AWS '24 Notes

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These are notes on talks given in "Arizona Winter School" which took place at University of Arizona. Unfortunately for the reader, these notes are live-texed and so their quality is upper bounded both by my (quite) limited ability to understand the material in real time and by my typing speed. With that in mind, they are doubtlessly missing content/insight present in the talks and certainly contain confusions not present in the talks. Despite all this, I hope that you can still find some use of them. Enjoy and happy mathing.

The website for this seminar is available here.

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1 Rachel Pries: The Torelli locus in the moduli space of abelian varieties, with applications to Newton polygons of curves

1.1 Lecture 1 (3/2): Torelli locus and Newton polygons

1.1.1 Families of Abelian varieties

arithmetic	geometry
Jacobians	moduli spaces
cyclic action	

In this talk, want to mention broad perspectives with open questions and examples. Hopefully have roughly half of the lectures over $\mathbb{C}$ and have over something like $\overline{\mathbb{F}}_p$. Get new invariants in positive characteristic.

Setup 1.1.1. Let X be an abelian variety (with a principal polarization) of dimension g.

(Today, all our examples will have $g = 2$)

Note that $X \cong \mathbb{C}^2/\Lambda$, so we need to think of how to write down this lattice Λ. Well, after choosing a basis for $\mathbb{C}^2$, we can represent a $\mathbb{Z}$-basis for Λ as a matrix of 4 column vectors, the `period matrix`. One can always assume this is in the form

$$\left[\begin{array}{c|cc} Z & 1 & 0 \\ & 0 & 1 \end{array}\right]$$

Since X is algebraic, one ends up showing that we must have $Z = Z^t$ (something something Riemann something something).

Lemma 1.1.2. *The moduli space $\mathcal{A}_2$ of principally polarizared abelian vareities of dimension 2 is 3-dimensional.*

(3 parameters needed to write down a symmetric 2×2 matrix).

Theorem 1.1.3 (Siegal)**.** *The moduli space $\mathcal{A}_g$ of PPAVs of dimension g is irreducible of dimension $\binom{g+1}{2} = g(g+1)/2$.*

Example 1.1.4. Let C be a curve of genus 2. Then, Riemann-Roch tells us that C must be hyperelliptic, so we have a double cover $\pi\colon C \to \mathbb{P}^1$. Thus,

$$C\colon y^2 = h(x) \text{ for some } \text{ polynomial } h(x).$$

By Riemann-Hurwitz, π has 6 branch points. We may assume that they are $\{0, 1, \infty, \lambda_1, \lambda_2, \lambda_3\}$ so

$$h(x) = x(x-1)(x-\lambda_1)(x-\lambda_2)(x-\lambda_3).$$

These $\lambda_1, \lambda_2, \lambda_3$ determine the curve. △

Lemma 1.1.5. *The moduli space $\mathcal{M}_2$ of smooth genus 2 curves is irreducible of dimension 3.*

Note that $\dim \mathcal{M}_2 = \dim \mathcal{A}_2$.

Theorem 1.1.6. *The moduli space $\mathcal{M}_g$ of smooth curves of genus g is irreducible of dimension $3g-3$, say for $g \geq 2$.*

Remark 1.1.7 (Recall from Barry's lecture)**.** We have a map

$$\begin{array}{rccc} \tau: & \mathcal{M}_g & \longrightarrow & \mathcal{A}_g \\ & C & \longmapsto & \mathrm{Jac}(C) \end{array}$$

○

Example 1.1.8. Say $g = 2$. We claim that every point on $\mathrm{Jac}(C)$ can be written in the form $Q_1 + Q_2 - 2Q_0$ (Q_0 fixed, but Q_1, Q_2 varying). △

Theorem 1.1.9 (Torelli)**.** *If $C_1 \ncong C_2$, then $\mathrm{Jac}(C_1) \ncong \mathrm{Jac}(C_2)$, i.e. $\tau : \mathcal{M}_g \to \mathcal{A}_g$ is injective on points.*

Generally, it's easier to study Jacobians than it is to study general abelian varieties.

Corollary 1.1.10. *For $g = 2$, almost every abelian surface is a Jacobian of a genus 2 curve.*

(the ones that aren't are direct sums $E_1 \oplus E_2$ of elliptic curves).
Similar story for $g = 3$ (notes that $3g - 3 = 6 = \binom{3+1}{2}$), but this is no longer true for $g \geq 4$.

Warning 1.1.11. Furthermore, not every abelian variety (say, over $\overline{\mathbb{Q}}$) is isogenous to a Jacobian. •

Question 1.1.12. *Given an abelian variety $A/\mathbb{C}$, when is it a Jacobian?*

Open Question 1.1.13 (Ekedahl–Serre)**.** *Fix $g \geq 2$. Does there exist a smooth curve C of genus g such that $\mathrm{Jac}(C) \sim E_1 \oplus \cdots \oplus E_g$ is isogenous to a product of elliptic curves?*

In many cases, the answer is yes! In fact, Ekedahl–Serre showed yes for $g = 1297$. It's also known to be yes for $g = 38$ (LMFDB). Sounds like the first open case is $g = 59$ and the second is $g = 66$.

1.1.2 Characteristic p

Let's go back to $g = 1$.

Setup 1.1.14. Fix $k = \overline{k}$ of characteristic p. Let E/k be an elliptic curve.

Definition 1.1.15. We say E is `supersingular` if $\mathrm{End}(E)$ is not commutative. Equivalently,

- $E[p](k) = 0$ (otherwise, $\#E[p](k) = p$ and we say E is `ordinary`)
- Say $E/\mathbb{F}_q$. Then, $\#E(\mathbb{F}_q) = (q + 1) - a$ and supersingular $\iff p \mid a$.
- Say $E/\mathbb{F}_q$ with zeta function (numerator called the `L-polynomial`)

$$Z(E/\mathbb{F}_q, T) = \frac{1 - aT + qT^2}{(1 - T)(1 - qT)}.$$

From numerator, get Newton polygon with vertices $(k, v_p(c_k))$ (c_k the coeffs of the L-polynomial). The Newton polygon here will have slopes 0,1 or slopes 1/2. Supersingular $\iff$ slopes 1/2.

◇

Theorem 1.1.16 (Deuring)**.** *For every prime p, there is a supersingular elliptic curve E defined over $\mathbb{F}_{p^2}$.*

Igusa: $y^2 = x(x-1)(x-\lambda)$ (internet troubles...) Cartier operatoar $C \curvearrowright \mathrm{d}x/y$ via $\mathrm{d}x/y \mapsto f(p, \lambda)\mathrm{d}x/y$ and supersingular if λ is a root of the polynomial or something like this?

Definition 1.1.17. For $g \geq 2$, we'll say X is `supersingular` if it is isogenous to a product of g supersingular elliptic curves. ◇

Open Question 1.1.18. *Given $g \geq 2$ and give p, does there exist sa smooth curve C of genus g such that $\mathrm{Jac}(C)$ is supersingular?*

(say above question over $\overline{\mathbb{F}}_p$)

Warning 1.1.19. No direct implication between this question and the earlier one of Ekedahl–Serre •

Remark 1.1.20. One can characterize supersingularity in terms of the Newton polygon of the characteristic polynomial of Frobenius. Supersingular means that this polygon is a line segment of slope 1/2. If is has slopes 0 and 1, one calls this being `ordinary`. For $g \geq 2$, there are more possibilities in between these two. ○

Question 1.1.21. *Does there exist a supersingular smooth curve of genus g over $\overline{\mathbb{F}}_p$?*

Answer (Van de Geer + Vlugt, up to spelling)**.** If $p = 2$, answer is yes for all g. ⋆

Answer (Serre)**.** Yes if $g = 2$ ⋆

Answer (Oort)**.** Yes if $g = 3$ ⋆

Answer (Kudo/Harashita/Senda, up-to-spelling)**.** Yes if $g = 4$ ⋆

(Give new proof of $g = 4$ case in lecture 2?)
We end today with a proof of Serre's result, but a different proof than the one that Serre gave.

Theorem 1.1.22. *For all primes $p > 5$, there exists a smooth curve of genus 2 that is supersingular.*

(Following proof do to Ibukiyama, Katsura, and Oort)

Proof. Consider the following families of curves of genus 2

(i) $y^2 = (x^3 - 1)(x^3 - t)$

This have action by an cube root of unity, so one has $S_3 \subset \mathrm{Aut}(C)$ (also have hyperelliptic involution)

(ii) $y^2 = x(x^2 - 1)(x^2 - t)$

These have $D_4 \subset \mathrm{Aut}(C)$ (D_4 = dihedral group of order 8)

Have two 1-parameter families in the 3-dimensional $\mathcal{M}_2$. What IKO show is that

$$\#\{\text{s.s. curves in these families}\} \sim \frac{p-1}{6} \text{ or } \frac{p-1}{8}$$

Their strategy is to show that the C's in these families dominator two elliptic curves $C \rightrightarrows E_1, E_2$, so $J_C \sim E_1 \oplus E_2$. They show that E_1 is supersingular $\iff$ E_2 is supersingular. They do this by studying the Cartier operator on $\mathrm{H}^0(C, \Omega^1)$; sounds like it's some polynomial $f(t,p)$ in t, p times a diagonal or anti-diagonal matrix, so one counts roots of this polynomial? ■

Example 1.1.23 (Miller, 1972)**.** $y^2 = x^{2g+1} + tx^{g+1} + x$ if $p \nmid g$ or $y^2 = x^{2g+2} + tx^{2g+1} + 1$ if $p \mid g$. Their Jacobians are ordinary for generic t, but not ordinary for some t's. △

1.2 Lecture 2 (3/3): Complete families over perfect fields

Recall 1.2.1. $\mathcal{M}_g$ is the moduli space of smooth curves of genus g, and $\mathcal{A}_g$ is the moduli space of principally polarized abelian varieties of dimension g. Between them is the Torelli map $\mathcal{M}_g \to \mathcal{A}_g$. ⊙

Remark 1.2.2. Neither $\mathcal{M}_g$ nor $\mathcal{A}_g$ is complete (= proper). There are families of smooth curves which degenerate to singular curves. ∘

Notation 1.2.3. We write $\overline{\mathcal{M}}_g$ for the Deligne–Mumford compactification of $\mathcal{M}_g$.

Seems there are lots of compactifications of $\mathcal{A}_g$. We'll try and mention two today.
Sounds like our examples will be in genus 4 today. Start with a discussion of singular curves.

Definition 1.2.4. A singular curve is `of compact type` if its dual graph is a tree. ⋄

One way to construct such a curve is to glue together two pointed curves, i.e. have a map

$$\kappa\colon \overline{\mathcal{M}}_{g_1,1} \times \overline{\mathcal{M}}_{g_2,1} \longrightarrow \overline{\mathcal{M}}_{g_1+g_2}.$$

The image of this map is called $\Delta_{g_1} = \Delta_{g_2}$. In this case, one has $\mathrm{Jac}(C) = \mathrm{Jac}(C_1) \oplus \mathrm{Jac}(C_2)$, so genera add, a-numbers add, p-ranks add, and Diedonné module is the product.

How to get a singular curve of non-compact type? Can, for example, glue two points on a single curve, i.e.

$$\kappa\colon \overline{\mathcal{M}}_{g_1,2} \longrightarrow \overline{\mathcal{M}}_{g_1+1}.$$

The image of this map is called Δ_0. In this case, one has

$$0 \longrightarrow T \longrightarrow \mathrm{Jac}(D) \longrightarrow \mathrm{Jac}(C) \longrightarrow 0,$$

where T is an algebraic torus (so $\mathrm{Jac}(D)$ is semi-abelian instead of abelian). Turns out that

$$\delta\mathcal{M}_g = \overline{\mathcal{M}}_g \setminus \mathcal{M}_g = \Delta_0 \cup \bigcup_{i=1}^{g/2} \Delta_i.$$

Question 1.2.5. *Let's say we want to avoid the boundary. What's the largest dimension of a complete subvariety (or substack) of $\mathcal{M}_g, \mathcal{M}_g^{\mathrm{ct}}, \mathcal{A}_g$?*

($\mathcal{M}_g^{\mathrm{ct}}$ = compact type curves)

Remark 1.2.6. Sounds like the following are known

- For $\mathcal{M}_g$, get subvarieties of dimension $\leq g-2$ (Diaz)
- For $\mathcal{M}_g^{\mathrm{ct}}$, goes up to $\leq 2g-3$ (Diaz)
- For $\mathcal{A}_g$, can get $g(g-1)/2 = \binom{g}{2}$ (van-der Gear, up to spelling)

  Sounds like you can decrease by 1 over $\mathbb{C}$.

