18.786 (Number Theory II) Notes

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These are my course notes for "Number Theory II" at MIT. Each lecture will get its own "chapter." These notes are live-texed or whatever, so there will likely to be some (but hopefully not too much) content missing from me typing more slowly than one lectures. They also, of course, reflect my understanding (or lack thereof) of the material, so they are far from perfect.[1] Finally, they contain many typos, but ideally not enough to distract from the mathematics. With all that taken care of, enjoy and happy mathing.

The instructor for this class is Wei Zhang.

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[1] In particular, if things seem confused/false at any point, this is me being confused, not the speaker

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1 Lecture 1 (2/17)

"You're not required to wear a mask for the class"[2]

1.1 Intro to Class

We would like to cover (mostly following Bump's "Automorphic forms and representations")

- Tate's thesis
- Automorphic forms on GL(2) (with a brief review of classical modular forms, then Hecke/Jacquet–Langlands theory of L-functions)
- Representation theory of GL(2) over p-adic fields and of local Galois groups (e.g. the formulation of Local Langlands correspondence for GL(2))
- Other topics: Arthur–Selberg Trace formula, Relative trace formula and applications (e.g. Jacquet–Langlands correspondence, Waldspurger/Gross–Zagier formula)

We will not cover everything in the book (in particular, sounds like we barley touch chapter 2, if at all).

We begin with Tate's thesis which gives the GL(1) story, and serves as good guidance/motivation for the generalizations in the higher GL(n) cases. At the end of the semester, we move to more advanced topics with the hope of giving more useful applications. The whole class is kinda centered on L-functions.

Here's one motivation for being interested in special values of L-functions: they're related to BSD. We'll see some of this if we talk about the Waldspurger/Gross–Zagier formula at the end.

One goal of the class is to make sure everybody gets used to computations using a(i)dèles. Also something about functional equation and meromorphic continuation for L-functions arising in the GL(1) and GL(2) cases.

Class stuff No final exam. Grade based on homeworks. Planning for 6/7 assignments, due every week or other week. Collaboration is allowed, but acknowledge this when writing solutions.

To ask a question, unmute yourself and speak up. Lecture notes all available on dropbox.[3] We'll spend first ~ 2 weeks on Tate's thesis.

1.2 Beginning of GL(1) Stuff

Notation 1.1. We let F denote a global field (e.g. $\mathbb{Q}$ or $\mathbb{F}_q(t)$).

Tate's thesis is a generalization of the following result probably already known to Riemann. Let $\zeta(s) = \sum_{n\geq 1} n^{-s}$ be the Riemann zeta function. Recall the Gamma function

Good references (IMO) include these notes by Poonen and the book 'Fourier Analysis on Number Fields' by Ramakrishnan and Valenza

$$\Gamma(s) = \int_0^\infty e^{-t}t^s\frac{\mathrm{d}t}{t}.$$

[2]it's via Zoom

[3]Not sure if I'm supposed to share this link, so don't tell them I sent you.

Then, $\Gamma\left(\frac{s}{2}\right)\zeta(s)$ has a nice integral representation; up to simple factor $(*)$, it looks like

$$\Gamma\left(\frac{s}{2}\right)\zeta(s) = (*)\int_0^\infty \left(\sum_{n\in\mathbb{Z}:n\neq 0} e^{-tn^2}\right) t^s \frac{\mathrm{d}t}{t}.$$

Let $\theta(\tau) = \sum_{n\in\mathbb{Z}} q^{n^2}$ where $q = e^{2\pi i\tau}$ and $\tau = x + iy \in \mathbb{C}$ with $y > 0$.

The main point is that we have an "integral representation of $\zeta(s)$." This is used to eventually give both the meromorphic continuation, and the functional equation of $\zeta(s)$. Tate's thesis gives a similar story for more general L-functions. Part of the appeal of Tate's thesis is that it gives a uniform treatment for general number fields.

Here are some sources of other L-functions

- Dirichlet characters $\chi : (\mathbb{Z}/N\mathbb{Z})^\times \to \mu_\infty \subset \mathbb{C}^\times$, where $\mu_\infty = \left\{e^{2\pi i q} : q \in \mathbb{Q}\right\}$ is the group of roots of unity.

- CM elliptic curves $E : dy^2 = x^3 - x = x(x^2-1)$ ($d \in \mathbb{Z}$ nonzero and squarefree). Note this curve has an order 4 automorphism $(x, y) \mapsto (-x, iy)$, and indeed $\mathrm{Aut}(E) \simeq \mathbb{Z}/4\mathbb{Z}$. In this case, can consider the Hasse-Weil zeta function. Set

  $$a_p = (p+1) - \#E(\mathbb{F}_p)$$

  (when p a prime of good reduction). It was know early on that from a CM elliptic curve, you can produce a Hecke grossencharacter. The punchline is that

  $$L(E, s) = \prod_p \left(1 - a_p p^{-s} + p^{1-2s}\right)^{-1} = L(\chi, s)$$

  for some Hecke character χ of $F = \mathbb{Q}(i)$.

  This shows that even if only want to study phenomena over $\mathbb{Q}$, you still need to understand what happens for more general number fields.

See section 5.3.4 (page 100) of the other Bump book (An intro to Langlands), I think

Remark 1.2. The references for Tate's thesis are Tate's original thesis and the notes by Kudla (shared in lecture notes). Kudla has a different treatment of the local theory than Tate, following an idea of Weil (who has a Bourbaki talk on Tate's thesis).

Recall 1.3 (Adeles/Ideles)**.** Let v be a place on a global field F. We let F_v denote the completion of F with respect to v, and when v is non-archimedean, we let $\mathscr{O}_{F_v}$ denote its valuation ring. The **Adeles** are the restricted direct product

$$\mathbb{A}_F = \prod_v{}' F_v \subset \prod_v F_v$$

consisting of the tuples $(x_v) \in \prod_v F_v$ such that $x_v \in \mathscr{O}_{F_v}$ for almost all v. We topologize this by giving it the smallest topology which contains the sets

$$\prod_{v\in S} U_v \times \prod_{v\notin S} \mathscr{O}_{F_v}$$

where $U_v \subset F_v$ is open for all $v \in S$ and S is a finite set of places and which makes $\mathbb{A}_F$ a topological ring. This is *not* the subspace topology.

The **Ideles**, as an abstract group, are the units

$$\mathbb{A}_F^\times = \prod_v{}' F_v^\times$$

of the adeles. The restricted direct product here is with respect to $\mathscr{O}_{F_v}^\times$. We give it the analogous restricted direct product topology (which is not the subspace topology).

Remark 1.4. Historically, ideles came first because of their connection to global class field theory.

Remark 1.5. Local class field theory gives the **local Artin map**

$$\varphi_v : F_v^\times \to \mathrm{Gal}(F_v^{\mathrm{ab}}/F_v)$$

(F_v^{ab} is the maximal abelian extension of F_v). This map is continuous with dense image. In the p-adic case, it is also injective.[4]

Global class field theory gives a continuous surjection

$$\varphi : \mathbb{A}_F^\times \overset{\prod_v \varphi_v}{\twoheadrightarrow} \mathrm{Gal}(F^{\mathrm{ab}}/F).$$

This map actually factors through the quotient $\mathbb{A}_F^\times/F^\times$ (this quotient called the **idèle class group**), i.e. $\varphi|_{F^\times} = 1$ (this is **Reciprocity**). Here $F^\times \hookrightarrow \mathbb{A}_F^\times$ via the diagonal embedding $x \mapsto (\dots, x, x, x, \dots)$.

Fact. The diagonal embedding $F^\times \hookrightarrow \mathbb{A}_F^\times$ has discrete image. The diagonal embedding $F \hookrightarrow \mathbb{A}_F$ has discrete image. These both follow from the "**product formula**"

$$x \in F^\times \implies \prod_v |x|_v = 1$$

(when using the normalized absolute value).

Note that global class field theory tells us that

$$\left\{\begin{matrix} \text{characters of} \\ \mathrm{Gal}(F^{\mathrm{ab}}/F) \end{matrix}\right\} \hookrightarrow \left\{\begin{matrix} \text{characters of} \\ \mathbb{A}_F^\times/F^\times \end{matrix}\right\}.$$

Definition 1.6. Say $\chi_v : F_v^\times \to \mathbb{C}^\times$ is a (quasi)character[5], i.e. a continuous homomorphism.

Definition 1.7. A **Hecke character** (or grossencharacter, up to spelling) is a continuous homomorphism

$$\chi : F^\times \backslash \mathbb{A}_F^\times \to \mathbb{C}^\times$$

Note that any character of the idele class group induces local characters on all $F_v^\times$.

[4] Can't be an iso since target compact but source is not

[5] Sometimes people use "character" to refer only to "unitary quasicharacters," i.e. those landing in S^1

Lemma 1.8. *Let $\chi : \mathbb{A}_F^\times / F^\times \to \mathbb{C}^\times$ be a Hecke character (really, enough to start with $\mathbb{A}_F \xrightarrow{\chi} \mathbb{C}^\times$). For each place v, let χ_v be the composition*

$$\chi_v : F_v^\times \to \mathbb{A}_F^\times \to \mathbb{A}_F^\times / F^\times \xrightarrow{\chi} \mathbb{C}^\times.$$

Then,

$$\chi_v|_{\mathscr{O}_{F_v}^\times} = 1$$

for almost all v.

something something kernel of χ is open something something?

By the above lemma, we can now see that

$$\chi = \prod_v \chi_v,$$

i.e. for $x = (x_v)_v \in \mathbb{A}_F^\times$, $\chi(x) = \prod_v \chi_v(x_v)$ (the product is secretly finite by the previous lemma + definition of $\mathbb{A}_F^\times$).

Definition 1.9. We say χ_v is **unramified** if $\chi_v|_{\mathscr{O}_{F_v}^\times} = 1$.

Example. Let Cl_F denote the class group of F. Then,

$$\mathrm{Cl}_F \xrightarrow{\sim} F^\times \backslash \mathbb{A}_F^\times \Bigg/ \left(\prod_{v<\infty} \mathscr{O}_{F_v}^\times \cdot \prod_{v|\infty} F_v^\times \right)$$

Hence, any character $\rho : \mathrm{Cl}_F \to \mathbb{C}^\times$ produces a character $\chi : F^\times \backslash \mathbb{A}_F^\times \to F^\times$ which is moreover unramified at all non-arch places and trivial at all arch places. Conversely, given any such Hecke character will give a character of the class group.

Example. Say $F = \mathbb{Q}$. Then, we can get an isomorphism

$$\mathbb{Q}^\times \backslash \mathbb{A}_\mathbb{Q}^\times / \mathbb{R}_+^\times \cdot \left(\prod_{p \nmid N} \mathbb{Z}_p^\times \right) \prod_{p|N} (1 + N\mathbb{Z}_p) \xrightarrow{\sim} (\mathbb{Z}/N\mathbb{Z})^\times$$

due to the fact that

$$\mathbb{A}_\mathbb{Q}^\times = \mathbb{Q}^\times \left(\mathbb{R}_+^\times \cdot \prod_p \mathbb{Z}_p^\times \right)$$

(since $\mathrm{Cl}_\mathbb{Q} = 1$ is trivial). Hence, LHS from before is isomorphic to

$$\prod_{p|N} \left(\frac{\mathbb{Z}_p^\times}{1 + N\mathbb{Z}_p} \right) \simeq (\mathbb{Z}/N\mathbb{Z})^\times .$$

This tells you that a Dirichlet character of conductor N gives rise to a Hecke character (over $\mathbb{Q}$) which is unramified away from $p \mid N$. Note that, in this case, χ_∞ is trivial or order 2 (depending on if Dirichlet character is even or odd)

Example. Say E is the CM elliptic curve from before, and $F = \mathbb{Q}(i)$. In this example, it is harder to write down the details, but the point is we get a character $\chi : F^\times \backslash \mathbb{A}_F^\times \to \mathbb{C}^\times$ such that[6] $\chi_\infty : \mathbb{C}^\times \to \mathbb{C}^\times$ has infinite order.

Remark 1.10. Hecke characters are (basically) the GL(1)-automorphic forms.

Next time we'll move onto L-functions, and give Tate's construction of zeta integrals.

2 Lecture 2 (2/22)

Note 1. *A few minutes late*

Definition 2.1. Let F be a global field. It's **idèle class group** if the quotient $\mathbb{A}^\times/F^\times$. A **Hecke character**

$$\chi : \mathbb{A}^\times/F^\times \longrightarrow \mathbb{C}^\times$$

is a continuous homomorphism.

Remark 2.2. Any Hecke character $\chi : \mathbb{A}^\times/F^\times \to \mathbb{C}^\times$ will decompose into a product $\chi = \prod_v \chi_v$ of local characters

$$\chi_v : F_v^\times \to \mathbb{C}^\times$$

which are continuous homomorphisms. This decomposition comes from the continuity of χ.

We will focus on local characters today.

Definition 2.3. To a Hecke character $\chi : \mathbb{A}^\times/F^\times \to \mathbb{C}^\times$, we attach the **$L$-function**

$$L(\chi, s) = \prod_v L(\chi_v, s),$$

when the local factors are tbd.

This is meant to generalize Dedekind zeta functions and Dirichlet L-functions.

Example. When χ_0 is the trivial character, we will get

$$L(\chi_0, s) = (*)\zeta_F(s),$$

the L-function will be the Dedekind zeta function, up to some archimedean factors. Recall,

$$\zeta_F(s) := \prod_{v<\infty} \frac{1}{1 - q_v^{-s}}.$$

Notation 2.4. Let v be a finite place. We set the notation

- $(\varpi_v) \subset \mathscr{O}_{F_v} \subset F_v$
- $k_v = \mathscr{O}_{F_v}/\varpi_v$
- $q_v = \#k_v$

[6]Note $F_\infty = \mathbb{C}$

2.1 Local Theory

Let F be a non-archimedean local field. It's valuation ring is $\mathscr{O}_F$, and we fix a uniformizer ϖ. Recall the short exact sequence

$$1 \longrightarrow \mathscr{O}_F^\times \longrightarrow F^\times \xrightarrow{\text{val}} \mathbb{Z} \longrightarrow 0.$$

Since we have chosen a uniformizer, this sequence is split, so $F^\times \simeq \mathscr{O}_F^\times \times \omega^{\mathbb{Z}}$. We know that $\mathscr{O}_F^\times = \varprojlim (\mathscr{O}_F/\varpi^n)^\times$ is a profinite abelian group (in particular, compact), so $F^\times$ is locally compact.

Let $\chi : F^\times \to \mathbb{C}^\times$ be a quasi-character. We will postpone discussion of characters on $\mathbb{R}, \mathbb{C}$ to the homework.

Remark 2.5. Set $\mathscr{O}_{F,c}^\times := 1 + (\varpi^c)$, where $1 + (\varpi^0) := \mathscr{O}_F^\times$, for $c \geq 0$. Hence, we get a filtration

$$\mathscr{O}_F^\times = \mathscr{O}_{F,0}^\times \supset \mathscr{O}_{F,1}^\times \supset \mathscr{O}_{F,2}^\times \supset \dots$$

with quotients $\mathscr{O}_{F,c}^\times / \mathscr{O}_{F,c+1}^\times \simeq k$ if $c \geq 1$ while $\mathscr{O}_{F,0}^\times / \mathscr{O}_{F,1}^\times \simeq k^\times$.

Lemma 2.6. *$\chi|_{\mathscr{O}_F^\times}$ has finite image. That is $\chi|_{1+(\varpi)^c} = 1$ for some $c \geq 0$. The smallest such c is called the* ***conductor*** *of χ.*

Proof. (No small subgroup argument) More generally, say

$$\chi : G \to \mathrm{GL}_n(\mathbb{C})$$

is a continuous homomorphism from a profinite group G to $\mathrm{GL}_n(\mathbb{C})$ (with its usual Euclidean topology). There exists an open neighborhood U of $1 \in \mathrm{GL}_n(\mathbb{C})$ such that U does not contain a nontrivial subgroup.[7] Now, $\chi^{-1}(U) \subset G$ is an open neighborhood of the identity, and so contains an open, finite-index subgroup (these give base of neighborhoods around identity) N. Thus, $\chi(N) \subset U$ is a subgroup contained in U, so $\chi(N) = 1$, so χ factors through the finite group G/N.

Applying this in the case $G = F^\times$ and using the filtration $\mathscr{O}_{F,c}$ gives the claim. ■

Definition 2.7. We call χ **unramified** if its conductor is $c(\chi) = 0$, i.e. $\chi|_{\mathscr{O}_F^\times} = 1$.

Note that if χ is unramified, then $\chi(\varpi) \in \mathbb{C}^\times$ is independent of the choice of uniformizer. We define local L-functions

$$L(\chi, s) := \begin{cases} \dfrac{1}{1 - \chi(\varpi) q^{-s}} & \text{if } \chi \text{ unramified} \\ 1 & \text{if } \chi \text{ ramified} \end{cases}$$

($s \in \mathbb{C}$). We now want to give a more complicated definition of this function.

Remark 2.8. When F is archimedean, the local factor will essentially be some Γ-function(s).

2.2 Local Zeta Integrals

Recall 2.9. Recall the **Gamma function**

$$\Gamma(s) = (*) \int_0^\infty \underbrace{e^{-t^2}}_{\text{func on } \mathbb{R}} \underbrace{t^{-s}}_{\text{char for } \mathbb{R}_{>0}^\times} \underbrace{\frac{\mathrm{d}t}{t}}_{\text{Haar measure}},$$

[7] Take a nontrivial element, and consider powers of it. It'll eventually escape from the neighborhood

up to some simple factor $(*)$ (do a change of variables to usual definition).

Let F be any local field (possibly archimedean).

Definition 2.10. We introduce the space of **Schwartz functions** $\mathcal{S}(F)$.

- When F is non-archimedean, this is $\mathcal{S}(F) = C_c^\infty(F)$, locally constant functions with compact support. Recall that F is totally disconnected.
- When F is archimedean, these are smooth functions $\varphi : F \to \mathbb{C}$ **of rapid decay**, i.e. (when $F = \mathbb{R}$)

$$\sup_{x\in\mathbb{R}} |P(x)| \left| \left(\frac{\partial}{\partial x}\right)^i \varphi(x) \right| < \infty$$

  for all i and polynomial $P \in \mathbb{C}[T]$. When $F = \mathbb{C}$, we get a similar definition using $\mathbb{C} \simeq \mathbb{R} \times \mathbb{R}$ (so take derivatives in each variable).

Remark 2.11. For any variety $X/\mathbb{R}$, can similarly define $\mathcal{S}(X(F))$. Then are functions $\varphi : X(\mathbb{R}) \to \mathbb{C}$ such that $\sup_{x\in X(F)} |D\varphi(x)| < \infty$ for every algebraic differential operator D on X.

Example. If $X = \mathbb{A}^1$, the differential operators are basically all $p(x)\left(\frac{\partial}{\partial x}\right)^i$.

Example. If $X = \mathbb{G}_m = \mathbb{A}^1 \setminus 0$, the differential operators look like $p(x, x^{-1})\left(\frac{\partial}{\partial x}\right)^i$.

Definition 2.12. Let $\chi : F^\times \to \mathbb{C}$ be a quasi-character, and let $\varphi \in \mathcal{S}(F)$ be a Schwarz function. The **local zeta integral** is the "Mellin transform"

$$\zeta(\chi, \varphi, s) = \int_{F^\times} \varphi(x)\chi(x) |x|^s \, \mathrm{d}^\times x$$

($s \in \mathbb{C}$), where $\mathrm{d}^\times x$ is a Haar measure of $F^\times$. This is (absolutely) convergent when $\mathrm{Re}(s) \gg 0$.

Remark 2.13. Given an additive Haar measure $\mathrm{d}x$ on F, one can set $\mathrm{d}^\times x = \frac{\mathrm{d}x}{|x|}$, where $|\cdot| : F^\times \to \mathbb{R}^\times$ is the normalize absolute value. When F is non-arch, this is characterized by $|\varpi| = q^{-1}$ (ϖ a uniformizer and $q = \#\mathscr{O}_F/(\varpi)$).

Note that $\zeta(\chi, \varphi, s)$ is convergent on some right half-place dependent on χ. Let $|\chi|$ be the character

$$|\chi| : F^\times \xrightarrow{\chi} \mathbb{C}^\times \xrightarrow{|\cdot|} \mathbb{R}_+^\times.$$

One can show that there exists some unique $c \in \mathbb{R}$ such that

TODO: Convince yourself this adds up

$$|\chi|(x) = |x|_F^c.$$

This c is called the **exponent** of χ.

Example. Say $F = \mathbb{R}$. Then, $\varphi(x) = e^{-\pi x^2}$ is a Schwartz function. If χ is trivial, then $\zeta(\chi, \varphi, s)$ is essentially a Γ-function (evaluated at $s/2$?)

Example. Say F is non-archimedean. Then, $\varphi(x) = \mathbf{1}_{\mathscr{O}_F}(x)$ is a Schwarz function (indicator function of $\mathscr{O}_F$). In fact, any Schwarz function will be a linear combination of indicator functions,

$$\varphi = \sum_i c_i \mathbf{1}_{a_i + \varpi^{n_i}\mathscr{O}_F}.$$

Note that

$$\begin{aligned}
\mathrm{vol}(\mathscr{O}_F^\times, \mathrm{d}^\times x) &= \int_{\mathscr{O}_F^\times} \frac{\mathrm{d}x}{|x|} \\
&= \int_{\mathscr{O}_F^\times} \mathrm{d}x \\
&= \int_{\mathscr{O}_F} \mathrm{d}x - \int_{\varpi\mathscr{O}_F} \mathrm{d}x \\
&= \mathrm{vol}(\mathscr{O}_F, \mathrm{d}x)(1 - q^{-1}).
\end{aligned}$$

We normalize $\mathrm{d}x$ so that $\mathrm{vol}(\mathscr{O}_F, \mathrm{d}x) = \left(1 - q^{-1}\right)^{-1}$ (so $\mathrm{vol}(\mathscr{O}_F^\times, \mathrm{d}^\times x) = 1$). Hence, for our earlier choice of $\varphi(x)$, we have

$$\begin{aligned}
\zeta(\chi, \varphi, s) &= \int_{F^\times} \mathbf{1}_{\mathscr{O}_F}(x)\, |x|_F^s\, \chi(s) \mathrm{d}^\times x \\
&= \sum_{n \geq 0} q^{-ns} \int_{\mathrm{val}(x) = n} \chi(x) \mathrm{d}^\times x \\
&\overset{\chi \text{ unram}}{=} \sum_{n \geq 0} q^{-ns} \chi(\varpi^n) \\
&= \frac{1}{1 - \chi(\varpi) q^{-s}}
\end{aligned}$$

(assuming χ unramified in second-to-last line), so we recover our definition of the local L-function.

In general, we get an interpretation of the L-function as a "gcd" of all these zeta integrals.

Recall 2.14. Given a Schwarz function $\varphi \in \mathcal{S}(F)$, it has a **Fourier transform** $\widehat{\varphi} \in \mathcal{S}(F)$. We fix some non-trivial additive character $\psi_0 : F \to \mathbb{C}^\times$. Then, we define

$$\widehat{\varphi}(x) := \int_F \varphi(y) \psi_0(xy) \mathrm{d}y.$$

We normalize $\mathrm{d}y$ so that it is self-dual w.r.t ψ_0, i.e.

$$\widehat{\widehat{\varphi}}(x) = c\varphi(-x) \text{ with } c = 1.$$

Theorem 2.15. *Fix a quasi-character χ on $F^\times$. Then,*

- ***(1)*** *$\zeta(\chi, \varphi, s)$ has a meromorphic continuation to $s \in \mathbb{C}$.*
- ***(2)*** *$L(\chi, s)$ is the GCD of $\zeta(\chi, \varphi, s)$ as we vary $\varphi \in \mathcal{S}(F)$. Say F is non-archimedean. What we mean is*

$$\frac{\zeta(\chi, \varphi, s)}{L(\chi, s)} \in \mathbb{C}[q^s, q^{-s}].$$

In particular, the poles of $\zeta(\chi, \varphi, s)$ are at worse the poles of $L(\chi, s)$.

(3) ***(local functional equation)***

$$\varepsilon(\chi, s) \cdot \frac{\zeta(\chi, \varphi, s)}{L(\chi, s)} = \frac{\zeta(\chi^{-1}, \widehat{\varphi}, 1-s)}{L(\chi^{-1}, 1-s)}.$$

Note the ***epsilon factor*** *does not depend on φ.*

Note 2. Got distracted a missed some stuff...

Remark 2.16. Apparently $\varepsilon(\chi, s) = \varepsilon(\chi, s, \psi) = 1$ if χ and ψ are both unramified. What does it mean for an additive character to be unramified? The **conductor** of ψ is

$$c(\psi) = \max \left\{c \in \mathbb{Z} : \psi|_{\varpi^{-c}\mathscr{O}_F} = 1\right\}.$$

We say ψ is **unramified** if $c(\psi) = 0$.

The bit about the ε-factor being trivial when χ, ψ are unramified comes from $\zeta(\chi, \mathbf{1}_{\mathscr{O}_F}, s) = L(\chi, s)$ (when χ unramified) + $\widehat{\mathbf{1}_{\mathscr{O}_F}} = \mathbf{1}_{\mathscr{O}_F}$ when ψ unramified (compare to $\varphi = e^{-\pi x^2}$ for $F = \mathbb{R}$ which also satisfies $\widehat{\varphi} = \varphi$).

Remark 2.17. The local L-factor $L(\chi, s)$ itself does not determine χ (e.g. it collapses all ramified characters to 1). However, the pair $(L(\chi, s), \varepsilon(\chi, s))$ contains more information about χ (though maybe still not enough to recover χ).

3 Lecture 3 (2/24)

In the previous two lectures, we discussed Hecke characters, and then moved to the local situation, and gave the key result.

Let F be a non-archimedean local field.

Theorem 3.1. *Fix a quasi-character χ on $F^\times$. Then,*

(1) *$\zeta(\chi, \varphi, s)$ has a meromorphic continuation to $s \in \mathbb{C}$.*

(2) *$L(\chi, s)$ is the GCD of $\zeta(\chi, \varphi, s)$ as we vary $\varphi \in \mathcal{S}(F)$. Say F is non-archimedean. What we mean is*

$$\frac{\zeta(\chi, \varphi, s)}{L(\chi, s)} \in \mathbb{C}[q^s, q^{-s}].$$

In particular, the poles of $\zeta(\chi, \varphi, s)$ are at worse the poles of $L(\chi, s)$.

(3) ***(local functional equation)***

$$\varepsilon(\chi, s) \cdot \frac{\zeta(\chi, \varphi, s)}{L(\chi, s)} = \frac{\zeta(\chi^{-1}, \widehat{\varphi}, 1-s)}{L(\chi^{-1}, 1-s)}.$$

Note the ***epsilon factor*** *does not depend on φ.*

We defined L-functions for local characters, and saw that they are the gcd's of local zeta functions defined in terms of Schwartz functions $\varphi \in \mathcal{S}(F) = C_c^\infty(F)$. We also see that we obtain a local functional equation involving the "**normalized ζ-integral**" $\zeta^\natural := \zeta(\chi, \varphi, s)/L(\chi, s)$.

Let's spend a little time discussing the proof of this statement (for non-archimedean fields). When F is archmidean, a similar statement is true, except $\zeta^\natural$ is no longer simply a polynomial in $q^{\pm s}$. Our argument follows the one given by Weil in his Bourbaki notes (also in Kudla's notes).

There is an action $F^\times \curvearrowright \mathcal{S}(F)$ induced by the natural action of $F^\times \curvearrowright F = F^\times \sqcup \{0\}$ by right translation. Recall

$$\zeta(\chi, \varphi, s) = \int_{F^\times} \varphi(x)\chi(x)\,|x|^s\,\mathrm{d}^\times x.$$

Let $r : F^\times \to \mathrm{Aut}(\mathcal{S}(F))$ denote the induced group action. That is, for $a \in F^\times$,

$$(r(a)\varphi)(x) = \varphi(ax).$$

Note that

$$\zeta(r(a)\varphi, \chi, s) = \chi^{-1}(a)\,|a|^{-s}\,\zeta(\varphi, \chi, s).$$

If we view the zeta integral as a linear function $\zeta : \mathcal{S}(F) \to \mathbb{C}$, then it is an eigenfunctional under the $F^\times$ action.

Continuing with this point of view, $\mathcal{S}(F)$ is an (infinite-dimensional) representation of the abelian group $F^\times$, and we have

$$\zeta(-, \chi, s) \in \mathrm{Hom}_{F^\times}\left(\mathcal{S}(F) \otimes \chi|\,\cdot\,|^s, \mathbb{C}\right),$$

i.e. it is a linear functional on this twisted representation. We want this to characterize the zeta integral.

Lemma 3.2. *Let χ be a quasi-character. We claim that*

$$\dim_{\mathbb{C}} \mathrm{Hom}_{F^\times}\left(\mathcal{S}(F) \otimes \chi, \mathbb{C}\right) \le 1.$$

(we'll later see it is 1-dimensional, but ≤ 1 suffices for uniqueness). One we have this, **(3)** of Theorem 3.1 is easy: both $\zeta^\natural(-, \chi, s)$ and $\zeta^\natural(\widehat{-}, \chi^{-1}, 1-s)$ live in a(n at most) 1-dimensional vector space.

Remark 3.3. There are three actions on $\mathcal{S}(F)$. There's the action of $F^\times$, there's the translation action of F, and there's the action fo the Fourier transform $\widehat{-}$. All three of these interact with each.

Proof of Lemma 3.2.

- The orbit decomposition $F^\times \overset{\text{open}}{\subset} F$ and $\{0\} \overset{\text{closed}}{\subset} F$ of $F^\times \curvearrowright F$ induces a short exact sequence (of $F^\times$-representations)

  $$0 \longrightarrow \mathcal{S}(F^\times) \longrightarrow \mathcal{S}(F) \longrightarrow \mathcal{S}(\{0\}) \longrightarrow 0.$$

  Only non-obvious thing is surjectivity of last map. Note that $\mathcal{S}(\{0\}) = \mathbb{C}$ and any $a \in \mathcal{S}(\{0\})$ can be lifted to $\varphi = a\mathbf{1}_{\mathscr{O}_F} \in \mathcal{S}(F)$.

- Now twist, dualize and take invariants, to get an exact sequence

  $$0 \to \mathrm{Hom}_{F^\times}(\mathcal{S}(\{0\}) \otimes \chi, \mathbb{C}) \to \mathrm{Hom}_{F^\times}(\mathcal{S}(F) \otimes \chi, \mathbb{C}) \to \mathrm{Hom}_{F^\times}(\mathcal{S}(F^\times) \otimes \chi, \mathbb{C})$$

  To simplify the argument, let's say we're in the case where $\chi \neq 1$. Then, $\mathrm{Hom}_{F^\times}(\mathcal{S}(\{0\}) \otimes \chi, \mathbb{C}) = 0$ by Schur's lemma ($\mathcal{S}(\{0\}) \simeq \mathbb{C}$ as $F^\times$-reps).

- Thus, it suffices[8] to prove $\dim_{\mathbb{C}} \mathrm{Hom}_{F^\times}(\mathcal{S}(F^\times) \otimes \chi, \mathbb{C}) = 1$. There is an $F^\times$-equivariant isomorphism $\mathcal{S}(F^\times) \otimes \chi \xrightarrow{\sim} \mathcal{S}(F^\times)$ e.g. by taking $\varphi \cdot \chi \hookleftarrow \varphi$. Hence, we only need show that $\dim_{\mathbb{C}} \mathrm{Hom}_{F^\times}(\mathcal{S}(F^\times), \mathbb{C}) = 1$. The point is that $\mathbf{1}_{\mathscr{O}_F^\times}$ generates $\mathcal{S}(F^\times)$ as a $F^\times$-representation. Something like if $\ell \in \mathrm{Hom}_{F^\times}(\mathcal{S}(F^\times), \mathbb{C})$, then $\mathscr{O}_F^\times = \bigsqcup_{a \in \mathscr{O}_F^\times/(1+\varpi^n} a \cdot (1 + (\varpi^n))$, so $\ell(\mathbf{1}_{a\cdot(1+\varpi^n)}) = \ell(\mathbf{1}_{\mathscr{O}_F^\times})/\#(*)$, where $\#(*)$ is the number of terms int he disjoint union from before.

$\blacksquare$

What goes into the proof of **(2)** of Theorem 3.1? Say $\varphi = \varphi_1 + c\mathbf{1}_{\mathscr{O}_F}$ with $\varphi_1(0) = 0$. Then,

$$\zeta(\varphi, \chi, s) = \zeta(\varphi_1, \chi, s) + c\zeta(\mathbf{1}_{\mathscr{O}_F}, \chi, s) = \zeta(\varphi_1, \chi, s) + cL(\chi, s).$$

Since φ_1 has support away from zero, we have $\varphi_1 \in \mathcal{S}(F^\times)$, so $\zeta(\varphi_1, \chi, s) \in \mathbb{C}[q^s, q^{-s}]$ using $F^\times = \bigsqcup_{n\in\mathbb{Z}} \varpi^n \mathscr{O}_F^\times$ (+ φ_1 compactly supported). Thus,

$$\frac{\zeta(\varphi, \chi, s)}{L(\chi, s)} \in \mathbb{C}[q^s, q^{-s}]$$

For surjectivity, use group action. This also shows the meromorphic continuation.

Corollary 3.4. *$\varepsilon(\chi, s) = \varepsilon(\chi, \frac{1}{2}) \cdot q^{\pm N(s-1/2)}$ with $\varepsilon(\chi, 1/2) \neq 0$.*

The point is that the functional equation tells you that the multiplication map

$$\varepsilon(\chi, s) : \mathbb{C}[q^{-s}, q^s] \xrightarrow{\sim} \mathbb{C}[q^{-s}, q^s]$$

is an isomorphism.

Remark 3.5. Tate gives a different proof in his Thesis. He shows it suffices to prove the functional equation for a single choice of φ,[9] and then does it directly for well-chosen φ.

The argument we sketched can be applied in more general settings. The key is the group action $F^\times \curvearrowright F$. This can be generalized e.g. to $\mathrm{GL}_n(F) \curvearrowright M_{n\times n}(F)$. This still has a unique open orbit, though there are more of them ($(n+1)$ of them correspond to rank $0, 1, \ldots, n$). The generalization is called Godement-Jacquet: given a (suitable) representation $\pi \in \mathrm{Rep}(\mathrm{GL}_n(R))$ and a schwartz function $\varphi \in \mathcal{S}(M_{n\times n}(F))$, they show[10]

$$\dim_{\mathbb{C}} \mathrm{Hom}_{\mathrm{GL}_n}(\mathcal{S}(M_{n\times n}) \otimes \pi \otimes \pi^\vee \otimes \chi, \mathbb{C}) \leq 1.$$

(something like this). This gives a similar result to Theorem 3.1.

3.1 Global Theory

Let F be a global field, and let $\chi : \mathbb{A}^\times/F^\times \to \mathbb{C}^\times$ be a character of the idèle class. We know $\chi = \prod_v \chi_v$ is a product of local quasi-characters.

[8] Always true, but only suffices in χ nontrivial case
[9] Using a Fubini trick
[10] $\mathrm{GL}_n \xrightarrow{\det} F^\times \xrightarrow{\chi} \mathbb{C}^\times$

Definition 3.6. We define the **Global L-function**

$$L(\chi, s) = \prod_v L(\chi_v, s),$$

convergent for $\mathrm{Re}(s) \gg 0$ ($\mathrm{Re}(s) > 1$ suffices when χ unitary).

Remark 3.7. There is a map

$$\begin{array}{cccc} |\cdot|: & \mathbb{A}^\times & \longrightarrow & \mathbb{R}_+^\times \\ & (\chi_v) & \longmapsto & \prod_v |\chi_v| \end{array}$$

where we use normalized absolute values at each place v. We let $\mathbb{A}^1 = \ker|\cdot| \subset \mathbb{A}^\times$. The **product formula** says $|F^\times| = 1$, so this descends to a map $|\cdot| : \mathbb{A}^\times/F^\times \to \mathbb{R}_+^\times$.

Fact. There is a splitting $\mathbb{A}^\times \xrightarrow{\sim} \mathbb{A}^1 \times \mathbb{R}_+^\times$. Given, $t \in \mathbb{R}_+^\times$, you can form $(\underbrace{t^{1/N}, t^{1/N}, \ldots, t^{1/N}}_{v|\infty}, \underbrace{1, 1, \ldots}_{v\nmid\infty}) \in \mathbb{A}^\times$. Basically, use appropriate powers of t in archimedean places to get an idele with norm t.

Fact. $\mathbb{A}^1/F^\times$ is compact (this is equivalent to Dirichlet's unit theorem + finiteness of class group).

Note that $\chi|_{\mathbb{A}^1/F^\times}$ is unitary since its source is compact. Thus, for any Hecke character χ, $|\chi| = |\cdot|^s$ for a unique $s \in \mathbb{R}$ denoted $s = \exp(\chi)$, the **exponent of** χ. Hence, $\chi|\cdot|^{-\exp(\chi)}$ is unitary, so assuming characters are unitary is not a big deal.

Theorem 3.8. *Assume χ is **unitary**, i.e. $|\chi| = 1$. Then,*

(1) *$L(\chi, s)$ has meromorphic continuation to all $s \in \mathbb{C}$*

(2) *There is a **functional equation***

$$L(\chi, s) = \varepsilon(\chi, s) L(\chi^{-1}, 1-s),$$

where $\varepsilon(\chi, s) = \prod_v \varepsilon(\chi_v, s)$ (independent of choice of nontrivial global additive character $\psi : \mathbb{A}/F \to \mathbb{C}^\times$).

Remark 3.9. Taking a product of local functional equations over all v gives

$$\varepsilon(\chi, s) \cdot \frac{\prod_v \zeta(\chi_v, \psi_v, s)}{L(\chi, s)} = \frac{\prod_v \zeta(\chi_v^{-1}, \widehat{\varphi}_v, 1-s)}{L(\chi^{-1}, 1-s)}.$$

Hence, to prove the global functional equation, it is enough to prove

$$\prod_v \zeta(\varphi_v, \chi_v, s) = \prod_v \zeta(\widehat{\varphi}_v, \chi_v^{-1}, 1-s).$$

We define the **global zeta integral** to be

$$\zeta(\varphi, \chi, s) = \int_{\mathbb{A}^\times} \varphi(x)\chi(x)\,|x|_{\mathbb{A}^\times}^s \,\mathrm{d}^\times x,$$

where we use Haar measure $\mathrm{d}^\times x = \prod_{v\le\infty} \mathrm{d}^\times x_v$, and we choose

$$\varphi \in \mathcal{S}(\mathbb{A}) = \bigotimes_{v\le\infty}{}' \mathcal{S}(F_v),$$

i.e. $\varphi = \otimes\varphi_v$ with $\varphi_v \in \mathcal{S}(F_v)$ and $\varphi_v = \mathbf{1}_{\mathscr{O}_{F_v}}$ for almost all v. Above $F_\infty := \prod_{v|\infty} F_v \simeq \mathbb{R}^{r_1} \otimes \mathbb{C}^{r_2}$.

Lemma 3.10. *Say χ is unitary ($\exp(\chi) = 0$). Then, $\zeta(\varphi, \chi, s)$ is absolutely convergent for $\mathrm{Re}(s) > 1$.*

(In the local case, when the exponent is 0, get convergence for $\mathrm{Re}(s) > 0$, instead of 1)

Proof. Write $\varphi = \varphi_\infty \otimes \varphi^\infty$ with $\varphi^\infty \in \mathcal{S}(\mathbb{A}_f)$, a Schwarz function on the finite adeles. Note that it is enough to consider $\varphi^\infty = \bigotimes_{v<\infty} \mathbf{1}_{\mathscr{O}_{F_v}}$ (every Schwartz function on $\mathbb{A}_f$ is more-or-less a finite linear combination of these guys).

$$\begin{aligned}\int_{\mathbb{A}^\times} |\varphi(x)|\,|x|^s\,\mathrm{d}^\times x &= \prod_{v<\infty} \int \varphi_v(x)\,|x|^s\,\mathrm{d}^\times x \cdot \prod_{v|\infty} blah \\ &= \prod_{v<\infty} \frac{1}{1-q_v^{-s}} \cdot \prod_{v|\infty} blah\end{aligned}$$

which converges absolutely if $\mathrm{Re}(s) > 1$ (think, Riemann-zeta). ∎

3.2 Some Fourier Theory

We still want the meromorphic continuation and functional equation of the global zeta integrals. Inspired by the usual proof of these properties for Riemann-zeta, we do some Fourier analysis. In particular, we want an analogue of Poisson summation.

Recall that $\mathbb{F} \hookrightarrow \mathbb{A}$ discretely, with compact quotient $\mathbb{A}/\mathbb{F}$. Fix some nontrivial additive character $\psi : \mathbb{A}/F \to \mathbb{C}$.

Notation 3.11. Let G be a locally compact Hausdorff abelian group. Its dual is

$$\widehat{G} := \mathrm{Hom}_{cts}(G, S^1)$$

with the compact-open topology.

Pontryagin duality tells us that $\widehat{\widehat{G}} \cong G$, and compares some properties between G and $\widehat{G}$.

G	$\widehat{G}$
compact	discrete
$H \subset G$ closed	$H^\perp = \{\chi : \chi\vert_H = 1\} \cong \widehat{(G/H)}$

Table 1: Pontryagin duality

Lemma 3.12.

$$\widehat{(\mathbb{A}/F)} \simeq F$$

via $\psi_a \leftarrow\!\shortmid a$, where $\psi_a(x) = \psi(ax)$.

Fact.

(1) $F_v \xrightarrow{\sim} \widehat{F_v}$ via $a \mapsto \psi_{v,a}$ upon fixing some nontrivial additive character ψ_v.

