} \sqcup \{(1,0)\} \arrow[r] \arrow[d] & \mathbb{A}^1 \arrow[d] \\
R \arrow[r] \arrow[d] & \mathbb{A}^1 \times \mathbb{A}^1 \arrow[d] \\
X \arrow[r] & X \times X
\end{tikzcd}$$

where the map $\mathbb{A}^1 \to \mathbb{A}^1 \times \mathbb{A}^1$ above is $z \mapsto (z, -z)$. So the Diagonal is not locally closed which means X is not a scheme.

Maybe you think this is contrived since we have a group acting on some non-separated space. However, consider the next description.

- This last description due to Mumford. Consider $\mathrm{SL}_2 \curvearrowright V_d := \mathrm{Sym}^d k^2$. Let

$$W = \left\{(L, F) : L \neq 0 \text{ and } F = Q^2 \text{ where } Q \text{ quad with disc} = 1\right\} \subset V_1 \times V_4.$$

Exercise. $X = W/\mathrm{SL}_2$. Note that W here is quasi-affine.

We have even more pathological examples.

Example. $\mathbb{Z}/2\mathbb{Z} \curvearrowright \mathbb{A}^1_{\mathbb{C}}$ via conjugation $z \mapsto \overline{z}$. Let $R = \mathbb{Z}/2\mathbb{Z} \times \mathbb{A}^1_{\mathbb{C}} \setminus \{-1, 0\}$. Then $R \rightrightarrows \mathbb{A}^1_{\mathbb{C}}$ is an étale equivalence relation. Let $X = \mathbb{A}^1_{\mathbb{C}}/R$, defined over $\mathbb{R}$.

Question: Do we not need to remove all real numbers?

Exercise. The basechange $X_{\mathbb{C}}$ is the non-separated affine line over $\mathbb{C}$. So

$$\begin{tikzcd}
X_{\mathbb{C}} \arrow[r] \arrow[d] & X \arrow[d] \\
\operatorname{spec} \mathbb{C} \arrow[r] & \operatorname{spec} \mathbb{R}
\end{tikzcd}$$

we have a $\mathbb{Z}/2\mathbb{Z}$-torsor of an algebraic space with total space a scheme.

Example. $\mathbb{Z}/2\mathbb{Z} \curvearrowright \operatorname{spec} k[x,y]/xy = U$ via $-1 \cdot (x,y) = (y,x)$. Let $X = U/R$ with $R = \mathbb{Z}/2\mathbb{Z} \times U \setminus \{(-1, 0)\}$. There are not two tangent directions at the origin in the quotient?

Example. $\operatorname{char} k = 0$. Consider $\underline{\mathbb{Z}}$ as a group scheme over k (e.g. $\underline{\mathbb{Z}} = \bigsqcup_{n \in \mathbb{Z}} \operatorname{spec} k$). Now, $\mathbb{Z} \curvearrowright \mathbb{A}^1_k$ via translation $n \cdot x = x + n$. Then, $X = \mathbb{A}^1_k/\mathbb{Z}$ is an algebraic space which turns out to not be a scheme and to not be quasi-separated.

Example. $\mathbb{Z}$ is a group scheme over k which is discrete, reduced, but not quasi-compact. Let $\mathcal{X} = B\mathbb{Z}$, a DM stack. $\mathcal{X}$ is even quasi-compact, but its diagonal $\Delta_{\mathcal{X}}$ is not quasi-compact.

Warning 7.9. We did not define $B\mathbb{Z}$ is this context. We were always working with smooth, affine group schemes before. But without too much effort, you can show $B\mathbb{Z}$ here is a DM stack.

Example. Let $G = \mathbb{A}^1_k/\mathbb{Z}$ a group algebraic space over k. Then, G is quasi-compact, but $\Delta_{G/k}$ is not. Let $\mathbb{Z} \subset \mathbb{G}_a = \mathbb{A}^1$. Then, $\mathcal{X} = BG$ quasi-compact, $\Delta_{\mathcal{X}}$ quasi-compact and local noetherian, but $\Delta_{\Delta_{\mathcal{X}}}$ is not quasi compacted.

Example. Take $G = \mathbb{A}^\infty \cup_{\mathbb{A}^\infty \setminus 0} \mathbb{A}^\infty$, a scheme that's not quasi-separated. Let $\mathcal{X} = BG$. Then, $\mathcal{X}$ is qc; $\Delta_\mathcal{X}$ is qc, but not loc noeth; and $\Delta_{\Delta_\mathcal{X}}$ is not qc.

Example. Let $G = \mathbb{A}^1 \cup_{\mathbb{A}^1 \setminus 0} \mathbb{A}^1$. This is a non-separated, relative group scheme over $\mathbb{A}^1$. Let $\mathcal{X} = B_{\mathbb{A}^1} G \to \mathbb{A}^1$. This is in fact a DM stack, and $\mathcal{X}, \Delta_\mathcal{X}, \Delta_{\Delta_\mathcal{X}}$ are all qc, but $\Delta_\mathcal{X}$ is not separated (here, $\Delta_\mathcal{X}$ is representable by schemes).

Fact. If $\mathcal{X}$ is DM with qc and sep. diagonal, then $\Delta_\mathcal{X}$ is quasi-affine (in particular, representable by schemes).

Example. Consider two group schemes over $\operatorname{spec} \mathbb{Z}_p$: $\mathbb{Z}/p\mathbb{Z} \to \operatorname{spec} \mathbb{Z}_p$ and $\mu_p \to \operatorname{spec} \mathbb{Z}_p$. These are isomorphic over $\mathbb{Q}_p$, but the fibers over $0 = (p)$ are different (p distinct points vs. this non-reduced scheme μ_p). Let $H := (\mathbb{Z}/p\mathbb{Z}) \setminus \{\text{non-id elements over } 0\}$. We now have a map $H \to \mu_p$ (over $\mathbb{Z}_p$) which is a bijective monomorphism (and an iso over $\mathbb{Q}_p$), but not a locally closed immersion. The quotient $Q := \mu_p / H$ is a group algebraic space which is not a scheme. Now $\mathcal{X} = B_{\mathbb{Z}_p} Q$ is a DM stack. It turns out that $\Delta_\mathcal{X}$ is quasi-compact, but neither separable nor representable by schemes.

Warning 7.10. For above example to work, need a pth root of unity. So either $p = 2$ or actually work over $\mathbb{Z}_p(\zeta_p)$.

7.3 The diagonal $\mathcal{X} \to \mathcal{X} \times \mathcal{X}$

Our goal is the following.

Theorem 7.11 (Representability of the Diagonal).

(1) The diagonal of an algebraic space is representable by schemes.

(2) The diagonal of an algebraic stack is representable.

Proof. **(1)** Let X be an algebraic space. Let $T \to X \times X$ be a map from a scheme. Consider

$$\begin{tikzcd} Q_T \arrow[r] \arrow[d] & T \arrow[d] \\ X \arrow[r] & X \times X. \end{tikzcd}$$

We need to show that Q_T is a scheme. Choose an étale presentation $U \twoheadrightarrow X$ with U a scheme. This is an epimorphism of sheaves[23], so there exists an étale cover $T' \to T$ fitting into a square

$U \times U \to X \times X$ is representable by schemes, so you can take T' to be the fiber product if you want (though I don't think this is necessary)

$$\begin{tikzcd} T' \arrow[r] \arrow[d] & U \times U \arrow[d] \\ T \arrow[r] & X \times X. \end{tikzcd}$$

Now, we once again form a cube (Below, $R \simeq U \times_X U$ is a scheme since $U \to X$ is representable by

[23]Being an epimorphism of sheaves just means that locally you can lift sections

schemes)

$$\begin{tikzcd}
& Q_{T'} \arrow[rr] \arrow[dl] \arrow[dd] & & T' \arrow[dl] \arrow[dd, two heads, "\text{ét}"] \\
R \arrow[rr] \arrow[dd] & & U \times U \arrow[dd] & \\
& Q_T \arrow[rr, crossing over] \arrow[dl] & & T \arrow[dl] \\
X \arrow[rr] & & X \times X &
\end{tikzcd}$$

Above, all faces except the left and right are Cartesian. We know that $T, T', Q_{T'}$ are schemes, that Q_T is a sheaf, and that $R \to U \times U$ is separated (since its a monomorphism) and locally quasi finite (since the compositions $R \to U \times U \rightrightarrows U$ are étale). Hence, $Q_{T'} \to T'$ is itself separated and locally quasi finite. Consider the back square. Effective descent for separated and locally quasi finite maps implies that Q_T is a scheme.[24]

(2) The argument here is largely the same. Let $\mathcal{X}$ be an algebraic stack and consider a map $T \to \mathcal{X} \times \mathcal{X}$ from a scheme. We want to show that the basechange Q_T is an algebraic space. Choose a smooth presentation $U \to \mathcal{X}$ with U a scheme.

Exercise. Maps can be lifted étale locally.

Hence, we get an étale cover $T' \to T$ with $T' \to U \times U$ over $T \to \mathcal{X} \times \mathcal{X}$. Once again consider a cube

$$\begin{tikzcd}
& Q_{T'} \arrow[rr] \arrow[dl] \arrow[dd] & & T' \arrow[dl] \arrow[dd, two heads, "\text{ét}"] \\
R \arrow[rr] \arrow[dd] & & U \times U \arrow[dd] & \\
& Q_T \arrow[rr, crossing over] \arrow[dl] & & T \arrow[dl] \\
\mathcal{X} \arrow[rr] & & \mathcal{X} \times \mathcal{X} &
\end{tikzcd}$$

Above, all faces except the left and right are Cartesian. $T' \to T$ is étale and representable by schemes, so $Q_{T'} \to Q_T$ is as well. Hence, $Q_{T'} \to Q_T$ is an étale presentation, so Q_T is an algebraic space. ∎

Corollary 7.12.

(1) Any map from a scheme to an algebraic space is representable by schemes.

(2) Any map from a scheme to an algebraic stack is representable.

Proof. Let $T \to \mathcal{X}$ be a map from a scheme to an algebraic space. Say $S \to \mathcal{X}$ is another map from a scheme. We want to show $T \times_{\mathcal{X}} S$ is a scheme. We know that we have a Cartesian diagram

$$\begin{tikzcd}
T \times_{\mathcal{X}} S \arrow[r] \arrow[d] & T \times S \arrow[d] \\
\mathcal{X} \arrow[r] & \mathcal{X} \times \mathcal{X}
\end{tikzcd}$$

[24]Descent gives you a scheme over T whose pullback is $Q_{T'}$. One still needs to show that the thing given by descent is (the sheaf) Q_T.

so we win since the diagonal is representable. ■

Exercise. If $\mathcal{X} \to \mathcal{Y}$ is a morphism of algebraic spaces (resp. algebraic stacks), the diagonal $\mathcal{X} \to \mathcal{X} \times_{\mathcal{Y}} \mathcal{X}$ is representable by schemes (resp. representable).

Corollary 7.13. *If $R \rightrightarrows U$ is an étale equivalence relation of schemes, then U/R is an algebraic space, and $U \to U/R$ is an étale presentation.*

Proof. Suffices to show that the diagonal of $X = U/R$ is representable by schemes. This would then imply that $U \to X$ is representable by schemes, and you get étale/surjectivity by descent. Consider once more a cube

$$\begin{tikzcd}[row sep=small, column sep=small]
 & Q_{T'} \arrow[rr] \arrow[dl] \arrow[dd] & & T' \arrow[dl] \arrow[dd, two heads, "\text{ét}"] \\
R \arrow[rr, crossing over] \arrow[dd] & & U \times U & \\
 & Q_T \arrow[rr] \arrow[dl] & & T \arrow[dl] \\
X \arrow[rr] & & X \times X \arrow[from=uu, crossing over] &
\end{tikzcd}$$

Above, all faces except the left and right are Cartesian. The upshot is that $R \to U \times U$ is separated and locally quasi finite, so $Q_{T'} \to T'$ is the same, so effect descent shows that Q_T is as well. ■

Remark 7.14. When $R \rightrightarrows U$ is an *equivalence relation*, one can show that $[U/R]$ is equivalent to a sheaf, and the above argument shows that that sheaf is an algebraic space.

Warning 7.15. Here are some things we *don't* know (yet):

- Sheaf + algebraic stack $\implies$ algebraic space.

  Remark 7.16. We do know that sheaf + DM stack $\implies$ algebraic space. Once we know this stronger statement, we'll also know that the quotient by a *smooth* equivalence relation of schemes is also an algebraic space.

- The diagonal of a quasi-separated algebraic space is quasi-affine.

7.4 Properties of the diagonal

Recall 7.17. The **stabilizer of** $x \in \mathcal{X}(k)$ is $G_x := \underline{\mathrm{Aut}}_k(x)$. We have a Cartesian diagram

$$\begin{tikzcd}
G_x \arrow[r] \arrow[d] & \operatorname{spec} k \arrow[d, "{(x,x)}"] \\
\mathcal{X} \arrow[r, "\Delta"] & \mathcal{X} \times \mathcal{X}.
\end{tikzcd}$$

Since Δ is representable, G_x is a group algebraic space.

Fact. If G is a qcqs group algebraic space over field $k = \overline{k}$, then G is actually an algebraic group over k (in particular, it is a scheme).

Show that there is an open, dense locus which is a scheme, then use group operation to

Definition 7.18. An **algebraic group** is a finite type group scheme over a field k.

Exercise. If $\mathcal{X}$ is an algebraic stack then $\mathcal{X} \to \mathcal{X} \times \mathcal{X}$ is locally of finite type.

Definition 7.19. The **inertia stack** of $\mathcal{X}$ is the fiber product

$$\begin{tikzcd} I_{\mathcal{X}} \arrow[r] \arrow[d] & \mathcal{X} \arrow[d, "\Delta"] \\ \mathcal{X} \arrow[r, "\Delta"] & \mathcal{X} \times \mathcal{X} \end{tikzcd}$$

Remember: The inertia stack is the pullback of the diagonal along the diagonal

Note that $I_{\mathcal{X}} \to \mathcal{X}$ is representable since $\mathcal{X} \to \mathcal{X} \times \mathcal{X}$ is. Furthermore, for $x \in \mathcal{X}(k)$, one has a Cartesian diagram

$$\begin{tikzcd} G_x \arrow[r] \arrow[d] & \operatorname{spec} k \arrow[d, "x"] \\ I_{\mathcal{X}} \arrow[r] & \mathcal{X} \end{tikzcd}$$

Exercise. Let G be a finite abelian group acting on a scheme U. Then,

$$I_{[U/G]} = \bigsqcup_{g \in G} [U^g/G]$$

where $U^g = \{u \in U : gu = u\}$.

7.5 Separation properties

Definition 7.20. A map $\mathcal{X} \to \mathcal{Y}$ of algebraic stacks is **quasi-separated** if $\mathcal{X} \to \mathcal{X} \times_Y \mathcal{X}$ is quasi-compact and quasi-separated (i.e. diagonal and second diagonal both quasi-compact).

Definition 7.21. Say $\mathcal{X}$ is **noetherian** if it is locally noetherian, quasi-compact, and quasi-separated.

Exercise. Let G be a smooth, affine algebraic group over a field k, and say G acts on a scheme U/k.

(1) If $x \in U(k)$, then the stabilizer of $\operatorname{spec} k \to [U/G]$ is the usual stabilizer G_x.

(2) Assume U is quasi-separated. Then, $[U/G]$ is quasi-affine.

(3) Assume U has affine diagonal (e.g. U separated). Then, $[U/G]$ has affine diagonal.

Example. We showed already that $\mathcal{M}_g = [H'/\mathrm{PGL}_n]$ (when $g \geq 2$). This H' is quasi-projective since its some locally closed subspace of a Hilbert scheme, so $\mathcal{M}_g$ has affine diagonal. We will later show it has finite diagonal.

8 Lecture 8 (2/3): Dimension, tangent spaces, and residual gerbes

8.1 Last time

Theorem 8.1 (Representability of the Diagonal)**.** *The diagonal of an algebraic space (resp. algebraic stack) is representable by schemes (resp. representable).*

Theorem 8.2 (**Algebraicity of quotients**)**.**

(1) *$R \rightrightarrows U$ smooth groupoid of schemes $\implies$ $[U/R]$ is an alg stack*

(2) *$R \rightrightarrows U$ étale groupoid of schemes $\implies$ $[U/R]$ is a DM stack*

(3) *$R \rightrightarrows U$ étale equiv relations of schemes $\implies$ U/R is an alg space*

Today, we want to talk about dimension, tangent spaces, and residual gerbes.

8.2 Dimension

Recall 8.3. If X is a scheme, then $\dim X$ is the Krull dimension of the topological space $|X|$. Given $x \in X$, we set $\dim_x X = \min_{U \ni x} \dim U$, the **local dimension** at x.

Warning 8.4. It is *not* true in general that, for a scheme X, one has $\dim_x X = \dim \mathscr{O}_{X,x}$.

Example. Imagine a plane union a line, e.g. $V(z) \cup V(x, y) \subset \mathbb{A}^3$. Most points on the line have local dimension 1, but the whole space has dimension 2.

For stacks, we will want to define $\dim \mathcal{X}$ using a smooth presentation $U \twoheadrightarrow \mathcal{X}$. We'll basically set $\dim \mathcal{X} = \dim U - \operatorname{reldim}(U \to \mathcal{X})$.

Definition 8.5. Let X be a noetherian algebraic space, and choose $x \in |X|$. Choose an étale presentation $(U, u) \to (X, x)$, and define

$$\dim_x X := \dim_u U \in \mathbb{Z}_{\geq 0} \cup \{\infty\}.$$

This is well-defined since étale maps preserve dimension (have rel dim 0).

Definition 8.6. Let $\mathcal{X}$ be an algebraic stack and choose $x \in |X|$. Choose a smooth presentation $(U, u) \to (\mathcal{X}, x)$, and let $s, t : R \rightrightarrows U$ be the smooth groupoid associated to this presentation, so $R = U \times_{\mathcal{X}} U$. Define

$$\dim_x \mathcal{X} := \dim_u U - \dim_{e(u)} R_u \in \mathbb{Z} \cup \infty,$$

where R_u is the fiber of $s : R \to U$ over u and $e : U \to R$ denotes the identity morphism in the groupoid.

Definition 8.7. If $\mathcal{X}$ is a noetherian algebraic space of stack, we define

$$\dim \mathcal{X} = \sup_{x \in |\mathcal{X}|} \dim_x \mathcal{X} \in \mathbb{Z} \cup \{\infty\}.$$

Checking that things are well-defined is best done on one's own.

Proposition 8.8. $\dim_x \mathcal{X}$ *is well-defined.*

We will use

Fact. Let $X \xrightarrow{f} Y$ be a smooth map of noetherian schemes, and say $f(x) = y$. Then,

$$\dim_x X = \dim_y Y + \dim_x X_y.$$

Proof Sketch of Proposition 8.8.

- Let $U' \to \mathcal{X}$ be another presentation with groupoid $R' \rightrightarrows U'$

- Consider $U'' = U \times_{\mathcal{X}} U' \ni u'' \mapsto u', u$.
- By symmetry, suffices to show

$$\dim_u U - \dim_{e(u)} R_U = \dim_{u''} U'' - \dim_{e''(u'')} R''_{U''}.$$

- Consider the picture

$$\begin{tikzcd}
R'_{U'} \arrow[r, equal] & U''_U \arrow[r] \arrow[d] & U'' \arrow[r] \arrow[d] & U' \arrow[d] \\
& \operatorname{spec} k(u) \arrow[r] & U \arrow[r] & \mathcal{X}
\end{tikzcd}$$

- Apply fact to $U'' \to U$ to see

$$\dim_{u''} U'' = \dim_u U + \dim_{u''} U''_u = \dim_u U + \dim_{e'(u')} R'_{U'}.$$

- Now that we know this, we only need show

$$\dim_{e''(u'')} R''_{U''} = \dim_{e(u)} R_U + \dim_{e'(u')} R'_{U'}.$$

- For simplicity assume u, u', u'' have same residue field k (dimension insensitive to field extension). Consider the Cartesian cube

$$\begin{tikzcd}
& R''_{U''} \arrow[rr] \arrow[dl] \arrow[dd] & & R'_{U'} \arrow[dl] \arrow[dd, two heads] \\
U'' \arrow[rr] \arrow[dd] & & U' \arrow[dd] & \\
& R_U \arrow[rr] \arrow[dl] & & \operatorname{spec} k \arrow[dl, "x"] \\
U \arrow[rr] & & \mathcal{X} &
\end{tikzcd}$$

  Finally, use additivity of dimension.

■

Example. Let G be a smooth, affine algebraic group over k, and let U be a k-scheme with G-action. Then,

$$\dim[U/G] = \dim U - \dim G.$$

In particular,

- $\dim BG = -\dim G$
- $\dim[\mathbb{A}^1/\mathbb{G}_m] = 0$
- $\dim[\mathbb{A}^2/\mathbb{G}_m] = 1$. Note $\mathbb{P}^1 \subset [\mathbb{A}^2/\mathbb{G}_m]$. Also have $B\mathbb{G}_m \hookrightarrow [\mathbb{A}^2/\mathbb{G}_m]$ of codimension 2.

- $\mathcal{M}_g = [H'/\mathrm{PGL}_n]$ with H' locally closed in Hilbert scheme, so $\dim \mathcal{M}_g = \dim H' - \dim \mathrm{PGL}_n$. Could try using deformation theory to compute these things, but we won't. We'll later show $\mathcal{M}_g$ is smooth and then compute the dimension of its tangent space instead.

8.3 Tangent spaces

Definition 8.9. Let $\mathcal{X}$ be an algebraic stack, and fix $x \in \mathcal{X}(k)$. The **Zariski tangent space** is defined as the set

$$T_{\mathcal{X},x} := \left\{ \begin{tikzcd} \operatorname{spec} k \arrow[d, hook] \arrow[dr, bend left, "x"] \arrow[dr, phantom, "\alpha"] & \\ \operatorname{spec} k[\varepsilon] \arrow[r, "\tau"] & \mathcal{X} \end{tikzcd} \right\} \Big/ \sim$$

where $(\tau, \alpha) \sim (\tau', \alpha')$ if $\exists$ iso $\beta : \tau \xrightarrow{\sim} \tau'$ in $\mathcal{X}(k[\varepsilon])$ compatible with α, α', i.e. $\alpha' = \beta|_{\operatorname{spec} k} \circ \alpha$.

Why is this a vector space?

(Scalar multiplication) For $c \in k$ on $(\tau, \alpha) \in T_{\mathcal{X},x}$, $c \cdot (\tau, \alpha)$ is defined as the composition

$$\operatorname{spec} k[\varepsilon] \xrightarrow{\varepsilon \mapsto c\varepsilon} \operatorname{spec} k[\varepsilon] \xrightarrow{\tau} \mathcal{X}$$

with same 2-iso α.

(Addition) There is a non-obvious equivalence[25]

$$\mathcal{X}(k[\varepsilon_1] \times_k k[\varepsilon_2]) \to \mathcal{X}(k[\varepsilon_1]) \times_{\mathcal{X}(k)} \mathcal{X}(k[\varepsilon_2]).$$

Define $(\tau_1, \alpha_1) + (\tau_2, \alpha_2)$ as the composition

$$\operatorname{spec} k[\varepsilon] \to \operatorname{spec}(k[\varepsilon_1] \times_k k[\varepsilon_2]) \to \mathcal{X}.$$

Proposition 8.10. *If $\mathcal{X}$ is an algebraic stack with affine diagonal, and $x \in \mathcal{X}(k)$, then $T_{\mathcal{X},x}$ is naturally a k-vector space.*

This is true more generally with affine diagonal hypothesis, but the proof (of the pushout property) is easier with this hypothesis.

Exercise. $T_{\mathcal{X},x}$ is a G_x-representation. Set-theoretically, given $g \in G_x(k)$ and $(T, \alpha) \in T_{\mathcal{X},x}$, we set $g \cdot (\tau, \alpha) = (\tau, g \circ \alpha)$.

Example. Take G a smooth and affine algebraic group. Take $\operatorname{spec} k \xrightarrow{x} BG$ be the canonical cover. Since G is smooth, $T_{BG,x} = 0$.

If $k = \overline{k}$, this maps corresponds to the trivial G-torsor and every G-torsor over the dual numbers is also trivial, or something

Example. Consider $\mathcal{X} = [\mathbb{A}^1/\mathbb{G}_m]$. Recall $|\mathcal{X}|$ consists of two points, one open and one closed. The open point is $\operatorname{spec} k \xhookrightarrow{1} [\mathbb{A}^1/\mathbb{G}_m]$ and the closed point is $B\mathbb{G}_m \hookrightarrow [\mathbb{A}^1/\mathbb{G}_m]$. Consider the composition

$$\begin{tikzcd} \operatorname{spec} k \arrow[r] \arrow[rr, bend left, "0"] & B\mathbb{G}_m \arrow[r] & {[\mathbb{A}^1/\mathbb{G}_m]}. \end{tikzcd}$$

[25] Amounts to showing that $\operatorname{spec} k[\varepsilon_1] \times_k k[\varepsilon_2]$ is a pushout of $\operatorname{spec} k[\varepsilon_2] \leftarrow \operatorname{spec} k \to \operatorname{spec} k[\varepsilon_1]$ in the category of algebraic stacks

One has $T_{\mathcal{X},1} = 0$ with $G_1 = \{1\}$ acting trivially, while $T_{\mathcal{X},0}$ is 1-dimensional with action of $G_0 = \mathbb{G}_m$.

This $\mathcal{X}$ will be an example of a smooth stack. Note that $\dim T_{\mathcal{X},x} - \dim G_x$ is constant.

Example. $B\mu_p$ over k of characteristic p. We have $\operatorname{spec} k \xrightarrow{x} B\mu_p$. Note we don't yet know $B\mu_p$ is algebraic, since we've focused on quotients of flat group schemes. In this case, one gets $T_{B\mu_p,x} = 1$. One way to think about this is to consider the exact sequence

$$1 \to \mu_p \to \mathbb{G}_m \xrightarrow{t \mapsto t^p} \mathbb{G}_m \to 1,$$

and use this to view $B\mu_p = [\mathbb{G}_m/\mathbb{G}_m]$ via the map $t \mapsto t^p$. This shows that $B\mu_p$ is a (smooth) algebraic stack, and that it has dimension 0, even though it has this 1-dimensional tangent space.

Example. Fix $g \geq 2$ and consider $\mathcal{M}_g$. Fix $\operatorname{spec} k \xrightarrow{[C]} \mathcal{M}_g$, where C smooth, projective curve of genus g. By definition,

$$T_{\mathcal{M}_g,[C]} = \left(\left\{ \begin{tikzcd} \operatorname{spec} k \arrow[d, hook] \arrow[dr, "{[C]}"] & \\ \operatorname{spec} k[\varepsilon] \arrow[r, "\tau"] & \mathcal{M}_g \end{tikzcd} \right\} \Big/ \sim \right) = \{\mathcal{C} \to \operatorname{spec} k[\varepsilon] \ni 0 \ \text{ and } \ \mathcal{C}_0 \cong C\}/\sim$$

It is a fact from infinitesimal deformation theory that this latter set can be identified with $\mathrm{H}^1(C, T_C)$. We now compute

$$h^1(C, T_C) = h^1(C, \Omega_C^\vee) = h^0(C, \Omega_C^{\otimes 2}) = 2(2g-2) - (g-1) = 3(g-1).$$

8.4 Residual gerbes

Recall 8.11. If X is a scheme and $x \in X$, then there exists a residue field $k(x)$, along with a monomorphism $\operatorname{spec} k(x) \hookrightarrow X$.

Goal. We want an analogous construction. Given $x \in |\mathcal{X}|$, we want to consider the smallest substack of $\mathcal{X}$ containing x.

Definition 8.12. We say $x \in |\mathcal{X}|$ is a **point (locally) of finite type** if there exists a representative $\operatorname{spec} k \to \mathcal{X}$ such that this morphism is (locally) of finite type.

Warning 8.13. We previously said that a map $\mathcal{X} \to \mathcal{Y}$ of algebraic stacks is quasi-compact if $|\mathcal{X}| \to |\mathcal{Y}|$ is. This not the right definition. Really, $\mathcal{X} \to \mathcal{Y}$ is **quasi-compact** if for every affine scheme T and map $T \to \mathcal{Y}$, the fiber product $T \times_{\mathcal{Y}} \mathcal{X}$ is a quasi-compact algebraic space (i.e. $|T \times_{\mathcal{Y}} \mathcal{X}|$ is a qc top space).

Fact. For X a scheme, $x \in X$ if of finite type $\iff$ it is locally closed (i.e. closed in some open set). As an example, if R is a dvr with $K = \operatorname{Frac} R$, then $\operatorname{spec} K \hookrightarrow \operatorname{spec} R$ is an open immersion, so of finite type.

For schemes of finite type over k, any k-point is closed. This will not be true for algebraic stacks, e.g. $\operatorname{spec} \mathbb{C} \xrightarrow{1} [\mathbb{A}^1_{\mathbb{C}}/\mathbb{G}_m]$ is not closed.

Definition 8.14. Let $\mathcal{X}$ be an algebraic stack and fix $x \in |\mathcal{X}|$. Choose a smooth presentation $(U, u) \twoheadrightarrow (\mathcal{X}, x)$. The **residual gerbe** of x is the substack $\mathcal{G}_x \subset \mathcal{X}$ defined as the stackification of the full subcategory $\mathcal{G}_x^{pre} \subset \mathcal{X}$ of objects $a \in \mathcal{X}$ over S which factor as $a : S \to \operatorname{spec} k(u) \to \mathcal{X}$.

Fact. If $\mathcal{X}$ is noetherian, then $\mathcal{G}_x$ is independent of the presentation.

Theorem 8.15. *Let $\mathcal{X}$ be a noetherian algebraic stack. Let $x \in |\mathcal{X}|$ be a finite type point with smooth and affine stabilizer. Then, $\mathcal{G}_x$ is an algebraic stack and $\mathcal{G}_x \hookrightarrow \mathcal{X}$ is a locally closed immersion.*

Moreover, if $(U, u) \to (\mathcal{X}, x)$ is a smooth morphism from a scheme U, then we have a Cartesian diagram

$$\begin{tikzcd} O(u) \arrow[r] \arrow[d] & U \arrow[d] \\ \mathcal{G}_x \arrow[r] & \mathcal{X} \end{tikzcd}$$

where $O(u)$ is the orbit $s(t^{-1}(u))$ of the induced groupoid $s, t : R := U \times_{\mathcal{X}} U \rightrightarrows U$. That is,

$$O(u) = s(t^{-1}(u)) = \left\{ v : \exists v \xrightarrow{\rho} u \text{ in } R \right\}$$

as a set.

Remark 8.16. We can give $O(u)$ the reduced scheme structure.

Missed stuff because zulipping

Fact. Say $G \curvearrowright U$ finite type over a field. Any G-orbit of $u \in U(k)$ is locally closed.

Example. He drew the diagram

$$\begin{tikzcd} & \mathbb{A}^1 \setminus 0 \arrow[r] \arrow[d] & \mathbb{A}^1 \arrow[d] & \operatorname{spec} k \arrow[l, "0"'] \arrow[d] & \\ \mathcal{G}_1 \arrow[r, equal] & \operatorname{spec} k \arrow[r, "1"] & \left[\mathbb{A}^1/\mathbb{G}_m\right] & B\mathbb{G}_m \arrow[l, "0"] \arrow[r, equal] & \mathcal{G}_0 \end{tikzcd}$$

Fact (I didn't really understand this)**.** In general, $\mathcal{G}_x$ is a gerbe over the residue field $k(x)$.

$$\begin{tikzcd} BG_{x'} \arrow[r] \arrow[d] & \mathcal{G}_x \arrow[r, hook] \arrow[d] & \mathcal{X} \\ \operatorname{spec} k' \arrow[r] \arrow[urr] & \operatorname{spec} k & \end{tikzcd}$$

Let's sketch a proof of a special case of the theorem.

Proof of special case of Theorem. Let's suppose $\mathcal{X}$ is of finite type over a field, and that $x \in \mathcal{X}(k)$.

(step 1) There exists $BG_x \to \mathcal{X}$ which is a finite type monomorphism. Recall BG_x^{pre} is the prestack whose fiber category $BG_x^{pre}(S)$ has one object with morphisms $G_x(S)$. Hence, we get a morphism of prestacks $BG_x^{pre}(\mathcal{X})$ where

$$\begin{array}{ccc} BG_x^{pre}(S) & \longrightarrow & \mathcal{X}(S) \\ * & \longmapsto & (S \to \operatorname{spec} k \xrightarrow{x} \mathcal{X}) \end{array}$$

and morphisms are dealt with in the obvious way. This map then factors through a map $BG_x \to \mathcal{X}$ from the stackification. The original map $BG_x^{pre} \to \mathcal{X}$ was a mono of prestacks, so $BG_x \to \mathcal{X}$ is a monomorphism of stacks. We also have a factorization

$$\begin{tikzcd} \operatorname{spec} k \arrow[r] \arrow[rr, bend left, "x"] & BG_x \arrow[r] & \mathcal{X} \end{tikzcd}$$

with x assumed of finite type, so $BG_x \to \mathcal{X}$ is also of finite type.

(step 2) Reduce to case where $BG_x \to \mathcal{X}$ is also flat and surjective. Can assume it has dense image by simply replacing $\mathcal{X}$ with the smallest closed substack containing BG_x. Since $\operatorname{spec} k \to BG_x \to \mathcal{X}$ has dense image, ???? implies that $\operatorname{spec} k \to \mathcal{X}$ is flat so $BG_x \to \mathcal{X}$ is flat. Flat and finite type morphisms are open, so image is open, so we can use that to assume that the map is surjective. Question: generic flatness?

(step 3) With reductions made, we now want to show that $BG_x \xrightarrow{\sim} \mathcal{X}$ is an isomorphism. Consider the fiber product

$$\begin{tikzcd} O \arrow[r] \arrow[d] & U \arrow[d] \\ BG_x \arrow[r] & \mathcal{X} \end{tikzcd}$$

By construction, O is an algebraic stack and also a sheaf (we don't know yet it is an algebraic space). Also recall that $BG_x \to \mathcal{X}$ is finite type, flat, surjective, and a monomorphism. We can and do assume that U is affine. From this, we know that Δ_O is affine.