Note for first and last bullet points, can have codimension g at lowest. These bounds are not known to be sharp. ∘

Example 1.2.7 ($g = 4$)**.** Not known if there is a complete surface (notes $4-2=2$) in $\mathcal{M}_4$. For $\mathcal{A}_4$, one knows can have at most dimension 5 over $\mathbb{C}$ (example not known), but at most 6 over $\overline{\mathbb{F}}_p$ (example known). For $\mathcal{M}_4^{\mathrm{ct}}$, 4 over $\mathbb{C}$ (no example) but 5 over $\overline{\mathbb{F}}_p$ (known example). △

Notation 1.2.8 (Rachel defined this notation earlier in lecture, but I was lazy.)**.** Write U_M for maximal dimension of a complete subvariety of M.

Theorem 1.2.9. *If $\mathcal{M}_g$, there is a complete curve, i.e. a 1-dimensional family of curves that does not hit the boundary.*

Note: can't look at hyperelliptic curves (get singular curve when branch points collide).

Proof for $g = 4$. We'll give two arguments.

- More concrete argument.

  Let $E\colon y^2 = x^3 - 1$, an elliptic curve, and let $C\colon\ : y^2 = x^6 - 1$, so $C \to E$ a double cover. Now, we'll choose and vary a double cover $Z \to C$ of C. Consider two points $0_E, Q$ in E (make sure Q is not 2-torsion in E). Consider $W \subset E^2 \setminus \Delta$ defined as the set of points $\{(P, P+Q)\}$. Note that $\mathrm{pr}_1 : W \xrightarrow{\sim} E$, so W gives a complete curve inside $E^2 \setminus \Delta$. Let $W_C \subset C^2 \setminus \Delta$ be the pullback of W, another complete curve. Now, we let $Z \to C$ be the double curve branched at the pair of points given by any element of W_C. Riemann-Hurwitz guarantees that $g(Z) = 4$, so this gives a complete family (parameterized by W_C) in $\mathcal{M}_4$.

  Remark 1.2.10. Apparently the curve in $\mathcal{M}_4$ that you product in this way is not connected. ○

- More abstract argument.

  We'll consider teh "minimal compactification" of $\mathcal{A}_4$. Sounds like this corresponds to something like jacobians of curves of compact type. One knows that the boundary here is codimension ≥ 2, so a general complete curve won't intersect it. ■

Note 1. Got distracted and missed some things.

Something about p-rank, $f = \dim_{\mathbb{F}_p} \mathrm{Hom}(\mu_p, X)$. This is positive for semi-abelian varieties which hve a nonzero toric part.

Let $\overline{\mathcal{M}}_g^0 \subset \mathcal{M}_g$ be the locus of smooth curves of genus g w/ p-rank 0, and let $\mathcal{A}_g^0 \subset \mathcal{A}_g$ be the p-rank 0 locus.

Theorem 1.2.11 (Norman–Oort)**.** $\operatorname{codim}(\mathcal{A}_g^0, \mathcal{A}_g) = g$.

Theorem 1.2.12 (Faber–Van der-Geer)**.** $\operatorname{codim}(\mathcal{M}_g^0, \mathcal{M}_g) = g$.

(note this locus only makes sense in characteristic p).

The idea behind these proofs is that each time you decrease the p-rank, the dimension should go down by one. So, the typical point is ordinary (use theorem of Katz that p-rank increases in deformation), and each time you increase the p-rank, the dimension goes down by 1.

Slogan. The Torelli locus is dimensinoally transverse to the p-rank stratification.

Theorem 1.2.13 (Kudo/Harashita/Senda)**.** *For all primes p, there exists a smooth curve of genus $g = 4$ that is supersingular (i.e. $\mathrm{Jac}(C) \sim E^4$ for E s.s.)*

(sounds like below proof due to Pries & Oort last summer. Sounds like old proof showed a-number ≥ 3)

Proof. Let $T \subset \mathcal{A}_4$ be the Torelli locus, the Jacobians of curves of compact type. This is a divisor ($\dim T = 12 - 3$ while $\dim \mathcal{A}_4 = \binom{5}{2}$). Consider $\mathcal{A}_4[ss] \subset \mathcal{A}_4^0 \subset \mathcal{A}_4$. One knows $\dim \mathcal{A}_4^0 = 6$ (codim g) and that $\dim \mathcal{A}_4[ss] = 4$. Want to show that $\Gamma \coloneqq T \cap \mathcal{M}_4[ss]$ contains a point coming from a smooth curve. Note that Γ is nonempty (e.g. take a chain of four supersingular elliptic curves). Let $\Gamma^{\text{sing}} \subset \Gamma$ be Jacobians of singular compact type curves. It will suffice to show that $\dim(\Gamma^{\text{sing}}) \le 2$. These look like jacobians of (a nodal union of a genus 1 curve w/ a genus 4 curve) or (a nodal union of 2 genus 2 curves). For such things, the choice of clutching point does not afftect the Jacobian. In the first case, the elliptic curve is ss (so 0-dim family) and the genus 3 curve is ss also (so 2-dim family apparently). In the second case, we have two ss genus 2 curves (so two 1-dim families, apparently), so one concludes that $\dim(\Gamma^{\text{sing}}) = 2$. On the other hand, $\dim \Gamma \ge 3 = \dim \mathcal{A}_4[ss] - \operatorname{codim} T$. ■

A similar idea shows that, for all p, there exists a smooth genus 5 curve with slopes $(1/4, 3/4) \oplus (1/2, 1/2)$.

1.3 Lecture 3 (3/3): Abelian varieties + curves with cyclic action

Goal ((not so) hidden agenda). Do smooth curves with interesting Newton polygons exist? Does the open Torelli locus (i.e. Jacobians of smooth curves only) intersect NP strata that have small dimension?

Division between arithmetic people and geometric people: arithmetic people will say "yes!" while the geometric people will say "well...". We'll discuss the arithmetic side today and the geometric side in the next time.

Remark 1.3.1. Today we'll look at examples in $g = 6$. ○

History. Consider $C\colon y^m = x(x-1)$ (say with m an odd (prime) number). This curve is famous for several reasons. It's naturally a cover of $\mathbb{P}^1$ branched at $0, 1, \infty$. It is also a quotient of a Fermat curve. Note that $g(C) = g_C = \frac{m-1}{2}$ and that C has an order m automorphism $(x, y) \mapsto (x, \zeta_m y)$. Hence, $\operatorname{Jac}(C)$ has CM by $\mathbb{Q}(\zeta_m)$. Weil looked at the eigenvalues of Frobenius, studying them using Jacobi sums. Let $f :=$ the order of p mod m. If f is even, then C is supersingular.

Example 1.3.2. For $m = 13$, $g = 6$ and C is supersingular if $p \not\equiv 1, 3, 9 \bmod 13$. △

⊖

Can we generalize this example? Keep the μ_m-action, but move away from the CM case. A `cyclic` μ_m`-cover` is a cover $C \to \mathbb{P}^1$ w/ Gal group μ_m. Can write them as

$$y^m = \prod_{i=1}^{N} (x - b_i)^{a_i}.$$

Assumption.

- Assume $\sum a_i \equiv 0 \mod m$, so $N = \#$ branch points (the b_i's are the branch points)

Above b_i, number of points in the fiber is $\gcd(N, a_i)$ and the ramification degree at each is $N/\gcd(N, a_i)$, assuming I heard correctly.

We may as well move 3 of the branch points to $0, 1, \infty$.

Example 1.3.3.

$M[16]\colon y^5 = x(x-1)(x-b_1)(x-b_2)$. This has $m = 5 = N$ and $g = 6$.

$M[19]\colon y^9 = x(x-1)(x-b_1)$. This has $m = 9$, $N = 4$, and $g = 7$. △

Definition 1.3.4. The data $\vec{a} = (a_1, \ldots, a_N)$, where $1 \le a_i < m$ with $\sum a_i \equiv 0 \mod m$, is called the `inertia type`. The dta $\gamma = (m, N, \vec{a})$ is called the `monodromy data`. ◇

We write $\mathcal{H}_\gamma$ for the Hurwitz space of cyclic μ_m-covers $C \to \mathbb{P}^1$ branched at N points, with inertia $\vec{a}$. It is irreducible, and naturally comes equipped with a map

$$\mathcal{H}_\gamma \longrightarrow \mathcal{M}_g.$$

Note that: have $N-3$ branch points (other than $0, 1, \infty$), so $\dim \mathcal{H}_\gamma = N - 3$.

Now, if $\mu_m \curvearrowright C$, can write $\mathrm{H}^0(C, \Omega^1) = \bigoplus L_i$, where τ acts on L_i by multiplication by ζ_m^i.

Fact. $\vec{a}$ determined $f_i = \dim L_i$.

The collection $\vec{f} = (f_1, \ldots, f_{m-1})$ is called the `signature type` (apparently, $f_0 = 0$).

Example 1.3.5. The sig type of $M[16]$ is $f = (3, 2, 1, 0)$. △

1.3.1 Abelian varieties

Let X be an abelian variety with μ_m-action with signature $\vec{f}$. What does the moduli space S_γ of these look like?

To summarize:

$$\begin{tikzcd} H_\gamma \arrow[r] \arrow[d] & S_\gamma \arrow[d] \\ \mathcal{M}_g \arrow[r, "T"] & \mathcal{A}_g \end{tikzcd}$$

This S_γ is a Deligne-Mostow Shimura variety (and is a unitary Shimura variety of PEL-type).

Fact. If m is odd, then

$$\dim S_\gamma = \sum_{i=1}^{\frac{m-1}{2}} f_i f_{m-i}.$$

Fact. Image of $\mathcal{M}_g \to \mathcal{A}_g$ is open and dense for $g = 2, 3$.

Note 2. I think I'm gonna stop paying attention...

2 Ben Moonen: Algebraic cycles on abelian varieties

2.1 Lecture 1 (3/2): Algebraic Cycles on Abelian Varieties

We will discuss Chow groups of abelian varieties in this course. In the background will be the theory of motives, but these will not play much of a role directly in the lectures.

We'll work over any field k.

Assumption. Symbols X, Y will always refer to smooth, projective varieties $/k$

Definition 2.1.1. Given $i, j \in \mathbb{Z}$, we define

$$Z^i(X) = \mathbb{Z} \cdot \left\{ \begin{matrix} \text{closed, irreducible subvars } Z \subset X \\ \text{of codimension } i \end{matrix} \right\},$$

the free abelian group on codim i subvarieties. We also define

$$Z_j(X) = \mathbb{Z} \cdot \left\{ \begin{matrix} \text{closed, irreducible subvars } Z \subset X \\ \text{of dimension } j \end{matrix} \right\}.$$

Note that $Z_j(X) = Z^{\dim X - j}(X)$. These groups are absolutely massive. $\diamond$

To get something more manageable, we mod out by rational equivalence.

Say $W \subset X$ is irreducible of dimension $i-1$ and some nonzero function $f \in K(W)$. Associated to f is a divisor $\operatorname{div}(f)$. Then, `rational equivalence` is the equivalence relation on cycles generated by things of the form $\operatorname{div}(f) \sim_{\text{rat]}} 0$. There's also a more geometric take on this definition. Consider $X \times \mathbb{P}^1 \to \mathbb{P}^1$. Imagine you have a flat family of cycles $V \subset X \times \mathbb{P}^1$. Then, for any $t_0, t_1 \in \mathbb{P}^1$, `rational equivalence` is, equivalently, generated by all relations of the form $V_{t_0} \sim_{\text{rat}} V_{t_1}$. We define

$$\mathrm{CH}^i(X) \coloneqq Z^i(X)/\sim_{\text{rat}} \quad \text{and} \quad \mathrm{CH}_j(X) = Z_j(X)/\sim_{\text{rat}}.$$

Furthermore, we set

$$\mathrm{CH}(X) \coloneqq \bigoplus_i \mathrm{CH}^i(X) = \bigoplus_j \mathrm{CH}_j(X).$$

One also considers $\mathrm{CH}(X)_{\mathbb{Q}} = \mathrm{CH}(X) \otimes \mathbb{Q}$.

Example 2.1.2 (X irreducible)**.**

- $\mathrm{CH}^0(X) = \mathbb{Z} \cdot [X]$.
- $\mathrm{CH}^1(X) = \mathrm{Cl}(X) \cong \mathrm{Pic}(X)$.