(2) $\widehat{\prod F_v} \xleftarrow{\sim} \prod F_v$ in the way you expect. Furthermore,

$$\widehat{\mathbb{A}} \xleftarrow{\sim} \mathbb{A}.$$

The facts more-or-less imply the lemma. We have $F \hookrightarrow \widehat{(\mathbb{A}/F)}$, and we know $\widehat{(\mathbb{A}/F)}$ is discrete subgroup of $\widehat{\mathbb{A}} = \mathbb{A}$. Something something have

$$\begin{tikzcd} F \arrow[r] \arrow[rr, bend left] & \widehat{(\mathbb{A}/F)} \arrow[r] & \mathbb{A} \end{tikzcd}$$

with $F, \widehat{(\mathbb{A}/F)}$ both discrete F-vector spaces and $\mathbb{A}/F$ compact. Thus, $F = \widehat{(\mathbb{A}/F)}$ something something.

4 Lecture 4 (3/1)

4.1 Global Functional Equation

Let $\chi : \mathbb{A}^\times/F^\times \to \mathbb{C}^\times$ be a global unitary character. Given some $\varphi \in \mathcal{S}(\mathbb{A})$, recall we had the global zeta integral

$$\zeta(\chi, \varphi, s) = \int_{\mathbb{A}^\times} \varphi(x)\chi(x)\,|x|^s\,\mathrm{d}^\times x.$$

In the range of convergence ($\mathrm{Re}(s) > 1$), we have

$$\zeta(\chi, \varphi, s) = \prod_{v \le \infty} \zeta(\chi_v, \varphi_v, s)$$

(recall $\varphi_v = \mathbf{1}_{\mathscr{O}_v}$ for almost all places).

Recall $\mathbb{A}^\times \xrightarrow{\sim} \mathbb{A}^1 \times \mathbb{R}_+^\times$ and $\mathbb{A}^1/F^\times$ is compact.

Theorem 4.1. *Assume that χ is additionally trivial on $\mathbb{R}_+^\times$, so it's really a character on $\mathbb{A}^1/F^\times$. Then,*

(1) *$\zeta(\chi, f, s)$ has a meromorphic continuation to all $s \in \mathbb{C}$*

(2) *$\zeta(\chi, f, s) = \zeta(\chi^{-1}, \widehat{f}, 1-s)$*

(3) *$\zeta(\chi, f, s)$ is entire unless $\chi = 1$ is trivial. When, $\chi = 1$, there are simple poles at $s = 0, 1$ with residue*

$$\mathrm{Res}_{s=1}\,\zeta(1, f, s) = f(0)\,\mathrm{vol}(\mathbb{A}^1/F^\times).$$

Global theory tells us that

$$\prod_v \zeta(\chi_v, f_v, s) = \prod_v \zeta(\chi_v^{-1}, \widehat{f}_v, 1-s),$$

while the local theory tells us that

$$\varepsilon(\chi_v, s)\frac{\zeta(\chi_v, f_v, s)}{L(\chi_v, s)} = \frac{\zeta(\chi_v^{-1}, \widehat{f}_x, 1-s)}{L(\chi_v^{-1}, 1-s)}.$$

Take a product over all places above, and then apply to global theory, and one sees that

$$L(\chi, s) = \varepsilon(\chi, s)L(\chi^{-1}, 1-s).$$

4.2 Global Theory: Proof Sketches

How does one proof something like theorem 4.1? Inspired by Riemann's proof of functional equation for Riemann zeta, we make use of Poisson summation.

Recall 4.2 (**Poisson summation**)**.** For $\varphi \in \mathcal{S}(\mathbb{R})$, one has

$$\sum_{n\in\mathbb{Z}} \varphi(n) = \sum_{n\in\mathbb{Z}} \widehat{\varphi}(n).$$

Proof Sketch. Formally, consider $\Phi(x) = \sum_{n\in\mathbb{Z}} \varphi(x+n) \in C^\infty(\mathbb{R}/\mathbb{Z})$. It's Fourier series expansion tells us that

$$\begin{aligned}\Phi(x) &= \sum_{n\in\mathbb{Z}} \left(\int_0^1 \Phi(y)e^{-2\pi i n y}\mathrm{d}y\right) e^{2\pi i n x}\\ &= \sum_{n\in\mathbb{Z}} \left(\int_{-\infty}^\infty \varphi(y)e^{2\pi i n y}\mathrm{d}y\right) e^{2\pi i n x}\\ &= \sum_{n\in\mathbb{Z}} \widehat{\varphi}(n)e^{2\pi i n x}\end{aligned}$$

In particular,

$$\sum_{n\in\mathbb{Z}} \varphi(n) = \Phi(0) = \sum_{n\in\mathbb{Z}} \widehat{\varphi}(n).$$

■

The same thing happens in the adelic world. Note that $\mathbb{Z} \hookrightarrow \mathbb{R}$ is discrete and cocompact. Similarly, $F \hookrightarrow \mathbb{A}$ is discrete and cocompact.

Theorem 4.3 (**Poisson Summation for** $F \hookrightarrow \mathbb{A}$)**.** *For* $\varphi \in \mathcal{S}(\mathbb{A})$*, one has*

$$\sum_{\xi\in F} \varphi(\xi) = \mathrm{vol}(\mathbb{A}/F) \sum_{\xi\in F} \widehat{\varphi}(\xi).$$

Recall 4.4. The measure on $\mathbb{A} = \prod' F_v$ we chose was $\mathrm{d}x = \prod_{v\le\infty} \mathrm{d}x_v$. Here, we have some additive character $\psi = \prod_v \psi_v : F\backslash\mathbb{A} \to \mathbb{C}^\times$, and each $\mathrm{d}x_v$ is self-dual with respect to ψ_v. On the discrete group F, we take the counting measure. These two choices induce a unique measure on the quotient $\mathbb{A}/F$ such that, for example, Fubini holds.

Secretly, the product $\mathrm{d}x$ is *independent* of ψ. This is a consequence of the product formula. Changing $\psi \rightsquigarrow \psi_a$ changes $\mathrm{d}x \rightsquigarrow |a|^{\pm 1/2}\,\mathrm{d}x = \mathrm{d}x$

Corollary 4.5. $\mathrm{vol}(\mathbb{A}/F) = 1$.

It is a non-negative number, and applying Poisson summation formula (PSF) twice shows that $\mathrm{vol}(\mathbb{A}/F)^2 = 1$.

Remark 4.6. $\mathrm{vol}(\mathbb{A}/F) = 1$ says that the *Tamagawa number* for $\mathbb{G}_a$ is 1. For G/F an algebraic group over a number field, can define a canonical measure on $G(F)\backslash G(\mathbb{A})$, and its volume $\mathrm{vol}(G(F)\backslash G(\mathbb{A})) =: \tau_G$ is called the **Tamagawa number** of G.

Notation 4.7. Given $\varphi \in \mathcal{S}(\mathbb{A})$ and $\alpha \in \mathbb{A}^\times$, let $\varphi_\alpha \in \mathcal{S}(\mathbb{A})$ denote $\varphi_\alpha(x) := \varphi(\alpha x)$.

Corollary 4.8.

$$\sum_{\xi \in F} \varphi(a\xi) = |a|^{-1} \sum_{\xi \in F} \widehat{\varphi}\left(\frac{\xi}{\alpha}\right).$$

Let's apply this to Tate's global integral. Recall it is defined by

$$\zeta = \int_{\mathbb{A}^\times} \varphi(x)\chi(x) |x|^s \, \mathrm{d}^\times x.$$

Note that $\chi(x) |x|^s$ above is invariant under the action of $F^\times$. Hence, we can write

This is Fubini (probably)

$$\zeta = \int_{\mathbb{A}^\times / F^\times} \left[\sum_{\xi \in F^\times} \varphi(\xi x)\right] \chi(x) |x|^s \, \mathrm{d}^\times x.$$

We now want to apply PSF:

$$\sum_{\xi \in F^\times} \varphi(\xi x) + \varphi(0) = |x|^{-1} \sum_{\xi \in F^\times} \widehat{\varphi}(\xi / x) + |x|^{-1} \widehat{\varphi}(0).$$

Applying this directly would given an integral over $\mathbb{A}^\times / F^\times$ of a constant function, which is worrisome since $\mathbb{A}^\times / F^\times$ is not compact. Hence, we break into two pieces (abs. val ≤ 1 or > 1):

$$\zeta = \int_{F^\times \backslash \mathbb{A}^{\leq 1}} (blah) + \int_{\mathbb{F}^\times \backslash \mathbb{A}^{\geq 1}} (blah) = \int_0^1 \int_{\mathbb{A}^1 / F^\times} (blah) + \int_1^\infty \int_{\mathbb{A}^1 / F^\times} (blah)$$

(we've used the splitting $\mathbb{A}^\times \simeq \mathbb{A}^1 \times \mathbb{R}_+^\times$).

Lemma 4.9. *The map*

$$\mathbb{A}^\times \ni x \mapsto \sum_{\xi \in F^\times} \varphi(x\xi)$$

has rapid decay as $|x| \to \infty$, i.e. it is bounded by $|x|^{-N}$ for all $N > 0$.

Example (Toy case)**.** $\mathbb{R} \ni x \mapsto \sum_{n \in \mathbb{Z} \setminus 0} e^{-nx^2}$ has rapid decay as $|x| \to \infty$.

Lemma implies that

$$\int_1^\infty \int_{\mathbb{A}^1 / F^\times} \left[\sum_{\xi \in F^\times} \varphi(\xi x)\right] (blah)$$

is an entire function in s. Now, PSF let's us rewrite

$$\int_0^1 \int_{\mathbb{A}^1 / F^\times} \leftrightarrow \int_1^\infty \int_{\mathbb{A}^1 / F^\times}$$

where we have $\widehat{f}$ on the right instead of f, i.e. by exchanging f with its Fourier transform, we get an integral which we know converges. In more detail, you get

$$\int_0^1 \int_{\mathbb{A}^1} \left[\sum_{\xi \in F^\times} \widehat{\varphi}(\xi / x)\right] |x|^{-1} \chi(x) |x|^s \, \mathrm{d}^\times x,$$

(plus contributions from $\varphi(0)$ and $|x|^{-1}\,\widehat{\varphi}(0)$) and then substitute $x \mapsto x^{-1}$, ending up with

$$\int_1^\infty \int_{\mathbb{A}^1/F^\times} \left[\sum_{\xi \in F^\times} \widehat{\varphi}(\xi x)\right] \chi^{-1}(x)\,|x|^{1-s}\,\mathrm{d}^\times x$$

(plus two additional terms). Combining what were originally the $\int_1^\infty$ and $\int_0^1$ parts, one obtains something like this

$$\zeta(\chi, \varphi, s) = \int_{F^\times \backslash \mathbb{A}^{\geq 1}} \left[\varphi(x)\,|x|^s\,\chi(x) + \widehat{\varphi}(x)\,|x|^{1-s}\,\chi^{-1}(x)\right] \mathrm{d}^\times x + \dots.$$

Exercise. Actually bother carrying out the integral manipulations sketched here to prove Theorem 4.1.

This basically finishes the discussion of Tate's thesis.

4.3 One Application: Class Number Formula

Let $\psi_F = \bigotimes \psi_v : \mathbb{A}/\mathbb{F} \to \mathbb{C}^\times$ be $\psi_F = \psi_{\mathbb{Q}} \circ \mathrm{Tr}_{F/\mathbb{Q}}$ where $\psi_{\mathbb{Q}} = \bigotimes_p \psi_{\mathbb{Q}_p}$ is determined by

$$\psi_{\mathbb{R}}(x) = e^{2\pi i x} \text{ and } \psi_{\mathbb{Q}_p}(x) = e^{-2\pi i (x \mod \mathbb{Z}_p)}.$$

Above, $\psi_{\mathbb{Q}_p}$ can be thought of as the composition

$$\mathbb{Q}_p \twoheadrightarrow \mathbb{Q}_p/\mathbb{Z}_p = \mathbb{Z}[1/p]/\mathbb{Z} \hookrightarrow \mathbb{Q}/\mathbb{Z} \xrightarrow{e^{-2\pi i x}} \mathbb{C}^\times.$$

Thus, we can a canonical choice of ψ_F.

We can also choose some $\varphi \in \mathcal{S}(\mathbb{A})$. Let φ_v be a Gaussian if $v \mid \infty$ – i.e. $\varphi_v(x) = e^{-\pi x^2}$ when v real and $\varphi_v(x) = e^{-\pi x \overline{x}}$ for v complex – and let $\varphi_v = \mathbf{1}_{\mathscr{O}_{F_v}}$ if $v \nmid \infty$. Then,

$$\xi(1, \varphi, s) = (*)\zeta_F(s)$$

with some simple constant $(*)$ related to $\mathrm{disc}(F/\mathbb{Q})$ and ζ_F the complete zeta function. Applying theorem tells us that $\mathrm{Res}_{s=1} = \mathrm{vol}(\mathbb{A}^1/\mathbb{F}^\times) \cdot \widehat{\varphi}(0)$. Computing this gives the class number formula which says

$$\mathrm{Res}_{s=1}\,\zeta_F(s) = (*) \cdot \text{regularator} \cdot \text{class } \#.$$

4.4 Algebraic Hecke characters

Remark 4.10. For a CM elliptic curve $E/\mathbb{Q}$, one obtains a Hecke character χ over some quadratic imaginary F, i.e. $\chi : \mathbb{A}^\times/F^\times \to \mathbb{C}^\times$. This has the property that $\chi_\infty : \mathbb{C}^\times \to \mathbb{C}^\times$ is nontrivial. However, this character is still special. At infinite, it is $\chi_\infty(z) = z$ (or something like this), which is an algebraic map.

Fix some embedding $\overline{\mathbb{Q}} \hookrightarrow \mathbb{C}$.

Question 4.11. *When is $L(\chi, s)$ "**algebraic**," i.e. $\chi(\varpi_v) \in \overline{\mathbb{Q}}^\times$ for $v < \infty$.*

Recall 4.12. When χ_v is unramified, we have

$$L(\chi_v, s) = \left(1 - \chi_v(\varpi_v) q_v^{-s}\right)^{-1},$$

so algebraicity is about requiring the coefficient $\chi_v(\varpi_v)$ above to be algebraic.

Remark 4.13. Note that $\chi_v|_{\mathscr{O}_{F_v}^\times}$ lands in μ_∞, the roots of unity, so algebraicity for one choice of uniformizer gives it for any choice.

Example (Weil)**.** Let F be **CM**, so F is purely imaginary with a totally real subfield $F_0 \hookrightarrow F$ such that F/F_0 is quadratic. Write

$$\chi = \prod_{v \leq \infty} \chi_v : \mathbb{A}^\times/F^\times \longrightarrow \mathbb{C}^\times.$$

For $v \mid \infty$, we have $F_v^\times \simeq \mathbb{C}^\times$, so let

$$\chi_v : F_v^\times \simeq \mathbb{C}^\times \to \mathbb{C}^\times$$

be given by $z \mapsto z^{n_v}\overline{z}^{m_v}$ for $n_v, m_v \in \mathbb{Z}$ ('weights') satisfying

$$n_v + m_v = N \text{ is independent of } v \in \operatorname{Hom}_{\mathbb{Q}}(F, \mathbb{C}).$$

Any Hecke character satisfying this condition (at ∞) is algebraic.

Later proved that these exhaust (almost?) all algebraic Hecke characters?

When χ is algebraic, $L(\chi, s \in \mathbb{Z})$ has special properties. These have been studied e.g. by Katz. One likes to construct "p-adic L-functions."

We'll next start studying modular forms.

5 Lecture 5 (3/3)

5.1 What comes next?

We finished Tate's thesis, so let's shift to modular forms. Topics for next 2–3 weeks:

- (holomorphic) Modular forms (examples: Eisenstein series, Theta functions)
- Hecke operators
- Non-holomorphic module (Maass) forms, Eisenstein series
- L-functions, and Rankin-Selberg convolution

Our main references will be Serre's 'A course in arithmetic' (chapter VII) and Bump (sections 1.2–4, 1.6, 1.9).

Remark 5.1. After studying Hecke characters $\chi : \mathbb{A}^\times/F^\times \to \mathbb{C}^\times$, a natural thing to look at would be continuous homomorphisms

$$\varphi : \mathrm{GL}_2(F)\backslash \mathrm{GL}_2(\mathbb{A}) \longrightarrow \mathbb{C}.$$

Historically, modular forms (which will give examples of such φ) were studied first, so it's good to have them in mind before looking at more abstract/general automorphic forms.

5.2 Holomorphic modular forms

The **modular group** is $\Gamma = \mathrm{SL}_2(\mathbb{Z}) \subset \mathrm{SL}_2(\mathbb{R})$. We like this group because there's a natural and interesting action $\mathrm{SL}_2(\mathbb{R}) \curvearrowright \mathfrak{H} = \{z \in \mathbb{C} : \mathrm{Im}(z) > 0\}$ via fractional linear transformations

$$\underbrace{\begin{pmatrix} a & b \\ c & d \end{pmatrix}}_{\in \mathrm{SL}_2(\mathbb{R})} \cdot \underbrace{z}_{\in \mathfrak{H}} := \frac{az+b}{cz+d} \in \mathfrak{H}.$$

This action realizes the isomorphism $\mathrm{Aut}(\mathfrak{H}) \cong \mathrm{PSL}_2(\mathbb{R})$.

Notation 5.2. I'm personally undecided on the 'right' notation for the upper half plane, so I may switch between $\mathfrak{H}$, $\mathbb{H}$, and $\mathcal{H}$ as I feel like until I decide to settle on one.

Definition 5.3. A **congruence subgroup** $\Gamma \subset \mathrm{SL}_2(\mathbb{Z})$ is one containing

$$\Gamma(N) = \left\{ \gamma \in \mathrm{SL}_2(\mathbb{Z}) : \gamma \equiv \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} \pmod N \right\}$$

for some $N \geq 1$.

Example.

$$\Gamma_0(N) = \left\{ \gamma \in \mathrm{SL}_2(\mathbb{Z}) : \gamma \equiv \begin{pmatrix} * & * \\ 0 & * \end{pmatrix} \pmod N \right\}$$

is a congruence subgroup.

Fact. As an abstract group, $\mathrm{SL}_2(\mathbb{Z})$ is generated by the two matrices

$$S = \begin{pmatrix} & 1 \\ -1 & \end{pmatrix} \text{ and } T = \begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix}.$$

Note that $S \cdot z = -1/z$ and $T \cdot z = z + 1$ for $z \in \mathbb{H}$. Inside $\mathrm{PSL}_2(\mathbb{Z})$, one has $S^2 = I$ and $(ST)^3 = I$.

Question: Are these the defining relations for $\mathrm{PSL}_2(\mathbb{Z})$?

Fact. The action $\mathrm{SL}_2(\mathbb{Z}) \curvearrowright \mathbb{H}$ has fundamental domain consisting of $z \in \mathbb{H}$ s.t. $|z| > 1$ and $|\mathrm{Re}(z)| \leq \frac{1}{2}$, pictured in Figure 1. In Figure 1, $\rho = \exp(2\pi i/3)$.

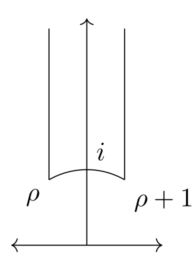

Figure 1: A fundamental domain for $\mathrm{SL}_2(\mathbb{Z}) \curvearrowright \mathfrak{H}$

Fact. For $N \geq 3$, $\Gamma(N)$ acts freely on $\mathbb{H}$, i.e. $\mathrm{Stab}_{\Gamma(N)}(z)$ is trivial (this means contained in ± 1, so $\Gamma(N)$'s image in $\mathrm{PSL}_2(\mathbb{Z})$ literally acts freely) for all $z \in \mathbb{H}$.

We call $z \in \mathbb{H}$ an **elliptic point** if $\operatorname{Stab} \mathrm{PSL}_2(\mathbb{Z})(z) \neq 1$. Up to $\mathrm{SL}_2(\mathbb{Z})$-translation, the only elliptic points are $i = \sqrt{-1}$ with stabilizer $\operatorname{Stab}_{\mathrm{PSL}_2(\mathbb{Z})}(i) = \langle S \rangle \simeq \mathbb{Z}/2\mathbb{Z}$, and $\rho = e^{2\pi i/3}$ with stabilizer $\operatorname{Stab}_{\mathrm{PSL}_2(\mathbb{Z})}(\rho) = \langle ST \rangle \simeq \mathbb{Z}/3\mathbb{Z}$.

Definition 5.4. A **modular function** $f : \mathbb{H} \to \mathbb{C}$ is a *meromorphic* function which is Γ-invariant.

Definition 5.5. A **holomorphic modular form of weight** $k \in \mathbb{Z}$ is a *holomorphic* map $f : \mathbb{H} \to \mathbb{C}$ satisfying

- $$f\left(\frac{az+b}{cz+d}\right) = (cz+d)^k f(z) \text{ for all } \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in \Gamma.$$

  Above, $(cz+d)^k$ is called a **weight k automorphy factor**. Note that, this is equivalent to requiring just
  $$f(z+1) = f(z) \text{ and } f\left(-\frac{1}{z}\right) = (-z)^k f(z).$$

  In particular, periodicity in x let's us write

  $$f(x+iy) = \sum_{n \in \mathbb{Z}} a_n(y) e^{2\pi i n x},$$

  but since f is holomorphic, one must actually have

  $$f(z) = \sum_{n \in \mathbb{Z}} a_n e^{2\pi i n z}$$

  (Cauchy-Riemann $\implies$ the terms $a_n(y)e^{2\pi i n x}$ must be holomorphic).

- (**holomorphic at** ∞) $a_n = 0$ if $n < 0$, i.e.

  $$f(z) = \sum_{n \geq 0} a_n q^n \text{ where } q = e^{2\pi i z}.$$

Remark 5.6. If $f : \mathbb{H} \to \mathbb{C}$ is holomorphic and satisfies the first bullet point above, then the second bullet point is equivalent to
$$|f(x+iy)| \leq C y^N \text{ for some } C, N > 0 \text{ as } y \to \infty$$
(i.e. **moderate growth**). Intuitively, terms like $e^{-2\pi i n z} = q^{-n}$ will grow exponentially as $\operatorname{Im} z \to \infty$.

Remark 5.7. If f is a weight k modular form, then taking $\begin{pmatrix} a & b \\ c & d \end{pmatrix} = \begin{pmatrix} -1 & 0 \\ 0 & -1 \end{pmatrix}$ shows $f(z) = (-1)^k f(z)$, so k better be even.

Notation 5.8. We let $M_k(\Gamma)$ denote the vector space of (holomorphic) modular forms of weight k for $\Gamma = \mathrm{SL}_2(\mathbb{Z})$.

Note that the sum
$$\bigoplus_{k \geq 0} M_k(\Gamma)$$

forms a graded ring in a natural way. Note that $M_0(\Gamma) = \mathbb{C}$. If f is holomorphic of weight 0, it is $\mathrm{SL}_2(\mathbb{Z})$-invariant and bounded/holomorphic at ∞, so it is constant by maximum modulus principle or whatever it's called.[11] Hence, $\bigoplus_{k\ge 0} M_k(\Gamma)$ is even a graded $\mathbb{C}$-algebra.

Example (**Eisenstein series**)**.** For any even integer $k \ge 4$, the function

$$G_k(z) = \sum_{m,n\in\mathbb{Z}}{}' \frac{1}{(mz+n)^k}$$

(the prime $'$ means sum doesn't include $(m,n) = (0,0)$) is holomorphic on $\mathbb{H}$ and Γ acts on it via weight k automorphy. To conclude it's a modular form, it suffices to show it is of moderate growth. For $z \in \mathcal{D}$, the fundamental domain, we use ($z = x + iy$ so $x^2 + y^2 \ge 1$ and $x \ge -1/2$)

$$|mz+n|^2 = (mx+n)^2 + m^2y^2 = m^2x^2 + 2mnx + n^2 + n^2y^2 = m^2(x^2+y^2) + 2mnx + n^2 \ge m^2 - mn + n^2$$

This is probably not the right phrasing, but whatever. It satisfies the functional equation you want it to satisfy

(with RHS a positive definite quadratic form) to see that

$$|G_k(z)| \le \sum_{m,n}{}' \frac{1}{|mz+n|^k} \le \sum_{m,n}{}' \frac{1}{m^2 - mn + n^2} < \infty.$$

Hence, G_k is indeed a modular form.

5.3 Structure of $\bigoplus M_k(\Gamma)$

Assumption. $\Gamma \simeq \mathrm{PSL}_2(\mathbb{Z})$ is the full modular group.

Notation 5.9. Let $M(\Gamma) := \bigoplus_{k\ge 0} M_k(\Gamma)$.

Eisenstein series look special, but they actually give all modular forms for $\mathrm{SL}_2(\mathbb{Z})$ in the following sense:

Theorem 5.10. $M_k(\Gamma) = \bigoplus_{4i+6j=k} \mathbb{C}G_4^iG_6^j$. *More concisely,*

$$M(\Gamma) = \mathbb{C}[G_4, G_6]$$

as graded $\mathbb{C}$-algebras.

Note that the above let's us determine $\dim_{\mathbb{C}} M_k(\mathrm{SL}_2(\mathbb{Z}))$. Secretly, the above is usually proved by first determining $\dim_{\mathbb{C}} M_k(\mathrm{SL}_2(\mathbb{Z}))$ in order to show that the natural map $\mathbb{C}[G_4, G_6] \hookrightarrow M(\mathrm{SL}_2(\mathbb{Z}))$ is an iso.

The actual tool for computing these dimensions is Riemann-Roch. The starting point is the isomorphism

$$\Gamma\backslash\mathcal{H} \cup \{\infty\} \xrightarrow{\sim} \mathbb{P}^1$$

as complex manifolds. To make this rigorous, one needs to carefully take care of the elliptic points i and $e^{2\pi i/3}$, but we won't sweat the details here. Let f be a modular function. It's divisor is $\operatorname{div} f = \sum_{[z]\in\Gamma\backslash\mathfrak{H}\cup\{\infty\}} \operatorname{ord}_z(f)[z]$. This will satisfy $\deg \operatorname{div}(f) = 0$ but again one needs to be careful about elliptic points. Think of this as motivation for the following.

[11] More geometrically, it extends to a holomorphic function of $\overline{\mathbb{H}/\mathrm{SL}_2(\mathbb{Z})} = \mathbb{P}^1$

Definition 5.11. The order of vanishing of $f(z) = \sum a_n q^n$ at ∞ is

$$v_\infty := \min \{n \in \mathbb{Z} : a_n \neq 0\}.$$

Note that $v_\infty(f) \geq 0 \iff f$ is holomorphic at ∞.

Theorem 5.12. *Let f be a nonzero modular function. Then,*

$$\sum_{[z] \in \Gamma \backslash \mathcal{H} \cup \{\infty\}} \frac{v_z(f)}{\#\Gamma_z} = 0,$$

where $\Gamma_z = \mathrm{Stab}_{\mathrm{PSL}_2(\mathbb{Z})}(z)$ is the stabilizer of z. Note that

$$\#\Gamma_z = \begin{cases} 2 & \text{if } [z] = [i] \\ 3 & \text{if } [z] = [e^{2\pi i/3}] \\ 1 & \text{otherwise.} \end{cases}.$$

More generally,

Theorem 5.13. *Let f be a nonzero weight k modular form. Then,*

$$\sum_{z \in \Gamma \backslash \mathcal{H} \cup \{\infty\}} \frac{v_z(f)}{\#\Gamma_z} = \frac{k}{12}.$$

Proof Idea. Let $\mathcal{D}$ be the usual fundamental domain. Apply the residue formula to the contour integral around the boundary of $\mathcal{D} \cap \{\mathrm{Im}\, z \leq T\}$, and take a limit as $T \to \infty$. ■

May need to do something about the cusps at $\rho, \rho + 1$. See Serre's book for details

Corollary 5.14.

- *When $k = 0$, one sees that $v_z(f) = 0$ for all z, so f is constant, i.e. $M_0(\mathrm{SL}_2(\mathbb{Z})) = \mathbb{C}$.*
- *When $k = 2$, $k/12 = 1/6$, so there's no possible solution, i.e. $M_2 = 0$.*
- *When $k = 4$, $k/12 = 1/3$ so $v_\rho(f) = 1$ (and $v_z(f) = 0$ for $[z] \neq [\rho]$), so $f/G_4 \in M_0 = \mathbb{C}$, i.e. $M_4 = \mathbb{C}G_4$.*
- *When $k = 6$, $k/12 = 1/2$ so $v_i(f) = 1$ (and $v_z(f) = 0$ otherwise), so $M_6 = \mathbb{C}G_6$.*
- *$G_8 = cG_4^2$ since $\dim M_8 = 1$.*
- *$G_{10} = cG_4G_6$ since $\dim M_{10} = 1$.*
- *$E_4^3 - E_6^2 = 1728\Delta$ is a weight 12 cusp form.*

Notation 5.15. One can compute that the constant term of $G_k(z)$ is $2\zeta(k)$, so we let

$$E_k(z) := \frac{1}{2\zeta(k)} G_k(z) = 1 + \sum_{n \geq 1} a_n q^n$$

be the **normalized Eisenstein series**.

Remark 5.16. $G_6(i) = 0$ coming from the functional equation applied to S.

Fact.

$$\Delta = \frac{E_4^3 - E_6^2}{1728} = q + \sum_{n\geq 2}^{\infty} a_n q^n = q \prod_{n\geq 1} (1-q^n)^{24}.$$

Definition 5.17. A modular form $f(z) = \sum_{n\geq 0} a_n q^n$ such that $a_0 = 0$ is called a **cuspidal modular form**. We let $S_k(\Gamma) \subset M_k(\Gamma)$ be the subspace of cusp forms.

Example. $\Delta \in S_{12}(\mathrm{SL}_2(\mathbb{Z}))$.

Lemma 5.18.

$$\dim M_k(\mathrm{SL}_2(\mathbb{Z})) = \begin{cases} \left\lfloor \frac{k}{12} \right\rfloor & \textit{if } k \equiv 2 \pmod{12} \\ \left\lfloor \frac{k}{12} \right\rfloor + 1 & \textit{otherwise.} \end{cases}$$

The main point is that multiplication by $\Delta \in S_{12}(\Gamma)$ will give an isomorphism $M_k(\Gamma) \xrightarrow{\sim} S_{k+12}(\Gamma)$ (and $M_k(\Gamma) = S_k(\Gamma) \oplus \mathbb{C}$ when k even).

Lemma 5.19.

(1)

$$v_z(\Delta) = \begin{cases} 1 & \textit{if } z = \infty \\ 0 & \textit{if } z \in \mathcal{H} \end{cases}$$

(2) $M_{k-12}(\Gamma) \xrightarrow{\sim} S_k(\Gamma)$ *via* $f \mapsto \Delta f$.

Corollary 5.20 (Theorem 5.10)**.** $M \simeq \mathbb{C}[E_4, E_6]$ *as graded* $\mathbb{C}$*-algebras.*

Corollary 5.21. *The field of modular functions is* $\mathbb{C}(j)$*, where*

$$j = E_4^3/\Delta = \frac{1}{q} + 744 + 196{,}884q + \dots$$

is holomorphic on $\mathcal{H}$ *with a simple pole at* ∞*. Furthermore,* j *induces an isomorphism* $j : \Gamma\backslash\mathcal{H}\cup\{\infty\} \xrightarrow{\sim} \mathbb{P}^1$ *sending* $z \mapsto j(z)$ *and* $\infty \mapsto \infty$*. In particular,*

$$j(e^{2\pi i/3}) = 0 \;\; \textit{and} \;\; j(i) = \frac{1728 E_4^3(i)}{E_4^3(i) - E_6^2(i)} = 1728.$$

6 Lecture 6 (3/9)

Started talking about modular forms last time. Let's keep doing that.

6.1 Arithmetic

Recall the Eisenstein series

$$G_k(\tau) = \sum_{(m,n)}{}' \frac{1}{(m\tau + n)^k}$$

for $k \geq 4$ (we now use $\tau \in \mathcal{H}$ as our upper half plane parameter). We can compute it's Fourier expansion. The constant term will be the limit as $q = e^{2\pi i t \tau} \to 0$ (i.e. $\tau \to i\infty$), so it's easy to determine. In the end one will get

Lemma 6.1.

$$G_k = 2\zeta(k) + (*)\pi^k \sum_{n \geq 1} \sigma_{k-1}(n) q^n,$$

where

$$\sigma_{k-1}(n) = \sum_{0 < d | n} d^{k-1}.$$

Recall we normalize by dividing to set the constant term to 1, i.e. we form

$$E_k = G_k/(2\zeta(k)) = 1 + c_k \sum_{n \geq 1} \sigma_{k-1}(n) q^n.$$

One can show that c_k above is a rational number.

Remark 6.2. We won't go over calculating this Fourier expansion. It is done e.g. in Serre's book (and also in the book by Diamond and Shurman). One makes use of some trigonmetric functions. Start with

$$\sin(\pi z) = \pi z \prod_{n \geq 1} \left(1 - \frac{z^2}{n^2}\right),$$

and take log-derivative to get something like

$$\frac{1}{\pi} \operatorname{cotan}(\pi z) = \frac{1}{z} + \sum_{n \geq 1} \left(\frac{1}{z-n} + \frac{1}{z+n}\right).$$

Taking the kth derivative will cause an Eisenstein series to show up.

We will see a 2nd proof later for non-holomorphic Eisenstein series.

Fact.

$$\zeta(k) \in \pi^k \cdot \mathbb{Q}^\times$$

for all $k \geq 2$.

Using this, one can show that $E_k \in \mathbb{Q}[\![q]\!]$ for all k, i.e. normalized Eisenstein series have rational coefficients. In fact, it is the case that $E_k \in \mathbb{Z}\left[\frac{1}{6}\right][\![q]\!]$. One can use the fact that each E_k is a polynomial in E_4, E_6 to reduce making showing this easier.

This let's you derive some arithmetic information from Eisenstein series. For example, dimension count + comparing constant terms shows that $E_8 = E_4^2$ and $E_{10} = E_4 E_6$. Expansing their power series then gives non-trivial identities involving divisor sums.

6.2 Connection to elliptic curves

Recall 6.3. A one-dimensional complex torus is always of the form $\mathbb{C}/\Lambda$ where $\Lambda \subset \mathbb{C}$ is a rank 2 lattice in $\mathbb{C}$, a discrete subgroup which is rank 2 (over $\mathbb{Z}$) and cocompact.

Definition 6.4 (**Homothety**)**.** We say $\Lambda \sim \Lambda'$ if $\exists \lambda \in \mathbb{C}^\times$ s.t. $\Lambda = \lambda\Lambda'$.

Exercise. Up to homothety,

$$\left\{ \begin{matrix} \text{rank 2 lattices} \\ \Lambda \subset \mathbb{C} \end{matrix} \right\}_{/\sim} \longleftrightarrow \mathrm{SL}_2(\mathbb{Z})\backslash\mathcal{H}$$

via $\mathbb{Z} + \mathbb{Z}\tau \leftarrow\!\shortmid \tau$.

Any one-dimensional complex torus has a canonical embedding $\mathbb{C}/\Lambda \hookrightarrow \mathbb{P}^2_{\mathbb{C}}$ via the theory theory of elliptic functions, given by $z \mapsto (\wp_\Lambda(z), \wp'_\Lambda(z), 1)$, with image the cubic curve

$$y^2 = 4x^3 - g_4(\Lambda)x - g_6(\Lambda),$$

where

$$g_k = c_k G_k = \sum_{\lambda\in\Lambda}{}' \frac{1}{\lambda^k}.$$

(we're ignoring the constants c_k).

Recall that j-invariant $j = E_4^3/\Delta$ where $\Delta = E_4^3 - E_6^2$ (up to constant; there's a 1728 somewhere). This gave an iso $j : \Gamma\backslash\mathbb{H} \cup \{\infty\} \xrightarrow{\sim} \mathbb{P}^1_{\mathbb{C}}$.

Theorem 6.5 ((A part of) Complex Multiplication)**.** *If* $[\mathbb{Q}(\tau) : \mathbb{Q}] = 2$*, then* $j(\tau) \in \overline{\mathbb{Q}}$ *is algebraic.*

(Apparently, this is corollary from stuff we've said so far).

Fact. If $\tau \in K$, some imaginary quadratic field (so $K = \mathbb{Q}(\tau)$), then $K(j(\tau))$ is an abelian extension of K.

Note that we have formed a family

$$E_\tau : y^2 - 4x^3 - g_4(\tau)x - g_6(\tau)$$

of elliptic curves (over $\mathbb{H}$) and the transformation $(g_4, g_6) \mapsto (\lambda^{-4}g_4, \lambda{-}6g_6)$ does not change the iso class of a memember of the family. Furthermore, this family contains (non-uniquely) all (complex) elliptic curves. Since $E_\tau \cong E_{\tau'}$ iff $(\mathbb{Z} \oplus \mathbb{Z}\tau) \sim (\mathbb{Z} \oplus \mathbb{Z}\tau')$, we see that $j = g_4^3/\Delta$ (up to a possibly missing 1728) is the unique invariant classifying complex elliptic curves, i.e. $j(E) = j(E') \iff E \simeq E'$ over $\mathbb{C}$.

Let's show that $j(\tau)$ is algebraic when τ lives in an imaginary quadratic. We have $\mathrm{Aut}(\mathbb{C}) \curvearrowright \mathbb{C}$, and we want to prove that the orbit $\mathrm{Aut}(\mathbb{C}) \cdot j(\tau)$ is finite (when $[\mathbb{Q}(\tau) : \mathbb{Q}] = 2$). To prove this is finite, it suffices to prove that $\mathrm{Aut}(\mathbb{C}) \cdot E_\tau$ is a finite set (of iso classes of elliptic curves).

Lemma 6.6. *Assume* τ *quadratic and* $\mathbb{Z}[\tau] = \mathscr{O}_K$ *(i.e.* $\mathbb{Z}[\tau]$ *is integrally closed/a maximal order). Then,* $\mathrm{End}(E_\tau) = \mathscr{O}_K$*. Furthermore, for any quadratic imaginary* $K \subset \mathbb{C}$*, there is a bijection*

By 'endomorphism' we mean 'self-isogeny'

$$\mathrm{Cl}_K \xrightarrow{\sim} \{E/\mathbb{C} : \mathrm{End}(E) = \mathscr{O}_K\}_{/\simeq}$$

via $\mathfrak{a} \mapsto \mathbb{C}/\mathfrak{a}$*(*$= (K \otimes \mathbb{R})/\mathfrak{a}$*).*

Remark 6.7. $\mathrm{Hom}(\mathbb{C}/\Lambda, \mathbb{C}/\Lambda') \xleftarrow{\sim} \{\lambda \in \mathbb{C} : \lambda\Lambda \subset \Lambda'\}$

As a consequence,

$$\mathrm{End}(\mathbb{C}/\mathfrak{a}) \simeq \{\lambda \in \mathbb{C} : \lambda\mathfrak{a} \subset \mathfrak{a}\} = \{\lambda \in K : \lambda\mathfrak{a} \subset \mathfrak{a}\} = \mathscr{O}_K.$$

In general, for $\Lambda \subset K$, the set $\{\lambda \in K : \lambda\mathfrak{a} \subset \mathfrak{a}\}$ is an order in $\mathscr{O}_K$.

For $\Lambda \subset \mathbb{C}$ not contained in an imaginary quadratic, one has $\operatorname{End}(\mathbb{C}/\Lambda) = \mathbb{Z}$. So $\operatorname{End}(\mathbb{C}/\Lambda_\tau)$ is $\mathbb{Z}$ when $[\mathbb{Q}(\tau) : \mathbb{Q}] \neq 2$, and is an order in an imaginary quadratic field when $[\mathbb{Q}(\tau) : \mathbb{Q}] = 2$.

Lemma 6.8. $[\mathbb{Q}(\tau) : \mathbb{Q}] = 2 \implies \operatorname{Aut}(\mathbb{C}) \cdot E_\tau$ *is finite.*

Proof. Imagine applying σ to coefficients of $y^2 = 4x^3 - g_4(\tau)x - g_6(\tau)$. Doing so does not change the endomorphism algebra, so the orbit $\operatorname{Aut}\mathbb{C} \cdot E_\tau$ is contained in the finite set of elliptic curves E with $\operatorname{End}(E) = \operatorname{End}(E_\tau)$. ∎

Remark 6.9. Above shows $j(\tau)$ is an algebraic number. In fact, $j(\tau) \in \overline{\mathbb{Z}}$ is an algebraic integer. Furthermore, if $j(\tau)$ is algebraic iff τ is quadratic.

Remark 6.10. We mentioned earlier that $K(j(\tau))/K$ is abelian. In fact, $\operatorname{Gal}(K(j(\tau))/K) \xrightarrow{\sim} \operatorname{Cl}(\mathscr{O}_\tau)$ (where $\mathscr{O}_\tau = \operatorname{End}(\mathbb{C}/\Lambda_\tau) \subset \mathscr{O}_K$). Furthermore,

$$K(j(\tau))\left((E_\tau)_{\text{tors}}\right) = K^{\text{ab}}$$

gives the maximal abelian extension of K.

(Reference: Serre in Cassels-Frohlich)

6.3 Analytic

Say $f(\tau) = \sum_{n\geq 0} a_n q^n \in M_k(\Gamma)$ is a holomorphic modular form of weight k. As a number theorists, when you see a sequence of numbers, you might think to turn it into a Dirichlet series $\sum_{n\geq 1} a_n n^{-s}$. For this to be useful, at minimum it needs to converge in a right half-plane, so we need some polynomial bound on the growth of the a_n.

Example. For Eisensten series we have $a_n \sim \sigma_{k-1}(n) \leq n^{k-1}\sigma_0(n) \leq O(n^k)$, so we're good there.

What about cuspidal modular forms?