Let's assume for a moment that we know that O, U are sheaves in $\mathrm{Sch}_{\mathrm{fppf}}$. A priori, we only know they are sheaves in the (big) étale topology. In this case, it suffices to show that for all $T \to U$, there is some fppf cover $T' \twoheadrightarrow T$ so that we get a square

$$\begin{tikzcd} T' \arrow[r] \arrow[d] & T \arrow[d] \\ O \arrow[r] & U \end{tikzcd}$$

Let $\widetilde{O} \to O$ be a smooth cover of O by a scheme. Consider the Cartesian squares

$$\begin{tikzcd} \widetilde{O}_T \arrow[r] \arrow[d] & O_T \arrow[r] \arrow[d] & T \arrow[d] \\ \widetilde{O} \arrow[r] & O \arrow[r] & U \end{tikzcd}$$

We can take $T' = \widetilde{O}_T$ since the composition $\widetilde{O}_T \to T$ is fppf ($\widetilde{O}_T \to O_T$ is smooth).

■

What about the missing ingredient?

Theorem 8.17. *Any algebraic space X is a sheaf in Sch_{fppf}.*

Fact. In fact, same proof shows that if $\mathcal{X}$ is an algebraic stack and a sheaf with diagonal representable by schemes, then $\mathcal{X}$ is a sheaf on Sch_{fppf}.

9 Lecture 9 (2/8): Characterization of DM stacks

Recall that our overall goal of the course is to prove the following.

Goal. The moduli space $\overline{\mathcal{M}}_g$ of stable curves of genus $g \geq 2$ is a smooth, proper and irreducible Deligne-Mumford stack of dimension $3g-3$ which admits a projective coarse moduli space.

After today, we will know that $\mathcal{M}_g$ is a DM stack which is smooth over $\operatorname{spec}\mathbb{Z}$ of relative dimension $3g-3$. On Wednesday, we'll move onto the geometry of DM stacks, with 2 lectures working towards to existence of coarse moduli spaces. After that, onto stable curves.

9.1 Recap

Dimension Let $\mathcal{X}$ be an algebraic stack with $x \in |\mathcal{X}|$. Choose a smooth presentation $(U,u) \to (\mathcal{X},x)$, and let $s,t : R \rightrightarrows U$ be the associated smooth groupoid. Then,

$$\dim_x \mathcal{X} := \dim_u U - \dim_{e(u)} R_u \in \mathbb{Z} \cup \{\infty\},$$

where R_u is the fiber of $s : R \to U$ over u, and $e : U \to R$ denotes the identity morphism in the groupoid. This is well-defined ultimately because dim is well-behaved (read: additive) for smooth morphisms.

Tangent space Let $\mathcal{X}$ be an algebraic stack and choose $x \in \mathcal{X}(k)$. The Zariski tangent space is defined as the set

$$T_{\mathcal{X},x} := \left\{\tau \in \mathcal{X}(k[\varepsilon]) \text{ with } \alpha : \tau|_{\operatorname{spec} k} \xrightarrow{\sim} x\right\} / \sim,$$

where $(\tau,\alpha) \sim (\tau',\alpha')$ if $\exists \beta \tau \xrightarrow{\sim} \tau'$ in $\mathcal{X}(k[\varepsilon])$ compatible with α, α'.

Fact. This is a k-vector space.

Example. If $[C] : \operatorname{spec} k \to \mathcal{M}_g$, then infinitesimal deformation theory gives $T_{\mathcal{M}_g,[C]} = \mathrm{H}^1(C,\Omega_C)$ or something. I missed it.

Residual gerbes First some clarifications.

Definition 9.1. An algebraic stack $\mathcal{X}$ is **quasi-compact** if $|\mathcal{X}|$ is. Equivalently, $\exists \operatorname{spec} A \twoheadrightarrow \mathcal{X}$ a smooth presentation.

A map $\mathcal{X} \to \mathcal{Y}$ is **quasi-compact** if for all $\operatorname{spec} A \to \mathcal{Y}$, the fiber product $\mathcal{X} \times_{\mathcal{Y}} \operatorname{spec} A$ is quasi-compact.

We say $\mathcal{X}$ is noetherian if $\mathcal{X}$ is locally noetherian, and $\mathcal{X}, \Delta_{\mathcal{X}}, \Delta_{\Delta_{\mathcal{X}}}$ are all quasi-compact (i.e. $\mathcal{X}$ qc and $\Delta_{\mathcal{X}}$ both qc and separated).

A point $x \in |\mathcal{X}|$ is **finite type** if there exists a representative $\operatorname{spec} k \to \mathcal{X}$ of x which is locally of finite type.

Remark 9.2. If $\mathcal{X}$ is noetherian, then $\operatorname{spec} k \to \mathcal{X}$ is locally of finite type iff it is of finite type.

We introduced residual gerbes last time. Essentially, the residual gerbe $\mathcal{G}_x \subset \mathcal{X}$ is the "smallest" substack containing x. In other words, we can always factor

$$\begin{tikzcd} \operatorname{spec} k \arrow[rr, "x"] \arrow[dr, twoheadrightarrow] & & \mathcal{X} \\ & \mathcal{G}_x \arrow[ur, hookrightarrow] & \end{tikzcd}$$

and this is universal (final?) among objects fitting into the above diagram. We showed last time that

Theorem 9.3. *Let $\mathcal{X}$ be a noetherian algebraic stack. Let $x \in |\mathcal{X}|$ be a finite type point with smooth stabilizer. Then $\mathcal{G}_x$ is an algebraic stack, and the inclusion $\mathcal{G}_x \hookrightarrow \mathcal{X}$ is a locally closed immersion.*

Remark 9.4. It is possible to relax the finite type and smooth stabilizer hypotheses above. If you remove finite type, the inclusion is a monomorphism but not necessarily a locally closed immersion. To remove the smooth stabilizer hypothesis, one would need to be able to take quotients by non-smooth group schemes (e.g. by flat group schemes/flat equivalence relations). We do not with to develop quotients by flat equivalence relations here.

Remark 9.5. In proof of this last time, we used that algebraic spaces are sheaves for the fppf topology, but this is not necessary for the proof since we have the smooth stabilizer assumption.

Remark 9.6. If $\mathcal{X}$ is finite type over a field k, and $x \in \mathcal{X}(k)$ (in context of above theorem), then $\mathcal{G}_x = BG_x \to \mathcal{X}$.

9.2 Miniversal presentations

Theorem 9.7 (**Existence of Miniversal Presentations**)**.** *Let $\mathcal{X}$ be a noetherian algebraic stack, and let $x \in |\mathcal{X}|$ be a finite type point with smooth stabilizer G_x. Then, there exists a smooth morphism $(U, u) \to (\mathcal{X}, x)$ from a scheme of relative dimension $\dim G_x$ s.t.*

$$\begin{tikzcd} \operatorname{spec} k(u) \arrow[r] \arrow[d] & U \arrow[d] \\ \mathcal{G}_x \arrow[r] & \mathcal{X} \end{tikzcd}$$

is Cartesian. In particular, if G_x is finite and reduced, there is an étale morphism $(U, u) \to (\mathcal{X}, x)$ from a scheme.

Definition 9.8. We say a smooth map $(U, u) \to (\mathcal{X}, x)$ is **miniversal** at u if it induces an isomorphism $T_{U,u} \xrightarrow{\sim} T_{\mathcal{X},x}$ on tangent spaces.

"It is a smooth presentation of the smallest possible dimension."

We will later see that the map $U \to \mathcal{X}$ in the theorem is miniversal. We could do it now, but it'll be more convienent to wait until we have the lifting property of smoothness.

Example. $\mathbb{G}_m \curvearrowright \mathbb{A}^2$ diagonally, $t \cdot (x, y) = (tx, ty)$.

$$\begin{tikzcd} \operatorname{spec} k \arrow[r, hook, "0"] \arrow[d] & \mathbb{A}^2 \arrow[d] \\ B\mathbb{G}_m \arrow[r, hook, "0"] & {[\mathbb{A}^2/\mathbb{G}_m]} \end{tikzcd}$$

Above $\mathcal{G}_0 = B\mathbb{G}_m$, and the map $\mathbb{A}^2 \to [\mathbb{A}^2/\mathbb{G}_m]$ is miniversal at 0. However, it is not miniversal everywhere, e.g. it is not at $(0,1)$.

$$\begin{tikzcd} \{(0,y) : y \neq 0\} \arrow[r, hook] \arrow[d] & \mathbb{A}^2 \arrow[d] \\ \operatorname{spec} k \arrow[r, "{(0,1)}"] & {[\mathbb{A}^2/\mathbb{G}_m]} \end{tikzcd}$$

This point has no stabilzer, so the residual gerbe is just $\operatorname{spec} k$, and the fiber product is the orbit of this point. This orbit is 1-dimensional. To get something miniversal, can take a slace transversal to the orbit, e.g.

$$\operatorname{spec} k[x] = \mathbb{A}^1 \xrightarrow{y=1} \mathbb{A}^2 \to [\mathbb{A}^2/\mathbb{G}_m]$$

is miniversal at 1.

Let's do the proof now, theorem restate for convenience.

Theorem 9.9 (**Existence of Miniversal Presentations**). *Let $\mathcal{X}$ be a noetherian algebraic stack, and let $x \in |\mathcal{X}|$ be a finite type point with smooth stabilizer G_x. Then, there exists a smooth morphism $(U,u) \to (\mathcal{X},x)$ from a scheme of relative dimension $\dim G_x$ s.t.*

$$\begin{tikzcd} \operatorname{spec} k(u) \arrow[r] \arrow[d] & U \arrow[d] \\ \mathcal{G}_x \arrow[r] & \mathcal{X} \end{tikzcd}$$

is Cartesian. In particular, if G_x is finite and reduced, there is an étale morphism $(U,u) \to (\mathcal{X},x)$ from a scheme.

Proof. Choose $(U,u) \to (\mathcal{X},x)$ a smooth presentation. Consider the fiber product

$$\begin{tikzcd} O(u) \arrow[r] \arrow[d] & U \arrow[d] \\ \mathcal{G}_x \arrow[r] & \mathcal{X} \end{tikzcd}$$

Note that $\dim \mathcal{G}_x = -\dim G_x$, so $\dim O(u) = c := n - \dim G_x$ where $n = \operatorname{reldim}(U \to \mathcal{X})$. If $c = 0$, we win. In general, we want to find a slice transverse to the orbit.

First observe that $O(u)$ is a smooth scheme of dimension c. Thus, there exists a regular sequence $f_1, \dots, f_c \in \mathfrak{m}_u \subset \mathscr{O}_{O(u),u}$ generating the maximal ideal $\mathfrak{m}_u$ at u. We know $O(u) \hookrightarrow U$ is a locally closed immersion. After shrinking, we can make this a closed immersion (even one with U affine), and then lift these to global sections $f_1, \dots, f_c \in \Gamma(U)$. Now define $W = V(f_1, \dots, f_c) \subset U$. By design, $W \cap O(u) = \{u\}$.

Question: Do we need u to be a closed point for this?

We want to show that W is flat over $\mathcal{X}$. We inductively apply

Fact (**local criterion of flatness**). Let $(A, \mathfrak{m}_A) \to (B, \mathfrak{m}_B)$ be a flat local ring homomorphism of local noetherian rings. Let $f \in \mathfrak{m}_B$ s.t. $f \otimes 1$ in $B \otimes_A A/\mathfrak{m}_A$ is not a zero divisor. Then, $A \to B \to B/f$ is flat.

We are in a situation to apply this since we have a regular sequence. We now have

$$\begin{tikzcd} \operatorname{spec} k(u) \arrow[r, hook] \arrow[d, "\mathrm{sm}"] & W \arrow[r, hook] \arrow[d, "\mathrm{flat}"] & U \arrow[dl] \\ \mathcal{G}_x \arrow[r, hook] & \mathcal{X} & \end{tikzcd}$$

We now have a flat morphism which is smooth at a point, so it is smooth in a neighborhood (?). We can shrink W further in order to have $W \to \mathcal{X}$ smooth, and so we win. ■

Remark 9.10. There are details not worked out above. e.g. we've maybe not yet quite shown that $\dim \mathcal{G}_x = -\dim G_x$ always.

Note that the above theorem is actually false without the smooth stabilizer assumption.

Corollary 9.11 (**characterizations of DM stacks**). *Let $\mathcal{X}$ be a noetherian algebraic stack. Then, TFAE*

(1) *$\mathcal{X}$ is DM*

(2) *every point of $\mathcal{X}$ has a finite, reduced stabilizer*

(3) *the diagonal $\mathcal{X} \to \mathcal{X} \times \mathcal{X}$ is unramified.*

(2) $\iff$ **(3)** above is formal.

Definition 9.12. A map $X \to Y$ of schemes is **unramified** if it is locally of finite type, and all fibers are discrete and reduced. This property is étale local on the source and target, so it naturally extends to morphisms of DM stacks.

The miniversal presentation theorem gives **(2)** $\implies$ **(1)**. This leaves **(1)** $\implies$ **(2)**.

Proof of (1) $\implies$ (2) in Corollary. Say $\mathcal{X}$ is DM, so there exists an étale presentation $U \overset{\text{ét}}{\twoheadrightarrow} \mathcal{X}$. Consider the diagram

$$\begin{tikzcd} R \arrow[r] \arrow[d] \arrow[rr, bend left] & U \times U \arrow[r, "\mathrm{pr}_1"] \arrow[d] & U \\ \mathcal{X} \arrow[r] & \mathcal{X} \times \mathcal{X} & \end{tikzcd}$$

The composition $R \to U$ is étale, so $R \to U \times U$ is unramified (condition on fibers). ■

Corollary 9.13. *Let $\mathcal{X}$ be a noetherian DM stack. Assume $\Delta_{\mathcal{X}}$ is representable by schemes. Then, TFAE*

(1) *$\mathcal{X}$ algebraic space*

(2) *every point has trivial stabilizer*

(3) *$\mathcal{X} \to \mathcal{X} \times \mathcal{X}$ is a monomorphism.*

Note that **(3)** above is easily equivalent to $\mathcal{X}$ being a sheaf.

Fact. A finite type group scheme $G \to S$ is trivial $\iff$ fibers are trivial.

Remark 9.14. By "every point" we mean "every field valued point" or even "every geometric point."

Remark 9.15. The diagonal being representable by schemes is not a necessary hypothesis above, but proving the corollary without this requires more work. That hypothesis not being necessary is basically telling us that quotients of (étale) equivalence relations of algebraic spaces are still algebraic spaces.

Fact. The finite type points form a dense subset of a noetherian stack.

Fact. If C is a smooth, projective curve over k (of genus $g \geq 2$), then $\operatorname{Aut}(C)$ is finite and reduced.

Proof Sketch. Two parts

(1) $\operatorname{Aut}(C)$ is an algebraic group.

Use Hilbert scheme to relate $\alpha : C \xrightarrow{\sim} C$ to its graph $\Gamma_\alpha : C \hookrightarrow C \times C$.

(2) Infinitesimal deformation theory identifies Lie algebra

$$T_e \operatorname{Aut}(C) = \mathrm{H}^0(C, T_C) = 0.$$

Have an algebraic group with trivial Lie algebra, so it must be finite and reduced. ■

Corollary 9.16. *$\mathcal{M}_g$ is a DM stack of finite type over $\mathbb{Z}$ with affine diagonal.*

Question: When did we show that the diagonal is affine?

Answer: We showed $\mathcal{M}_g$ is the quotient stack of something quasi-projective mod the (affine) group PGL_n

9.3 Smoothness

We know what it means for a morphism of algebraic stacks to be smooth, since smoothness is smooth-local on both the source and target. We want a criterion for checking smoothness more easily.

Theorem 9.17 (Formal lifting Criterion for smoothness). *Let $f : \mathcal{X} \to \mathcal{Y}$ be a morphism of algebraic stacks. Then, f is **smooth** if and only if f is locally of finite presentation, and for every diagram*

$$\begin{tikzcd} \operatorname{spec} A_0 \arrow[r] \arrow[d] & \mathcal{X} \arrow[d, "f"] \\ \operatorname{spec} A \arrow[r] \arrow[ur, dashed] & \mathcal{Y} \end{tikzcd}$$

of solid arrows where $A \twoheadrightarrow A_0$ is a surjection of rings with nilpotent kernel, there exists a lifting.

If $\mathcal{X}, \mathcal{Y}$ are notherian, then it suffices to consider diagrams where A and A_0 are local artinian rings.

Remark 9.18. To be explicit, a **lifting** of

$$\begin{tikzcd} \operatorname{spec} A_0 \arrow[r, "x"] \arrow[d, "g"'] & \mathcal{X} \arrow[d, "f"] \arrow[dl, phantom, "\Swarrow\alpha"] \\ \operatorname{spec} A \arrow[r, "y"] & \mathcal{Y} \end{tikzcd}$$

is a map $\widetilde{x} : \operatorname{spec} A \to \mathcal{X}$ and 2-isomorphisms β, γ

$$\begin{tikzcd}
\operatorname{spec} A_0 \arrow[r, "x"] \arrow[d, "g"'] & \mathcal{X} \arrow[d, "f"] \\
\operatorname{spec} A \arrow[r, "y"'] \arrow[ur, "\widetilde{x}"] \arrow[u, phantom, shift left=4, "\Uparrow\beta"] \arrow[r, phantom, shift left=4, "\Downarrow\gamma"] & \mathcal{Y},
\end{tikzcd}$$

such that

$$\begin{tikzcd}
& f \circ x \arrow[dl, "\alpha"'] & \\
y \circ g & & f \circ \widetilde{x} \circ g \arrow[ul, "f(\beta)"'] \arrow[ll, "g^*\gamma"]
\end{tikzcd}$$

commutes

Remark 9.19. $\mathcal{X} \to \mathcal{Y}$ is smooth $\iff$ for all such diagrams, $\mathcal{X}(A) \to \mathcal{X}(A_0) \times_{\mathcal{Y}(A_0)} \mathcal{Y}(A)$ is essentially surjective.

We skip the proof of this criterion. We do mention that there are similar criteria for étale (unique lift) and unramified (at most one lift) morphisms.

Proposition 9.20. *Let $\mathcal{X}$ be a noetherian algebraic stack. Let $x \in |\mathcal{X}|$ be a finite type point with smooth stabilizer. Suppose we have*

$$\begin{tikzcd}
\operatorname{spec} k(u) \arrow[r] \arrow[d] & U \arrow[d] \\
\mathcal{G}_x \arrow[r] & \mathcal{X}
\end{tikzcd}$$

a Cartesian diagram. Then $U \to \mathcal{X}$ is miniversal at u, i.e.

$$T_{U,u} \xrightarrow{\sim} T_{\mathcal{X},f(u)}$$

as $k(u)$-vector spaces.

Proof. Formal lifting criterion applied to tangent vectors ($k = k(u)$)

$$\begin{tikzcd}
\operatorname{spec} k \arrow[r, "u"] \arrow[d] & U \arrow[d] \\
\operatorname{spec} k[\varepsilon] \arrow[r] \arrow[ur, dashed] & \mathcal{X}
\end{tikzcd}$$

exactly says that we can lift tangent vectors, we have a surjection $T_{U,u} \twoheadrightarrow T_{\mathcal{X},f(u)}$.

We still need injectivity. Say we have a tangent vector $\tau : \operatorname{spec} k[\varepsilon] \to U$ in $T_{U,u}$ which maps to 0. The map to $\mathcal{X}$ factors through $\operatorname{spec} k$. By definition of the residual gerbe, this means that $\operatorname{spec} k[\varepsilon] \xrightarrow{\tau} U \to \mathcal{X}$ factors through $\mathcal{G}_x \hookrightarrow \mathcal{X}$. Thus, τ factors through the fiber product $\mathcal{G}_x \times_{\mathcal{X}} U = \operatorname{spec} k(u)$, so τ is itself trivial.

$$\begin{tikzcd}
\operatorname{spec} k[\varepsilon] \arrow[dr] \arrow[drr, bend left, "\tau"] \arrow[ddr, bend right] & & \\
& \operatorname{spec} k(u) \arrow[r] \arrow[d] & U \arrow[d] \\
& \mathcal{G}_x \arrow[r] & \mathcal{X}
\end{tikzcd}$$

■

Example. $\mathbb{A}^1 \to [\mathbb{A}^1/\mathbb{G}_m]$ is smooth of relative dimension 1, but nevertheless induces an iso on tangent spaces *at the origin.*

Corollary 9.21. *Let $\mathcal{X}$ be a noetherian algebraic stack which is smooth over a field k. Let $x \in \mathcal{X}(k)$ (in particular, x is finite type?) have smooth stabilizer. Then,*

$$\dim_x \mathcal{X} = \dim T_{\mathcal{X},x} - \dim G_x.$$

Proof. Take a miniversal presentation

$$\begin{tikzcd}
\operatorname{spec} k(u) \arrow[r] \arrow[d] & U \arrow[d] \\
\mathcal{G}_x \arrow[r] & \mathcal{X}.
\end{tikzcd}$$

We know the relative dimension of $\operatorname{spec} k(u) \to \mathcal{G}_x$ is $\dim G_x$, so

$$\dim_x \mathcal{X} = \dim_u U - \operatorname{reldim}(\operatorname{spec} k(u) \to \mathcal{G}_x) = \dim T_{U,u} - \dim G_x = \dim T_{\mathcal{X},x} - \dim G_x$$

where we used above that U is smooth over $\operatorname{spec} k$ (so $\dim_u U = \dim T_{U,u}$). ■

Example. We can now show that $\mathcal{M}_g \to \operatorname{spec} \mathbb{Z}$ is smooth of relative dimension $3g - 3$, when $g \geq 2$.

(smoothness) Say we have a diagram

$$\begin{tikzcd}
\operatorname{spec} A_0 \arrow[r] \arrow[d] & \mathcal{M}_g \arrow[d] \\
\operatorname{spec} A \arrow[r] & \operatorname{spec} \mathbb{Z}.
\end{tikzcd}$$

In the formal lifting criterion, it suffices to assume that A, A_0 are local, artinian rings (so have $\operatorname{spec} k \hookrightarrow \operatorname{spec} A_0$) with $\ker(A \twoheadrightarrow A_0) = k$. Let $[C] : \operatorname{spec} k \hookrightarrow \operatorname{spec} A_0 \to \mathcal{M}_g$. We know want to fill in the diagram

$$\begin{tikzcd}
C \arrow[r] \arrow[d] & \mathcal{C}_0 \arrow[r, dashed] \arrow[d] & \mathcal{C} \arrow[d, dashed] \\
\operatorname{spec} k \arrow[r, hook] & \operatorname{spec} A_0 \arrow[r, hook] & \operatorname{spec} A
\end{tikzcd}$$

Infitesimal deformation theory gives a cohomology class $ob \in \mathrm{H}^2(C, T_C)$ s.t. $ob = 0 \iff \exists \mathcal{C} \to \operatorname{spec} A$ extension. We're on a curve so $\mathrm{H}^2(C, T_C) = 0$ and we win.

(dimension) For a field k, we know $\dim_{[C]} \mathcal{M}_{g,k} = \dim \mathrm{H}^1(C, T_C) = 3g - 3$.

9.4 Properness

Many definitions...

Definition 9.22. For algebraic stacks $\mathcal{X}, \mathcal{Y}$, we define

- A morphism $\mathcal{X} \to \mathcal{Y}$ is **universally closed** if for every morphism $\mathcal{Y}' \to \mathcal{Y}$, the morphism $\mathcal{X} \times_{\mathcal{Y}} \mathcal{Y}' \to \mathcal{Y}'$ induces a closed map on topological spaces.
- A representable morphism $\mathcal{X} \to \mathcal{Y}$ is **separated** if the diagonal $\mathcal{X} \to \mathcal{X} \times_{\mathcal{Y}} \mathcal{X}$ (which is representable by schemes) is proper.
- A representable morphism $\mathcal{X} \to \mathcal{Y}$ is **proper** if it is universally closed, separated and of finite type.
- A morphism $\mathcal{X} \to \mathcal{Y}$ of algebraic stacks is **separated** if the diagonal $\mathcal{X} \to \mathcal{X} \times_{\mathcal{Y}} \mathcal{X}$ (which is representable) is proper.
- A morphism $\mathcal{X} \to \mathcal{Y}$ is algebraic stacks is **proper** if it is universally closed, separated and of finite type.

Remark 9.23. If X is a scheme, the diagonal is a monomorphism so Δ_X is a closed immersion $\iff$ Δ_X is proper.

Example. BG for G finite group is proper, but its diagonal is not a closed immersion.

Theorem 9.24 (**Valuative Criteria for Univ. Closed/Proper/Separated**)**.** *Let $f : \mathcal{X} \to \mathcal{Y}$ be a finite type morphism of algebraic stacks, and consider a 2-commutative diagram*

$$\begin{tikzcd} \operatorname{spec} K \arrow[r] \arrow[d] & \mathcal{X} \arrow[d, "f"] \arrow[dl, phantom, "\swarrow\alpha"] \\ \operatorname{spec} R \arrow[r] & \mathcal{Y} \end{tikzcd}$$

where R is a valuation ring with fraction field K. Then,

(1) *f is universally closed $\iff$ for all such diagrams, there exists an extension $R \to R'$ of valuation rings and $K \to K'$ of fraction fields together with a lifting*

$$\begin{tikzcd} \operatorname{spec} K' \arrow[r] \arrow[d] & \operatorname{spec} K \arrow[r] \arrow[d, dashed] & \mathcal{X} \arrow[d, "f"] \\ \operatorname{spec} R' \arrow[r] \arrow[urr, dashed] & \operatorname{spec} R \arrow[r] & \mathcal{Y}. \end{tikzcd}$$

(2) *f is separated $\iff$ any 2 liftings are isomorphic*

(3) *f is proper $\iff$ every diagram has a lifting after an extension $R \to R'$ and any 2 liftings are isomorphic.*

Moreover, if $f : \mathcal{X} \to \mathcal{Y}$ is a finite type morphism of noetherian algebraic stacks, then it suffices to consider DVRs R and extensions such that $K \to K'$ is of finite transcendence degree.

10 Lecture 10 (2/10): Geometry of DM stacks

10.1 Where are we?

Today begins the third part of the course: geometry of DM stacks (~ 2 lectures). After this is the part of stable curves and their moduli ($\sim 6--7$ lectures). The first part was sites, sheaves and stacks, while the second part was algebraic spaces and stacks.

Note 5. There are ~ 50 people here today.

10.2 Recap

Miniversal presentations Our first theorem last time was the existence of miniversal presentations.

Theorem 10.1 (**Existence of Miniversal Presentations**). *Let $\mathcal{X}$ be a noetherian algebraic stack, and let $x \in |\mathcal{X}|$ be a finite type point with smooth stabilizer G_x. Then, there exists a smooth morphism $(U, u) \to (\mathcal{X}, x)$ from a scheme of relative dimension $\dim G_x$ s.t.*

$$\begin{tikzcd} \operatorname{spec} k(u) \arrow[r] \arrow[d] & U \arrow[d] \\ \mathcal{G}_x \arrow[r] & \mathcal{X} \end{tikzcd}$$

is Cartesian. In particular, if G_x is finite and reduced, there is an étale morphism $(U, u) \to (\mathcal{X}, x)$ from a scheme.

We showed that the induced map $T_{U,u} \xrightarrow{\sim} T_{\mathcal{X}, f(u)}$ on tangent spaces, at u, was an isomorphism, in the above situation. We used this to show that if $\mathcal{X}$ is finite type over a field k, and $\mathcal{X}$ is smooth at x, then

$$\dim_x \mathcal{X} = \dim T_{\mathcal{X},x} - \dim G_x.$$

We also obtained the below corollay, which we used to show that $\mathcal{M}_g$ is DM.

Corollary 10.2 (**characterizations of DM stacks**). *Let $\mathcal{X}$ be a noetherian algebraic stack. Then, TFAE*

(1) *$\mathcal{X}$ is DM*

(2) *every point of $\mathcal{X}$ has a finite, reduced stabilizer*

(3) *the diagonal $\mathcal{X} \to \mathcal{X} \times \mathcal{X}$ is unramified.*

Smoothness We then talked about smoothness, and gave the following lifting criteria for smoothness.

Theorem 10.3 (**Formal lifting Criterion for smoothness**). *Let $f : \mathcal{X} \to \mathcal{Y}$ be a morphism of algebraic stacks. Then, f is* ***smooth*** *if and only if f is locally of finite presentation, and for every diagram*

$$\begin{tikzcd} \operatorname{spec} A_0 \arrow[r] \arrow[d] & \mathcal{X} \arrow[d, "f"] \\ \operatorname{spec} A \arrow[r] \arrow[ur, dashed] & \mathcal{Y} \end{tikzcd}$$

of solid arrows where $A \twoheadrightarrow A_0$ is a surjection of rings with nilpotent kernel, there exists a lifting.

If $\mathcal{X}, \mathcal{Y}$ are noetherian, then it suffices to consider diagrams where A and A_0 are local artinian rings.

There is often (read: always) an "obstruction" to the existence of such a lift $\operatorname{spec} A \to \mathcal{X}$ which is given by the element of some cohomology group.

For $\mathcal{M}_g$, it sufficed to check the lifting criterion on local Artinian rings, where we would restrict to the residue field and ultimately obtain an obstruction class in $\mathrm{H}^2(C, T_C) = 0$. We used this to conclude that $\mathcal{M}_g$ is smooth.

10.2.1 Separated and Properness

We talked about properness/separatedness last time, but we were kinda rushed, so let's say a little more this time.

Definition 10.4. We say $\mathcal{X} \to \mathcal{Y}$ is

(1) separated if $\Delta : \mathcal{X} \to \mathcal{X} \times_{\mathcal{Y}} \mathcal{X}$ is proper

(2) proper if $\mathcal{X} \to \mathcal{Y}$ is finite type, universally closed, and separated.

Remark 10.5. This is not circular since the diagonal is representable, so has a notion of properness coming from properness of maps of schemes.

Example. If X is a scheme, then Δ_X is always a locally closed immersion. Hence, X separated $\iff$ Δ_X closed immersion $\iff$ Δ_X finite $\iff$ Δ_X proper.

Example. Let G be a finite group. Then, BG is separated since we have a Cartesian diagram

$$\begin{tikzcd} G \arrow[r, "fin"] \arrow[d, "\text{ét}"'] & \operatorname{spec} k \arrow[d, "\text{ét}"] \\ BG \arrow[r, "\Delta"] & BG \times BG \end{tikzcd}$$

Example. Consider $\mathbb{Z}/2\mathbb{Z} \curvearrowright U$ with U the non-separated affine line. Consider the action which swaps the two origins, but *fixes* everything else, and let $\mathcal{X} = [U/(\mathbb{Z}/2\mathbb{Z})]$. This looks like the affine line, except there's a generic $\mathbb{Z}/2\mathbb{Z}$ stabilizer away from the origin (and the stabilizer at the origin is trivial). We have a map $U \to \mathcal{X}$, and can choose an affine line $\mathbb{A}^1 \to U$ over U. Let $f : \mathbb{A}^1 \to U \to \mathcal{X}$ be the composition. What is $\underline{\mathrm{Aut}}_{\mathbb{A}^1}(f)$?

It is not finite over $\mathbb{A}^1$ (look at the origin), so $\mathcal{X}$ is not separated. One can show that $G = \underline{\mathrm{Aut}}_{\mathbb{A}^1}(f)$ is a group scheme over $\mathbb{A}^1$, and $\mathcal{X} = B_{\mathbb{A}^1} G$.

Example. $B\mathbb{G}_m$ is *not* separated. The reason is that $\mathbb{G}_m \to \operatorname{spec} k$ is an affine group scheme, but not proper. This is the basechange of the diagonal along $\operatorname{spec} k \to B\mathbb{G}_m \times B\mathbb{G}_m$.

Fact. If $\mathcal{X}$ has affine diagonal, then $\mathcal{X}$ is separated $\iff$ $\Delta_{\mathcal{X}}$ is finite.

This is because proper + affine = finite. As a consequence, algebraic stacks with affine diagonal and positive dimensional stabilizers are never separated.

Example. $[\mathbb{A}^1/\mathbb{G}_m]$, $\mathrm{Bun}_{r,d}(C)$ are not separated.

We will show that $\mathcal{M}_g$ is separated. We do not know this yet.

Recall the valuative criterion.

Theorem 10.6 (**Valuative Criteria for Univ. Closed/Proper/Separated**)**.** *Let $f : \mathcal{X} \to \mathcal{Y}$ be a finite type morphism of algebraic stacks, and consider a 2-commutative diagram*

$$\begin{tikzcd} \operatorname{spec} K \arrow[r] \arrow[d] \arrow[dr, phantom, "\swarrow\alpha"] & \mathcal{X} \arrow[d, "f"] \\ \operatorname{spec} R \arrow[r] & \mathcal{Y} \end{tikzcd}$$

where R is a valuation ring with fraction field K. Then,

(1) *f is universally closed $\iff$ for all such diagrams, there exists an extension $R \to R'$ of valuation rings and $k \to K'$ of fraction fields together with a lifting*

$$\begin{tikzcd} \operatorname{spec} K' \arrow[r] \arrow[d] & \operatorname{spec} K \arrow[r] \arrow[d, dashed] & \mathcal{X} \arrow[d, "f"] \\ \operatorname{spec} R' \arrow[r] \arrow[urr, dashed] & \operatorname{spec} R \arrow[r] & \mathcal{Y}. \end{tikzcd}$$

(2) *f is separated $\iff$ any 2 liftings are isomorphic*

(3) *f is proper $\iff$ every diagram has a lifting after an extension $R \to R'$ and any 2 liftings are isomorphic.*

Moreover, if $f : \mathcal{X} \to \mathcal{Y}$ is a finite type morphism of noetherian algebraic stacks, then it suffices to consider DVRs R and extensions such that $K \to K'$ is of finite transcendence degree.

Note 6. Got distracted for a minute. He said something about later showing $\overline{\mathcal{M}}_g$ is proper, but I missed it.

Example. We can use this criterion to show $B\mathbb{G}_m$ is not separated. Let R be a valuation ring with fraction field K. Any $\mathbb{G}_m$-torsor over $\operatorname{spec} K$ is trivial. Say we have two trivial torsors $h_1, h_2 : \operatorname{spec} R \rightrightarrows B\mathbb{G}_m$ over R, along with an isom $(h_1|_{\operatorname{spec} k} \xrightarrow{\sim} h_2|_{\operatorname{spec} K}) \in \mathbb{G}_m(k)$. Properness (the *uniqueness* of lifts) requires this isomorphism to lift to one over R (i.e. to lift to an element of $\mathbb{G}_m(R)$). However, $\mathbb{G}_m(R) \to \mathbb{G}_m(K)$ is *not* surjective (e.g. if R is a dvr, let π be a uniformize and let $1/\pi \in \mathbb{G}_m(K)$ be the iso you start with), so $B\mathbb{G}_m$ is not proper.

10.3 Quasi-coherent sheaves

We won't need much of the theory of quasi-coherent sheaves for a while, but we introduce them now. For DM stacks, this theory works just like it does for schemes, so we want give a taste of it. Extending things to algebraic stacks is harder, and we do not do that.

Let $\mathcal{X}$ be a Deligne-Mumford stack.