"These are basically on the only two cases where we sort of understand what we are doing." $\triangle$

Construction 2.1.3. By passing to rational equivalence classes, we win a multitude of operations.

(pushforward) $f\colon X \to Y$ induces $f_*\colon \mathrm{CH}(X) \to \mathrm{CH}(Y)$. Given $Z \subset X$, have closed (b/c proper) $f(Z) \subset Y$. If $Z \to f(Z)$ is generically finite of degree d, we set $f_*[Z] = d[f(Z)]$; else, $f_*[Z] = 0$.

Remark 2.1.4. This preserves the grading by dimension of cycles, which can be suggested by writing $f_*\colon \mathrm{CH}_*(X) \to \mathrm{CH}_*(Y)$. $\circ$

(pullback) $f\colon X \to Y$ induces $f^*\colon \mathrm{CH}^*(Y) \to \mathrm{CH}^*(X)$ which preserves codimension grading. This map is harder to develop than the pushforward. In the special case that f is flat, for $W \subset Y$, one has $f^*[W] = [f^{-1}(W)]$ (this implicitly takes into account the multiplicity of its components).

(Intersection product) $\mathrm{CH}^*(X)$ is a commutative graded ring, i.e. have

$$\cdot\colon\ \mathrm{CH}^i(X)\times\mathrm{CH}^j(X)\longrightarrow\mathrm{CH}^{i+j}(X).$$

As a special case, saw we have subvarieties $W,Z\subset X$ which intersect transversally. In this case, $[W]\cdot[Z]=[W\cap Z]$.

(Exteriod product) Say we have $X,Y/k$ along with classes $\alpha\in\mathrm{CH}^i(X)$ and $\beta\in\mathrm{CH}^j(Y)$. Then, can form $\alpha\times\beta\in\mathrm{CH}^{i+j}(X\times Y)$.

Idea: if $\alpha=[W]$ and $\beta=[Z]$, then $\alpha\times\beta[W\times Z]$. ○

(Above, we liberally (but implicitly) used that X,Y are smooth, projective over a field)
Let's mention various relations among the above constructions

- $\alpha\cdot\beta=\Delta^*(\alpha\times\beta)$, where $\Delta\colon X\to X\times X$ is the diagonal.
- $\alpha\times\beta=\mathrm{pr}_1^*(\alpha)\cdot\mathrm{pr}_2^*(\beta)$.
- `projection formula`: given $f\colon X\to Y$, have

$$f_*(f^*(\alpha)\cdot\beta)=\alpha\cdot f_*(\beta)$$

f_* is a $\mathrm{CH}(Y)$-module map (or something like this)

Setup 2.1.5. Let X/k be an abelian variety of dimension g. Let $m\colon X\times X\to X$ be the group law.

Observation 2.1.6. $\mathrm{CH}(X)$ has a second ring structure! In addition to the intersection product, there's also the Pontryagin product

$$\star\colon\ \mathrm{CH}_i(X)\times\mathrm{CH}_j(X)\longrightarrow\mathrm{CH}_{i+j}(X)$$

given by $\alpha\star\beta=m_*(\alpha\times\beta)$. This makes $\mathrm{CH}(X)$ into a commutative graded ring, where now one uses the *homological* grading!

Moreover, consider $X\xrightarrow{\Delta}X\times X\xrightarrow{m}X$. Say $\alpha=[W]$ and $\beta=[Z]$. Consider the map $W\times Z\xrightarrow{m}W+Z$.[1] If this is generically finite of degree d, then $\alpha\star\beta=d[W+Z]$; otherwise, it is zero. The unit for $\star$ is the class $[e]$ of the origin.

Notation 2.1.7. We write X^t for the dual abelian variety of X: $X^t=\mathrm{Pic}^0_{X/k}$.

Note that $X\sim X^t$ (isogenous) always, but generally, they are not isomorphic. The duality between them is witnessed by the `Poincaré line bundle` $\mathcal{P}$ on $X\times X^t$. Give $\xi\in X^t$, the restriction $\mathcal{P}|_{X\times\{\xi\}}$ is the line bundle on X corresponding to the point ξ. Note that $\mathcal{P}$ also gives a family of line bundles on X^t and in this way realizes $X\simeq(X^t)^t$. Furthermore, $\mathcal{P}_{X^t}$ on $X^t\times X$ is just $\mathrm{sw}^*\mathcal{P}$, where $\mathrm{sw}\colon X^t\times X\to X\times X^t$ is the swap map. We'll use this $\mathcal{P}$ to construct an operation called "Fourier duality."

Notation 2.1.8. Write $p=c_1(\mathcal{P})\in\mathrm{CH}^1(X\times X^t)$ for the class of the line bundle $\mathcal{P}$.

We would like to consider its Chern character as well, so we move to $\mathbb{Q}$-coefficients:

$$\mathrm{ch}(\mathcal{P})=\exp(p)=1+p+\frac{1}{2}p^2+\frac{1}{6}p^3+\cdots+\ldots\frac{1}{(2g)!}p^{2g}\in\mathrm{CH}(X\times X^t)_{\mathbb{Q}}$$

(using intersection product in above expression).

[1] $W+Z=\{P+Q:P\in W,Q\in Z\}$

Definition 2.1.9. The `Fourier transform` is the map

$$\mathcal{F} = \mathcal{F}_X \colon \operatorname{CH}(X)_{\mathbb{Q}} \longrightarrow \operatorname{Ch}(X^t)_{\mathbb{Q}}$$

given by

$$\mathcal{F}(\alpha) = \operatorname{pr}_{X^t,*}(\operatorname{pr}_X^*(\alpha)\cdot \operatorname{ch}(\mathcal{P}))\,. \qquad \diamond$$

Keep in mind the diagram

$$\begin{tikzcd} & X \times X^t \arrow[dl, "\operatorname{pr}_X"'] \arrow[dr, "\operatorname{pr}_{X^t}"] & \\ X & & X^t. \end{tikzcd}$$

Remark 2.1.10. Of course, one can reverse the roles to obtain $\mathcal{F}^t = \mathcal{F}_{X^t} \colon \operatorname{CH}(X^t)_{\mathbb{Q}} \to \operatorname{CH}(X)_{\mathbb{Q}}$. ∘

Theorem 2.1.11 (Mukai, Beauville)**.**

(i) $\mathcal{F}^t \circ \mathcal{F} = (-1)^g \cdot [-1]_* \colon \operatorname{CH}(X)_{\mathbb{Q}} \to \operatorname{CH}(X)_{\mathbb{Q}}$.

Notation 2.1.12. If $n \in \mathbb{Z}$, write $[n]_X \colon X \to X$ for multiplication by n.

In particular, $\mathcal{F} \colon \operatorname{CH}(X)_{\mathbb{Q}} \xrightarrow{\sim} \operatorname{CH}(X^t)_{\mathbb{Q}}$ *is an isomorphism of vector spaces.*

(ii) $\mathcal{F}(\alpha \star \beta) = \mathcal{F}(\alpha)\cdot\mathcal{F}(\beta)$ *and* $\mathcal{F}(\alpha\cdot\beta) = (-1)^g \mathcal{F}(\alpha) \star \mathcal{F}(\beta)$.

Thus, $\mathcal{F}$ is an isomoprhism of rings where the LHS uses the Pontryagin product and the RHS uses intersection product.

Remark 2.1.13. I missed this, but something about how duality of abelian varieties exchanges Δ and m. ∘

Warning 2.1.14. This Fourier transform does not, in any simple way, preserve the grading by dimension or codimension. Moonen claims this is spectacularly good news. •

2.2 Lecture 2 (3/3):

Note 3. Missed beginning bit, but looks like he recapped the description of the Fourier transform $\mathcal{F}$.

Setup 2.2.1. X/k abelian variety of dimension g, X^t its dual, and Fourier transform $\mathcal{F}$.

Note for $[m] : X \to X$, get $[m]^*, [m]_* \colon \operatorname{CH}(X) \to \operatorname{CH}(X)$. Note that the projection formula tells you that $[m]_*[m]^* = n^{2g}$ (multiplication by degree).

Definition 2.2.2. Let $\mathscr{L}$ be a line bundle on X. Call it `symmetric` if $[-1]^*\mathscr{L} \simeq \mathscr{L}$ and `antisymmetric` if $[-1]^*\mathscr{L} \simeq \mathscr{L}^{-1}$. Note that

$$[n]^*\mathscr{L} \simeq \begin{cases} \mathscr{L}^{n^2} & \text{if } \mathscr{L} \text{ symmetric} \\ \mathscr{L}^{n} & \text{if } \mathscr{L} \text{ antisymmetric.} \end{cases} \qquad \diamond$$

Recall $\operatorname{Pic}(X) = \operatorname{CH}^1(X)$, so think about translating this to the Chow setting.

Remark 2.2.3. For any line bundle $\mathscr{L}$, can always write

$$\mathscr{L}^2 \simeq (\mathscr{L} \otimes [-1]^*\mathscr{L}) \otimes \big(\mathscr{L} \otimes [-1]^*\mathscr{L}^{-1}\big)$$

as a combination of something symmetric and something antisymmetric. ∘

Thus, $[-1]^* \curvearrowright \mathrm{CH}(X)_{\mathbb{Q}}$ and we have

$$\mathrm{CH}(X)_{\mathbb{Q}} = \mathrm{CH}(X)_{\mathbb{Q}}^{\mathrm{sym}} \oplus \mathrm{CH}(X)_{\mathbb{Q}}^{\mathrm{antisym}}$$

(could even replace $\mathbb{Q}$ with $\mathbb{Z}[1/2]$). $[n]$ acts by n^2 on symmetric part and by n on the antisymmetric point. We'll soon rename these parts to $\mathrm{CH}^1_{(0)}(X)$ and $\mathrm{CH}^1_{(1)}(X)$.

Definition 2.2.4. Given integers i, j, s, we define $\mathrm{CH}^i_{(s)}(X) \subset \mathrm{CH}^i(X)_{\mathbb{Q}}$ to be

$$\mathrm{CH}^i_{(s)}(X) \coloneqq \left\{\alpha \in \mathrm{CH}^i(X)_{\mathbb{Q}} \mid \forall n\colon [n]^*\alpha = n^{2i-s}\alpha\right\}.$$

We also define

$$\mathrm{CH}_j(X) \supset \mathrm{CH}_j^{(s)}(X) \coloneqq \left\{\alpha \in \mathrm{CH}_j(X)_{\mathbb{Q}} \mid \forall n\colon [n]_*\alpha = n^{2j+s}\alpha\right\} = \mathrm{CH}^{g-j}_{(s)}(X). \qquad \diamond$$

Theorem 2.2.5 (Beauville)**.**

(i) $\mathrm{CH}^i_{(s)}(X) = \left\{\alpha \in \mathrm{CH}^i(X)_{\mathbb{Q}} \mid \mathcal{F}(\alpha) \in \mathrm{CH}^{g-i+s}(X^t)_{\mathbb{Q}}\right\}$*, and* $\mathcal{F}\colon \mathrm{CH}^i_{(s)}(X) \xrightarrow{\sim} \mathrm{CH}^{g-i+s}_{(s)}(X^t)$*. Similarly,* $\mathrm{CH}_j^{(s)}(X) = \{\alpha \in \mathrm{CH}_j(X)\mathbb{Q} \mid \mathcal{F}(\alpha) \in \mathrm{CH}_{g-j-s}(X^t)\}$*.*

(ii) $\mathrm{CH}^i_{(s)}(X) \cdot \mathrm{CH}^j_{(t)}(X) \subset \mathrm{CH}^{i+j}_{(s+t)}(X)$ *and* $\mathrm{CH}_{i,(s)}(X) \star \mathrm{CH}_{j,(t)}(X) \subset \mathrm{CH}_{i+j,(s+t)}(X)$*.*

(iii) $\mathrm{CH}^i(X)_{\mathbb{Q}} = \bigoplus_{s=i-g}^{i} \mathrm{CH}^i_{(s)}(X)$ *and* $\mathrm{CH}_j(X)_{\mathbb{Q}} = \bigoplus_{s=-j}^{g-j} \mathrm{CH}_{j,(s)}(X)$*.*

Note 4. Experienced strange technical difficulties (possessed computer), so see notes for statement.

Have a bigrading, so any linear combination of it gives a new bigrading.

Definition 2.2.6. Consider $2i - s$ to be the `weight`. $\diamond$

(See notes for pictures)

One way to study cycles is via their cohomology classes. Let H be any Weil cohomology theory on smooth projective k-varieties.