Lemma 6.11 (trivial bound). *Say $f \in S_k(\Gamma)$ is a weight k cusp form. Then,*

$$a_n = O\left(n^{\frac{k}{2}}\right),$$

i.e.

$$\left\{\left|\frac{a_n}{n^{k/2}}\right| : n \geq 1\right\} \text{ is bounded.}$$

Proof. First note that $\tau \mapsto |f(\tau)|\, y^{k/2}$ is Γ-invariant on the nose (this only requires f modular of weight k). Since f is cuspidal, $|f(\tau)|\, y^{k/2}$ is in fact continuous (bounded?) near ∞. Hence, it gives a continuous function on the compact domain $\Gamma\backslash\mathbb{H}\cup\{\infty\}$, so $\left|f(\tau)y^{k/2}\right|$ is bounded, say by C. Thus, $|f(\tau)| \leq Cy^{-k/2}$. Cauchy[12] tells us that

$$a_n = \frac{1}{2\pi i}\int_{|q|=\delta>0} \frac{f(\tau)}{q^{n+1}}\mathrm{d}q$$

so $|a_n| \leq Cy^{-k/2}e^{2\pi ny}$ for any y. Take $y = 1/n$ to get

$$|a_n| \leq C'n^{k/2}$$

[12] Cauchy/Fourier?

as desired. ■

Remark 6.12. Above bound is not optimal. Deligne (via Weil conjectures) proved that

$$|a_n| = O\left(n^{\frac{k-1}{2}+\varepsilon}\right)$$

for any $\varepsilon > 0$. This was earlier conjectured by Ramanujan (in the case of the Δ function). Writing $\Delta = \sum_{n\geq 1} \tau(n)q^n$ (the unique, up to scaling, weight 12 cusp form for $\mathrm{SL}_2(\mathbb{Z})$), he conjectured

(1) $\tau(mn) = \tau(n)\tau(m)$ when $(n, m) = 1$

(2) $\tau(p^i)$ is given by some (precise) formula of $\tau(p^i), \tau(p^{i-1})$

(3) $|\tau(p)| \leq 2p^{11/2}$

Deligne's proof of the improved bound is the only one known. Note that Ramanujan's conjecture above is purely analytic in its statement. There's an analogue of it for non-holomorphic modular forms which is still open. One of Langland's motivation for formulating his conjecture(s) was to allow for a purely analytic proof of Ramanujan's conjecture.

We will later see the Rankin-Selberg method which will give an analytic improvement to the trivial estimate (though still not optimal). Langland's functoriality conjecture would allow one to achieve the optimal bound via analytic methods (for both holomorphic and non-holomorphic forms?).

The trivial bound shows that $L(s, f) = \sum_{n\geq 1} a_n n^{-s}$ converges when $\mathrm{Re}(s) > k/2 + 1$ (by Deligne, it even converges for $\mathrm{Re}(s) > (k-1)/2 + 1$). We'll talk more about this L-function next time, and in particular about when it has an Euler product.

7 Lecture 7 (3/10)

Recall the weight 12 cusp form

$$\Delta = q\prod_{n\geq 1}(1-q^n)^{24} = \sum_{n\geq 1}\tau(n)q^n \in S_{12}(\Gamma)$$

($\Gamma = \mathrm{SL}_2(\mathbb{Z})$).

Remark 7.1. If you ever want examples, check out the LMFDB. Currently, the Δ function is second in its 'Hall of Fame' (behind Riemann zeta).

Conjecture 7.2 (**Ramanujan's Conjecture**)**.**

(1) $\tau(nm) = \tau(n)\tau(m)$ *when* $(n, m) = 1$.

(2) $\tau(p)\tau(p^i) = \tau(p^{i+1}) + p^{11}\tau(p^{i-1})$ *for* $i \geq 1$.

Hence, the collection $\{\tau(n)\}_{n\in\mathbb{Z}}$ *is really no more data than* $\{\tau(p)\}_{p\ prime}$.

This was first proven by Mordell ca. 1920s.

The modern understanding of these facts comes from the theory of Hecke operators.

7.1 Hecke Operators

There are multiple ways to describe these. Let's start with one of them. Set

$$\mathcal{R} = \left\{ \begin{matrix} \text{lattices} \\ \Lambda \subset \mathbb{C} \end{matrix} \right\},$$

and note that $\mathbb{C}^\times \curvearrowright \mathcal{R}$ via scaling (the orbits of this action are the homothety classes from last time). Say a function $f : \mathcal{R} \to \mathbb{C}$ **has weight** k if

$$f(\lambda\Lambda) = \lambda^{-k} f(\Lambda) \text{ for } \lambda \in \mathbb{C}^\times \text{ and } \Lambda \in \mathcal{R}.$$

If $k = 0$, this says that f is invariant under homothety, so it descends to a map on $\mathcal{R}/\mathbb{C}^\times \xleftarrow{\sim} \Gamma\backslash\mathbb{H}$.

Remark 7.3. The usual map

$$\begin{array}{ccc} \mathbb{H} & \longrightarrow & \mathcal{R} \\ \tau & \longmapsto & \mathbb{Z} + \mathbb{Z}\tau \end{array}$$

(the one inducing the iso $\Gamma\backslash\mathbb{H} \xrightarrow{\sim} \mathcal{R}/\mathbb{C}^\times$) allows you to pullback the definition of 'weight k' from functions on $\mathcal{R}$ to functions on $\mathbb{H}$, and this agrees with our earlier definition. That is $f : \mathcal{R} \to \mathbb{C}$ if of weight k iff $\mathbb{H} \hookrightarrow \mathcal{R} \xrightarrow{f} \mathbb{C}$ is.

The point of the remark is that we can freely move between making definitions in terms of lattices and in terms of the upper half-plane. We'll use this to define Hecke operators using lattices.

Definition 7.4. Fix a positive integer $n \geq 1$. We define a correspondence

$$\begin{tikzcd} & \mathcal{T}_n \arrow[dl] \arrow[dr] & \\ \mathcal{R} & & \mathcal{R} \end{tikzcd}$$

where

$$\mathcal{T}_n = \{(\Lambda, \Lambda') : \Lambda \supset \Lambda' \text{ of index } n\}.$$

Note that the fibers of the two maps $\mathcal{T}_n \rightrightarrows \mathcal{R}$ are finite, so given $f : \mathcal{R} \to \mathbb{C}$ we can define the **Hecke operator** $T(n) = T_n$ by sending f to the function $T_n f : \mathcal{R} \to \mathbb{C}$ given by

$$(T_n f)(\Lambda) = \sum_{(\Lambda, \Lambda') \in \mathcal{T}_n} f(\Lambda') = \sum_{\Lambda' \underset{n}{\subseteq} \Lambda} f(\Lambda').$$

Note that defining this operator does not require f being holomorphic.

Given $\lambda \in \mathbb{C}^\times$, we also define the operator

$$(R_\lambda f)(\Lambda) = f(\lambda\Lambda).$$

Proposition 7.5.

- $R_\lambda R_\mu = R_\mu R_\lambda$ *for* $\lambda, \mu \in \mathbb{C}^\times$
- $R_\lambda T_n = T_n R_\lambda$ *for all* $n \in \mathbb{Z}_{\geq 1}$ *and* $\lambda \in \mathbb{R}^\times$

- *$T_mT_n = T_{mn}$ when $(m,n)=1$.*[13]
- *$T_pT_{p^i} = T_{p^{i+1}} + pT_{p^{i-1}}R_p$ for prime p and $i \geq 1$.*

Only the last property requires some work.

Corollary 7.6. *T_{p^i} is a polynomial of T_p and R_p.*

We can repackage the previous proposition using a generating function. Consider

$$g(x) := \sum_{i\geq 0} T_{p^i}x^i$$

For convenience, set $T_{p^{-1}} = 0$. Then,

$$T_p \cdot \sum_{i\geq 0} T_{p^i}X^i = \sum_{i\geq 0} T_{p^{i+1}}X^i + pR_p \sum_{i\geq 0} T_{p^{i-1}}X^i \implies T_pg(x) = x^{-1}\left(g(x)-1\right) + pR_pxg(x).$$

Thus, we can solve

Corollary 7.7.

$$\sum_{i\geq 0} T_{p^i}x^i = g(x) = \frac{1}{1 - T_px + pR_px^2}.$$

At some point, we need to translate everything back to the language of modular forms and Fourier coefficients and all that mess...

Definition 7.8. For any $\gamma = \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in \mathrm{GL}_2^+(\mathbb{Q})$ (+ denotes positive determinant), we can define a **weight k action of γ**

$$\left(f|_{\gamma,k}\right)(\tau) := (\det\gamma)^{k-1}(c\tau+d)^{-k}f(\gamma\tau).$$

Then, f is **of weight** k iff $f|_{\gamma,k} = f$ for all $\gamma \in \Gamma = \mathrm{SL}_2(\mathbb{Z})$.

The above gives a group actions in the sense that

$$f|_{\gamma\gamma',k} = \left(f|_{\gamma,k}\right)|_{\gamma',k}.$$

This requires checking the "automorphy factor" $(c\tau+d)$ satisfies some cocycle condition. This leads into the second definition of the Hecke operators T_n.

Consider $\mathcal{T}_n := M_{2\times 2}(\mathbb{Z})_{\det=n}$, the set of 2×2 integral matrices of determinant n. Note that $\mathcal{T}_n$ is bi-Γ-invariant (as suual, $\Gamma = \mathrm{SL}_2(\mathbb{Z})$), i.e. $\Gamma\mathcal{T}_n\Gamma = \mathcal{T}_n$. Furthermore, the corresponding double coset space is finite, i.e.

$$\#\Gamma\backslash\mathcal{T}_n/\Gamma < \infty.$$

We can (re)define the **Hecke operator** as the (finite!) sum

$$T_nf := \sum_{\gamma\in\Gamma\backslash\mathcal{T}_n} f|_{\gamma,k}.$$

[13] $\Lambda \simeq \mathbb{Z}^2$, so sublattices of index n are in bijection with subgroups $G \subset (\mathbb{Z}/n\mathbb{Z})$ of order n. With this in mind, this part of the Proposition basically follows from Chinese Remainder Theorem

We want to show this operator preserves the space of modular forms (a priori it's an operator on all functions $\mathbb{H} \to \mathbb{C}$)

Recall 7.9. In the lattice formalism, the T_n came from the correspondence $\mathcal{R} \leftarrow \mathcal{T}_n \to \mathcal{R}$. In the current context, the relevant correspondences are coming from (finite index) subgroups of $\mathrm{SL}_2(\mathbb{Z})$. For example, we have the correspondence

$$\begin{tikzcd} & \Gamma(N)\backslash\mathbb{H} \arrow[dl] \arrow[dr] & \\ \Gamma\backslash\mathbb{H} & & \Gamma\backslash\mathbb{H} \end{tikzcd}$$

(I'm not sure if this is the correspondence giving T_n. See chapter 5 of Diamond and Shurman. I think they talk about the correspondence connection).

We want to understand how T_n acts on a modular form f by seeing what it does to its Fourier expansion.

Lemma 7.10. *Let $\Gamma = \mathrm{SL}_2(\mathbb{Z})$ as usual. Then,*

$$\mathcal{T}_n = \bigsqcup_{\substack{a>0 \\ ad=n \\ 0\le b<d}} \Gamma \begin{pmatrix} a & b \\ 0 & d \end{pmatrix}$$

Example. If $n = p$ is prime, then

$$\mathcal{T}_p = \Gamma \begin{pmatrix} p & 0 \\ 0 & 1 \end{pmatrix} \sqcup \bigsqcup_{0\le b<p} \Gamma \begin{pmatrix} 1 & b \\ & p \end{pmatrix}.$$

Hence, $\#\Gamma\backslash\mathcal{T}_p = (p+1)$.

Using this description, one can simply work out by hand that

Proposition 7.11. *For $f = \sum_{n\in\mathbb{Z}} a_n(f)q^n$ which is holomorphic on $\mathbb{H}$ (but not nec. at ∞) of weight k, one has*

$$a_m(T_nf) = \sum_{0<d|(m,n)} d^{k-1}a_{mn/d^2}(f)$$

for all $m \in \mathbb{Z}$ (above $n \ge 1$).

Corollary 7.12.

(1) $a_0(T_nf) = \sigma_{k-1}(n)a_0(f)$

(2) $a_1(T_nf) = a_n(f)$

(3) $a_m(T_nf) = a_{mn}(f)$ *when* $(m,n) = 1$

(4) When p prime, $a_m(T_pf) = a_{mp}(f) + p^{k-1}a_{m/p}(f)$ where $a_{m/p} = 0$ if $p \nmid m$.

Corollary 7.13. *If $f = \sum_{n\in\mathbb{Z}} a_n(f)q^n$ is holomorphic at ∞ (i.e. $a_n(f) = 0$ for $n < 0$), then T_nf is also holomorphic at ∞.*

So we have an action $T_n \curvearrowright M_k(\Gamma)$ preserving the subspace of cuspidal modular forms. Furthermore, we have a Hermitian structure, an inner product on the cuspidal subspace.

Definition 7.14. The **Petersson inner product** on $S_k(\Gamma)$ is given by

$$\langle f, g\rangle_{\mathrm{Pet}} = \int_{\Gamma\backslash\mathbb{H}} f(\tau)\overline{g(\tau)}y^k \frac{\mathrm{d}x\mathrm{d}y}{y^2}.$$

It is positive and Hermitian.

Remark 7.15. $f(\tau)\overline{g(\tau)}y^k$ is Γ-invariant, and $\mathrm{d}x\mathrm{d}y/y^2$ is a Γ-invariant measure.

Lemma 7.16. *The Hecke operators T_n are all self-adjoint w.r.t $\langle -, -\rangle_{Pet}$, i.e.*

$$\langle T_n f, g\rangle_{Pet} = \langle f, T_n g\rangle_{Pet}$$

Corollary 7.17. *Since $\{T_n\}_{n\geq 1}$ are self-adjoint operators on the finite-dimensional vector space $S_k(\Gamma)$, they are each diagonalizable. In fact, since the T_n are pairwise commute, they can actually be simultaneously diagonalized!*

Definition 7.18. Let

$$\mathbb{T} := \mathbb{C}[T_n : n \geq 1] \subset \mathrm{End}_{\mathbb{C}}(S_k(\Gamma))$$

be the **Hecke algebra**. Note that it is a commutative $\mathbb{C}$-algebra.

The previous corollary tells us that we have a decomposition

$$S_k(\Gamma) \simeq \bigoplus_{\lambda:\mathbb{T}\to\mathbb{C}} S_k(\Gamma)[\lambda] \;\text{ where }\; S_k(\Gamma)[\lambda] := \{f \in S_k(\Gamma) : T_n f = \lambda(T_n) f\}$$

of cusp forms into literal eigenspaces.

Recall 7.19. $a_1(T_m f) = a_m(f)$.

If $f \in S_k(\Gamma)[\lambda]$ is an **eigenform** with eigenvalue λ, then $a_n(f) = \lambda_n a_1(f)$ (where we've set $\lambda_n := \lambda(T_n)$), so the eigenvalues determine the Fourier coefficients $a_n(f)$, up to constant. Specifically,

$$f = a_1 \sum_{n\geq 1} \lambda_n q^n \;\text{ and }\; \lambda_1 = 1.$$

We call $f \in S_k(\Gamma)[\lambda]$ a **normalized eigenform** if $a_1(f) = 1$. In this case

$$f = \sum_{n\geq 1} \lambda_n q^n = \sum_{n\geq 1} \lambda(T_n) q^n.$$

Remark 7.20. Recall that $S_{12}(\Gamma) = \mathbb{C}\Delta$ is a 1-dimensional space. Hence, Δ is automatically an eigenform. In fact, it is a normalized eigenform since the linear term of $\Delta = q\prod_{n\geq 1}(1-q^n)^{24}$ is $1\cdot q$.

Recall 7.21. $T_m T_n = T_{mn}$ when $(m, n) = 1$ and

$$\sum_{i\geq 0} T_{p^i} x^i = g(x) = \frac{1}{1 - T_p x + pR_p x^2}.$$

Corollary 7.22. *For f a normalized eigenform, one has*

$$\sum_{i\geq 0} a_{p^i} x^i = \frac{1}{1 - a_p x + pp^{-k} x^2}$$

*(implicitly above, $R_p f = p^{-k} f$). Recalling the **L-function** $L(f,s) = \sum_{n\geq 1} a_n n^{-s}$. It has an Euler product*

$$L(f,s) = \prod_{p \text{ prime}} \frac{1}{1 - a_p p^{-s} + p^{-2s+1-k}}.$$

Seems there was a mistake at some point. The $1-k$ in the exponent should really be a $k-1$

Corollary 7.23. *Ramanujan's conjecture holds.*

We see that the 'Hecke theory' is in some sense equivalent to the existence of the Euler product for the L-function.

Recall Ramanujan conjectured $|a_p| \leq 2p^{(k-1)/2}$. The Hecke polynomial at p is $1 - a_p x + p^{k-1} x^2$. This has two complex roots both of absolute value $p^{\frac{k-1}{2}}$. This is sounding more like the Weil conjectures, and more like the Hecke polynomial at p behaving like a characteristic polynomial of Frobenius.

8 Lecture 8 (3/15)

8.1 The story of the incorrect factor from last time

Recall at the end of last lecture, we ended up with a $p^{-2s+1-k}$ that was supposed to be a $p^{-2s+k-1}$.

It turns out our two descriptions of Hecke operators were not quite one in the same. In the second description, there was the additional factor $(\det\gamma)^{k-1}$ in

This factor eliminates annoying denominators in describing coefficients of Hecke operators applied to modular forms

$$(f|_{\gamma,k})(\tau) := (\det\gamma)^{k-1}(c\tau + d)^{-k} f(\gamma\tau).$$

Recall T_n was attached to 2×2 matrices with determinant n, so this factor adds an additional factor of n^{k-1}. That is $T_n^{\text{2nd descrip}} = n^{k-1} T_n^{\text{1st descript}}$. The series

$$\sum_{i\geq 0} T_{p^i} x^i = g(x) = \frac{1}{1 - T_p x + pR_p x^2}$$

for the 1st (lattice) description of Hecke operators becomes

$$\sum T_p x^i = \frac{1}{1 - T_p p^{k-1} + TBD}$$

for the 2nd (analytic) description of them.

TODO: Finish this tale

8.2 Today's material

Picking up from last time, recall the Hecke polynomial at p is $x^2 - a_p x + p^{k-1}$, which has 2 roots each of absolute value $p^{(k-1)/2}$ (by Weil conjecture). Let α_p, β_p be the roots of the Hecke polynomial at p, so

$$L(f,s) = \prod_p \frac{1}{(1-\alpha_p p^{-s})(1-\beta_p p^{-s})}$$

Note 3. Got distracted and missed some stuff.

Recall from yesterday that the Fourier coefficients of a normalized eigenform are determined by its Hecke eigenvalues.

Theorem 8.1 (Multiplicity one). *Let $\lambda : \mathbb{T} \to \mathbb{C}$ be some character of the Hecke algebra. Then,*

$$\dim_{\mathbb{C}} S_k(\Gamma)[\lambda] = 1.$$

This is just a restatement of eigenvalues determining coefficients, i.e. $a_n(f) = a_1(f)\lambda(T_n)$. In fact, something stronger is true.

Theorem 8.2 (Strong Multiplicity One). *Say f, g are two nonzero eigenforms. If $a_p(f) = a_p(g)$ for almost all p, then $f = cg$ for some constant $c \in \mathbb{C}$.*

Might prove this later. It's not easy.

Slogan. The right building blocks for modular forms are eigenforms.[14]

8.3 Connection to Galois representations

Fix a prime ℓ.

Theorem 8.3 (due to Deligne and/or Eichler-Shimura it sounds like)**.** *Let $f = \sum_{n \geq 1} a_n q^n$ be a weight k eigenform for $\mathrm{SL}_2(\mathbb{Z})$. Then, there exists a unique ℓ-adic (continuous) Galois representations*

$$\rho_f : \mathrm{Gal}(\overline{\mathbb{Q}}/\mathbb{Q}) \longrightarrow \mathrm{GL}_2(\overline{\mathbb{Q}}_\ell)$$

such that

(1) *$\rho_{f,\ell}$ is unramified away from ℓ.*

(2) *The characteristic polynomial of Frob_p^{-1} (**geometric Frobenius**[15]) is the Hecke polynomial at p*

(3) *ρ_f is "de Rham" at ℓ*

Something something can make a twist by some character to use arithmetic frobenius instead something something

We have the above theorem for every ℓ, so really what we get is a collection $\{\rho_{f,\ell}\}_\ell$.

Remark 8.4. I version of Chebotarev density says that the Galois group is generated by (all but finitely many) conjugacy classes of Frobenius, so the above theorem suggests the truth of strong multiplicity one.[16]

Remark 8.5. We've been working this whole time with $\Gamma = \mathrm{SL}_2(\mathbb{Z})$. We can replace this with a congruence subgroup, e.g.

$$\Gamma_1(N) = \left\{ \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in \mathrm{SL}_2(\mathbb{Z}) \middle| \begin{pmatrix} a & b \\ c & d \end{pmatrix} \equiv \begin{pmatrix} 1 & * \\ 0 & 1 \end{pmatrix} \pmod{N} \right\}$$

or

$$\Gamma_0(N) = \left\{ \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in \mathrm{SL}_2(\mathbb{Z}) \middle| \begin{pmatrix} a & b \\ c & d \end{pmatrix} \equiv \begin{pmatrix} * & * \\ 0 & * \end{pmatrix} \pmod{N} \right\}$$

[14] I think I heard Wei say this.
[15] i.e. instead of inducing the pth power map on residue fields, it induces the inverse of that
[16] and maybe has it as a corollary? I'd need to think about this

or

$$\Gamma(N) = \ker\left(\mathrm{SL}_2(\mathbb{Z}) \twoheadrightarrow \mathrm{SL}_2(\mathbb{Z}/N\mathbb{Z})\right).$$

In any case, N is called the **level**.

Remark 8.6. Note that $\Gamma(N) \subset \Gamma_1(N) \subset \Gamma_0(N)$. Furthermore, $\Gamma(N) \sim \Gamma_1(N^2)$ are conjugate inside $\mathrm{GL}_2(\mathbb{Q})$.

One can tell the same story with these congruence subgroups. You get spaces $S_k(\Gamma_*(N)) \subset M_k(\Gamma_*(N))$ of (cuspidal) modular forms. You also get a Hecke algebra $\mathbb{T} = \mathbb{Z}[T_n]_{n\geq 1}$ (same definition if $(n, N) = 1$. More annoying to defining if n, N are not coprime). And so one...

8.4 Old and New Forms

We first observe that you can 'artificially' raise the level of a modular form.

Example. If $f \in S_k(\Gamma_0(N/n))$, then $f(n\tau) \in S_k(\Gamma_0(N))$.

Hence, inside of any space $S_k(\Gamma_0(N))$ of cusp forms, there is a large subspace $S_k^{\mathrm{old}}(\Gamma_0(N))$ of **oldforms** coming from forms of lower levels. The orthocomplemnt of this space w.r.t. the Peterssen inner product is the space $S_k^{\mathrm{new}}(\Gamma_0(N))$ of **newforms**.

Theorem 8.7 (Multiplicity One). *Let f, g be nonzero newforms of level N for $\mathbb{T}$, with the same eigenvalues. Then, $f = cg$ for some $c \in \mathbb{C}$.*

This is not true if f, g are not required to be newforms because of trivial counterexamples coming from the existence of oldforms (this isn't in issue for level 1, i.e. for $\mathrm{SL}_2(\mathbb{Z})$).

8.5 Lattice and Theta Functions

We return to limiting ourselves to the level 1 case.

Example. The **Jacobi theta function** is

$$\theta(\tau) = \sum_{n\in\mathbb{Z}} q^{n^2}.$$

This satisfies[17] $\theta(i/y) = \sqrt{y}\theta(iy)$ (for $y \in \mathbb{R}_{>0}$), which can be proven using (classical) Poisson summation.

Recall $S = \begin{pmatrix} & 1 \\ -1 & \end{pmatrix} \in \mathrm{SL}_2(\mathbb{Z})$. More generally, one has

$$\theta(-1/\tau) \overset{\bullet}{=} \sqrt{-\tau}\theta(\tau)$$

(ambiguity in definition of square root[18]), so θ looks like a "modular form of weight 1/2".

Can we observe this phenomena more generally?

[17] If $\tau = iy$, then $-1/\tau = i/y$

[18] τ in upper half-plane, so can easily fix a particular branch. However, still unclear (to me) if this should be an equality or only an equality up to $\pm$ depending on τ

8.5.1 Lattices

Let Λ be free $\mathbb{Z}$-module of finite rank n. Say Λ comes equipped with a symmetric bilinear pairing $\langle -,- \rangle : \Lambda \times \Lambda \to \mathbb{Z}$. Choosing a basis, $\Lambda = \bigoplus_{i=1}^{k} \mathbb{Z}e_i$, write $\langle e_i, e_j \rangle = a_{ij} \in \mathbb{Z}$ and form the $n \times n$ symmetric matrix

$$A = (a_{ij}) \in M_{n \times n}(\mathbb{Z}).$$

Definition 8.8. We call the form **nondegenerate** if $\det A \neq 0$. We call it **unimodular** if $\det A = \pm 1$, i.e. A is invertible. We call it **positive definite** if A is positive definite, i.e. $\langle a, a \rangle \geq 0$ for all $a \in \Lambda$ with equality iff $a = 0$.

Remark 8.9. Attached to our pairing is a quadratic form $q(x) = \frac{\langle x,x \rangle}{2}$. Why bother dividing by 2? Without it, we will end up with a more complicated transformation formula later on.

Definition 8.10. A symmetric bilinear pairing is called **even** if $\langle x, x \rangle \equiv 0 \pmod 2$ always.

Being even forces the associated quadratic form to be integer-valued.

Question 8.11. *Are there any unimodular, positive definite, even quadratic forms?*

Yes, but the smallest example lives in dimension 8.

Non-example. $\mathbb{Z} \times \mathbb{Z} \to \mathbb{Z}, (x, y) \mapsto 2xy$ is positive definite and even, but not unimodular.

Remark 8.12. Being unimodular is a condition at (finite) primes. In the previous example, the form is unimodular at all primes away from 2 (i.e. $2 \in \mathbb{Z}_p^\times$ when $p \neq 2$). Being positive definite should be thought of as a condition at the infinite place of $\mathbb{Q}$.

Definition 8.13. Let

$$\mathscr{L}at_n := \left\{ \begin{matrix} \text{unimodular, even, positive definite} \\ \text{lattices } \Lambda \text{ of rank } n \end{matrix} \right\}_{/\simeq}$$

Example. The E_8-lattice is one of these things. First note that the standard lattice $\mathbb{Z}^n \subset \mathbb{R}^n$ is positive definite and unimodular (the matrix representing it is the identity), but is not even. To make it even, could consider the sublattice

$$\Lambda_1 := \left\{ \sum_{i=1}^{n} x_i \equiv 0 \pmod 2 \right\} \subset \mathbb{Z}^n.$$

This lattice is now even, but it has index 2 in $\mathbb{Z}^n$ so it will not be unimodular. To fix this, consider

$$\Lambda = \Lambda_1 + \mathbb{Z} \underbrace{\left(\frac{1}{2}, \frac{1}{2}, \ldots, \frac{1}{2} \right)}_{f}$$

which contains Λ_1 as an index 2 sublattice. This is again positive definite and unimodular. Is it even? We've only added one new vector, so we only need

$$q(f) = \sum_{i=1}^{n} \left(\frac{1}{2} \right)^2 = \frac{n}{4}$$

to be even, i.e. we only need $8 \mid n$.

Thus, $E_n = \Lambda \subset \mathbb{R}^n$ is a unimodular, even, positive definite lattice when $8 \mid n$.

Example. $E_8 \oplus E_8 \in \mathscr{L}at_{16} \ni E_{16}$ and these turn out to be legitimately different lattices.

Why are these lattices interesting?

8.5.2 Theta functions

Fix some $\Lambda \in \mathscr{L}at_d$. We define the **theta function**

$$\theta_\Lambda(\tau) := \sum_{x\in\Lambda} q^{Q(x)} = \sum_{x\in\Lambda} q^{\frac{\langle x,x\rangle}{2}}$$

(Above, we use $Q(x)$ for the quadratic form attached to Λ instead of $q(x)$ to avoid a notation clash). After choosing a basis and so obtaining a matrix A representing your form, this is the same as

$$\sum_{n\in\mathbb{Z}^n} q^{\frac{1}{2}x^t Ax} = \sum_{n\geq 0} r_n(\Lambda)q^n,$$

where

$$r_n(\Lambda) = \#\left\{x \in \Lambda : (x,x) = 2n\right\}.$$

Remark 8.14. Our insistence of focusing on $\mathscr{L}at_d$ (coming from restricted ourselves to $\mathrm{SL}_2(\mathbb{Z})$) means we miss interesting lattices like $A = I$. This correspond to the form $\sum_{i=1}^d x_i^2$ and so to the function

$$\theta(\tau) = \left(\sum_{n\in\mathbb{Z}} q^{1/2n^2}\right)^d$$

whose coefficients count the number of ways of writing a number as the sum of d squares (or something like this). This will give a modular form of weight $d/2$ and level 2.

Theorem 8.15.

(1) *If $\Lambda \in \mathscr{L}at_d$, then*

$$\theta_\Lambda \in M_{d/2}(\mathrm{SL}_2(\mathbb{Z})).$$

(2) $\mathscr{L}at_d \neq \emptyset \iff 8 \mid d$.

Proof Sketch. **(1)** Only need to check the transformation laws under $T = \begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix}$ and $S = \begin{pmatrix} & 1 \\ -1 & 0 \end{pmatrix}$. Things work out for T since the lattice is even, and they work out for S by using Poisson summation (+ the lattice being unimodular). Specifically, Poisson summation will give (for $y > 0$)

$$\theta_\Lambda\left(\frac{i}{y}\right) = y^{d/2}\theta_\Lambda(iy)$$

which gives the desired transformation law in general by analytic continuation.

(2) Note $(ST)^3 = 1 \in \mathrm{PSL}_2(\mathbb{Z})$. This should be all you need to know for this part apparently. ∎

Application to $r_\Lambda(n)$ Note that the constant term of θ_Λ is always 1 since there's a unique way to represent 0 using the attached quadratic form. Fix some $\Lambda \in \mathscr{L}at_d$ (so $8 \mid d$). Then,

$$\theta_\Lambda = 1 + \sum_{n \geq 1} r_n(\Lambda) q^n \in E_{d/2} + S_{d/2}(\mathrm{SL}_2(\mathbb{Z})),$$

where $E_{d/2}$ is the normalized Eisenstein series. Recall that

$$E_{\frac{d}{2}}(\tau) = 1 + C_{d/2} \sum_{n \geq 1} \sigma_{d/2-1}(n) q^n,$$

and that the trivial estimate for cusp forms says their coefficients are always $O(n^{k/2}) = O(n^{d/4})$ (since $k = d/2$ in the present case). Note that $\sigma_{k-1}(n) = O(n^{k-1+\varepsilon})$. The upshot of all of this is the following.

Wei wrote $n^{\frac{k}{2}+\varepsilon}$ instead, but I didn't understand why

Corollary 8.16.

$$r_n(\Lambda) = C_{\frac{d}{2}} \sigma_{\frac{d}{2}-1}(n) + O(n^{\frac{d}{4}+\varepsilon})$$

for any $\varepsilon > 0$.

This argument can be generalized to not necessarily unimodular lattices as long as you understand Eisenstein series of higher levels.

8.5.3 Connection to Eisenstein series

We've defined the set $\mathscr{L}at_d$ for $8 \mid d$ and given a function

$$\mathscr{L}at_d \longrightarrow M_{\frac{d}{2}}(\mathrm{SL}_2(\mathbb{Z})).$$

What is the image of this map?

Theorem 8.17 (**Siegel(-Weil) formula**)**.** *The weighted average*

$$\sum_{\Lambda \in \mathscr{L}at_d} \frac{\theta_\Lambda}{\# \operatorname{Aut}(\Lambda)} = c_d E_{\frac{d}{2}}$$

is a multiple of an Eisenstein series with explicit constant c_d (given by some product of ζ values).

(Originally due to Siegal but generalized by Weil?)

Remark 8.18. You should think of $\mathscr{L}at_d$ not as a set, but as a groupoid. Note that $\operatorname{Aut}(\Lambda)$ is always finite since it is compact and discrete (e.g. in $O(n)$ and $\mathrm{GL}_n(\mathbb{Z})$).

Corollary 8.19 (of Siegal-Weil)**.**

$$\sum_{\Lambda \in \mathscr{L}at_d} \frac{1}{\# \operatorname{Aut}(\Lambda)} = c_d.$$

Example. One can show that $\mathscr{L}at_8 = \{E_8\}$ and compute the cardinality of its automorphism group.

One can show $\mathscr{L}at_{16} = \{E_8^{\oplus 2}, E_{16}\}$.

One can show $\# \mathscr{L}at_{24} = 24$

One can show $\# \mathscr{L}at_{32} > 80$ million.

9 Lecture 9 (3/17): Non-holomorphic world

We've spent time studying holomorphic modular forms, so let's shift to the next type of automorphic form: non-holomorphic modular forms.

Even if we only care about holomorphic modular forms, we'll see it still makes sense to study non-holomorphic ones. We begin with a key example.

Recall in the holomorphic case we had Eisenstein series E_k indexed by integers.

Example (Non-holomorphic Eisenstein series). Write $z = x + iy \in \mathbb{H}$. Consider the real-valued function $\operatorname{Im} : \mathbb{H} \to \mathbb{R}$ sending $z \mapsto y$. As usual, let $\Gamma = \mathrm{SL}_2(\mathbb{Z})$. Note that

$$\operatorname{Im}(\gamma z) = \frac{\operatorname{Im} z}{|cz+d|^2} \left(= \frac{\operatorname{Im}(z)}{(cz+d)(c\overline{z}+d)} \right) \text{ when } \gamma = \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in \Gamma.$$

This looks like it's "holomorphic of weight 1 and antiholomorphic of weight 1." Inspired by this, we average (kill the subgroup preserving the imaginary part)

$$E(z,s) := \frac{1}{2}\pi^{-s}\Gamma(s)\zeta(2s) \sum_{\gamma \in \Gamma_\infty \backslash \Gamma} \operatorname{Im}(\gamma z)^s$$

to form a function which is (by definition) invariant under the $\mathrm{SL}_2(\mathbb{Z})$ action (at least, when it converges). Above, the factors out front are there to get a nice functional equation later[19], and

$$\Gamma_\infty = \left\{ \begin{pmatrix} 1 & * \\ 0 & 1 \end{pmatrix} \right\} \subset \Gamma$$

is subgroup of matrices preserving the imaginary part. Note we have

$$E(z,s) := \frac{1}{2}\pi^{-s}\Gamma(s)\zeta(2s) \sum_{\substack{(c,d)\in\mathbb{Z}^2 \\ \gcd(c,d)=1}} \frac{y^s}{|cz+d|^{2s}} = \frac{1}{2}\pi^{-s}\Gamma(s) \sideset{}{'}\sum_{(c,d)\in\mathbb{Z}^2} \frac{y^s}{|cz+d|^{2s}}.$$

This converges absolutely when $\operatorname{Re}(s) > 1$ and is invariant under Γ.

Recall 9.1. For comparison, recall the (non-normalizes) holomorphic Eisenstein series were defined by

$$G_k(z) = \sideset{}{'}\sum_{(m,n)} \frac{1}{(mz+n)^k}$$

Remark 9.2. Apparently one can also p-adically interpolate the Eisenstein series E_k ($k \in \mathbb{Z}$).

Remark 9.3. Can we get the holomorphic Eisenstein series from the non-holomorphic ones? Recall we observed that $\operatorname{Im}(\gamma z)$ has a holomorphic part and an anti-holomorphic part. Applying combinations of $\frac{\partial}{\partial z}$ and $y\frac{\partial}{\partial \overline{z}}$ let's you obtain expressions of the form

$$\sum \frac{y^s}{|cz+d|^{2s}(cz+d)^m(c\overline{z}+d)^n}$$

[19] The $\frac{1}{2}$ is coming from the difference between $\mathrm{SL}_2(\mathbb{Z})$ and $\mathrm{PSL}_2(\mathbb{Z})$

with $m, n \geq 0$. Then evaluating at $s = 0$ gives various flavors of Eisenstein series, including the holomorphic ones from before. Details left as an exercise.

Theorem 9.4. *$E(z, s)$ has a meromorphic continuation to all of $s \in \mathbb{C}$ which satisfies the functional equation*

$$E(z, s) = E(z, 1 - s).$$

*It has a simple poles at $s = 0, 1$ both with residue $1/2$ (independent of z). Moreover, it has "**moderate growth**" at infinity, i.e.*

$$|E(z, s)| = O\left(y^{\max\{\mathrm{Re}(s), 1-\mathrm{Re}(s)\}}\right) \quad \text{as} \quad \mathrm{Im}(z) \to \infty.$$

Remark 9.5. For holomorphic modular forms, the "holomorphy at ∞" was essentially equivalent to "moderate growth".

9.1 Fourier expansion of $E(z, s)$

We didn't detail how to find the Fourier expansion of holomorphic Eisenstein series, but doing it for the non-holomorphic ones is more general (get holomorphic ones via differential operators).

The full computation here is kinda tedious, so we just give a sketch. Start with

$$E(z, s) = \sum_{n \in \mathbb{Z}} a_n(y, s) e^{2\pi i n x}$$

coming from invariance under $T = \begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix}$. These coefficients are

$$\begin{aligned} a_n(y, s) &= y^s \Gamma(s) \int_0^1 \sum_{c,d}{}' \frac{1}{|c(x + iy) + d|^s} e^{-2\pi i n x} \mathrm{d}x \\ &= y^s \Gamma(s) \int_0^1 \sum_{c,d}{}' \frac{1}{((cx + d)^2 + c^2 y^2)^s} e^{-2\pi i n x} \mathrm{d}x \end{aligned}$$

so the main point will be to compute the integral of something like

$$\frac{1}{((cx + d)^2 + c^2 y^2)^s} e^{-2\pi i n x}.$$

One can do some integral trickery to end up with something like (when[20] $\mathrm{Re}(s) \gg 0$)

$$\int_{-\infty}^{\infty} \frac{1}{(1 + x^2)^s} e^{-2\pi i y x} \mathrm{d}x.$$

This is still not easy to compute, but wait, we have a Γ-factor which simplifies things. Throw that in

[20] $\mathrm{Re}(s) > 1$ probably works

there and change the order of integration. You end up with

$$\begin{cases} \sqrt{\pi}\Gamma\left(s-\frac{1}{2}\right) & \text{if } y=0 \\ 2\pi^s y^{s-\frac{1}{2}} K_{s-\frac{1}{2}}(2\pi\left|y\right|) & \text{if } y\neq 0 \end{cases}$$

(this is a different y because of substitutions/shifts taken in the interim). Above, the new notation is the **K-Bessel function**

$$K_s(y) := \frac{1}{2}\int_0^\infty e^{-(t+\frac{1}{t})y} t^s \frac{\mathrm{d}t}{t} \text{ where } y>0$$

which is entire in $s\in\mathbb{C}$.

In the end, one obtains

Theorem 9.6. *The Fourier coefficients of $E(z,s)$ are*

$$a_0(y,s) = \pi^{-s}\Gamma(s)\zeta(2s)y^s + \pi^{s-1}\Gamma(1-s)\zeta(2-2s)y^{1-s}$$

and

$$a_n(y,s) = 2\left|n\right|^{s-\frac{1}{2}}\sigma_{1-2s}(|n|)y^{\frac{1}{2}}K_{s-\frac{1}{2}}\left(2\pi\left|n\right|y\right)$$

for $n\neq 0$.

Remark 9.7. Note that these coefficients all have meromorphic continuations to $s\in\mathbb{C}$ and are all invariant under the change $s\leftrightarrow 1-s$.

I guess you need to know the functional equation of the K Bessel function to verify this for $n \neq 0$. I don't know it

Lemma 9.8. *For $y>4$,*

$$|K_s(y)| < e^{-y/2}K_{\mathrm{Re}(s)}(2)$$

so the K-Bessel function has exponential decay.

"It's easy to prove this asymptotic just by looking at the definition of the Bessel function."

This implies the "moderate growth" stated in an earlier theorem.

One could have predicted that something like K Bessel should appear by looking at the differential equation satisfies by the Eisenstein series.

Remark 9.9. Consider the **Hyperbolic Laplacian**

$$\Delta = -y^2\left(\frac{\partial^2}{\partial x^2}+\frac{\partial}{\partial^2}y^2\right)$$

(which is $\mathrm{SL}_2(\mathbb{R})$-invariant). Note that $\mathrm{Im}(z)^s$ is an eigenfunction with eigenvalue $s(1-s)$, i.e.

$$\Delta\,\mathrm{Im}(z)^s = s(1-s)\,\mathrm{Im}(z)^s.$$

This property is invariant under addition and $\mathrm{SL}_2(\mathbb{Z})$-translation, so $E(z,s)$ is a Δ-eigenfunction with eigenvalue $s(1-s)$. In fact, taking Fourier transforms does not invalidate this property, so also

$$\Delta\left(a_n(y,s)e^{2\pi nix}\right) = s(1-s)a_n(y,s)e^{2\pi inx}.$$

If we write $a_n(y,s) = y^{1/2} b_n(2\pi |n| y)$, then this mysterious new b_n satisfies the differential equation

$$\left(y^2 \frac{\partial^2}{\partial y^2} + y\frac{\partial}{\partial y} - (y^2+\nu^2)\right) b_n = 0 \text{ where } \frac{1}{4} - \nu^2 = (1-s)s.$$

Apparently this equation has two solutions, one involving K Bessel and one which does not. The one which does not does not have rapid decay.

9.2 An Application: Prime Number Theorem

Let $\varphi : \Gamma\backslash\mathbb{H} \to \mathbb{C}$ be a smooth function on the upper half plane which is invariant under $\mathrm{SL}_2(\mathbb{Z})$. Suppose it has **rapid decay** as $y \to \infty$, i.e.

e.g. If $f(z)$ is a weight k holomorphic cusp form, then we could take $\varphi(z) = |f(z)|\, y^{k/2}$

$$|\varphi(z)| = O\left(y^{-N}\right) \text{ for all } N \text{ as } \mathrm{Im}(z) \to \infty.$$

Write it's Fourier expansion

$$\varphi(z) = \sum_{n\in\mathbb{Z}} \varphi_n(y) e^{2\pi i n x}$$

(note each φ_n has rapid decay as $y \to \infty$).