Definition 10.7. The **small étale site** of $\mathcal{X}$ is the category $\mathcal{X}_{\text{ét}}$ of schemes étale over $\mathcal{X}$. A covering is a collection $\{U_i \xrightarrow{\text{ét}} U\}$ such that $\bigsqcup U_i \twoheadrightarrow U$.

This gives a notion of sheaves on $\mathcal{X}_{\text{ét}}$, so let $\mathrm{Sh}(\mathcal{X}_{\text{ét}})$ denote the category of sheaves on $\mathcal{X}_{\text{ét}}$. Spelled out a bit, given $\mathscr{F} \in \mathrm{Sh}(\mathcal{X}_{\text{ét}})$, for any $U \xrightarrow{\text{ét}} \mathcal{X}$, we get a set of sections $\mathscr{F}(U \to \mathcal{X})$. In fact, can extend this to étale maps $\mathcal{U} \to \mathcal{X}$ of DM stacks; the idea is to choose a presentation $U \to \mathcal{U}$ with corresponding groupoid $R \rightrightarrows U$, and then define

$$\mathscr{F}(\mathcal{U} \xrightarrow{\text{ét}} \mathcal{X}) := \mathrm{Eq}\left(\mathscr{F}(U \to \mathcal{X}) \rightrightarrows \mathscr{F}(R \to \mathcal{X})\right)$$

(Eq for equalizer). If you want, you could enlarge your site to consist of all étale coverings by DM stacks, and then what we're calling $\mathcal{X}_{\text{ét}}$ would be like a basis for this larger site. We don't do this, but we introduced this in order to make sense of **global sections**

$$\Gamma(\mathcal{X}, \mathscr{F}) := \mathscr{F}(\mathcal{X} \xrightarrow{\text{id}} \mathcal{X}).$$

Fact. For any map $f : \mathcal{X} \to \mathcal{Y}$ of DM stacks, there are adjoint functors

$$f_* : \mathrm{Sh}(\mathcal{X}_{\text{ét}}) \rightleftarrows \mathrm{Sh}(\mathcal{Y}_{\text{ét}}) : f^{-1}.$$

The pushforward above is

$$(f_*\mathscr{F})(V \xrightarrow{\text{ét}} \mathcal{Y}) = \mathscr{F}(V \times_{\mathcal{Y}} \mathcal{X} \to \mathcal{X}).$$

The inverse image is trickier (as usual); it is

$$(f^{-1}\mathcal{G})(U \xrightarrow{\text{ét}} \mathcal{X}) = \lim \mathcal{G}(V \to \mathcal{Y})$$

where the limit is over diagrams

This should be something like limit over sections of the $V \supset f(U)$

$$\begin{tikzcd} U \arrow[r] \arrow[d, "\text{ét}"'] & V \arrow[d, "\text{ét}"] \\ \mathcal{X} \arrow[r, "f"] & \mathcal{Y} \end{tikzcd}$$

Definition 10.8. Let $\mathcal{X}$ be a DM stack. Its **structure sheaf** is

$$\mathscr{O}_{\mathcal{X}}(U \xrightarrow{\text{ét}} \mathcal{X}) := \Gamma(U, \mathscr{O}_U),$$

a sheaf of rings. This gives rise to a notion of **$\mathscr{O}_{\mathcal{X}}$-modules**.

Fact. If $f : \mathcal{X} \to \mathcal{Y}$ is a map of DM stacks, we get adjoint functors

$$f_* : \mathrm{Mod}(\mathscr{O}_{\mathcal{X}}) \rightleftarrows: \mathrm{Mod}(\mathscr{O}_{\mathcal{Y}}) : f^*$$

with f_* as before, and

$$f^*(-) := f^{-1}(-) \otimes_{f^{-1}\mathscr{O}_{\mathcal{Y}}} \mathscr{O}_{\mathcal{X}}.$$

Definition 10.9. We say an $\mathscr{O}_{\mathcal{X}}$-module $\mathscr{F}$ is **quasi-coherent** if for all $U \xrightarrow{\text{ét}} \mathscr{X}$ (U a scheme), the restriction $\mathscr{F}|_U$ to U_{zar} (small Zariski site on U) is quasi-coherent.

See warning at end of lecture

Fact. Let $\mathcal{X} \xrightarrow{f} \mathcal{Y}$ be any map of DM stacks. Then,

(1) f^* preserves quasi-coherence

(2) If f is qcqs, f_* also preserves quasi-coherence

These facts "take work, but are manageable."

Example. Let G be a finite group. Then,

$$\mathrm{QCoh}(BG) = \mathrm{Rep}(G).$$

Let $V \in \mathrm{Rep}(G)$ be a G-rep. Consider

$$\operatorname{spec} k \xrightarrow{p} BG \xrightarrow{\pi} \operatorname{spec} k$$

with p the natural projection and π the structure map. Then,

$$p^*V = V \text{ and } \pi_*V = V^G.$$

If you have W on $\operatorname{spec} k$, then π^*W is the trivial G-rep while $p_*W = W \otimes p_*k$ with p_*k the regular representation $\Gamma(G)$.

This is $\mathrm{Ind}_1^G W$, I'm pretty sure

10.3.1 Another perspective

Say $\mathscr{F} \in \mathrm{QCoh}(\mathcal{X})$. This can equivalently viewed as a qcoh sheaf $\mathscr{F}_S$ on S (a scheme) for each $S \to \mathcal{X}$ (not nec. étale) such that for any diagram

$$\begin{tikzcd} S \arrow[rr, "f"] \arrow[dr] & & T \arrow[dl] \\ & \mathcal{X} & \end{tikzcd}$$

there si a canonical iso $f^*\mathscr{F}_T \cong \mathscr{F}_S$.

Example. Define $\mathcal{H} \in \mathrm{QCoh}(\mathcal{M}_g)$ for $S \to \mathcal{M}_g$ (i.e. $\mathcal{C} \xrightarrow{\pi} S$ smooth) via $\mathcal{H}_S := \pi_*\Omega_{\mathcal{C}/S}$. This is called the **Hodge bundle**.

Other notions...

- A **vector bundle** is a locally free sheaf of finite rank, and a rank 1 vector bundle is a **line bundle**.
- Can discuss coherent sheaves (at least when $\mathcal{X}$ noetherian)
- Can make sense of $\mathscr{O}_{\mathcal{X}}$-algebras $\mathscr{A}$
- There exists relative spectra $\operatorname{spec} \mathscr{A} \to \mathcal{X}$ (so the map is affine)

10.4 Local structure of DM stacks

Theorem 10.10 (Local Structure of DM Stacks). *Let $\mathcal{X}$ be a separated DM stack, and $x \in \mathcal{X}(k)$ a **geometric point** (i.e. $k = \overline{k}$) with stabilizer G_x. Then there exists an affine, étale map*

$$f : ([\operatorname{spec} A/G_x], w) \xrightarrow{\text{ét}} (\mathcal{X}, x)$$

such that f induces an isomorphism of stabilizer groups at w.

Something something DM stacks are like algebraic orbifolds something something

Above, we can choose some $w : \operatorname{spec} k \to \operatorname{spec} A$.

Remark 10.11. This tells us that we can view DM stacks as quotients $[\operatorname{spec} A/G]$ of affine schemes by finite groups, glued étale locally.

Let's prove this. We first fix some notation

Notation 10.12. Let $U \xrightarrow{\text{ét}} \mathcal{X}$ with U a scheme, and let

$$(U/\mathcal{X})^d = U \times_{\mathcal{X}} \ldots \times_{\mathcal{X}} U$$

with d factors. Note that a map $S \to (U/\mathcal{X})^d$ is the same as the data of an object $S \xrightarrow{x} \mathcal{X}$ along with d sections $s_1, \ldots, s_d : S \to U_S$ of the fiber product

$$\begin{tikzcd} U_S \arrow[r] \arrow[d] & U \arrow[d] \\ S \arrow[r] & \mathcal{X} \end{tikzcd}$$

We set $(U/\mathcal{X})^d_0 \subset (U/\mathcal{X})^d$ the open substack which is the complement of all diagonals. Hence, a map $S \to (U/\mathcal{X})^d_0$ is the same as the data of an object $S \xrightarrow{x} \mathcal{X}$ along with d disjoint sections $s_1, \ldots, s_d : S \to U_S$ of the fiber product.

Proof of Theorem 10.10. Choose

$$\begin{tikzcd} \operatorname{spec} k \arrow[r, "u"] \arrow[d] & U \arrow[d, "\text{ét}"] \\ BG_x \arrow[r] & \mathcal{X} \end{tikzcd}$$

with U an affine scheme. We've assumed $\mathcal{X}$ is separated, so the diagonal is affine, which means $U \to \mathcal{X}$ is also affine.[26] We can extend the above diagram to ($d = |G_x|$)

Question: Diagonal of a DM stack is always unramified. finite = proper + quasi-finite. Does this do it?

$$\begin{tikzcd} G_x \arrow[r] \arrow[d] & \operatorname{spec} k \arrow[r, "u"] \arrow[d, "\substack{\text{fin, ét} \\ \deg = d}"] & U \arrow[d, "\substack{\text{affine} \\ \text{ét}}"] \\ \operatorname{spec} k \arrow[r] & BG_x \arrow[r] & \mathcal{X} \end{tikzcd}$$

Note that $(U/\mathcal{X})^d = U \times_{\mathcal{X}} \ldots \times_{\mathcal{X}} U$ is an affine scheme, so $W := (U/\mathcal{X})^d_0$ is quasi-affine. Given $S \to (U/\mathcal{X})^d_0$, let $s_1, \ldots, s_d : S \to U_S$ be the corresponding (disjoint) sections. We can consider $\bigsqcup s_i(U) \hookrightarrow U_S$, and the composition $\bigsqcup s_i(U) \to S$ is finite étale of degree d. There's an action $S_d \curvearrowright (U/\mathcal{X})^d_0 = W$, and so we form $[W/S_d]$. A map $S \to [W/S_d]$ is the same as $Z \hookrightarrow U_S$ s.t. $Z \hookrightarrow U_S \to S$ is finite, étale of degree d, so we earlier created an object $w : \operatorname{spec} k \to [W/S_d]$ (or something). Let $Z = G_x$. A choice of ordering elements in G_x gives a lift $\widetilde{w} : \operatorname{spec} k \to W$.

Answer: I think so (since unramified morphisms are quasi-finite)

Exercise. Under the S_d-action, the stabilizer of $\widetilde{w}$ is precisely $G_x \subset S_d$.

[26] To get the pullback to be a point, just start with some affine U and then remove points of the fiber if there are more than one, or something

The picture is something like

$$\begin{tikzcd}
 & W \arrow[d, equal] & \\
S_d \arrow[r, phantom, "\curvearrowright"] & (U/\mathcal{X})^d_0 \arrow[r, phantom, "\subset"] \arrow[dd, "\text{ét}"] & (U/\mathcal{X})^d \arrow[d, "\text{ét}"] \\
 & & U \arrow[d, "\text{ét}"] \\
 & {[W/S_d]} \arrow[r, "f", "\text{ét}"'] & \mathcal{X}
\end{tikzcd}$$

Need to check that f is representable and étale (note everything in this diagram is étale). The exercise implies that f induces an iso on stabilizers at w. We still have more work. We have W which is quasi-affine, but not yet affine. We have an action of S_d, not of the stabilizer. Finally, we don't know the map is affine.

To fix the action issue (the second one), just consider $[W/G_x] \to [W/S_d] \to \mathcal{X}$ which is étale and preserves the stabilizer at w. For the affine issue, choose an affine open $W' \subset W$ containing w, and let $W'' = \bigcap_{g \in G} gW' \ni w$. This is now affine and G_x-invariant, so this is out $\operatorname{spec} A$. Finally, why is the map affine? We now have

$$\operatorname{spec} A \to [\operatorname{spec} A/G_x] \to \operatorname{repr}\mathcal{X}.$$

Since $\mathcal{X}$ has affine diagonal, this composition is affine. Serre's criterion for affineness implies that affineness descends under finite morphisms, so this suffices to show that $[\operatorname{spec} A/G_x] \to \mathcal{X}$ is affine. ■

Remark 10.13. This argument actually works with separatedness replace with "affine diagonal."

Can do this proof when the diagonal is quasi-affine (the resulting map is no longer affine). It is true (but we don't know this yet) that under mild hypotheses (e.g. the diagonal being separated + something else?), the diagonal of a DM stack is always quasi-affine.

Warning 10.14. There's maybe an issue in our definition of quasi-coherence (unclear if there is or not). Let $\mathscr{F}$ be an $\mathscr{O}_{\mathcal{X}}$-module.

- We said that $\mathscr{F}$ is quasi-coherent if $\mathscr{F}|_U$ is quasi-coherent on U_{zar} for all étale maps $U \to \mathcal{X}$ from a scheme.

- One usually requires that $\mathscr{F}|_U$ be quasi-coherent on $U_{\text{ét}}$ for all étale maps $U \to \mathcal{X}$. Here (assuming I'm understanding the discussion correctly), $\mathscr{F}$ is étale on $U_{\text{ét}}$ if there exists a qcoh sheaf $\mathscr{G}$ on U_{zar} s.t.

  $$\mathscr{F}(V \xrightarrow{f} U) = \Gamma(V, f^*\mathscr{G})$$

  for all $V \to U$ in $U_{\text{ét}}$.

  It sounds like, using étale descent for qcoh sheaves, one can show that this is equivalent to requiring $\mathscr{F}|_U$ being qcoh on U_{zar} (what we said) *plus* requiring that for any $f : V \to U$ (unclear to me if this necessary étale), the restrictions $\mathscr{F}|_V$ and $f^*(\mathscr{F}|_U)$ are isomorphic (via the natural map between them).

11 Lecture 11 (2/17): Existence of coarse moduli spaces

The goal for today is

Theorem 11.1 (**Keel-Mori Theorem**). *A separated DM stack $\mathcal{X}$ admits a coarse moduli space $\pi : \mathcal{X} \to X$ where X is a separated algebraic space.*

Last time we talked about

- Quasi-coherent sheaves on DM stacks
- local structure of DM stacks

There were a couple of audience questions I didn't write down, but maybe should have since they were not bad questions... Oh well

11.1 Review

11.1.1 Local Structure of DM Stacks

Theorem 11.2. *Let $\mathcal{X}$ be a separated DM stack and let $x \in \mathcal{X}(k)$ be a geometric point with stabilizer G_x. Then there is an affine, étale map*

$$f : ([\operatorname{spec} A/G_x], w) \to (\mathcal{X}, x)$$

inducing an isomorphism of stabilizer groups at w.

This ends our review. Onto the new stuff.

11.2 Definition

Definition 11.3. A map $\mathcal{X} \xrightarrow{\pi} X$ (with $\mathcal{X}$ an algebraic stack and X an algebraic space) is a **coarse module space** (or **cms**) if both of the following hold

(1) For all $k = \overline{k}$, the map $\mathcal{X}(k)/\sim \to X(k)$ from iso classes of objects of the groupoid $\mathcal{X}(k)$ to the set $X(k)$ is a bijection.

(2) π is universal (initial) for maps to algebraic spaces, i.e. if $\mathcal{X} \to Y$ is a map to an algebraic space, then it uniquely factors as

$$\begin{tikzcd} \mathcal{X} \arrow[r, "\pi"'] \arrow[rr, bend left] & X \arrow[r, "\exists!"'] & Y \end{tikzcd}$$

We view the coarse moduli space X as the closest approximation to $\mathcal{X}$ which is an algebraic space. There's a tradeoff here. $\mathcal{X}$ has universal properties (e.g. supports a universal family), but X is a more familiar sort of space (ideally, it's projective).

Note 7. Got distracted for a minute.

Strategy to show existece of cms

- special case: if $\mathcal{X} = [\operatorname{spec} A/G]$, then

$$\mathcal{X} = [\operatorname{spec} A/G] \xrightarrow{cms} \operatorname{spec} A^G.$$

- Use local structure theorem to glue these in étale topology to construct X.

$$\begin{tikzcd} {[\operatorname{spec} A/G_x]} \arrow[r, "\text{ét}"] \arrow[d, "cms"] & \mathcal{X} \arrow[d, dashed] \\ \operatorname{spec} A^{G_x} \arrow[r, dashed] & X \end{tikzcd}$$

In practice, we desire stronger properties than what are guaranteed by the theorem. For us, if $\mathcal{X}$ is a separated DM stack, we sill construct a coarse module space $\mathcal{X} \xrightarrow{\pi} X$ satisfying additional properties

- stable under flat base change
- $\pi_* \mathscr{O}_{\mathcal{X}} = \mathscr{O}_X$
- π is proper (in particular, separated)
- π is a universal homeomorphism (in particular, $|\mathcal{X}| \xrightarrow{\sim} |X|$)

Remark 11.4. The first condition above implies the second one. Consider maps to $\mathbb{A}^1$, the fact that we have

$$\begin{tikzcd} \mathcal{X} \arrow[r] \arrow[rr, bend left] & X \arrow[r, "\exists!"'] & \mathbb{A}^1 \end{tikzcd}$$

tells us that $\Gamma(X, \mathscr{O}_X) = \Gamma(\mathcal{X}, \mathscr{O}_{\mathcal{X}})$. If $X' \to X$ is étale (even just flat) then the basechange $\mathcal{X}' \to X'$ is a coarse moduli space, so also $\Gamma(X' \to X, \pi_* \mathscr{O}_{\mathcal{X}}) = \Gamma(\mathcal{X}', \mathscr{O}_{\mathcal{X}'}) = \Gamma(X', \mathscr{O}_{X'})$. Hence, $\mathscr{O}_X \xrightarrow{\sim} \pi_* \mathscr{O}_{\mathcal{X}}$.

Lemma 11.5 (descent lemma). *Let $\mathcal{X} \xrightarrow{\pi} X$ be a map. If $\{X_i \xrightarrow{\text{ét}} X\}$ is an étale cover (even fppf) s.t. $\mathcal{X} \times_X X_i \to X_i$ is a cms, then $\mathcal{X} \to X$ is.*

Exercise. Prove this lemma.

11.3 Quotients by finite groups

Let G be a finite group acting on $\operatorname{spec} A$. Define

$$A^G = \{a \in A : ga = a \,\forall g\}.$$

Lemma 11.6. *Let R be a noetherian ring. If A is a finitely generated R-algebra, then $A^G \to A$ is finite and A^G is a finitely generated R-algebra.*

Note G acts via R-algebra homomorphisms

Proof. First note that $A^G \to A$ is integral since any $a \in A$ is a root of the monic poly

$$\prod_{g \in G} (x - ga) \in A^G[x].$$

Now, $A^G \to A$ is finitely generated and integral, so $A^G \to A$ is actually finite. Since R is noetherian, this implies A^G is finitely generated as an R-algebra. ■

An R-subalgebra of A which A is finite over is automatically

Lemma 11.7. *Let $A^G \to B$ be a ring map, so G acts on $\operatorname{spec}(B \otimes_{A^G} A)$. Consider*

$$\begin{tikzcd}
 & \operatorname{spec}(B \otimes_{A^G} A) \arrow[r] \arrow[dl] \arrow[d] & \operatorname{spec} A \arrow[d] \\
\operatorname{spec}(B \otimes_{A^G} A)^G \arrow[r, "\psi"] & \operatorname{spec} B \arrow[r] & \operatorname{spec} A^G
\end{tikzcd}$$

(1) $A^G \to B$ *flat* $\implies \psi^* : B \xrightarrow{\sim} (B \otimes_{A^G} A)^G$.

(2) In general, ψ is integral, and is a universal homeomorphism.

Proof. **(1)** We have an equalizer diagram

$$A^G \to A \rightrightarrows \prod_{g \in G} A$$

with one map the diagonal and one map multiplication by elements of G. Since $- \otimes_{A^G} B$ is exact, we then get an equalizer diagram

$$B \to (A \otimes_{A^G} B) \rightrightarrows \prod_{g \in G} (A \otimes_{A^G} B)$$

which exactly says $B \xrightarrow{\sim} (A \otimes_{A^G} B)^G$.

(2) Exercise. ■

Theorem 11.8. *Let G be a finite group acting on an affine scheme $\operatorname{spec} A$ of finite type over a noetherian ring R. Then,*

$$\pi : [\operatorname{spec} A/G] \to \operatorname{spec} A^G$$

is a coarse moduli space such that

(1) A^G is finitely generated over R;

(2) π is a proper universal homeomorphism; and

(3) the base change of π along any flat map of noetherian algebraic spaces is a coarse moduli space

Proof. **(Step 1)** π is a proper universal homeo ($\implies$ π bijective on geometric points). Consider

$$\begin{tikzcd}
\operatorname{spec} A \arrow[d, "\text{finite}"', "\pi"] \arrow[dr, two heads, "\substack{\text{finite} \\ \text{dominant}}"] & \\
{[\operatorname{spec} A/G]} \arrow[r, two heads] & \operatorname{spec} A^G
\end{tikzcd}$$

and see that bottom map is proper. We claim that π is injective on geometric points. Assume $R = k = \overline{k}$. Let $x, x' \in \operatorname{spec} A$ be closed points with $Gx \neq Gx'$ (distinct orbits). Since $Gx \cap Gx' = \emptyset$, there exists some $f \in A$ such that $f|_{Gx} = 1$ and $f|_{Gx'} = 0$. Then, $f' = \prod_{g \in G} gf \in A^G$ is an invariant function with $f'|_{Gx} = 1$ and $f'|_{Gx'} = 0$. Hence, $\pi(x) \neq \pi(x')$ since f' separated them. Therefore, π is bijective on geometric points. Since π is proper, it is closed, so it is a homeomorphism.

Question: Prime avoidance lemma?

We still need to show that it is a universal homeomorphism. For $A^G \to B$, we have

$$\begin{tikzcd}
 & \operatorname{spec}(B \otimes_{A^G} A) \arrow[dl] \arrow[d] \arrow[r] & \operatorname{spec} A \arrow[d] \\
\operatorname{spec}(B \otimes_{A^G} A)^G \arrow[r, "\psi"] & \operatorname{spec} B \arrow[r] & \operatorname{spec} A^G
\end{tikzcd}$$

some diagram chase shows that $\operatorname{spec} A \to \operatorname{spec} A^G$ is a universal homeomorphism.

(Step 2) We wish to show π is universal for maps to algebraic spaces. Let $f : \mathcal{X} = [\operatorname{spec} A/G] \to Y$ be a map to an algebraic space. We want a factorization

$$\begin{tikzcd}
\mathcal{X} \arrow[r] \arrow[rr, bend left, "f"] & X \arrow[r, "\exists!"'] & Y
\end{tikzcd}$$

Remark 11.9. If Y is affine, then f gives $\Gamma(Y, \mathscr{O}_Y) \to \Gamma(\mathcal{X}, \mathscr{O}_{\mathcal{X}}) = \Gamma(X, \mathscr{O}_X)$, and so gives a map $X = \operatorname{spec} A^G \to Y$. Hence, the affine case is easy.

Let's first show uniqueness. Suppose we have two maps $h_1, h_2 : X \to Y$ through which f factors. Let E be the equalizer of $X \rightrightarrows Y$, i.e. the pullback

$$\begin{tikzcd}
E \arrow[r] \arrow[d] & X \arrow[d, "(h_1,h_2)"] \\
Y \arrow[r] & Y \times Y.
\end{tikzcd}$$

By definition of the equalizer, we have a diagram

$$\begin{tikzcd}
 & \mathcal{X} \arrow[dl] \arrow[d] \arrow[dr, "f"] & \\
E \arrow[r] & X \arrow[r, shift left, "h_1"] \arrow[r, shift right, "h_2"'] & Y
\end{tikzcd}$$

Since Y is an algebraic space, the diagonal is a monomorphism locally of finite type, so the same is true for $E \to X$. We know $\mathcal{X} \to X$ is proper and schematically dominant, so the same are true for $E \to X$. That is, $E \to X$ is a monomorphism, it is locally of finite type, it is proper, and it is schematically dominant. The first three imply that it is a closed immersion, and then the last one implies that it is actually an isomorphism $E \xrightarrow{\sim} X$, so $h_1 = h_2$.

We now do existence. We claim that existence is étale-local on X. This is a combination of étale descent and the universal property (+ uniqueness). Hence, we may assume that A^G is strictly henselian, and we may also assume that Y is quasi-compact (since $\mathcal{X}$ is). Since Y is quasi-compact, we can choose an étale presentation $Y' \overset{\text{ét}}{\twoheadrightarrow} Y$ with Y' affine. Consider the diagram with leftmost squares Cartesian

$$\begin{tikzcd}
Y' \times_Y \operatorname{spec} A \arrow[r] \arrow[d] & \operatorname{spec} A \arrow[d] \arrow[dr] & \\
\mathcal{X}' \arrow[r, "\text{ét}"] \arrow[d] & \mathcal{X} \arrow[r] \arrow[d, "f"] & X = \operatorname{spec} A^G \arrow[dl, dashed, "?"] \\
Y' \arrow[r, "\text{ét}"] & Y &
\end{tikzcd}$$

Since $\operatorname{spec} A$ is (strictly) henselian, we get a section $s : \operatorname{spec} A \to Y' \times_Y \operatorname{spec} A$ which then descends to a section $\mathcal{X} \to \mathcal{X}'$ of $\mathcal{X}' \to \mathcal{X}$. This gives us a map $\mathcal{X} \to \mathcal{X}' \to Y' =$ affine, so since the affine case is easy, we know there's a unique map $X \to Y'$ through which this factors. Hence, we get $X \to Y' \to Y$ which is the factorization we wanted, proving existence. ■

I think this is not obvious. Need G-invariance or something

11.4 Reducing to Quotient Stacks

Let's recall our strategy. We're after a coarse moduli space for a general separated DM stack $\mathcal{X}$. We have the local structure map $W = [\operatorname{spec} A/G_x] \xrightarrow[aff]{\text{ét}} \mathcal{X}$. Since this map is affine, we'll have $W \times_{\mathcal{X}} W = [\operatorname{spec} B/G_x]$. By the finite group quotient case, we have coarse moduli spaces $\operatorname{spec} B^{G_x}$ and $\operatorname{spec} A^{G_x}$, and the two maps $W \times_{\mathcal{X}} W = [\operatorname{spec} B/G_x] \rightrightarrows [\operatorname{spec} A/G_x] = W$ give rise to maps $\operatorname{spec} B^{G_x} \rightrightarrows \operatorname{spec} A^{G_x}$. We want this to give an étale equivalence relation, and then argue that the quotient is the coarse moduli space for $\mathcal{X}$.

Question 11.10. *If $f : \operatorname{spec} A \to \operatorname{spec} B$ is G-equiv and étale, when is $\overline{f} : \operatorname{spec} A^G \to \operatorname{spec} B^G$ étale?*

Heuristically, say $R = k = \overline{k}$, and let $x \in \operatorname{spec} A$.

Fact. $\left(\widehat{A}\right)^{G_x} = \widehat{A^G}$ where we're taking completions at x.

If f is étale at x, then $\widehat{A} \xrightarrow{\sim} \widehat{B}$. $\overline{f}$ is étale at $\pi_A(x) \iff \widehat{A^G} \xrightarrow{\sim} \widehat{B^G} \iff \widehat{A}^{G_x} \xrightarrow{\sim} \widehat{B}^{G_{f(x)}}$. The upshot here is that if $G_x = G_{f(x)}$, then we win.

More formally,

Proposition 11.11. *Let G be a finite group. Let $f : \operatorname{spec} A \to \operatorname{spec} B$ be a G-equivariant map of schemes of finite type over a noetherian ring R. Let $x \in \operatorname{spec} A$ be a closed point. Assume that*

(a) *f is étale at x; and*

(b) *the map $G_x \xrightarrow{\sim} G_{f(x)}$ of stab groups is bijective.*

Then there is an open affine neighborhood $W \subset \operatorname{spec} A^G$ of $\pi_A(x)$ such that $W \to \operatorname{spec} A^G \to \operatorname{spec} B^G$ is étale and the outer square in

$$\begin{tikzcd}
\pi_A^{-1}(W) \arrow[r] \arrow[d] & {[\operatorname{spec} A/G]} \arrow[r, "f"] \arrow[d, "\pi_A"] & {[\operatorname{spec} B/G]} \arrow[d, "\pi_B"] \\
W \arrow[r] \arrow[rr, bend right, "\acute{e}t"'] & \operatorname{spec} A^G \arrow[r] & \operatorname{spec} B^G
\end{tikzcd}$$

is Cartesian.

Proof. Question is étale local around $\pi_B(y)$

$$\begin{tikzcd} [\operatorname{spec} A/G] \arrow[r] \arrow[d] & [\operatorname{spec} B/G] \arrow[d] \\ \operatorname{spec} A^G \arrow[r] & \operatorname{spec} B^G \end{tikzcd}$$

Showing the question is étale local is left as an exercise.

Now we may assume that B^G is strictly Henselian. Consider

$$\begin{tikzcd} \operatorname{spec} A \arrow[r, "\text{ét}"] \arrow[d] & \operatorname{spec} B \arrow[d] \\ [\operatorname{spec} A/G] \arrow[r] \arrow[d] & [\operatorname{spec} B/G] \arrow[d] \\ \operatorname{spec} A^G \arrow[r] & \operatorname{spec} B^G \end{tikzcd}$$

We now B is Henselian, so there's a section $\operatorname{spec} B \to \operatorname{spec} A$ (which is an open and closed immersion?). One show this is G-invariant, so descends to a section $s : [\operatorname{spec} B/G] \to [\operatorname{spec} A/G]$ (which is open and closed immersion). Then shrink to $\operatorname{im}(s)$ in order to get $\operatorname{im}(s) \to \operatorname{spec} B^G$ to be étale or something. ■

Corollary 11.12. *If in addition, **(a)** and **(b)** hold at all points, then*

$$\begin{tikzcd} [\operatorname{spec} A/G] \arrow[r] \arrow[d] & [\operatorname{spec} B/G] \arrow[d] \\ \operatorname{spec} A^G \arrow[r, "\acute{e}t"] & \operatorname{spec} B^G \end{tikzcd}$$

is Cartesian.

11.5 Keel-Mori

Theorem 11.13 (**Keel-Mori Theorem**)**.** *Let $\mathcal{X}$ be a DM stack which is separated of finite type over a noetherian algebraic space S. Then there exists a coarse moduli space $\pi : \mathcal{X} \to X$ with $\mathscr{O}_X = \pi_* \mathscr{O}_{\mathcal{X}}$ such that*

(1) *X is separated and of finite type over S*

(2) *π is a proper universal homeomorphism, and*

(3) *for any flat map $X' \to X$ of noetherian algebraic spaces, $\mathcal{X} \times_X X' \to X'$ is a coarse moduli space.*

Proof. Assume $S = \operatorname{spec} R$ is affine.[27] Then argue that the question is Zariski-local on $\mathcal{X}$. Hence it suffices to show that for a closed point $x \in |\mathcal{X}|$, there exists an open neighborhood of X with a coarse moduli space.

[27]Can reduce to this case. Reduction argument similar to proof we're about to give

Let $\operatorname{spec} k \to \mathcal{X}$ be a representative of x with $k = \overline{k}$, and set $G = G_{\overline{x}}$ to be the stabilizer. The local structure theorem gives us $[\operatorname{spec} A/G_x] \to \mathcal{X}$ both étale and affine coming with a point $w \mapsto x$. We know that $\operatorname{Aut}(w) \xrightarrow{\sim} \operatorname{Aut}(x)$. We need this to hold also in a neighborhood of w. Since $\mathcal{X}$ is separated, $\mathcal{X} \to \mathcal{X} \times \mathcal{X}$ is finite, so the Inertia map $I_{\mathcal{X}} \to \mathcal{X}$ is finite too. Consider

$$\begin{tikzcd} I_W \arrow[r] \arrow[d] & W \times_{\mathcal{X}} I_{\mathcal{X}} \arrow[r] \arrow[dl] & I_{\mathcal{X}} \arrow[d] \\ W & & \mathcal{X} \arrow[ll] \end{tikzcd}$$

The right square is Cartesian, so $W \times_{\mathcal{X}} I_{\mathcal{X}} \to W$ is finite. Furthermore,

$$\begin{tikzcd} I_W \arrow[r] \arrow[d] & W \times_{\mathcal{X}} I_{\mathcal{X}} \arrow[d] \\ W \arrow[r] & W \times_{\mathcal{X}} W \end{tikzcd}$$

is also Cartesian. Note that the top arrow is a map of group schemes over W s.t. for a $w \in W(k)$, the fiber is $\operatorname{Aut}(w) \to \operatorname{Aut}(f(w))$, so it is the 'right' map to study. Since $W \to \mathcal{X}$ is affine, $W \to W \times_{\mathcal{X}} W$ is a closed immersion. Since $W \to \mathcal{X}$ is étale, $W \to W \times_{\mathcal{X}} W$ is an open immersion. Hence, $I_W \to W \times_{\mathcal{X}} I_{\mathcal{X}}$ is also a closed an open immersion (union of connected components). Let p_1 be the projection $p_1 : W \times_{\mathcal{X}} I_{\mathcal{X}} \to W$. Then, $p_1(|W \times_{\mathcal{X}} I_X| \setminus I_W) \subset W$ is precisely where $W \to \mathcal{X}$ is not stabilzer preserving, and is closed! Thus, it's complement (which is nonempty since it contains w!) is an open on which our map is stabilizer preserving. Hence, we can arrange that $W \to \mathcal{X}$ be stabilizer preserving (and even surjective if we want).

Now we have

$$\begin{tikzcd} {[\operatorname{spec} B/G] = R = W \times_{\mathcal{X}} W} \arrow[r, shift left] \arrow[r, shift right] \arrow[d, "cms"] & W = [\operatorname{spec} A/G] \arrow[r, "\text{ét}", "aff"'] \arrow[d, "cms"] & \mathcal{X} \\ R \arrow[r, shift left] \arrow[r, shift right] & W = \operatorname{spec} A^G & \end{tikzcd}$$

Since $W \to \mathcal{X}$ is stabilizer preserving, so is $R \rightrightarrows W$. Both squares are Cartesian (previous lemma?), so $R \rightrightarrows W$ is an étale groupoid of affine schemes[28] Check that $R \to W \times W$ is a monomorphism. Then we can take $X = W/R$ to be the algebraic space quotient. Finally, étale descent gives a morphism $\mathcal{X} \to X$, shows that the resulting square is Cartesian, and is used to show that X is a coarse moduli space and satisfies all additional properties. ■

Remark 11.14. Above, why is X separated? Because $\mathcal{X}$ is separated and $\mathcal{X} \to X$ is proper.