Example 2.2.7. If $k = \mathbb{C}$, can just take singular/Betti/topological cohomology of $X(\mathbb{C})$. $\triangle$

Example 2.2.8. For any k and any prime $\ell \neq \operatorname{char} k$, could take ℓ-adic cohomology. $\triangle$

Example 2.2.9. For any k, could take de Rham cohomology. $\triangle$

There are cycle class map $\mathrm{cl}\colon \mathrm{CH}^i(X) \to \mathrm{H}^{2i}(X)$. For X/k an abelian variety, in any theory, have $\mathrm{H}^m(X) = \bigwedge^m \mathrm{H}^1(X)$ so $[n]^*$ is multiplication by n^m on H^m. So, passing to cohomology kills everything with $s \neq 0$. Conjecturally, cl should be injective on $\mathrm{CH}^i_{(0)}$ (at least, for most theories. Sounds like de Rham cohomology maybe shouldn't play well with p-power torsion, for example).

If $\alpha \in \mathrm{CH}^i(X)_{\mathbb{Q}}$ with $\mathrm{cl}(\alpha) = 0$, then try an Abel-Jacobi map; the target space of such a thing, in whatever version you do, will be "built out of H^{2i-1}." If, again $\mathrm{AJ}(\alpha) = 0$, then, at least in ℓ-adic cohomology, go on using higher Abel-Jacobi maps.

2.3 Lecture 3 (3/4)

Note 5. Like 8ish minutes late. Seems like he began by summarizing the rep theory of $\mathfrak{sl}_2$ for some reason.

Setup 2.3.1. Let X/k be a g-dimensional AV. Today we'll choose a polarization $\theta\colon X \to X^t$, say coming from some ample symmetric bundle L. Let $\ell = c_1(L) \in \mathrm{CH}^1_{(0)}(X)$.

We'll use this polarization to cook up an $\mathfrak{sl}_2$ action on $\mathrm{CH}(X)$.

Remark 2.3.2. θ induces (an iso) $\theta_*\colon \mathrm{CH}(X)_{\mathbb{Q}} \to \mathrm{CH}(X^t)_{\mathbb{Q}}$. We define $\lambda \in \mathrm{CH}^{g-1}_{(0)}(X)$ to be the unique class such that

$$\theta_*\lambda = \mathcal{F}(\ell),$$

i.e. it is "the Fourier dual of ℓ." ◦

Theorem 2.3.3. *Define the following operators on* $\mathrm{CH}(X)_{\mathbb{Q}}$*:*

$$\begin{aligned} e(\alpha) &= \ell \cdot \alpha \\ h(\alpha) &= (2i - s - g) \cdot \alpha && \text{if } \alpha \in \mathrm{CH}^i_{(s)} \\ f(\alpha) &= \lambda \star \alpha. \end{aligned}$$

(recall $\mathrm{CH}^i_{(s)}$ *is weight* $2i - s$*, so* h *is multiplication by a shifted weight). These operators define an representation of* $\mathfrak{sl}_2$ *on* $\mathrm{CH}(X)_{\mathbb{Q}}$*.*

Recall 2.3.4. On $\mathfrak{sl}_2$, have commutator relations $[e, h] = 2e$, $[f, h] = -2f$ and $[e, f] = 0$. ⊙

This is probably wrong. Look up the right thing later

Remark 2.3.5. For every s, the subspace

$$\mathrm{CH}^*_{(s)} = \bigoplus \mathrm{CH}^i_{(s)}$$

is stable under $\mathfrak{sl}_2$. More generally, there are $\mathbb{Q}$-subspaces $P_{j,s} \subset \mathrm{CH}_{(s)}$ such that

$$\mathrm{CH}^*_{(s)}(X) = \bigoplus_{\substack{j=0 \\ j \equiv g-s \mod 2}}^{g-|s|} P_{j,s} \otimes \mathrm{Sym}^j(\mathtt{std})$$

(possibly mis-copied the above). ◦

2.3.1 0-cycles

Setup 2.3.6. Now work over $k = \overline{k}$ and, as much as possible, work integrally.

Goal. We want to study $\mathrm{CH}_0(X)$ which is itself a ring under pontryagin product.

Note there is a degree map $\deg\colon \mathrm{CH}_0(X) \to \mathbb{Z}$ which has a section (given e.g. by the origin).

Notation 2.3.7. Write $I\colon \ker(\deg\colon \mathrm{CH}_0(X) \to \mathbb{Z})$, so $\mathrm{CH}_0(X) = I \oplus \mathbb{Z}$ and I is an ideal in $\mathrm{CH}_0(X)$.

Remark 2.3.8. As a $\mathbb{Z}$ module, I is generated by classes of the form $(P) - (0)$ for $P \in X(k)$. ◦

We filter CH_0 by powers of I:

$$\mathrm{CH}_0 \supset I \supset I^{\star 2} \supset I^{\star 3} \supset \cdots.$$

There is a summation map $S\colon I \to X(k)$ given by summing zero cycles in the group law (note this makes sense on all of CH_0, but we'll really only care about it in I). That is,

$$S\Big(\sum m_i(P_i)\Big) = \sum m_i P_i \in X(k).$$

Proposition 2.3.9. *There is a short exact sequence*

$$0 \longrightarrow I^{\star 2} \longrightarrow I \xrightarrow{S} X(k) \longrightarrow 0.$$

Note that $I^{\star 2}$ is generated by elements of the form $((P)-(0))\star((Q)-(0)) = (P+Q)-(P)-(Q)+(0)$. This makes it clear that $I^{\star 2}$ is in the kernel of S. This computation also shows that

$$((P)-(0)) + ((Q)-(0)) \equiv (P+Q)-(0) \mod I^{\star 2},$$

so every class in $I/I^{\star 2}$ is represented by an expression of the form $(P)-(0)$ at which point it is clear that $I/I^{\star 2} \simeq X(k)$.

Fact. Two 0-cycles are algebraically equivalent iff they have the same degree (move all points to the origin)

Thus, $I = \{\alpha \in \mathrm{CH}_0 \mid \alpha \sim_{\mathrm{alg}} 0\}$. This in fact implies that I is a divisible group[2] (so $I^{\star n}$ is a divisible group). Thus, our ealier short exact sequence splits (for abstract reasons; hard to write down a section), i.e.

$$I = I^{\star 2} \oplus X(k).$$

Theorem 2.3.10 (Roitman)**.** *The summation map S induces an isomorphism $I_{tors} \xrightarrow{\sim} X(k)_{\mathrm{tors}}$.*

Corollary 2.3.11. *$I^{\star 2}$ is torsion-free and divisible, so it is a $\mathbb{Q}$-vector space.*

Thus, we get a filtration

$$\mathrm{CH}_0(X)_{\mathbb{Q}} \supset I_Q \supset I^{\star 2} \supset I^{\star 3} \supset \dots$$

(no need to tensor with $\mathbb{Q}$ after second piece). Let's compare this with another filtration we have:

$$\mathrm{CH}_0(X)_{\mathbb{Q}} \supset \bigoplus_{s\ge 1} \mathrm{CH}_{0,(s)} \supset \bigoplus_{s\ge 2} \mathrm{CH}_{0,(s)} \supset \dots$$

Theorem 2.3.12. *The above two filtrations are the same, i.e.*

$$\bigoplus_{s\ge k} \mathrm{CH}_{0,(s)} = I_{\mathbb{Q}}^{\star k}$$

for all $k \ge 1$.

Note $I_{\mathbb{Q}}^{\star k} = I^{\star k}$ if $k \ge 2$

Corollary 2.3.13. $I^{\star(g+1)} = 0$.

Example 2.3.14 ($g=1$)**.** For all points $P, Q \in X(k)$ (X an elliptic curve), there is some point R such that $(R)+(0) \sim_{\mathrm{rat}} (P)+(Q)$. Of course, R is simply $P+Q$. △

Example 2.3.15 ($g=2$)**.** In this case, the statement ends up being that

$$(P+Q+R)+(P)+(Q)+(R) \sim_{\mathrm{rat}} (P+Q)+(P+R)+(Q+R)+(0). \quad \triangle$$

This is somehow reminiscent of the theorem of the cube

For general g, the vanishing $I^{\star(g+1)} = 0$ is some sort of "hypercube statement" (some alternating sum of sums of points is 0).

Proof Sketch of Theorem 2.3.12. One first checks that $I_{\mathbb{Q}} = \bigoplus_{s\ge1} \mathrm{CH}_{0,(s)}$. This is not hard to show. Now, recall that the Beauville decomposition is compatible w/ $\star$-products. This is enough to deduce that $I^{\star r} \subset \bigoplus_{s\ge r} \mathrm{CH}_{0,(s)}$ for $r \ge 2$ (note that this is already enough to prove the corollary). Next, observe that $[n]_* I \subset I$, so also $[n]_* I^{\star r} \subset I^{\star r}$. Now one needs to check/convince thyself that

[2] Sounds like this uses $k = \overline{k}$?

- $[n]_*$ induces multiplication by n on $I/I^{\star 2} \simeq X(k)$.

  Since there is a surjection $\left(I/I^{\star 2}\right)^{\otimes r} \twoheadrightarrow I^{\star r}/I^{\star(r+1)}$, $[n]_* \curvearrowright I^{\star r}/I^{\star r+1}$ by multiplication by n^r.

- A "weight argument" then will imply that the two filtrations are the same. ■

2.4 Lecture 4 (3/5)

Recall the cycle class maps

$$\mathrm{cl}_\ell \colon \mathrm{CH}^m(X) \otimes \mathbb{Q}_\ell \longrightarrow \mathrm{H}^{2m}_{\text{ét}}(X_{\overline{k}}, \mathbb{Q}_\ell(m))$$

or, over $\mathbb{C}$,

$$\mathrm{cl} \colon \mathrm{CH}^m(X)_{\mathbb{Q}} \longrightarrow \mathrm{H}^{2m}(X(\mathbb{C}), \mathbb{Q}(m)).$$

Conjecture 2.4.1. *This should be injective on the* 0*th layer* $\mathrm{CH}^*_{(0)}$.

This would give that $\mathrm{CH}^*_{(0)}$ are f.dim $\mathbb{Q}$-vector spaces. Also, it would give some bound for how big this 0th layer is which is independent of the base field. In fact, potentially after some finite extension of the base field, the 0th layer doesn't growing with field extensions.

Recall 2.4.2. In the first layer ($s = 1$), you find $X^t(k) \otimes \mathbb{Q}$ and $X(k) \otimes \mathbb{Q}$, so you get different behavior. That is, in general, $\mathrm{CH}^1_{(1)}$ and $\mathrm{CH}^g_{(1)}$ are not f.dim (imagine $k = \mathbb{C}$) and do depend on k. However, still "managable" because these are more-or-less points of some algebraic variety. ⊙

Theorem 2.4.3 (Soulé, Künnemon, up to spelling)**.** *If* $k \subset \overline{\mathbb{F}}_p$*, then* $\mathrm{CH}^m_{(s)}(X) = 0$ *for all* $s \neq 0$.

Proof idea. First step is a "routine" reduction to the case $k = \mathbb{F}_q$. Let $\varphi \in \mathrm{End}(X)$ denote the q-Frobenius endomorphism. One can check that φ_* is multiplication by q^d in $\mathrm{CH}_d(X)$, so φ^* is multiplication by q^m on $\mathrm{CH}^m(X)$.

The next step is harder to explain, since it is of a more motivic nature. Let $P_m \in \mathbb{Z}[X]$ denote the characteristic polynomial of $\varphi^* \curvearrowright \mathrm{H}^m(X)$. A motivic argument (using e.g. tensor products in the category of motives) allows you to deduce that $P_m(\varphi^*) = 0$ on $\mathrm{CH}^i_{(s)}(X)$ in weight $2i - s = m$. Thus, multiplication by q^i is a root of the polynomial P_m, but by the Weil conjectures, this is possible only if $m = 2i$ (i.e. $s = 0$). ■

Corollary 2.4.4. *Say* $X/\overline{\mathbb{F}}_p$*. Then,*

$$\mathrm{CH}_0(X) \cong \mathbb{Z} \oplus X(\overline{\mathbb{F}}_p).$$

Setup 2.4.5. Let $Y/k = \overline{k}$ be any smooth projective variety (we'll later re-specialize to abelian varieties).

Let $\mathrm{CH}_0(X)_{\mathrm{hom}} \subset \mathrm{CH}_0(X)$ denote those abelian varieties which are homologically trivial (have degree 0?).