Definition 9.10. The **Mellin transform** of $\varphi_0(y)$ above is

$$M(s, \varphi_0(y)) = \int_0^\infty \varphi_0(y) t^s \frac{\mathrm{d}t}{t}.$$

Recall 9.11. The **Petersson inner product** is

$$\langle f, g\rangle_{\mathrm{Pet}} = \int_{\Gamma\backslash\mathbb{H}} f(z)\overline{g(z)} \frac{\mathrm{d}x\mathrm{d}y}{y^2}$$

which is well-defined whenever $f, g : \mathbb{H} \to \mathbb{C}$ are Γ-invariant and *either* of f, g have rapid decay as $y \to \infty$ (with the other being of moderate growth).

Lemma 9.12. *Suppose φ is as described in the beginning of this (sub)section. Then,*

$$\langle E_s, \overline{\varphi}\rangle_{Pet} = c(s) M(s-1, \varphi_0(y)),$$

where $c(s) = \pi^{-s}\Gamma(s)\zeta(2s)$.

Slogan. Inner products with Eisenstein series pick up the constant term.

The main point of the proof of the lemma is that this inner product (up to outside factors) is

$$\begin{aligned}
\int_{\Gamma\backslash\mathbb{H}} \left(\sum_{\gamma\in\Gamma_\infty\backslash\Gamma} \mathrm{Im}(\gamma z)^s\right) \varphi(z) \frac{\mathrm{d}x\mathrm{d}y}{y^2} &= \int_{\Gamma_\infty\backslash\mathbb{H}} \mathrm{Im}(\gamma z)^s \overline{\varphi}(z) \frac{\mathrm{d}x\mathrm{d}y}{y^2} \\
&= \int_0^\infty \left(\int_0^1 \varphi(x+iy)\mathrm{d}x\right) y^s \frac{\mathrm{d}y}{y^2} \\
&= \int_0^\infty \varphi_0(y) y^{s-1} \frac{\mathrm{d}y}{y}
\end{aligned}$$

$$= M(1-s, \varphi_0(y)).$$

The version of the '**Prime Number Theorem**' we will prove with this is that Riemann zeta $\zeta(s)$ is nonvanishing when $\mathrm{Re}(s) = 1$. This is equivalent to the usual 'Prime Number Theorem' that $\pi(x) \sim x/\log x$.

Recall 9.13. The zeta function shows up in the constant term

$$a_0(y,s) = \pi^{-s}\Gamma(s)\zeta(2s)y^s + \pi^{s-1}\Gamma(1-s)\zeta(2-2s)y^{1-s}$$

of $E(z,s)$.

Proof of 'Prime Number Theorem'. Suppose $\zeta(1+it_0) = 0$ for some $t_0 \in \mathbb{R}$. By taking complex conjugation, we see that also $\zeta(1-it_0) = 0$. Let $s_0 = \frac{1+it_0}{2}$, so $E_{s_0} = E(-, s_0)$ has vanishing constant term, which is strange for an Eisenstein series. Looking at the remaining terms of its Fourier expansion, this implies that E_{s_0} has rapid decay as $y \to \infty$ (as if it were a cusp form). Now, the previous lemma tells us that

$$\langle E_s, E_{s_0}\rangle_{\mathrm{Pet}} = c(s)M(s-1, 0) = 0.$$

In particular, $\langle E_{s_0}, E_{s_0}\rangle = 0$ which implies $E_{s_0}(z) = 0$. So all its Fourier coefficients vanish, but this is certainly not the case because we calculated them earlier (the K Bessel function won't vanish identically as a function of y). ■

The Petersson inner product is positive definite by construction

Remark 9.14. This same idea can work to prove non-vanishing results for other L-functions. Some people like to try to use the connection between ζ and E to get progress on zeta functions.

Next class we do Rankin-Selberg.

10 Lecture 10 (3/24)

We talked about non-holomorphic Eisenstein series last time, and saw them applied to a non-vanishing result for Riemann-zeta. We will see one or two more serious applications today.

Recall

$$E(z,s) = \frac{1}{2}\pi^{-s}\Gamma(s)\zeta(2s)\sum_{\gamma \in \Gamma_\infty\backslash\Gamma} \mathrm{Im}(\gamma z)^s,$$

where $\Gamma = \mathrm{SL}_2(\mathbb{Z})$ and Γ_∞ is the subgroup (of translations) fixing ∞.

10.1 Rankin-Selberg

Let $f \in S_k(\Gamma)$ be a holomorphic cuspidal modular form of weight k. Expand

$$f = \sum_{n\geq 1} a_n q^n.$$

Recall that associated to f is an L-function $L(f,s) = \sum_{n\geq 1} a_n n^{-s}$. If f is a normalized (i.e. $a_1 = 1$) Hecke eigenform, then we get an Euler product

$$L(f,s) = \prod_p \frac{1}{1 - a_p p^{-s} + p^{-2s+k-1}}.$$

Recall 10.1 (Trivial bound)**.** $|a_n| = O(n^{\frac{k}{2}})$.

Ramanujan conjectured and Deligne proved $|a_n| = O(n^{\frac{k-1}{2}})$. Equivalently, if we factor

Note (α_p, β_p) well-defined up to $\mathbb{Z}/2\mathbb{Z}$-action.

$$1 - a_p x + p^{k-1}x^2 = (1-\alpha_p x)(1-\beta_p x),$$

then we see f gives rise to $\begin{pmatrix} \alpha_p & \\ & \beta_p \end{pmatrix} \in \mathrm{GL}_2(\mathbb{C})^{ss}//\mathrm{conj}$ for each p. That is, f gives rise to a collection

semisimple element well-defined up to conjugation

$$\left\{ \begin{pmatrix} \alpha_p & \\ & \beta_p \end{pmatrix} \right\}_{p \text{ prime}}$$

of semisimple conjugacy classes in $\mathrm{GL}_2(\mathbb{C})$.

History. (1930∼1940) People wanted to improve the trivial bound (this was before Deligne's proof). One idea (due to Rankin[21]) was to consider the modified L-function $\sum_{n\geq 1} a_n \overline{a}_n n^{-s}$. If you can prove nice analytic properties for this, this could help bound the square of the absolute value of a_n.

This idea leads one to consider the **Rankin-Selberg convolution**. Given $f, g \in S_k(\Gamma)$, write $f = \sum a_n q^n$ and $g = \sum b_n q^n$. Say they are both normalized eigenforms. We can then form a new Dirichlet series

Could take forms of different weights, and only one needs to be cuspidal, but let's keep it simple

$$\text{“}\sum_{n\geq 1} a_n b_n n^{-s}\text{”}.$$

This will not be quite the correction definition; we will need to modify it by some zeta-factors.

Recall f, g each have a collection of semisimple conjugacy classes (one for each prime p) attached to them.

Fact (See Theorem 8.3)**.** Attached to f is a 2-dimensional continuous Galois representation

$$\rho_f : \mathrm{Gal}(\overline{\mathbb{Q}}/\mathbb{Q}) \to \mathrm{GL}_2(\overline{\mathbb{Q}}_\ell)$$

which is unramified at all primes $p \neq \ell$. Further, the image of Frob_p, up to semimplication, is

$$\begin{pmatrix} \alpha_p & \\ & \beta_p \end{pmatrix}.$$

Above, we've implicitly fixed and made use of some abstract isomorphism $\overline{\mathbb{Q}}_\ell \xrightarrow{\sim} \mathbb{C}$

One expects there to be a bijection between such f and such Galois representations.[22] Roughly, there

[21] and independently to Selberg?

[22] Buzzwords: “Fontaine-Mazur” and “Langlands” and “Taniyama-Shimura”

should be a 'natural' bijection

$$\left\{\begin{matrix}\text{Normalized Hecke}\\ \text{eigenforms}\end{matrix}\right\} \longleftrightarrow \left\{\begin{matrix}\text{2-dim Galois reps}\\ \rho : \mathrm{Gal}(\overline{\mathbb{Q}}/\mathbb{Q}) \to \mathrm{GL}_2(\overline{\mathbb{Q}}_\ell)\end{matrix}\right\}$$

such that the conjugacy class of Frobenius matches the conjugacy class coming from Hecke polynomials. Put another way, both sides have L-functions

$$L(f,s) = \prod_p L_p(f,s) \text{ and } L(\rho_f,s) = \prod_p L(\rho_f|_{\mathrm{Gal}(\overline{\mathbb{Q}}_p/\mathbb{Q}_p)}, s),$$

and this bijection is characterized by demanding L-functions to match up, i.e. $L(f,s) = L(\rho_f,s)$.

Slogan. Operations of Galois representations should give rise to reasonable L-functions.

Example. Say V, W are two 2-dimensional ℓ-adic Galois representations. Can form their tensor product $V \otimes W$, now 4-dimensional.[23] There should be some nice L-function attached to this tensor product.

This gives a way to see the 'right' definition of the Rankin L-function instead of just guessing that you should multiply the coefficients termwise.

What, specifically, does this philosophy predict? Do things in terms of the local factors. Let $\begin{pmatrix}\alpha_p & \\ & \beta_p\end{pmatrix}$ be the semisimple conjugacy classes attached to f, and let $\begin{pmatrix}\alpha'_p & \\ & \beta'_p\end{pmatrix}$ be the semisimple conjugacy classes attached to g. The "semisimple conjugacy class attached to $f \times g$" should then be the tensor product

$$\begin{pmatrix}\alpha_p & \\ & \beta_p\end{pmatrix} \otimes \begin{pmatrix}\alpha'_p & \\ & \beta'_p\end{pmatrix},$$

so the correct definition of the **Rankin-Selberg L-function** is

$$L(f \times g, s) := \prod_p \frac{1}{(1-\alpha_p\alpha'_p p^{-s})(1-\alpha_p\beta'_p p^{-s})(1-\beta_p\alpha'_p p^{-s})(1-\beta_p\beta'_p p^{-s})}.$$

Lemma 10.2.

$$\sum_{i \geq 0} a_{p^i} X^i = \frac{1}{(1-\alpha_p X)(1-\beta_p X)} \quad \textit{and} \quad \sum b_{p^i} X^i = \frac{1}{(1-\alpha'_p X)(1-\beta'_p X)}.$$

And

$$\sum_{i \geq 1} a_{p^i} b_{p^i} X^i = \frac{(1-\alpha_p\alpha'_p\beta_p\beta'_p X^2)}{(1-\alpha_p\alpha'_p X)(1-\alpha_p\beta'_p X)(1-\beta_p\alpha'_p X)(1-\beta_p\beta'_p X)}$$

Thus, we see that the Rankin-Selberg L-function differs from the naive guess by a zeta factor:

$$L(f \times g, s) = \zeta(2s+2k-2) \sum_{n \geq 1} a_n b_n n^{-s}$$

(the arguments given to ζ above may be incorrect).

[23] Could also take $\mathrm{Sym}^n V$ which is $(n+1)$-dimensional, and could do other sorts of linear algebraic things if you want

Conjecture 10.3 (**Langlands**)**.** *Given any $f_i \in S_{k_i}(\Gamma)$, can define $L(f_1 \times f_2 \times \ldots \times f_n, s)$ using an Euler product analogous to what was done above. These should all have meromorphic continuations to all $s \in \mathbb{C}$ and satisfy a function equation.*

This is known (in much greater generality[24]) for $n \leq 3$. Sounds like it's known for all n in the generality stated above.

Rankin-Selberg corresponds to the case $n = 2$, and we will prove it using Eisenstein series.

Lemma 10.4 (Lemma 9.12)**.** *Let $\varphi : \Gamma\backslash\mathbb{H} \to \mathbb{C}$ be a smooth function on the upper half plane which is invariant under $\mathrm{SL}_2(\mathbb{Z})$. Suppose it has rapid decay as $y \to \infty$, i.e.*

$$|\varphi(z)| = O\left(y^{-N}\right) \ \textit{for all } N \ \textit{as } \operatorname{Im}(z) \to \infty.$$

Write it's Fourier expansion

$$\varphi(z) = \sum_{n\in\mathbb{Z}} \varphi_n(y)e^{2\pi inx}$$

(note each φ_n has rapid decay as $y \to \infty$). Then,

$$\langle E_s, \overline{\varphi}\rangle_{Pet} = c(s)M(s-1, \varphi_0(y)),$$

where $c(s) = \pi^{-s}\Gamma(s)\zeta(2s)$ and M denotes the Mellin transform

$$M(s-1, \varphi_0(y)) = \int_0^\infty \varphi_0(y)t^{s-1}\frac{\mathrm{d}t}{t}.$$

Note 4. Got distracted and missed some of what he was saying.

Say we have $f, g \in S_k(\Gamma)$. Consider $\varphi(z) = f(z)\overline{g(z)}y^k$ which invariant under $\Gamma = \mathrm{SL}_2(\mathbb{Z})$. It is also smooth and has rapid decay since f, g are cuspidal (in fact, enough for one of them to be cuspidal and the other one having moderate growth). Then,

$$\langle E_s, \overline{\varphi}\rangle_{\mathrm{Pet}} = c(s)M(s-1, \varphi_0).$$

Claim 10.5. $M(s-1, \varphi_0) \sim \sum a_n\overline{b}_n n^{-s}$.

Note: if g is a normalized eigenform, its eigenvalues/coefficients will be real numbers so $\overline{b}_n = b_n$. This follows from the Hecke operators being self-adjoint.

We can just compute this. Write $f = \sum a_n e^{2\pi in(x+iy)}$ and $\overline{g} = \sum \overline{b}_n e^{-2\pi in(x-iy)}$. Then the constant term of $\varphi(z) = f(z)\overline{g}(z)y^k$ is

$$\varphi_0(y) = y^k \sum_{n\geq 1} a_n\overline{b}_n e^{2\pi in(iy)-2\pi in(-iy)} = y^k \sum_{n\geq 1} a_n\overline{b}_n e^{-4\pi ny}.$$

Hence,

$$\begin{aligned} M(s-1, \varphi_0) &= \int_0^\infty \left(y^k \sum (blah)\right) y^{s-1}\frac{\mathrm{d}y}{y} \\ &= \sum_{n\geq 1} a_n\overline{b}_n \int_0^\infty e^{-4\pi ny}y^{s-1+k}\frac{\mathrm{d}y}{y} \end{aligned}$$

[24] i.e. for automorphic forms on GL_2 over a general number field

$$= (*)\Gamma(s+k-1)\sum_{n\geq 1} a_n\overline{b}_n$$

(for $\mathrm{Re}(s) \gg 0$). This gives

$$\langle E_s, \overline{\varphi}\rangle_{\mathrm{Pet}} = (*)L(f\times g, s+k-1) =: \Lambda(f\times g, s+k-1),$$

so we recover the Rankin-Selberg L-function up to some 'simple factors' (Γ-factors, π-powers, ζ-factors, etc.). This is the **complete Rankin-Selberg L-function** (recall from Tate's thesis that Γ-factors are local L-factors at archimedean places). Hence,

$$\Lambda(f\times g, s+k-1) = \int_{\Gamma\backslash\mathbb{H}} f(z)\overline{g}(z)y^k E_s(z)\frac{\mathrm{d}x\mathrm{d}y}{y^2}.$$

Since $\varphi(z) = f(z)\overline{g}(z)y^k$ has rapid decay, this represents a holomorphic function (except possibly at the poles of $E_s(z)$ when $s = 0, 1$). The residues at these two (simple) poles are

$$\mathrm{Res}_{s=0,1}\,\Lambda(f\times g, s+k-1) = \frac{1}{2}\langle f, g\rangle_{\mathrm{Pet}}.$$

If f, g are normalized eigenforms, then they must be orthogonal or equal. When $f = g$, $\mathrm{Res} \neq 0$. If $f \neq g$, then $\langle f, g\rangle = 0$ so $\mathrm{Res} = 0$ and the complete L-function is entire. In either case, the equality $E_s(z) = E_{1-s}(z)$ implies the functional equation

$$\Lambda(f\times g, s) = \Lambda(f\times g, 2k-1-s)$$

centered at $s = (2k-1)/2$. This is the expected center. Recall the local factors

$$L(f\times g, s)_p = \frac{1}{(1-\alpha_p\alpha_p' p^{-s})(1-\alpha_p\beta_p' p^{-s})(1-\beta_p\alpha_p' p^{-s})(1-\beta_p\beta_p' p^{-s})}.$$

By Deligne, we know $|\alpha_p| = |\alpha_p'| = |\beta_p| = |\beta_p'| = p^{\frac{k-1}{2}}$. Whence $|\alpha_p\alpha_p'| = p^{k-1}$ (and so on), so $\prod_p L_p(f\times g, s)$ converges absolutely for $\mathrm{Re}(s) > 1 + (k-1) = k$. This leads to the prediction that the center should be at $k - 1/2$.

We'll later see a generalization to automorphic reps on $\mathrm{GL}_n \times \mathrm{GL}_m$ (widely open if you take $\mathrm{GL}_{n_1}\times\mathrm{GL}_{n_2}\times\mathrm{GL}_{n_3}$ unless $n_1 = n_2 = n_3 = 2$).

A bit of history, I think Langlands wanted to prove Ramanujan's conjecture. In one form, this says $|\alpha_p| = 1$ when you normalize α_p in an appropriate way. Doing this normalization makes things look nicer (e.g. eliminates awkward k's showing up in complete Rankin-Selberg L-functions).

Divide by $p^{(k-1)/2}$, I think

Note that these come in pairs (the reciprocal is also a root), so enough to show $|\alpha_p| \leq 1$. To show this, one might try to show $|\alpha_p^n| \leq p^{\varepsilon}$ for all n. Since

$$\mathrm{Sym}^n\begin{pmatrix}\alpha_p & \\ & \beta_p\end{pmatrix} \sim \mathrm{diag}\left(\alpha_p^i\beta_p^{n-1}\right)_{0\leq i\leq n},$$

this is the same as saying that $L(\mathrm{Sym}^n f, s)$ is absolutely convergent for $\mathrm{Re}(s) > \varepsilon$ for all n.

The actual conjecture made was $L(\mathrm{Sym}^n f, s) = L(\pi_n, s)$ where π_n is some automorphic representation on GL_{n+1}.

Note 5. Wei said more stuff, but I was distracted and didn't write it down.

Sounds like Deligne learned about Rankin-Selberg from a class by Serre on modular forms. He then adapted this technique as an ingredient in his proof of the Weil conjectures.

11 Lecture 11 (3/29)

Note 6. A minute or two (or more?) late.

11.1 Maass (wave) form

Something something **Hyperbolic Laplacian operator** (Note that Δ is $\Gamma = \mathrm{SL}_2(\mathbb{Z})$-invariant, so descends to an operator of $\Gamma\backslash\mathbb{H}$)

$$\Delta = -y^2\left(\frac{\partial}{\partial x^2} + \frac{\partial}{\partial y^2}\right).$$

The sign is chosen to make this semipositive, $\langle \Delta f, f\rangle_{\mathrm{Pet}} \geq 0$. It is also self-adjoint w.r.t the Petersson inner product: $\langle \Delta f, f\rangle = \langle f, \Delta f\rangle$. One can easily compute

$$\Delta y^2 = -s(s-1)y^2 = s(1-s)y^2.$$

Thus, the non-holomorphic Eisenstein series $E_s(z) = \frac{1}{2}\pi^{-s}\Gamma(s)\zeta(2s)\sum_{\gamma\in\Gamma_\infty\backslash\Gamma}\mathrm{Im}(\gamma z)^s$ also satisfies $\Delta E_s = s(1-s)E_s$. If we write $s = \frac{1}{2}+\nu$, then the Δ-eigenvalue of E_s is $\frac{1}{4}-\nu^2 = s(1-s)$. We know the non-holomorphic Eisenstein series satisfies a functional equation relating $s \leftrightarrow 1-s$, so the "central critical line" occurs at $\mathrm{Re}(\nu) = 0$, i.e. $\nu \in i\mathbb{R}$ (these will be "tempered" in a sense to be make precise later).

To generalize this situation, we drop the explicit description of E_s, but keep these eigenfunction properties.

Definition 11.1. Let $f : \mathbb{H} \to \mathbb{C}$ be a smooth function. We call is a **Maass form** if it satisfies

(1) Γ-invariant

(2) Δ-eigenfunction: we write $\Delta f = \left(\frac{1}{4}-\nu^2\right) f$ with $\nu \in \mathbb{C}$.

(3) moderate growth as $y \to \infty$, i.e. $|f(z)| = O(y^N)$ for some N as $y \to \infty$.

Consider also the condition

(3') **rapid decay**, i.e. $|f(z)| = O(y^{-N})$ for all N as $y \to \infty$.

If f satisfies **(1)**,**(2)**,**(3')**, then we call is a **cuspidal Maass form**.

Compare above definition to that of (cuspidal) holomorphic modular forms.

11.1.1 Fourier expansion

If $f(z)$ is a Maass form, then Γ-invariant means we can write $f(z) = \sum_{n\in\mathbb{Z}} a_n(y)e^{2\pi inx}$ where $z = x+iy$. The individual terms must satisfy

$$\Delta\left(a_n(y)e^{2\pi inx}\right) = \left(\frac{1}{4}-\nu^2\right)a_n(y)e^{2\pi inx}.$$

When $n=0$, this says that

$$-y^2\left(\frac{\partial}{\partial y}\right)^2 a_n(y) = \left(\frac{1}{4}-\nu^2\right)a_n(y).$$

One can see that this has two linearly independent solutions $y^{\frac{1}{2}\pm\nu}$, so we must have $a_0(y) = \lambda y^{\frac{1}{2}-\nu} + \mu u^{\frac{1}{2}+\nu}$ for some $\lambda,\mu\in\mathbb{C}$. This is the same as what happened with the Eisenstein series.

What about when $|n|\geq 1$? The corresponding differential equation again has two independent solutions, but now one of them is distinguished since it's the only one of the two to have moderate growth (the other has exponential growth).

Recall 11.2. Recall the K-Bessel function

$$K_s(y) := \frac{1}{2}\int_0^\infty e^{-(t+\frac{1}{t})y}t^s\frac{\mathrm{d}t}{t} \text{ where } y>0.$$

Since $f(z)$ satisfies moderate growth, one must have

$$a_n(y) = y^{\frac{1}{2}}K_\nu(2\pi|n|y) \text{ for some } a_n\in\mathbb{C}$$

when $|n|\geq 1$ (note $s-\frac{1}{2}=\nu$). The upshot is that we can write

$$f = \left(\lambda y^{\frac{1}{2}-\nu} + \mu y^{\frac{1}{2}+\nu}\right) + \sum_{n\geq 0} a_n y^{\frac{1}{2}}K_\nu(2\pi|n|y)e^{2\pi inx}.$$

Compare to the holomorphic case where the terms are (multiples of) $e^{2\pi inz} = e^{-2\pi iny}e^{2\pi inx}$. We have a similar expression with $e^{-2\pi ny}$ replaced by $y^{1/2}K_\nu(2\pi|n|y)$.

Notation 11.3. Set $a_0^+ := \mu$ and $a_0^- := \lambda$.

We see f is determined by the coefficients $\{a_n\}_{n\in\mathbb{Z}}$ (implicitly $a_0 = (a_0^+, a_0^-)$).

Lemma 11.4. *A Maass form f is cuspidal* $\iff$ $a_0^\pm = 0$.

Warning 11.5. It is hard to give any explicit example of cuspidal Maass forms. Nevertheless, they do exist.

Notation 11.6. We let $S(\Gamma,\nu)$ denote the space of cusp forms with paramter ν.

Question 11.7 (Audience)**.** *Are there known dimension counts for space of cuspidal Maass forms?*

Answer. More of this later, but for a quick answer: there is the *Weyl law*, something like $\dim S(\Gamma,\nu)\sim c\left|\nu^2\right|^2$ (exponent might be missing a factor of 1/2?). The proof of this is difficult; Selberg established this using what's now called the Selberg trace formula.

Lemma 11.8 (**Trivial estimate**)**.** *Let f be a cusp form. $|a_n| = O(n^{1/2})$ with implied constant depending on ν and $\langle f, f\rangle_{Pet}$.*

(Cuspidal Maass forms behave like weight 1 cusp forms).

On $S(\Gamma, \nu)$, there is an involution $\sigma : f \mapsto \sigma(f)$ where $\sigma(f)(z) = f(-\overline{z})$. We say f is **even** if $\sigma(f) = f$ and is **odd** if $\sigma(f) = -f$. Equivalently, f is even/odd $\iff$ $a_n = \pm a_{-n}$. Let's fix a parity $\varepsilon = \{\pm 1\}$, so only the positive coefficients of f matter. We then define the **L-function**

$$L(f, s) := \sum_{n \geq 1} a_n n^{-s}.$$

Theorem 11.9. *Let f be cuspidal with $\sigma(f) = \varepsilon f$. Then, $L(f, s)$ extends to an entire function on $\mathbb{C}$ with functional equation: For*

$$\Lambda(f, s) := \pi^{-s}\Gamma\left(\frac{s+\varepsilon+\nu}{2}\right)\Gamma\left(\frac{s+\varepsilon-\nu}{2}\right)L(f, s),$$

one has

$$\Lambda(f, s) = (-1)^{\varepsilon}\Lambda(f, 1-s).$$

Recall 11.10. The functional equation for a weight k holomorphic modular form relates $s \leftrightarrow k - s$, so we again see that Maass forms behave like weight 1 modular forms.

Proof Idea. Γ-invariant applied to $\begin{pmatrix} & 1 \\ -1 & \end{pmatrix} \in \Gamma$ gives $f(-1/z) = f(z)$, so $f(i/y) = f(iy)$. As usual, we then take the Mellin transform

$$\int_0^{\infty} f(iy)y^s \frac{\mathrm{d}y}{y} = \int_0^{\infty} \left(\sum_{n \neq 0} a_n y^{\frac{1}{2}} K_{\nu}(2\pi |n| y) y^s \frac{\mathrm{d}y}{y}\right).$$

To compute this, the essential part is understanding

$$\int_0^{\infty} K_{\nu}(y) y^{\frac{1}{2}+s} \frac{\mathrm{d}y}{y}.$$

Plug in the integral representation of the K-Bessel function $K_{\nu}(y) = \int_0^{\infty} e^{-y(t+1/t)} t^{\nu} \frac{\mathrm{d}t}{t}$, do some change of variables, and watch the Γ-factors fall out. ■

Remark 11.11. When proving the prime number theorem, we claimed at the end that the K-Bessel function is nonvanishing. This follows from the below computation.

Lemma 11.12.

$$\int_0^{\infty} K_v(y) y^s \frac{\mathrm{d}y}{y} = 2^{s-2}\Gamma\left(\frac{s+\nu}{2}\right)\Gamma\left(\frac{s+\nu}{2}\right)$$

Corollary 11.13. *$K_{\nu}(y) \not\equiv 0$ in y for any choice of $\nu \in \mathbb{C}$.*

Remark 11.14. If f is cuspidal, we can also do Hecke theory. Hence, it's possible to define Hecke eigenforms f for Maass forms. These will again satisfy $T_n f = a_n f$ for $n \geq 1$. Hence, (if f normalized?), its L-function will have an Euler product

$$L(f,s) = \prod_p \frac{1}{1 - a_p p^{-s} + p^{-2s}}$$

(compare last term in denom with $p^{-2s+k-1}$).

Conjecture 11.15 (**Generalized Ramanujan Conjecture**). *If you decompose the Hecke polynomials $1 - a_p x + x^2 = (1-\alpha_p x)(1-\beta_p x)$, then $|\alpha_p| = |\beta_p| = 1$ for all p.*

There is an archimedean analogue due to Selberg.

Unclear to me if this was meant to be under the same header as 'generalized Ramanujan'

Conjecture 11.16. *If f is a cuspidal eigenform, then $\nu \in i\mathbb{R}$ (i.e. $|e^\nu| = 1$).*

These are both still quite open. These Maass forms don't have an algebro-geometric interpretation, so Deligne's argument (i.e. the Weil conjectures) for holomorphic modular forms does not apply.

11.2 Spectral Decomposition

What does the space of all Maass forms look like?

Note 7. Got distracted and missed some of what he was saying.

Sounds like we want to consider some space of functions (e.g. $C^\infty(\Gamma\backslash\mathbb{H})$ or $L^2(\Gamma\backslash\mathbb{H})$) and then decompose it somehow...

Let $C_0^\infty(\Gamma\backslash\mathbb{H}) \subset C^\infty(\Gamma\backslash\mathbb{H})$ denote the subspace of **cuspidal** (smooth) functions, i.e. those for which

$$\int_0^1 f(x+iy)\mathrm{d}x = 0 \text{ for all } y.$$

Similarly, let $L_0^2(\Gamma\backslash\mathbb{H}) \subset L^2(\Gamma\backslash\mathbb{H})$ denote the subspace of cuspidal (square-integrable) functions.

Fact. $C^\infty(\Gamma\backslash\mathbb{H})$ is dense in $L^2(\Gamma\backslash\mathbb{H})$.

We have $\Delta \curvearrowright C^\infty(\Gamma\backslash\mathbb{H})$ and this action extends to one on $L^2(\Gamma\backslash\mathbb{H})$.

Question 11.17. *How do we decompose $L^2(\Gamma\backslash\mathbb{H})$ into Δ-eigenspaces?*

Let's look at some classical examples.

Example. Say we have a compact space like $S^1 = \mathbb{R}/\mathbb{Z}$. Set $\Delta = \left(\frac{\partial}{\partial x}\right)^2$. This descends to an operator on S^1, and the eigenfunctions are $x \mapsto e^{2\pi i n x}$. Here,

$$L^2(S^1) = \bigoplus_{n\in\mathbb{Z}} \mathbb{C}e^{2\pi i n x}$$

(Hilbert space direct sum). It's not true that every $f \in L^2(S^1)$ is of the form $\sum_{n\in\mathbb{Z}} a_n e^{2\pi i n x}$, but that they are in L^2 (equality away from a measure zero subset). If $f \in C^\infty(S^1)$, this we do have $f(x) = \sum_{n\in\mathbb{Z}} a_n e^{2\pi i n x}$ pointwise.

Example. Say we have a noncompact space like $\mathbb{R}$. The eigenfunctions here are $x \mapsto e^{2\pi i x \nu}$ for $\nu \in \mathbb{C}$. The theory of Fourier transforms gives

$$L^2(\mathbb{R}) \text{``}='' \int_{y \in \mathbb{R}} \mathbb{C} e^{2\pi i x y}.$$

What we mean by this is that any $f \in L^2(\mathbb{R})$ can be written as

$$f(x) = \int_{\mathbb{R}} \widehat{f}(y) e^{2\pi i x y} \mathrm{d}y$$

inside $L^2(\mathbb{R})$ (if $f \in \mathcal{S}(\mathbb{R})$ is Schwartz, for example, above equality holds as honest functions, not just in L^2).

In the present case, for $\Gamma\backslash\mathbb{H}$, we'll get a situation which is a mix of these two. First note that $\Gamma\backslash\mathbb{H} \simeq \mathbb{C}$ is noncompact (Recall $\Gamma = \mathrm{SL}_2(\mathbb{Z})$).

Theorem 11.18.

$$L^2(\Gamma\backslash\mathbb{H}) = \mathbb{C}1 \oplus L^2_0(\Gamma\backslash\mathbb{H}) \oplus \int_{i\mathbb{R}} E\left(\frac{1}{2} + \nu, z\right) \mathrm{d}\nu.$$

Further, the cuspidal part is

$$L^2_0(\Gamma\backslash\mathbb{H}) = \bigoplus_{i \geq 1} \mathbb{C} f_i \;\; \textit{where } \Delta f_i = \lambda_i f_i \;\; \textit{and } \lambda_0 = 0 < \lambda_1 \leq \lambda_2 \leq \cdots$$

(with multiplicity). Above, we normalize f_i s.t. $\langle f_i, f_i \rangle = 1$.

There are the constant functions, the cuspidal functions, and the Eisenstein series. We say that

$$\mathbb{C}\mathbf{1}_{\Gamma\backslash\mathbb{H}} \oplus L^2_0(\Gamma\backslash\mathbb{H})$$

is the "**discrete part**," a Hilbert direct sum of 1-dimensional spaces. The other part

$$\int_{i\mathbb{R}} E\left(\frac{1}{2} + \nu, z\right) \mathrm{d}\nu$$

is called the "**continuous part**," given by a direct integral instead of direct sum.

Fact. Say $f : \Gamma\backslash\mathbb{H} \to \mathbb{C}$ is smooth of rapid decay. Then, there is an on-the-nose equality

$$f = \frac{\langle f, 1 \rangle}{\langle 1, 1 \rangle} + \sum_{i \geq 1} \langle f, f_i \rangle f_i + \int_{i\mathbb{R}} E\left(\frac{1}{2} + \nu, -\right) \left\langle f, E\left(\frac{1}{2} + \nu, -\right)\right\rangle \mathrm{d}\nu$$

(note we normalized by the volume).

Note that we only need the "tempered" Eisenstein series to describe the spectral decomposition.

Remark 11.19. Implicitly, the decomposition tells us that for $\lambda = \frac{1}{4} - \nu^2$, $\dim S(\Gamma, \lambda) < \infty$. In fact, it even says there are only countably many possible eigenvalues for cuspidal Mass forms. What ones do appear is very mysterious.

The **Weyl law** says

$$\dim\{f : \Delta f = \lambda f \;\text{ with }\; |\lambda| \le T\} \approx cT^?$$

(with some explicit exponent).

Remark 11.20. $\Gamma\backslash\mathbb{H}$ is non-compact. This was a source of complications. There exists some "arithmetic subgroups" $\Gamma \le \mathrm{SL}_2(\mathbb{R})$ such that $\Gamma\backslash\mathbb{H}$ *is* compact. The spectral decomposition in these cases is easier; it only has discrete parts: $L^2(\Gamma\backslash\mathbb{H}) = \bigoplus_{i\ge 0}\mathbb{C}f_i$ with $\Delta f_i = \lambda_i f_i$ and

$$\lambda_0 = 0 < \lambda_1 \le \lambda_2 \le \dots,$$

so the Eisenstein series are an effect of the non-compactness.

In the future, we'll try to incorporate/investigate these co-compact subgroups more.

12 Lecture 12 (3/31)

Note 8. *A little late again*

We will see that holomorphic modular forms and Maass wave forms both give examples of automorphic forms on $\mathrm{GL}_2(\mathbb{R})$ (to be defined). More generally, one can consider automorphic forms on $\mathrm{GL}_2(\mathbb{A})$ with $\mathbb{A}$ = ideles. What sort of properties/whatnot do these display?

I kinda jumped in in the middle of this, so I'm not sure what he's going for right now

- Analytic conditions (e.g. holomorphic or Laplacian eigenvector)
- "discrete" group theoretic conditions (e.g. automorphy for some $\Gamma \subset \mathrm{SL}_2(\mathbb{Z})$)

We want to put everything under one umbrella.

12.1 Automorphic forms on $\mathrm{GL}_2(\mathbb{R})$

Assumption. As usual, we make the simplifying assumption $\Gamma = \mathrm{SL}_2(\mathbb{Z})$.

Assumption. In the spirit of keeping things simple, let's only consider the connected component $G = \mathrm{GL}_2(\mathbb{R})^+$.

For the next part, we mostly follow chpt. 2, sect. 1–2 of Bump. We won't spend a ton of time on this chapter in this course.

Remark 12.1. We will need a bit of Lie theory. We don't want to assume too much, and we don't want to get sidetracked developing Lie theory, so we stick with the simplest case.

12.1.1 Crash Course on Lie Theory of GL_2

Fact. The *Lie algebra* $\mathfrak{g} = \mathfrak{gl}_2(\mathbb{R})$ of $G = \mathrm{GL}_2(\mathbb{R})^+$ is the set of 2×2 real matrices with Lie bracket $[X, Y] = XY - YX$ given by the commutator.

Note that $\mathfrak{g}$ acts (by differential operators) on any vector space on which G acts.

Example. Let $C^\infty(G)$ be the space of smooth functions on G. Then, $G \curvearrowright C^\infty(G)$ by right translations, called the **right regular action**. This induces a $\mathfrak{g}$-action. Given $X \in \mathfrak{g}$, one has

Could have used $L^2(G)$ instead, but we want to keep things easy

$$(X.f)(g) = \frac{\partial}{\partial t}\bigg|_{t=0} f\left(ge^{tX}\right).$$

Notation 12.2. We let $U(\mathfrak{g})$ denote the *universal enveloping algebra* of $\mathfrak{g}$.

Note that any $\mathfrak{g}$-action extends to an $U(\mathfrak{g})$-action. Concretely, this is just saying that you can compose differential operators, i.e. consider something like $(X_1X_2\ldots X_n)\cdot f$ (X_1X_2 does not refer to matrix multiplication).

Of particular important is the center $Z(\mathfrak{g}) := Z(U(\mathfrak{g}))$ of the universal envoloping algebra.

Fact. For $\mathfrak{g} = \mathfrak{gl}_2(\mathbb{R})$ as we have here, $Z(\mathfrak{g}) = \mathbb{C}[Z, \Delta]$, where

$$Z = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} \in \mathfrak{g} \text{ and } \Delta = \text{Casimir element}.$$

Note that Casimir elements exist for any reductive Lie algebra, and are built from a orthonormal basis of the Killing form (restricting to the semisimplification of $\mathfrak{g}$).

note that $\mathfrak{gl}_2(\mathbb{R}) = \mathbb{C}Z \oplus \mathfrak{sl}_2(\mathbb{R})$ where $Z = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}$ as above. Here $\mathfrak{sl}_2(\mathbb{R})$ consists of trace 0 matrices, and has a standard basis

$$h = \begin{pmatrix} 1 & \\ & -1 \end{pmatrix}, \quad e = \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix}, \quad \text{and } f = \begin{pmatrix} 0 & 0 \\ 1 & 0 \end{pmatrix}$$

which satisfies

$$[e, f] = h, \quad [h, e] = 2e, \quad \text{and } [h, f] = -2f.$$

In this case (up to scaling), the Killing form is simply $(X, Y) = \operatorname{Tr}(XY)$ (here, XY is matrix multiplication). If $e^*, f^*, h^* \in \mathfrak{sl}_2(\mathbb{R})$ is a dual basis w.r.t the Killing form ($e^* = f, f^* = e$, and $h^* = \frac{1}{2}h$), then the **Casimir element** is

$$\Delta = ee^* + ff^* + hh^* = ef + fe + \frac{1}{2}h^2.$$

Example. Let $K = \mathrm{SO}(2, \mathbb{R})$ be a maximal compact subgroup of $\mathrm{GL}_2(\mathbb{R})^+$.[25] Then, $\mathrm{GL}_2(\mathbb{R})^+/(\mathbb{R}_+^\times \cdot K) \xrightarrow{\sim} \mathcal{H}$ via $g \mapsto g \cdot \sqrt{-1}$. Here, $\operatorname{Stab}(\sqrt{-1}) = \mathbb{R}_+^\times \cdot K$.

Example. The Casimir element Δ gives rise to the Hyperbolic Laplacian $y^2\left(\left(\frac{\partial}{\partial x}\right)^2 + \left(\frac{\partial}{\partial y}\right)^2\right)$ (maybe up to scaling). By this we mean, under the natural 'restriction' map $C^\infty(\mathcal{H}) \to C^\infty(G)$ (recall $\Delta \curvearrowright C^\infty(G)$), the two actions line up.

We want to give a(n ad-hoc) definition of heights on $G = \mathrm{GL}_2(\mathbb{R})^+$. Note that we have a closed embedding $G \hookrightarrow \mathbb{R}^{4+1}$ via $g \mapsto (G, \det(g)^{-1})$. This is actually a closed embedding as varieties over $\mathbb{R}$.

[25] If $K \leq H$ with H a compact subgroup, then $K = H$. Maybe do something like take the H-average of the standard inner product on $\mathbb{R}^2$ to get H conjugate to $\mathrm{SO}(2, \mathbb{R})$?

There's a natural norm on $\mathbb{R}^5$, so we pull this back to define the **norm**

$$\begin{array}{rccc} \|\cdot\|: & \mathrm{GL}_2(\mathbb{R}) & \longrightarrow & \mathbb{R}_+ \\ & g & \longmapsto & \left(\sum |g_{ij}|^2 + (\det g)^{-2}\right)^{1/2} \end{array}$$

Fact. This norm is bi-K-invariant.

Note that this will never be 0 (in fact, it sounds like it is bounded away from 0). Furthermore, the set

$$\{g \in G : \|g\| \leq T\}$$

is compact for all $T > 0$.

12.1.2 Back to automorphic forms

As before, $G = \mathrm{GL}_2(\mathbb{R})^+$.

Definition 12.3. A smooth function $f : G \to \mathbb{C}$ is an **automorphic form** if it is

(0) (**central character**) The center $Z_G \cong \mathbb{R}_+^\times$ acts by a character $f(gz) = \omega(z)f(g)$ for all $z \in Z_G$ and $g \in G$.

For simplicity, we'll assume $\omega = 1$ is trivial.

(1) Γ-invariant

(2) K-finite, $Z(\mathfrak{g})$-finite

(3) of **moderate growth**, i.e. $|f(G)| = O\left(\|g\|^N\right)$ for some N.

What are these finiteness conditions in **(2)**?

Definition 12.4. We say f is K**-finite** if the set

$$\{k \cdot f \mid k \in \mathrm{SO}(2, \mathbb{R}) = K\}$$

generates a finite dimensional $\mathbb{C}$-vector space.

Example. If f is a K-eigenvector, i.e. $(k \cdot f)(g) = f(gk) = \chi(k)f(g)$ for some character $\chi : K \to \mathbb{C}^\times$, then f is K-finite.

Example. Working with $C^\infty(S^1)$. Consider $f = \sum_{n\in\mathbb{Z}} a_n e^{2\pi inx}$. Let $K = S^1$. Then, f is K-finite $\iff a_n = 0$ for all but finitely many n. Thus,

$$C^\infty(S^1)^{K\text{-fin}} = \bigoplus_{n\in\mathbb{Z}} \mathbb{C}e^{2\pi inx},$$

where above we take the algebraic (not Hilbert) direct sum. Hence, we see K-finiteness is an algebraic condition. We want to restrict to a subspace more amenable to purely algebraic techniques (w/o having to worry about topologies).