Question 11.15 (Audience)**.** *Why is R of the form $[\operatorname{spec} B/G]$?*

Answer. The map $R \to W$ is itself affine. Since $\operatorname{spec} A \to W$ is a G-torsor, we see that the pullback $\operatorname{spec} A \times_W R \to R$ is a G-torsor and is affine, so $\operatorname{spec} A \times_W R = \operatorname{spec} B$ and $R = [\operatorname{spec} B/G]$.

Remark 11.16. The Keel-Mori theorem is actually more general than what we stated. In particular, you don't need a DM stack. The important thing is that inertia is finite.

[28]Need to worry about inverse and composition and whatnot, but that's more annoying than difficult it sounds.

12 Lecture 12 (2/22): Nodal Curves

Today marks the start of part IV of the course: moduli of stable curves. The plan for today is

- Recap of Keel-Mori
- Refresher on smooth curves
- Nodal curves

12.1 Recap

Recall 12.1. A map $\pi : \mathcal{X} \to X$ from an algebraic stack to an algebraic space is a *coarse moduli space* if

(1) for all $k = \overline{k}$, $\mathcal{X}(k)/\sim \xrightarrow{\sim} X(k)$

(2) π is initial for maps to algebraic spaces.

Theorem 12.2 (Keel-Mori)**.** *Let $\mathcal{X}$ be a DM stack with is separated and of finite type over a noetherian algebraic space S. Then there exists a coarse moduli space $\pi : \mathcal{X} \to X$ with $\mathscr{O}_X = \pi_*\mathscr{O}_{\mathcal{X}}$ such that*

(1) X is separated and of finte type over S;

(2) π is a proper universal homeomorphism; and

(3) for any flat map $X' \to X$ of noetherian algebraic spaces, $\mathcal{X} \times_X X' \to X'$ is a coarse moduli space.

Remark 12.3. Recall that an algebraic stack is separated iff its diagonal is proper. Also recall that the diagonal of a DM stack is quasi-finite. Thus, for a DM stack $\mathcal{X}$,

$$\mathcal{X} \text{ is separated} \iff \mathcal{X} \to \mathcal{X} \times \mathcal{X} \text{ is finite.}$$

We will later show that $\mathcal{M}_g$ and $\overline{\mathcal{M}}_{g,n}$ are separated, and so get coarse moduli spaces for them by Keel-Mori.

We want to discuss what we did last as well as how to generalize things. Throughout, assume $\mathcal{X}$ is finite type over a noetherian ring. The next two (subsub)sections best viewed side-by-side, but figuring out how to do that live sounds like a bad idea...

12.1.1 DM case of Keel-Mori

Theorem 12.4. *If $\mathcal{X} \to \mathcal{X} \times \mathcal{X}$ is unramified (i.e. $\mathcal{X}$ DM) and finite (i.e. $\mathcal{X}$ separated), then there exists a coarse moduli space $\mathcal{X} \to X$.*

Proof Sketch. Let $x : \operatorname{spec} k \to \mathcal{X}$ be a geometric point with closed image.

(1) Let $(U, u) \xrightarrow{sm} (\mathcal{X}, x)$ be a smooth neighborhood of our point. Then, "slice" U to arrange that we have $U \to \mathcal{X}$ étale with a Cartesian diagram

See proof that unramified diagonal gives DM

$$\begin{tikzcd} \operatorname{spec} k \arrow[r, "u"] \arrow[d] & U \arrow[d, "\text{ét}"] \\ BG_x \arrow[r] & \mathcal{X} \end{tikzcd}$$

(2) Let $W \subset (U/\mathcal{X})^d := U \times_{\mathcal{X}} \ldots \times_{\mathcal{X}} U$ be the complement of the diagonal(s). Note that $S_d \curvearrowright W$, and $[W/S_d]$ parameterizes diagrams

$$\begin{tikzcd} & U_S \arrow[r] \arrow[d] \arrow[dr, phantom, "\ulcorner", very near start] & S \arrow[d] \\ Z \arrow[ur, hook, "\text{closed}"] \arrow[r, "\substack{\text{fin, ét}\\ \deg = d}"'] & S \arrow[r] & \mathcal{X} \end{tikzcd}$$

We showed that there's a map $[W/S_d] \to \mathcal{X}$ sending $w \mapsto x$ which is étale, representable, and identifies $\mathrm{Aut}(w) = \mathrm{Aut}(x)$. Finally, shrink to arrange W to be affine and replace S_d with G_x. We know $\exists [W/G_x] \xrightarrow{\text{ét}, repr} \mathcal{X}$ with W affine.

(3) Show that $[\operatorname{spec} A/G] \to \operatorname{spec} A^G$ is a coarse moduli space (note A^G is global sections of $[\operatorname{spec} A/G]$).

(4) Use $I_{\mathcal{X}} \to \mathcal{X}$ finite to arrange that $[\operatorname{spec} A/G_x] \to \mathcal{X}$ preserves all stabilizers.

(5) Show

$$\begin{tikzcd} \mathcal{X} \arrow[r, "f"] \arrow[d, "cms"'] & \mathcal{Y} \arrow[d, "cms"] \\ X \arrow[r, "\overline{f}"] & Y \end{tikzcd}$$

with $\mathcal{X}, \mathcal{Y}$ DM, if f étale and preserves stabilizer groups, then $\overline{f}$ is étale and the diagram is Cartesian.

(6) Find $W = [\operatorname{spec} A/G] \xrightarrow{\text{ét}} \mathcal{X} \ni x$. Get diagram

$$\begin{tikzcd} W \times_{\mathcal{X}} W \arrow[r, shift left] \arrow[r, shift right] \arrow[d] & W = [\operatorname{spec} A/G] \arrow[r, two heads, "\text{ét}"] \arrow[d, "cms"] & \mathcal{X} \ni x \arrow[d, dashed, "cms"] \\ R \arrow[r, shift left] \arrow[r, shift right] & \operatorname{spec} A^G = V \arrow[r, dashed] & V/R \end{tikzcd}$$

with $R \rightrightarrows \operatorname{spec} A^G =: V$ an étale equivalence relation. Hence, V/R is an algebraic space. This is the space we're after.

■

12.1.2 General Keel-Mori Theorem

Theorem 12.5. *If inertia $I_{\mathcal{X}} \to \mathcal{X}$ is finite, then there exists a coarse moduli space $\mathcal{X} \to X$.*

Recall 12.6. Inertia is given by the pullback square

$$\begin{tikzcd} I_{\mathcal{X}} \arrow[r] \arrow[d] & \mathcal{X} \arrow[d] \\ \mathcal{X} \arrow[r] & \mathcal{X} \times \mathcal{X} \end{tikzcd}$$

so $\mathcal{X}$ separated (and DM?) $\implies I_{\mathcal{X}} \to \mathcal{X}$ is finite.

Proof Sketch. Let $x : \operatorname{spec} k \to \mathcal{X}$ be a geometric point with closed image.

(1) Same argument as **(1)** of DM case can be used to arrange a Cartesian diagram

$$\begin{tikzcd} \operatorname{spec} k \arrow[r] \arrow[d] & U \arrow[d] \\ BG_x \arrow[r] & \mathcal{X} \end{tikzcd}$$

with $U \to \mathcal{X}$ quasi-finite and flat.

(2) Consider relative Hilbert scheme $\mathcal{H}$ parameterizing

$$\begin{tikzcd} & U_S \arrow[r] \arrow[d] & U \arrow[d] \\ Z \arrow[ur, hook] \arrow[r, "\substack{\text{fin, flat}\\ \deg = d}"'] & S \arrow[r] & \mathcal{X} \end{tikzcd}$$

(right square Cartesian). Universal family gives

$$\begin{tikzcd} & U_{\mathcal{X}} \arrow[r] \arrow[d] & U \arrow[d] \\ W \arrow[ur, hook] \arrow[r, "\text{fin, flat}"'] & \mathcal{H} \arrow[r, "\text{ét}", "\text{repr}"'] & \mathcal{X} \end{tikzcd}$$

with W a scheme, and you can arrange for it to be affine. Get $w \mapsto x$ with $\operatorname{Aut}(w) \xrightarrow{\sim} \operatorname{Aut}(x)$.

Know $\exists$

$$\begin{tikzcd} W \arrow[r, equal] \arrow[d, "\substack{\text{fin}\\ \text{flat}}"] & \text{affine} \\ \mathcal{H} \arrow[r, "\text{ét,repr}"] & \mathcal{X} \end{tikzcd}$$

(3) Take $R = W \times_{\mathcal{X}} W \rightrightarrows W = \operatorname{spec} A \to \mathcal{H}$, a finite flat groupoid. Show that $\mathcal{H} \to \operatorname{spec} \Gamma(\mathcal{H}, \mathscr{O}_{\mathcal{H}}) = A^R$ is a coarse moduli space.

(4) Same as **(4)** in DM case

(5) Same as **(5)** in DM case.

(6) Same as **(6)** in DM case.

■

12.2 Refresher on smooth curves

Definition 12.7. A **curve** over a field k is a pure 1-dimensional scheme C of finite type over k. If C is proper, then we define its **genus** to be $g = g(C) = h^1(X, \mathscr{O}_X)$.

Fact.

- If C is proper, then C is projective.
- If C is a separated algebraic space of dimension 1, then C is a scheme. We may never actually use this second fact, but good to know.

Theorem 12.8 (**Easy Riemann-Roch**). *Let C be an integral projective curve of genus g. If L is a line bundle on C, then*

$$\chi(C, L) = \deg L + 1 - g.$$

Theorem 12.9 (**Serre-Duality**). *If C is a smooth projective curve over k, then Ω_C is a **dualizing sheaf**, i.e. there is a linear map $\mathrm{tr} : \mathrm{H}^1(C, \Omega_C) \to k$ such that for any coherent sheaf $\mathscr{F}$, the natural pairing*

$$\mathrm{Hom}(\mathscr{F}, \Omega_C) \times \mathrm{H}^1(C, \mathscr{F}) \to \mathrm{H}^1(C, \Omega_C) \xrightarrow{\mathrm{tr}} k$$

is perfect.

Corollary 12.10 (**Riemann-Roch**). *In the setting of easy Riemann-Roch,*

$$h^0(L) - h^0(\omega_C \otimes L^{-1}) = \deg L + 1 - g.$$

Corollary 12.11. *Let C be a smooth, projective curve over k, and let L be a line bundle on C. Then,*

(1) $\deg L < 0 \implies h^0(C, L) = 0$

(2) $\deg L > 0 \implies$ *L ample*

(3) $\deg L \geq 2g \implies$ *L basepoint free*

(4) $\deg L \geq 2g + 1 \implies$ *L very ample*

Assume C is geometrically connected over k. If $g(C) \geq 2$, then

- $h^0(C, \omega_C) = h^1(C, \mathscr{O}_C) = g$
- $h^1(C, \omega_C) = h^0(C, \mathscr{O}_C) = 1$
- *RR applies to ω_C gives $\deg \omega_C = 2g - 2$ so the canonical bundle is ample.*

For $k > 1$

- $h^0(C, \omega_C^{\otimes k}) = (2k - 1)(g - 1)$ *and* $h^1(C, \omega_C^{\otimes k}) = 0$.
- $\deg \omega_C^{\otimes k} = k(2g - 2)$
- $\omega_C^{\otimes k}$ *is very ample if $k \geq 3$*

12.3 Families of smooth curves

Definition 12.12. A **family of smooth curves of genus** g over a scheme S is a smooth and proper morphism $\mathcal{C} \to S$ of schemes s.t. every geometric fiber is a connected curve of genus g.

Note any such family has a sheaf $\Omega_{\mathcal{C}/S}$ of relative differentials, and for $s \in S$, one has $\Omega_{\mathcal{C}/S}|_{\mathcal{C}_s} = \Omega_{\mathcal{C}_s/\kappa(s)}$.

Proposition 12.13. *Let $\mathcal{C} \to S$ be a family of smooth curves of genus $g \geq 2$. Then, for $k \geq 3$, $\Omega_{\mathcal{C}/S}^{\otimes k}$ is relatively very ample, and $\pi_* \Omega_{\mathcal{C}/S}^{\otimes k}$ is a vector bundle of rank $(2k-1)(g-1)$.*

Uses this to show $\mathcal{M}_g$ is a stack.

12.4 Nodal curves

Definition 12.14. Let C be a curve over an arbitrary field k.

(1) If $k = \overline{k}$, then $p \in C$ is a **node** is

$$\widehat{\mathscr{O}}_{C,p} \cong k\llbracket x, y \rrbracket / (xy),$$

where

$$\widehat{\mathscr{O}}_{C,p} := \varprojlim \mathscr{O}_{C,p}/\mathfrak{m}_p^n$$

is the completion of the local ring.

(2) In general, $g \in C$ is a **node** if there exists a node $p' \in C_{\overline{k}}$ over p.

Exercise. Say $p \in C$ is a node and C a curve over k. Then, $\kappa(p)$ is separable over k.

Exercise. If $p \in C$ is a node over k, then there exists a finite, separable field extension $k \hookrightarrow k'$ and a point $p' \in C_{k'}$ over p such that

$$\widehat{\mathscr{O}}_{C_{k'},p'} \cong k' \llbracket x, y \rrbracket / (xy).$$

Note that happens over the algebraic closure by definition; this is saying only need a finite, separable extension.

Exercise. If $p \in C$ is a node, then you form a diagram

$$C \xleftarrow{\text{ét}} U \xrightarrow{\text{ét}} \operatorname{spec} k[x,y]/(xy) \subset \mathbb{A}^2$$

and a point $u \in U$ such that

$$p \longleftarrow\!\shortmid\, u \longmapsto 0.$$

We'll later see a relative version of this.

12.4.1 Genus

Let C be a connected, nodal[29], projective curve over $k = \overline{k}$. Let $p_1, \ldots, p_\delta \in C$ be the nodes, let $C_1, \ldots, C_\nu$ be the irreducible components of C, and let $g_i = g(\widetilde{C}_i)$ be the genus of the normalization of C_i.

Example. See the slides for this class.

Note that the normalization of C is $\widetilde{C} = \bigsqcup \widetilde{C}_i$; let $\pi : \widetilde{C} \to C$ denote the normalization map. This is birational (iso away from nodes), so we get an exact sequence

$$0 \longrightarrow \mathscr{O}_C \longrightarrow \pi_* \mathscr{O}_{\widetilde{C}} \longrightarrow \bigoplus_i \kappa(p_i) \longrightarrow 0.$$

[29] By a 'nodal curve' we mean each point is either smooth or a node. This implies that it is geometrically reduced

This induces the long exact sequence

$$0 \to \underbrace{\mathrm{H}^0(C, \mathscr{O}_C)}_{1} \to \underbrace{\mathrm{H}^0(\widetilde{C}, \mathscr{O}_{\widetilde{C}})}_{\nu} \to \underbrace{\bigoplus_i \kappa(p_i)}_{\delta} \to \underbrace{\mathrm{H}^1(C, \mathscr{O}_C)}_{g} \to \underbrace{\mathrm{H}^1(\widetilde{C}, \mathscr{O}_{\widetilde{C}})}_{\sum g_i} \to 0$$

with the numbers under each group denoting its dimension. Thus,

$$g = \sum g_i + \delta - \nu + 1.$$

12.4.2 Nodal Curve

A nodal curve C is a local complete intersection, so it has a dualizing sheaf ω_C (e.g. see Hartshorne's chapter on Serre duality). We can also get an explicit description of ω_C.

Consider $\pi : \widetilde{C} \to C$ the normalization, and let $\Sigma = C^{sing}$ be the set of nodes. Let $\widetilde{\Sigma} := \pi^{-1}(\Sigma)$ be the points above the nodes. If $z_i \in \Sigma$, we let p_i, q_i be its two preimages in $\widetilde{C}$. Consider

$$0 \longrightarrow \Omega_{\widetilde{C}} \longrightarrow \Omega_{\widetilde{C}}(\widetilde{\Sigma}) \longrightarrow \mathscr{O}_{\widetilde{\Sigma}} \longrightarrow 0,$$

and note that

$$\mathscr{O}_{\widetilde{\Sigma}} = \bigoplus_{y \in \widetilde{\Sigma}} \kappa(y).$$

Remark 12.15. You can interpret $\Omega_{\widetilde{C}}(\widetilde{\Sigma})$ as the sheaf of meromorphic/rational sections of Ω with a pole of order ≤ 1 at each point above a node. The map $\Omega_{\widetilde{C}}(\widetilde{\Sigma}) \to \bigoplus_{y \in \widetilde{\Sigma}} \kappa(y)$ above just sends such a differential to its residues at the points $y \in \widetilde{\Sigma}$.

Definition 12.16. The subsheaf $\omega_C \subset \pi_* \Omega_{\widetilde{C}}(\widetilde{\Sigma})$ is defined on $V \subset C$ as

$$\Gamma(V, \omega_C) = \left\{ s \in \Gamma\left(\pi^{-1}(V), \Omega_{\widetilde{C}}(\widetilde{\Sigma})\right) \middle| \forall z_i \in \Sigma : \mathrm{res}_{p_i}(s) + \mathrm{res}_{q_i}(s) = 0 \right\}.$$

Remark 12.17. This definition gives rise to the two exact sequences

$$0 \longrightarrow \omega_C \longrightarrow \pi_* \Omega_{\widetilde{C}}(\widetilde{\Sigma}) \longrightarrow \bigoplus_{z_i \in \Sigma} \kappa(z_i) \longrightarrow 0,$$

where the second map is $s \mapsto (\mathrm{res}_{p_i}(s) - \mathrm{res}_{q_i}(s))_i$, and

$$0 \longrightarrow \pi_* \Omega_{\widetilde{C}} \longrightarrow \omega_C \longrightarrow \bigoplus_{z_i \in k} \kappa(z_i) \longrightarrow 0$$

with latter map $s \mapsto (\mathrm{res}_{p_i}(s))_i$.

Example. Let $C = \operatorname{spec} k[x,y]/(xy) \subset \mathbb{A}^2$, the union of the two axes. It's normalization is $\widetilde{C} = \operatorname{spec}(k[x] \times k[y])$, two copies of $\mathbb{A}^1$. Let $\pi : \widetilde{C} \to C$ be the normalization map. Then, there's one node $\Sigma = \{0\} \subset C$, and we let $\widetilde{\Sigma} = \{p, q\}$ be its preimages. Note that $k[x,y]/(xy) \hookrightarrow k[x] \times k[y]$ is the subring consisting of $(f(x), g(y))$ such that $f(0) = g(0)$. Note that

$$\nu = \left(\frac{\mathrm{d}x}{x}, -\frac{\mathrm{d}y}{y}\right) \in \Gamma(C, \omega_C)$$

(residue at x is 1 and residue at y is -1, so this is a global section). Say we have

$$\left(f(x)\frac{\mathrm{d}x}{x}, -g(y)\frac{\mathrm{d}y}{y}\right) \in \Gamma(C, \omega_C)$$

is a global section (all of this form, since poles at worse simple). Since the residues are negation of each other, we see that $f(0) = g(0)$. From this, one sees that

$$\left(f(x)\frac{\mathrm{d}x}{x}, -g(y)\frac{\mathrm{d}y}{y}\right) = (f(0) + f(x) + g(y)) \cdot \nu.$$

Thus, $\omega_C \cong \mathscr{O}_C$ with generator ν.

The importance of the above example is that (by a previous exercise), étale-locally, all nodes look like that.

Exercise. If $f : C' \xrightarrow{\text{ét}} C$ is étale, then $f^*\omega_C \cong \omega_{C'}$.

As a consequence, ω_C is always a line bundle on a nodal curve.

Exercise. Use Serre-Duality for smooth curves to show that ω_C is a dualizing sheaf as we defined it above.

12.4.3 Local Structure of Nodes

Recall 12.18 (**Local structure of smooth points**). Let $\mathcal{C} \to S$ be a smooth family of curves. Then, for any $p \in \mathcal{C}$, there exists a diagram

$$\begin{tikzcd} \mathcal{C} \arrow[d] & \mathcal{C}' \arrow[l, "\text{ét}"'] \arrow[r, "\text{ét}"] \arrow[d] & \mathbb{A}^1_{A'} \arrow[dl] \\ S & S' \arrow[l, "\text{ét}"] & \end{tikzcd}$$

(square not necessarily Cartesian). That is, étale-locally this looks like $\mathbb{A}^1$. This is generally true just for smooth morphisms. Above, there exists $p' \in \mathcal{C}'$ mapping to $p \in \mathcal{C}$ and to $0 \in \mathbb{A}^1_{S'}$.

Theorem 12.19 (**Local Structure of Nodes**). *Let $\pi : \mathcal{C} \to S$ be a flat of finite presentation map s.t. every fiber is a curve. If $p \in \mathcal{C}$ is a node in a fiber $\mathcal{C}_s$, there exists*

$$\begin{tikzcd} (\mathcal{C}, p) \arrow[d] & (\mathcal{U}, u) \arrow[l, "\acute{e}t"'] \arrow[r, "\acute{e}t"] \arrow[d] & \left(\frac{\operatorname{spec} A[x,y]}{(xy-f)}, (s', 0)\right) \arrow[dl] \\ (S, s) & (\operatorname{spec} A, s') \arrow[l, "\acute{e}t"'] & \end{tikzcd}$$

where $f \in A$ is a function vanishing at s'.

Remark 12.20. The diagonal map above is a basechange of

$$\begin{tikzcd} & \operatorname{spec} A \arrow[d] \\ \operatorname{spec} \mathbb{Z}[x,y,t]/(xy-t) \arrow[r] & \operatorname{spec} \mathbb{Z}[t] \end{tikzcd}$$

where $A \ni f \mapsto t \in \mathbb{Z}[t]$.

We end with a proof sketch.

Proof Sketch.

(step 1) Reduce to case that S is finite type over $\mathbb{Z}$ using absolute noetherian approximation.

(step 2) Reduce to case where

$$\widehat{\mathscr{O}}_{\mathcal{C}_s,p} \cong k(s)\,[\![x,y]\!]\,/(xy)$$

Apply earlier exercise to show there exists a separable field extension $k(s) \to k'$ and a point $p' \in C_{k'}$ with above property. Then choose an étale neighborhood $(S', s') \xrightarrow{\text{ét}} (S, s)$ s.t. $k(s) \to k(s')$ is the map $k(s) \to k'$ we want.

(step 3) Show

$$\widehat{\mathscr{O}}_{\mathcal{C},p} \cong \widehat{\mathscr{O}}_{S,s}\,[\![x,y]\!]\,/(xy-f)$$

for $f \in \widehat{\mathfrak{m}}_s$ using formal deformation theory. Use Schlessinger's (up to spelling) theorem applied to local deformation functor of a node; this says that given $\mathcal{C}_s \to \operatorname{spec} k(s)$ (say $k(s) = k$ for convenience) with a deformation, i.e. Cartesian

$$\begin{tikzcd} \mathcal{C}_s \arrow[r] \arrow[d] & \mathcal{D} \arrow[d, "flat"] \\ \operatorname{spec} k \arrow[r] & \operatorname{spec} B \end{tikzcd}$$

(with B local artinian or complete), then both vertical maps above are pulled back from $\operatorname{spec} k\,[\![t,x,y]\!]\,/(xy-t) \to \operatorname{spec} k\,[\![t]\!]$. We apply this to

$$\begin{tikzcd} \mathcal{C}_s \arrow[r] \arrow[d] & \mathcal{C} \times_S \widehat{\mathscr{O}}_{S,s} \arrow[d] \\ \operatorname{spec} k \arrow[r] & \operatorname{spec} \widehat{\mathscr{O}}_{S,s} \end{tikzcd}$$

This gives

$$\begin{tikzcd} \mathcal{C}_s \arrow[r] \arrow[d] & \mathcal{C} \times_S \widehat{\mathscr{O}}_{S,s} \arrow[r] \arrow[d] & \operatorname{spec} k\,[\![t,x,y]\!]\,/(xy-t) \arrow[d] \\ \operatorname{spec} k \arrow[r] & \operatorname{spec} \widehat{\mathscr{O}}_{S,s} \arrow[r] & \operatorname{spec} k\,[\![t]\!] \end{tikzcd}$$

with both square Cartesian. This finishes the step.

(step 4) At this stage, we know
$$\widehat{\mathscr{O}}_{\mathcal{C},p} \cong \widehat{\mathscr{O}}_{S,s}[\![x,y]\!]/(xy-f)$$
for some $f \in \widehat{\mathfrak{m}}_s$. We now apply Artin Approximation (Theorem 2.3). Define the functor

$$\begin{array}{cccc} F: & \mathrm{Sch}/S & \longrightarrow & \mathrm{Set} \\ & (T \to S) & \longmapsto & \left\{ \begin{tikzcd} \mathcal{C}_s \arrow[d] & C_T \arrow[l] \arrow[r] \arrow[d] & \operatorname{spec}\mathbb{Z}[t,x,y]/(xy-t) \arrow[d] \\ S & T \arrow[l] \arrow[r] & \operatorname{spec}\mathbb{Z}[t] \end{tikzcd} \right\} \end{array}$$

We constructed an element $\widehat{\xi} \in F(\widehat{\mathscr{O}}_{S,s})$ in the previous step. Artin approximation (with $N = 2$) then gives us our desired diagram. To check that the diagonal map is étale, show that it induces an iso on completions[30].

■

13 Lecture 13 (2/24): Stable Curves

Note 8. See slides for a picture of a 3-pointed stable curve of genus 14.

13.1 Review of nodes

Definition 13.1. Let C be a curve over k.

- If $K = \overline{k}$, then $p \in C$ is a *node* if $\widehat{\mathscr{O}}_{C,p} \cong k[\![x,y]\!]/(xy)$.
- In general, $p \in C$ is a *node* if there exists a node $p' \in C_{\overline{k}}$ over p

Example. $0 \in C = \operatorname{spec}\mathbb{R}[x,y]/(x^2+y^2)$ is a node. After $\mathbb{R} \to \mathbb{C}$, node becomes 'split' (completion is what you want).

Definition 13.2. Say C/k is a **nodal curve** if all $p \in C$ are either smooth or nodal.

Theorem 13.3 (**Local Structure of Nodes**). *Let $\pi : \mathcal{C} \to S$ be a flat of finite presentation map s.t. every fiber is a curve. If $p \in \mathcal{C}$ is a node in a fiber $\mathcal{C}_s$, there exists*

$$\begin{tikzcd} (\mathcal{C},p) \arrow[d] & (\mathcal{U},u) \arrow[l, "\acute{e}t"'] \arrow[r, "\acute{e}t"] \arrow[d] & \left(\frac{\operatorname{spec} A[x,y]}{(xy-f)}, (s',0)\right) \arrow[dl] \\ (S,s) & (\operatorname{spec} A, s') \arrow[l, "\acute{e}t"'] & \end{tikzcd}$$

where $f \in A$ is a function vanishing at s'.

Remark 13.4. When $S = \operatorname{spec} k$, the theorem implies that there's a finite, separable extension $k \to k'$ and a point $p' \in C_{k'}$ such that $\widehat{\mathscr{O}}_{C_{k'},p'} \cong k'[\![x,y]\!]/(xy)$.

[30]a surjective endomorphism of noetherian rings is an isomorphism.

Example. Consider the family

$$\operatorname{spec} \mathbb{Z}[x,y,t]/(xy-t) \to \operatorname{spec} \mathbb{Z}[t].$$

When $t=0$, fiber looks like union of axes; away from 0, the fibers look like hyperbolas which are smooth.
Theorem says that every deformation of a node étale-locally is the pullback of this example.

We can get a variant of the theorem when $\mathcal{C}_s$ is non-reduced and $p \in (\mathcal{C}_s)_{\text{red}}$ is nodal.

Exercise. Let R be a dvr with uniformizer $t \in R$. Suppose $\mathcal{C} \to \operatorname{spec} R$ is flat of finite presentation, and $\mathcal{C}$ is regular.

(1) If $p \in (\mathcal{C}_0)_{\text{red}}$ smooth, there exists $R \to R'$ étale and $\operatorname{spec} R'[x,y]/(x^a - t) \xrightarrow{\text{ét}} \mathcal{C}$ sending $0 \mapsto p$.

(2) If $p \in (\mathcal{C}_0)_{\text{red}}$ is a node, then there exists $R \to R'$ étale and $\operatorname{spec} R'[x,y]/(x^a y^b - t) \xrightarrow{\text{ét}} \mathcal{C}$ with $0 \mapsto p$.

13.2 Applications

What are some applications of the local structure theorem?

Corollary 13.5. *Let $\pi : \mathcal{C} \to S$ be as in the theorem. Then,*

$$\mathcal{C}^{\leq node} := \left\{p \in \mathcal{C} : p \in \mathcal{C}_{\pi(p)} \text{ is smooth or a node}\right\} \overset{open}{\subset} \mathcal{C}.$$

Proof. We know the smooth locus is open. If $p \in \mathcal{C}_s$ is a node ($s = \pi(p)$), apply structure theorem to get $p \in g(\mathcal{U}) \overset{\text{open}}{\subset} \mathcal{C}^{\leq \text{node}}$, where $g : (\mathcal{U}, u) \xrightarrow{\text{ét}} (\mathcal{C}, p)$ is the map guaranteed by the theorem. ∎

Corollary 13.6. *If in addition, $\pi : \mathcal{C} \to S$ is proper, then*

$$S^{\leq node} := \{s \in S : \mathcal{C}_s \text{ nodal}\} \overset{open}{\subset} S.$$

Proof. $S^{\leq \text{node}} = S \setminus \pi(\mathcal{C} \setminus \mathcal{C}^{\leq \text{node}})$ and proper maps are closed. ∎

We will apply this result next time to show that the stack $\mathcal{M}_g^{\leq \text{nodal}}$ of nodal curves is algebraic; the objects here are maps $\mathcal{C} \to S$ which are flat, proper and finitely presented with fibers $\mathcal{C}_s$ which are all nodal curves.

There are problems with this stack, even though it is algebraic. It is not separated, and note bounded (i.e. not finite type). This comes from valuative criterion. The point is if you have a smooth family over $\operatorname{spec} R$ (a dvr), then you can blowup a node in the special fiber to get a different family (of nodal curves) over $\operatorname{spec} R$ extending the family over the generic fiber. This new family will still have nodes, so you can keep blowing up to get infinitely many limits of the family over $\operatorname{spec} \operatorname{Frac} R$.

13.3 Stable curves

Definition 13.7. An ***n*-pointed curve** $\mathcal{C}$ over k is a curve C along with an ordered set of rational points $p_1, \ldots, p_n \in C(k)$.

Definition 13.8. Say $q \in C$ is **special** if q is marked or a node.

Definition 13.9. An n-pointed curve $(C, p_1, \dots, p_n)$ over k is a **stable curve** if C is connected, nodal, projective curve, and $p_1, \dots, p_n \in C$ are *distinct smooth* points such that

A reference for much of what's in this lecture (and presumably later ones) is this paper

(1) every smooth rational subcurve $\mathbb{P}^1 \subset C$ contains at least 3 special points; and

(2) C is not of genus 1 without marked points.

Example. See lecture slides.

Remark 13.10. There are no stable curves if

$$(g, n) = (0,0), (0,1), (0,2) \text{ or } (1,0) \iff 2g - 2 + n \leq 0.$$

So sometimes we impose $2g - 2 + n > 0$.

Definition 13.11. A **semistable curve** C is a stable curve except we've replace the number '3' with '2' in condition **(1)**. It is **prestable** if neither condition **(1)** nor condition **(2)** are required, i.e. prestable = connected, nodal, projective + distinct, smooth marked points.

Let $(C, p_1, \dots, p_n)$ be an n-pointed prestable curve over k. Take normalization $\widetilde{C} \xrightarrow{\pi} C$, and we let $\widetilde{p_i} \in \widetilde{C}$ be the unique preimage of $p_i \in C$. We denote $\pi^{-1}(C^{sing}) = \{\widetilde{q}_1, \dots, \widetilde{q}_m\}$, the points over nodes.

Exercise. Show $(C, \{p_i\})$ is stable $\iff$ every connected component of $\left(\widetilde{C}, \{\widetilde{p}_i\}, \{\widetilde{q}_i\}\right)$, the **pointed normalization**, is stable.

Example. *See lecture slides*

Fact. The only smooth n-pointed curves $(C, p_1, \dots, p_n)$ with $\#\operatorname{Aut}(C, \{p_i\}) = \infty$ are

- $C = \mathbb{P}^1$ with $n = 0, 1, 2$.
- $g = 1$ and $n = 0$.

Note these are the same curves (those satisfying $2g - 2 + n \leq 0$) we saw earlier.

Proposition 13.12. *Let $(C, \{p_i\})$ be an n-pointed prestable curve. Then, TFAE*

(1) *$(C, \{p_i\})$ is stable*

(2) *$\operatorname{Aut}(C, \{p_i\})$ is finite*

(3) *$\omega_C(p_1 + \cdots + p_n)$ is ample*

Proof. (**(1)** $\iff$ **(2)**) follows from previous fact + exercise before it.

(**(1)** $\iff$ **(3)**) We need a property of the dualizing sheaf not mentioned last time.

Fact. If C is a nodal curve, and $T \subset C$ is a subcurve, then $\omega_C|_T = \omega_T(T \cap T^c)$, where T^c is the closure of $C \setminus T$.

Therefore, $\omega_C(p_1 + \cdots + p_n)$ ample $\iff$ $\pi^*\omega_C(p_1 + \cdots + p_n)$ is ample $\iff$ $\forall T \subset \widetilde{C}$, $\omega_C(p_1 + \cdots + p_n)|_T = \omega_T\left(\sum_{p_i \in T} p_i + T \cap T^c\right)$ (is ample?) $\iff$ $\forall T \subset \widetilde{C}$ the pointed curve $\left(T, \sum_{p_i \in T} p_i + T \cap T^c\right)$ is stable. ■

Exercise. If $(C, \{p_i\})$ is stable, then for $k \geq 3$,

$$\omega_C(p_1 + \cdots + p_n)^{\otimes k}$$

is very ample.

Hint(assuming $n = 0$ for simplicity): Need to show $\omega_C^{\otimes k}$ separates points and tangent vectors. That is,

- for all $x, y \in C$
$$\mathrm{H}^0(C, \omega_C^{\otimes k}) \twoheadrightarrow (\omega_C^{\otimes k} \otimes \kappa(x)) \otimes \left(\omega_C^{\otimes k} \otimes \kappa(y)\right).$$

- for all $x \in C$
$$\mathrm{H}^0(C, \omega_C^{\otimes k}) \twoheadrightarrow \omega_C^{\otimes k} \otimes \mathscr{O}_{C,x}/\mathfrak{m}_x^2.$$

Both of these come from the exact sequence

$$0 \to \omega_C^{\otimes k} \otimes \mathfrak{m}_x\mathfrak{m}_y \to \omega_C^{\otimes k} \to \omega_C^{\otimes k} \otimes \mathscr{O}_{C,x}/\mathfrak{m}_x\mathfrak{m}_y \to 0,$$

so suffices to show $\mathrm{H}^1(C, \omega_C^{\otimes k} \otimes \mathfrak{m}_x\mathfrak{m}_y) = 0$ for all $x, y \in C$ (possibly equal). By Serre duality, this is cohomology group is $\mathrm{Hom}(\mathfrak{m}_x\mathfrak{m}_y, \omega_C^{\otimes k})$. Show this vanishes with case analysis of $x, y \in C$ being nodes or smooth.