Question 2.4.6. *What would it mean to say that* CH_0 *is "small/managable" in some sense?*

- Define `property A`:

  there is some m such that every class in $\mathrm{CH}_0(X)_{\mathrm{hom}}$ can be represented as $(P_1 + \cdots + P_m) - (Q_1 + \cdots + Q_m)$.

- Consider map $\gamma_m \colon Y^m \times Y^m \to \mathrm{CH}_0(Y)_{\mathrm{hom}}$ given by taking differences as above. Some Hilbert scheme argument tells you that the fibers of this map are countable unions of algebraic subvarieties. Set

$$d_m := 2m \dim(Y) - \text{max dim of a subvar contained in a fiber.}$$

Think of this as "the dimension of $\mathrm{CH}_0(X)_{\mathrm{hom}}$." This leads to `property B`: the map $m \mapsto d_m$ is bounded.

- `property C`:

  There exists a nonsingular curve C along with a map $j\colon C \to Y$ such that $j_*\colon \mathrm{CH}_0(X)_{\mathrm{hom}} \to \mathrm{CH}_0(Y)_{\mathrm{hom}}$ is surjective.

- Choose $y_0 \in Y(k)$ base point. `property D`:

  The Albanese map $\mathrm{CH}_0(Y)_{\mathrm{hom}} \to \mathrm{Alb}_Y(k)$ is an isomorphism.

Theorem 2.4.7. *Properties A through D are all equivalent.*

Theorem 2.4.8 (Mumford '69, Roitman)**.** *Suppose $Y/\mathbb{C}$ is a smooth, projective variety such that properties A–D hold. Then, $\mathrm{H}^0(Y, \Omega^i_Y) = 0$ for all $i \geq 2$.*

Upshot: it almost never happens that these properties hold.

Corollary 2.4.9. *If $X/\mathbb{C}$ is an abelian variety of dimension ≥ 2, then A – D do not hold.*

Moonen reccomends reading Mumford's paper.[3]

Theorem 2.4.10 (Bloch)**.** *Let $k = \overline{k}$ be uncountable of characteristic 0. Let X/k be a g-dimensional AV, and let $I = \mathrm{CH}_0(X)_{hom}$ as before. Then, $I^{\star g} \neq 0$ (recall: $I^{\star(g+1)} = 0$). Equivalently, $\mathrm{CH}_{0,(s)}$ is nonzero for $s = 0, \ldots, g$.*

(using $\mathfrak{sl}_2$ action, get that all boxes are not trivial.)

Remark 2.4.11. This is false in char p. For example, if X is supersingular, then all $\mathrm{CH}^i_{(s)}$ with $s > 2$ vanish. ○

Moonen then pulled up the paper "Chow groups are finite dimensional, in some sense" by Kimura. This leads to questions about "complexity" of CH.

- Kimura/O'Sullivan: f.dim Chow motives. We know "motives of abelian type" are f.dim'l.

- `Voevodsky's smash-nilpotence conjecture`:

  Given $\alpha \in \mathrm{CH}(Y)$, if $\alpha \sim_{\mathrm{num}} 0$ ($\impliedby \alpha \sim_{\mathrm{hom}} 0$), then there exists N such that $0 = \alpha^{\otimes N} = \alpha \times \alpha \ldots \alpha \in \mathrm{CH}(Y^N)$ (exterior product on Chow) is equal to 0 in Chow (so rationally equivalent to 0).

  This is known (and apparently not that hard) if $\alpha \sim_{\mathrm{alg}} 0$. It's also known for 1-cycles on abelian varieties.

[3] Apparently quite geometric. Something something lots of rational equivalnce means lots of rational curves in $\mathrm{Sym}^n Y$ so now can try to pullback differentials forms and restrict them to these rational curves something something

3 Valentijn Karemaker: Geometry and arithmetic of moduli spaces of abelia(n varieties in positive characteristic

3.1 Lecture 1 (3/2): Abelian Varieties in char. p

Setup 3.1.1. Fix a prime $p > 0$ as well as some power $q = p^r$. Set $k = \overline{\mathbb{F}}_p \supset \mathbb{F}_q \supset \mathbb{F}_p$ and let K denote any of these fields.

Definition 3.1.2 (1.13)**.** Any abelian variety $X/\mathbb{F}_q$ has an $\mathbb{F}_q$-morphism, the `Frobenius endomorphism` $\pi_X = F^r_{X/\mathbb{F}_q}$. It acts by $f \mapsto f^q$ on regular functions. $\diamond$

Numbers refer to her notes

Note that $X(\mathbb{F}_{q^n})$ consists of the points fixed by π_X^n. Note $\pi_X : (x_1, \dots, x_n) \mapsto (x_1^q : \cdots : x_n^q)$.

Definition 3.1.3 (1.14)**.** π_X has a characteristic polynomial $h_{\pi_X} \in \mathbb{Z}[x]$. $\diamond$

Theorem 3.1.4 (1.15)**.**

- $\deg h_{\pi_X} = 2 \dim X$
- *All roots have absolute value* $\sqrt{q}$
- α *root* $\iff \overline{\alpha} = q/\alpha$ *is a root.*

Theorem 3.1.5 (1.17)**.** *If* $h_{\pi_X}(x) = \prod_{i=1}^{2g}(x - \alpha_i)$ *over* $\overline{\mathbb{Q}}$, *then*

$$\#X(\mathbb{F}_{q^n}) = \prod_{i=1}^{2g}(1 - \alpha_i^n) \text{ for all } n \geq 1.$$

Write $X \sim Y$, X is `isogenous` to Y, if $\exists \varphi\colon X \twoheadrightarrow Y$ with finite kernel. This is an equivalence relation on abelian varieties.

Theorem 3.1.6 (1.10, Tate)**.** *Two abelian varieties* $X, Y/\mathbb{F}_q$ *are isogenous if and only if* $h_{\pi_X} = h_{\pi_Y}$.

Also, for every h_π (satisfying properties listed in Theorem 3.1.4), there exists an abelian variety $Z/\mathbb{F}_q$ wit $h_\pi = h_{\pi_Z}$.

3.1.1 p^n-torsion in char p

Recall 3.1.7. Let E/K be an elliptic curve. Then,

$$E[p](K) = \begin{cases} 0 & \text{if supersingular} \\ \mathbb{Z}/p\mathbb{Z} & \text{if ordinary} \end{cases} \qquad \odot$$

Definition 3.1.8 (1.19)**.** One always have $\#X[p](k) = p^f$ for some $0 \leq f \leq g$. This ic called the p-`rank` $f(X) = f$ of X. $\diamond$

Definition 3.1.9 (1.20)**.** We say X is `ordinary` if it has maximal p-rank, $f(X) = g$. $\diamond$

Definition 3.1.10 (1.22)**.** We say X/K is `supersingular` (ss) if $X_k \sim E^g$ for a supersingular elliptic curve E. It is `superspecial` if it is isomorphic to such a product. $\diamond$

Remark 3.1.11. X ss $\implies p(X) = 0$, but the converse can fail if $g \geq 3$. $\circ$

Remark 3.1.12. p-rank is an isogeny invariant. $\circ$

Coming up next

- generalize: $p \rightsquigarrow p^n, p^\infty$
- isogeny $\rightsquigarrow$ isomorphism

Definition 3.1.13 (1.2.5)**.** The p-`divisible group` of X/K is the group scheme $X[p^\infty] := \varinjlim X[p^n]$. $\diamond$

Theorem 3.1.14 (2.2.1, Dieudonné–Manin)**.** *There is a direct sum decomposition*

$$X[p^\infty] \sim_k \sum_i (G_{m_i,n_i} \oplus G_{n_i,m_i}) \oplus G_{1,1}^{\oplus s} \oplus (G_{1,0} \oplus G_{0,1})^f ,$$

where

- *for* $\gcd(m_i, n_i) = 1$, $0 \le s, f$ *and* $f \le g$ *(not sure if I copied this correctly).*
- *all* $G_{m,n}$ *are simple and of dimension* m *and height* $m+n$
- $G_{m,n}^\vee = G_{n,m}$
- *Sounds like last two factors are ss part and orindary part*

Say $G_{m,n}$ has `slope` $\lambda = m/(m+n)$ and multiplicity $m+n$.

Question: Is this correct?

Definition 3.1.15. The `Newton polygon` $\mathcal{N}(X)$ of X is formed out of slopes λ of $X[p^\infty]$ in non-decreasing order. $\diamond$

Remark 3.1.16. The p-rank is equal to the number of zero slopes. $\circ$

Definition 3.1.17. Define an ordinary on the Newton polygons. Say $\sigma < \rho$ if no point of σ lies below a point of ρ. $\diamond$

W.r.t. this ordering, the max is the Newton poly in the supersingular case (straight line of slope 1/2) and the min is the Newton poly in the ordinary case (line segments $(0,0) \to (g,0) \to (2g,g)$).

Example 3.1.18. Consider NP $\left(\frac{1}{3}, \frac{1}{3}, \frac{1}{3}, \frac{2}{3}, \frac{2}{3}, \frac{2}{3}\right)$ with $g = 3$. This has p-rank 0 (no zero slopes), but it is not ss (not all slopes are 1/2). $\triangle$

Theorem 3.1.19 (Honda–Serre)**.** *Every symmetric*[4] *NP occurs for some abelian variety.*

Fact. There are three non-isomorphic finite k-group schemes of rank p:

$$\mu_p \qquad \mathbb{Z}/p\mathbb{Z} \qquad \alpha_p.$$

Definition 3.1.20. The a-`number` of X/K is

$$a(X) := \dim_K \operatorname{Hom}(\alpha_p, X)\,. \qquad \diamond$$

Note that $0 \le a(X) + f(X) \le g$.

Example 3.1.21. X ordinary $\implies a(X) = 0$. For non-ordinary cases, generically one has $a(X) = 1$. $\triangle$

Example 3.1.22 (Oort)**.** X is superspecial $\iff a(X) = g$. $\triangle$

[4] Slope λ appears as often as $1 - \lambda$. Also multiplicity condition (should match denominator?)

Fact (2.3.0)**.** On $X[p]$ (group scheme), one has $0 = [p] = F + V = V + F$, where F is Frobenius and V is Verschiebung.

Get filtration

$$G \supset V(G) \supset V^2(G) \supset V^3(G) \supset \dots$$

and apply F^{-1} to each, e.g.

$$V(G) \subset F^{-1}V(G) \subset F^{-2}V(G) \subset \dots$$

and apply V's to these, e.g.

$$F^{-1}V(G) \supset VF^{-1}V(G) \supset \dots$$

and continue to get some horrible multi-dimensional diagram.

Theorem 3.1.23. *Rank G is finite (p^r) $\implies$ stabilizers after $\leq 2(r-1)$ steps.*

Definition 3.1.24. The `canonical filtration` of $X[p] = G$ is

$$0 = G_0 \subset \dots \subset G_s = V(G) \subset \dots \subset G_t = G.$$

$\diamond$

Definition 3.1.25. Encoded filtration in `canonical type` $\tau = (v, f, \rho)$ where $V(G_i) = G_{v(i)}$, $F^{-1}(G_i) = G_{f(i)}$, and $\operatorname{rank} G_i = p^{\rho(i)}$. $\diamond$

Theorem 3.1.26. *Canonical type determines $X[p]$ up to k-isomorphism.*

For us, $s = g$ and $t = 2g$.

Example 3.1.27 ($g = 3$)**.** Have $0 = G_0 \subset \dots \subset G_6 = G$. Apparently can get

$$0 \subset V^2(G) \subset VF^{-1}V(G) \subset V(G) \subset F^{-1}V^2(G) \subset F^{-1}V(G) \subset G.$$

One has $v(0) = v(1) = v(2) = 0$, $v(3) = v(4) = 1$, $v(5) = 2$, $v(6) = 3$, $f(0) = 3$, $f(1) = 4$, $f(2) = f(3) = 5$, and $f(4) = f(5) = f(6) = 6$. Finally, $\rho(i) = i$ for all i. $\triangle$

From canonical type, can get an '`elementary sequence`' $\varphi(0) = 0, \varphi(1), \dots, \varphi(g)$. This is defined inductively via

- From $\varphi(0), \dots, \varphi(\rho(i))$ with $\rho(i) < \rho(i+1)$, get $\varphi(0), \dots, \varphi(\rho(i+1))$ via

$$\begin{cases} \varphi(\rho(i+1)) = \dots = \varphi(\rho(i)+1)\varphi(\rho(1)) & \text{if } v(i) = v(i+1) \\ \varphi(\rho(i+1)) > \dots > \varphi(\rho(i)+1) > \varphi(\rho(i)) & \text{if } v(i) < v(i+1). \end{cases}$$

(always jump by 1, I think?)