Note that $\mathrm{SO}(2,\mathbb{R}) \cong S^1$ is a torus, so we know all its characters. Write $\kappa_\theta := \begin{pmatrix} \cos\theta & \sin\theta \\ -\sin\theta & \cos\theta \end{pmatrix}$ where $\theta \in \mathbb{R}/2\pi\mathbb{Z}$. Then the characters of K are precisely

$$\chi_n(\kappa_\theta) = e^{2\pi i n\theta} \text{ for } n \in \mathbb{Z}.$$

Above, χ_n is the "**weight n character**."

$Z(\mathfrak{g})$-finite has the same sort of definition.

Recall 12.5. $Z(\mathfrak{g}) = \mathbb{C}[Z,\Delta]$ and the action of Z is determined by the central character (since the center of the Lie algebra $\mathfrak{g}$, $\mathfrak{z}(\mathfrak{g}) = \mathbb{C}Z$, is generated by Z).

Definition 12.6. We say f is **$Z(\mathfrak{g})$-finite** if there exists a nonzero polynomial $p(\Delta) \in \mathbb{C}[\Delta]$ such that $p(\Delta)f = 0$. Equivalently, there is a ideal $I \subset Z(\mathfrak{g})$ of finite codimension which kills f.

Definition 12.7. We'll say f is **$(K, Z(\mathfrak{g}))$-finite** if it is K-finite and $Z(\mathfrak{g})$-finite.

Example (Mass forms are automorphic)**.** Let $f : \Gamma\backslash\mathcal{H} \to \mathbb{C}$ be a Mass more. Consider its pullback

$$\widetilde{f} : G \twoheadrightarrow \Gamma\backslash G/K \cdot Z_G \xrightarrow{\sim} \Gamma\backslash\mathcal{H} \xrightarrow{f} \mathbb{C}.$$

Then, $\widetilde{f}$ is automorphic. It is visibly Γ-invariant. We see K acts trivial so it is K-finite. It is $Z(\mathfrak{g})$-finite since f is a Δ-eigenvector. Finally, it inherits moderate growth from f.

Notation 12.8. We let $\mathcal{A}(G,\Gamma)$ denote the space of automorphic forms on $G = \mathrm{GL}_2(\mathbb{R})^+$.

Embedding holomorphic forms into $\mathcal{A}(G,\Gamma)$ is a little more involved (have to translate holomorphic condition and the weight k condition), so we postpone this until next time.

12.2 Adelic version of automorphic forms

Recall that for modular forms, you have the ability to change the level of the form.

Let $F/\mathbb{Q}$ be some number field (one can make sense of the below for function fields too), and let $\mathbb{A} := \mathbb{A}_F$ be its adeles. Let G be any algebraic group of F (e.g. $G = \mathrm{GL}_1$ or $G = \mathrm{GL}_2$). Once can give $G(\mathbb{A})$ its adelic topology so that $G(\mathbb{A}) = \prod_v' G(F_v)$ (i.e. (g_v)'s s.t. $g_v \in G(\mathscr{O}_{F_v})$ for almost all places v), and then consider the quotient $G(F)\backslash G(\mathbb{A})$.

Recall 12.9. A *Hecke character* was a continuous homomorphism

$$\chi : \mathrm{GL}_1(F)\backslash \mathrm{GL}_1(\mathbb{A}) \to \mathbb{C}^\times.$$

Notation 12.10. We let K_∞ denote a maximal compact of $G(F_\infty)$, where $F_\infty = \prod_{v|\infty} F_v$.

Example. When $F = \mathbb{Q}$, $K_\infty = O(2,\mathbb{R})$.

Fact. $G(F) \hookrightarrow G(\mathbb{A})$ is a discrete subgroup

Definition 12.11. We say a continuous homomorphism $\varphi : G(\mathbb{A}) \to \mathbb{C}$ is an **automorphic form** if

(0) (**central character**)

$$\varphi(zg) = \omega(z)\varphi(g)$$

for all $z \in \mathbb{A}^\times$, the center of $G(\mathbb{A})$, where $\omega : F^\times \backslash \mathbb{A}^\times \to \mathbb{C}^\times$ is a Hecke character.

(1) it is $G(F)$-invariant (on the left)

(2) it is $(K_\infty, Z(\mathfrak{g}_\infty))$-finite, and is right-invariant under some compact open $K_f \subset G(\mathbb{A}_f)$. Here $\mathbb{A}_f = \prod'_{v\nmid\infty} F_v$ and $G(\mathbb{A}_f) = \prod'_{v<\infty} G(F_v)$.

This is the non-arch analogue of the finiteness conditions

(3) it is of **moderate growth**, i.e. for any $g_f \in G(\mathbb{A}_f)$

$$|\varphi(g_\infty g_f)| = O_{g_f}\left(\|g_\infty\|^N\right) \text{ for some } N$$

(N depending on g_f).

Notation 12.12. We let $\mathcal{A}(G,\omega)$ denote the space of all automorphic forms φ on $G(\mathbb{A})$ with central character ω. Note that

$$\mathcal{A}(G,\omega) = \bigcup_{K_f} \underbrace{\mathcal{A}(G,\omega,K_f)}_{\substack{K_f\text{-invariant}\\ \text{auto forms}}}$$

where K_f ranges over all compact opens in $G(\mathbb{A}_f)$.

Remark 12.13. If $\varphi \in \mathcal{A}(G,\omega,K_f)$ then it descends to a function

$$\varphi : G(F)\backslash G(\mathbb{A})/K_f \longrightarrow \mathbb{C}$$

Remark 12.14. "K_f-invariant for some compact open K_f" can equivalently be replaced by φ is finite under some choice of maximal open compact K_f°. A standard choice is

$$K_f^\circ = \prod_{v<\infty} \mathrm{GL}_2(\mathscr{O}_{F_v}).$$

Fact. Any two maximal compact opens are conjugate.

Say φ is K_f-invariant. Then, $K_f \subset K_f^\circ$ is finite index. This is because K_f°/K_f is compact (since K_f° is) and discrete (since K_f is open).

Can we relate automorphic forms on $G(\mathbb{A})$ with those on $G(\mathbb{R})$? That is, can we relate ${}_{G(F)}\backslash^{G(\mathbb{A})}/_{K_f}$ to ${}_{\Gamma}\backslash^{\mathrm{GL}_2(\mathbb{R})}$?

One day I'll figure out a nice-looking way to display double cosets

The key is strong approximation (more on this next time).

Strong approximation As before $F/\mathbb{Q}$ some number field. Recall we have an embedding $F \hookrightarrow \mathbb{A}_f$. One version of **strong approximation** says that $\mathrm{SL}_2(F) \hookrightarrow \mathrm{SL}_2(\mathbb{A}_f)$ is dense, i.e. for any open $K_f \subset \mathrm{SL}_2(\mathbb{A}_f)$, we have

$$\prod_v{}' \mathrm{SL}_2(F_v) = \mathrm{SL}_2(\mathbb{A}_f) = \mathrm{SL}_2(F)K_f.$$

We will give a proof of this next time. Here's how it relates to the question we asked before.

Application. Take $F = \mathbb{Q}$ and $K_f = \prod_{p<\infty} \mathrm{SL}_2(\mathbb{Z}_p)$. Then,

$$\mathrm{SL}_2(\mathbb{Z})\backslash \mathrm{SL}_2(\mathbb{R}) \xrightarrow{\sim} \mathrm{SL}_2(\mathbb{Q})\backslash \mathrm{SL}_2(\mathbb{A}_{\mathbb{Q}})/\prod_{p<\infty} \mathrm{SL}_2(\mathbb{Z}_p).$$

Question 12.15 (Audience)**.** *Which kinds of groups does strong approximation hold for?*

Answer. Simply-connected, semisimple algebraic groups G for which $G(F_\infty)$ is noncompact.

Question: $\pi_1^{\text{ét}}(G) = 1$?

Answer: No. A different notion of fundamental group based on isogenies instead.

Question: Over $\mathscr{O}_F$?

Assuming I heard correctly

13 Lecture 13 (4/5)

13.1 Strong approximation

Let F be a global field (actually, say it's a number field $F/\mathbb{Q}$). Let $\mathbb{A} = \mathbb{A}_F = \prod'_v F_v$ be its adeles, and let $\mathbb{A}_f = \prod'_{v\nmid\infty} F_v$ be the finite adeles. We want to show $\mathrm{SL}_2(F) \hookrightarrow \mathrm{SL}_2(\mathbb{A}_f)$ has dense image.

In fact, let's show something stronger. Fix a finite set S of places. Define

$$\mathbb{A}^S = \prod_{v\notin S}{}' F_v.$$

Theorem 13.1 (**Strong approximation**)**.** *Assume $S \neq \emptyset$. Then,*

(i) *Let $G = \mathrm{SL}_N$. Then, $G(F)$ is dense in $G(\mathbb{A}^S)$.*

(0) *Let $G = \mathbb{G}_a$ be the additive group. Then, $F = G(F)$ is dense in $G(\mathbb{A}^S) = \mathbb{A}^S$.*

Remark 13.2. There is also **weak approximation**: for S a finite set, the embedding $F \hookrightarrow \prod_{v\in S} F_v$ has dense image. If S consists solely of non-archimedean places, then this is basically Chinese Remainder Theorem.

In the proof of strong approximation, one can reduce **(i)** to **(0)**. This is based on the following linear algebraic lemma.

Lemma 13.3. *Fix a field k. Any $g \in \mathrm{SL}_N(k)$ can written as a product of **unipotent matrices**, where X is unipotent $\iff$ $(1-X)^m = 0$ for some m ($\iff$ after conjugating it's upper triangular with 1's along the diagonal).*

Question: Does this follow from $\exp : \mathfrak{sl}_N \to \mathrm{SL}_N$ having image generating SL_N?

There's apparently at least one proof of this which writes things as products of $(1+\lambda E_{ij})$ where E_{ij} is the elementary matrix with a single nonzero entry, a 1 in slot ij.

*Proof that **(0)** $\implies$ **(i)** in Strong approximation.* May assume $g = 1 + \lambda E_{ij}$ for some $\lambda \in \mathbb{A}^S$. **(0)** then says λ can be approximated by F-points. ∎

Remark 13.4. The linear algebra lemma says SL_N is generated by unipotent elements. Any group with such properties can have a version of strong approximation.

Non-example. $\mathbb{G}_m = \mathrm{GL}_1$ does not satisfy strong approximation. $F^\times \hookrightarrow \left(\mathbb{A}^S\right)^\times$ might not have dense image, e.g. if $S = \{\text{arch. places}\}$, then $\mathbb{A}_f^\times/\left(F^\times \cdot \prod_{v<\infty} \mathscr{O}_{F_v}^\times\right) \simeq \mathrm{Cl}_F$ which might not be trivial. If the image were trivial, this image would be trivial.

In fact, $F^\times \hookrightarrow \left(\mathbb{A}^S\right)^\times$ never has a dense image. This is because there's always some nontrivial ray class group.

What does a general version of strong approximation look like? Let G be a simple F-algebraic group (no proper (positive dimensional) normal F-algebraic subgroup). Also assume G is semisimple (trivial (geometric) radical, i.e. no solvable part).

Proposition 13.5. *Under the above hypotheses, G has strong approximation (i.e. $G(F) \hookrightarrow G(\mathbb{A}^S)$ has dense image for all finite $S \neq \emptyset$) $\iff$ $G(F_S)$ is noncompact and G is* ***simply connected****, i.e. any surjective isogeny $G' \twoheadrightarrow G$ (so finite kernel) of F-groups is an isomorphism.*[26]

Above, I think, $F_S = \prod_{v \in S} F_v$, and you want this noncompact for all choices of S?

Example. If S is the set of all arch. places, then $\prod_{v \in S} G(F_v)$ compact $\iff$ $G(F_v)$ compact for all $v \in S$.

Example. $G = \mathrm{SL}_N$ works ('type A').

Could also consider other types. $G = \mathrm{Sp}_{2n}$ ('type C') is simply connected. Orthogonal groups are not simply connected, but spin groups $G = \mathrm{Spin}_{2n+1}$ ('type B') and $G = \mathrm{Spin}_{2n}$ ('type D') are.

Example. Any forms/twists of above examples also work.

We still haven't proved strong approximation for the additive group. Let's give a proof in a simple case.

Proof of Strong Approximation for $\mathbb{G}_a$ when $S = \{$arch. places$\}$. We want to show $F \hookrightarrow \mathbb{A}_f$ has dense image. It suffices to show that for any compact open subgroup $U \subset \mathbb{A}_f$, $F + U = \mathbb{A}_f$. WLOG, we may assume

$$U = \prod_{v \in T} \varpi_v^{n_v} \mathscr{O}_{F_v} \times \prod_{v \notin T} \mathscr{O}_{F_v}$$

for some finite set T. In fact, it suffices to show this for $U = \prod_{v < \infty} \mathscr{O}_{F_v}$ (any compact open subgroup will contain aU for some $a \in F^\times$). Thus, we only need show the induced map

$$F \longrightarrow \mathbb{A}_f \Big/ \prod_{v \nmid \infty} \mathscr{O}_{F_v}$$

is surjective. In fact, we can reduce to case $F = \mathbb{Q}$ (since you apply $- \otimes_{\mathbb{Z}} \mathscr{O}_F$ to get from this case to the general case), i.e. we need show

$$\mathbb{Q} \longrightarrow \frac{\mathbb{A}_{\mathbb{Q},f}}{\prod_p \mathbb{Z}_p} = \bigoplus_{p \nmid \infty} \frac{\mathbb{Q}_p}{\mathbb{Z}_p}$$

is surjective. This is the case because it's the natural quotient map

$$\mathbb{Q} \twoheadrightarrow \mathbb{Q}/\mathbb{Z} \xrightarrow{\sim} \bigoplus \mathbb{Q}_p/\mathbb{Z}_p.$$

∎

[26] This implies $\pi_1(G(\mathbb{C})) = 0$, and it sounds like this is actually equivalent to $\pi_1(G(\mathbb{C})) = 0$.

If S does not contain all archimedean places, use that $\mathscr{O}_F \hookrightarrow \prod_{v|\infty} F_v$ is a (full rank) lattice, so $\mathscr{O}_F \to \prod_{\substack{v|\infty \\ v\neq v_0}} F_v$ has dense image after removing any single place.

If S has no archimedean places, use/show that $\mathscr{O}_F[1/S]$ is dense in $F_\infty = \prod_{v|\infty} F_v$. For example, $\mathbb{Z}[1/p] \hookrightarrow \mathbb{R}$ has dense image.

■

Corollary 13.6. $\pi_0(\mathrm{SL}_N(F)\backslash \mathrm{SL}_N(\mathbb{A}_f)/K_f) = 0$ *for any compact open* $K_f \subset \mathrm{SL}_N(\mathbb{A}_f)$.

This is because, this quotient is $(K \cap \mathrm{SL}_N(F))\backslash \mathrm{SL}_N(F_\infty)$ which is the quotient of a connected topological space.

Example. Let B be a quaternion algebra over $\mathbb{Q}$. In fact, let $B = \mathbb{H}$ be the Hamilton quaternions (over $\mathbb{Q}$).[27] Note that $B \otimes_{\mathbb{Q}} \overline{\mathbb{Q}} \simeq M_{2\times 2}$ (i.e. B is a form of 2×2 matrices). Let $\widetilde{G} = B^\times$ as $\mathbb{Q}$-algebraic groups, so $\widetilde{G}$ is a form of GL_2. In fact it's a so-called "inner form of GL_2." Here $\widetilde{G}(\mathbb{R}) = B_{\mathbb{R}}^\times$. Note there is a "determinant/reduced norm" map $\widetilde{G} \xrightarrow[\mathrm{Nm}]{\det} \mathbb{G}_m$. This is surjective, so let G be its kernel, i.e.

I'm not convinced I'll ever know the actual definition of a quaternion algebra

Answer: It's a central simple algebra of dimension 4

$$1 \longrightarrow G \longrightarrow \widetilde{G} \longrightarrow \mathbb{G}_m \longrightarrow 1.$$

Then, $G \otimes \overline{\mathbb{Q}} \simeq \mathrm{SL}_2$, so G is an inner form of SL_2. Note that $G(\mathbb{R}) \simeq \mathrm{SU}(2)$ (the unit sphere in the quaternions over $\mathbb{R}$) is compact. This G we denote $\mathrm{SL}(B) := G$.

In this case, strong approximation still applies with extra conditions. If you choose S s.t. $G(F_S)$ is non-compact, then $G(F) \hookrightarrow G(\mathbb{A}^S)$ has dense image.

If B is a quaternion algebra over F, can perform an analogous construction to get all "inner forms" of GL_2. More generally, if you have an n^2-dimensional central simple algebra over F, this construction will give all inner forms of GL_n. In any case, you set $G = B^\times$ and then $G \otimes \overline{F} \simeq \mathrm{GL}_{n,\overline{F}}$. One can use this to define a notion of automorphic form $\varphi : G(F)\backslash G(\mathbb{A}) \to \mathbb{C}$.

Fact. If B is 'division at one place,' i.e. $B \otimes_F F_v$ is a division algebra, then $G(F)\backslash G(\mathbb{A})$ is compact where now $G = \mathrm{SL}(B)$.

Recall 13.7. $\mathrm{SL}_2(\mathbb{Z})\backslash \mathrm{SL}_2(\mathbb{R}) \simeq \mathrm{SL}_2(\mathbb{Q})\backslash \mathrm{SL}_2(\mathbb{A}_{\mathbb{Q}})/\prod_{p<\infty} \mathrm{SL}_2(\mathbb{Z}_p)$ (by strong approximation) is non-compact. Furthermore, if we divide by $K_\infty = \mathrm{SO}(2,\mathbb{R})$, then $\Gamma\backslash \mathrm{SL}_2(\mathbb{R})/K_\infty \simeq \Gamma\backslash\mathcal{H}$ is still non-compact, but can be compactified by adding a single cusp point. This non-compactness complexifies things, e.g. it caused the spectral decomposition of the L^2 space to have a continuous part in addition to its discrete part.

Compare the above to the following situation:

Fact (from global cft)**.**

$$\left\{\begin{matrix}\text{quaternion} \\ \text{alg. } /F\end{matrix}\right\}_{\simeq} \longleftrightarrow \left\{\Sigma \,\middle|\, \begin{matrix}\text{finite set of places w/} \\ \#\Sigma = \text{even}\end{matrix}\right\}.$$

Above $B \mapsto \Sigma_B = \{v : B_v \text{ division}\}$.

[27] $\mathbb{H} = \mathbb{Q} \oplus \mathbb{Q}i \oplus \mathbb{Q}j \oplus \mathbb{Q}k$ where $i^2 = j^2 = k^2 = ijk = -1$.

Example. Hamilton quaternions correspond to $\{\infty, 2\}$. Given two non-archimidean primes p, q, get some corresponding quaternion algebra $B = B(p, q)$.

Now, strong approximate applies to show that

$$\mathrm{SL}_2(B)(\mathbb{Q})\backslash \mathrm{SL}(B)(\mathbb{A})/K_f \times K_\infty \simeq \Gamma\backslash \mathrm{SL}(B)(\mathbb{R})/K_\infty \simeq \Gamma\backslash\mathcal{H},$$

where $\Gamma = K_f \cap \mathrm{SL}(B)(\mathbb{Q})$ for $K_f \subset \mathbb{A}_f$ any compact open. Now, Γ is co-compact (i.e. the above 3 equivalent spaces are compact) and the L^2-decomposition is purely discrete:

$$L^2(\Gamma\backslash\mathcal{H}) \simeq \widehat{\bigoplus_{\varphi_i}} \mathbb{C}\varphi_i$$

with sum taken over Δ-eigenfunctions.

One of the things we'll talk about in the future is the following.

Goal (**Jacquet-Langlands correspondence**). Each φ_i as above can be transformed into an eigenfunction φ for SL_2.

Automorphic forms attached to different quaternion algebras are not all that different.

14 Lecture 14 (4/7): $(\mathfrak{g}, K)$-modules; spectral decomp revisited

We defined automorphic forms over real numbers and over adeles.

Let $G = \mathrm{GL}_2(\mathbb{R})^+$ (or $\mathrm{SL}_2(\mathbb{R})$). We can consider $L^2(\Gamma\backslash G)$ or just the automorphic forms $\mathcal{A}(\Gamma\backslash G)$; these are actually not in L^2, but we set

$$\mathcal{A}^2(\Gamma\backslash G) := L^2(\Gamma\backslash G) \cap \mathcal{A}(\Gamma\backslash G).$$

Note that this contains cuspidal forms (but e.g. not Eisenstein series). Let $\mathcal{A}_0(\Gamma\backslash G) \subset \mathcal{A}^2(\Gamma\backslash G)$ denote the subspace of **cuspidal automorphic forms**, i.e. those automorphic $f : \Gamma\backslash G \to \mathbb{C}$ such that

$$\mathrm{const}(f)(g) := \int_{\Gamma_\infty\backslash N(\mathbb{R})} f(ng)\mathrm{d}n = 0 \text{ for all } g$$

(i.e. the **constant term** vanishes). Here, $\Gamma_\infty = \left\{\begin{pmatrix} 1 & n \\ 0 & 1 \end{pmatrix} \in \Gamma = \mathrm{SL}_2(\mathbb{Z}) : n \in \mathbb{Z}\right\}$, and N *is the unipotent radical of the Borel subgroup*

$$N = \left\{\begin{pmatrix} 1 & * \\ 0 & 1 \end{pmatrix}\right\} \subset G.$$

If $G = \mathrm{GL}_2(F)$ for a number field F, we say $f : G(F)\backslash G(\mathbb{A}) \to \mathbb{C}$ is a **cuspidal automorphic form** if

$$\mathrm{const}(f)(g) := \int_{N(F)\backslash N(\mathbb{A})} f(ng)\mathrm{d}n = 0 \text{ for all } g.$$

Remark 14.1. Why do we call this thing the 'constant term'? Say we have $f : N(F)\backslash G(\mathbb{A}) \to \mathbb{C}$. Consider

$$\begin{array}{rccc} \varphi_{f,g} : & N(F)\backslash N(\mathbb{A}) & \longrightarrow & \mathbb{C} \\ & n & \longmapsto & f(ng) \end{array}$$

Above, $N(F)\backslash N(\mathbb{A})$ is abelian and compact, so it has a Fourier expansion

$$\varphi_{f,g}(n) = \sum_{\xi \in F} \widehat{\varphi}_{f,g}(-)\psi_\xi(-)$$

(Pontryagin dual of $F\backslash\mathbb{A}$ is F). The 'constant term' is precisely the coefficient associated to the trivial character (i.e. $\xi = 0 \in F$). That is

$$\mathrm{const}(f) = \widehat{\varphi_{f,g}}(0) = \int_{N(F)\backslash N(\mathbb{A})} \varphi_{f,g}(n)\mathrm{d}n = \int_{N(F)\backslash N(\mathbb{A})} f(ng)\mathrm{d}n.$$

Remark 14.2. Say $G = \mathrm{GL}_2(\mathbb{R})^+$ (or $\mathrm{SL}_2(\mathbb{R})$) and $K = \mathrm{SO}(2)$. We have embeddings

$$\begin{array}{ccccc} L^2(\Gamma\backslash G) & \overset{\text{dense}}{\supset} & \mathcal{A}^2(\Gamma\backslash G) & \subset & \mathcal{A}(\Gamma\backslash G) \\ \cup & & \cup & & \\ L^2_0(\Gamma\backslash G) & \overset{\text{dense}}{\supset} & \mathcal{A}_0(\Gamma\backslash G) & & \end{array}$$

Note that $L^2(\Gamma\backslash G)$ and $L^2_0(\Gamma\backslash G)$ are Hilbert spaces. The spaces of automorphic forms though have no topology; they are just $\mathbb{C}$-vector spaces. On the LHS, there is a G-action. On the RHS there is no G-action, but there is an action by the Lie algebra $\mathfrak{g}_{\mathbb{C}} = \mathrm{Lie}\, G \otimes_{\mathbb{R}} \mathbb{C}$. There's also (on the RHS) an action by the maximal compact K.

14.1 $(\mathfrak{g}, K)$-modules

Definition 14.3. A $(\mathfrak{g}, K)$**-module** (really should be $\mathfrak{g}_{\mathbb{C}}$) is a $\mathbb{C}$-vector space V along with representations $\pi_{\mathfrak{g}} : \mathfrak{g} \to \mathrm{End}(V)$ (a Lie algebra homomorphism) and $\pi_K : K \to \mathrm{GL}(V)$ (a group homomorphism) such that

(a) K-action is **locally finite**, i.e. any $v \in V$ is K-finite. Let $V(v) = \mathbb{C}K \cdot v$ be the (f.dim) vector space generated by v under the K-action. We require that $K \curvearrowright V(v)$ be a **smooth** action.

(b) Let $\mathfrak{k} := \mathrm{Lie}(K)$. The induced action $\mathrm{d}\pi_K : \mathfrak{k} \to \mathrm{End}(V)$ must be the restriction $\pi_{\mathfrak{g}}|_{\mathfrak{k}}$.

(c) We have the following compatibility condition: for any $X \in \mathfrak{g}$ and $k \in K$,

$$\pi_K(k)\pi_{\mathfrak{g}}(X)\pi_K(k)^{-1} = \pi_{\mathfrak{g}}(\mathrm{Ad}(k)X).$$

Example. Say G acts on some separable (so countable basis) Hilbert space $\mathbf{V}$. Can consider the subspace $\mathbf{V}^\infty$ of **smooth vectors**, i.e. those v for which $\varphi_{v'} : G \to \mathbb{C}, g \mapsto (gv, v')$ is smooth for all $v' \in \mathbf{V}$. Inside $\mathbf{V}^\infty$ is $V := \mathbf{V}_{K\text{-fin}}$ consisting of the K-finite vectors. If $\mathbf{V}$ is an **admissible** G**-rep** in the sense that $\mathbf{V}|_K = \widehat{\bigoplus}_{\sigma_i \in \mathrm{Irr}(K)} \sigma_i^{m_i}$ with $m_i < \infty$, then V will be a $(\mathfrak{g}, K)$-module.

Definition 14.4. A $(\mathfrak{g}, K)$-module V is **admissible** if

$$V|_K \simeq \bigoplus_{\sigma_i \in \mathrm{Irr}(K)} \sigma_i^{m_i}$$

(algebraic direct sum) with $m_i < \infty$.

A natural question is to try and classify all irreducible $(\mathfrak{g}, K)$-modules. Bump's book (section 2.5) does this for $\mathfrak{g} = \mathfrak{sl}_2$ and $K = \mathrm{SO}(2)$; in this case, one is in luck since K is abelian (so all its irreps are 1-dim/characters). In this case, $V = \bigoplus_{k\in\mathbb{Z}} V_k$ has a '*weight decomposition*' for the K-action (here, $\dim V_k \le 1$).

I'm not sure where we're going, but coming up next...

Recall 14.5. Recall vectors $f \in \mathcal{A}(\Gamma\backslash G)$ are required to be K-finite and finite under $Z(U(\mathfrak{g}_{\mathbb{C}}))$.

Theorem 14.6.

(1) *For any $f \in \mathcal{A}(\Gamma\backslash G)$, $U(\mathfrak{g}_{\mathbb{C}}) \cdot f$ is an admissible $(\mathfrak{g}, K)$-module.*

(2) *Fix a finite codimensional ideal $I \subset Z(\mathfrak{g}_{\mathbb{C}})$ and an irrep $\sigma \in \mathrm{Irr}(K)$. Then,*

$$\mathcal{A}(\Gamma\backslash G, (I, \sigma)) := \{f \in \mathcal{A}(\Gamma\backslash G) : I.f = 0 \ \textit{ and } \ \mathbb{C}K \cdot f \simeq \sigma^m \ \textit{ as } \ K\textit{-reps}\}$$

is finite-dimensional.

Example. Say $\sigma = \mathbf{1}$ is the trivial rep, and $I = (\Delta - \lambda) \subset Z(\mathfrak{g}_{\mathbb{C}})$. Then, **(2)** says that $\mathcal{A}(\Gamma\backslash G/K)^{\Delta=\lambda}$ is finite-dimensional (i.e. $\mathcal{A}(\Gamma\backslash G/K)$ has f.dim Δ-eigenspaces). Compare with finite dimensionality of spaces of Maass forms.

Say $G = \mathrm{SL}_2$ (and $K = \mathrm{SO}(2)$). For Maass forms, we have $\mathcal{A}(\Gamma\backslash G/K)^{\Delta=\lambda} \subset \mathcal{A}(\Gamma\backslash G)$ and the larger space is a $(\mathfrak{g}, K)$-module. Given $f \in \mathcal{A}(\Gamma\backslash G/K)^{\Delta=\lambda}$, it turns out that $U(\mathfrak{g}_{\mathbb{C}}) \cdot f$ is an irreducible $(\mathfrak{g}, K)$-module π_λ. One can show that, as a K-rep, one has $\pi_\lambda|_K = \bigoplus_{i\equiv 0 \pmod 2} \pi_\lambda(i)$ with $\dim \pi_\lambda(i) = 1$.

What about for holomorphic modular forms of weight k? By one of the homework problems, this is identified with $\mathcal{A}(\Gamma\backslash G)^{\text{weight } k, L=0}$ where $L \in U(\mathfrak{g}_{\mathbb{C}})$ is some differential operator ('weight lowering differential operator'). If you take some f in here, then $U(\mathfrak{g}_{\mathbb{C}})f$ is again an irreducible $(\mathfrak{g}, K)$-module π_k, but one of a different flavor. Again $\Delta \curvearrowright \pi_k$ by a scalar; this time the scalar is $\frac{k}{2}\left(\frac{k}{2} - 1\right)$ (or something like this), and we get a decomposition of K-reps (assume k even)

$$\pi_k = \bigoplus_{\substack{i\equiv 0 \pmod 2 \\ i<2-k \text{ or } i>k-2}} \pi_k(i).$$

Again, $\dim \pi_k(i) \le 1$.

Note 9. Got distracted and missed a little.

Remark 14.7. Non-trivial analysis goes into proving some of the claims we made, but we want to keep the discussion as algebraic as we can. We also don't want to get distracted, so that we have time later to talk more about automorphic representations and applications to L-functions.

Remark 14.8. One can study automorphic forms over function fields (e.g. $F = \mathbb{F}_p(t)$). In this case, there's no analysis or Lie groups, and things are often easier.

14.2 More adelic situation (?)

Let $F/\mathbb{Q}$ be a number field with adeles $\mathbb{A} = \mathbb{A}_F$. Let $G = \mathrm{GL}_2$ and consider $G(\mathbb{A})$. Fix a central character $\omega : \mathbb{A}^\times/F^\times \to \mathbb{C}^\times$. We have defined $\mathcal{A}(G(F)\backslash G(\mathbb{A}), \omega)$ previously. This space is huge, but we can write

$$\mathcal{A}(G(F)\backslash G(\mathbb{A}), \omega) = \bigcup_{K_f \subset G(\mathbb{A}_f)} \mathcal{A}\left(G(F)\backslash G(\mathbb{A})/K_f, \omega\right)$$

where $K_f \subset G(\mathbb{A}_f)$ runs over open, compacts.

Remark 14.9. By strong approximation, $\mathcal{A}(G(F)\backslash G(\mathbb{A})/K_f, \omega)$ is basically $\mathcal{A}(\Gamma\backslash G(\mathbb{R}), \omega)$. We saw this for SL_2 at the end of Lecture 12. For GL_2, use the exact sequence

$$1 \longrightarrow \mathrm{SL}_2 \longrightarrow \mathrm{GL}_2 \xrightarrow{\det} \mathbb{G}_m \longrightarrow 1.$$

By strong approximation,

$$\mathrm{GL}_2(F)\backslash \mathrm{GL}_2(\mathbb{A}_f)/K \overset{\det}{\xrightarrow{\sim}} F^\times \mathbb{A}_f^\times / \det K_f =: I$$

and the RHS (MHS?) is finite. Thus, in actuality,

$$\mathcal{A}(G(F)\backslash G(\mathbb{A})/K_f, \omega) = \bigoplus_{i \in I} \mathcal{A}(\Gamma_i\backslash G(\mathbb{R}), \omega)$$

(I think there's a literal iso as above).

Note that $\mathcal{A}(G(F)\backslash G(\mathbb{A}), \omega)$ is a $G(\mathbb{A}_f) \times (\mathfrak{g}, K)$-module (a $(\mathfrak{g}, K)$-module and has an action by $G(\mathbb{A}_f)$ (and these actions commute I guess?)); this bigger space has the advantage of having more symmetries, and we'll see more concretely how this is helpful in the future.

Definition 14.10 (Quite possibly the most important one)**.** Write $[G] = G(F)\backslash G(\mathbb{A})$, and call this the **automorphic quotient**.

Can look at $L^2([G], \omega)$. This is a Hilbert space with a $G(\mathbb{A})$-action. Recall $\mathcal{A}_0([G], \omega) \subset L^2_0([G], \omega)$.

Theorem 14.11 (**spectral decomposition**)**.** *Let's just state the result for the cuspidal part.*

$$L^2_0([G], \omega) = \widehat{\bigoplus}_{\pi \in \mathrm{Irr}(G(\mathbb{A}))} \pi^{m(\pi)},$$

*where $m(\pi) \leq 1$ (**multiplicity one**). Similarly,*

$$\mathcal{A}_0([G], \omega) = \bigoplus_{\substack{\pi \text{ irr. adm.} \\ (\mathfrak{g}, K) \times G(\mathbb{A}_f)\text{-module}}} \pi^{m(\pi)}.$$

In particular, the spectrum (for the cuspidal part) is discrete.

There's a version for the full L^2-space which involves Eisenstein series.

Recall 14.12. Let $F = \mathbb{Q}$, $K_f = \prod_{p<\infty} G(\mathbb{Z}_p)$, and $K = K_\infty \times K_f$. Taking K-invariants,

$$\bigoplus_{\varphi_i \ \Delta\text{-Eigenform}} \mathbb{C}\varphi_i = L_0^2(\Gamma\backslash\mathcal{H}) \simeq L_0^2([G],\omega)^K.$$

Above decomposition should match up with

$$L_0^2([G],\omega)^K = \left(\widehat{\bigoplus}_{\pi\in\operatorname{Irr}(G(\mathbb{A}))} \pi^{m(\pi)}\right)^K.$$

Let $\mathbb{T} = \mathbb{C}[T_n : n \geq 1]$ be the Hecke algebra. If I heard correctly, φ_i should be a $(\Delta, \mathbb{T})$-eigenform. Wei said a little more about this, but I did not follow... sounds like we define a **Hecke algebra** $\mathbb{T}_{K_f} := C_c^\infty(G(\mathbb{A}_f)//K_f)$.

We'll say more about this Hecke stuff another time. We'd like to prove (a simple version of) the spectral decomposition at some point. We'd like to discuss Selberg trace formula at some point.

15 Lecture 15 (4/12): A bit on the proof of the spectral decomposition

We state the spectral decomposition last time (at least the discrete part).

Corollary 15.1 (*spectral decomposition for cuspidal part*)**.**

$$L_0^2([G],\omega) = \widehat{\bigoplus}_{\pi\in\operatorname{Irr}(G(\mathbb{A}))} \pi^{m(\pi)},$$

where $m(\pi) \leq 1$. Similarly,

$$\mathcal{A}_0([G],\omega) = \bigoplus_{\substack{\pi \text{ irr. adm.} \\ (\mathfrak{g},K)\times G(\mathbb{A}_f)\text{-module}}} \pi^{m(\pi)}$$

(algebraic direct sum above).

First version more topological while second more algebraic, but the two are related. Taking K-finite vectors of first version will more-or-less give second version.

We would like to say something of the proof. The whole thing is long/complicated, so we won't go through it all, but we can at least prove a simplified version.

15.1 Proof of Simplified Spectral Decomposition

Instead of working with the adeles, let's just do something over $\mathbb{R}$. Note that in the statement for $\mathrm{GL}_2(\mathbb{A})$, we got 'multiplicity one' ($m(\pi) \leq 1$). It sounds like you don't get multiplicity one if you use $\mathrm{SL}_2(\mathbb{A})$ instead, but that this difficulty disappears when working with $\mathbb{R}$ instead of all of $\mathbb{A}$ (in the sense that you don't get multiplicity one in either case).

Let $G = \mathrm{SL}_2(\mathbb{R})$, and let $\Gamma \subset G$ be a cocompact, discrete subgroup (e.g. coming from a quaternion division algebra over $\mathbb{Q}$). Let $X = \Gamma\backslash G$ which is compact.

Non-example. $\Gamma = \mathrm{SL}_2(\mathbb{Z})$ is not cocompact, $\mathrm{SL}_2(\mathbb{Z})\backslash\mathrm{SL}_2(\mathbb{R})$ is not compact (e.g. since it has $\mathbb{C} = \mathrm{SL}_2(\mathbb{Z})\backslash\mathrm{SL}_2(\mathbb{R})/\mathrm{SO}(2)$ as a quotient).

Note that $G \curvearrowright X$ and so on $L^2(X)$. We will prove the following.

Theorem 15.2 (**Simplified Spectral Decomposition**).

$$L^2(X) = \widehat{\bigoplus_\pi} \pi^{m(\pi)},$$

where π ranges over irreducible representations of $G = \mathrm{SL}_2(\mathbb{R})$, and $0 \le m(\pi) < \infty$ always.

(Note there's no multiplicity one here).

Recall 15.3. Say H is a separable Hilbert space (so has a countable basis). We speak of **continuous G-actions** $G \curvearrowright H$ in the sense that $\pi : G \times H \to H$ is continuous. Equivalently, for any $v \in H$, the function $G \to H, g \mapsto \pi(g, v)$ is continuous[28]. Equivalently, for any $u, v \in H$ the **matrix coefficient**

Question: Can G be any topological group here?

$$\begin{array}{rccc} \Phi_{uv}: & G & \longrightarrow & \mathbb{C} \\ & g & \longmapsto & \langle gu, v\rangle \end{array}$$

is continuous.

Recall 15.4. Let H be a (separable) Hilbert space. An operator $T : H \to H$ is called **compact** if it sends the unit ball to a (relative) compact in H, i.e. the closure of the image of the unit ball is compact.

Example. Any **bounded** (i.e. image of unit ball is bounded) **finite rank operator** (i.e. $\dim \mathrm{im}(T) < \infty$) is compact.

More generally, if T_n is a sequence of finite rank operators, and $T_n \to T$ (in the strong/norm topology, i.e.[29] $\|T_n - T\| \to 0$), then T is compact.

Example (**Hilbert-Schmidt Operator**). Write $H = \widehat{\bigoplus}_{i \ge 1} \mathbb{C} e_i$. If $T \in \mathrm{End}(H)$ with

$$\sum_{i,j} |\langle Te_i, e_j\rangle|^2 < \infty$$

(T an infinite matrix with square-integrable entries), then T is compact. Equivalently, if $\{e_i\}$ form an orthonormal basis, then this says

$$\|T\|_{HS}^2 := \sum_i \|Te_i\|_H^2 < \infty.$$

Call $\|T\|_{HS}$ (the square root of the above sum) the **Hilbert-Schmidt norm**.

Why do we care about this compact operators?

Fact (**spectral theory of compact operators**). Say $T : H \to H$ is a **self-adjoint** (i.e. $\langle Tu, v\rangle = \langle u, Tv\rangle$ for all $u, v \in H$), compact operator. Then, H has a decomposition

$$H = \bigoplus_{\lambda \in \mathbb{R}} H_\lambda$$

into eigenspaces s.t. $\lambda \neq 0 \implies \dim H_\lambda < \infty$.

Say we have $G \curvearrowright H$ and H some Hilbert space. We want to decompose this G-action, so we'd really like some compact operator. We know the fact cannot apply to $g \in G$ it's unitary (hence only eigenvalues

Part of definition of acting on a Hilbert space, I believe

[28] This is equivalent using uniform boundedness

[29] Recall that the norm of a bounded operator T is $\|T\| = \sup_{x \in B_1} \|T(x)\|_H$

of $|\cdot|=1$, so only eigenvalues ± 1 if self-adjoint so f.dim eigenspaces). The key is to consider $C_c^\infty(G)$ with the convolution product. This algebra has no unit, but it does have an **approximation of a unit** in the sense that there are functions "$\varphi_n \to \delta_1$" (we'll make sense of this later, see Lemma 15.12).

Construction 15.5. Given $\varphi \in C_c^\infty(G)$, we may define

$$\pi(\varphi) := \int_G \varphi(g)\pi(g)\mathrm{d}g \in \mathrm{End}(H)$$

Concretely, given $v \in H$,

$$\pi(\varphi)v = \int_G \varphi(g)(\pi(g)v)\mathrm{d}g \in H$$

(convergent since φ has compact support).

These will be compact, and even Hilbert-Schmidt, operators.

Example. Recall $X = \Gamma\backslash G$. Say we have $f \in L^2(X)$ and $\varphi \in C_c^\infty(G)$. Given $x = \Gamma h \in X$ $(h \in G)$, we have

$$(\pi(\varphi)f)(x) = \int_G \varphi(g)f(xg)\mathrm{d}g = \int_G \varphi(h^{-1}g)f(\Gamma g)\mathrm{d}g = \int_{\Gamma\backslash G}\left(\sum_{\gamma\in\Gamma}\varphi(h^{-1}\gamma g)\right)f(\Gamma g)\mathrm{d}g.$$

At this point, we see that this operator attached to φ is actually an 'integral operator' in the sense that we may write

$$(\pi(\varphi)f)(x) = \int_X K_\varphi(x,y)f(y)\mathrm{d}y \text{ where } K_\varphi(x,y) = \sum_{\gamma\in\Gamma}\varphi(h^{-1}\gamma g) \text{ if } x = \Gamma h \text{ and } y = \Gamma g.$$

Note that $K_\varphi(x,y)$ above is (left) Γ-invariant separately in each variable. We say K_φ is a **kernel function** attached to $\varphi \in C_c^\infty(G)$.