13.4 Families of stable curves

Definition 13.13.

(1) A **family of n-pointed nodal curves** is a flat, proper and finitely presented morphism $\mathcal{C} \to S$ of schemes with n sections $\sigma_1, \ldots, \sigma_n : S \to \mathcal{C}$ such that every geometric fiber is a (reduced) connected nodal curve.

(2) A **family of n-pointed stable curves** is a family $\mathcal{C} \to S$ of n-pointed nodal curves such that every geometric fiber $(\mathcal{C}_s, \sigma_1(s), \ldots, \sigma_n(s))$ is stable.

Can similarly define familes of semi- or pre-stable curves.

Definition 13.14. We define $\overline{\mathcal{M}}_{g,n}$ as the prestack over Sch with

- objects $(\mathcal{C} \to S, \sigma_1, \ldots, \sigma_n)$ families of n-pointed stable curves (this is an object over S).
- morphism = cartesian diagrams preserving the sections.

Fact. If $\mathcal{C} \to S$ is prestable (or even just nodal), then $\mathcal{C} \to S$ is a local complete intersection. This is enough to guarantee the existence of a relative dualizing sheaf $\omega_{\mathcal{C}/S}$. This is e.g. in Hartshorne's 'Residue and Duality' book. It's also in Liu's 'AG and arith. curves'.

The main property you need to know is that if you have $T \to S$, then

$$\omega_{C/S}|_{C_T} = \omega_{C_T/T}.$$

In particular, $\omega_{\mathcal{C}/S}|_{\mathcal{C}_s} = \omega_{\mathcal{C}_s/\kappa(s)}$.

Proposition 13.15 (**Properties of Families of Stable Curves**). *Let $(\mathcal{C} \to S, \{\sigma_i\})$ be a family of n-pointed stable curves of genus g, and set $L := \omega_{\mathcal{C}/S}(\sum_i \sigma_i)$. If $k \geq 3$, then $L^{\otimes k}$ is relatively very ample, and $\pi_* L^{\otimes k}$ is a vector bundle of rank $(2k-1)(g-1) + kn$.*

Relative very ampleness can be checked on fibers, the pushforward being a vector bundles comes from cohomology and base change, and the rank is Riemann-Roch.

Proposition 13.16 (**Openness of Stability**). *Let $(\mathcal{C} \to S, \{\sigma_i\})$ be a family of n-pointed nodal curves. Then, the locus of points $s \in S$ such that $(\mathcal{C}_s, \{\sigma_i(s)\})$ is stable is open.*

Proof. First note that the locus of $s \in S$ where $\sigma_1(s), \ldots, \sigma_n(s)$ are distinct and smooth is open. Hence, we may assume $(\mathcal{C} \to S, \{\sigma_i\})$ is prestable. We now give two arguments.

(1) Consider $\underline{\mathrm{Aut}}(\mathcal{C}/S, \sigma_1, \ldots, \sigma_n) \to S$. This is a finite type group scheme with identity section $e : S \to \underline{\mathrm{Aut}}(\mathcal{C}/S, \sigma_1, \ldots, \sigma_n)$. Note that the map

$$s \mapsto \dim \mathrm{Aut}(\mathcal{C}_s, \{\pi_i(s)\})$$

is upper semi-continuous, so

$$\{s \in S : (\mathcal{C}_s, \{\sigma_i(s)\}) \text{ stable}\} = \{s \in S : \dim \mathrm{Aut}(\mathcal{C}_s, \{\sigma_i(s)\}) = 0\}$$

is open.

(2) Let $L = \omega_{\mathcal{C}/S}(\sigma_1 + \cdots + \sigma_n)$. Ampleness is also an open condition, so

$$\{s \in S : L_s \text{ ample on } \mathcal{C}_s\} = \{s \in S : (\mathcal{C}_s, \{\sigma_i(s)\}) \text{ stable}\}$$

is open.

■

13.5 Automorphisms, deformations and obstructions

The automorphisms, deformations, and obstructions of a stable curve C are governed by $\mathrm{Ext}^i(\Omega_C, \mathscr{O}_C)$ for $i = 0, 1, 2$.

Proposition 13.17. *Let $(C, p_1, \ldots, p_n)$ be an n-pointed stable curve of genus g over k. Then,*

$$\dim_k \mathrm{Ext}^i\left(\Omega_C\left(\sum_i p_i\right), \mathscr{O}_C\right) = \begin{cases} 0 & \text{if } i = 0 \\ 3g - 3 + n & \text{if } i = 1 \\ 0 & \text{if } i = 2 \end{cases}$$

Remark 13.18. We will later apply

- $\mathrm{Ext}^0 = 0 \implies \overline{\mathcal{M}}_{g,n}$ is DM.
- $\mathrm{Ext}^2 = 0 \implies \overline{\mathcal{M}}_{g,n}$ is smooth.

- $\dim_k \operatorname{Ext}^1 = 3g-3+n \implies \dim \overline{\mathcal{M}}_{g,n} = 3g-3+n$ over a field.

Proof of Proposition 13.17. ($i=0$) We do the $n=0$ case for notational simplicity. Keep in mind the diagram

$$\begin{tikzcd} \widetilde{\Sigma} = \pi^{-1}(\Sigma) \arrow[d] & \subset & \widetilde{C} \arrow[d, "\pi"] \\ \Sigma & \subset & C \end{tikzcd}$$

where $(\widetilde{C}, \widetilde{\Sigma})$ is the pointed normalization (Σ the set of nodes in C).

Claim 13.19. $\operatorname{Hom}(\Omega_C, \mathscr{O}_C) = \operatorname{Hom}(\Omega_{\widetilde{C}}(\widetilde{\Sigma}), \mathscr{O}_{\widetilde{C}})$.

That is, regular vector fields on C are the same as regular vector fields on $\widetilde{C}$ vanishing on preimages of nodes. We skip the proof of this claim.[31]

Note $(\widetilde{C}, \widetilde{\Sigma})$ is stable and $\Omega_{\widetilde{C}}$ is a line bundle. Hence,

$$\operatorname{Hom}(\Omega_{\widetilde{C}}(\widetilde{\Sigma}), \mathscr{O}_{\widetilde{C}}) = \mathrm{H}^0(T_{\widetilde{C}}(-\widetilde{\Sigma})) = 0,$$

where we've used that $(\widetilde{C}, \widetilde{\Sigma})$ is stable to see that the degree of the bundle above is < 0 (hence no global sections). This finishes the $i=0$ case.

($i=2$) Use the local-to-global spectral sequence

$$E_2^{p,q} = \mathrm{H}^p(C, \mathscr{E}xt^q(\Omega_C, \mathscr{O}_C)) \implies \operatorname{Ext}^{p+q}(\Omega_C, \mathscr{O}_C).$$

Since $\dim C = 1$, $E_2^{p,q} = 0$ for $p > 1$, so this is a two-column spectral sequence. We claim that $\mathscr{E}xt^1(\Omega_C, \mathscr{O}_C)$ has 0-dimensional support because Ω_C is a line bundle away from the nodes and (for $z \in C$) $\mathscr{E}xt^1(\Omega_C, \mathscr{O}_C)_z = \operatorname{Ext}^1(\Omega_{c,z}, \mathscr{O}_{C,z})$ vanishes when z smooth. Hence, $E_2^{1,1} = \mathrm{H}^1(C, \mathscr{E}xt^1(\Omega_C, \mathscr{O}_C)) = 0$. Now we compute $E_2^{0,2}$. Since C is a locally complete intersection, there exists a locally free resolution

$$0 \longrightarrow E_1 \longrightarrow E_0 \longrightarrow \Omega_C \longrightarrow 0.$$

This tells us that $E_2^{0,2} = \mathscr{E}xt^2(\Omega_C, \mathscr{O}_C) = 0$. Thus, the 2-diagonal on the E_2-page vanishes, so $\operatorname{Ext}^2(\Omega_C, \mathscr{O}_C) = 0$ as desired.

($i=1$) The low degree exact sequence of the local-to-global spectral sequence looks like

$$\begin{tikzcd} 0 \arrow[r] & E_2^{1,0} \arrow[r] \arrow[d, equal] & \operatorname{Ext}^1(\Omega_C, \mathscr{O}_C) \arrow[r] & E_2^{0,1} \arrow[r] \arrow[d, equal] & E_2^{2,0} \arrow[d, equal] \\ & \mathrm{H}^1(\mathscr{H}om(\Omega_C, \mathscr{O}_C)) & & \mathrm{H}^0(\mathscr{E}xt^1(\Omega_C, \mathscr{O}_C)) \arrow[d, equal] & 0 \\ & & & \prod_{z\in\Sigma} \operatorname{Ext}^1(\Omega_{C,z}, \mathscr{O}_{C,z}) \arrow[d, equal] & \\ & & & \prod_{z\in\Sigma} \operatorname{Ext}^1(\widehat{\Omega}_{C,z}, \widehat{\mathscr{O}}_{C,z}) & \end{tikzcd}$$

[31] Use knowledge of dualizing sheaf?

We know $\mathrm{Ext}^1(\Omega_C, \mathscr{O}_C)$ classifies 1st order deformations of C, i.e. Cartesian diagrams

$$\begin{tikzcd} C \arrow[r] \arrow[d] & \mathcal{C} \arrow[d, "flat"] \\ \operatorname{spec} k \arrow[r] & \operatorname{spec} k[\varepsilon]/(\varepsilon^2) \end{tikzcd}$$

Similarly, $\mathrm{Ext}^1(\widehat{\Omega}_{C/z}, \widehat{\mathscr{O}}_{C,z})$ classifies first order deformations of $\widehat{\mathscr{O}}_{C,z}$ (note $\widehat{\Omega}_{C,z} = \Omega_{\widehat{\mathscr{O}}_{C,z}}$), and the image of a deformation of C is

$$\begin{tikzcd} \operatorname{spec} \widehat{\mathscr{O}}_{C,z} \arrow[r] \arrow[d] & \operatorname{spec} \widehat{\mathscr{O}}_{\mathcal{C},z} \arrow[d] \\ \operatorname{spec} k \arrow[r] & \operatorname{spec} k[\varepsilon] \end{tikzcd}$$

This let's us identity the kernel with $\mathrm{Ext}^1(\Omega_{\widetilde{C}}(\widetilde{\Sigma}), \mathscr{O}_{\widetilde{C}})$, so the sequence looks like

$$0 \to \mathrm{Ext}^1(\Omega_{\widetilde{C}}(\widetilde{\Sigma}), \mathscr{O}_{\widetilde{C}}) \to \mathrm{Ext}^1(\Omega_C, \mathscr{O}_C) \to \prod_{z \in \Sigma} \mathrm{Ext}^1(\Omega_{\widehat{\mathscr{O}}_{C,z}}, \widehat{\mathscr{O}}_{C,z}) \to 0.$$

Each Ext group on the right is 1-dimensional appartently, so the quotient has dimension equal to the number of nodes. On the left, write $\widetilde{C} = \bigsqcup \widetilde{C}_i$ and $\widetilde{\Sigma}_i = \widetilde{\Sigma} \cap \widetilde{C}_i$, so the dimension of the kernel is

$$\begin{aligned} \sum_i \mathrm{Ext}^1(\Omega_{\widetilde{C}_i}(\widetilde{\Sigma}_i), \mathscr{O}_{\widetilde{C}_i}) &= \sum_i h^1(T_{\widetilde{C}_i}(-\widetilde{\Sigma}_i)) \\ &= \sum_i h^0(\Omega_{\widetilde{C}_i}^{\otimes 2}(\widetilde{\Sigma}_i)) \\ &= \sum_i \left(\deg\left(\Omega_{\widetilde{C}_i}^{\otimes 2}(\widetilde{\Sigma}_i)\right) + 1 - \widetilde{g}_i\right) \\ &= \sum_i \left(3\widetilde{g}_i - 3 + \#\widetilde{\Sigma}_i\right) \\ &= 3\sum_i \widetilde{g}_i - 3\#\mathrm{comp} + 2\#\mathrm{nodes} \end{aligned}$$

(every node has 2 preimages in the normalization). Doing one final addition, this shows that

$$\dim \mathrm{Ext}^1(\Omega_C, \mathscr{O}_C) = 3\sum_i \widetilde{g}_i - 3\#\mathrm{comp} + 3\#\mathrm{nodes} = 3g - 3,$$

where we've used the genus formula

$$g = \sum \widetilde{g}_i - \#\mathrm{comp} + \#\mathrm{nodes} + 1.$$

■

14 Lecture 14 (3/1): The stack of all curves

It's been a while since we reminded ourselves of the goal of this class.

Goal. The moduli space $\overline{\mathcal{M}}_g$ of stable curves of genus $g \geq 2$ is a smooth, proper and irreducible Deligne-

Mumford stack of dimension $3g-3$ which admits a projective coarse moduli space.

Where are we?

- we've introduced stable curves, and so defined the prestack $\overline{\mathcal{M}}_g$.
- We almost know that
  - $\overline{\mathcal{M}}_g$ is a DM stack which is smooth over $\operatorname{spec}\mathbb{Z}$ of rel. dim $3g-3$ (we secretly just need to show that it is algebraic).
  - there exists a coarse moduli space $\overline{\mathcal{M}}_g \to \overline{M}_g$ (need to know $\overline{\mathcal{M}}_g$ is separated to apply our version of Keel-Mori)

Remark 14.1. We could show that $\overline{\mathcal{M}}_g$ is algebraic by replicating the Hilbert scheme proof we used for $\mathcal{M}_g$, but we do something else instead.

14.1 Six steps towards projective moduli

This is a general plan of attack we want to quickly outline.

Setup. Let $\mathcal{X}$ be the moduli stack of interest, and let $\mathcal{M} \supset \mathcal{X}$ be some "enlargement."

Example. Take $\mathcal{X} = \overline{\mathcal{M}}_g$ and $\mathcal{M} = \mathcal{M}_g^{all}$.

Definition 14.2. Call $x \in \mathcal{M}(k)$ **stable** if $x \in \mathcal{X}(k)$.

The six steps are

(Step 1) Algebraicity: show $\mathcal{M}$ is an algebraic stack locally of finite type over the base. This is today's goal in our case of interest.

(Step 2) Openness of stability: given $E \in \mathcal{M}(T)$, show that

$$\left\{t \in T : E|_{\operatorname{spec}\kappa(t)} \text{ stable}\right\} \subset T \text{ is open.}$$

This implies that $\mathcal{X}$ is algebraic and locally of finite type.

Example. For $\overline{\mathcal{M}}_g \subset \mathcal{M}_g^{all}$, this will follow from openess of nodal locus and openess of stability for a family of nodal curves.

(Step 3) Boundedness of stability: show $\mathcal{X}$ is of finite type ($\iff$ quasi-compact)

Example. For us, this will follow from $\mathcal{C} \to S$ stable $\implies \omega_{\mathcal{C}/S}^{\otimes 3}$ is very ample.

(Step 4) Existence of coarse moduli space: Show $\exists \mathcal{X} \xrightarrow{cms} X$ (e.g. show $\mathcal{X}$ is separated and DM).

(Step 5) Stable reduction: show $\mathcal{X}$ is proper ($\implies$ X is proper). We'll do this next time (probably).

(Step 6) Projectivity: show X is projective. This will be our final lecture, following Kollár.

14.2 Recap on stable curves

Recall 14.3. An n-pointed curve $(C, p_1, \dots, p_n)$ over k (recall $p_i \in C(k)$) is a *stable curve* if C is connected, nodal, projective curve, and $p_1, \dots, p_n \in C$ are *distinct smooth* points such that

(1) every smooth rational subcurve $\mathbb{P}^1 \subset C$ contains at least 3 special (i.e. nodal or marked) points; and

(2) C is not of genus 1 without marked points.

Proposition 14.4. *Let $(C, p_1, \dots, p_n)$ be an n-pointed stable curve of genus g over k. Then,*

$$\dim_k \operatorname{Ext}^i\left(\Omega_C\left(\sum_i p_i\right), \mathscr{O}_C\right) = \begin{cases} 0 & \text{if } i = 0 \\ 3g - 3 + n & \text{if } i = 1 \\ 0 & \text{if } i = 2 \end{cases}$$

Recall 14.5.

(1) A *family of n-pointed nodal curves* is a flat, proper and finitely presented morphism $\mathcal{C} \to S$ of schemes with n sections $\sigma_1, \dots, \sigma_n : S \to \mathcal{C}$ such that every geometric fiber is a (reduced) connected nodal curve.

(2) A *family of n-pointed stable curves* is a family $\mathcal{C} \to S$ of n-pointed nodal curves such that every geometric fiber $(\mathcal{C}_s, \sigma_1(s), \dots, \sigma_n(s))$ is stable.

Proposition 14.6 (**Properties of Families of Stable Curves**)**.** *Let $(\mathcal{C} \to S, \{\sigma_i\})$ be a family of n-pointed stable curves of genus g, and set $L := \omega_{\mathcal{C}/S}(\sum_i \sigma_i)$. If $k \geq 3$, then $L^{\otimes k}$ is relatively very ample, and $\pi_* L^{\otimes k}$ is a vector bundle of rank $(2k-1)(g-1) + kn$.*

Proposition 14.7 (**Openness of Stability**)**.** *Let $(\mathcal{C} \to S, \{\sigma_i\})$ be a family of n-pointed nodal curves. Then, the locus of points $s \in S$ such that $(\mathcal{C}_s, \{\sigma_i(s)\})$ is stable is open.*

14.3 More on stability: contraction morphisms

Definition 14.8. Let $(C, p_1, \dots, p_n)$ be an n-pointed prestable curve. We say that a smooth rational subcurve $E \cong \mathbb{P}^1 \subset C$ is

- a **rational tail** if $E \cap E^c = 1$ and E contains no marked pointed.
- a **rational bridge** if either $\#E \cap E^c = 2$ and E contains no marked points, or $\#E \cap E^c = 1$ and E contains one marked point.

Recall 14.9. E^c is the union of irreducible components other than E.

Note that $(C, \{p_i\})$ is stable $\iff \nexists$ rational tails or bridges, and it is semistable $\iff \nexists$ rational tails.

We would like to be able to contract the rational tails and bridges.

Definition 14.10. If $(C, \{p_i\})$ is prestable, its **contraction/stable model** is the proper curve C^{st} obtained by contracting all rational tails and bridges E_i.

Remark 14.11. In some simple cases, one has

$$C^{st} = \overline{C \setminus \bigcup E_i}$$

If $\pi : C \to C^{st}$ is the natural map and $p_i \mapsto p_i'$, then $(C^{st}, p_1', \ldots, p_n')$ is stable.

Example. See slides for examples.

Can we do this in families? Yes, but it requires a lot of work. The end result is the following.

Proposition 14.12 (**Stable Models in Families**). *If $(\mathcal{C} \to S, \sigma_1, \ldots, \sigma_n)$ is a family of prestable curves, there exists a map $\pi : \mathcal{C} \to \mathcal{C}^{st}$ over S such that*

(1) $(\mathcal{C}^{st} \to S, \{\sigma_i'\})$ *is a faimly of stable curves where* $\sigma_i' = \pi \circ \sigma_i$*;*

(2) for all $s \in S$*,* $(\mathcal{C}_s, \{\sigma_i(s)\}) \to (\mathcal{C}_s^{st}, \{\sigma_i'(s)\})$ *is the map contracting rational tails and bridges;*

(3) $\mathscr{O}_{\mathcal{C}^{st}} = \pi_* \mathscr{O}_{\mathcal{C}}$ *and* $R^1\pi_*\mathscr{O}_{\mathcal{C}} = 0$ *(and this remains true after base change); and*

(4) If $\mathcal{C} \to S$ *is semistable, then* $\omega_{\mathcal{C}/S}(\sum_i \sigma_i) = \pi^*\omega_{\mathcal{C}^{st}/S}(\sum_i \sigma_i')$*.*

Example. Say $\mathcal{C} \to S$ is a family of prestable curves with $\mathcal{C}_s$ stable for $s \neq 0$ and $\mathcal{C}_0 = E + E^c$ (with E rational). Then, $0 = \mathcal{C}_0 \cdot E = E^2 + E^c \cdot E$. If E is a rational bridge, then $E^2 = -1$. If E is a rational bridge, then $E^2 = -2$. Can contract by Castelnuovo (spelling) maybe, I think.

Vague Proof Sketch. (local to global) See Stacks tag 0E8A

- Use noetherian approximation to reduce to S finite type over $\mathbb{Z}$
- Show uniqueness of $\mathcal{C}^{st} \to S$
- Given $s \in S$, there exists $\mathcal{C}_s \to \mathcal{C}_s^{st} = Y_0$. Something about infinitesimal thickenings

$$\mathcal{C}_s = \mathcal{C}_0 \hookrightarrow \mathcal{C}_1 \hookrightarrow \mathcal{C}_2 \hookrightarrow \ldots$$

  and a Cartesian diagram

$$\begin{tikzcd} \widehat{\mathcal{C}} \arrow[r] \arrow[d] & \mathcal{C} \arrow[d] \\ \operatorname{spec} \widehat{\mathscr{O}}_{S,s} \arrow[r] & S \end{tikzcd}$$

- Use deformation theory to extend $\mathcal{C}_0 \to Y_0$ to $\mathcal{C}_n \to Y_n$
- Algebraize to $\widehat{\mathcal{C}} \to \widehat{Y}$ (formal schemes?)
- Artin approximation gives $\exists S' \xrightarrow{\text{ét}} S$ and $\mathcal{C}_{S'} \to Y'$.
- Use uniqueness to descend to a map $\mathcal{C} \to Y$.

■

14.4 Stack of all curves

We begin by redefining our notion of curve.

Definition 14.13. A **curve** is a 1-dimensional scheme C of finite type over a field k. In particular, we *do not assume* pure dimension 1 or connected.

Example. Consider $\mathbb{P}^1 \to \mathbb{P}^3$ given by $[x, y] \mapsto [x^3, x^2y, xy^2, ty^3]$. As $t \mapsto 0$, you get a plane nodal curve with an embedded point at the origin with degenerates (has as a deformation?) to a reduced nodal curve with a disjoint rational point.

Definition 14.14. Let S be a scheme.

- A **family of curves over** S is a flat, proper and finitely presented morphism $\mathcal{C} \to S$ of *algebraic spaces* such that every fiber is a curve.
- A **family of** n**-pointed curves over** S is a family of curves $\mathcal{C} \to S$ with n (arbitrary) sections $\sigma_1, \ldots, \sigma_n : S \to \mathcal{C}$.

Remark 14.15 (Fulghesu). There exists a family of genus 0 nodal curves $\mathcal{C} \to S$ with S a smooth projective surface and $\mathcal{C}$ a smooth 3 dimensional algebraic space which is not a scheme. Can even have generic fiber $\mathcal{C}_{k(S)} = \mathbb{P}^1_{k(S)}$.

Remark 14.16 (Raynaud). There exists a family of smooth genus 1 curves $\mathcal{C} \to S$ with S a normal surface and $\mathcal{C}$ not a scheme (so $\mathcal{M}_1$ defined way back when is not a stack).

Remember: Families of genus 1 curves do not always glue, as schemes

Remark 14.17. If $\mathcal{C} \to S$ is a stable family, then $\omega_{\mathcal{C}/S}$ is ample, so $\mathcal{C}$ is projective over S (and in particular, a scheme).

Proposition 14.18. *If $\mathcal{C} \to S$ is a family of curves, then $\exists S' \twoheadrightarrow S$ étale s.t. $\mathcal{C}_{S'} \to S'$ is projective.*

We give two sketches

- (first sketch: local to global) Same sort of picture as last time

$$\begin{tikzcd}
\mathcal{C}_s = \mathcal{C}_0 \arrow[r, hook] \arrow[d] & \mathcal{C}_1 \arrow[r, hook] \arrow[d] & \ldots \arrow[r, hook] & \widehat{\mathcal{C}} \arrow[r] \arrow[d] & \mathcal{C} \arrow[d] & \\
\operatorname{spec} \kappa(s) \arrow[r, hook] & \operatorname{spec} \widehat{\mathscr{O}}_{S,s}/\mathfrak{m}_s^2 \arrow[r, hook] & \ldots \arrow[r, hook] & \operatorname{spec} \widehat{\mathscr{O}}_{S,s} \arrow[r] & S & \ni \quad s
\end{tikzcd}$$

We proceed in cases/steps

(1) ($\mathcal{C}_0$). Need to use the following.

Fact. Separated, 1-dimensional algebraic spaces are schemes. Furthermore, proper 1-dimensional schemes are projective.

This gives us a line bundle L_0 of $\mathcal{C}_s = \mathcal{C}_0$ which is ample.

(2) ($\mathcal{C}_n = \mathcal{C} \times_S \operatorname{spec} \mathscr{O}_{S,s}/\mathfrak{m}_s^{n+1}$) Deformation theory says the obstructions to deforming line bundle L_n on $\mathcal{C}_n$ to L_{n+1} on $\mathcal{C}_{n+1}$ live in $\mathrm{H}^2(\mathcal{C}_0, \mathscr{O}_{\mathcal{C}_0}) = 0$, so get L_n on $\mathcal{C}_n$ for all n.

(3) ($\widehat{\mathcal{C}}$ over $\widehat{\mathscr{O}}_{S,s}$) Assume $\widehat{\mathscr{O}}_{S,s}$ is complete, local and noetherian. Use **Grothendieck Existence Theorem**[32]

$$\operatorname{Coh}(\widehat{\mathcal{C}}) \xrightarrow{\sim} \varprojlim \operatorname{Coh}(\mathcal{C}_n)$$

(need to generalize to proper algebraic spaces over a complete local noetherian ring) "If you remember how you show Grothendieck's existence theorem, which you probably don't..." Rough strategy is to show it in projective case (use niceties of $\mathscr{O}(1)$) and then reduce to projective case using Chow's lemma (smth smth devissage at some point smth smth).

This gives a line bundle $\widehat{L}$ mapping to the compatible family of line bundles L_n constructed in previous step.

(4) (S f.type $/\mathbb{Z}$) Apply Artin approximation to

$$\begin{array}{ccc} \operatorname{Sch}/S & \longrightarrow & \operatorname{Set} \\ (T \to S) & \longmapsto & \operatorname{Pic}(\mathcal{C}_T) \end{array}$$

(5) (S general) Use noetherian approximation.

- (second sketch: explicitly extend line bundle). Say we have

$$\begin{tikzcd} \mathcal{C}_s \arrow[r] \arrow[d] & \mathcal{C} \arrow[d] \\ \operatorname{spec} k(s) \arrow[r] & S \end{tikzcd}$$

with L_0 an ample line bundle on $\mathcal{C}_s$. Let's assume all fibers are generically reduced ($\implies$ generically smooth). Let $\mathcal{C}^0 \subset \mathcal{C}$ be the smooth locus, and note that $\mathcal{C}^0 \twoheadrightarrow S$ is smooth and surjective. Choose $p_1, \dots, p_n \in \mathcal{C}_s$ s.t. every irreducible dimension 1 component contains some p_i. We can and do take $L_0 = \mathscr{O}_{\mathcal{C}_s}(p_1 + \cdots + p_n)$. Use étale-local structure of smooth theorems to get $S' \xrightarrow{\text{ét}} S$ and sections $\sigma_i : S' \to \mathcal{C}^0_{S'} \to \mathcal{C}^0$ extending p_i. Then, $\mathscr{O}_{\mathcal{C}}(\sigma_1 + \cdots + \sigma_n)$ is ample in an open neighborhood of s.

14.5 Algebraicity of the stack of all curves

Let $\mathcal{M}^{all}_{g,n}$ be the prestack with

- objects: $(\mathcal{C} \to S, \sigma_1, \dots, \sigma_n)$ families of curves
- morphisms: Cartesian diagrams compatible with sections.

Lemma 14.19. *$\mathcal{M}^{all}_{g,n}$ is a stack over* $\operatorname{Sch}_{\acute{E}t}$.

Proof. (We'll only do the case $n = 0$). Say we have

$$\begin{tikzcd} R = p_1^*\mathcal{C}' \arrow[r, shift left, "p_2 \circ \alpha"] \arrow[r, shift right, "p_1"'] \arrow[d] & \mathcal{C}' \arrow[d] \\ S' \times_S S' \arrow[r, shift left, "p_2"] \arrow[r, shift right, "p_1"'] & S' \arrow[r] & S \end{tikzcd}$$

[32]Need $\widehat{\mathcal{C}} \to \operatorname{spec} \widehat{\mathscr{O}}_{S,s}$ proper morphism over a complete local noetherian ring

Above, $\alpha : p_1^*\mathcal{C}' \xrightarrow{\sim} p_2^*\mathcal{C}'$. $R \rightrightarrows \mathcal{C}'$ is an étale equivalence relation, so we can simply form the quotient $\mathcal{C} = \mathcal{C}'/R$ which is an algebraic space over S. ∎

Lemma 14.20. *The diagonal*

$$\mathcal{M}_{g,n}^{all} \xrightarrow{\Delta} \mathcal{M}_{g,n}^{all} \times \mathcal{M}_{g,n}^{all}$$

is is representable ($\implies$ any map $S \to \mathcal{M}_{g,n}^{all}$ from a scheme is representable).

Proof. (We only do $n = 0$ case). Let T be a scheme fitting into the Cartesian diagram

$$\begin{tikzcd}
\underline{\mathrm{Isom}}_T(\mathcal{C}_1, \mathcal{C}_2) \arrow[r] \arrow[d] & T \arrow[d, "{(\mathcal{C}_1,\mathcal{C}_2)}"] \\
\mathcal{M}_g^{all} \arrow[r, "\Delta"] & \mathcal{M}_g^{all} \times \mathcal{M}_g^{all}
\end{tikzcd}$$

We need to show that $\underline{\mathrm{Isom}}_T(\mathcal{C}_1, \mathcal{C}_2)$ is an algebraic space. We first reduce to $\mathcal{C}_1, \mathcal{C}_2$ both projective. We know $\exists T' \overset{\text{ét}}{\twoheadrightarrow} T$ such that the base change $\mathcal{C}_{1,T'}, \mathcal{C}_{2,T'}$ are projective over T'. If $\underline{\mathrm{Isom}}_{T'}(\mathcal{C}_{1,T'}, \mathcal{C}_{2,T'})$ is an algebraic space, then use the Cartesian diagram

$$\begin{tikzcd}
\underline{\mathrm{Isom}}_{T'}(\mathcal{C}_{1,T'}, \mathcal{C}_{2,T'}) \arrow[r] \arrow[d] & T' \arrow[d] \\
\underline{\mathrm{Isom}}_T(\mathcal{C}_1, \mathcal{C}_2) \arrow[r] & T
\end{tikzcd}$$

to see that $\underline{\mathrm{Isom}}_T(\mathcal{C}_1, \mathcal{C}_2)$ is an algebraic space (left vertical map is étale, representable, and surjective).

So now we assume $\mathcal{C}_1, \mathcal{C}_2 \rightrightarrows T$ are projective. We have an inclusion of functors

$$\underline{\mathrm{Isom}}_T(\mathcal{C}_1, \mathcal{C}_2) \subset \underline{\mathrm{Mor}}_T(\mathcal{C}_1, \mathcal{C}_2) \subset \mathrm{Hilb}(\mathcal{C}_1 \times_T \mathcal{C}_2/T)$$

with latter map sending a morphism $(\mathcal{C}_1 \xrightarrow{\alpha} \mathcal{C}_2)$ to its graph $(\mathcal{C}_1 \xrightarrow{\Gamma_\alpha} \mathcal{C}_1 \times_T \mathcal{C}_2)$. We know the Hilbert scheme is projective, so suffices to show these inclusions are represented by open immersions.

So $\underline{\mathrm{Isom}}_T(\mathcal{C}_1, \mathcal{C}_2)$ is even a scheme (!) when $\mathcal{C}_1, \mathcal{C}_2 \rightrightarrows T$ are projective.

Fact. Given

$$\begin{tikzcd}
X \arrow[rr] \arrow[dr] & & Y \arrow[dl] \\
& T &
\end{tikzcd}$$

with $X \to T \leftarrow Y$ both proper, then there exists an open $T^0 \subset T$ such that for all $S \to T$,

$$X_S \xrightarrow{\sim} Y_S \iff S \to T \text{ factors through } T_0.$$

This facts implies $\underline{\mathrm{Isom}}_T(\mathcal{C}_1, \mathcal{C}_2) \subset \underline{\mathrm{Mor}}_T(\mathcal{C}_1, \mathcal{C}_2)$ is an open immersion. To know that $\underline{\mathrm{Mor}}_T(\mathcal{C}_1, \mathcal{C}_2) \subset \mathrm{Hilb}(\mathcal{C}_1 \times_T \mathcal{C}_2/T)$ is an open immersion, know that a subscheme $Z \subset \mathcal{C}_1 \times_T \mathcal{C}_2$ is the image of a map $\mathcal{C}_1 \to \mathcal{C}_2$ iff the composition $Z \hookrightarrow \mathcal{C}_1 \times_T \mathcal{C}_2 \xrightarrow{p_1} \mathcal{C}_1$ is an isomorphism. Hence, the fact implies that we again get an open immersion. ∎

Theorem 14.21. $\mathcal{M}_{g,n}^{all}$ *is an algebraic stack locally of finite type over* $\mathbb{Z}$

Proof. We start with some inductions.

- it suffices to assume $n = 0$.

  This is because $\mathcal{M}^{all}_{g,n+1} \to \mathcal{M}^{all}_{g,n}$ is the universal family, so the target being algebraic shows the source is

  Question: Why?

  Answer: If $U \to \mathcal{M}^{all}_{g,n}$ is a representable, smooth cover by an algebraic space, then so is its pullback to $\mathcal{M}^{all}_{g,n+1}$

- Suffices to show that for all projective curves $\mathcal{C}_0/k$, there exists a scheme U with smooth presentation $U \to \mathcal{M}^{all}_g$ w/ $[\mathcal{C}_0]$ in the image.

  Question: Why?