Example 3.1.28 (continued from previous example)**.** $\varphi = (0, 0, 0, 1)$. Sometimes ignore the first 0, so more like $\varphi = (\cancel{0}, 0, 0, 1)$. $\triangle$

Such φ's are also called `Ekedahl-Oort types`.

Remark 3.1.29. We can read off the a-number from φ as $a(X) = g - \varphi(g)$. Can read off the p-rank from φ as $f(X) = \max\{i : \varphi(i) = i\}$. It's not as easy to read off the Newton polygon from the Ekedahl–Oort type. $\circ$

Example 3.1.30 (continuation of same one)**.** Recall $\varphi = (0, 0, 1)$, $g = 3$. So $a(X) = 3 - 1 = 2$. Note that $f(X) = 0$. Is this supersingular? $\triangle$

Example 3.1.31. Other φ for $g = 3$:

- with a-number 2, p-rank 0: $(\cancel{0}, 0, 1, 1)$.
- with a-number 2, p-rank 1: $(1, 1, 1)$.
- with a-number 1, p-rank 0: $(0, 1, 2)$.

$\triangle$

4 Joe Silverman: Canonical heights on abelian varieties

Note 6. Joe got sick, so via zoom. He sent a big stuffed bear (Joe Silverbear) in his place.

4.1 Lecture 1 (3/2): Construct & Properties

Topics for 4 lectures

- Construction & Properties
- Applications/Local Hts
- Lower Bounds
- Heights in Families

Setup 4.1.1. Let K be a field (always of characteristic 0). Usually it will be a number field or sometimes a 1-dimensional function field.

4.1.1 Heights on projective space

Start with the usual function $h\colon \mathbb{P}^N(K) \to [0, \infty)$ (won't give formal definition in this lecture; it's ugly if you haven't seen it). Informally,

$$h(P) = \text{“\#bits needed to store } P\text{.”}$$

Slogan. heights measure arithmetic complexity

Key property: only f.many objects of bounded complexity, e.g.

$$\#\left\{P \in \mathbb{P}^N(K)\colon h(P) \le B\right\} < \infty.$$

We'll later extend to a function $h\colon \mathbb{P}^N(\overline{K}) \to [0, \infty)$ on points defined over the algebraic closure.

Remark 4.1.2. In general, for $P \in \mathbb{P}^N(L)$, $h(P) \approx \#(\text{of bits})/[L : K]$. ∘

Slogan. The Weil Height Machine is a tool for converting geometry into arithmetic.

Construction 4.1.3. Let X/K be a smooth projective variety. For any embedding $\varphi\colon X \hookrightarrow \mathbb{P}^N$, can define $h_{\varphi,X} = h \circ \varphi$.

- If D is an ample divisor $\in \operatorname{Div}(X)$, we'll choose some n s.t. nD is very ample as well as choose some basis for its sections (in order to get an embedding $\varphi_{nD}\colon X \hookrightarrow \mathbb{P}^N$), and then set
$$h_{X,D} = \frac{1}{n} h_{\varphi_{nD},X}.$$
- If $D \in \operatorname{Div}(X)$ is arbitrary, we'll write $D = D_1 - D_2$ as a difference of (very) ample divisors, and then set
$$h_{X,D} = h_{X,D_1} - h_{X,D_2}.$$

So every $D \in \operatorname{Div}(X)$ now has a Weil height $h_D\colon X(K) \to \mathbb{R}$. ○

Theorem 4.1.4 (Weil's Height Machine)**.**

(1) *If D is very ample, with $\varphi_D : X \hookrightarrow \mathbb{P}^N$, then*

$$h_D = h \circ \varphi_D + O(1),$$

i.e. all the choices made cause h_D to be well-defined only up to bounded function.

(2) $D \sim D' \implies h_D = h_{D'} + O(1)$.

(3) *Given $\varphi\colon X \to Y$, $h_{X,\varphi^* D} = h_{X,D} \circ \varphi + O(1)$*

"These need to be morphisms, not rational maps. You can get into a lot of trouble if you try to do things with rational maps. Actually, it's interesting."

(4) $h_{D+D'} = h_D + h_{D'} + O(1)$.

(5) `Northcott`*: If D is ample, then*

$$\#\{P \in X(K)\colon h_D(P) \leq B\} < \infty$$

(In fact, still finite if you look at points of bounded degree)

Think of properties **(2)**–**(4)** as "geometry to arithmetic."

4.1.2 Canonical Heights

Notation 4.1.5. $A =$ abelian variety.

Definition 4.1.6. If $D \in \mathrm{Div}(A)$ is a divisor, say D is `symmetric` (resp. `antisymmetric`) if $[-1]^* D \sim D$ (resp. $[-1]^* \sim -D$). ⋄

For a general divisor D, $D + [-1]^* D$ is symmetric. We'll mainly work with symmetric divisors in this course.

Abelian varieties are groups, and one always wants to exploit that group structure as much as possible. Write $[m]\colon A \to A$ for the multiplication-by-m map on A.

Theorem 4.1.7.

$$[m]^* D \sim \frac{m^2+m}{2} D + \frac{m^2-m}{2}[-1]^* D.$$

In particular, if D is symmetric (resp. antisymmetric), then $[m]^* D \sim m^2 D$ (resp. $[m]^* D \sim mD$).

Example 4.1.8. On an elliptic curve E, $[m]^* 0 = E[m]$ is a divisor of degree m^2. △

Assumption. From now on, assume D symmetric.

Saying something like $[m]^* D \sim m^2 D$ is geometry. Converted to arithmetic, this says

$$h_D([m]P) = h_{[m]^* D}(P) + O_{A,D,m}(1) = h_{m^2 D}(P) + O(1) = m^2 h_D(P) + O(1).$$

This says that "$[m]P$ is m^2 more complex than the original point."

Observation 4.1.9. The $O(1)$ is annoying.

Sounds like Néron's motivation for getting rid of the $O(1)$ was point counting; apparently the error term you get with this $O(1)$ around is not so good. He suggested that there was an especially nice choice of height function h_D on an abelian variety. Tate gave a clever construction, and Néron himself gave a complicated construction.

Theorem 4.1.10 (Néron–Tate, 1960s)**.**

$$\widehat{h}_D(P) \coloneqq \lim_{n\to\infty} \frac{h_D([2^n]P)}{4^n h_D(P)}$$

exists and is called the canonical height *(or the* Néron-Tate height*).*

Remark 4.1.11. If you replace this 4^n by 5^n, it's easy to show the limit exists, but it's always 0. If you divide by 3^n, it diverges to ∞. Even for 4^n, it's conceivable that it could always converge to 0. ∘

This satisfies the properties

(1) $\widehat{h}_D(P) = h_D(P) + O(1)$.

Canonical height still measures arithmetic complexity.

(2) $\widehat{h}_D\big([m]P^2\big) = m^2\widehat{h}_D(P)$.

(3) $D' \sim D \implies \widehat{h}_D = \widehat{h}_{D'}$

Proof. Need to show $(4^{-n}h_D(2^nP))_{n\geq 1}$ is Cauchy. Choose constant C such that $|h_D(2Q) - 4h_D(Q)| \leq C$ for all Q. Now observe (Tate's telescoping sum trick)

$$\begin{aligned}
\left|4^{-n}h(2^NP) - 4^{-k}h(2^kP)\right| &\leq \sum_{i=k}^{n-1} \left|4^{-(i+1)}h(2^{i+1}P) - 4^{-i}h(2^iP)\right| \\
&= \sum_{i=k}^{n-1} 4^{-(i+1)}\left|h(2^{i+1}P) - 4h(2^iP)\right| \\
&\leq \sum_{i=k}^{n-1} 4^{-(i+1)}C \\
&\leq \sum_{i=k}^{\infty} 4^{-(i+1)}C \\
&= \frac{C}{3\cdot 4^k} \xrightarrow{k\to\infty} 0,
\end{aligned}$$

so the sequence is Cauchy. Note that if you put $k = 0$, you get that

$$\left|4^{-n}h(2^nP) - h(P)\right| \leq \frac{C}{3},$$

so letting $n \to \infty$ shows that $\left|\widehat{h}_D(P) - h_D(P)\right| \leq C/3$. This proves existence of the limit and proves **(1)**. Other parts prove in the notes... ■

Sounds like there are no known examples where this height is proven to be rational (or irrational) when it's nonzero.

Theorem 4.1.12. *Say D is ample and symmetric. Then, the canonical height $\widehat{h}_D\colon A(K) \to \mathbb{R}$ satisfies*

(1) *$\widehat{h}_D$ is a* quadratic form*, i.e.*

$$(P,Q) \mapsto \frac{1}{2}\Big(\widehat{h}(P+Q) - \widehat{h}(P) - \widehat{h}(Q)\Big) \eqqcolon \langle P,Q\rangle_D$$

is bilinear.

(2) $\widehat{h}_D$ *is positive-definite, i.e.* $\widehat{h}_D(P) \geq 0$ *for all* $P \in A(\overline{K})$*.*

(3) $\widehat{h}_D(P) = 0 \iff P$ *is torsion.*

(4) $\widehat{h}_D$ *extends to a positive-definite quadratic form on* $A(K) \otimes \mathbb{R} \cong \mathbb{R}^{\operatorname{rank} A(K)}$*.*

Definition 4.1.13. The Néron-Tate regulator is $\operatorname{Reg}_D(A/K) := \det\big(\langle P_i, P_j\rangle_D\big)_{i,j=1}^r > 0$ for some choice of basis $P_1, \ldots, P_r$ of $A(K)/$mod torsion. $\diamond$

This measures the arithmetic complexity of (any choice of basis for) the Mordell–Weil group.
Let's discuss some of the properties.

Question 4.1.14. *Why is* $\widehat{h}_D \geq 0$*?*

Answer. If D is ample, one can check that $h_D(Q) \geq -C$ for some C for all Q. Thus,

$$\widehat{h}_D(P) = \lim_{n\to\infty} \frac{h(2^n P)}{4^n} \geq \lim_{n\to\infty} \frac{-C}{4^n} = 0. \qquad \star$$

Question 4.1.15. *Why is* $\widehat{h}_D(P) = 0$ *if* P *is torsion?*

Answer. $2^n P$ only takes finitely many values, so is bounded (while 4^n is not). $\star$

Remark 4.1.16. Sounds like Tate was hoping this torsion characterization via heights would be useful in proving uniform boundedness (for torsion in elliptic curves/abelian varieties), but this hasn't panned out so far (instead, Barry gave a proof using very different methods). $\circ$

Question 4.1.17. *Why* $\widehat{h}(P) = 0 \implies P$ *torsion?*

Answer. This implies that $h(mP) < C$ (b/c $\widehat{h}(mP) = 0$) so P has only finitely many multiplies, so P is torsion. $\star$

4.2 Lecture 2 (3/3)

4.2.1 Néron's Counting Theorem

Setup 4.2.1. K a number field, A/K an abelian variety, and $D \in \operatorname{Div}(A)$ ample and symmetric. Write $\widehat{h}_D : A(K) \to \mathbb{R}_{\geq 0}$ for the canonical height.

Theorem 4.2.2. $\widehat{h}_D$ *is a positive definite quadratic form on* $A(K) \otimes \mathbb{R} =: A(K)_{\mathbb{R}}$*. In particular,*

$$\langle P, Q\rangle_D = \frac{1}{2}\Big(\widehat{h}_D(P+Q) - \widehat{h}_D(P) - \widehat{h}_D(Q)\Big)$$

is a bilinear form.

Warning 4.2.3. Being positive-definite on $A(K)/A(K)_{\text{tors}}$ does not automatically mean it will be positive-definite on $A(K) \otimes \mathbb{R}$. The proof in this cases uses the Northcott property. $\bullet$

Remark 4.2.4. An element of $A(K)_{\mathbb{R}}$ looks like $\sum_{i=1}^n P_i \otimes c_i$, where $c_i \in \mathbb{R}$. What does the canonical height of such a point look like? It's determined by the bilinear form

$$\left\langle \sum_{i=1}^n P_i \otimes a_i, \sum_{j=1}^m Q_j \otimes b_j \right\rangle_D := \sum_{i,j} a_i b_j \cdot \langle P_i, Q_j\rangle_D. \qquad \circ$$

Notation 4.2.5. Define $\|\cdot\|_D\colon A(K)_{\mathbb{R}} \to [0,\infty)$ via $\|P\|_P = \sqrt{\langle P,P\rangle_D}$. Note that $(A(K)_{\mathbb{R}}, \|\cdot\|_D)$ is a Euclidean $\mathbb{R}$-vector space. Write

$$A(K)_{\mathbb{Z}} \coloneqq \operatorname{im}(A(K) \to A(K)_{\mathbb{R}}) \cong A(K)/A(K)_{\text{tors}},$$

a full rank lattice in $A(K)_{\mathbb{R}}$.