Lemma 15.6. *$\pi(\varphi)$ is Hilbert-Schmidt.*

Proof. By the most recent example, we see that

$$\pi(\varphi)f(x) = \langle f, K_\varphi(x,-)\rangle_X\,.$$

We want to show that $\|\pi(\varphi)\|_{HS} < \infty$. One can compute that

$$\|\pi(\varphi)\|_{HS}^2 = \sum_i \|\pi(\varphi)e_i\| = \int_{X\times X}|K_\varphi(x,y)|^2\,\mathrm{d}x\mathrm{d}y = \|K_\varphi(-,-)\|_{L^2(X\times X)}.$$

This is finite since $K_\varphi(-,-)$ is continuous[30] on the compact space $X \times X$, so bounded. ■

Note 10. Our treatment here is based off of section 2.3 of Bump's book, but hopefully a little more streamlined (Bump wants to unify the proofs of the L^2 and automorphic versions).

We're starting to run of out time, so we'll try to pick up the pace a bit.

[30]This should follow from φ being continuous with compact support

Notation 15.7. π is a little overloaded, so we switch up notation. From now one, let

$$R(\varphi) := \int_G \varphi(g)\pi(g)\mathrm{d}g \in \mathrm{End}(H)$$

denote the action of $C_c^\infty(G)$ we've been talking about.

Here's a natural question: what is the adjoint of $R(\varphi)$?

Proposition 15.8. *$R(\varphi)^* = R(\varphi^*)$ where $\varphi^*(g) := \overline{\varphi}(g^{-1})$, so * gives an involution on $C_c^\infty(G) =: C$.*

We also want a notion of trace.

Definition 15.9. Let $T \in \mathrm{End}(H)$ and let $\{e_i\}$ be an orthonormal basis of H. Then, the **trace** of T is

$$\mathrm{trace}(T) = \mathrm{Tr}(T) = \sum_i \langle Te_i, e_i\rangle$$

(when this converges).

Remark 15.10. If $T = AB$ with A, B both Hilbert-Schmidt, then $\mathrm{Tr}(T)$ converges absolutely. Indeed,

$$\begin{aligned}
\sum_i |\langle Te_i, e_i\rangle| &= \sum_i |\langle ABe_i, e_i\rangle| \\
&= \sum_i |\langle Be_i, A^*e_i\rangle| \\
&\le \sum_i \left(\|Be_i\|^2 + \|Ae_i\|^2\right) \\
&\le \|B\|_{HS}^2 + \|A^*\|_{HS}^2 \\
&< \infty
\end{aligned}$$

Let's remind ourselves of what we're after. We have a representation $G \curvearrowright H = L^2(X)$ inducing an action of $C = C_c^\infty(G)$ by HS operators. We want to decompose H into irreps with finite multiplicities. We claim that these properties are all that is needed.

Theorem 15.11. *Let H be a G-rep such that $C = C_c^\infty(G)$ acts by Hilbert-Schmidt operators (e.g. $H = L^2(X)$). Then,*

$$H = \widehat{\bigoplus_\pi} \pi^{m(\pi)}$$

with π ranging over irreducible G-reps, and $m(\pi) < \infty$.

In fact, C above does not necessarily have to be $C_c^\infty(G)$.

Lemma 15.12 (**Approximation Lemma**, Bump Lemma 2.3.2)**.** *Fix a nonzero vector $v \in H$. For any $\varepsilon > 0$, there exists a self-adjoint function $\varphi \in C$ (i.e. $\varphi^* = \varphi$) such that $\|R(\varphi)v - v\| < \varepsilon$.*

Proof of Theorem 15.11. We want to do induction. We need to show that any nonzero such H has a (nontrivial) irreducible G-invariant closed subspace.

Question: Does this really mean apply Zorn's lemma?

Fix some nonzero $v \in H$. By the approximation lemma, there exists $\varphi = \varphi^* \in C$ such that $R(\varphi) \neq 0$. Then we can apply spectral theory of self-adjoint, compact operators to write $H = \bigoplus$(eigenspaces of $R(\varphi)$). Let $\lambda_0 \neq 0$ be some eigenvalue of $R(\varphi)$, so $\dim H_{\lambda_0} < \infty$.

Let $V \subset H$ be a G-invariant subspace such that $V_{\lambda_0} \neq 0$ and $\dim V_{\lambda_0}$ minimal (among G-invariant subspaces W with $W_{\lambda_0} \neq 0$). Choose some nonzero $v_0 \in V_{\lambda_0}$, and let W be the closed subspace (subrep?) generated by v_0. We claim W is in fact irreducible (even as a C-module). We have $V = W \oplus W'$, so also $V_{\lambda_0} = W_{\lambda_0} \oplus W'_{\lambda_0}$. By minimality, $W_{\lambda_0} = 0$ or $W'_{\lambda_0} = 0$; by construction, $W_{\lambda_0} \neq 0$; hence, $V_{\lambda_0} = W_{\lambda_0}$. Hence, we may take $V = W$ (i.e. we may assume V is generated by a single element as a C-rep).

Now, suppose $W = W_1 \oplus W_2$. Again, take eigenspaces: $W_{\lambda_0} = W_{1,\lambda_0} \oplus W_{2,\lambda_0}$. Again, by minimality, we have $W_{\lambda_0} = W_{1,\lambda_0}$ (swap W_1, W_2 if it's the other way), but W is generated by $v_0 \in W_{\lambda_0} = W_{1,\lambda_0}$, so we must have $W_1 \supset W$. Thus, $W = W_1$, so W is irreducible.

Remember: Our representations are unitary

This suffices to conclude the existence of a decomposition

$$H = \widehat{\bigoplus_\pi} \pi^{m(\pi)}.$$

We still need finite multiplicity. For this, we use traces. Consider $\varphi = \varphi_1 * \varphi_2 \in C_c^\infty(G)$. Then,

$$\infty > \operatorname{Tr}(R(\varphi)) = \sum_\pi \operatorname{Tr}(\pi(\varphi)) \cdot m(\pi)$$

(with first equality since φ_1, φ_2 act by HS operators). Now, there exists $\varphi = \varphi_1 * \varphi_2$ such that $\pi(\varphi) \neq 0$. This forces $m(\pi) < \infty$. ■

Remark 15.13 (Audience). Here's an alternate proof of finite multiplicity (w/o needing to introduce trace). Say we have an irrep $\pi \subset H$, and fix some $\varphi = \varphi^* \in C_c^\infty(G)$ with $R(\varphi) \neq 0$. Then, since φ is HS, there exists λ s.t. $\pi_\lambda \neq 0$ but $\dim \pi_\lambda < \infty$ and $\dim H_\lambda < \infty$. Thus, $m(\pi) \leq \dim H_\lambda / \dim \pi_\lambda < \infty$ too.

Non-example. Apparently there's no irreducible invariant subspace when $\mathbb{R} \curvearrowright L^2(\mathbb{R})$ (sounds like we've seen this example before?)

Sounds like we'll talk in the future out understanding/computing $\operatorname{Tr} R(\varphi)$.

16 Lecture 16 (4/14): Automorphic representations, tensor product theorem, etc.

We want to introduce the concept of automorphic representations and introduce the fundamental tensor product theorem. Then we will outline the plan for the rest of the course (no class on Monday).

On Monday's class, we saw a bit about automorphic functions giving representations of archimedean groups. Today we move to the adelic language; the archimedean places will have some complications not arising for the non-arch places, but we will brush these under the rug.

16.1 Auto reps

Note 11. This corresponds roughly to Bump 3.3 (and a bit from 4.2).

Let $G = \mathrm{GL}_2$, fix a global field F, and consider the finite ideals $\mathbb{A}_f \subset \mathbb{A} = \prod'_v F_v$. We fix a choice of maximal compact

$$K = \prod_{v<\infty} G(\mathscr{O}_{F_v}) \cdot \prod_{v\mid\infty} \begin{cases} O(2,\mathbb{R}) & \text{if } v \text{ real} \\ U(2) & \text{if } v \text{ complex} \end{cases}$$

in $G(\mathbb{A})$. We write this as $K_f \times K_\infty = K = \prod_v K_v$ with K_v as above.

We want to define so-called '**inner forms**' of GL_2. These are $B^\times$ for B a quaternion algebra.

Recall 16.1. $[G] = G(F)\backslash G(\mathbb{A})$, I think.

Consider $L^2([G], \omega)$ for some fixed central character

$$\omega : Z_G(F)\backslash Z_G(\mathbb{A}) = F^\times\backslash\mathbb{A}^\times \xrightarrow{\text{unitary}} S^1 \subset \mathbb{C}^\times.$$

Note 12. Got distracted and missed some of what Wei said. Something about relating Eisenstein series to parabolic subgroups, I think...

Remark 16.2. There was something about cuspidal $L^2_0([G], \omega)$ that I didn't quite get. Also apparently $B^\times$ has no parabolic subgroups defined over F if B is a division algebra?

Consider $\mathcal{A}_0([G], \omega) \subset L^2_0([G], \omega)$ (cuspidal implies square-integrable); elements here are $(Z(\mathfrak{g}_\mathbb{C}), K_\infty)\times K_f$-finite. There is a (discrete) spectral decomposition

$$L^2_0([G], \omega) = \widehat{\bigoplus_\pi} \pi^{m(\pi)}$$

(each π a unitary rep (of $[G]$?)).

We still haven't gotten to what is usually called an 'automorphic representation.' Let π be an irreducible, unitary representation of $G(\mathbb{A})$. We want to only care about the k-finite vectors in π. This is no longer a rep of $G(\mathbb{A})$ but is a $(\mathfrak{g}_\infty, K_\infty) \times G(\mathbb{A}_f)$-module. Recall

$$G(\mathbb{A}_f) = \prod_{v<\infty}{}' G(F_v)$$

with $G(F_v)$ locally compact and totally disconnected.

Keep in mind that, for automorphic stuff, we want to "forget the topology."

Recall 16.3. In Tate's thesis, we considered continuous characters $\chi : F_v^\times \to \mathbb{C}^\times$. The continuity condition on these homomorphisms was equivalent to the kernel being open. That is χ is a continuous character $\iff$ it is invariant for some open compact; these forgets about the topology of the target so gives a condition that makes sense for any target (e.g. replace $\mathbb{C}^\times$ with $\overline{\mathbb{Q}}_\ell^\times$).

Notation 16.4. To simplify notation, (temporarily?) let G refer to either $G(\mathbb{A}_f)$ or $G(F_v)$. Also, fix a maximal open compact $K_{\max} \subset G$.

Definition 16.5. Let V be a vector space, and let $\pi : G \to \mathrm{GL}(V)$ be an abstract representation. We say (π, V) is a **smooth representation** if $V = \bigcup_K V^K$ with K ranging over (compact) open subgroups of G, i.e. any $v \in V$ has open stabilizer.

Example. A character $\chi : F_v^\times \to \mathbb{C}^\times$ is continuous iff it is smooth in the above sense.

Definition 16.6. A smooth (π, V) is furthermore **admissible** if $\dim V^K < \infty$ for all K (compact open).

Lemma 16.7. *(π, V) is admissible iff for any irrep $\rho \in \widehat{K_{max}} :=$ {irr (smooth?) reps of $K_{\max}$}, the ρ-isotypic component $V(\rho)$ is finite dimensional.*

Recall 16.8. K profinite (e.g. $\varprojlim_m \mathrm{GL}_2(\mathscr{O}_{F_v}/\varpi_v^m)$). Then, $\rho \in \widehat{K} \iff \rho$ factors through a finite quotient. Hence,

$$V = \bigoplus_{\rho \in \widehat{K}_{\max}} V(\rho) \text{ where } V(\rho) := \sum (\text{all subreps} \simeq \rho) \subset V.$$

Hence, the condition above is saying every ρ appears with finite multiplicity.

Definition 16.9. A $\rho \in \widehat{K}$ is called a K**-type**.

Proof of Lemma. Exercise. Note you can take K to be a subgroup of $K_{\max}$ (at least up to conjugation). ■

We want to study smooth, admissible representations at non-arch places.

Definition 16.10. A smooth (admissible) **"representation"** of $G(\mathbb{A})$ is a smooth (admissible) $(\mathfrak{g}_\infty, K_\infty) \times G(\mathbb{A}_f)$-module.

These are not really representations of $G(\mathbb{A})$ because of the archimedean places.

Definition 16.11. We say (π, V) is an **irreducible automorphic representation** of $G(\mathbb{A})$ if it is a smooth $(\mathfrak{g}_\infty, K_\infty) \times G(\mathbb{A}_f)$-module appearing as a subquotient of $\mathcal{A}([G], \omega)$.

Can have a 3-step filtration $0 \leq W_1 \subset W_2 \subset \mathcal{A}([G], \omega)$ with $V \simeq W_2/W_1$.

Theorem 16.12. *Say $(\pi, V) \subset L^2_0([G], \omega)$ is an irreducible sub $G(\mathbb{A})$-rep (this is a literal representation). Then, $(\pi, V_{K\text{-fin}})$ is an irreducible, admissible automorphic $(\mathfrak{g}_\infty, K_\infty) \times G(\mathbb{A}_f)$-module.*

Apparently, the non-arch part of this it not too difficult.

Proof Sketch of Admissibility. Recall $K = K_\infty \times K_f$. Any $\rho \in \widehat{K}$ is of the form $\rho = \rho_f \otimes \rho_\infty$.[31] The finite part ρ_f must factor through some finite quotient of K_f. Let $K_{f,1} := \ker \rho_f$, so this is a compact open. Note that

$$V(\rho) \subset V^{K_{f,1}}(\rho_\infty).$$

By strong approximation, we have (pretend[32] $G = \mathrm{SL}_2$)

$$G(F)\backslash G(\mathbb{A})/K_{f,1} \simeq \Gamma\backslash G(F_\infty) \text{ where } \Gamma = K_{f,1} \cap G(F).$$

Hence, we have $V^{K_{f,1}} \subset L^2_0(\Gamma\backslash G(F_\infty))$ and these are reps of the archimedean $G(F_\infty)$. This is annihilated by the a finite codimension ideal $I \subset Z(\mathfrak{g}_\mathbb{C})$ in the center of the universal enveloping algebra and contained in the K-type at infinite, so

$$V^{K_{f,1}} \subset L^2_0(\Gamma\backslash G(F_\infty))(\rho_\infty, I)$$

and one can show the latter space is finite. ■

From now on, we will simple say "$G(\mathbb{A})$**-representation**" to mean $(\mathfrak{g}_\infty, K_\infty) \times G(\mathbb{A}_f)$-module (maybe unless we're talking about L^2).

Fact. A smooth, irreducible $G(\mathbb{A})$-representations is automatically admissible.

[31] Bump's book has a proof that every rep of K is actually a tensor product

[32] For GL_2, make use of the center and the exact sequence $1 \to Z(\mathrm{GL}_2) \to \mathrm{GL}_2 \to \mathrm{SL}_2 \to 1$

16.2 Tensor product theorem

We would like to write representations as tensor products over places.

Recall 16.13. Consider $\chi : \mathbb{A}_F^\times \to \mathbb{C}^\times$ an idèle class character. When studying Tate's thesis, we showed this is always of the form $\chi = \prod_v \chi_v$ with χ_v unramified for almost all places v (so the product is finite for any given element).

Fix representations π_v of $G(F_v)$ with each π_v smooth admissible (when $v \nmid \infty$?). We want to make sense of $\bigotimes'_v \pi_v$. This requires a choice of distinguished vector for almost all places.

Fix some $v < \infty$ (i.e. non-arch) place. $G(F_v) = \mathrm{GL}_2(F_v) \supset K_v := \mathrm{GL}_2(\mathscr{O}_{F_v})$.

Assumption. Assume all our reps are smooth.

Definition 16.14. We say a (smooth) representation π_v is **unramified** (or **spherical**) if $\pi_v^{K_v} \neq 0$, i.e. there's a nonzero vector fixed by the maximal compact.

Fact (to be proved later)**.** If π_v is irreducible, then $\dim \pi_v^{K_v} \leq 1$.

Hence, the unramified case is the 'maximal' case.

Recall 16.15. π a smooth $G(\mathbb{A})$-rep, so $\pi = \bigcup_{K_f \text{ compact open}} \pi^{K_f}$.

Construction 16.16. Assume we have a collection of irreducible (admissible) representations $\{\pi_v\}_v$ such that π_v is unramified for almost all v, with $0 \neq e_v^0 \in \pi_v^{K_v}$. We define the **restricted tensor product**

$$\bigotimes_v{}' \pi_v = \operatorname{span}\left\{\otimes_v e_v : e_v = e_v^0 \text{ for almost all } v\right\}.$$

This is a (smooth admissible) $G(\mathbb{A})$-module. If $(g_v) \in G(\mathbb{A})$, then $g_v \in G(\mathscr{O}_{F_v})$ for almost all v (i.e. g_v is in the compact open for almost all v), so (g_v) preserves this space.

Theorem 16.17 (**Tensor Product Theorem**)**.** *Let π be an irreducible, admissible $G(\mathbb{A})$-module. Then, $\pi = \bigotimes'_v \pi_v$ with π_v unramified for almost all v, and the π_v's are determined uniquely, up to iso, by π.*

We haven't done enough preparation to give the proof. Also, the proof techniques themselves do not really reappear in our later discussion, so we can safely ignore it.

16.3 What's next?

- One can consider (cuspidal) automorphism representations for $G = \mathrm{GL}_2$ and $G_B = B^\times$ (B a quaternion algebra). We'd like to prove the Jacquet-Langlands correspondence.

- We'd also like to study the L-functions $L(\pi, s)$. We've looked at the case of holomorphic modular/-Maass forms before; we'd like to prove corresponding results for all π (cuspidal automorphic rep of GL_2 over any global field).

  This will require some local representation theory (i.e. rep theory of $\mathrm{GL}_2(F)$ for F a non-arch local field).

- "Langlands functoriality" via trace formula. We'd like to go over two examples. One is Jacquet-Langlands. The other is automorphic induction.

- Relative trace formula and special values of L-functions (related to things like BSD).

How much of this we get through depends on how fast we go in our last month.

We can already formulate some of this stuff already with what we've seen so far.

Theorem 16.18 (**Jacquet-Langlands correspondence**). *Let $G_B = B^\times$,[33] and also consider GL_2. Then there is an injective map*

$$\begin{array}{ccc} \left\{\begin{matrix} \text{not-1-dim cuspidal} \\ \text{auto rep of } G_B \end{matrix}\right\} & \hookrightarrow & \left\{\begin{matrix} \text{cuspidal auto rep} \\ \text{of } \mathrm{GL}_2 \end{matrix}\right\} \\ \pi_B = \bigotimes_v{}' \pi_{B,v} & \longmapsto & \pi = \bigotimes_v{}' \pi_v \end{array}$$

characterised by the fact that for almost all v, $B_v = M_{2\times 2}(F_v)$ (so $G_{B_v} \simeq \mathrm{GL}_{2,F_v}$) and $\pi_{B,v} \simeq \pi_v$.

Note 13. If you need more time on the homework, feel free to send an email; no issue with making an extension.

17 Lecture 17 (4/21): GL(2) L-functions

For the last month, we first want to give a GL(2) analogue of Tate's thesis, understanding L-functions attached to cuspidal automorphic representations. Sounds like this is due to Hecke (then generalized by Jacquet-Langlands),

Recall 17.1 (Tate's thesis)**.** Let F be a global field, and say we have $\chi : \mathbb{A}^\times/F^\times \to \mathbb{C}^\times$. To this, we can attach an L-function

$$L(\chi, s) := \prod_v L(\chi_v, s) \text{ where } \chi = \prod_v \chi_v.$$

This satisfies a functional equation

$$L(\chi, s) = \varepsilon(\chi, s) L(\chi^{-1}, 1-s).$$

The product formula for $L(\chi, s)$ is initially only defined for $\mathrm{Re}(s) \gg 0$. To get a meromorphic continuation to $\mathbb{C}$ (e.g. so that the functional equation makes sense), the key was to consider the auxilliary integrals

$$\zeta(\varphi, \chi, s) := \int_{\mathbb{A}^\times} \varphi(x)\chi(x)\,|x|^s\,\mathrm{d}x = \prod_v \zeta(\varphi_v, \chi_v, s),$$

for $\varphi \in \mathcal{S}(\mathbb{A})$ a Schwarz function, and then cleverly applying Poisson summation.

What are the analogues in the GL_2-case?

Say $G = \mathrm{GL}_2(\mathbb{A})$. Consider some cuspidal automorphic representation

$$\pi = \bigotimes_v{}' \pi_v \subset \mathcal{A}_0([G], \omega) \text{ where } [G] = G(F)\backslash G(\mathbb{A}),$$

and fix a central character $\omega_\pi = \omega : \mathbb{A}^\times/F^\times \to \mathbb{C}^\times$.

[33]Really the algebraic group with functor of points $G_B(R) = (R \otimes B)^\times$

The archimedean and non-archimedean local factors require separate treatments. We will only go over the non-archimedean case.[34] We want to define a global L-function

$$L(\pi, s) := \prod_v L(\pi_v, s)$$

as a product of local L-functions.

17.1 The (non-arch) local case

Let $G = \mathrm{GL}_2(F)$ where F is a non-arch local field.

Assumption. All our G-reps are assumed to be smooth.

Remark 17.2. Note that G is a totally disconnected, locally compact topological group. Everything we say will work in this generality.

A basic technique is '**parabolic induction**'. The basic idea is that representations can be inducted from parabolic (co-compact?) subgroups. In the present case, consider the Borel subgroup

$$B = \left\{ \begin{pmatrix} a & b \\ 0 & d \end{pmatrix} \right\} \subset G.$$

Remark 17.3. We know $G = BK^0$ by Iwasawa, where $K^0 = \mathrm{GL}_2(\mathscr{O}_F)$. Hence, G/B is compact e.g. since $K^0 \twoheadrightarrow G/B$.

Note that $B = AN = NA$ where

$$A = \left\{ \begin{pmatrix} * & \\ & * \end{pmatrix} \right\} \simeq F^\times \times F^\times \text{ and } N = \left\{ \begin{pmatrix} 1 & * \\ & 1 \end{pmatrix} \right\} \simeq F.$$

Note we have an exact sequence

$$1 \longrightarrow N \longrightarrow B \longrightarrow A \longrightarrow 1.$$

Consider a character $\chi : A \to \mathbb{C}^\times$. We can *inflate* this to a character $B \to A \xrightarrow{\chi} \mathbb{C}^\times$ of B, also denoted χ.

Definition 17.4 (*smooth induction*)**.** Define the representation $\mathrm{Ind}_B^G(\chi)$ from B to G is

$$\mathrm{Ind}_B^G(\chi) := \{f : G \to \mathbb{C} : f(bg) = \chi(b)f(g)\}$$

with f above (implicitly) required to be **"locally constant"**, i.e. there exists K such that $f(gk) = f(g)$ for all $g \in G$ and $k \in K$.

This has a G-action given by

$$(\pi(g)f)(x) = f(xg),$$

and it is a smooth representation of G

Recall 17.5. A **smooth representation** V of G is one where $V = \bigcup_{K \text{ cmpt. open}} V^K$.

[34] If you want, F is a function field, $F = k(X)$ for some smooth X/k ($k = \mathbb{F}_q$), so all its places are non-archimedean

In turns out to be useful to normalize our induction. For this, consider the (modulous?) character $\delta_B : B \to \mathbb{R}^\times$ given by $\begin{pmatrix} a & b \\ 0 & d \end{pmatrix} \mapsto |a/d|$.

Definition 17.6 (**smooth induction**, normalized)**.** Define the representation $\mathrm{Ind}_B^G(\chi)$ from B to G is

$$\mathrm{Ind}_B^G(\chi) := \left\{ f : G \to \mathbb{C} : f(bg) = \delta_B(b)^{1/2}\chi(b)f(g) \right\}$$

with f above (implicitly) required to be **"locally constant"**, i.e. there exists K such that $f(gk) = f(g)$ for all $g \in G$ and $k \in K$.

This has the benefit that the induction of a unitary representation will still be unitary, assuming I heard correctly.

Note that Ind_B^G is smooth and admissible (for admissible, uses that $B\backslash G/K$ is finite).

17.1.1 There are two special cases

For the first case, take $K = \mathrm{GL}_2(\mathscr{O}_F)$, and consider

Question: What's going on here?

$$\pi^K = \left\{ f(bgk) = \delta(b)^{1/2}\chi(b)f(g) \text{ for } b \in B, k \in K \right\}.$$

Since $G = BK$ (Iwasawa decomposition), we see that $\dim \pi^K \leq 1$ and that

Question: Why?

$$\dim \pi^K = 1 \iff \chi \text{ unramified}.$$

Since χ is a character of $A = F^\times \times F^\times$, we can write $\chi = (\chi_1, \chi_2)$ and then χ unramified $\iff$ χ_1, χ_2 both unramified.

In the second case, consider $\widetilde{\eta} =$ "$\eta \circ \det$" $: G \to \mathbb{C}^\times$ with $\widetilde{\eta}(bg) = \widetilde{\eta}(b)\widetilde{\eta}(g)$ and

$$\widetilde{\eta}\begin{pmatrix} a & b \\ & d \end{pmatrix} = \eta(a)\eta(d) = \left|\frac{a}{d}\right|^{1/2} \chi_1(a)\chi_2(d)$$

where

$$\chi_1(a) = \eta(a)\,|a|^{1/2} \text{ and } \chi_2(d) = \eta(d)\,|d|^{-1/2}.$$

That is, $\chi_1/\chi_2 = |\cdot|^{\pm 1}$ is the absolute value character. When this happens, $\mathrm{Ind}_B^G(\chi)$ is reducible, with a 1-dimensional subspace or a 1-dimensional quotient.

Exercise. If $\chi_1/\chi_2 = |\cdot|^{-1}$, then $\mathrm{Ind}_B^G(\chi)$ has a one-dimensional invariant quotient.

17.1.2 Classification of reps

Theorem 17.7 (**Classification of Irreducible, Smooth, Admissible Representations of** G)**.**

(0) *There are the 1-dimensional reps* $\widetilde{\eta} = \eta \circ \det$

(1) *(**principal series**)* $\mathrm{Ind}_B^G(\chi)$ *where* $\chi_1/\chi_2 \neq |\cdot|^{\pm 1}$.

*(2) (Twisted by η **Steinberg representations**)* St_η *defined as*

$$0 \longrightarrow \widetilde{\eta} \longrightarrow \mathrm{Ind}_B^G\left(\eta|\cdot|^{1/2}, \eta|\cdot|^{-1/2}\right) \longrightarrow \mathrm{St}_\eta \longrightarrow 0.$$

*(3) ("**supercuspidal representations**") everything else.*

What about the other case, where $\chi_1/\chi_2 = |\cdot|^{-1}$? Can show that the reps arising here are isomorphic to the Steinberg reps.

Remark 17.8. Can check $\mathrm{St}_\eta = \mathrm{St}_1 \otimes \eta \circ \det$

We can also classify irreducible, unramified representations.

Theorem 17.9 (**Classification of irreducible, unramified G-reps**)**.** *Let π be unramified and irreducible. Then either*

- *π is 1-dimensional, $\pi = \widetilde{\eta} = \eta \circ \det$ for η unramified; or*
- *$\pi = \mathrm{Ind}_B^G \chi$ with χ unramified and $\chi_1/\chi_2 \neq |\cdot|^{\pm 1}$.*

Remark 17.10. In general, $\mathrm{Rep}\,\mathrm{GL}_n$ has parabolics in bijection with partitions $n = n_1 + n_2 + \cdots + n_r$ of n. For each partition, you get a parabolic P consisting of matrices preserving a flag with spaces whose dimensions are given by the partition. Each P has a decomposition $P = MN$ where $M \cong \prod_i \mathrm{GL}_{n_i}(F)$ is "the diagonal" (and N is the "unipotent part"). On M, get reps $\bigotimes \sigma_i$ with σ_i cuspidal reps of GL_{n_i}, and then can consider the reps $\mathrm{Ind}_P^G\left(\bigotimes_{i=1}^r \sigma_i\right)$ of $G = \mathrm{GL}_n$. The reps you cannot get like this, the primitive ones, will be the cuspidal representations.

Note 14. Wei said more that I did not quite get.

Remark 17.11. Apparently there is a natural bijection between supercuspidal representations of $\mathrm{GL}_2(F)$ and irreducible, 2-dimensional representations of $\mathrm{Gal}(\overline{F}/F)$. The other smooth, irreducible, admissible reps will give non-semisimple reps of $\mathrm{Gal}(\overline{F}/F)$. Requiring a compatibility with induction in the local Langlands correspondence let's you reduce things to the primitive case (where the Galois reps are nicer) though. We'll talk about local Langlands correspondence later when we have more definitions, and can say more than "there is a bijection."

17.1.3 Back to L-functions

Say π an irreducible representation of $G = \mathrm{GL}_2(F)$ of infinite dimension.[35] Then, we define the **L-function**

$$L(\pi, s) = \begin{cases} L(\chi_1, s)L(\chi_2, s) & \text{if } \pi = \mathrm{Ind}_B^G \chi \text{ and } \chi_1/\chi_2 = |\cdot|^{\pm 1} \\ L(\eta, s + 1/2) & \text{if } \pi = \mathrm{St}_\eta \\ 1 & \text{if } \pi = \text{supercuspidal.} \end{cases}$$

Recall 17.12 (Tate's thesis)**.**

$$L(\chi, s) = \begin{cases} \dfrac{1}{1 - \chi(\varpi)q^{-s}} & \text{if } \chi \text{ unram} \\ 1 & \text{if } \chi \text{ ram.} \end{cases}$$

[35] Can also define L-functions in the 1-dimensional case

Globally, if $\pi = \bigotimes' \pi_v$, then its L**-function** is simply the product $L(\pi, s) := \prod_v L(\pi_v, s)$.

Later, we will define an ε-factor $\varepsilon(\pi, s) = \prod_v \varepsilon(\pi_v, \varphi_v, s)$. Then obtain the theorem.

Theorem 17.13 (Hecke, Jacquet-Langlands)**.** *Say π is a cuspidal automorphic representation. Then, $L(\pi, s)$ has a holomorphic continuation to $s \in \mathbb{C}$ (i.e. no poles) and satisfies a functional equation*

$$L(\pi, s) = \varepsilon(\pi, s) L(\pi^\vee, 1 - s)$$

for a suitable notion of dual/contragredient.

Note that (π, V) is a smooth representations iff $V = \bigoplus_{\rho \in \widehat{K}} V(\rho)$ with $K = \mathrm{GL}_2(\mathscr{O}_F)$ a maximal compact. Its **contragredient** is

$$V^\vee = \{\text{“smooth” } \ell : V \to \mathbb{C}\} = \bigoplus_{\rho \in \widehat{K}} V(\rho)^* \subset \mathrm{Hom}(V, \mathbb{C})$$

where * denotes the usual dual.

Fact. If (π, V) is admissible (i.e. $\dim V(\rho) < \infty$ always), then $(\pi^\vee)^\vee \simeq \pi$.

While we're at it, let's also state a converse theorem. We know $\pi = \bigotimes'_v \pi_v$ is a tensor product. Here's a natural question: if we take some local places $\{\pi_v\}_v$ (with π_v unramified for almost all v, and $\omega = \prod \omega_{\pi_v} : \mathbb{A}^\times / F^\times \to \mathbb{C}^\times$ is an idele class character), then when will the restricted tensor product $\bigotimes'_v \pi_v$ be automorphic? If it always is, our studied can purely be reduced to the local case.

However, there must be some obstruction (e.g. since only countably many cuspidal auto reps once you fix a central character).

Remark 17.14. The modularity conjecture amounts to saying that L-functions of elliptic curves are automorphic.

Theorem 17.15 (**Converse Theorem**, Weil, Jacquet-Langlands)**.** *π defined above is cuspidal automorphic iff $L(\pi \otimes \eta, s)$ is “nice” for all $\eta : \mathbb{A}^\times / \mathbb{F}^\times \to \mathbb{C}^\times$.*

Here, $(\pi \otimes \eta)_v = \pi_v \otimes (\eta_v \circ \det)$. Furthermore, “*nice*” means ‘entire, satisfies a functional equation, and some more analytic properties on critical strip $0 \leq \mathrm{Re} \leq 1$.’

For a long time, using this theorem was the main strategy for proving something was an automorphic form. There are more strategies now, e.g. based on Wile's work on modularity.

18 Lecture 18 (4/26)

Last time we stated the global functional equation. This week we give an overview of the proof, skipping some details in order to not take too much time.

18.1 Fourier expansion

Say π is a cuspidal automorphic representation of $\mathrm{GL}_2(\mathbb{A}) = G(\mathbb{A})$, and write

$$\pi = \bigotimes' \pi_v.$$

It is realized as a subspace $\pi \subset \mathcal{A}_0([G])$ so an element $\varphi \in \pi$ is a function

$$\varphi : [G] = G(F)\backslash G(\mathbb{A}) \longrightarrow \mathbb{C}.$$

Question: Is this an assumption or by definition?

Answer: A theorem (specifically, theorem 14.11). $\mathcal{A}_0([G])$ is semisimple so all subquotients are subreps

We let F denote our global field (so $\mathbb{A} = \mathbb{A}_F$).

Notation 18.1. $AN = B \subset G = \mathrm{GL}_2$ where

$$N = \left\{\begin{pmatrix} 1 & * \\ 0 & 1 \end{pmatrix}\right\}, \quad A = \left\{\begin{pmatrix} * & 0 \\ 0 & * \end{pmatrix}\right\}, \quad \text{and} \quad B = \left\{\begin{pmatrix} * & * \\ 0 & * \end{pmatrix}\right\}.$$

Note that $[N] \cong F\backslash\mathbb{A}$ is compact, so $\varphi|_{[N]}$ has a Fourier expansion. Note that

$$\widehat{N(F)\backslash N}(\mathbb{A}) \simeq \widehat{F\backslash\mathbb{A}} \simeq F.$$

Fix a choice of nontrivial character

$$\psi = \prod_v \psi_v : \mathbb{A}/F \to \mathbb{C}^\times,$$

so the characters of $[N]$ are all of the form $\psi_a(x) = \psi(ax)$ for some (unique) $a \in F$. Thus, we may write

$$\varphi|_{[N]} = \sum_{a\in F} \underbrace{\left\langle \varphi|_{[N]}, \psi_{-a}\right\rangle}_{W_{\varphi,a}} \psi_a.$$

The pairing above is given by

$$W_{\varphi,a} = \int_{[N]} \varphi(n)\psi_{-a}(n)\mathrm{d}n.$$

Restricting to $[N]$ loses some information. To combat this, note (when F a function field) there's an action[36] $G(\mathbb{A}) \curvearrowright \mathcal{A}_0([G])$ ($G(\mathbb{A})$ acts by right translation). Thus, we can decompose $\pi(g)\varphi_{[N]}$ as well and get

$$W_{\pi(g)\varphi,a} = \int_{[N]} \pi(g)\varphi(n)\psi_{-a}(n)\mathrm{d}n = \int_{[N]} \varphi(ng)\psi_{-a}(n)\mathrm{d}n.$$

To keep notation simpler, we write $W_{\varphi,a}(g) := W_{\pi(g)\varphi,a}$ for $g \in G(\mathbb{A})$. Note this satisfies $W_{\varphi,a}(ng) = \psi_a(n)W_{\varphi,a}(g)$ for all $n \in N(\mathbb{A})$. Let's name this property.

Definition 18.2. Let[37]

$$\mathcal{W}_\psi := \left\{ w : G(\mathbb{A}) \to \mathbb{C} \,\middle|\, \begin{array}{c} w(ng) = \psi(n)W(g) \text{ for all } n \in N(\mathbb{A}), g \in G(\mathbb{A}) \\ \text{invariant under some compact open } K \end{array} \right\}.$$

In archimedean case, second condition should be K-finite

We call this the space of **Whittaker functions**. Thinking back to induced representations from last time, we see

$$\mathcal{W}_\psi = \mathrm{Ind}_{N(\mathbb{A})}^{G(\mathbb{A})} \psi.$$

[36] In the number field, say the phrase $(\mathfrak{g}, K)$-module instead

[37] Invariance under K on the right, i.e. $w(gk) = w(g)$ for all $k \in K$ and $g \in G(\mathbb{A})$

Thus, we get a $G(\mathbb{A})$-equivariant map

$$\begin{array}{ccc} \pi & \longrightarrow & \mathcal{W}_\psi \\ \varphi & \longmapsto & (g \mapsto W_{\varphi,\psi}(g)) \end{array}$$

where we've slightly changed notation by now saying

$$W_{\varphi,\psi}(g) = \int_{[N]} \varphi(ng)\overline{\psi(n)}\mathrm{d}n.$$

This map will be an embedding, and one will be able to recover π from its image.

Note we have an evaluation map $\mathrm{ev}_1 : W_\psi \to \mathbb{C}$. The composition $\pi \to W_\psi \xrightarrow{\mathrm{ev}_1} \mathbb{C}$ sends $\varphi \mapsto W_{\varphi,\psi}(1)$. Note that

$$W_\psi : \varphi \mapsto \int_{[N]} \varphi(n)\overline{\psi(n)}\mathrm{d}n$$

defines an element of $\mathrm{Hom}_{N(\mathbb{A})}(\pi, \mathbb{C}_\psi)$. That is the composition is $N(\mathbb{A})$-equivariant where $N(\mathbb{A}) \curvearrowright \mathbb{C}$ via the character ψ. By Frobenius reciprocity, one has

$$\mathrm{Hom}_{N(\mathbb{A})}(\pi|_{N(\mathbb{A})}, \mathbb{C}_\psi) \cong \mathrm{Hom}_{G(\mathbb{A})}(\pi, \mathrm{Ind}_N^{G(\mathbb{A})} \psi).$$

Recall 18.3 (**Frobenius reciprocity** for smooth induction)**.** Given a closed subgroup $H \subset G$ (totally disconnected) and a representation $\sigma \in \mathrm{Rep}(H)$, we can form the induced representation $\mathrm{Ind}_H^G(\sigma)$. Reciprocity says that for any $\pi \in \mathrm{Reg}(G)$, there is a natural isomorphism

$$\mathrm{Hom}_H(\pi|_H, \sigma) \xrightarrow{\sim} \mathrm{Hom}_G(\pi, \mathrm{Ind}_H^G \sigma).$$

The upshot is that the maps $\pi \to \mathcal{W}_\psi$ and $\pi \to \mathbb{C}_\psi$ are more-or-less equivalent (I think).

Question 18.4. *Can we characterize the image $\pi \hookrightarrow \mathcal{W}_\psi$?*

I missed/didn't follow some stuff, but think about the symbols

$$W_\psi \in \mathrm{Hom}_{N(\mathbb{A})}(\pi|_{N(\mathbb{A})}, \mathbb{C}_\psi) \stackrel{?}{=} {\bigotimes_v}' \mathrm{Hom}_{N(F_v)}(\pi_v, \mathbb{C}_{\psi_v})$$

18.2 Local Theory

Now say F is a local non-archimedean field, and let $\pi \in \mathrm{Rep}(G)$ be some smooth representation. We would like some sort of 'local Fourier expansion.'

Formally, we would like something like

$$\text{“}\sum_{a \in F} \langle \pi|_N, \psi_a \rangle \, \psi_a\text{”}$$

Define

$$\pi_{N,\psi} := V_{N,\psi} = \frac{V}{\langle \pi(n)v - \psi(n)v \mid n \in N, v \in V \rangle},$$

the maximal quotient where N acts by ψ.

Example. When ψ = trivial, $\pi_N := \pi_{N,\text{triv}}$ is the N-coinvariants, the maximal quotient where N acts trivially. One calls this a **Jacquet functor** I think.

Remark 18.5. A (the diagonal matrices) normalizes N (i.e. $aNa^{-1} = N$ for $a \in A$), so the coinvariants π_N has an A-action. When ψ is nontrivial, you simply get a vector space.

Recall (parabolic induction?) we saw how to take a representation of A, inflate it to one of B, and then get an induced rep of G. Frobenius reciprocity tells us that taking N-coinvariants and induction give an adjoint pair,

$$\mathrm{Hom}_G(\pi, \mathrm{Ind}_B^G \sigma) \simeq \mathrm{Hom}_B(\pi|_B, \sigma) \simeq \mathrm{Hom}_A(\pi_N, \sigma)$$

(recall $B = AN$ and $B/N = A$) for any $\sigma \in \mathrm{Rep}(A)$.

What about the twisted version? If $\psi \neq 1$, then we still get

$$\mathrm{Hom}_G(\pi, \mathrm{Ind}_N^G \psi) \simeq \mathrm{Hom}_N(\pi|_N, \psi) \cong \mathrm{Hom}_{\mathrm{Vect}}(\pi_{V,\psi}, \mathbb{C}).$$

Let's state the main result.

Theorem 18.6.

(a) *If $\psi = 1$ is trivial, the Jacquet functor is an exact functor*

Rep(−) is smooth representations of −

$$\mathrm{Rep}(G) \to \mathrm{Rep}(A)$$

which sends admissible representations to admissible representations.

(b) *If ψ is nontrivial, still get an exact functor*

$$\mathrm{Rep}(G) \longrightarrow \mathrm{Vect}_{\mathbb{C}}\,.$$

Moreover, if π is an irrep of G, then $\dim \pi_{N,\psi} \leq 1$.

Definition 18.7. An irreducible π is called "**ψ-generic**" if $\dim \pi_{N,\psi} = 1$ for ψ nontrivial.

Lemma 18.8. *For $G = \mathrm{GL}_2$, irreducible π is generic iff π is ∞-dimensional. Equivalently, irreducible π is non-generic iff it is 1-dimensional.*

Fact: Any f.dim rep of GL_2 over a (non-arch) local field is 1-dimensional

Lemma 18.9. *Say π unramified generic and ψ "unramified" (i.e. trivial on $\mathscr{O}_F$, but non-trivial on $\varpi^{-1}\mathscr{O}_F$). Then, any nonzero linear function $\ell \in \mathrm{Hom}_N(\pi, \mathbb{C}_\psi) \simeq \mathbb{C}$ is nonzero on K-invariants, i.e. $\ell|_{\pi^K} \neq 0$.*

Recall that $\pi^K \subset \pi$ is only 1-dimensional.

Remark 18.10. If you want $\ell|_{\pi^K} \neq 0$, this forces $\psi|_{N(\mathscr{O}_F)} = 1$. This is because $N(\mathscr{O}_F)$ acts on ℓ by ψ and something else I didn't catch...