  Answer: I guess just cover by the disjoint union of all the U's

The argument now (I think)

- Choose embedding $\mathcal{C}_0 \hookrightarrow \mathbb{P}^N$ such that $h^1(\mathcal{C}_0, \mathscr{O}(1)) = 0$. Let $P(t)$ be the Hilbert polynomial, and consider the Hilbert scheme $H := \mathrm{Hilb}^P(\mathbb{P}^n_{\mathbb{Z}})$ (which is projective over $\mathbb{Z}$). There's a universal family

$$\begin{tikzcd}
\mathcal{C}_{h_0} \arrow[r, equal] & \mathcal{C}_0 \arrow[r, hook] \arrow[d, mapsto] & \mathcal{C} \arrow[r, hook] \arrow[d] & \mathbb{P}^N_H \arrow[dl] \\
& h_0 & \in \; H &
\end{tikzcd}$$

- Cohomology and base change implies that $\exists H' \overset{\text{open}}{\subset} H$ of h_0 such that for all $s \in H'$, $h^1(\mathcal{C}_s, \mathscr{O}(1)) = 0$.

- We have a map $H' \to \mathcal{M}^{all}_g$ sending $[C \subset \mathbb{P}^N] \mapsto [C]$.

  Claim 14.22. *$H' \to \mathcal{M}^{all}_g$ is smooth.*

  For this, use formal lifting criterion. Say $A \twoheadrightarrow A_0$ local artinian with residue field k s.t. $\ker(A \to A_0) = k$. The lifting criterion translates to

$$\begin{tikzcd}
\operatorname{spec} k \arrow[d, hook] \arrow[dr, "{[C \subset \mathbb{P}^N_k]}"] & \\
\operatorname{spec} A_0 \arrow[r, "{[\mathcal{C}_0 \subset \mathbb{P}^N_{A_0}]}"] \arrow[d, hook] & H' \arrow[d] \\
\operatorname{spec} A \arrow[r, "{[\mathcal{C}_0]}"] \arrow[ur, dashed, "{[\mathcal{C} \subset \mathbb{P}^N_A]}"] & \mathcal{M}^{all}_g
\end{tikzcd}$$

  This further translates to

$$\begin{tikzcd}
& \mathbb{P}^N_k & \mathbb{P}^N_{A_0} & \mathbb{P}^N_A \\
C \arrow[ur, hook] \arrow[r, hook] \arrow[d] & \mathcal{C}_0 \arrow[ur, hook] \arrow[r] \arrow[d] & \mathcal{C} \arrow[ur, dashed, hook] \arrow[d] & \\
\operatorname{spec} k \arrow[r, hook] & \operatorname{spec} A_0 \arrow[r, hook] & \operatorname{spec} A &
\end{tikzcd}$$

  Assumption (simplifying assumption)**.** Let's assume C is a local complete intersection.

  Over the central fiber, we have $C \hookrightarrow \mathbb{P}^N_k$ and some deformation of this $\mathcal{C}_0 \hookrightarrow \mathbb{P}^N_{A_0}$. We also have a deformation $\mathcal{C}$ of C over A, and we want to know if we can extend this into a deformation of the embedding $C \hookrightarrow \mathbb{P}^N_k$. Deformation theory tells us that such extensions are classified by

If C were smooth, then Ω_C would be a line bundle, and we'd have $\operatorname{Ext}^1(\Omega_C, \mathscr{O}_C)$ $\operatorname{Ext}^1(\mathscr{O}_C, \Omega_C^\vee)$ $\mathrm{H}^1(C, \mathcal{T}_C)$ where $\mathcal{T}_C = \Omega_C^\vee$ is the tangent bundle

$$\left\{ \text{extensions} \quad \begin{tikzcd} \mathcal{C}_0 \arrow[r, hook, dashed] \arrow[d] & \mathcal{C} \arrow[d, dashed] \\ \operatorname{spec} A_0 \arrow[r, hook] & \operatorname{spec} A \end{tikzcd} \right\} = \operatorname{Ext}^1(\Omega_C, \mathscr{O}_C)$$

and

$$\left\{ \text{extensions} \quad \begin{tikzcd} & \mathbb{P}^N_{A_0} & \mathbb{P}^N_A \\ \mathcal{C}_0 \arrow[ur, hook] \arrow[r, hook, dashed] \arrow[d] & \mathcal{C} \arrow[ur, dashed] \arrow[d, dashed] & \\ \operatorname{spec} A_0 \arrow[r, hook] & \operatorname{spec} A & \end{tikzcd} \right\} = \operatorname{Hom}(I/I^2, \mathscr{O}_C) = \mathrm{H}^0(\mathcal{N}_{C/\mathbb{P}^n}),$$

where $I = \mathscr{I}_{\mathbb{P}^N_k/C}$ is the ideal sheaf of $C \hookrightarrow \mathbb{P}^N_k$. Because we assume C is a local complete intersection, it is locally cut out by a regular sequence, and we even have an exact sequence

$$0 \longrightarrow I/I^2 \longrightarrow \Omega_{\mathbb{P}^n}|_C \longrightarrow \Omega_C \longrightarrow 0.$$

Applying $\operatorname{Hom}(-, \mathscr{O}_C)$ gives

$$\operatorname{Hom}(I/I^2, \mathscr{O}_C) \longrightarrow \operatorname{Ext}^1(\Omega_C, \mathscr{O}_C) \longrightarrow \operatorname{Ext}^1(\Omega_{\mathbb{P}^n}|_C, \mathscr{O}_C) = \mathrm{H}^1(C, T_{\mathbb{P}^n}|_C)$$

(the first map above takes an embedded deformation to an abstract deformation). Now, it suffices to show the first map above is surjective. To see this, we appeal to the **Euler sequence**

$$0 \longrightarrow \mathscr{O}_C \longrightarrow \mathscr{O}_C(1)^{\oplus(n+1)} \longrightarrow T_{\mathbb{P}^n}|_C \longrightarrow 0.$$

Taking cohomology of this shows that $\mathrm{H}^1(C, T_{\mathbb{P}^n}|_C) = 0$ (note we're using that we restricted to H' where $h^1(\mathscr{O}(1)) = 0$).

■

Remark 14.23. We have inclusions

$$\mathcal{M}_{g,n} \subset \overline{\mathcal{M}}_{g,n} \subset \mathcal{M}^{ss}_{g,n} \subset \mathcal{M}^{pre}_{g,n} \subset \mathcal{M}^{\leq \text{nodal}}_{g,n} \subset \mathcal{M}^{all}_{g,n}$$

We've just seen the right end is algebraic. Each of these is an open inclusion, so all of these are algebraic stacks.

- We showed nodal open in all last time (use étale descent to reduce to scheme case)
- prestable is open in nodal since it's the open locus where n sections are disjoint and smooth
- ss is the open locus in prestable where $\omega_{\mathcal{C}/S}$ is nef
- We've shown the stable locus is open last time

- Finally, $\mathcal{M}_{g,n} \subset \overline{\mathcal{M}}_{g,n}$ is the smooth locus

Remark 14.24. Contraction map gives a map

$$\begin{tikzcd} \mathcal{M}^{pre}_{g,n} \arrow[r] & \overline{\mathcal{M}}_{g,n} \\ \cup & \\ \overline{\mathcal{M}}_{g,n} \arrow[uur, equal] & \end{tikzcd}$$

It is true that $\overline{\mathcal{M}}_{g,n}$ is proper and that $\mathcal{M}^{pre}_{g,n}$ is universally closed (over $\mathbb{Z}$), but not separated.

15 Lecture 15 (3/3): Stable reduction

UW quarter ends next week, but we'll give 4 more lectures (including this one). The rought schedule is

- Today: stable reduction (4/3 lectures)
  - $\overline{\mathcal{M}}_{g,n}$ proper
- Monday: gluing and forgetful morphisms (2/3 lecture)
  - $\overline{\mathcal{M}}_{g_1,n_1} \times \overline{\mathcal{M}}_{g_2,n_2} \to \overline{\mathcal{M}}_{g_1+g_2,n_1+n_2-2}$
  - $\overline{\mathcal{M}}_{g,n} \to \overline{\mathcal{M}}_{g+1,n-2}$
  - $\overline{\mathcal{M}}_{g,n} \to \overline{\mathcal{M}}_{g,n-1}$ universal family
- Wednesday: irreducibility (1 lecture)
  - $\overline{\mathcal{M}}_{g,n}$ irreducible
- Monday after: projectivity (1 lecture)
  - The coarse moduli space $\overline{M}_{g,n}$ is projective

15.1 Recap

Recall 15.1. Let $(C, p_1, \ldots, p_n)$ be an n-pointed stable curve of genus g over k. Then,

$$\dim_k \operatorname{Ext}^i\left(\Omega_C\left(\sum_i p_i\right), \mathscr{O}_C\right) = \begin{cases} 0 & \text{if } i = 0 \\ 3g-3+n & \text{if } i = 1 \\ 0 & \text{if } i = 2 \end{cases}$$

Theorem 15.2. *If $2g-2+n>0$, then $\overline{\mathcal{M}}_{g,n}$ is a quasi-compact DM stack which is smooth over* $\operatorname{spec}\mathbb{Z}$ *of relative dimension $3g-3+n$>*

Proof.

- First we need to know $\overline{\mathcal{M}}_{g,n}$ is algebraic and locally of finite type over $\mathbb{Z}$.

  We showed last time that the stack $\mathcal{M}^{all}_{g,n}$ of all curves is algebraic and locally of finite type over $\mathbb{Z}$. We've also showed that $\overline{\mathcal{M}}_{g,n} \overset{\text{open}}{\subset} \mathcal{M}^{all}_{g,n}$ so it inherits algebraicity.

- We next need to know that $\overline{\mathcal{M}}_{g,n}$ is DM.

  For an n-pointed stable curve $(C, p_1, \dots, p_n)$, the abstract automorphism group $\mathrm{Aut}(C, \{p_i\})$ is finite when $(C, \{p_i\})$ is stable. We need more than this. We know that $T_{\mathfrak{C}}\underline{\mathrm{Aut}}(C, \{p_i\}) = \mathrm{Ext}^0(\Omega_C(\sum p_i), \mathscr{O}_C) = 0$, so the scheme $\underline{\mathrm{Aut}}(C, \{p_i\})$ is finite and reduced. By earlier characterization of DM stacks, we can now conclude $\overline{\mathcal{M}}_{g,n}$ is DM.

- $\overline{\mathcal{M}}_{g,n} \to \operatorname{spec} \mathbb{Z}$ is smooth.

  Use the formal lifting criterion. Consider diagrams

$$\begin{array}{ccc} \operatorname{spec} A_0 & \longrightarrow & \overline{\mathcal{M}}_{g,n} \\ \downarrow & & \downarrow \\ \operatorname{spec} A & \longrightarrow & \operatorname{spec} \mathbb{Z}. \end{array}$$

  We want to know that we can get a lifting $\operatorname{spec} A \to \overline{\mathcal{M}}_{g,n}$. We may assume $A \twoheadrightarrow A_0$ map of local artinian rings with kernel k. Hence, we have $\operatorname{spec} k \hookrightarrow \operatorname{spec} A_0$, and so a stable curve $(C, \{p_i\}) \in \overline{\mathcal{M}}_{g,n}(k)$. Deformation theory now tells us there's some element of $\mathrm{Ext}^2(\Omega_C(\sum p_i), \mathscr{O}_C)$ which vanishes iff a lift exists, but this whole group is 0, so we win.

- For a field k, $\dim(\overline{\mathcal{M}}_{g,n} \times_{\mathbb{Z}} k) = 3g - 3 + n$.

  Use deformation theory again to identify

$$T_{\overline{\mathcal{M}}_{g,n} \times_{\mathbb{Z}} k, [C, \{p_i\}]} \cong \mathrm{Ext}^1\left(\Omega_C(\sum_i p_i), \mathscr{O}_C\right)$$

- All that remains is boundedness: $\overline{\mathcal{M}}_{g,n}$ being quasi-compact

  We use the following fact.

  Fact. If $(C, \{p_i\})$ is stable, then $L := \omega_C(p_1 + \cdots + p_n)^{\otimes 3}$ is very ample, so gives an embedding $|L| : C \hookrightarrow \mathbb{P}^N$, where $N = h^0(L) - 1$.

  Let $P(t)$ be the Hilbert polynomial, and consider the Hilbert scheme $\mathrm{Hilb}^P(\mathbb{P}^N_{\mathbb{Z}})$ which is projective. In fact, consider more. We have n marked points $p_1, \dots, p_n$, so consider

$$H := \left\{(C \hookrightarrow \mathbb{P}^N, p_1, \dots, p_n) : p_i \in C\right\} \subset \mathrm{Hilb}^P(\mathbb{P}^N_{\mathbb{Z}}) \times (\mathbb{P}^N_{\mathbb{Z}})^n.$$

  We know that H is quasi-compact (since it's quasi-projective?). There's a natural map $f : H \to \overline{\mathcal{M}}^{all}_{g,n}$ sending $(C \hookrightarrow \mathbb{P}^N, p_i) \mapsto (C, p_i)$, and the fact tells us that its image contains $\overline{\mathcal{M}}_{g,n}$, so $\overline{\mathcal{M}}_{g,n}$ is quasi-compact.[33] ■

15.2 Overview of stable reduction

Goal. $\overline{\mathcal{M}}_{g,n} \to \operatorname{spec} \mathbb{Z}$ is proper.

[33] $\mathrm{Hilb}^p(\mathbb{P}^N_{\mathbb{Z}}) \times (\mathbb{P}^N_{\mathbb{Z}})^n$ is projective so noetherian so *all* its subspaces are quasi-compact, including $H \times_{\mathcal{M}^{all}_{g,n}} \overline{\mathcal{M}}_{g,n}$ (which is open subscheme since $\overline{\mathcal{M}}_{g,n} \hookrightarrow \mathcal{M}^{all}_{g,n}$ is an open immersion) which surjects onto $\overline{\mathcal{M}}_{g,n}$

Recall the valuative criterion for properness.

Theorem 15.3 (**Valuative Criterion for Properness**)**.** *Let $f : \mathcal{X} \to \mathcal{Y}$ be a finite type morphism of notherian algebraic stacks. Then, f is proper if and only if for every dvr R with fraction field K and 2-commutative diagram*

$$\begin{tikzcd} \operatorname{spec} K \arrow[r] \arrow[d] & \mathcal{X} \arrow[d, "f"] \\ \operatorname{spec} R \arrow[r] \arrow[ur, phantom, "\swArrow"] & \mathcal{Y} \end{tikzcd}$$

one has

(1) *there exists an extension $R \to R'$ of dvrs (with $K \to K'$ of fraction fields) together with a lifting*

$$\begin{tikzcd} \operatorname{spec} K' \arrow[r] \arrow[d] & \operatorname{spec} K \arrow[r] \arrow[d, dashed] & \mathcal{X} \arrow[d, "f"] \\ \operatorname{spec} R' \arrow[r] \arrow[urr, dashed] & \operatorname{spec} R \arrow[r] & \mathcal{Y}. \end{tikzcd}$$

(2) *any two liftings are isomorphic.*

Notation 15.4. Think of $\operatorname{spec} R =: \Delta$ as a unit disk, and $\operatorname{spec} K := \Delta^*$ as a punctured unit disk.

In the present case, existence becomes

Theorem 15.5 (**Stable Reduction**)**.** *If $(\mathcal{C}^* \to \Delta^*, s_1^*, \dots, s_n^*)$ is a family of n-pointed stable curves of genus g, then there exists a finite cover $\Delta' \to \Delta$ of spectra of dvrs, and a family $(\mathcal{C}' \to \Delta', s_1', \dots, s_n')$ of stable curves extending $\mathcal{C}^* \times_{\Delta^*} \Delta'^* \to \Delta'^*$.*

and uniqueness becomes

Theorem 15.6 (**Separatedness of $\overline{\mathcal{M}}_{g,n}$**)**.** *If $(\mathcal{C} \to \Delta, \sigma_1, \dots, \sigma_n)$ and $(\mathcal{D} \to \Delta, \tau_1, \dots, \tau_n)$ are families of n-pointed stable curves, then any isomorphism $\alpha^* : \mathcal{C}^* \to \mathcal{D}^*$ over Δ^* with $\tau_i^* = \alpha^* \circ \sigma_i^*$ of the generic fibers extends to a unique isomorphism $\alpha : \mathcal{C} \to \mathcal{D}$ over Δ with $\tau_i = \alpha \circ \sigma_i$.*

15.3 First examples

Example (Colliding marked points)**.** Let C be smooth, projective and connected curve over K. Consider $C \times \Delta \xrightarrow{p_2} \Delta$ with sections $\sigma_1, \sigma_2, \sigma_3$ given (étale locally) by $t \mapsto (t^2, -t^2, 4t)$ (e.g. $\sigma_2(t) = -t^2$). Hence, $(\mathcal{C}, \sigma_1, \sigma_2, \sigma_3)$ is stable away from the identity $t = 0$.

See slides for details Here you end up blowing up the intersection point in the central fiber which introduces a rational bridge, so then you contract that bridge, and then you win.

Example (Node degenerating to a cusp)**.** Suppose we have a family $\mathcal{C} \to \Delta$ with local equations $y^2 = x^3 + tx^2$. Here, $\Delta = \operatorname{spec} R$ and $t \in R$ a uniformizer. The central fiber is a cusp $y^2 = x^3$, while when $t \neq 0$ you get a node $y^2 = x^2(x + t)$. What is the stable limit?

These examples are hard to write notes for... I'm just gonna drop a link to the course website which has slides/videos/notes: `https://sites.math.washington.edu/ jarod/math582C.html`. This example also appears at the beginning of section 3.C of Harris-Morrison's 'Moduli of Curves' book.

15.4 Stable reduction: basic strategy

(Reference: Harris and Morrison)

We're given a stable curve $\mathcal{C}^\times \to \Delta^\times = \operatorname{spec} K \hookrightarrow \operatorname{spec} R = \Delta$.

(Step 1) Find some extension $\mathcal{C} \to \Delta$ to a *flat* family with possible very singular central fiber $\mathcal{C}_0$.

Consider embedding $\left|\omega_{\mathcal{C}^\times}^{\otimes 3}\right| : \mathcal{C}^\times \hookrightarrow \mathbb{P}_K^N$. We can let $\mathcal{C} := \overline{\mathcal{C}^\times} \subset \mathbb{P}_R^N$, the closure/scheme theoretic image of $\mathcal{C}^\times$ under $\mathbb{P}_K^N \hookrightarrow \mathbb{P}_R^N$. The map $\mathcal{C} \to \Delta = \operatorname{spec} R$ is flat.

Question: Why?

Answer: See e.g. Proposition III.9.8 in Hartshorne

(Step 2) Reduce to the case where the generic fiber $\mathcal{C}^\times \to \Delta^\times$ is smooth.

The idea is that if $\mathcal{C}^\times$ has k nodes, take the pointed normalization $(\widetilde{\mathcal{C}^\times}, \widetilde{p}_1, \ldots, \widetilde{p}_{2k})$. Can perform stable reduction to each component, and then take nodal union of sections (glue them back together). For this to work, need pointed case.

(Step 3) Use embedded resolutions. There exists

$$\begin{tikzcd} \widetilde{\mathcal{C}} \arrow[rr, "\text{proj}", "\text{birational}"'] \arrow[dr] & & \mathcal{C} \arrow[dl, "\substack{\text{generically}\\ \text{smooth}}"] \\ & \Delta & \end{tikzcd}$$

s.t. $\widetilde{\mathcal{C}} \to \mathcal{C}$ is a finite sequence of blow-ups at closed points, $\widetilde{\mathcal{C}}$ is regular, and $\widetilde{\mathcal{C}}_0$ is/has (?) set-theoretic normal crossings (i.e. $(\widetilde{\mathcal{C}}_0)_{\text{red}}$ nodal).

Note 9. There was an example here I missed.

(Step 4) Take ramified base extension $\Delta' = \operatorname{spec} R \to \Delta = \operatorname{spec} R$, $t \mapsto t^m$ such the central fiber of the normalization $\widetilde{\widetilde{\mathcal{C}} \times_\Delta \Delta}$ is reduced and nodal.

(Step 5) Take minimal resolution and contract rational tails/bridges.

We'll give more details on steps 4/5.

15.5 Birational geometry of surfaces

Slogan. Understanding the moduli of n-dimensional varieties (specifically, stable reduction) requires birational geometry and minimal model program in dimension $n+1$.

Definition 15.7. A **surface** will be an integral scheme of finite type over $k = \overline{k}$ of dimension 2.

(References: Hartshorn Ch. V and Kollár 'Lectures on Resolutions of Singularities')

Theorem 15.8 (**Embedded Resolutions**). *Let X be a surface and $X_0 \subset X$ be a curve. There is a projective birational morphism $\widetilde{X} \to X$ obtained as a finite sequence of blow-ups at reduced points of X_0 yielding such that $\widetilde{X}$ is smooth and the preimage $\widetilde{X}_0$ of X_0 has set-theoretic normal crossings, i.e. $(\widetilde{X}_0)_{\text{red}}$ is nodal.*

Theorem 15.9 (**Minimal Resolutions**). *Let X be a surface. There exists a unique projective birational morphism $\widetilde{X} \to X$ from a smooth surface s.t. every other resolution $Y \to X$ factors as $Y \to \widetilde{X} \to X$.*

Theorem 15.10 (**Castelnuovo's Contraction Theorem**). *Let X be a smooth projective surface, and E a smooth rational curve with $E^2 = -1$. Then there is a projective birational morphism $X \to Y$ to a smooth projective surface and a point $y \in Y$ s.t. $X_y = E$ and $X \setminus E \to Y \setminus \{y\}$ is an isomorphism.*

Corollary 15.11 (**Existence of Relative Minimal Models**). *A smooth surface X admits a projective birational morphism $X \to X_{min}$ to a smooth surface such that every projective birational morphism $X_{min} \to Y$ to a smooth surface is an isomorphism. In particular, X_{min} has no smooth rational (-1)-curves.*

All of these are in Hartshorne.

15.6 Stable reduction (in characteristic zero)

Assume R is a dvr over $\mathbb{Q}$. We're given a stable curve $\mathbb{C}^\times \to \Delta^\times = \operatorname{spec} K$, and we write $\Delta = \operatorname{spec} R$.

(Steps 1–3) Reduce to $\mathcal{C}^\times \to \Delta^\times$ smooth, find some limit, and apply embedded resolutions.

This gives $\mathcal{C} \to \Delta$ with $(\mathcal{C}_0)_{\text{red}}$ nodal and $\mathcal{C}$ regular.

(Step 4) Take ramified base extension

$$\begin{array}{ccc} \mathcal{C}' & \longrightarrow & \mathcal{C} \\ \downarrow & & \downarrow \\ \Delta & \xrightarrow{t \mapsto t^m} & \Delta \end{array}$$

s.t. central fiber of the normalization $\widetilde{\mathcal{C}}'$ is reduced and nodal.

Can take m to be the lcm of the multiplicities of the components.

Recall 15.12. We have an étale-local description of $\mathcal{C} \to \Delta$ near $p \in \mathcal{C}_0$.

(1) If $p \in (\mathcal{C}_0)_{\text{red}}$ is smooth, we get something looking like $(x, y) \mapsto x^a$, i.e. central fiber looks like $x^a = 0$ near p.

(2) If $p \in (\mathcal{C}_0)_{\text{red}}$ is nodal, get something looking like $(x, y) \mapsto x^a y^b$, i.e. central fiber looks like two thick irred components $x^a y^b = 0$ near p.

For **(1)**, $p \in \mathcal{C}$ has local equation $x^a - t$ ($t \in R$ uniformizer local coordinate of base), and there exists a unique preimage $p' \in \mathcal{C}' = \mathcal{C} \times_{\Delta, t \mapsto t^m} \Delta$ which has local equation given by $x^a - t^m$. We took m s.t. $a \mid m$, so this factors

$$x^a - t^m = \prod_{i=0}^{a-1} \left(x - \rho^i t^{m/a} \right),$$

where ρ a primitive ath root of unity. Since we're in char. 0, this is reduced.[34] Now let $\widetilde{\mathcal{C}}' \to \mathcal{C}'$ be the normalization. This has a preimage $\widetilde{p}' \mapsto p'$ with local equation $x - \rho^i t^{m/a}$, one of these factors. Hence, $\widetilde{p}' \in \widetilde{\mathcal{C}}'_0$ is smooth?

For **(2)**, $p \in \mathcal{C}$ has local equation $(x, y) \mapsto x^a y^b$.

[34] In char. p, if you take a ramified base change $t \mapsto t^p$, things don't improve

Exercise. Under $\widetilde{\mathcal{C}}' \to \mathcal{C}$, each preimage $\widetilde{p}'$ of p has a local equation of the form $t^k = xy$.

This tells us that $\widetilde{p}' \in \widetilde{\mathcal{C}}'_0$ is reduced and nodal, and it tells you that $\widetilde{p} \in \widetilde{\mathcal{C}}'$ is an A_{k-1}-singularity.

The upshot is we have $\widetilde{\mathcal{C}}' \to \Delta$ a nodal family whose total space may have singularities.

(Step 5) Take a minimal resolution of $\mathcal{C}$ (what we called $\widetilde{\mathcal{C}}'$ in the previous step) to get $\mathcal{C} \to \Delta$ a prestable family with regular total space. Now, take the stable model by contracting rational tails and bridges to get

$$\begin{tikzcd} \mathcal{C} \arrow[rr] \arrow[dr] & & \mathcal{C}^{st} \arrow[dl] \\ & \Delta & \end{tikzcd}$$

Explicitly, we get $\mathcal{C}^{st}$ by contracting rational tails (by Castelnuovo) which gives the relative minimal model $\mathcal{C} \to \mathcal{C}_{min}$ with $\mathcal{C}_{min} \to \Delta$ semistable family with regular total space (could stop here and call it semistable reduction), and then now contract rational bridges.

Remember: Rational tails are (-1)-curves, and rational bridges are (-2)-curves

15.7 Summary

$\Delta = \operatorname{spec} R$ and R a dvr. In characteristic 0, we have proved

Theorem 15.13 (**Semistable Reduction**). *If $\mathcal{C}^* \to \Delta^* = \operatorname{spec} K$ is a smooth, projective and geometrically connected curve, then there exists a cover $\Delta' \to \Delta$ of spectrums of dvrs and a family $\mathcal{C}' \to \Delta'$ of semistable curves extending $\mathcal{C}^* \times_{\Delta^*} \Delta'^* \to \Delta'^*$ such that $\mathcal{C}'$ is regular.*

Theorem 15.14 (**Stable Reduction**). *If $(\mathcal{C}^* \to \Delta, s_1^*, \dots, s_n^*)$ is a family of n-pointed stable curves of genus g, then there exists a finite cover $\Delta' \to \Delta$ of spectra of dvrs and a family $(\mathcal{C}' \to \Delta', s_1', \dots, s_n')$ of stable curves extending $\mathcal{C}^* \times_{\Delta^*} \Delta'^* \to \Delta'^*$.*

History. Deligne-Mumford ('69) showed stable reduction in positive/mixed characteristic by using stable reduction for abelian varieties.

Artin-Winters ('71) gave a proof (again, working in positive/mixed characteristic) roughly along the same lines as what we presented.

16 Lecture 16 (3/8): Gluing and forgetful morhpisms

Today's outline

- Recap
- Explicit stable reduction (compute stable limit of $y^2 = x^5 + t$)
- Uniqueness of the stable limit (i.e. $\overline{\mathcal{M}}_{g,n}$ separated)
- Gluing morphisms
  - $\overline{\mathcal{M}}_{g_1,n_1} \times \overline{\mathcal{M}}_{g_2,n_2} \to \overline{\mathcal{M}}_{g_1+g_2,n_1+n_2-2}$ (will allow us to define boundary divisors)
  - $\overline{\mathcal{M}}_{g,n} \to \overline{\mathcal{M}}_{g+1,n-2}$
- Forgetful morphisms
  - $\overline{\mathcal{M}}_{g,n+1} \to \overline{\mathcal{M}}_{g,n}$ (will show this is a universal family)

16.1 Recap of stable reduction

Theorem 16.1 (Stable Reduction). *If $(\mathcal{C}^* \to \Delta, s_1^*, \ldots, s_n^*)$ is a family of n-pointed stable curves of genus g, then there exists a finite cover $\Delta' \to \Delta$ of spectra of dvrs and a family $(\mathcal{C}' \to \Delta', s_1', \ldots, s_n')$ of stable curves extending $\mathcal{C}^* \times_{\Delta^*} \Delta'^* \to \Delta'^*$.*

In the $n = 0$ case, Stable reduction says that given a stable curve $\mathcal{C}^\times \to \Delta^\times = \operatorname{spec} K$, the fraction field of a dvr $\operatorname{spec} R = \Delta$, possibly after an extension $\Delta' \to \Delta, t \mapsto t^n$, there exists a completion of $\mathcal{C}^\times$ to a stable family $\mathcal{C} \to \Delta = \operatorname{spec} R$ s.t.

$$\begin{tikzcd} \mathcal{C}^\times \arrow[r] \arrow[d] & \mathcal{C} \arrow[d] \\ \Delta^\times \arrow[r] & \Delta \end{tikzcd}$$

commutes. This is the existence part of the valuative criterion of properness.

See notes from last lecture for an idea of the proof (especially when $\operatorname{char} K = 0$).

(Step 1) Reduce to case where the generic fiber $\mathcal{C}^\times \to \Delta^\times$ is smooth.

(Step 2) Find some extension $\mathcal{C} \xrightarrow{flat} \Delta$

(Step 3) Use embedded resolutions to get

$$\begin{tikzcd} \widetilde{\mathcal{C}} \arrow[rr] \arrow[dr] & & \mathcal{C} \arrow[dl] \\ & \Delta & \end{tikzcd}$$

with $\widetilde{\mathcal{C}}$ regular, $\widetilde{\mathcal{C}} \to \mathcal{C}$ projective + birational (finite sequence of blowups at closed points) and $\widetilde{\mathcal{C}}_0$ set-theoretic normal crossings (i.e. $(\widetilde{\mathcal{C}}_0)_{red}$ a nodal curve).

(Step 4) Take ramified base extension $\Delta' = \operatorname{spec} R \to \operatorname{spec} R = \Delta$, $t \mapsto t^m$ s.t. the central fiber of the normalization $\widetilde{\widetilde{\mathcal{C}} \times_\Delta \Delta'}$ is reduced and nodal.

(Step 5) Take minimal resolution and contract rational tails/bridges (i.e. take stable model).

16.2 Explicit stable reduction

The biggest challenge in computing the stable limit is in 'step 4,' computing the normalization $\widetilde{\mathcal{C} \times_\Delta \Delta'}$ after base change $\Delta' \to \Delta$, $t \mapsto t^n$. In practice, it is useful to factor $\Delta' \to \Delta$ as a composition of prime order base changes $\Delta' \to \Delta, t \mapsto t^p$.

Proposition 16.2.

- *Let $\mathcal{C} \to \Delta$ be a generically smooth family of curves such that $(\mathcal{C}_0)_{\text{red}}$ is nodal.*
- *Define the divisor $\mathcal{C}_0 = \sum a_i D_i$ on $\mathcal{C}$*
- *Let $\Delta' \to \Delta$ be defined by $t \mapsto t^p$, and set $\mathcal{C}' := \mathcal{C} \times_\Delta \Delta'$ with normalization $\widetilde{\mathcal{C}}'$*

Then, $\widetilde{\mathcal{C}}' \to \mathcal{C}$ is a branched cover ramified over $\sum (a_i \mod p) D_i$.[35]

Example. Suppose $\mathcal{C} \to \Delta = \operatorname{spec} R$ with local equation $y^2 = x^5 + t$. Here, $t \in R$ is a uniformizer. When $t = 0$, central fiber $\mathcal{C}_0$ is not stable and $p \in \mathcal{C}_0$ is ramiphoid (spelling?) cusp. Central fiber looks like $y^2 = x^5$. What is the stable limit?

See also Harris-Morrison pg. 122

Since we already have some limit, begin with step 3: blow-up points in central fiber until $(\mathcal{C}_0)_{red}$ is nodal.

Notation 16.3. We'll let $\mathcal{C}$ be the surface we're blowing up, and let $\widetilde{\mathcal{C}} = \operatorname{Bl}_p \mathcal{C}$ with exceptional divisor E above p. There are two charts $U_1, U_2 \subset \widetilde{\mathcal{C}}$ s.t. $U_1 \to \mathcal{C}$ looks like $(\widetilde{x}, \widetilde{y}) \mapsto (\widetilde{x}, \widetilde{x}\widetilde{y})$ and $U_2 \to \mathcal{C}$ looks like $(\widetilde{x}, \widetilde{y}) \mapsto (\widetilde{x}\widetilde{y}, \widetilde{y})$. Note that $E|_{U_1} : \widetilde{x} = 0$ and $E|_{U_2} : \widetilde{y} = 0$.

- Blowup 1.
  - 1st chart: $y^2 - x^5 = (\widetilde{x}\widetilde{y})^2 - \widetilde{x}^5 = \widetilde{x}^2(\widetilde{y}^2 - \widetilde{x}^3)$ so one (reduced) component looks like E_1 and the other is cuspidal.
- Blowup 2
  - 1st chart: $x^2(y^2 - x^3) = \widetilde{x}^2((\widetilde{x}\widetilde{y})^2 - \widetilde{x}^3) = \widetilde{x}^4(\widetilde{y}^2 - \widetilde{x})$ with components the new exceptional divisor E_2 and also $D : \widetilde{y}^2 - \widetilde{x}$, a nodal curve (?).
  - 2nd chart: $x^2(y^2 - x^3) = \widetilde{x}^2\widetilde{y}^4(1 - \widetilde{x}^3\widetilde{y})$. The $\widetilde{x}^2$ is the first exceptional divisor E_1, the $\widetilde{y}^4$ is E_2, and the $(1 - \widetilde{x}^3\widetilde{y})$ is D.
- Blowup 3
  - 2nd chart: $x^4(y^2 - x) = \widetilde{x}^4\widetilde{y}^4(\widetilde{y}^2 - \widetilde{x}\widetilde{y}) = \widetilde{x}^4\widetilde{y}^5(\widetilde{y} - \widetilde{x})$. Our latest exceptional divisor is $F : \widetilde{y} = 0$. We also have an E_2 component (the $\widetilde{x}^4$ factor) and $D : \widetilde{y} - \widetilde{x}$, the normalization of the original central fiber.
- Blowup 4
  - 1st chart: $x^4y^5(y - x) = \widetilde{x}^{10}\widetilde{y}^5(\widetilde{y} - 1)$. So we have a new exceptional divisor G with multiplicity 10, we have F from the third blowup with multiplicity 5, and we have $D : \widetilde{y} - 1$.

Now we have a family[36] $\mathcal{C} \to \Delta$ with $(\mathcal{C}_0)_{\text{red}}$ nodal and

$$\widetilde{\mathcal{C}}_0 = D + 10G + 5F + 4E_2 + 2E_1.$$

We base change by $t \mapsto t^5$ and normalize to get $\widetilde{\mathcal{C}}' \xrightarrow{\pi} \mathcal{C}$ ramified over $D + E_2 + E_1$ (they have coefficients not divisible by 5. Use previous prop). What are the preimages of the other components? Let $G' = \pi^{-1}(G) \xrightarrow{5:1} G = \mathbb{P}^1$. This map is branched over 2 points, each with ramification index 5, so Riemann-Hurwitz gives

Question: Why?