Theorem 4.2.6 (Néron)**.** *Define*

$$N(A(K), h_D, T) \coloneqq \#\left\{P \in A(K)\colon h_D(P) \le T\right\}.$$

Then,

$$N(A(K), h_D, T) = \alpha(A/K, D)\, T^{r/2} + O\Big(T^{\frac{r-1}{2}}\Big),$$

where $\alpha(A/K, D) > 0$*. In fact,*

$$\alpha(A/K, D) = \#A(K)_{tors} \cdot \frac{\operatorname{vol}(\textit{unit ball in } \mathbb{R}^n)}{\operatorname{Reg}_D(A/K)^{1/2}}$$

"This was, to a certain extent, Néron's motivation in trying to construct canonical heights."

Proof (sketch).

- Step 1

  We know $\widehat{h}_D = h_D + O(1)$, so (exercise) it suffices to prove the same formula for counting function $N(A(K), \widehat{h}_D, T)$ relative to the canonical height.

- Step 2

  Instead of counting in $A(K)$ itself, we'd prefer to count in $A(K)_{\mathbb{Z}} \subset A(K)_{\mathbb{R}}$. For this we use

  $$N(A(K), \widehat{h}_D, T) = \#A(K)_{\text{tors}} \cdot N\Big(A(K)_{\mathbb{Z}}, \widehat{h}_D, T\Big),$$

  which holds because $\widehat{h}_D(P) = 0 \iff P \in \#A(K)_{\text{tors}}$ and because $\widehat{h}_D$ is a quadratic form.

- Step 3

  At this point, we can reduce to a geometric problem of counting lattice points in a Euclidean space:

  $$N\Big(A(K)_{\mathbb{Z}}, \widehat{h}_D, T\Big) = N\big(A(K)_{\mathbb{Z}}, \|\cdot\|^2, T\big) = N\Big(A(K)_{\mathbb{Z}}, \|\cdot\|, \sqrt{T}\Big).$$

  "And now we've achieved the basic goal: anytime you're working with elliptic curves, you want to get to a point where there are no elliptic curves at all, because elliptic curves are hard."

- Step 4

  Apply a standard theorem like the following.

  Theorem 4.2.7. *Let V be an $\mathbb{R}$-vector space, let $\|\cdot\|$ be a Euclidean norm, and let $L \subset V$ be a (full rank) lattice. Then,*

  $$\#\left\{v \in L\colon \|x\| \le T\right\} = \alpha T^{\dim V} + O\big(T^{\dim V - 1}\big),$$

  where

  $$\alpha = \frac{\operatorname{vol}(\textit{unit ball})}{\operatorname{vol}(\textit{fundamental domain for } L)}.$$

∎

4.2.2 Local Heights

Before talking about canonical local heights, let's talk about Weil local heights.

Notation 4.2.8. M_K denotes the set of places of K. We'll write $v \in M_K$ for a place of K and write K_v for the completion.

Consider a point $P \in \mathbb{P}^N(K_v)$ along with an effective divisor $D \in \mathrm{Div}(\mathbb{P}^N)$. We would like to define a "*local height*" $\lambda_{D,v}(P)$.

Definition 4.2.9. The `v-adic local height of P with respect to the divisor D` is

$$\lambda_{D,v}(P) \coloneqq \text{worry about this later},$$

which one thinks of as $-\log(\text{"the } v\text{-adic distance from } P \text{ to } D\text{"})$. ◇

Note that $\lambda_{D,v}(P)$ is *large* when P is v-adically close to D (i.e. distance is small, so $-\log$ is big).

Remark 4.2.10. $\lambda_{D,v}(P) = \infty$ if $P \in |D| \coloneqq \operatorname{supp} D$. ○

Example 4.2.11. Let's give an actual formula in $\mathbb{P}^N$. Say $D = \{F = 0\}$ for some homogeneous degree d polynomial $F \in K_v[x_0, \dots, x_N]$. One would like to say that they are close to the hypersurface if $F(P)$ is small, but there's the wrinkle that $F(P)$ isn't well-defined. However, $F(P)/x_0(P)^d$ is well-defined. One sets

$$\lambda_{D,v}(P) \coloneqq -\log \min_i \left| \frac{F(p)}{x_i(P)^d} \right|_v.$$

Note this involves some choices (e.g. F well-defined only up to multiplication by a constant), so this business is still only well-defined up to constant. △

Fact. If you normalize the absolute values properly, then

$$h_{\mathbb{P}^N,D}(P) = \sum_{v \in M_K} \lambda_{D,v}(P) + O(1)$$

for all $P \in \mathbb{P}^N(K) \setminus D$. The local heights at points on D are infinite.

In general, one can define local Weil heights $\lambda_{X,D,v} \colon X(K_v) \setminus |D| \to \mathbb{R}$ on any variety X, and one will have

$$h_D = \sum_v \lambda_{D,v} + O(1)$$

(away from D).

Néton constructed canonical local heights on abelian varieties.

Theorem 4.2.12 (Néron)**.** *There exists unique up to constant* `canonical local heights`

$$\widehat{\lambda}_{D,v} \colon A(K_v) \setminus |D| \to \mathbb{R}$$

such that

(a) *$\widehat{\lambda}_{D,v}$ is continuous for the v-adic topology on $A(K_v)$.*

(b) *$\widehat{\lambda}_{D+D',v} = \widehat{\lambda}_{D,v} + \widehat{\lambda}_{D',v} + \gamma_{D,D',v}$ where $\gamma_{D,D',v} = 0$ for almost all v (when D, D' fixed).*

(c) *Given $\varphi \colon A \to A'$, have $\widehat{\lambda}_{A,\varphi^* D,v} = \widehat{\lambda} A', D, v \circ \varphi + \gamma_{A,A',D,D',v}$ (with constant zero for almost all v again)* *(d)* $\widehat{h}_{A,D} = \sum_{v \in M_K} \widehat{\lambda}_{A,D,v} - \kappa(A, D)$.

You can try to normalize the local heights to make $\kappa = 0$. There is a nice way to do this in the elliptic curves case, but things are more complicated for higher-dimensional abelian varieties.

Néron gives a view constructions:

(1) via limit formulas

(2) If v is non-archimedean, via intersection theory

(3) If v archimedean, $A(\mathbb{C}) = \mathbb{C}^g/L$, via $-\log|\text{theta function, slightly modified}|$.

4.3 Lecture 3 (3/4)

Note 7. Roughly 10 minutes late

Question 4.3.1. *How small can $0 \neq \widehat{h}_{A,D}(P)$ be?*

Two variants

- Fix K and vary A/K
- Fix A and vary the field of definition for $P \in A(\overline{K})$

(Asked by analogy w/ similar question for heights of non-roots of unity)

4.3.1 Part I: Fix A/K, vary field of def of P

Example 4.3.2 (analogy)**.** Say $\alpha \in \overline{\mathbb{Q}}^\times$ is not a root of unity. $h(\alpha) \geq ?$

Well, $h(2^{1/n}) = \frac{1}{n} h(2) = \frac{\log 2}{n} \xrightarrow{n \to \infty} 0$ (in general $h(\alpha^m) = |m|\, h(\alpha)$. So get arbitrary small, but need increasingly large degrees, $[\mathbb{Q}(2^{1/n}) : \mathbb{Q}] = n$. △

Notation 4.3.3. Write $\widehat{h} = \widehat{h}_{A,D}$ to keep notation simple.

Example 4.3.4. Can do same trick for $P \in A(K) - \text{tors}$. For each n, can find $Q \in A(\overline{K})$ such that $nQ = P$. Then, $\widehat{h}(Q) = \frac{1}{n^2} \widehat{h}(P) \xrightarrow{n \to \infty} 0$. What's $[K(Q) : K]$? Technically this depends on A, P, n,etc., but generically

$$[K(Q) : K] \approx \#[m]^{-1}(P) = \#A[m](\overline{K}) = m^{2g},$$

where $g = \dim A$. So we get something like

$$\widehat{h}(Q) \approx \frac{1}{[K(Q) : K]^{1/g}}.$$

△

Upshot: to make the height small, need to pass to a field of relatively high degree.

Conjecture 4.3.5 (`Lehmer's Conjecture`)**.** *There is some $C > 0$ such that for all $\alpha \in \overline{\mathbb{Q}}^\times \setminus \mu$,*

$$h(\alpha) \geq \frac{C}{[\mathbb{Q}(\alpha) : \mathbb{Q}]}.$$

History. Lehmer did not conjecture, he asked a question. In fact, based on the way he asked the question, it sounds like he was likely more inclined to think the above is false instead of true. ⊖

Theorem 4.3.6 (Dobrowolski 1979)**.** *Write $d = [\mathbb{Q}(\alpha) : \mathbb{Q}]$. Then, with assumptions as in Lehmer's conjecture,*

$$h(\alpha) \geq \frac{C}{d} \cdot \left(\frac{\log\log d}{\log d} \right)^3$$

Conjecture 4.3.7 (Lehmer's Conjecture for AVs, Masser 1984)**.** *There exists a constant $C = C(A/K, D) > 0$ such that, for all non-torsion $P \in A(\overline{K})$,*

$$\widehat{h}_D(P) \geq \frac{C}{[K(P) : K]^{1/g}}.$$

Let's mention some known results for this conjecture. Use the following notation

$$\varepsilon > 0 \qquad P \in A(\overline{K}) \setminus \text{tors} \qquad d = [K(P) : K] \qquad C = C(A/K, D, \varepsilon) > 0.$$

(a) $\widehat{h}(P) \geq C/d^{2g+1+\varepsilon}$.

This is close for elliptic curves, but still quite far in general. However, it's non-trivial to even get a polynomial in d in the denominator.

(b) If A has CM, $\widehat{h}(P) \geq C/d^{1+\varepsilon}$.

(c) If $g = 1$ and $j(A) \notin \mathscr{O}_K$, then $\widehat{h}(P) \geq C/d^{2+\varepsilon}$.

(Above d^ε really means some power of $\log d$)

The two main proof methods that have been most successful are

(1) Transcendence Theory

Take L/K of degree d. Consider

$$A(L, B) := \left\{ P \in A(L) \colon \widehat{h}_D(P) \leq B \right\}$$

(might take $B = C/d$ or C/d^2 or something like that).

Goal. Show that $\#A(L, C/d) \leq C_2 d^{\text{some power}}$.

Apply this to $\{mP \colon m \leq \text{some bound}\}$ if $\widehat{h}(P)$ is small.

To get bound on $A(L, B)$ exploit group law by considering

$$A(L, B)^{(m)} := \{P_1 + \cdots + P_m : P_i \in A(L, B)\}.$$

Note that $\#A(L, B)^{(m)} \approx \#A(L, B)^m/m!$, but also $A(L, B)^{(m)} \subset A(L, m^2 B)$. Now

- construct a (small) theta function F such that F vanishes at all points in $A(L, B)^{(m)}$ to high order.
- Prove upper bound for $|\partial_\eta F(Q)|$ for $Q \in A(L, B)^{(m)}$.
- hard part: zero estimate to get nonzero lower bound for $|\partial_\eta F(Q)|$.
- Previous two contradict if $\#A(L, B)$ is big.

(2) Fourier averaging (Harmonic analysis)

Maybe later if time...

4.3.2 Part II: Fix K, vary A/K

Intuition: $\widehat{h}(P) \neq 0 \implies \widehat{h}(P) >$ complexisty of A/K itself.

Slogan. Complicated abelian varieties have complicated points.

To make sense of this, we need some notion of the complexity/height of the abelian variety.