18.3 Back to Global case

Returning to the question at the end of the Fourier expansion section, we indeed have

Theorem 18.11. *Consider* $\pi = \bigotimes' \pi_v$. *Then,*

$$\mathrm{Hom}_{N(\mathbb{A})}(\pi, \psi) \cong \bigotimes{}' \mathrm{Hom}_{N(F_v)}(\pi_v, \psi_v).$$

Let's make sense of this.

Recall 18.12. For almost all v, π_v is unramified and ψ_v is unramified.

Recall 18.13.

$$\pi = \bigotimes{}' \pi_v = \bigcup_{S \text{ finite}} \otimes_{v \in S} \pi_v = \left\{ \bigotimes_v \varphi_v : \varphi_v = \varphi_v^\circ \in \pi_v^K \text{ for almost all } v \right\}$$

(something like this)

Given $\ell_v \in \mathrm{Hom}_{N(F_v)}(\pi_v, \psi_v)$, it might not make sense to talk about the infinite product $\bigotimes_v \ell_v$. However, we can normalize so the $\ell_v(\varphi_v^0) = 1$ for almost all v. Then,

$$\bigotimes_v \ell_v \left(\bigotimes_v \varphi_v \right) = \prod_v \ell_v(\varphi_v)$$

makes sense since $\varphi_v = \varphi_v^0$ for almost all v. Thus, we get a map

$$\bigotimes_v{}' \mathrm{Hom}_{N(F_v)}(\pi_v, \psi_v) \to \mathrm{Hom}_{N(\mathbb{A})}(\pi, \psi).$$

Checking this is an iso is not too hard.

Note 15. This is only for $F =$ function field. For the number field case, also need to muck around with $(\mathfrak{g}, K)$-modules.

Corollary 18.14.

$$\dim \mathrm{Hom}_{N(\mathbb{A})}(\pi, \psi) \leq 1.$$

It might happen that one of the factors is trivial.

Theorem 18.15. *If* π *is cuspidal automorphic, then*

$$\mathrm{Hom}_{N(\mathbb{A})}(\pi, \psi) \neq 0.$$

In particular, $\pi = \bigotimes' \pi_v$ *with* π_v *infinite dimensional for every* v.

Recall the Whittaker functional

$$\mathrm{Hom}_N(\pi, \psi) \simeq \mathrm{Hom}_G(\pi, \underbrace{\mathrm{Ind}_N^G \psi}_{\mathcal{W}_\psi}).$$

We see that there is only one way to embed π inside this space $\mathcal{W}_\psi$ of Whittake functions. Hence, π can be recovered from its image.

Definition 18.16. Given, $\ell \in \mathrm{Hom}_G(\pi, \mathcal{W}_\psi)$, **Whittaker model of** π is $\mathrm{Im}(\ell) \subset \mathcal{W}_\psi$ with its G-action.

This exists when π generic.

Suppose we have a function $\varphi \in \pi \subset \mathcal{A}_0([G])$. Recall

$$W_{\varphi,\psi}(g) = \int \varphi(ng)\psi(n)\mathrm{d}n$$

which gives a map $\pi \to \mathcal{W}_\psi, \varphi \mapsto W_{\varphi,\psi}$.

Lemma 18.17 (**Fourier-Whittaker expansion**). *One has (when π cuspidal automorphic)*

$$\varphi(g) = \sum_{a \in F^\times} W_{\varphi,\psi}\left(\begin{pmatrix} a & \\ & 1 \end{pmatrix} g\right).$$

Proof. By $G(\mathbb{A})$-equivariance, it's enough to prove this when $g = 1$, i.e. that

$$\varphi(1) = \sum_{a \in F^\times} W_{\varphi,\psi}\begin{pmatrix} a & \\ & 1 \end{pmatrix}.$$

By Fourier expansion, we know that

$$\varphi(1) = \sum_{a \in F} W_{\varphi,\psi_a}(1)$$

$[N] \xrightarrow{\sim} F\backslash\mathbb{A}$ via $\begin{pmatrix} 1 & n \\ & 1 \end{pmatrix} \mapsto n$ so $1 \in [N]$ corresponds to $0 \in \mathbb{A}/F$. Hence, there's no "$\psi_a(1)$" factor above because it's really $\psi_a(0) = 1$.

Since φ cuspidal, when $a = 0$, $W_{\varphi,\psi_0}(1) = 0$. When $a \neq 0$,

$$\begin{aligned}
W_{\varphi,\psi_a}(1) &= \int_{[N]} \varphi(n)\psi_{-a}(n)\mathrm{d}n \\
&= \int_{\mathbb{A}/F} \varphi\begin{pmatrix} 1 & n \\ & 1 \end{pmatrix}\psi(-an)\mathrm{d}n \\
&\overset{n \mapsto n/a}{=} \int_{\mathbb{A}/F} \varphi\begin{pmatrix} 1 & n/a \\ & 1 \end{pmatrix}\psi(-n)\mathrm{d}n \\
&= \int_{\mathbb{A}/F} \varphi\left(\begin{pmatrix} a^{-1} & \\ & 1 \end{pmatrix}\begin{pmatrix} 1 & n \\ & 1 \end{pmatrix}\begin{pmatrix} a & \\ & 1 \end{pmatrix}\right)\psi(-n)\mathrm{d}n \\
&= \int_{\mathbb{A}/F} \varphi\left(\begin{pmatrix} 1 & n \\ & 1 \end{pmatrix}\begin{pmatrix} a & \\ & 1 \end{pmatrix}\right)\psi(-n)\mathrm{d}n \\
&= W_{\varphi,\psi}\begin{pmatrix} a & \\ & 1 \end{pmatrix}
\end{aligned}$$

∎

As a result, we see that $\varphi \mapsto W_\varphi$ is nonvanishing, and so an embedding (since π irreducible). We can also recover $\pi \subset \mathcal{A}_0([G]) = \bigoplus_\pi \pi^{m(\pi)}$.

Corollary 18.18 (**multiplicity one**). $m(\pi) = 1$ *when π cuspidal automorphic.*

Note 16. I missed the reasoning on why this is the case, but hopefully it's recoverable from what I did manage to write down. Also, something about $W_\psi[\pi] \to \mathcal{A}_0([G])$.

19 Lecture 19 (4/28): Local theory

Note 17. Wei's notability background is black instead of white today.

And every day afterwards

Working with $G = \mathrm{GL}_2$ as always. Let $\pi = \bigotimes_v' \pi_v$ be a cuspidal automorphic representation with central character ω_π. We want to prove nice properties of the L-function $L(\pi, s) = \prod_v L(\pi_v, s)$. Let F be our global field. We let $A \subset G$ denote the diagonal torus (so $A \simeq \mathbb{G}_m^2$) which contains the center Z_G ($\simeq \mathbb{G}_m$ the scalars).

Construction 19.1. Given any $\varphi \in \pi \subset \mathcal{A}_0([G])$, we can define the **zeta integral**

$$\zeta(\varphi, s) = \int_{\mathbb{A}^\times/F^\times} \varphi\begin{pmatrix} a & \\ & 1 \end{pmatrix} |a|^{s-\frac{1}{2}}\,\mathrm{d}a.$$

Note this is holomorphic for all $s \in \mathbb{C}$ because of the rapid decreasing property of the cuspidal form φ.

Think of this as an integral over A/Z_G, I think? Like, you want to integrate over the diagonal, but don't want the center there artificially inflating things or something?

Remark 19.2. This is in some sense 'simpler' than Tate's thesis since we do not need to introduce auxiliary Schwartz functions.

Remark 19.3. $\varphi \mapsto \zeta(\varphi, s)$ defines an element of $\mathrm{Hom}_{A(\mathbb{A})}(\pi \otimes |\cdot|^s, \mathbb{C}_{\omega_\pi})$ (I think. This is at least true if $\omega_\pi = 1$. Not sure if there should be any sort of inverting or anything in general).

Compare this with the Whittaker model involving $\mathrm{Hom}_{N(\mathbb{A})}(\pi, \mathbb{C}_\psi)$ which we know is at most 1-dimensional (recall: π generic $\iff$ exactly 1-dimensional).

Fact. $\dim \mathrm{Hom}_{A(\mathbb{A})}(\pi \otimes |\cdot|^s, \mathbb{C}) \le 1$ as well.

This discussion has been global so far. We would like a local theory, and hopefully also an Euler product.

19.1 Global to local

Fix a nontrivial additive character $\psi : \mathbb{A}/F \to \mathbb{C}^\times$.

Recall 19.4. $\mathbb{A}/F \simeq N(\mathbb{A})/N(F)$, and for $\varphi \in \pi$, we have the "Fourier coefficient"

$$W_\varphi(g) = \int_{[N]} \varphi(ng)\psi(n)\mathrm{d}n.$$

We proved that these satisfy

$$\varphi(g) = \sum_{a \in F^\times} W_\varphi\left(\begin{pmatrix} a & \\ & 1 \end{pmatrix} g\right)$$

Among other things, this shows cuspidal automorphic reps are generic.

This let's us write (at least formally)

$$\begin{aligned}\zeta(\varphi, s) &= \int_{\mathbb{A}^\times/F^\times}\left(\sum_{a\in F^\times} W_\varphi\begin{pmatrix} a & \\ & 1\end{pmatrix}\right)|a|^{s-\frac{1}{2}}\,\mathrm{d}a \\ &= \int_{\mathbb{A}^\times} W_\varphi\begin{pmatrix} a & \\ & 1\end{pmatrix}|a|^{s-\frac{1}{2}}\,\mathrm{d}a\end{aligned}$$

Notation 19.5. Let $\mathcal{W}_\psi(\pi)$ denote the image of π in $\operatorname{Ind}_{N(\mathbb{A})}^{G(\mathbb{A})}\psi$ under $\varphi \mapsto W_\varphi$, so in particular, $\pi \xrightarrow{\sim} \mathcal{W}_\psi(\pi)$.

We can write

$$\mathcal{W}_\psi(\pi) = \bigotimes_v{}' \mathcal{W}_{\psi_v}(\pi_v)$$

normalized so that for unramified v, there is a distinguished vector $W_v^0 \in \mathcal{W}_{\psi_v}(\pi_v)^{K_v}$ such that $W_v^0(1) = 1$. Keep in mind W_v^0 is a function $G(F_v) \to \mathbb{C}$.

If $\varphi = \otimes_v \varphi_v$ is a pure tensor, then we can write

$$W_\varphi(g) = \prod_v W_{\varphi_v}(g_v) \text{ for } g \in G(\mathbb{A})$$

as a product. Furthermore,

$$\zeta(\varphi, s) = \prod_v \zeta(W_{\varphi_v}, s)$$

where the **local zeta integral** is

$$\zeta(W_{\varphi_v}, s) := \int_{F_v^\times} W_{\varphi_v}\begin{pmatrix} a & \\ & 1\end{pmatrix}|a|_v^{s-\frac{1}{2}}\,\mathrm{d}a_v.$$

This gets us to working locally.

Question 19.6. *What is the space* $\{\zeta(W, s) : W \in \mathcal{W}(\pi_v)\}$*?*

Maybe, like in Tate's thesis, the "gcd" will be the local L-function. Maybe we'll also get some sort of functional equation involving an ε-factor.

19.2 Local

Fix F a local field.

Assumption. Let's identity a representation π with its (unique) Whittaker model $\mathcal{W}(\pi) \subset \operatorname{Ind}_N^G \psi$.

Remember: Elements of $\operatorname{Ind}_N^G \psi$ are certain $\mathbb{C}$-valued functions on G

Notation 19.7. Let

$$A_0 := \left\{\begin{pmatrix} a & \\ & 1\end{pmatrix}\right\} \subset A$$

What are the functions $\{W|_{A_0} : W \in \mathcal{W}(\pi)\}$?

Definition 19.8 ("Miraculous parabolic"). We define the **mirabolic group**

$$B_0 := \left\{ \begin{pmatrix} a & b \\ & 1 \end{pmatrix} \in B \right\} \subset B.$$

This is not a parabolic, but is close to one.

Remark 19.9. Think of B as the stabilizer of some line in $\mathbb{C}^2$. Then, B_0 is the stabilizer of a fixed element of that line. Note $B_0 \simeq N \cdot A_0$.

Since we know how N acts (it transforms functions via additive character) understanding $\{W|_{A_0} : W \in \mathcal{W}(\pi)\}$ is equivalent to understanding $\{W|_{B_0} : W \in \mathcal{W}(\pi)\}$.

Lemma 19.10. *Identify $A_0 \simeq F^\times$. If π is generic irreducible, the map*[38]

$$\begin{array}{ccccc} \pi & \simeq & \mathcal{W}(\pi) & \longrightarrow & C^\infty(F^\times) \\ & & W & \longmapsto & W|_{A_0} \end{array}$$

is injective, and its image $\mathcal{K}(\pi)$ is called the ***Kirillov model*** *of π.*

Theorem 19.11 ("local Fourier expansion"). *Let π be a smooth representation of GL_2. Then the natural map*

$$\pi \longrightarrow \bigotimes_{x \in F} \pi_{N,\psi_x},$$

to the product of all of π's twisted Jacquet modules, is injective. That is,

$$\bigcap_{x \in F} \ker\left(\pi \to \pi_{N,\psi_x}\right) = 0.$$

Note 18. Proof in Wei's lecture notes (on the dropbox).

This of RHS as space of Fourier coefficients, one for each additive character. This theorem proves the injectivity in Lemma 19.10.

Lemma 19.12 (Reformulation of Lemma 19.10). *Say we have nonzero $\Lambda \in \mathrm{Hom}_N(\pi, \mathbb{C}_\psi)$. Consider the map*

$$\pi \ni v \longmapsto \Lambda(\pi(g)v) \in \mathcal{W}(\pi).$$

Then, This is equiv to Lemma 19.10

$$\text{“}\Lambda\left(\pi\begin{pmatrix} a & \\ & 1 \end{pmatrix} v\right) = 0 \textit{ for all } a\text{”} \implies v = 0.$$

Proof. The linear functional

$$\Lambda_a : v \mapsto \Lambda\left(\pi\begin{pmatrix} a & \\ & 1 \end{pmatrix} v\right)$$

belongs to $\mathrm{Hom}_N(\pi, \mathbb{C}_{\psi_a}) = (\pi_{N,\psi_a})^*$ (up to possibly needing to replace a by a^{-1} somewhere).

If $\Lambda_a(v) = 0$ for all $a \neq 0$, then image of v in π_{N,ψ_a} vanishes for all $a \neq 0$. If we had vanishing for all a (including $a = 0$), we could conclude $v = 0$. Consider $v^0 = v - \pi(n)v$ for any $n \in N$; then the image of

[38] smooth means locally constant when F non-arch (as you should think of it as being here since we've been ignoring arch places)

v^0 in π_{N,ψ_a} is equal to 0 for all $a \in F$. Hence, $v^0 = 0$, so v is N-invariant (can choose any $n \in N$). Since $v \in \pi$ with π smooth, we know v is invariant under some open $K \subset G$.

Question: Why?

Exercise. Let H be the group generated by N, K. Then, $H \supset \mathrm{SL}_2(F)$.

This forces π to be ≤ 1-dimensional, which is a contradiction (π assumed generic in Lemma 19.10). ∎

We can now identify π with its Kirillov model

$$\begin{array}{ccccc} \pi & \simeq & \mathcal{W}(\pi) & \hookrightarrow & C^\infty(F^\times) \\ & & W & \longmapsto & W|_{A_0} \end{array}$$

What is the image $\mathcal{K}(\pi) \subset C^\infty(F^\times)$?

Theorem 19.13. *There is an exact sequence*

$$0 \longrightarrow C_c^\infty(F^\times) \longrightarrow \mathcal{K}(\pi) \longrightarrow \pi_N \longrightarrow 0$$

for any generic, irreducible π.

That is, the kernel of the natural map $\mathcal{K}(\pi) \simeq \pi \twoheadrightarrow \pi_N$ from π to its Jacquet module is independent of π.

Warning 19.14. There's no simple formula for the group action $G \curvearrowright \mathcal{K}(\pi)$ thought of as a space of smooth functions, i.e. as a subspace of $C^\infty(F^\times)$.

Remark 19.15.

$$\dim_{\mathbb{C}} \pi_N = \begin{cases} 2 & \text{if principal} \\ 1 & \text{if Steinberg twist} \\ 0 & \text{if supercuspidal.} \end{cases}$$

In particular, if π is supercuspidal, then $\mathcal{K}(\pi) = C_c^\infty(F^\times)$.

Description of Π_N First recall the Jacquet functor

$$\begin{array}{ccc} \mathrm{Rep}(G) & \longrightarrow & \mathrm{Rep}(A) \\ \pi & \longmapsto & \pi_N. \end{array}$$

What does this look like, say when $\pi = \mathrm{Ind}_B^G \chi$ for $\chi = (\chi_1, \chi_2) : A \to \mathbb{C}^\times$. By Frobenius reciprocity, we know

We initially forgot to include the normalization $\delta^{1/2}$ below, so some of the stuff after it is missing a $\delta^{1/2}$ factor

$$\mathrm{Hom}_G(\sigma, \mathrm{Ind}_B^G \chi) = \mathrm{Hom}_B(\sigma|_B, \chi\delta^{1/2}) \simeq \mathrm{Hom}_A(\sigma_N, \chi\delta^{1/2}).$$

Hence if σ irreducible, then $\sigma_N \neq 0 \iff \sigma \subset \mathrm{Ind}_B^G \chi$ for some χ.

Exercise. Any admissible representation of $A \simeq F^\times \times F^\times$ has an irreducible quotient (reps of A are not always semisimple).

Recall 19.16. We defined σ **supercuspidal** if σ is not a subrepresentation of any $\mathrm{Ind}_B^G \chi$.

However, we have just shown the following.

Proposition 19.17. *σ supercuspidal iff $\sigma_n = 0$*

("constant term of σ vanishes")

Say $\sigma = \operatorname{Ind}\chi$, so $\operatorname{Hom}(\sigma_N, \chi) \neq 0$. Hence, χ is a qutoeint of σ_N.

Notation 19.18. Given $\chi = (\chi_1, \chi_2)$ its transpose is $\chi^t = (\chi_2, \chi_1)$.

Lemma 19.19.

(1) If $\chi_1 \neq \chi_2$ ($\chi \neq \chi^t$), then

$$\left(\operatorname{Ind}_B^G \chi\right)_N \simeq \delta^{1/2}\chi \oplus \delta^{1/2}\chi^t$$

as reps of A.

(2) If $\chi_1 = \chi_2$ ($\chi = \chi^t$), then

Question: What?

$$(\operatorname{Ind}_\chi)_N \simeq \delta^{\frac{1}{2}}\chi \begin{pmatrix} 1 & \operatorname{val}(t_1/t_2) \\ & 1 \end{pmatrix} \quad \textit{where } (t_1, t_2) \in A \simeq F^\times \times F^\times.$$

How does one prove Theorem 19.13? We want to show $\ker(\mathcal{K}(\pi) \to \pi_N) = C_c^\infty(F^\times)$. Here's an observation probably due to Kirillov. Recall the mirabolic B_0, and consider the *compact induction* $\operatorname{Ind}_{N,c}^{B_0}(\psi) \simeq C_c^\infty(F^\times)$.[39]

Something like consider only functions with compact support in definition of (smooth) induction

Lemma 19.20. $\operatorname{Ind}_{N,c}^{B_0}(\psi)$ *is an irreducible B_0-rep.*

Proof. Exercise (apparently 'easy') ∎

Now one wants to show that $\ker(\mathcal{K}(\pi) \to \pi_N) \subset \operatorname{Ind}_N^{B_0}(\psi) \simeq C^\infty(F^\times)$, and is invariant under B_0. We know by definition that

$$\ker(\pi \to \pi_N) = \langle \pi(n)v - v : n \in N, v \in \pi \rangle$$

so it is easily seen to be invariant under $B_0 = NA_0$ (N invariance is immediate. Sounds like maybe a little bit of work for A_0 invariance). By invariance/irreducibility, it's now enough to show $\ker(\mathcal{K}(\pi) \to \pi_N) \subset C_c^\infty(F^\times)$ (since this kernel must be nonzero). We can check this just for vectors of the form $\pi(n)v - v$. That is, we want to show that

$$F^\times \ni a \longmapsto W\left(\begin{pmatrix} a & \\ & 1 \end{pmatrix}\begin{pmatrix} 1 & b \\ & 1 \end{pmatrix}\right) - W\begin{pmatrix} a & \\ & 1 \end{pmatrix} = (\psi(ba) - 1)W\begin{pmatrix} a & \\ & 1 \end{pmatrix}$$

has compact support. This will always have compact support as $a \to \infty$ by the group action or something. When $a \to 0$, the $(\psi(ba) - 1)$ factor above goes to 0. We conclude that $\pi\begin{pmatrix} 1 & b \\ & 1 \end{pmatrix} v - v$ gets sent to a function with compact support, and so this finishes the proof of Theorem 19.13.

20 Lecture 20 (5/3)

Let's continue the discussion on GL(2) L-functions. Recall we want to prove meromorphic continuation, functional equation, etc.

[39] There's a split exact sequence $1 \to N \to B_0 \to F^\times \to 1$. In other words $B_0 = N \rtimes A_0$

Fix $\pi = \bigotimes' \pi_v$ a cuspidal automorphic form. Say $G = \mathrm{GL}_2$ over a global field F. Consider a pure tensor

$$\otimes_v \varphi_v = \varphi \in \pi \subset \mathcal{A}_0([G]).$$

We defined the global zeta integral ($s \in \mathbb{C}$)

$$\zeta(\varphi, s) = \int_{\mathbb{A}^\times/F^\times} \varphi \begin{pmatrix} a & \\ & 1 \end{pmatrix} |a|^{s-\frac{1}{2}} \, \mathrm{d}^\times a.$$

Remark 20.1. This function is entire because φ is cuspidal (decays quickly).

The hard part is relating this integral to L-functions.

Remark 20.2. If F is a function field, e.g. $F = \mathbb{F}_q(t)$, then any cuspidal automorphic form on GL_2 has compact support. That is, $\varphi \in \mathcal{A}_0([G]) \implies \mathrm{supp}(\varphi)$ is compact modulo center. In this case, the ζ-integral is essentially a finite sum, $\zeta(\varphi, s) \in \mathbb{C}[q^{-s}, q^s]$ with $q = \#(\text{residue field})$.

Recall 20.3 (Fourier expansion w.r.t. $\psi : \mathbb{A}/\mathbb{F} \to \mathbb{C}^\times$)**.** Define

$$W_\varphi(g) := \int_{\mathbb{A}/F} \varphi \left(\begin{pmatrix} 1 & x \\ & 1 \end{pmatrix} g \right) \psi(-x) \mathrm{d}x.$$

Then, $W_\varphi(g) \in \mathrm{Ind}_{N(\mathbb{A})}^{G(\mathbb{A})} \psi$ (strictly speaking, this is only true for function fields. Need to be more careful when archimedean places are involved). Then,

$$\varphi(g) = \sum_{a \in F^\times} W_\varphi \left(\begin{pmatrix} a & \\ & 1 \end{pmatrix} g \right).$$

Remark 20.4. When $\varphi = \otimes_v \varphi_v$ is a pure tensor, we get a decomposition

$$W_\varphi(g) = \prod_v W_{\varphi_v}(g_v).$$

These remarks/recalls combine to give

$$\zeta(\varphi, s) = \prod_v \zeta(W_{\varphi_v}, s) \text{ where } \zeta(W_v, s) = \int_{F_v^\times} W_v \begin{pmatrix} a & \\ & 1 \end{pmatrix} |a|^{s-\frac{1}{2}} \, \mathrm{d}a$$

(note $\zeta(w_v, s)$ convergent for $\mathrm{Re}(s) > 0$). Finally, recall that we have previously defined the local L-functions $L(\pi_v, s)$.

Theorem 20.5. *Let v be a non-arch place, and let $F = F_v$ (so π a local rep). Then,*

(0) *There is a meromorphic continuation to all $s \in \mathbb{C}$.*

(1) *The space*

$$\{\zeta(w, s) : w \in \mathcal{W}(\pi_v)\} = L(\pi, s) \cdot \mathbb{C}[q_v^{-s}, q_v^s]$$

of ζ-integrals "has gcd $L(\pi, s)$".

(2) We have the functional equation

$$\frac{\zeta(w,s)}{L(\pi,s)}\varepsilon(s,\pi) = \frac{\zeta(w^\vee,1-s)}{L(\pi^\vee,1-s)},$$

where $\varepsilon(s,\pi) = cN^{s-\frac{1}{2}}$ for some $c \in \mathbb{C}^\times$ and $N \in \mathbb{N}$.

Remark 20.6. Above, $\pi^\vee$ is the contragradient representation (defined in Lecture 17?). Furthermore,

$$W^\vee(g) := W(gw) \text{ where } w := \begin{pmatrix} & 1 \\ -1 & \end{pmatrix}.$$

If $W \in \mathcal{W}(\pi)$, then $W^\vee \in \mathcal{W}(\pi^\vee)$.

In the present case (for GL(2)), $\pi^\vee \cong \pi \otimes \omega_\pi$, π twisted by the central character.

Example. $\operatorname{Ind}_B^G(\chi)^\vee \simeq \operatorname{Ind}_B^G(\chi^{-1})$. When $\pi = \operatorname{Ind}_B^G \chi$, the central character is $\omega_\pi = \chi_1\chi_2$ (where $\chi = (\chi_1,\chi_2)$). One sees that $\operatorname{Ind}_B^G(\chi^{-1}) \simeq \operatorname{Ind}_B^G(\chi) \otimes \omega_\pi$.

Use $\pi \simeq \operatorname{Ind}_B^G(\chi^t)$ where $\chi^t = (\chi_2,\chi_1)$ and that twisting here just multiplites the two characters.

Remark 20.7. Say $\chi : F^\times \to \mathbb{C}^\times$. Can consider the operation $\pi \rightsquigarrow \pi \otimes \chi$. This leads us to consider the 3-variable ζ-integral

$$\zeta(W,\chi,s) := \int_{F^\times} W\begin{pmatrix} a & \\ & 1\end{pmatrix}\chi(a)\,|a|^{s-\frac{1}{2}}\,\mathrm{d}^\times a.$$

Note that $W\begin{pmatrix} a & \\ & 1\end{pmatrix}\chi(a) \in \mathcal{A}(\pi\otimes\chi)$, so can apply same result to this ζ integral by using the representation $\pi \otimes \chi$. Explicitly, one gets

$$\frac{\zeta(W,\chi,s)}{L(\pi\otimes\chi,s)}\varepsilon(\pi\otimes\chi,s) = \frac{\zeta(W^\vee,\chi^{-1}\omega_\pi^{-1},1-s)}{L(\pi^\vee\otimes\chi^{-1},1-s)}.$$

Recall 20.8. Recall the Kirillov model

$$\mathcal{K}(\pi) = \left\{W\begin{pmatrix} a & \\ & 1\end{pmatrix} : W \in \mathcal{W}(\pi)\right\} \subset C^\infty(F^\times).$$

This sits in an exact sequence

$$0 \longrightarrow C_c^\infty(F^\times) \longrightarrow \mathcal{K}(\pi) \longrightarrow \pi_N \longrightarrow 0$$

with $\dim \pi_N \in \{0,1,2\}$.

Remark 20.9. The "**Mellin transform**"

$$\begin{array}{ccc} C^\infty(F^\times) & \longrightarrow & \text{functions on } \mathbb{C} \\ W\begin{pmatrix} a & \\ & 1\end{pmatrix} & \longmapsto & \displaystyle\int_{F^\times} W\begin{pmatrix} a & \\ & 1\end{pmatrix}|a|^{s-\frac{1}{2}}\,\mathrm{d}^\times a. \end{array}$$

After taking this Mellin transform, the space of ζ-integrals is more-or-less the underlying space of the Kirillov model (?)

Consider $\varphi(a) = W\begin{pmatrix} a & \\ & 1 \end{pmatrix} \in \ker\left(\mathcal{K}(\pi) \twoheadrightarrow \pi_N\right) = C_c^\infty(F^\times)$. Then its Mellin transform is (compact support $\implies$ bounded above and below, so supported on some annulus)

$$\int_{F^\times} \varphi(a)\,|a|^{s-\frac{1}{2}}\,\mathrm{d}^\times a = \sum_{i=-N}^{N} \int_{\mathrm{val}(a)=i} (blah) \in \mathbb{C}[q^s, q^{-s}].$$

Any such polynomial can be realized. This proves the result for supercuspidal representations (those for which $\pi_N = 0$ so $\mathcal{K}(\pi) = C_c^\infty(F^\times)$). Recall $L(\text{supercuspidal}, s) = 1$.

To extend proof to other two cases (Steinberg and principal series representations) is to extend things to π_N. Since $\dim \pi_N \le 2$, can just choose one or two representations spanning this quotient space and check things there.

Fact (Representatives of π_N in $\mathcal{K}(\pi)$)**.**

(1) When $\pi = \mathrm{Ind}_B^G \chi$ irreducible with $\chi_1/\chi_2(t) \neq |t|^{\pm 1}$. Then,

- If $\chi_1 \neq \chi_2$, then can take $|t|^{1/2}\chi_1(t)$, $|t|^{1/2}\chi_2(t)$ ($|t| \le 1$ and 0 outside this range)
- If $\chi_1 = \chi_2$, can take
$$|t|^{1/2}\chi_1(t) \text{ and } |t|^{1/2}\chi_1(t)\,\mathrm{val}(t)$$
(again for $|t| \le 1$ and vanishing outside this range)

(2) For $\pi = \mathrm{St}_\eta$, can take $|t|\,\eta(t)$ for $|t| \le 1$

What's important above is the behaviour as $\to 0$. We see that the behaviour of $\varphi(t)$, for $\varphi \in \mathcal{K}(\pi)$, as $t \to 0$ is determined by π (?).[40]

Let's consider the principal series case, so $\pi = \int_B^G \chi$, say with $\chi_1 \neq \chi_2$. Note first that

$$\int_{|t|\le 1} |t|^{1/2}\chi_1(t)\,|t|^{s-1/2}\,\mathrm{d}^\times t = \int_{|t|\le 1} \chi_1(t)\mathrm{d}t^\times = L(\chi_1, s)$$

with last equality if measure normalized so that $\mathrm{vol}(\mathscr{O}_F^\times) = 1$. Similarly for the other representation. We see from this that

$$\gcd\left\{\zeta(W,s) : W \in \mathcal{W}(\pi)\right\} = \gcd L(\chi_1, s), L(\chi_2, s) = L(\chi_1, s)L(\chi_2, s) =: L(\pi, s).$$

Remark 20.10. Any thing in the Kirillov model is of the form $\varphi(a) = \varphi_0(a) + \lambda_1\varphi_1(a) + \lambda_2\varphi_2(a)$ with φ_1 the χ_1 guy and φ_2 the χ_2 guy. Hence,

$$\int \varphi(a)\,|a|^{s-\frac{1}{2}}\,\mathrm{d}^\times a \in \mathbb{C}[q^s, q^{-s}] + \lambda_1 L(\chi_1, s) + \lambda_2 L(\chi_2, s) = L(\chi_1, s)L(\chi_2, s)\mathbb{C}[q^{-s}, q^s]$$

where we recall that

$$L(\chi_b, s) := \begin{cases} \dfrac{1}{1 - \chi_b(\varpi)q^{-s}} & \text{if } \chi \text{ unramified} \\ 1 & \text{if } \chi \text{ ramified} \end{cases}$$

[40]Something like, when π not supercuspidal, can recover χ_1, χ_2 or η from this behavior

is the reciprocal of a polynomial ($b \in \{0,1\}$)

Let's transition to talking about the functional equation. To show it, we'd like to apply a uniqueness result. First note that we can view the ζ-integral as a linear functional

$$\begin{array}{rccc} \zeta_s : & \pi & \longrightarrow & \mathbb{C} \\ & W & \longmapsto & \displaystyle\int_{F^\times} \begin{pmatrix} a & \\ & 1 \end{pmatrix} |a|^{s-\frac{1}{2}} \,\mathrm{d}^\times a \end{array}$$

where we've identified π with its Whittaker model. This defines $\zeta_s \in \mathrm{Hom}_A(\pi, \mathbb{C}_{s-\frac{1}{2}})$ where $A = \left\{\begin{pmatrix} * & \\ & 1 \end{pmatrix}\right\}$.

The subscript $s-1/2$ is missing after what's written below. It should be there

Lemma 20.11. *$\dim \mathrm{Hom}_A(\pi_s \mathbb{C}) \leq 1$ and is $= 1$ if π generic. Here, $\pi_s = \pi \otimes |\cdot|^s$.*

When $\dim \pi = 1$, this may be 0-dimensional, but the 1-dimensional π case is overall easier, so can handle it separately

Let's prove a weaker version of this.

Lemma 20.12. *Say π generic. Then, $\dim \mathrm{Hom}_A(\pi_s, \mathbb{C}) = 1$ for all $s \in \mathbb{C}/(2\pi i \log q)\mathbb{Z}$ with at most 2 exceptions*

Proof. Use Kirillov model. Recall $\mathcal{K}(\pi)$ does not really remember how G acts, but does remember how the diagonal elements act; they act by translation. We have

$$\mathrm{Hom}_A(\mathcal{K}(\pi), \mathbb{C}) \subset C^\infty(F^\times) \text{ where } A \simeq F^\times \curvearrowright F^\times.$$

Note that $\dim_{\mathbb{C}}(C_c^\infty(F^\times), \mathbb{C}) = 1$. This proves things in the supercuspidal case.

From Tate's thesis?

For the other two cases, we know for $\zeta^\circ(W, s) := \zeta(W, s)/L(\pi, s) \in \mathbb{C}[q^s, q^{-s}]$ that $0 \neq \zeta_s^\circ \in \mathrm{Hom}_A(\pi, \mathbb{C})$ so $\dim \mathrm{Hom}_A(\pi_s, \mathbb{C}) \geq 1$. Rest of the details omitted... ■

We have $\zeta_s^\circ \in \mathrm{Hom}_A(\pi_s, \mathbb{C})$.

Claim 20.13.
$$W \mapsto \frac{\zeta(W^\vee, 1-s)}{L(\pi^\vee, s)}$$
is also invariant for the A-action.

Corollary 20.14. *There must be some constant $\varepsilon(\pi, s) \in \mathbb{C}^\times$ so that*

$$\varepsilon(\pi, s)\zeta^\circ(W, s) = \zeta^\circ(W^\vee, 1-s).$$

This holds for all $W \in \mathcal{W}(\pi)$. Can use this to show $\varepsilon(\pi, s) = c_\pi q^{N(s-1/2)}$ for some $c_\pi \in \mathbb{C}^\times$.

Remark 20.15. ($G = \mathrm{GL}_2, A$) is a '*Galfand pair*' in the sense that $\dim \mathrm{Hom}_A(\pi, \mathbb{C}) \leq 1$ for all π.

This more-or-less finishes our discussion of the local theory.

20.1 Back to Global Theory

Say F a global field, and we have $\varphi \in \pi = \otimes'_v \pi_v$ a pure tensor. Recall

$$\zeta(\varphi, s) = \prod_v \zeta(W_{\varphi_v}, s) = \prod_v L(\pi_v, s) \left(\prod_v \zeta^\circ(W_{\varphi_v}, s)\right).$$

We need some global input (recall Poisson summation in Tate's thesis). Note that φ is invariant by $\mathrm{GL}_2(F)$ (automorphic form on $[G]=G(F)\backslash G(\mathbb{A})$). Get global function equation ($w=\begin{pmatrix} & 1\\ -1 & \end{pmatrix}$ and we assume $\omega_\pi=1$ for simplicity)

$$\begin{aligned}
\zeta(\varphi,s) &= \int_{\mathbb{A}^\times/F^\times}\varphi\begin{pmatrix} a & \\ & 1\end{pmatrix}|a|^{s-\frac{1}{2}}\,\mathrm{d}^\times a\\
&= \int_{\mathbb{A}^\times/F^\times}\varphi\left(w\begin{pmatrix} a & \\ & 1\end{pmatrix}\right)|a|^{s-\frac{1}{2}}\,\mathrm{d}^\times a\\
&= \int\varphi\left(a\begin{pmatrix} a^{-1} & \\ & 1\end{pmatrix}w\right)|a|^{s-\frac{1}{2}}\,\mathrm{d}^\times a\\
&\overset{a\mapsto 1/a}{=} \int\varphi\left(\begin{pmatrix} a & \\ & 1\end{pmatrix}w\right)|a|^{(1-s)-\frac{1}{2}}\,\mathrm{d}^\times a\\
&= \prod_v \zeta(W^\vee_{\varphi_v},1-s)\\
&= L(\pi^\vee,1-s)\prod_v\zeta^\circ(W^\vee_{\varphi_v},1-s)
\end{aligned}$$

(Note this doesn't really use that π is automorphic, just the Fourier expansion which just used left-invariancy under Borel).

Making use of the local function equations as well now implies that

$$L(\pi,s)=L(\pi^\vee,1-s)\varepsilon(\pi,s),$$

so we have the global functional equation. Note this is entire since

$$\zeta(\varphi,s)=L(\pi,s)\prod_v\zeta^\circ(W_{\varphi_v},s)$$

is entire (for any φ cuspidal) and $\exists\varphi\in\pi$ s.t. $\prod_v\zeta^\circ=1$.

Next time we'll start a new topic. For the remaining 5 lectures, we try to say something about some more advance topics.

21 Lecture 21 (5/5)

We start by making a remark we forgot last time.

Recall 21.1. We had the decomposition

$$\zeta(\varphi,s)=\prod_v L(\pi_v,s)\prod_v\zeta^\circ(W_{\varphi_v},s).$$

We should mention that the product over the normalized zeta integral is finite (since things are unramified almost everywhere). Specifically, $\zeta^\circ(W_v,s)=1$ for "unramified data," i.e. π_v unramified, ψ_v unramified, and $W_v\in\pi_v^{K_v^\circ}$ ($K_v^\circ=\mathrm{GL}_2(\mathscr{O}_{F_v})$ maximal compact) with normalizations $W_v(1)=1$ and

$\mathrm{vol}(\mathscr{O}_{F_v}^\times) = 1$. This is something to prove. It does not follow immediately from our discussion.

Lemma 21.2 (Homework)**.** *$\zeta^\circ(W_v, s) = 1$ for unramified data.*

There is an explicit formula for W_v in terms of "Satake parameters." $\pi_v = \mathrm{Ind}_B^G \chi$ with $\chi = (\chi_1, \chi_2)$. These parameters are the pair

$$\begin{pmatrix} \chi_1(\varpi) & \\ & \chi_2(\varpi) \end{pmatrix} \in \mathrm{GL}_2(\mathbb{C})/\text{conjugacy}.$$

You can switch the order of the characters, so only this pair is well-defined.

In the remaining lectures (including today), we want to look at two applications of trace formulas (TFs).

- Arthur-Selberg TF

  Has applications to Jacquet-Langlands correspondence as well as (cyclic) base change for GL_2.

- Relative TF

  Applications to Waldspurger's formula.

21.1 Jacquet-Langlands

Let $G' = \mathrm{GL}_2$ over a global field F, and let $\pi' = \otimes'_v \pi'_v$ be an irreducible cuspidal automorphic representation. To simplify life, we'll say the central character $\omega_{\pi'} = 1$ is trivial.

Jacquet-Langlands wants to relate automorphic forms on GL_2 to those on inner forms of GL_2. Let $\mathrm{PGL}_2 = \mathrm{GL}_2 / Z(\mathrm{GL}_2)$ be the 'adjoint group' of GL_2. Then inner forms of GL_2 are characterized by the pointed set $\mathrm{H}^1(F, \mathrm{PGL}_2)$. Elements of this set are in bijection with (iso classes) quaternion algebras B over F, i.e. $\mathrm{H}^1(F, \mathrm{PGL}_2) \simeq \mathrm{Br}(F)[2]$ (so it secretly is a group after all).

Remark 21.3. The distinguished element of the set of quaternion algebras over F is $B \cong M_{2\times 2}(F)$, so this corresponds to $G' = \mathrm{GL}_2$. In general, the quaternion algebra B corresponds to $G = B^\times$, viewed as an algebraic group over F, i.e.

$$G(R) = (B \otimes_F R)^\times$$

for any F-algebra R.

Remark 21.4. We can give an even more explicit description of these inner forms. By class field theory, $\mathrm{Br}(F)[2] = \mathrm{H}^2(F, \mathbb{G}_m) = \mathrm{H}^2(\mathrm{Gal}(F^s/F), (F^s)^\times)$ sits in an exact sequence

$$0 \longrightarrow \mathrm{Br}(F) \longrightarrow \bigoplus_v \mathrm{Br}(F_v) \xrightarrow{\sum_v \mathrm{Inv}_v} \mathbb{Q}/\mathbb{Z} \longrightarrow 0,$$

where Inv_v is an invariance map

$$\mathrm{Inv}_v : \mathrm{Br}(F_v) \to \mathbb{Q}/\mathbb{Z}\,.$$

This is an isomorphism if v non-archimedean while $\mathrm{Br}(\mathbb{R}) \xrightarrow{\sim} \frac{1}{2}\mathbb{Z}/\mathbb{Z}$ and $\mathrm{Br}(\mathbb{C}) \xrightarrow{\sim} 0$.

We see from this that $\mathrm{Br}(F_v)[2] \simeq \mathbb{Z}/2\mathbb{Z}$ (unless v complex), so exactly two choices for quaternion algebra over F_v (the split one and some division quaternion algebra).

Continuing trying to make these inner forms feel more familiar, we get a third description of $\mathrm{H}^1(F, \mathrm{PGL}_2)$; it is in bijection with

Question: Finite set of non-complex places?

$$\left\{ \begin{matrix} \text{finite set of places} \\ \text{w/ even cardinality} \end{matrix} \right\}.$$

Fact. $[G] = G(F)\backslash G(\mathbb{A})$ is compact modulo center $\iff$ B is a division algebra (i.e. $B \not\simeq M_{2\times 2}$).

Say $\pi = \otimes_v' \pi_v$ cuspidal with trivial center character and is infinite dimensional.

Remark 21.5. The one-dimensional reps are all of the form $\pi \simeq \chi \circ \det$ where $\det : B^\times \to \mathbb{G}_m$ is the reduced norm map and $\chi : \mathbb{A}^\times/F^\times \to \mathbb{C}^\times$ a Hecke character. These are simpler and can be handled separately.