Answer: It's branched over the points where G intersects any of D, E_2, D_1. G intersects D, E_2 each at one point. Still not

$$2g(G') - 2 = 5(-2) + 8 = -10 + 8 \implies g(G') = 0 \implies G' = \mathbb{P}^1.$$

Similarly, let $F' = \pi^{-1}(F) \xrightarrow{5:1} F = \mathbb{P}^1$ unramified. Since $\mathbb{P}^1$ has no non-trivial unramified covers, we must have $F' = F_1 \sqcup \cdots \sqcup F_5$ with $F_i = \mathbb{P}^1$.

[35] set-theoretically. This expression does not indicate multiplicity beyond $= 0$ or $\neq 0$

[36] See slides for a picture

Over Δ, new central fiber is $\mathcal{C}'_0 = 5D + 10G' + 5(F_1 + \cdots + F_5) + 20E_2 + 10E_1$. Over Δ', we divide by 5 to get $\mathcal{C}'_0 = D + 2G' + (F_1 + \cdots + F_5) + 4E_2 + 2E_1$. We've decreased some multiplicities. Now we compute base change $t \mapsto t^2$ and normalize to get $\pi : \widetilde{\mathcal{C}}' \xrightarrow{2:1} \mathcal{C}$ ramified over $D + F_1 + \cdots + F_5$. We do same sort of Riemann-Hurwitz calculations, e.g. let $H := \pi^{-1}(G') \xrightarrow{2:1} G' = \mathbb{P}^1$ ramified over 6 points and RH gives $g(H) = 2$. We also know $q = H \cap D$ is a ramification point of $H \to \mathbb{P}^1$ which some people call a "Weierstrass point." The central fiber will have everything with multiplicity 1 except E_2 with multiplicity 2.

Question: Why?

Answer: G' intersects each of $D, F_1, \ldots, F_5$

After this, one needs to repeat $t \mapsto t^2$ one last time to get a reduced central fiber. Then the final step is to contract rational tails. You end up with a genus two curve H along with D, the normalization of the original central fiber (the two intersect at a Weierstrass point), and nothing else.

Question 16.4. *Which genus* 2 *curve is it? More generally, what happens to*

$$y^2 = x^5 + a_3(t)x^3 + \cdots + a_0(t)?$$

Remark 16.5. Apparently Timothy Dokchitser has implemented an algorithm for doing such computations.

16.3 Uniqueness of stable limit

Proposition 16.6 (Separatedness)**.** *If $(\mathcal{C} \to \Delta, \sigma_1, \ldots, \sigma_n)$ and $(\mathcal{D} \to \Delta, \tau_1, \ldots, \tau_n)$ are families of n-pointed stable curves, then any isomorphism $\alpha^* : \mathcal{C}^* \to \mathcal{D}^*$ over Δ^* with $\tau_i^* = \alpha^* \circ \sigma_i^*$ of generic fibers extends to a unique iso $\alpha : \mathcal{C} \to \mathcal{D}$ over Δ with $\tau_i = \alpha \circ \sigma_i$.*

Proof. Assume $n = 0$ and $\mathcal{C}^\times = \mathcal{D}^\times$ smooth for simplicity.

We're given stable families $\mathcal{C}, \mathcal{D} \rightrightarrows \Delta = \operatorname{spec} R \ni t$ which restrict to isomorphic families

$$\begin{tikzcd} \mathcal{C}^\times \arrow[rr, "\sim"] \arrow[dr, "smooth"'] & & \mathcal{D}^\times \arrow[dl, "smooth"] \\ & \Delta^\times & \end{tikzcd}$$

over the generic point $\Delta^\times = \operatorname{spec} K$. We want to extend this isomorphism. We know the local structure of $z \in \mathcal{C}$; if $z \in \mathcal{C}_0$ is a node, it looks like $xy = t^{n+1}$, an A_n-singularity.

(Step 1) Take minimal resolutions of $\mathcal{C}, \mathcal{D}$. Since the node in the central fiber meant we had a surface with an A_n-singularity, this is resolved by a collection

$$\pi_{\mathcal{C}}^{-1}(z) = E_1 \cup \cdots \cup E_n \text{ with } E_i^2 = -2$$

of (-2)-curves.

(Step 2) In order to compare these minimal resolutions, we want to compare them by another smooth surface. To do this, we take the closure of the image of the graph of $\alpha^\times : \mathcal{C}^\times \xrightarrow{\sim} \mathcal{D}^\times$, and then take the minimal resolution of that. That is, we consider a minimal resolution $\Gamma_{min} \to \Gamma$ of

$$\Gamma = \operatorname{im}\left(\mathcal{C}^\times \to \mathcal{C}_{min} \times_\Delta \mathcal{D}_{min}\right)$$

(scheme-theoretic image above). At this point, we have a diagram like so

$$\begin{tikzcd}
 & \Gamma_{min} \arrow[dl] \arrow[dr] & \\
\mathcal{C}_{min} \arrow[d] & & \mathcal{D}_{min} \arrow[d] \\
\mathcal{C} \arrow[rr, dashed, "\sim"] \arrow[dr, "\text{stable}"'] & & \mathcal{D} \arrow[dl, "stable"] \\
 & \Delta &
\end{tikzcd}$$

(everything with a $_{min}$ subscript is smooth).

(Step 3) Both maps $\Gamma_{min} \rightrightarrows \mathcal{C}_{min}, \mathcal{D}_{min}$ are birational maps of smooth projective surfaces, so we have an equality

$$\Gamma\left(\omega^{\otimes k}_{\mathcal{C}_{min}/\Delta}\right) = \Gamma\left(\omega^{\otimes k}_{\mathcal{D}_{min}/\Delta}\right) = \Gamma\left(\omega^{\otimes k}_{\Gamma_{min}/\Delta}\right).$$

of pluricanonical sections.

(Step 4) $\mathcal{C}, \mathcal{D}$ are relative stable models of $\mathcal{C}_{min}, \mathcal{D}_{min}$. Apparently by uniqueness of stable models, we know that

$$\mathcal{C} = \operatorname{Proj} \bigoplus_{k \geq 0} \Gamma\left(\omega^{\otimes k}_{\mathcal{C}_{min}/\Delta}\right) \quad \text{and} \quad \mathcal{D} = \operatorname{Proj} \bigoplus_{k \geq 0} \Gamma\left(\omega^{\otimes k}_{\mathcal{D}_{min}/\Delta}\right),$$

Question: Why do we have these Proj descriptions?

Answer: I think maybe this is a consequence of Proposition 14.12 (4) and the canonical bundle being ample

and so $\mathcal{C} \cong \mathcal{D}$.

■

Above argument works more generally for moduli of higher dimensional varieties. In the case of moduli of curves, where you're just dealing with surfaces, things can be made more explicit.

Alternate Proof Sketch. Let's give a more explicit argument that $\Gamma_{min} \xrightarrow{\sim} \mathcal{C}_{min}$ and $\Gamma_{min} \xrightarrow{\sim} \mathcal{D}_{min}$. If not, there exists $E = \mathbb{P}^1 \subset \Gamma_{min}$ that is contracted under $\Gamma_{min} \to \mathcal{C}_{min}$, but not to $\Gamma_{min} \to \mathcal{D}_{min}$ (since minimal resolutions are built via successive blowups). Let $E_{\mathcal{D}} = \pi_{\mathcal{D}}(E)$ with total transform $\pi_{\mathcal{D}}^{-1}(E_{\mathcal{D}}) = E \cup F$ (so F is contracted by $\pi_{\mathcal{D}}$). We know how blowing up affects self-intersection; by projection formula

$$E_{\mathcal{D}}^2 = E \cdot (E + F) \geq E^2 = -1.$$

The Hodge index theorem for exceptional curves tells us that $E_{\mathcal{D}}^2 < 0$, so $E_{\mathcal{D}}^2 = -1$. Hence, $E_{\mathcal{D}} \subset \mathcal{D}_{min}$ is singular. As $\Gamma_{min} \to \Gamma$ resolves the singularity, we must have $E \cdot F \geq 1$ which gives $E_{\mathcal{D}}^2 \geq 0$, a contradiction. ■

The upshot is that $\overline{\mathcal{M}}_g$ is proper. We've shown this in characteristic 0, but it's actually true that $\overline{\mathcal{M}}_g \to \operatorname{spec} \mathbb{Z}$ is proper. By Keel-Mori, we then get a coarse moduli space $\overline{\mathcal{M}}_g \xrightarrow{cms} \overline{M}_g$ with $\overline{M}_g$ a proper algebraic space.

16.4 Gluing morphisms

(Reference: Knudsen, Projectivity II (1983))

Proposition 16.7. *There are morphisms of algebraic stacks*

$$\overline{\mathcal{M}}_{g,n} \times \overline{\mathcal{M}}_{g',n'} \to \overline{\mathcal{M}}_{g+g',n+n'-2} \text{ and } \overline{\mathcal{M}}_{g,n} \to \overline{\mathcal{M}}_{g+1,n-2}.$$

The first one comes from gluing two curves together at a couple of marked points (and then not marking the resulting glued point). The second comes from gluing together two marked points on the same curve (and again not marking the glued point).

We need this to work in families.

Proof sketch of first map. Assume $n = n' = 1$ we we have $\mathcal{C} \rightleftarrows S \leftrightarrows \mathcal{C}'$ with $\sigma, \sigma' : S \rightrightarrows \mathcal{C}, \mathcal{C}'$ the sections.

- (Approach 1) use pushout

$$\begin{tikzcd} S \arrow[r, "\sigma'"] \arrow[d, "\sigma"'] & \mathcal{C}' \arrow[d] \\ \mathcal{C} \arrow[r] & \widetilde{\mathcal{C}} \end{tikzcd}$$

  which exists by Ferrand. Pushouts in AG scary in general, but well-behaved when one map is a closed immersion and the other is finite (here, both are closed immersions).

  We need to show $\widetilde{\mathcal{C}} \to S$ is the desired family of stable curves. We will use

  (1) Pushout is étale local on $S, \mathcal{C}, \mathcal{C}'$

  (2) local structure of smooth maps ($\mathcal{C}, \mathcal{C}'$ are smooth along σ, σ').

  Reduce pushout computation to

$$\begin{tikzcd} \operatorname{spec} A \arrow[r, "\sigma'=0"] \arrow[d, "\sigma=0"'] & \operatorname{spec} A[y] \arrow[d] \\ \operatorname{spec} A[x] \arrow[r] & \operatorname{spec} A[x] \times_A A[y], \end{tikzcd}$$

  where

$$A[x] \times_A A[y] = \{(f(x), g(y)) : f(0) = g(0)\} = A[x,y]/(xy)$$

  so $\widetilde{\mathcal{C}} \to S$ is a nodal family. One can check stability on fibers.

- (Approach 2) Use Proj construction. Same initial setup. We know $\omega_{\mathcal{C}/s}(\sigma)$ and $\omega_{\mathcal{C}'/S}(\sigma')$ are ample. Use exact sequence

$$0 \longrightarrow \omega_{\mathcal{C}} \longrightarrow \omega_{\mathcal{C}}(\sigma) \longrightarrow \mathscr{O}_\sigma \longrightarrow 0.$$

  We get $\omega_{\mathcal{C}}(\sigma)^{\otimes k} \to \mathscr{O}_\sigma$ for all k. Form the sheaf fiber product

$$\begin{tikzcd} \mathscr{A}_k \arrow[r] \arrow[d] & \pi'_*(\omega_{\mathcal{C}'/S}(\sigma')^{\otimes k}) \arrow[d] \\ \pi_*\left(\omega_{\mathcal{C}/S}(\sigma)^{\otimes k}\right) \arrow[r] & \mathscr{O}_S \end{tikzcd}$$

  Check that $\widetilde{\mathcal{C}} = \operatorname{Proj} \bigoplus_{k \geq 0} \mathscr{A}_k \to S$ is desired stable family.

■

Proof sketch of second map. Assume $n = 2$ so we're given $(\mathcal{C} \to S, \sigma_1, \sigma_2)$.

- (Approach 1) use pushout again

$$\begin{tikzcd} S \sqcup S \arrow[r, "\sigma_1 \sqcup \sigma_2"] \arrow[d, "\text{finite}"'] & \mathcal{C} \arrow[d] \\ S \arrow[r] & \widetilde{\mathcal{C}} \end{tikzcd}$$

  Local calculation now looks like

$$\begin{tikzcd} \operatorname{spec}(A \times A) \arrow[r, "{(0,1)}"] \arrow[d] & \operatorname{spec} A[t] \arrow[d] \\ \operatorname{spec} A \arrow[r] & \operatorname{spec} B \end{tikzcd}$$

  where $(x = t^2 - 1,\ y = t^3 - t)$

$$B = \{f \in A[t] : f(0) = f(1)\} = A\left\langle t^2 - 1, t^3 - t\right\rangle = A[x, y]/(y^2 - x^2(x + 1)).$$

- (Approach 2) Proj construction is similar.

■

Application (Boundary divisors). Let $\delta_i = \operatorname{im}\left(\overline{\mathcal{M}}_{i,1} \times \overline{\mathcal{M}}_{g-i,1} \to \overline{\mathcal{M}}_g\right)$ for $i = 1, \ldots, \lfloor g/2 \rfloor$, and let $\delta_0 = \operatorname{im}\left(\overline{\mathcal{M}}_{g-1,2} \to \overline{\mathcal{M}}_g\right)$.

Note 10. Not being able to draw these curves on the fly really negatively impacts these notes.

Fact (Important)**.** $\delta = \delta_0 + \cdots + \delta_{\lfloor g/2 \rfloor}$ is a divisor (e.g. codim 1) and is furthermore a simple[37] normal crossing (snc) divisor (e.g. intersect transversally).

16.5 The universal family $\overline{\mathcal{M}_{g,n+1}} \to \overline{\mathcal{M}_{g,n}}$

Warning 16.8. Forgetting a marked point can make the curve unstable!

Let $\mathcal{U}_g \to \overline{\mathcal{M}}_g$ be the universal family (coming form 2-Yoneda). As a stack, $\mathcal{U}_g$ parameterizes

$$\begin{tikzcd} \mathcal{C} \arrow[d, shift right] \\ S \arrow[u, bend right, "\sigma"'] \end{tikzcd}$$

with $\sigma : S \to \mathcal{C}$ arbitrary (could land in singular locus).

Definition 16.9. $\mathcal{U}_g \to \overline{\mathcal{M}}_g$ a universal family means given any other family of curves $\mathcal{C} \to S$, there's a Cartesian diagram

$$\begin{tikzcd} \mathcal{C} \arrow[r] \arrow[d] & \mathcal{U}_g \arrow[d] \\ S \arrow[r, "{[\mathcal{C}]}"] & \overline{\mathcal{M}}_g \end{tikzcd}$$

[37]This word might should not be here

In particular, $\mathcal{U}_g$ is an algebraic stack, and $\mathcal{U}_g \to \overline{\mathcal{M}}_g$ is a proper, flat family of nodal curves.

We have $\overline{\mathcal{M}}_{g,1} \to \mathcal{U}_g$ sending a stable 1-pointed family $(\mathcal{C} \to S, \sigma)$ to $(\mathcal{C}^{st} \to S, \sigma^{st})$, the stable model obtained by contracting rational bridges.

Proposition 16.10. *$\overline{\mathcal{M}}_{g,1} \xrightarrow{\psi} \mathcal{U}_g$ is an isomorphism.*

Proof Sketch.

- (Proj construction, follow Knudsen) We explicitly construct an inverse $\mathcal{U}_g \to \overline{\mathcal{M}}_{g,1}$. Let $\mathcal{C} \to S$ be a stable curve with section $\sigma : S \to \mathcal{C}$. Let $I_\sigma \subset \mathscr{O}_{\mathcal{C}}$ be the ideal sheaf of σ. Define K as

Question: Is this right?

$$0 \longrightarrow \mathscr{O}_{\mathcal{C}} \longrightarrow I_\sigma^\vee \oplus \mathscr{O}_{\mathcal{C}} \longrightarrow K \longrightarrow 0,$$

and define $\widetilde{\mathcal{C}} = \operatorname{Proj} \operatorname{Sym} K$. Use that there's a surjection

$$\sigma^* K \twoheadrightarrow \sigma^* \left(K/\mathscr{O}_C\right) = \sigma^* \left(I_\sigma^\vee/\mathscr{O}_{\mathcal{C}}\right).$$

One needs to show that $I_\sigma^\vee/\mathscr{O}_C$ is a line bundle, so we get a section $\widetilde{\sigma} : S \to \widetilde{\mathcal{C}}$ and show that $(\widetilde{\mathcal{C}} \to S, \widetilde{\sigma})$ is stable.

- (deformation theory[38]) We have $\overline{\mathcal{M}}_{g,1} \xrightarrow{\psi} \mathcal{U}_g$ sending $(C', p') \mapsto (C, p)$ with $C = (C')^{st}$ the stable model of C. To show it is an iso, we need to show that

Potentially $\operatorname{spec} k \to B(\mathbb{Z}/2\mathbb{Z})$ is a counterexample. Sounds like you need to add that the morphism all preserves stabilizers

  - ψ is proper and representable
  - ψ separates points
  - ψ separates tangent vectors

  By stable reduction, we know both sides are proper, so we get that ψ is proper for free. One can check directly that ψ separates points. The challenge is in having it separate tangent vectors.

  For simplicity, assume only one rational bridge is contracted and consider

$$\begin{array}{ccc} T_{\overline{\mathcal{M}}_{g,1},[C',p']} & \longrightarrow & T_{\mathcal{U}_g,[C,p]} \\ \| & & \| \\ \operatorname{Ext}^1(\Omega_{C'}, I_{p'}) & \longrightarrow & \operatorname{Ext}^1(\Omega_C, I_p) \end{array}$$

Question: What are I_p and $I_{p'}$?

  ($I_{p'}$ is a line bundle, but I_p is not). Here are some properties of contraction map $C' \xrightarrow{\pi} C$

Answer: ideal sheaf of a point

  (1) $\pi_* \mathscr{O}_{C'} = \mathscr{O}_C$ and $R^1\pi_*\mathscr{O}_{C'} = 0$.

  (2) $\pi_* I_{p'} = I_p$ and $R^1\pi_* I_{p'} = 0$.

  (3) $0 \to \kappa(p) \to \Omega_C \to \pi_*\Omega_{C'} \to 0$ exact.

Question: Because it's a node or something?

Answer: Yes, so the its stalk at p (i.e. the maximal ideal $\mathfrak{m}_p \subset \mathscr{O}_{C,p}$)

  The map on Ext's is

$$\operatorname{Ext}^1(\Omega_{C'}, p') \ni [0 \to I_{p'} \to E' \to \Omega_{C'} \to 0] \mapsto [0 \to I_p \to E \to \Omega_C] \in \operatorname{Ext}^1(\Omega_C, I_p)$$

[38]"I'm sort of confident that this argument works, but I have not checked all the details myself, and I am not following a reference here" (paraphrased)

where

$$E = \Omega_C \times_{\pi_*\Omega_{C'}} \pi_* E'$$

(take pushforward of original sequence then pull it back to one ending in Ω_C). Suppose E is the trivial extension, so there's a section $s : \Omega_C \to E$. One can check that s descends to a section $\widetilde{s} : \pi_*\Omega_{C'} \to \pi_* E'$. Then use adjunction

$$\mathrm{Hom}(\pi_*\Omega_{C'}, \pi_* E') = \mathrm{Hom}(\pi^*\pi_*\Omega_{C'}, E'),$$

and check that the map $\pi^*\pi_*\Omega_{C'} \to E'$ obtained from it descends to a section $s' : \Omega_{C'} \to E'$, so E' is also trivial and the map of Ext groups is injective (so an iso, meaning we win).

■

In the above proof, the correct fact to use is

Fact. Let $f : \mathcal{X} \to \mathcal{Y}$ be a proper map of algebraic stacks such that

(1) For all $k = \overline{k}$, the induced map $\mathcal{X}(k)/\sim \longrightarrow \mathcal{Y}(k)/\sim$ is injective

(2) f induces isomorphisms $\mathrm{Aut}(x) \to \mathrm{Aut}(f(x))$

(3) f separates tangent vectors

Then, $\mathcal{X} \to \mathcal{Y}$ is a closed immersion.

This is an analogue of proper monomorphisms being closed immersions it sounds like.

17 Lecture 17: Irreducibility

Recall the goal of the course is to prove

Goal. The moduli space $\overline{\mathcal{M}}_g$ of stable curves of genus $g \geq 2$ is a smooth, proper and irreducible DM stack of dimension $3g - 3$ which admits a projective coarse moduli space.

We know everything [39] except irreducibility and projectivity. We do irreducibility today and summarize projectivity on Monday.

Note 11. ∼40 people today.

Today's outline

(1) Background on branched covers[40]

(2) Clebsch-Hurwitz argument (1872 and 1891), a char $= 0$ argument with some non-algebraic input

(3) Fulton's appendix to Harris and Mumford's paper "On the Kodaira Dimension of $\mathcal{M}_g$" (1982), a char $= 0$ argument which is completely algebraic

(4) Some arguments which work in char $= p$ via reducing to the char $= 0$ case (Fulton's argument here also needs $p \gg 0$)

[39] We only proved properness in char 0, but it's true over $\mathbb{Z}$ and we'll even use that today

[40] Comes up in arguments **(2)**,**(3)**, and Fulton's part of **(4)**

- Deligne-Mumford's 2 arguments in "On the irreducibility of $\mathcal{M}_g$" (1969)
- Fulton's argument in "Hurwitz schemes and irreducibility of $\mathcal{M}_g$" (1969)

Note 12. The lecture notes for the course are caught up to Monday. It may be another week or two before the stuff from these last two lectures are added.

Remark 17.1. Argument **(3)** and the DM arguments in **(4)** will exploit the compactification $\mathcal{M}_g \subset \overline{\mathcal{M}}_g$. Heuristically, if you have a smooth curve $[C] \in \mathcal{M}_g$, you want to show you can degenerate it to a something on the boundary $\overline{\mathcal{M}}_g \setminus \mathcal{M}_g$, and then show that any two curves on the boundary can be degenerated to each other (picture this as giving a path between any two points).

Remark 17.2. $\mathcal{M}_g$ itself is irreducible.

17.1 The goal

Goal. $\overline{\mathcal{M}}_{g,n}$ is irreducible.

Remark 17.3. As $\overline{\mathcal{M}}_{g,n}$ is smooth, this is $\iff$ $\overline{\mathcal{M}}_{g,n}$ is connected $\iff$ $\overline{\mathcal{M}}_g$ connected $\iff$ $\mathcal{M}_g$ connected and dense in $\overline{\mathcal{M}}_g$.

Remark 17.4. We have coarse moduli spaces $\overline{\mathcal{M}}_{g,n} \xrightarrow{cms} \overline{M}_{g,n}$ with $|\mathcal{M}_{g,n}| = \left|\overline{M}_{g,n}\right|$ at topological spaces. Hence the previous statements on stacks are equivalent to corresponding statements on coarse moduli spaces.

Why do we care about any of this?

- $\mathcal{M}_g$ connected $\iff$ genus is the only discrete invariant.
- $\mathcal{M}_g \subset \overline{\mathcal{M}}_g$ dense $\iff$ $\nexists$ (irreducible) component of $\overline{\mathcal{M}}_g$ consisting entirely of singular curves.

We will try to summarize algebraic approaches. There are other topological/analytic arguments (e.g. using Teichmüller).

17.2 Background on branched covers

Definition 17.5. A **branched cover of** $\mathbb{P}^1$ is the a finite morphism $f : C \to \mathbb{P}^1$ with C a smooth, connected curve and $K(\mathbb{P}^1) \to K(C)$ separable.

Remark 17.6. f above is étale at points p where $\left(\Omega_{C/\mathbb{P}^1}\right)_p = 0$.

Definition 17.7. We say f is **ramified at** p **of index** e if $\text{length}(\Omega_{C/\mathbb{P}^1})_p = e - 1$.

Example. $X = \mathbb{A}^1 \to \mathbb{A}^1 = Y$ via $x \mapsto x^d$. Then,

$$\Omega_{X/Y} = \frac{k[x]\mathrm{d}x}{d \cdot x^{d-1}\mathrm{d}x = 0}.$$

If char $\nmid d$, then $\text{length}(\Omega_{X/Y})_0 = d - 1$.

Definition 17.8. The **ramification divisor** is

$$R = \sum_{p \in C} \text{length}\left(\Omega_{C/\mathbb{P}^1}\right)_p \cdot p.$$

The short exact sequence

$$0 \longrightarrow f^*\Omega_{\mathbb{P}^1} \longrightarrow \Omega_C \longrightarrow \Omega_{C/\mathbb{P}^1} \longrightarrow 0$$

implies $K_C = f^*K_{\mathbb{P}^1} + R$ as divisors. Taking degrees gives **Riemann-Hurwitz**

$$2g - 2 = d(-2) + \deg R \implies \deg R = 2d + 2g - 2.$$

where $d = \deg f$.

Definition 17.9. A branched cover $C \to \mathbb{P}^1$ is called **simply branched** if

(1) every ramification point has index 2; and

(2) there exists at most one ramification point in every fiber.

Riemann-Hurwtiz implies that a simply branched covering $C \xrightarrow{d} \mathbb{P}^1$ is ramified over $b := 2d + 2g - 2$ *distinct* points in $\mathbb{P}^1$.

Lemma 17.10. *Let C be a smooth, projective, connected curve of genus g, and let L be a line bundle of degree $d \gg 0$. Then for a general $V \subset \mathrm{H}^0(L)$ of dimension 2, the induced map $|V| : C \to \mathbb{P}^1$ is simply branched.*

Proof. Do a dimension count: $h^0(L) = d + 1 - g$. Also $\dim \mathrm{Gr}(2, \mathrm{H}^0(L)) = 2(h^0(L) - 2) = 2(d - g - 1)$. If $C \xrightarrow{V} \mathbb{P}^1$ is not simply branched, then either

(1) V has a base point

(2) there's a ramification point of index > 2

(3) there's 2 ramification points in the same fiber

Want to show these all describe lower dimensional subspace of the Grassmannian. Let's just look at case **(2)** in detail.

Case **(2)**. There's a point with ramification index at least 3, so $\exists s \in V$ s.t. $s \in \mathrm{H}^0(L(-3p))$ for some $p \in C$. At the same time

$$\begin{aligned} \dim \left\{ V \in \mathrm{Gr}(2, \mathrm{H}^0(L)) : C \xrightarrow{V} \mathbb{P}^1 \text{ satisfies } \mathbf{(2)} \right\} &= \dim \mathbb{P}\mathrm{H}^0(L(-3p)) + \dim \mathbb{P}(\mathrm{H}^0(L)/s) \\ &= d - 3 + (1 - g) - 1 + d - g - 1 \\ &= 2d - 2g - 4 < 2(d - g - 1) - 1 \end{aligned}$$

(need one section $s \in \mathrm{H}^0(L(-3p))$ and one section in the quotient). We added the -1 to the RHS since there's an extra degree of freedom from choosing the point $p \in C$. ■

Exercise. Do the other two cases in the previous proof.

Warning 17.11. It's possible above proof (sketch) will break down in positive characteristic. We only need it in characteristic 0.

Lemma 17.12. *If $C \to \mathbb{P}^1$ is a simply branched cover of degree $d > 2$, then* $\mathrm{Aut}(C/\mathbb{P}^1) = \{1\}$.

The main point is an automorphism $\alpha : C \to C$ over $\mathbb{P}^1$ would fix the $2d+2g-2$ branched points, and a classical fact about curves is that there does not exist any non-trivial automorphisms fixing more than $2g+2$ points.

Question: Why?

Answer: Briefly, Lefschetz fixed point will say any non-identity automorphism has $\leq 2+2g$ fixed points

Define the (topological or algebraic) space

algebraic space since automorphisms are trivial

$$H_{d,b} := \left\{ C \xrightarrow{d} \mathbb{P}^1 \text{ simply branched over } b \text{ points} \right\} \quad \text{where } \ b := 2g - 2d - 2.$$

We have maps

$$\begin{tikzcd} & H_{d,b} \arrow[dl] \arrow[dr] & \\ \mathcal{M}_g & & \mathrm{Sym}^b \, \mathbb{P}^1 \setminus \Delta = \mathbb{P}^b \setminus \Delta \end{tikzcd}$$

The left map is $[C \to \mathbb{P}^1] \mapsto [C]$. The right map is $[C \to \mathbb{P}^1] \mapsto \{b \text{ branched points}\}$. Can use right map to define topology if you want.

Lemma 17.13. *In* char $=0$, $H_{d,b} \to \mathrm{Sym}^b \, \mathbb{P}^1 \setminus \Delta$ *is finite and étale, i.e. a covering space.*

This implies that any $C \to \mathbb{P}^1$ can be deformed so that the branched locus is general.

Proof sketch of étaleness.

(Topological) Say we're given $C \to \mathbb{P}^1$ and have $p_1, p_2 \in \mathbb{P}^1$. Say we want to move p_1 to q_1 (nearby) and deform the cover along with it. Everything is topological, so we can take a small open U around p_1, q_1 and just deform things above small open. Then just glue everything back in.

(Algebraic) $\mathrm{Def}^{1st}(C \xrightarrow{f} \mathbb{P}^1) \to \mathrm{Def}^{1st}(\{p_i\}_{i=1}^b \subset \mathbb{P}^1)$. This right space is b-dimensional. The left space is $\mathrm{H}^0(C, N_f)$ where N_f is the **normal bundle**

$$0 \longrightarrow T_C \longrightarrow f^* T_{\mathbb{P}^1} \longrightarrow N_f \longrightarrow 0.$$

We have a short exact sequence

$$0 \longrightarrow \mathrm{H}^0(f^* T_{\mathbb{P}^1}) \longrightarrow \mathrm{H}^0(N_f) \longrightarrow \mathrm{H}^1(T_C) \longrightarrow 0$$

(the quotient above is secretly the forget map from deformations of the cover to deformations of the curve[41]). Note $\deg(f^* T_{\mathbb{P}^1}) = 2d$, so one can use Riemann-Roch to see that $h^0(N_f) = b$. This tells you should expect $\mathrm{Def}^{1st}(C \xrightarrow{f} \mathbb{P}^1) \to \mathrm{Def}^{1st}(\{p_i\}_{i=1}^b \subset \mathbb{P}^1)$ to be a bijection. Showing that it actually is a bijection requires more work. ∎

Remark 17.14. We're always taking d to be sufficiently large for our purposes.

17.3 Clebsch-Hurwitz proof over $\mathbb{C}$

(References: Clebsch (1872), Hurwitz (1891), and Fulton "Hurwitz schemes and irreducibility..." (1969)) Need the following non-algebraic input:

[41] It is also secretly the induced map of tangent spaces $H_{d,b} \to \mathcal{M}_g$

Theorem 17.15 (**Riemann Existence Theorem**). *There are bijections*

$$\left\{\begin{matrix} C \to \mathbb{P}^1 \text{ algebraic} \\ \text{branched covers}\end{matrix}\right\} \leftrightarrow \left\{\begin{matrix} C \to \mathbb{P}^1 \text{ topological} \\ \text{branched covers}\end{matrix}\right\} \leftrightarrow \left\{\begin{matrix} C \to \mathbb{P}^1 \text{ holomorphic} \\ \text{branched cover}\end{matrix}\right\}$$

We also need monodromy action. Given $C \xrightarrow{f} \mathbb{P}^1$, let $B \subset \mathbb{P}^1$ be the ramification locus. Say $p \in \mathbb{P}^1$ under some $q \in C$. Take a loop $\gamma \in \pi_1(\mathbb{P}^1_{\mathbb{C}} \setminus B, p)$ based at p around some branch point(s). We can trace q under the lifting of the path γ to C to get another point $q' \in f^{-1}(p)$. That is, we have a **monodromy action**

$$\pi_1(\mathbb{P}^1_{\mathbb{C}} \setminus B, p) \curvearrowright f^{-1}(p)$$

(note $\# f^{-1}(p) = d$), i.e. we have a group homomorphism

$$\pi_1(\mathbb{P}^1_{\mathbb{C}} \setminus B, p) \xrightarrow{\rho} S_d.$$

Note that $\pi_1(\mathbb{P}^1_{\mathbb{C}} \setminus B, p) = \langle \sigma_1, \ldots, \sigma_b \mid \sigma_1\sigma_2 \ldots \sigma_b = 1\rangle$ with σ_i a simple loop around the ith point in B.

Remark 17.16. C is connected $\iff$ $\mathrm{im}(\rho) \subset S_d$ is a transitive subgroup.

(Keep in mind in our definition of (simply) branched cover, we assumed C connected. This isn't need in this analytic story above, but we have still been assuming C smooth above)

The upshot is that for a subset $B \subset \mathbb{P}^1$ of b points, there's a bijection

$$\left\{\begin{matrix} C \xrightarrow{d} \mathbb{P}^1 \\ \text{branched covers}\end{matrix}\right\}_{/\sim} \longleftrightarrow \left\{\begin{matrix} \text{group homomorphisms } \pi_1(\mathbb{P}^1 \setminus B) \xrightarrow{\rho} S_d \\ \text{s.t. } \mathrm{im}\,\rho \subset S_d \text{ is a transitive subgroup}\end{matrix}\right\}_{/\sim}$$

where the objects on the right are defined up to inner automorphisms. This restricts to a bijection

$$\{\text{simply branched covers}\} \longleftrightarrow \{\rho(\sigma_i) \in S_d \text{ transpositions}\}.$$

Recall 17.17. We have a diagram

$$\begin{tikzcd} & H_{d,b} \arrow[dl, two heads] \arrow[dr, "\text{f.étale}"] & \\ \mathcal{M}_g & & \mathrm{Sym}^b\,\mathbb{P}^1 \setminus \Delta \end{tikzcd}$$

Theorem 17.18 (**Clebsch-Hurwitz**). *$H_{d,b}$ is connected over $\mathbb{C}$, so $\mathcal{M}_g$ is connected too.*

Proof sketch.