Example 4.3.8. Say $E\colon y^2 = x^3 + ax + b$ an elliptic curve. Can define

$$h(E/k) = \min_{u \in K^\times} h([1, u^4 a, u^6 b]). \qquad \triangle$$

Example 4.3.9. A_g is quasi-projective, so fix an embedding $j\colon A_g \hookrightarrow \mathbb{P}^N$ and define

$$h(A/K) = h(j(A/K)).$$

This is ok, but it's not Northcott. Abelian varieties which are isomorphic over $\overline{K}$ will give the same point in the moduli space, twists have the same height. Hence, people usually modify this by setting something like

$$h(A/K) = h(j(A/K)) + \log \operatorname{Nm}_{K/\mathbb{Q}}(N_A),$$

where N_A is the conductor of A. $\triangle$

Example 4.3.10. Faltings height. $\triangle$

Conjecture 4.3.11 (Demjaneko-Lang(-Silverman))**.** *Let $P \in A(K)$ such that $\mathbb{Z} \cdot P$ is Zariski dense in A.*[5] *Then,*

$$\widehat{h}_{A,D}(P) \geq C_1(A/K, g) h(A/K) - C_2(A/K, g)$$

($g = \dim A$).

(Silverman was modest and didn't include himself)

Results:

(1) True if $j(A) \in \mathcal{A}_g(\mathbb{C})$ is ε-distance from the geometrically simple locus in $\mathcal{A}_g(\mathbb{C})$.

(2) True if you fix A_0 and let $A = A_0^\chi$ run over twists.

(3) If $g = 1$, ABC conjecture $\implies$ Demjaneko–Lang conjecture.

4.3.3 Fourier averaging, very quickly

Say we have distinct $P_1, \ldots, P_N$ of small height on elliptic curve $A = E$. Consider

$$\sum_{i \neq j} \widehat{h}(P_i - P_j) = \sum_{v \in M_K} \sum_{i \neq j} \lambda_v(P_i - P_j).$$

One partitions the inner sum depending on if v is a place of good reduction (≥ 0), additive reduction (handlable), archimedean (we won't discuss), etc.

Say v is non-archimedean of multiplicative reduction. Consider the map $t\colon E(K) \to E(K)/E_0(K) \cong \frac{1}{M}\mathbb{Z}/\mathbb{Z}$ (component group of Néron model at v), where $M = \operatorname{ord}_v(\Delta_E)$. In this case,

$$\lambda_v(P) = \frac{1}{2}\mathbb{B}_2(t(P)) + \underbrace{\text{formal group part}}_{\geq 0}.$$

[5] Worry about something like $(0, P) \in A \times A$

Above, the `second Bernoulli polynomial` is $\mathbb{B}_2(T) = T^2 - T + \frac{1}{6}$ on $[0,1]$ and then make 1-periodic. Has Fourier series

$$\mathbb{B}_2(T) = \frac{1}{2T} + \sum_{n\neq 0} \frac{e(nT)}{n^2} \text{ where } e(x) = \exp(2\pi i x).$$

Plug this in to get

$$\sum_{i\neq j} \lambda_v(P_i - P_j) \geq \sum_{i\neq j} \frac{1}{2}\mathbb{B}_2(t_(P_i) - t(P_j)).$$

Plug in Fourier series and switch sums to get

$$\frac{1}{4\pi^2} \sum_{n\neq 0} \frac{1}{n^2} \sum_{i\neq j} e(t_i - t_j).$$

This inner sum is interesting (see notes for the manipulations at this point...).

Note 8. See recording/posted notes on AWS for more discussion...

4.4 Lecture 4 (3/5): Canonical Heights in Families and Specialization Theorems

Example 4.4.1 (Family of ECs)**.** Consider $E_T : y^2 = x^3 + T^2 x - 1$. Note that we have $P_T = (1, T) \in E_T(\mathbb{Q}(T))$. Can start plugging in values for T. Choose some $t \in \mathbb{Q}$ and compute $\widehat{h}_{E_t,2(0_t)}(P_t)$. One gets numbers like

t	$\approx \widehat{h}_{E_t,2(0_t)}(P_t)$
0	0
2	0.93
17	2.51
1729	7.11
22/7	3.24
355/113	5.68

Seems like maybe when you specialize the height of P_t grows as the height of t grows. △

Notation 4.4.2.

- K a number field
- C/K a smooth projective K-curve
- $A/K(C)$ an abelian variety over C's function field
- $(\mathcal{A}, \pi)$ a spreading out of $A/K(C)$, so $\pi : \mathcal{A} \to C$.

  $\mathcal{A}$ is a family of abelian varieties (except for some singular fibers) parameterized by C.
- $P \in A(K(C))$ a point with associated section $\mathcal{P} : C \to \mathcal{A}$.

Remark 4.4.3. For $t \in C(\overline{K})$, get specialization map

$$S_t : A(\overline{K}(C)) \longrightarrow \mathcal{A}_t(\overline{K}),$$

i.e. $S_t(P) = \mathcal{P}(t)$. This is homomorphism. ○

Question 4.4.4. *To what extent does S_t send independent points to independent points?*

Theorem 4.4.5 (Specialization Theorem)**.** *Assume that $\mathcal{A} \to C$ has no "constant part." Then,*

$$\left\{t \in C(\overline{K}) : \ker S_t \neq 0\right\}$$

is a set of bounded height. In particular, $\#\{t \in C(K) : \ker S_t \neq 0\} < \infty$.

This theorem follows from a 'height limit theorem'.

Let $D \in \mathrm{Div}(A/K(C))$ be a divisor, and thicken it to a divisor $\mathcal{D} \in \mathrm{Div}(\mathcal{A}/K)$ (take closure of D). Let h_C be a Weil height on C for some degree 1 divisor.

Theorem 4.4.6 (Height limit theorem)**.**

$$\lim_{\substack{t \in C(\overline{K}) \\ h_C(t) \to \infty}} \frac{\widehat{h}_{\mathcal{A}_t, \mathcal{D}_t}(\mathcal{P}(t))}{h_C(t)} = \widehat{h}_{A,D}(P).$$

Note the above theorem involves 3 different height functions.

Recall 4.4.7 (canonical height pairing)**.** $\langle P, Q\rangle = \frac{1}{2}\left(\widehat{h}(P+Q) - \widehat{h}(P) - \widehat{h}(Q)\right).$ ⊙

Corollary 4.4.8.

$$\lim_{\substack{t \in C(\overline{K}) \\ h_C(t) \to \infty}} \frac{\langle \mathcal{P}(t), \mathcal{Q}(t)\rangle_{\mathcal{A}_t, \mathcal{D}_t}}{h_C(t)} = \langle P, Q\rangle_{A,D}.$$

Proof of Theorem 4.4.5, assuming Theorem 4.4.6. Let $P_1, \ldots, P_r$ generate $A(\overline{K}(C))/\mathrm{tors}$. Then,

$$\lim_{h_C(t) \to \infty} \frac{\mathrm{Reg}_{\mathcal{D}_t}(\mathcal{P}_1(t), \ldots, \mathcal{P}_r(t))}{h_C(t)^r} = \mathrm{Reg}_D(P_1, \ldots, P_r) > 0.$$

Thus, $h_C(t) \gg 1 \implies \mathrm{Reg}_{\mathcal{D}_t}(\mathcal{P}_1(t), \ldots, \mathcal{P}_r(t))$, so these points are linearly independent in the fiber above t. ■

(One still needs an additional argument to deal with the torsion subgroup. I guess it's enough to note that torsion injects if you reduce mod v where $v \nmid \#A(\overline{K}(C))_{\mathrm{tors}}$)

Generalizations

(1) $\dim C \geq 2$

In this case, specialization will fail along curves (if $\dim C = 2$). There are still statements that say that for "most" points on your base, the specialization map will be injective.

(2) $h(t) \gg 1 \implies \mathrm{rank}\, \mathcal{A}_t(K) \geq \mathrm{rank}\, A(K(C))$ (say $t \in C(K)$). This leads to question for '*rank jumps*', e.g. when can you get $\geq +1$ or $\geq +2$?

(3) Theorem 4.4.6 shows $\widehat{h}(P_t) = \widehat{h}(t)h(t) + o(h(t))$. Even more is known though.

Theorem 4.4.9 (Patrick Ingram, up to spelling, and Tate)**.** *$o(h(t))$ can be replaced by $O(h(t)^{2/3})$. If $C = \mathbb{P}^1$ or $\dim A = 1$, then you can further replace it by $O(h(t)^{1/2})$. If both $C = \mathbb{P}^1$ or $\dim A = 1$, then can replace it by $O(1)$.*

(Sounds like in some cases, you can do even better)

Let's end by sketching a proof of the height limit theorem.

Proof Sketch of Theorem 4.4.6. Consider difference

$$\overbrace{\left|\widehat{h}_{\mathcal{A}_t,\mathcal{D}_t}(\mathcal{P}(t)) - \widehat{h}_{A,D}(P)h_C(t)\right|}^{\textbf{(a)}}$$
$$\le \underbrace{\left|\widehat{h}_{\mathcal{A}_t,\mathcal{D}_t}(\mathcal{P}(t)) - h_{\mathcal{A}_t,\mathcal{D}_t}(\mathcal{P}(t))\right|}_{\textbf{(b)}} + \underbrace{\left|h_{\mathcal{A}_t,\mathcal{D}_t}(\mathcal{P}(t)) - h_{A,D}(P)h_C(t)\right|}_{\textbf{(c)}} + \underbrace{\left|h_{A,D}(P)h_C(t) - \widehat{h}_{A,D}(P)h_C(t)\right|}_{\textbf{(d)}}.$$

(b) One knows that $\widehat{h}(\mathcal{P}(t)) - h(\mathcal{P}(t)) = O(1)$, but with implicit constant depending on the abelian variety and on the divisor. If one carefully does this proof, keeping track of how constants vary with t, they can show that
$$\widehat{h}(\mathcal{P}(t)) - h(\mathcal{P}(t)) = O(h(t)).$$

(c) This is the hardest summand to deal with. Divide by $h_C(t)$ to look at
$$\left|\frac{h(\mathcal{P}_t)}{h(t)} - h(P)\right|.$$
By functoriality applied to $\mathcal{P}\colon C \to \mathcal{A}$, we know $h_{\mathcal{A},\mathcal{D}}(\mathcal{P}_t) = h_{C,\mathcal{P}^*\mathcal{D}}(t) + O(1)$ and $\deg \mathcal{P}^*\mathcal{D} = h_D(P) + O(1)$. Thus, we essentially have $h_{\mathcal{A},\mathcal{D}}(\mathcal{P}_t) = h_C(P)h(t)$. Hence, up to being a bit sloppy, we have
$$\left|\frac{h(\mathcal{P}_t)}{h(t)} - h(P)\right| \le O(1).$$
More accurately, one ends up with
$$\limsup_{h(t)\to\infty}\left|\frac{\widehat{h}(\mathcal{P}_t)}{h(t)} - \widehat{h}(P)\right| \le O(1)$$
with constant independent of P. Now, replace P by mP in order to show that this constant is 0, so the above $\limsup$ is really a $\lim$ and is equal to 0.

(d) Factoring out the $h_C(t)$, we're interested in $\left|h_{A,D}(P) - \widehat{h}_{A,D}(P)\right|$. This is simply $O(1)$ (constant independent of P). Thus, this summand is again $O(h(t))$. ■

5 Barry Mazur

5.1 Lecture 1 (3/2): Abelian Varieties

"It's wonderful to see this wonderful, enormous crowd of people and bear." (paraphrase)

Note 9. Slide talk, so we'll see how well I keep up

Question 5.1.1 (theme of lectures)**.** *What are abelian varieties? Why are they interesting/useful?*

We'll answer this, in part, by discussing recent work/conjectures/questions regarding, for example, uniformity and statistics of their Diophantine behavior.

Example 5.1.2 (Vague example of utility)**.** One of the most useful tools in understanding rational points on curves is to use abelian varieties (e.g. the curve's jacobian). $\triangle$

Also, the subject is moving fast. Barry claims that many of the theorems he will mention have already been furthered/expanded upon and that it's hard to keep up. Sounds like Barry is interested in databases of abelian varieties (even just of abelian surfaces as a start).

5.1.1 Start with elliptic curves over K

(K just a field for now)

Note 10. This is slow moving, so I'm gonna make my notes scant (and/or not take notes) starting now...

Barry recommends reading Poincaré's paper "Sur les propri'et'es arithm'etiques des courbes alg'ebriques" when elliptic curves got started. He also recommends looking at the paper where his Poincaré conjecture first appears.

Theorem 5.1.3. *If $d < \delta_C$, the natural map*

$$S_C(K; d) \longrightarrow J_C(K)$$

is injective.

Above,

- $S_C(K; d)$ is the set of K-conjugacy classes of algebraic points on C of degree $\leq d$
- δ_C is the $\overline{K}$-gonality of C
- J_C is the Jacobian of C
- j sends an algebraic point to its trace

5.2 Lecture 2 (3/5)

6 List of Marginal Comments

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