- $\pi_1(\mathbb{P}^1 \setminus B) = \langle \sigma_1, \ldots, \sigma_b \,|\prod_i \sigma_i = 1\rangle$
- $\pi_1(\mathbb{P}^1 \setminus B) \curvearrowright$ fibers of $C \to \mathbb{P}^1$ simply branched over B.
- Similarly, $\pi_1(\mathrm{Sym}^b\,\mathbb{P}^1 \setminus \Delta, B)$ acts on the fibers of $\beta : H_{d,b} \to \mathrm{Sym}^b\,\mathbb{P}^1 \setminus \Delta$.
- Let's define

$$H_{d,B} := \beta^{-1}(B) = \{C \to \mathbb{P}^1 \text{ simply branched over } B\} = \left\{(\tau_1, \ldots, \tau_b) \in (S_d)^b : \tau_i \text{ transposition and } \prod_i \tau_i = 1\right\}$$

(implicitly above we require $\langle \tau_i \rangle \leq S_d$ be a transitive subgroup)

We want the action $\pi_1(\operatorname{Sym}^b \mathbb{P}^1 \setminus \Delta, B) \curvearrowright H_{d,B}$ to be transitive (gives $H_{d,b}$ connected).

Strategy: find loops in $\operatorname{Sym}^b \mathbb{P}^1 \setminus \Delta$ that act on $(\tau_1, \ldots, \tau_b) \in H_{d,B}$ in a controlled way so that we can show each orbit contains

$$\tau^* = \left(\underbrace{(12), (12), (13), (13), \ldots, (1\,d-1)(1\,d-1)}_{2(d-1)}, \underbrace{(1d)(1d), \ldots, (1d)}_{2g+2} \right).$$

Define

$$\begin{array}{rccc} \Gamma_i : & [0,1] & \longrightarrow & \operatorname{Sym}^b \mathbb{P}^1 \setminus \Delta \\ & t & \longmapsto & (p_1, \ldots, p_{i-1}, \gamma_i(t), \gamma_i'(t), p_{i+2}, \ldots) \end{array}$$

Check

One day I'll actually understand the connection between Hurwitz spaces and braid groups. Today is not that day

(1) $\Gamma_i \cdot (\tau_1, \ldots, \tau_b) = \left(\tau_1, \ldots, \tau_{i-1}, \tau_i^{-1}\tau_{i+1}\tau_i, \tau_i, \tau_{i+2}, \ldots\right)$

(2) By using Γ_i's in some order, can move any $(\tau_1, \ldots, \tau_b)$ to τ^*

■

17.4 Fulton's 1982 appendix to Harris and Morrison's admissible covers

(Reference: Harris and Mumford "On the Kodaira Dimension of $\mathcal{M}_g$")

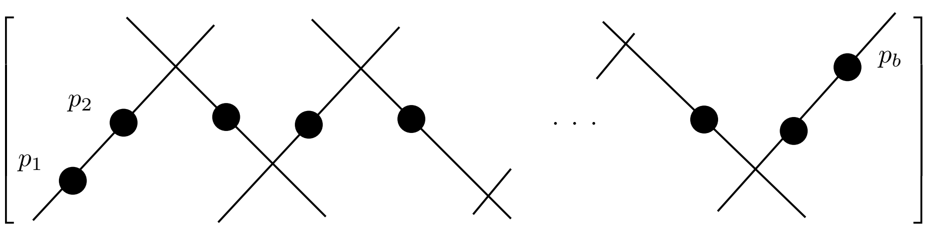
Completely algebraic argument in characteristic 0.

Key input

Proposition 17.19. *Every smooth curve C degenerates to a singular stable curve.*

In other words, there exists some curve T and map $T \to \overline{\mathcal{M}}_g$ s.t. $t \mapsto [C]$ (C the given smooth curve) and $0 \mapsto [C_0]$, a singular stable curve ($0, t \in T$ any two points).

Proof Sketch. Lemma 17.10 tells us there's some simply branched cover $C \to \mathbb{P}^1$. Choose an ordering $p_1, \ldots, p_b \in \mathbb{P}^1$ of branched points. This defines a b-pointed curve $B \in \mathcal{M}_{0,b}$. Lemma 17.13 let's us assume that $B \in \mathcal{M}_{0,b}$ is general. Then, B degenerates to

$D_0 =$ 

$\in \overline{\mathcal{M}}_{0,b}$.

We have

$$\begin{tikzcd} \mathcal{C}^\times \arrow[d, "\substack{\text{simply}\\\text{branched}}"] & & \\ B = \mathcal{D}^\times \arrow[r, hook] \arrow[d] & \mathcal{D} \arrow[d, "\in \overline{\mathcal{M}}_{0,n}(R)"] & \\ \operatorname{spec} K \arrow[r, hook] & \operatorname{spec} R \arrow[r, equal] & \Delta \end{tikzcd}$$

Define $\mathcal{C}$ as the integral closure of $\mathscr{O}_{\mathcal{D}}$ in $K(\mathcal{C}^\times)$. Purity of the branched locus (branched locus always a divisor) implies (one of?)

- the ramification of $\mathcal{C}$ over $\mathcal{D}$ is a divisor in the relative smooth locus of $\mathcal{D}$
- $\mathcal{C}_0 \to \mathcal{D}_0$ ramified over $\sigma_1(0), \ldots, \sigma_b(0)$ and possibly over an entire component of $\mathcal{C}_0$

As in stable reduction, after base change by $\Delta' \to \Delta, t \mapsto t^m$ (m multiplicity of some component) and replace $\mathcal{C}$ with $\widetilde{\mathcal{C} \times_\Delta \Delta'}$ to arrange that $\mathcal{C}_0 \to \mathcal{D}_0$ is ramified only over $\sigma_i(0)$ and nodes. Check that $\mathcal{C}_0$ is nodal, so $\mathcal{C} \to \Delta$ is a family of nodal curves.

If $\mathcal{C}_0$ is stable, we win (can't be smooth since maps to union of $\mathbb{P}^1$'s). Otherwise, take the stable model (contract rational tails/bridges) $\mathcal{C}^{st} \to \Delta$, and check that $\mathcal{C}^{st}$ is not smooth. Let $T \subset \mathcal{C}_0$ be an irreducible component. Since $\mathcal{C}_0 \to \mathcal{D}_0$, the union of $\mathbb{P}^1$'s, we have $T \to \mathbb{P}^1$ (since T irreducible). Now, RH $\implies 2g(T) - 2 = -2d + R$. If $\mathbb{P}^1$ is a tail (at the end), then $R \le 2 + (d-1)$. If $\mathbb{P}^1$ is a bridge (in the middle), then $R \le 1 + 2(d-1)$. In either case,

$$2g(T) - 2 \le -2 + 1 + 2(d-1) = -1 \implies g(T) = 0.$$

Thus, every component T of $\mathcal{C}_0$ is a $\mathbb{P}^1$ and this implies that $\mathcal{C}_0^{st}$ is nodal, which is what we wanted. ■

Second key prop is

Proposition 17.20. $\overline{\mathcal{M}}_g \setminus \mathcal{M}_g$ *is connected.*

Proof. $\delta := \overline{\mathcal{M}}_g \setminus \mathcal{M}_g = \delta_0 \cup \delta_1 \cup \cdots \cup \delta_{\lfloor g/2 \rfloor}$ where $\delta_0 = \operatorname{im}(\overline{\mathcal{M}}_{g-1,2} \to \overline{\mathcal{M}}_g)$ and $\delta_i = \operatorname{im}\left(\overline{\mathcal{M}}_{i,1} \times \overline{\mathcal{M}}_{g-1,i} \to \overline{\mathcal{M}}_g\right)$. By induction on the genus, we know each $\delta_0, \delta_1, \ldots$ is irreducible. We also know $\delta_i \cap \delta_j \neq \emptyset$ for all i, j (can come up with examples). ■

Corollary 17.21. $\overline{\mathcal{M}}_g$ *is connected.*

Given two curves, first proposition let's you degenerate them to the boundary. The second proposition then let's you degenerate them further (along the boundary) to each other.

This argument in fact shows more. The limit $\mathcal{C}_0 \to \mathcal{D}_0$ we constructed is what's called an **admissible cover**, i.e. it is simply branched away from nodes, and over the nodes the branching looks like $\operatorname{spec} k[x,y]/xy \to \operatorname{spec} k[x,y]/xy$ via $(x,y) \mapsto (x^m, y^m)$. Can define a stack $\overline{\mathcal{H}}_{d,b}$ of admissible covers and get a compactified diagram

$$\begin{tikzcd} & \overline{\mathcal{H}}_{d,b} \arrow[dl] \arrow[dr, "\text{finite}"] & \\ \overline{\mathcal{M}}_g & & \overline{\mathcal{M}}_{0,b} \end{tikzcd}$$

17.5 Two irreducibility papers in 1969

(References: Deligne and Mumford 'The irreducibility of the space of curves of given genus' and Fulton 'Hurwitz schemes and Irreducibility of Moduli of Algebraic Curves')

Both papers show that $\overline{\mathcal{M}}_g$ is irreducible in positive characteristic ($p > g + 1$ in Fulton) relying on char $= 0$ case.

DM #1 Uses $\overline{\mathcal{M}}_g \to \operatorname{spec} \mathbb{Z}$ smooth and proper.

Fact. If $X \to Y$ smooth and proper, the function $y \mapsto \#$ connected components of X_y is constant.

If you have proper and flat, this map is lower semicontinuous (assuming I heard correctly)

Since $\overline{\mathcal{M}}_g \times_{\mathbb{Z}} \mathbb{C}$ is connected, this let' you conclude $\overline{\mathcal{M}}_g \times_{\mathbb{Z}} \overline{\mathbb{F}}_p$ is connected.

DM #2 Several steps.

(Step 1) For any field k of char $= p$, show there are no proper connected components of $\mathcal{M} := \mathcal{M}_g \times_{\mathbb{Z}} k$.

Uses existence of coarse moduli space $\mathcal{M}_g \to M_g$ over $\mathbb{Z}$ using GIT. Also use a compactification $M_g \subset X$ with X projective over $\mathbb{Z}$.[42] Using result like in DM #1, use $X \times_{\mathbb{Z}} \mathbb{C}$ connected to conclude that $X \times_{\mathbb{Z}} \overline{\mathbb{F}}_p$ is connected. Showed $\mathcal{M}_g$ not proper by using degneration of Jacobians.

(Step 2) No connected component of $\overline{\mathcal{M}}_g \times_{\mathbb{Z}} k$ consisting entirely of smooth curves.

This is step 1 + stable reduction.

(Step 3) $\overline{\mathcal{M}}_g \setminus \mathcal{M}_g$ connected.

Steps 1 and 2 give the key proposition we say when looking at Fulton's appendix. That + this step finishes the argument

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Fulton The Hurwitz scheme $H_{d,b}$ is defined over $\mathbb{Z}$, and there is a diagram

$$\begin{tikzcd} & H_{d,b} \arrow[dl] \arrow[dr, "\beta"] & \\ \mathcal{M}_G & & \operatorname{Sym}^b \mathbb{P}^1 \setminus \Delta \end{tikzcd}$$

with β étale always and finite when $p > g + 1$. He established a "reduction theorem": since $H_{d,b} \to \operatorname{Sym}^b \mathbb{P}^1 \setminus \Delta$ is finite étale, connectedness of $H_{d,b} \times_{\mathbb{Z}} \mathbb{C}$ implies connectedness of $H_{d,b} \times_{\mathbb{Z}} \overline{\mathbb{F}}_p$ for $p > g + 1$.

18 Lecture 18 (3/15): Projectivity (Last Lecture)

(References: Kollár's 'Projectivity of complete moduli' and Viehweg's 'Quasi-projective moduli for polarized manifolds)

Goal. The moduli space $\overline{\mathcal{M}}_g$ of stable curves of genus $g \geq 2$ is a smooth, proper and irreducible Deligne-Mumford stack of dimension $3g - 3$ which admits a projective coarse moduli space.

We finish this today.

Today's outline

(0) Recap of how we got here

(1) Setup for $\overline{M}_g$

[42] projectivity not important. Could have used Nagata compactification to get something proper

(2) Survey of projectivity methods

(3) Nef vector bundles

(4) The ampleness lemma

(5) Application to $\overline{M}_g$

There will be missing details, but we'll try to cover the main ideas.

Recall 18.1. $\overline{M}_g$ is the coarse moduli space of $\overline{\mathcal{M}}_g$.

18.1 Recap

Recall the 6 steps towards projective moduli. In our case, we have $\overline{\mathcal{M}}_g \subset \mathcal{M}_g^{all}$.

(Step 1: algebraicity) $\mathcal{M}_g^{all}$ is algebraic and locally of finite type.

We used a Hilbert scheme to construct a smooth neighborhood $\mathrm{Hilb} \to \mathcal{M}_g^{all}$ around any particular curve.

(Step 2: Openness of stability) $\overline{\mathcal{M}}_g \subset \mathcal{M}_g^{all}$ is an open substack, so it inherits algebraicity from $\mathcal{M}_g^{all}$.

Openness translates to: if $\mathcal{C} \to S$ is a family of arbitrary curves, then $\{s \in S : \mathcal{C}_s \text{ is stable}\} \subset S$ is open in the base. We showed this in two steps.

– Nodal locus is open (using local structure of nodes)

– The stable locus within the nodal locus is open (C stable $\iff$ $\mathrm{Aut}(C)$ finite $\iff$ ω_C ample).

(Step 3: Boundedness of stability) $\overline{\mathcal{M}}_g$ is of finite type (in particular, it's quasi-compact).

We used: if $\mathcal{C} \to S$ is a stable family, then $\omega_{\mathcal{C}/S}^{\otimes 3}$ is relatively very ample and $\mathrm{Hilb}^P(\mathbb{P}^{5g-6})$ is finite type.

(Step 4: Existence of coarse moduli space) $\exists \overline{\mathcal{M}}_g \xrightarrow{cms} \overline{M}_g$

We showed $\overline{\mathcal{M}}_g$ is a separated DM stack, and applied the Keel-Mori theorem.

Can interchange steps 4 and 5 since separatedness is shown when showing properness

(Step 5: Stable reduction) $\overline{\mathcal{M}}_g$ is proper ($\implies$ $\overline{M}_g$ proper)

We verified the valuative criterion for properness. In lecture, we only did the characteristic 0 case (using results from birational geometry and minimal model program of surfaces).

(Step 6: Projectivity) $\overline{M}_g$ is projective.

Today!

Remark 18.2. Sometimes you can work directly with the stack, but you have many more tools (e.g. intersection theory and Hodge theory) if you know it has a projective coarse moduli space.

18.2 Setup

- Let $\mathcal{U}_g \xrightarrow{\pi} \overline{\mathcal{M}}_g$ be the universal family
- Define the coherent sheaf
$$E_k := \pi_* \left(\omega_{\mathcal{U}_g/\overline{\mathcal{M}}_g}^{\otimes k}\right)$$
on $\overline{\mathcal{M}}_g$
- For $[\mathcal{C}] : S \xrightarrow{f} \overline{\mathcal{M}}_g$, note one has
$$f^* E_k = \pi_{S,*}(\omega_{\mathcal{C}/S}^{\otimes k}).$$
- E_k is a vector bundle by cohomology and base change.

This is really our first time making use of the notion of a coherent sheaf.[43] Why consider $\pi_* \left(\omega_{\mathcal{U}_g/\overline{\mathcal{M}}_g}^{\otimes k}\right)$?

(1) We get line bundles
$$\lambda_k := \det \pi_* \left(\omega_{\mathcal{U}_g/\overline{\mathcal{M}}_g}^{\otimes k}\right)$$
on $\overline{\mathcal{M}}_g$. Note: to show projectivity, you want existence of an ample line bundle.

For $S \to \overline{\mathcal{M}}_g$ corresponding to $\pi_S : \mathcal{C} \to S$, one has $\lambda_k|_S = \det \pi_* \left(\omega_{\mathcal{C}/S}^{\otimes k}\right)$.

(2) We get **multiplication maps**: for $\mathcal{C} \to S$ a family of curves, there's the natural map
$$\operatorname{Sym}^d \pi_* \left(\omega_{\mathcal{C}/S}\right) \longrightarrow \pi_* \left(\omega_{\mathcal{C}/S}^{\otimes d}\right).$$
For $C \to \operatorname{spec} k$, this is
$$\operatorname{Sym}^d \mathrm{H}^0(\omega_C) \to \mathrm{H}^0(\omega_C^{\otimes d}),$$
and its kernel is identified with
$$\ker = \left\{\text{degree } d \text{ equations defining } C \xrightarrow{|\omega_C|} \mathbb{P}^{g-1}\right\},$$
so (if C no hyperelliptic), can recover it from the data of this map.

More generally, given two integers k and d, we have a map
$$\operatorname{Sym}^k \pi_* \left(\omega_{\mathcal{C}/S}^{\otimes k}\right) \to \pi_* \left(\omega_{\mathcal{C}/S}^{\otimes dk}\right)$$
with kernel given by $\left\{\text{degree } k \text{ equations cutting out } \mathcal{C} \xrightarrow{|\omega_C^{\otimes k}|} \mathbb{P}^N\right\}$. When $k \geq 3$, the kth pluri-canoncial bundle is very ample, so can recover $\mathcal{C}$ from this multiplication map.

18.3 Survey of projectivity methods

Geometric Invariant Theory (GIT) The construction in this case depends on two integers
$$k \geq 5 \text{ and } d \gg 0.$$

[43] the structure sheaf played a role in Keel-Mori, but we didn't need more general coherent sheaves

A stable curve C is pluri-canonically embedded

$$C \overset{|\omega_C^{\otimes k}|}{\hookrightarrow} \mathbb{P}^{r(k)-1} \text{ where } r(k) = h^0(\omega_C^{\otimes k}) = (2k-1)(g-1).$$

Let $P(t)$ be the Hilbert polynomial of this embedding. Consider the locally closed locus

$$H' := \left\{ \left[C \hookrightarrow \mathbb{P}^{r(k)-1} \right] \middle| \begin{array}{c} C \text{ stable} \\ C \text{ embedded via } \omega_C^{\otimes k} \end{array} \right\} \subset \operatorname{Hilb}^P(\mathbb{P}^{r(k)-1},$$

and define $H := \overline{H'}$ so H is projectivity. For $d \gg 0$, we get an embedding

$$\begin{array}{ccc} H & \longrightarrow & \operatorname{Gr}(P(d), h^0(\mathbb{P}^{r(k)-1}, \mathscr{O}(d))) \\ {[C \hookrightarrow \mathbb{P}^{r(k)-1}]} & \longmapsto & \left[\Gamma(\mathbb{P}^{r(k)-1}, \mathscr{O}(d)) \twoheadrightarrow \Gamma(C, \mathscr{O}(d))\right] \end{array}$$

Note that

$$\Gamma(\mathbb{P}^{r(k)-1}, \mathscr{O}(d)) = \operatorname{Sym}^d \mathrm{H}^0(\omega_C^{\otimes k}) \text{ and } \Gamma(C, \mathscr{O}(d)) = \mathrm{H}^0(\omega_C^{\otimes dk}),$$

so the embedding of H into the Grassmannian associates an embedded curve to its multiplication map.

Furthermore, we have the Plucker embedding $\operatorname{Gr}(P(d), \Gamma(\mathbb{P}^{r(k)-1}, \mathscr{O}(d))) \hookrightarrow \mathbb{P}\Lambda^{P(d)}\Gamma(\mathbb{P}^{r(k)-1}, \mathscr{O}(d))$. Also $\mathrm{PGL}_{r(k)-1}$ acts on the Grassmian and the map from H is equivariant w.r.t this action.[44]

Note 13. Missed a couple things.

Remark 18.3. It is easy to show that $\overline{\mathcal{M}}_g = [H'/\mathrm{PGL}_{r(k)-1}]$.

The hard part (where GIT comes in) is showing that given $h = [C \subset \mathbb{P}^{r(k)-1}] \in H$, C is stable (as a curve) $\iff$ $h \in H$ is GIT stable with respect to L_d (i.e. $\exists s \in \Gamma(H, L_d^N)^{\mathrm{PGL}}$ s.t. $s(h) \neq 0$). Use Hilbert-Mumford criterion to production sections.

Question: Is this the pullback of $\mathscr{O}(1)$ coming from the Plucker embedding?

Answer: yes

In the end, one shows

Proposition 18.4. *The coarse moduli space of $\overline{\mathcal{M}}_g$ is*

$$\overline{M}_g = \operatorname{Proj}\left(\bigoplus_{n \geq 0} \Gamma(H, L_d^n)^{\mathrm{PGL}_{r(k)-1}} \right)$$

with the graded ring above finitely generated. Hence, $\overline{M}_g$ is projective.

Remark 18.5. Hilbert-Mumford criterion let's you produce sections, and ampleness of a line bundle is about showing existence of lots of sections.

For $k \geq 5$ and $d \gg 0$, you can compute the class of the ample line bundle that you get. It is $r(k)\lambda_{dk} - r(dk)\lambda_k$ on $\overline{M}_g$. As $d \to \infty$, the asymptotic limit is $\sim \left(12 - \frac{4}{k}\right)\lambda_1 - \delta$ with δ the boundary divisor. Can get even more:

Theorem 18.6 (Comalla-Harris). *$a\lambda_1 - \delta$ ample $\iff$ $a > 11$*

(GIT w/ $k = 5$ gives 11.2)

[44]This is how you know the line bundle L_d on H descends to one on the quotient stack $\overline{\mathcal{M}}_g$

Projectivity via Griffith's period maps This is a complex-analytic approach. The main idea is to consider the map from C to its Jacobian $\mathrm{Jac}(C) = \mathrm{H}^0(C, \omega_C)/\mathrm{H}_1(C, \mathbb{Z})$ (alternatively, consider map to Hodge structure $C \mapsto [\mathrm{H}^1(C, \omega_C) \subset \mathrm{H}^1(C, \mathbb{C})]$). This gives

$$\mathcal{M}_g \to h_g/\mathrm{Sp}_{2g}$$

(need to compatify this map, but let's not worry about that right now). The strategy is to show projectivity of h_g/Sp_{2g} and infer projectivity of $\overline{M}_g$ (via some Torelli theorem?).

We note that $\pi_*\omega_{\mathcal{C}/S}$ and $R^1\pi_*\mathbb{C}$ both play a role here.

Projectivity via positivity We have a coarse moduli space

$$\overline{\mathcal{M}}_g \xrightarrow{\pi} \overline{M}_g.$$

Fact. For n sufficiently divisible, each λ_k descends to $\overline{M}_g$, i.e. $\lambda_k = \pi^*\overline{\lambda}_k$.

We want to show that $\overline{\lambda}_k$ is ample. Here are some approaches

- Suppose

  (a) λ_k is **semiample** (i.e. $\lambda_k^{\otimes n}$ basepoint free for $n \gg 0$)

  (b) For every $T \to \overline{\mathcal{M}}_g$ non-trivial, $\deg \lambda_K|_T > 0$.

  Then λ_k is ample.

  Consider $\overline{\mathcal{M}}_G \to X \subset \mathbb{P}(\mathrm{H}^0(\lambda_k^{\otimes n}))$. **(a)** says the map is well-defined and **(b)** says it doesn't contract curves, so it must factor through the coarse moduli space $\overline{M}_g \xrightarrow{\exists!} X$. This map is quasi-finite and proper so finite over X (which is projective), so $\overline{M}_g$ is projective.

  Sadly, semiampleness is tough to check.

- Basepoint-free theorems can imply semiampleness.

  Example. big, nef, and $\bigoplus_{n \geq 0} \Gamma(\lambda_k^{\otimes n})$ finitely generated together imply semiample.

  These properties are not themselves easy to show. Can get finite generation using BCMM (some paper from the past 10 years. Apparently kind of a big deal).

- There are other ampleness criteria

  Theorem 18.7 (**Nakai-Moishezon criterion**). *If X a proper algebraic space with line bundle L, then L ample $\iff$ $\forall Z \subset X$ irreduce and closed $L^{\dim Z} \cdot Z > 0$.*

  Theorem 18.8 (**Kleiman's criterion**). *L ample $\iff$ $\forall C \in \overline{NE(X)}$, $C \cdot L > 0$. Here, $\overline{NE(X)}$ is the closure of the cone of curves.* Question: What's that

  Theorem 18.9 (**Seshadir's criterion**). *L ample $\iff$ $\exists \varepsilon > 0$ s.t. for all curves C, $C \cdot L > \varepsilon m(C)$ where $m(C)$ is the 'multiplicity of the singularities of C.'* Question: What's that mean?

18.4 Nefness

Definition 18.10. A vector bundle E on a scheme X is **nef** (or **semipositive**) if for all proper curves $T \xrightarrow{f} X$ and all line bundle quotients $F^* \twoheadrightarrow L$, $\deg L \geq 0$.

Remark 18.11. This is the case iff $\mathscr{O}_{\mathbb{P}E}(1)$ is nef on $\mathbb{P}E$ in the usual line bundle sense of the word.

Here are some properties

(1) Quotients and extensions of nef vector bundles are nef.

(2) Nefness is open in flat families.

(3) E nef $\implies \operatorname{Sym}^k E$ is nef.

Remark 18.12. We have this multiplication map

$$\operatorname{Sym}^d\left(\pi_*\omega_{\mathcal{C}/S}^{\otimes k}\right) \longrightarrow \pi_*\left(\omega_{\mathcal{C}/S}^{\otimes dk}\right).$$

If this is a surjection, the the source being nef implies that $\pi_*(\omega_{\mathcal{C}/S}^{\otimes dk})$ is nef.

Theorem 18.13. *Suppose we know that*

- $\overline{\mathcal{M}}_g$ *is a peopre DM stack (we do know this)*
- $\exists k_0 > 0$ *s.t. for all* $\mathcal{C} \to T$ *stable families,*

$$\pi_*(\omega_{\mathcal{C}/T}^{\otimes k}) \text{ nef for } k \geq k_0.$$

Then, $\lambda_k = \det \pi_*(\omega_{\mathcal{C}/T}^{\otimes k})$ *is ample for* $k \gg 0$.

Remark 18.14. Kollar motivation in proving this was not to construct $\overline{M}_g$, but to study moduli of surfaces and other higher dimensional varieties. His method is very much still relevant and useful.

This is closer to what was written down

- generalizes to any moduli of polarized varieties
  - [Kovais-Patakfolvi '17] Moduli of stable varieties in any dimension is projective.
  - [Codogni-Patakfilui '20] and [Xu-Zhang '20] Moduli of K-polystable Fano varities is projective
- Nefness of $\pi_*\left(\omega_{\mathcal{C}/T}\right)$ is easier and is classical
- Harder to show nefness for $\pi_*\left(\omega_{\mathcal{C}/T}^{\otimes k}\right)$ (despite the fact that they are actually more positive)

Sounds like this is enough for $\overline{M}_g$ using some ad-hoc argument. However, for other moduli spaces, you really wanna show nefness for push-forwards of higher pow-ers

18.5 The ampleness lemma

Setup. Let X be a proper algebraic space. Let W be a rank w vector bundle with reductive structure group $G \to \mathrm{GL}_w$ (i.e. its transition functions live in G). Let $W \twoheadrightarrow Q$ be a quotient bundle of rank q.

There is a classifying map

$$X \longrightarrow [\mathrm{Gr}(q,w)/G]$$

taking a point x to the quotient of the fibers $x \mapsto [W_x \twoheadrightarrow Q_x]$. To get a point in the Grassmannian, need to choose bases $W_x \cong k^w$ and $Q_x \cong k^q$; this choice is well-defined up to G.

Lemma 18.15 (**Ampleness Lemma**, char 0 version)**.** *If in addition,*

(1) *W is nef*

(2) *$X \to [\mathrm{Gr}(q,w)/G]$ is quasi-finite*

Then $\det Q$ *is ample.*

Remark 18.16. The easy case is W trivial, so the structure group $G = \{1\}$ is also trivial. Hence, get $X \to \mathrm{Gr}(q,w)$ between proper spaces. Since we've assumed it is quasi-finite, this map is quasi-finite and proper (so finite) which implies that X is projective.

Not that we are not assuming the image of X lands in the G-stable locus. But if it did, you would get

$$X \to [\mathrm{Gr}(q,w)^{ss}/G] \twoheadrightarrow \mathrm{Gr}(q,w)//G$$

where $\mathrm{Gr}(q,w)//G$ is the projective GIT quotient. The map $X \to \mathrm{Gr}(q,w)//G$ is quasi-finite, so finite, so X is projective. You get that $\det Q$ is ample, and even that $(\det Q)^w \otimes (\det W)^{-q}$ is ample.

The main idea of the proof of the ampleness lemma is to use nefness to verify Nakai-Moishezon.

18.6 Application to $\overline{\mathcal{M}}_g$

We restate the results we need.

Lemma 18.17 (**Ampleness Lemma**, char 0 version)**.** *If in addition,*

(1) *W is nef*

(2) *$X \to [\mathrm{Gr}(q,w)/G]$ is quasi-finite*

Then $\det Q$ *is ample.*

Theorem 18.18. *Suppose we know that*

- *$\overline{\mathcal{M}}_g$ is a peopre DM stack (we do know this)*
- *$\exists k_0 > 0$ s.t. for all $\mathcal{C} \to T$ stable families,*

$$\pi_*(\omega_{\mathcal{C}/T}^{\otimes k}) \text{ nef for } k \geq k_0.$$

Then, $\lambda_k = \det \pi_*(\omega_{\mathcal{C}/T}^{\otimes k})$ is ample for $k \gg 0$.

Proof Sketch. Assume the ampleness lemma. Consider the universal curve

$$\mathcal{C} = \mathcal{U}_g \xrightarrow{\pi} S = \overline{\mathcal{M}}_g.$$

Choose k and d such that

- $\omega_{\mathcal{C}/S}^{\otimes k}$ is relatively very ample

- $R^1\pi_*\omega_{\mathcal{C}/S}^{\otimes k} = 0$

- Every curve $C \overset{|\omega_C^{\otimes k}|}{\hookrightarrow} \mathbb{P}^w$ is cut out by degree $\le d$ equations

- $\pi_*(\omega_{\mathcal{C}/S}^{\otimes k})$ is nef

- The multiplication map

$$\underbrace{\operatorname{Sym}^d \pi_*\left(\omega_{\mathcal{C}/S}^{\otimes k}\right)}_{W} \twoheadrightarrow \underbrace{\pi_*\left(\omega_{\mathcal{C}/S}^{\otimes dk}\right)}_{Q}$$

  is surjective.

Note that W has rank w and Q has rank q. Also note that the structure group of W is $G = \mathrm{PGL}_{r(k)}$. We know W is nef by hypothesis. We claim

Claim 18.19. *The map*

$$\overline{\mathcal{M}}_g \to [\mathrm{Gr}(q, w)/G]\,,$$

taking a curve to its multiplication map, is injective (so quasi-finite among other things).

Why is this true? Note that $W = \mathrm{H}^0(\mathbb{P}^N, \mathscr{O}(d))$ and $Q = \mathrm{H}^0(C, \mathscr{O}(d))$. The kernel of this map determines C, so that give injectivity.

To apply the Ampleness lemma, we need a scheme (or algebraic space?), not an algebraic stack, so take a finite cover $U \xrightarrow{\text{fin}} \overline{\mathcal{M}}_g$ with U a scheme. ■

Here's the logical order of Kollár's argument

- Use Nakai-Moishezon to proof ampleness lemma

- Use ampleness lemma to proof the above theorem

- Show the last remaning theorem below

Theorem 18.20. *Let $\mathcal{C} \xrightarrow{\pi} T$ (T a smooth projective curve) be a stable family of curves of genus $g \ge 2$ over a field k. Then, $\pi_*\left(\omega_{\mathcal{C}/T}^{\otimes k}\right)$ is nef for $k \ge 2$.*

Remark 18.21. Similar strategy can show $\overline{M}_{g,n}$ is projective.

The first reductions: one may assume

- $\mathcal{C}$ smooth and minimal surface

- $\mathcal{C} \to T$ is generically smooth

- genus of T is ≥ 2 ($\implies$ $\mathcal{C}$ is general type)

Can reduce to positive characteristic case. Suppose $\operatorname{char} k = 0$. Since everything is finite type, $\exists A \subset k$ f.g. $\mathbb{Z}$-subalgebra so $\mathcal{C}, T$ spread out to $\widetilde{T}, \widetilde{\mathcal{C}}$ defined over A. After enlarging A, can arrange the $\widetilde{\mathcal{C}} \to \widetilde{T}$ and $\widetilde{T} \to \operatorname{spec} A$ are flat. Can also arranges that all fibers satisfy the conditions of the first reduction. Now, the closed points of $\operatorname{spec} A$ are in positive characteristic. Since Nefness is open in flat families, we get it also in characteristic 0.[45]

[45]Completment of nef locus is a closed set containing no closed points, but this is impossible since T quasi-compact

Assumption. To simply things, still assume $p \neq 2$. Can make this works with some modification even if $p = 2$

We now need some birational input.

Theorem 18.22 (Ekedahl). *In* char $= p > 0$, *if*

- *S smooth projective minimal surface of general type*
- *D effective divisor with $D^2 = 0$*

Then $\mathrm{H}^1(S, \omega_S^{\otimes n}(D)) = 0$ *for* $n \geq 2$.

Remark 18.23. Ekedahl actually showed $\mathrm{H}^1(S, \omega_S^{\otimes(-n)}) = 0$ for $n \geq 1$. It's not hard to show this implies the version we stated (use Serre dual and a short exact sequence involving the divisor).

Proof Sketch of Theorem 18.20. If not nef, $\exists \pi_* \left(\omega_{\mathcal{C}/T}^{\otimes k}\right) \twoheadrightarrow M^\vee$ with $d := \deg M > 0$. Consider Frobeinus

$$\begin{tikzcd} \mathcal{C} \arrow[r, "F"] \arrow[d] & \mathcal{C} \arrow[d] \\ T \arrow[r, "F"] & T \end{tikzcd}$$

Then,

$$F^* \pi_* \left(\omega_{\mathcal{C}/T}^{\otimes k}\right) = \pi_* \left(\omega_{\mathcal{C}/T}^{\otimes k}\right) \text{ and } \deg F^* M = pd.$$

Hence we can arrange $d \gg 0$!

So we may assume $M = \omega_T^{\otimes k} \otimes L$ with L very ample. Thus we get a surjection

$$\pi_* \omega_{\mathcal{C}/T}^{\otimes k} \otimes \omega_T^{\otimes k} \otimes L \twoheadrightarrow \mathscr{O}_T.$$

Note $h^1(\mathscr{O}_T) = g(T) \geq 2$. Apparently also $h^1(LHS) \geq 2$. Use Leray spectral sequence to relate

$$\mathrm{H}^1\left(\pi_* \omega_{\mathcal{C}/T}^{\otimes k} \otimes \omega_T^{\otimes k} \otimes L\right) \text{ and } \mathrm{H}^1(\mathcal{C}, \omega_{\mathcal{C}}^{\otimes k} \otimes \pi^* L).$$

In particular, you get that the right hand space has dimension ≥ 2. This contradicts Ekedahl. Unravelling the chain of implications, this shows projectivity of $\overline{M}_g$, so we win. ■

"Man, that was some course. I'm surprised you're still here."

19 List of Marginal Comments

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