What are we after?

Theorem 21.6 (Local Jacquet-Langlands). *Let F be a local field, and let $G = B^\times$ with B a division quaternion algebra of F (so $G/F^\times$ is compact). Then there is an injective map*

$$JL : \mathrm{Rep}(G) \hookrightarrow \mathrm{Rep}(G')$$

form smooth, irreducible reps of G into those of G'. Say this sends $\pi \mapsto \pi'$. This injection satisfies

(1) *Image = **discrete series** of G' (i.e.*[41] St_η *or supercuspidal)*

On the LHS, everything is discrete since G basically compact, so we have a bijection $\mathrm{Rep}^{disc}(G) \xrightarrow{\sim} \mathrm{Rep}^{disc}(G')$.

(2) *(characterization of JL) $JL(\pi) = \pi'$ iff*

$$\theta_\pi(\gamma) = -\theta_{\pi'}(\gamma')$$

for all regular semisimple $\gamma \leftrightarrow \gamma'$, where θ_π is the character of π (note G compact mod center), and $\theta_{\pi'}$ is the Harish-Chandra character, characterized by

$$f \in C_c^\infty(G') \implies \mathrm{tr}(\pi'(f)) = \int_{G'} \theta_{\pi'}(\gamma') f(\gamma') \mathrm{d}\gamma'$$

with $\theta_{\pi'}$ conjugate-invariant, locally L^1 on $G' = \mathrm{GL}_2$, and smooth on $G'_{r.s.}$ (elements whose characteristic polys have distinct roots). Note $G'_{r,s}$ is open,dense in G'.

We also need to explain this matching $\gamma \leftrightarrow \gamma'$. Note $\gamma \in G$ has a characteristic polynomial in $F[x]_{\deg=2}$ (same def with $\det$ *replaced by reduced norm). We write $\gamma \leftrightarrow \gamma'$ to denote* $\mathrm{char}(\gamma) = \mathrm{char}(\gamma')$, *their char polys agree.*

Remark 21.7. $\mathrm{char} : G' \to F[x]_{\deg=2}$ is surjective while $\mathrm{char} : G \to F[x]_{\deg=2}$ is not. For the division algebra, only elliptic elements appear in the image.[42]

Question: What does elliptic mean here?

(3) *Say $\chi : F^\times \to \mathbb{C}^\times$ is a character. Then,*

$$JL(\chi \circ \det) = \mathrm{St}_\chi.$$

[41] There's also an intrinsic definition of discrete series. Equivalently, matrix coefficients are all in $L^2(G')$. Recall matrix coefficients are $g \mapsto (g\varphi_1, \varphi_2)$ with $\varphi_1, \varphi_2 \in \pi' \otimes (\pi')^\vee$

[42] Related to not being able to embed $F \times F \hookrightarrow B$ or something like this?

*This explains the − sign in **(2)**.*

Remark 21.8. Recall we have a short exact sequence

$$0 \to \mathrm{St}_\chi \to \mathrm{Ind}\,\widetilde{\chi} \to \chi \circ \det \to 0.$$

Get embedding $G/\mathrm{conj} \hookrightarrow G'/\mathrm{conj}$ with image the elliptic elements. Can check that

$$\theta_{\mathrm{St}_\chi}(\gamma') = -\chi\det(\gamma').$$

Trace of middle rep must vanish because of something about parabolic v. elliptic having trivial intersection or something? I'm still not sure what elliptic is here...

Note this gives a recipe for obtaining supercuspidal representations of G'. Note that, since $G/F^\times$ compact, elements of $\mathrm{Rep}(G)$ (which we require are smooth and irreducible) are f.dim.

Theorem 21.9 (**Global Jacquet-Langlands**). *Let F be a global field. Let*

$$\mathcal{A}_0(G) = \{\pi : \pi\ \infty\text{-dimensional cuspidal automorphic}\}.$$

Then, there is a unique injection

$$\mathcal{A}_0(G) \hookrightarrow \mathcal{A}_0(G')$$

sending $\otimes'\pi_v = \pi \mapsto \pi' = \otimes'\pi'_v$ characterized by

(1) *$\pi'_v = JL_v(\pi_v)$ for all v.*[43]

(2) *$\otimes\pi'_v = \pi'$ is in the image iff π'_v is discrete for all $v \in \Sigma_B$, where $\Sigma_B = \{v : B_v \text{ non-split}\}$.*

This is obviously unique if it exists. The content of this is that the tensor product $\pi' := \otimes' JL_v(\pi_v)$ is automorphic.

Remark 21.10. Let B, B' be division algebras. If $\Sigma_{B'} \subset \Sigma_B$, then you also get a Jacquet-Langlands correspondence $JL : \mathcal{A}_0(G) \to \mathcal{A}_0(G')$ where $G = B^\times$ and $G' = (B')^\times$.

21.2 Trace formula

Recall the time we spent talking about Hilbert-Schmidt stuff (the spectral theorem stuff). Given π and $f \in C_c^\infty(G(\mathbb{A}))$, we want to understand $\mathrm{tr}(\pi(f))$.

Recall 21.11. $G(\mathbb{A})$ acts on $L^2([G], \omega_\pi = 1)$. Say $G = B^\times$ with B nonsplit so we're in the cocompact case. Let $X = [G]/\text{center}$. We have the integral operator

$$R(f)\varphi(x) = \int_X K_f(x,y)\varphi(y)\mathrm{d}y$$

for $f \in C_c^\infty(G(\mathbb{A}))$ where the **kernel function** is

$$K_f(x,y) = \sum_{\gamma \in G(F)} f(x^{-1}\gamma y).$$

[43] If B_v non-split apply local Jacquet-Langlands. If it is split, then $G_V \simeq G'_v \simeq \mathrm{GL}_2$ and we want $\pi'_v \simeq \pi_v$

This is a finite sum for any given x, y since it only involves $\gamma \in \left(x\,\mathrm{supp}(f)y^{-1}\right) \cap G(F)$ (compact and discrete set), so $K_f \in C(X \times X) \subset L^2(X \times X)$ (X compact). Hence, $R(f)$ is Hilbert-Schmidt.

Remark 21.12. Say T_1, T_2 are Hilbert-Schmidt (HS). Let $T = T_1T_2$ be their product. Then the trace

$$\mathrm{Tr}(T) = \sum_{i \in I} (Te_i, e_i)$$

makes sense and is independent of the choice of basis $\{e_i\}_{i \in I}$. Note that

$$\mathrm{Tr}(T) = \sum_i (T_2e_i, T_1^*e_i) = \sum_{i,j \in I} (T_2e_i, e_j)\overline{(T_1^*e_i, e_j)} \le \frac{1}{2}\sum_{i,j} |(T_1e_i, e_j)|^2 + |(T_1^*e_i, e_j)|^2 < \infty.$$

Lemma 21.13. *Say $f \in C_c^\infty(G(\mathbb{A}))$. Then, $\mathrm{tr}(R(f))$ if of trace class, i.e. this trace makes sense. Furthermore,*

$$\mathrm{tr}(R(f)) = \int_X K_f(x,x)\mathrm{d}x.$$

Proof sketch. The convolution map

$$\begin{array}{ccc} C_c^\infty(G(\mathbb{R})) \otimes C_c^\infty(G(\mathbb{R})) & \longrightarrow & C_c^\infty(G(\mathbb{R})) \\ f_1 \otimes f_2 & \longmapsto & f_1 * f_2 \end{array}$$

is surjective (**Dixmier-Malliavin Theorem**). Hence, any $f \in C_c^\infty(G(\mathbb{R}))$ can be written as

$$f = \sum_{i \in I} f_1^{(i)} * f_2^{(i)} \text{ with } \#I < \infty.$$

This + the previous remark shows that $\mathrm{tr}(R(f))$ is well-defined. Specifically, if $f = f_1 * f_2$, we see that

$$\mathrm{Tr}(R(f)) = \langle K_{f_1}, K_{f_2} \rangle_{X \times X} = \int_X K_f(x,x)\mathrm{d}x.$$

■

Recall 21.14.

$$L^2([G]) = \bigoplus_\pi m_\pi \pi$$

Question: Should this be $L^2([G], \omega_\pi = 1) = L^2(X)$ on the left?

This gives $\mathrm{tr}(R(f)) = \sum_\pi m_\pi \,\mathrm{tr}(\pi(f))$. This is sort of the "spectral" side. For more info, we need to turn to the "geometric" side (look at conjugacy classes).

Recall $K_f = \sum_{\gamma \in G(F)} f(x^{-1}\gamma y)$. The Lemma says to compute the trace we can compute an integral

$$\mathrm{tr}(R(f)) = \int_{X = [G]} K_f(x,x)\mathrm{d}x$$

(we're ignoring issues with the center, but will state a correct result in the end).

Notation 21.15. For a conjugacy class $\gamma \in G(F)/\mathrm{conj}$,

$$G(\mathbb{A})_\gamma := \left\{g \in G(\mathbb{A}) : g^{-1}\gamma g = \gamma\right\}.$$

Fix a measure on $G(\mathbb{A})$ as well as on $G(\mathbb{A})_\gamma$ (assuming existence of (nice) measure on this stabilizer).

Definition 21.16. For a conjugacy class $\gamma \in G(F)/\text{conj}$. The **orbital integral** is

$$\text{Orb}(\gamma, f) := \text{vol}(G_\gamma) \int_{G(\mathbb{A})/G(\mathbb{A})_\gamma} f(g^{-1}\gamma g)\text{d}g$$

where

$$\text{vol}(G_\gamma) := \text{vol}\left(G(F)_\gamma \backslash G(\mathbb{A})_\gamma\right).$$

We'll justify later that these are well-defined.

Remark 21.17. The easiest way to not have to deal with center issues is to just use $G = PB^\times = B^\times/F^\times$, so let's do this.

Theorem 21.18. *Fix some $f \in C_c^\infty(G(\mathbb{A}))$. Then,*

$$\text{tr}(R(f)) = \sum_{\gamma \in G(F)/conj} \text{Orb}(\gamma \cdot f).$$

This is (the simplest case?) of the **Arthur-Selberg Trace formula**; more generally, for any $f \in C_c^\infty(G(\mathbb{A}))$ in the **Hecke algebra**, one has

$$\sum_\pi m(\pi)\,\text{tr}(\pi(f)) = \sum_{\gamma \in G(F)/\text{conj}} \text{Orb}(\gamma \cdot f)$$

(LHS 'spectral' and RHS 'geometric').

Remark 21.19. Compare this with the fact from finite group rep theory that the number of irreps of a finite group is the same as its number of conjugacy classes.

The Jacquet-Langlands correspondence is a statement of the spectral side. To use it, we'll compare the geometric sides for G and G', and then use this trace formula to obtain a comparison on the spectral sides.

22 Lecture 22 (5/10): Jacquet-Langlands, continued

We want to outline the proof of Jacquet-Langlands via the Arthur-Selberg trace formula.

Let B/F be a quaternion division algebra, and let $G = PB^\times = B^\times/F^\times$. Hence, $[G] = G(F)\backslash F(\mathbb{A})$ is compact. Let $G' = \text{PGL}_2/F$, so $[G']$ is not compact. For this lecture, we will ignore the technical difficulties caused by $[G']$ not being compact (e.g. stuff involving Eisenstein series/the continuous part of its spectrum).

Consider some $f \in C_c^\infty(G(\mathbb{A}))$ acting on $L^2([G])$. Recall the automorphic kernel function

$$K_f(x,y) = \sum_{\gamma \in G(F)} f(x^{-1}\gamma y) \text{ for } x, y \in [G]. \tag{1}$$

Note this is smooth in both variables. Last time we established that

$$\text{Tr}\,R(f) = \int_{[G]} K_f(x,x)\text{d}x.$$

Recall the spectral decomposition

$$L^2([G]) = \bigoplus_{\pi} \pi^{\oplus m(\pi)}$$

(using compactness of $[G]$), and that the trace is given by

$$\operatorname{Tr} R(f) = \sum_{\pi} m(\pi) \operatorname{tr} \pi(f).$$

On the RHS (1), we have

$$\int_{[G]} K_f(x,x) \mathrm{d}x = \sum_{\gamma \in G(F)/\mathrm{conj}} \operatorname{vol}([G_\gamma]) \operatorname{Orb}(\gamma, f),$$

where $G_\gamma := \left\{g \in G : g^{-1}\gamma g = \gamma\right\}$ is the centralizer of γ (stabilizer under conjugation action), $\operatorname{vol}[G_\gamma] = \operatorname{vol}\left(\frac{G_\gamma(\mathbb{A})}{G_\gamma(F)}\right)$, and

$$\operatorname{Orb}(\gamma, f) := \int_{G(\mathbb{A})_\gamma \backslash G(\mathbb{A})} f(g^{-1}\gamma g) \mathrm{d}g$$

(implicitly, we've chosen a Haar measure on $G(\mathbb{A})$ and on $G_\gamma(\mathbb{A})$, and then the corresponding measure of $G(\mathbb{A})/G_\gamma(\mathbb{A})$).

Warning 22.1. There's some subtly with choosing measure consistently everywhere. To make life a little easier, it sounds like the product $\operatorname{vol}([G_\gamma])\operatorname{Orb}(\gamma, f)$ is independent of the choice of measure of G_γ.

The upshot is we have

$$\sum_{\pi} m(\pi) \operatorname{tr} \pi(f) = \sum_{\gamma \in G(F)/\mathrm{conj}} \operatorname{vol}([G_\gamma]) \operatorname{Orb}(\gamma, f). \tag{2}$$

We see that

$$\int_{[G]} K_f(g,g) \mathrm{d}g = \int_{G(F)\backslash G(\mathbb{A})} \left(\sum_{G(F)} f(g^{-1}\gamma g) \right) \mathrm{d}g$$

We can form a sum over conjugacy classes:

$$\int_{[G]} \sum_{\gamma \in G(F)/\mathrm{conj}} \sum_{\delta \in G(F)/F_\gamma(F) \cong G(F)\cdot\gamma} f(g^{-1}\delta^{-1}\gamma\delta g) \mathrm{d}g$$

Put another way

$$K_f(g,g) = \sum_{\gamma \in G(F)/\mathrm{conj}} K_{f,\gamma}(g,g) \text{ where } K_{f,\gamma}(g,g) = \sum_{\delta \in G(F)/G_\gamma(F)} f(g^{-1}\delta^{-1}\gamma\delta g).$$

Let's interchange the order (worry about convergency later)

$$\begin{aligned} \int_{[G]} K_f(g,g) \mathrm{d}g &= \sum_{\gamma \in G(F)/\mathrm{conj}} \int_{[G]} K_{f,\gamma}(g,g) \mathrm{d}g \\ &= \sum_{\gamma \in G(F)/\mathrm{conj}} \int_{G_\gamma(F)\backslash G(\mathbb{A})} f(g^{-1}\gamma g) \mathrm{d}g \end{aligned}$$

$$= \sum_{\gamma \in G(F)/\mathrm{conj}} \mathrm{vol}\left(G_\gamma(F)\backslash G_\gamma(\mathbb{A})\right) \underbrace{\int_{G_\gamma(\mathbb{A})\backslash G(\mathbb{A})} f(g^{-1}\gamma g)\mathrm{d}g}_{\mathrm{Orb}(\gamma, f)}.$$

Recall our case of interest is $G = PB^\times$ or $G' = \mathrm{PGL}_2$. Here conjugacy classes can more-or-less be characterized by characteristic polynomials. There's a normalization issue for PGL_2, so let's pretend we're working with GL_2. We have

$$\mathrm{char} : \mathrm{GL}_2 \longrightarrow F[x]_{\deg = 2} \simeq \mathbb{A}^2.$$

Definition 22.2. γ is **regular semisimple** if $\mathrm{char}(\gamma)$ has distinct roots, i.e. the discriminant $\Delta(\gamma)$ is nonzero.

Remark 22.3. For GL_2, the elements which are not (regular semisimple) look like

$$\begin{pmatrix} \lambda & \\ & \lambda \end{pmatrix} \text{ or } \begin{pmatrix} 1 & \lambda \\ & 1 \end{pmatrix},$$

up to conjugacy.

Example. Say $\lambda = \begin{pmatrix} a & \\ & d \end{pmatrix} \in \mathrm{GL}_2(F)$ with $a \neq d$. Its centralizer is the diagonal torus

$$G_\gamma \cong A = \left\{ \begin{pmatrix} * & 0 \\ 0 & * \end{pmatrix} \right\}.$$

In this case, we look at the integral

$$\int_{A(\mathbb{A})\backslash G(\mathbb{A})} f(g^{-1}\gamma g)\mathrm{d}g$$

If we assume $f = \otimes f_v$ is a pure tensor, we can write this as an Euler product

$$\int_{A(\mathbb{A})\backslash G(\mathbb{A})} f(g^{-1}\gamma g)\mathrm{d}g = \prod_v \int_{A(F_v)\backslash G(F_v)} f_v(g_v^{-1}\gamma g_v)\mathrm{d}g_v.$$

So the question is local. Locally, $f_v = \mathbf{1}_{\mathrm{GL}_2(\mathscr{O}_{F_v})}$ for almost all v (note we've been writing $G = \mathrm{GL}_2$ for this example). Recall the Iwasawa decomposition $G(F_v) = A(F_v)N(F_v)K$ where $K = \mathrm{GL}_2(\mathscr{O}_{F_v})$ (and N unipotent upper triangular matrices?). This decomposition is useful because of the left-invariance by $A(F_v)$. If we write $g = nak$ under such a decomposition, then $\mathrm{d}g = \mathrm{d}a\mathrm{d}n\mathrm{d}k$. Let's choose these so $\mathrm{vol}(K) = 1$. Then (second equality since f_v is bi-k-invariant[44])

$$\begin{aligned} \mathrm{Orb}(\gamma, f_v = \mathbf{1}_K) &= \int_N \int_K f_v(k^{-1}n^{-1}\gamma n k)\mathrm{d}k\mathrm{d}n \\ &= \int_{N \simeq F_v} f_v\left(\begin{pmatrix} 1 & -x \\ & 1 \end{pmatrix}\begin{pmatrix} a & \\ & d \end{pmatrix}\begin{pmatrix} 1 & x \\ & 1 \end{pmatrix}\right)\mathrm{d}n \end{aligned}$$

[44]In general, K is compact anyways so the integral over K being there doesn't really affect convergence much

$$= \int_{F_v} f_v \begin{pmatrix} a & (a-d)x \\ 0 & d \end{pmatrix} \mathrm{d}x$$

This being nonzero $\implies a, d \in \mathscr{O}_{F_v^\times}$ and $(a-d)x \in \mathscr{O}_{F_v}$. Under this assumptions, this is

$$\mathrm{Orb}(\gamma, f_v = \mathbf{1}_K) = \mathrm{vol}\left\{x \in F_v : x \in \frac{1}{a-d}\mathscr{O}_{F_v}\right\} = \begin{cases} |a-d|_v^{-1} & \text{if } a, d \in \mathscr{O}_{F_v}^\times \\ 0 & \text{otherwise.} \end{cases}$$

Remark 22.4. First note that maximal tori in GL_2 are parameterized by semisimple quadratic algebras over F, i.e. E a quadratic field extension of F or $E = F \times F$. For such an E, its multiplicative group $E^\times \hookrightarrow \mathrm{GL}_2$ gives a maximal torus (embedding unique up to conjugation). Given γ regular semisimple, we can consider $E = F[x]/(\operatorname{char}\gamma)$ to get a (potentially non-split) torus.

When $G = PB^\times$ for B a quaternion division algebra, recall there is a finite set $\Sigma_B = \{v : B \otimes_F F_v \text{ nonsplit}\}$. Can consider

$$E(B) = \{E : E \text{ quad and } E \hookrightarrow B\}.$$

Exercise.

$$E(B) = \{E : E \otimes F_v \text{ nonsplit at all } v \in \Sigma_B\}.$$

In particular, there are "more" tori in $G' = \mathrm{PGL}_2$ than in $G = PB^\times$.

Example. For $\mathrm{GL}_2(F_v)$ and $\gamma \in E^\times \hookrightarrow \mathrm{GL}_2(F_v)$ with E nonsplit, one has $G_\gamma \simeq E^\times$. Can compute, for example,

Question: Implicitly assuming γ regular semisimple?

$$\int_{G_\gamma \backslash G(F_v)} f_v(g^{-1}\gamma g)\mathrm{d}g \text{ where } f_v = \mathbf{1}_{G(\mathscr{O}_{F_v})} = \mathbf{1}_K.$$

The point of all these integral calculations is to get a sense of where convergence comes from in our earlier formal manipulations. To summarize, we get

Theorem 22.5 (**Arthur-Selberg trace formula**). *When $[G]$ is compact,*

$$\sum_\pi m(\pi)\,\mathrm{tr}\,\pi(f) = \sum_\gamma \mathrm{vol}(G_\gamma)\,Orb(\gamma, f).$$

What about the non-compact case?

Say $G' = \mathrm{PGL}_2$, so $[G']$ is non-compact. Can still get "simple trace formulas" by imposing conditions on $f \in C_c^\infty(G(\mathbb{A}))$, e.g. the formula still holds if $f = 0$. Ideally, we can satisfy it for enough f to not lose two much information. Say $f = \otimes_v f_v$ and fix a non-arch place v_0. Assume $f_{v_0} =$ matrix coefficient of some supercuspidal representation σ_{v_0} of $G(F_v)$.

Recall 22.6. Given $\varphi \in \sigma$ and $\varphi^\vee \in \sigma^\vee$, the give rise to the **matrix coefficient**

$$f_{\varphi,\varphi^\vee} : g \mapsto \langle \sigma(g)\varphi, \varphi^\vee\rangle.$$

Fact. A rep σ_v of $G(F_v)$ is supercuspidal iff all matrix coefficients have compact support (modulo center)[45]

[45] true for general p-adic reductive groups

Apparently, $\operatorname{tr}\pi(f) \neq 0 \implies \pi_{v_0} \simeq \sigma_{v_0}$ (π cuspidal automorphic representation).

Warning 22.7. Globally being cuspidal $\not\Rightarrow$ there's a place where local representation is supercuspidal.

Hence, this condition does eliminate some representations on the LHS (of trace formula). Nevertheless, this condition is still useful.

Definition 22.8. Let's call $f = \otimes_v f_v$ a **cuspidal function** if f_{v_0} is a matrix coefficient of a supercuspidal representation σ_{v_0} of $G(F_{v_0})$.

Theorem 22.9. *When f cuspidal, we have*

$$\sum_\pi m(\pi)\operatorname{tr}\pi(f) = \sum_{\gamma\in G(F)/conj} \operatorname{vol}(blah)\operatorname{orb}(\gamma, f).$$

Corollary 22.10. *Any supercuspidal representation globalizes. That is, for σ_{v_0} supercuspidal, there exists a global cuspidal automorphic representation π (of PGL_2?) such that $\pi_{v_0} \simeq \sigma_{v_0}$.*

Fact. $\operatorname{orb}(\gamma, f_{\varphi,\varphi^\vee})$ is more-or-less the character $\theta_{\sigma_{v_0}}(\gamma)$ (when γ regular semi-simple) and so is nonzero.

You can use this fact to show the RHS of the theorem is nonzero which then implies the corollary (the existence of some π such that $\operatorname{tr}\pi(f) \neq 0$, i.e. so that $\pi_{v_0} \simeq \sigma_{v_0}$).

Remark 22.11. There is a more sophisticated simple trace formula s.t. you can keep all cuspidal π. Using this version, can strengthen corollary to say that all discrete series representations of $\mathrm{PGL}_2(F_{v_0})$ can be globalized.

Fact (to be proven later)**.** The mutiplicities $m(\pi)$ are all equal to 1.

We're now in position in prove (a version of) Jacquet-Langlands.

Claim 22.12. *Say π a cuspidal automorphic rep of $G = PB^\times$. Then, there exists a cuspidal automorphic rep π' of $G' = \mathrm{PGL}_2$ s.t.*

$$\pi'_v = \pi_v \text{ for all } v \notin \Sigma_B.$$

Proof. For $f \in C_c^\infty(G(\mathbb{A}))$, apply trace formula

$$\sum_\pi \operatorname{tr}\pi(f) = \sum_\gamma \operatorname{orb}(\gamma, f).$$

For $f' \in C_c^\infty(G'(\mathbb{A}))$ chosen appropriately, we similarly have

$$\sum_{\pi'} \operatorname{tr}\pi'(f') = \sum_{\gamma'} \operatorname{orb}(\gamma, f).$$

Let's identity $G(F_v) \simeq G'(F_v)$ for $v \notin \Sigma_B$. For $v \in \Sigma_B$, these are genuinely different (e.g. $G(F_v)$ compact while $G'(F_v) = \mathrm{PGL}_2(F_v)$ is not). We'll need to introduce 'transfer'

Definition 22.13. Locally at $v \in \Sigma_B$, we say f_v, f'_v are **transfers** of each other, $f_v \leftrightarrow f'_v$, if $\operatorname{orb}(\gamma, f_v) = \operatorname{orb}(\gamma', f'_v)$ whenever $\operatorname{char}(\gamma) = \operatorname{char}(\gamma')$ (in this case, we say γ, γ' are **transfer** of each other as well) and γ, γ' regular semisimple. We should probably also require $\operatorname{orb}(\gamma', f'_v) = 0$ when γ' split.

Fact. transfers exist

Back to comparison. Given $f = \otimes_v f_v$. Define $f' = \otimes_v f'_v$ where $f'_v = f_v$ if $v \notin \Sigma_B$ and $f'_v \leftrightarrow f_v$ if $v \in \Sigma_B$. Then, orbit integrals match up, so we have the identity

$$\sum_{\pi} \operatorname{tr} \pi(f) = \sum_{\pi'} \operatorname{tr}(\pi'(f')).$$

Pick a set $S \supset \Sigma_B$ s.t. π_v is unramified for all $v \notin S$. Write $f = f_S \otimes f^S$ with $f_S = \otimes_{v \in S} f_v$ and $f^S = \otimes_{v \notin S} f_v$. Consider $f^S \in \mathcal{H}(G(\mathbb{A}^S)//K^S) = \bigotimes'_{v \notin S} \mathcal{H}(G(F_v)//K_v)$ in the Hecke algebra (note this is commutative). Let $\pi^S = \bigotimes'_{v \notin S} \pi_v$. We get a character

$$\begin{array}{cccc} \lambda_{\pi^S}: & \mathcal{H}^S & \longrightarrow & \mathbb{C} \\ & f^S & \longmapsto & \operatorname{tr} \pi^S(f^S). \end{array}$$

By Strong multiplicity one for G', these $\lambda_{\pi'^S}$'s are linearly independent linear functions on $\mathcal{H}^S$.

Let's pretend this sum over π only has finitely many terms... Given π', get an identity like (recall $f^S = f'^S$)

$$\sum \operatorname{tr}(\pi_S(f_S)) \operatorname{tr} \pi^S(f^S) = \sum \operatorname{tr}(\pi'_S(f'_S)) \operatorname{tr} \pi^S(f^S).$$

Now we want to use linear independence. We conclude that, given σ^S, we must have

$$\sum_{\pi : \pi^S \simeq \sigma^S} \operatorname{tr} \pi_S(f_S) = \sum_{\pi' : \pi'^S \simeq \sigma^S} \operatorname{tr} \pi'_S(f_S).$$

Given π on G, showing nonvanishing on LHS will allow us, via transfer, to conclude that there exists π' s.t. $\pi'^S \simeq \pi^S$. ■

We'll go over the argument again next time since the idea is one that shows up repeatedly.

23 Lecture 23 (5/12)

23.1 Comparison of trace formulas, i.e. Jacquet-Langlands argument but more slowly

Let $G = B^\times / F^\times$ and $G' = \mathrm{PGL}_2$, where B is a quaternion division algebra over F. Recall that $[G]$ is compact.

Recall that we have been looking at trace formulas of the form

$$\sum_{\pi \text{ cuspidal}} m(\pi) \operatorname{tr} \pi(f) = \sum_{\gamma \in G(F)/\text{conj}} \operatorname{Orb}(\gamma, f)$$

(for $f \in C_c^\infty(G(\mathbb{A}))$). The LHS is the 'spectral' side while the RHS is the 'geometric' side. This holds for all f in the case of G (the co-compact case).

Recall 23.1. For $G' = \mathrm{PGL}_2$, the multiplicities $m(\pi')$ are all 1. We proved this before using (uniqueness of) the Whittaker model. This existed as a consequence of having a Borel defined over the ground field (assuming I heard correctly). The co-compact groups G are not quasi-split, so no Borel with which to define a Fourier expansion, so no Whittaker model argument. We will still be able to deduce that $m(\pi) = 1$ (for G) using Jacquet-Langlands.

For G', we have similar trace formulas

$$\sum_{\pi'} \operatorname{tr} \pi'(f') = \sum_{\pi'} m(\pi') \operatorname{tr} \pi'(f') = \sum_{\gamma \in G'(F)/\text{conj}} \operatorname{Orb}(\gamma', f')$$

for some $f' \in C_c^\infty(G(\mathbb{A}))$. One of the more important ingredients for JL was the notion of transfer.

- We can match characteristic polynomials between (regular semi-simple elements of) $G(F)/\text{conj}$ and $G'(F)/\text{conj}$, writing $\gamma \leftrightarrow \gamma'$ when they're char pols agree. A priori, there are more characteristic polynomials in $G'(F)$ than in $G(F)$, but we don't make use of all of them.
- (**(smooth) transfer**) Say

  $$C_c^\infty(G(F_v)) \ni f_v \longleftrightarrow f'_v \in C_c^\infty(G'(F_v))$$

  have transfer if they have the same orbit integral, i.e.

  $$\operatorname{Orb}(\gamma', f'_v) = \begin{cases} \operatorname{Orb}(\gamma, f_v) & \text{if } \gamma \leftrightarrow \gamma' \\ 0 & \text{otherwise.} \end{cases}$$

Fact (transfer exists)**.** For all f_v, there exists (non-unique) f'_v such that above identity holds.

Starting from a pure tensor $f = \otimes f_v$, we can choose transfers $f'_v \leftrightarrow f_v$ for $v \in \Sigma_B$ (the places where B is division), and choose $f'_v = f_v$ for $v \notin \Sigma_B$ (places where $G'(F_v) \simeq G(F_v)$ not division). We let $f' = \otimes f'_v$. Then, the geometric sides of the two trace formulas match, so we get an equality

$$\sum_{\pi} m(\pi) \operatorname{tr} \pi(f) = \sum_{\pi'} \operatorname{tr} \pi'(f').$$

over the spectral side, for such choices of f, f'.

For $\pi = \otimes' \pi_v$, note we have $\operatorname{tr} \pi(f) = \prod_v \operatorname{tr} \pi_v(f_v)$. For unramified π_v (with $v \notin \Sigma_B$?), we have[46] (or want?) $f_v \in \mathcal{H}_v := C_c^\infty(G(F_v)//K_v)$, the **Hecke algebra at** v (I think). This is a commutative algebra $\mathcal{H}_v = \mathbb{C}[T_v]$ with

$$T_v = \mathbf{1}_{E_v} \text{ where } E_v := K_v \begin{pmatrix} \varpi_v & \\ & 1 \end{pmatrix} K_v,$$

the **Hecke operator at** v. Note that $\pi_v(f_v) \subset \pi_v^{K_v}$ where we recall

$$\pi_v(f_v)\varphi = \int_{G(F_v)} f_v(g)\pi(g)\varphi \mathrm{d}g \text{ for } \varphi \in \pi_v.$$

Recall $\pi_v^{K_v} = \mathbb{C}\varphi_v^\circ$ is 1-dimensional. Thus, for $f_v \in \mathcal{H}_v$, the trace is simply the eigenvalue of $f_v \curvearrowright \mathbb{C}\varphi_v^\circ$,

[46] Here K_v is a maximal compact

so we name the homomorphism

$$\begin{array}{rccc} \lambda_{\pi_v}: & \mathcal{H}_v & \longrightarrow & \mathbb{C} \\ & f_v & \longmapsto & \text{eigenvalue of } \pi_v(f_v) \text{ on } \pi_v^{K_v}. \end{array}$$

We see this depends only on the Satake parameters[47] of $\pi_v = \operatorname{Ind}_B^G \chi$.

Question: Why is π_v a principal series here?

Fix a finite set $S \supset \Sigma_B$ of places (containing all the 'bad' places). Write $f = f_S \otimes f^S$ as we did last semester. We fix f_S but vary

$$f^S \in \mathcal{H}^S := \bigotimes_{v \notin S}{}' \mathcal{H}_v.$$

Consider the linear functional

$$f^S \mapsto \sum_{\pi} \operatorname{tr}(\pi_S(f_S))\lambda_{\pi^S}(f^S).$$

as well as the linear functional

This makes sense since f_S fixed and f^S contained only in the good places

$$f^S \mapsto \sum_{\pi'} \operatorname{tr} \pi'_S(f'_S)\lambda_{\pi'^S}(f^S)$$

on the same space.

Warning 23.2. To avoid dealing with infinite sums, we will *pretend* that both sums are finite.

Lemma 23.3 (Linear independence of characters). *Given (finitely many) distinct irreducible unramified representations π_i^S of $G(\mathbb{A}^S) = \prod'_{v \notin S} \mathrm{PGL}_2(F_v)$, the $\lambda_{\pi_i^S} \in \operatorname{Hom}(\mathcal{H}^S, \mathbb{C})$ are linearly independent.*

(Exercise: prove this)

Fix any $\sigma^S = \bigotimes'_{v \notin S} \sigma_v$ irreducible and unramified. The spectral identity

$$\sum_{\pi} \operatorname{tr}(\pi_S(f_S))\lambda_{\pi^S}(f^S) = \sum_{\pi} \operatorname{tr}(\pi_S(f_S))\lambda_{\pi^S}(f^S)$$

along with linear independence implies that we must have

$$\sum_{\pi : \pi \simeq \sigma^S} \operatorname{tr} \pi_S(f_S) = \sum_{\pi' : \pi'^S \simeq \sigma^S} \operatorname{tr} \pi'_S(f'_S) \tag{3}$$

for all transfers $f_S \leftrightarrow f'_S$.

Example. Can choose $\sigma^S = \pi_0'^S$ for π'_0 cuspidal on $[G']$. By strong multiplicity one for G' (proved using Kirillov and Whittaker models. See lecture notes on the dropbox), the RHS of (3) has at most one term:

$$\sum_{\pi : \pi^S \simeq \pi'^S} \operatorname{tr}(\pi_S(f_S)) = \operatorname{tr} \pi'_S(f'_S).$$

Note that both sides only involve finitely many places $v \in S$. This means we're now reduced to local theory. What can we deduce from this equality?

- If $\operatorname{tr} \pi_S(f'_S) \neq 0$ for some f'_S, then $LHS \neq 0$. Thus, $\exists \pi$ cuspidal such that $\pi^S \simeq \pi'^S$. This is already non-trivial.

[47] The set $\{\chi_1(\varpi), \chi_2(\varpi)\}$

To get the whole of Jacquet-Langlands, you want linear independence for functions

$$\operatorname{tr}\pi_v : C_c^\infty(G(F_v)) \to \mathbb{C}.$$

This will imply, for example, that given $\pi = \pi_S \otimes \pi^S$, there exists π' cuspidal on G' s.t.

- $\pi^S \simeq \pi'^S$ and for all $v \in S$; and
- $\prod_v \operatorname{tr}\pi_v(f_v) = \prod_v \operatorname{tr}\pi'_v(f'_v)$ for all transfers $f_v \leftrightarrow f'_v$

This second property in particular will imply the local Jacquet-Langlands.[48]

Remark 23.4. The key of all of this is the existence of transfers. This is what gave the bridge between G and G'. That combined with their individual trace formulas is what allows one to compare reps between the two.

23.2 What's next?

The next two examples of applications of trace formulas we want to see are

- base change for GL_2
- Waldspurger formula

These will need to make use of a "relative trace formula."

Base change for GL_n The motivation for this is **Langlands reciprocity** which (morally, at least) conjectures the existence of a bijection

$$\left\{\begin{matrix}\text{cuspidal automorphic}\\ \text{reps of } \mathrm{GL}_{n,F}\end{matrix}\right\} \longleftrightarrow \left\{\begin{matrix}\text{irred } n\text{-dimensional}\\ \text{reps of } \operatorname{Gal}(\overline{F}/F)\end{matrix}\right\}.$$

This predicts a lot of things for cuspidal automorphic reps. On the Galois side, there are many operations one can apply to representations (restriction, tensor product, etc.) which should have analogs in for cuspidal automorphic reps.

Example. Say E/F be a finite extension. Given an n-dimensional Gal_F-rep ρ, one can form the n-dimensional Gal_E-rep $\rho|_{\operatorname{Gal}_E}$ given by restriction.

By reciprocity, one conjectures the existence of a base change operation BC s.t. given by π for $\mathrm{GL}_{n,E}$ one can form $\Pi = BC(\pi)$ for $\mathrm{GL}_{n,F}$ so that $\rho_\Pi = \rho_\pi|_{\operatorname{Gal}_E}$.

Remark 23.5. By strong multiplicity one for GL_n, $\Pi = BG(\pi)$ is characterized by the following (recall E/F finite): let v be an unramified place of F, and write $E_v \simeq \prod_{w|v} E_w$; then, we want $\Pi_v \in \operatorname{Rep}_n(\mathrm{GL}_n(E_v))$ $(\mathrm{GL}_n(E_v) = \prod_{w|v}\mathrm{GL}_n(E_w))$. Say π_v is **unramified**, i.e. $\pi_v = \operatorname{Ind}_B^G \chi$ for $\chi = (\chi_1, \dots, \chi_n)$ with each $\chi_i : F_v^\times \to \mathbb{C}^\times$ unramified. We have a norm map $E_v^\times \xrightarrow{\mathrm{Nm}} F_v^\times$, so can lift χ_i to a character of $E_v^\times$. We require

$$\Pi_v = \operatorname{Ind}_{B(E_v)}^{G(E_v)}(\chi \circ \mathrm{Nm}_v).$$

This condition characterizes base change.

For writing down this characterization, we don't actually need v unrami-

[48] Really need to get rid of the product and in fact prove $\operatorname{tr}\pi_v(f_v) = -\operatorname{tr}\pi'_v(f'_v)$

The point is strong multiplicity one allows us to ignore finitely many places (e.g. ignore all the ramified ones).

Theorem 23.6. *Suppose E/F is cyclic.*

(a) *For any cuspidal auto π of $\mathrm{GL}_{n,F}$, there exists a unique Π automorphic for $\mathrm{GL}_{n,E}$ such that*

$$\Pi_v = BC(\pi_v)$$

for unramified π_v (more generally, for principal series).

Warning 23.7. The basechange may no longer be cuspidal.

(b) *Say Π cuspidal auto for $\mathrm{GL}_{n,E}$. Then Π is a basechange iff it is $\mathrm{Gal}(E/F)$-invariant.*[49]

Warning 23.8. Describing non-cuspidal reps in the image is more difficult.

(c) *The fiber of BC is precisely $\left\{\pi \otimes \eta : \eta|_{\mathrm{Nm}\,\mathbb{A}_E^\times}\right\}$ so one has*

$$\begin{tikzcd}
\mathbb{A}_F^\times/F^\times \arrow[r, "\eta"] \arrow[d] & \mathbb{C}^\times \\
\mathbb{A}_F^\times/F^\times \cdot \mathrm{Nm}\,\mathbb{A}_E^\times \arrow[r, "\sim", "CFT"'] & \mathrm{Gal}(E/F) \arrow[u]
\end{tikzcd}$$

The number of such η is precisely $\#\,\mathrm{Gal}(E/F) = [E : F]$.

Remark 23.9. Base change for general (non-Galois) extensions is still open. From the cyclic case, though, one can at least deduce the solvable Galois case.

For GL_2, this is due to Langlands. For GL_n, it is due to Arthut-Clozel.

Remark 23.10. The GL_n case implies Artin's conjecture for $\rho : \mathrm{Gal}_{\overline{F}/F} \to \mathrm{GL}_n(\mathbb{C})$ (finite image) with solvable image.

Question: ?

Langlands-Tunnel proved Artin conjecture for GL_2 when the image is solvable. This was a key ingredient in Wiles' proof of modularity. Given an elliptic curve $E/\mathbb{Q}$, looking at 3-torsion gives a rep

$$\overline{\rho}_{E,3} : \mathrm{Gal}_{\mathbb{Q}} \longrightarrow \mathrm{GL}_2(\mathbb{F}_3) \cong S_4$$

(and S_4 is solvable). Can embed $\mathrm{GL}_2(\mathbb{F}_3) \hookrightarrow \mathrm{GL}_2(\mathbb{C})$ to get an Artin representation with solvable image. From here, Wiles can apply his modularity lifting result and eventually prove modularity.

Let's say a bit about the proof strategy. Say $G = \mathrm{GL}_{n,F}$ and $G' = \mathrm{GL}_{n,E}$. Given π on G we want $\Pi = BC(\pi)$ on G'. We have (simple, since GL_n not compact) trace formulas

$$\sum_{\pi} \mathrm{tr}\,\pi(f) = \sum_{\gamma} \mathrm{Orb}(\gamma, f) \;\text{ and }\; \sum_{\Pi} \mathrm{tr}\,\Pi(f') =$$

for some $f \in C_c^\infty(G(\mathbb{A}))$ and some $f' \in C_c^\infty(G'(\mathbb{A}))$. We should not expect these to be equal; the RHS should have 'more' automorphic representations. Can we instead design a different trace formula so that the spectral side only contains Π which are Gal-invariant?

[49] $\mathrm{Gal}(E/F) \curvearrowright \mathrm{GL}_n(\mathbb{A}_E)$ so given $\sigma \in \mathrm{Gal}(E/F)$ gives rise to $\pi \rightsquigarrow \sigma(\pi)$

Next time: "twisted trace formula" s.t. spectral side on $\mathrm{GL}_{n,E}$ only contains Π s.t. $\sigma(\Pi) = \Pi$.

24 Lecture 24 (5/17): Base change

The remaining two lectures will be about base change for GL_n and about a result of Waldspurger.

Setup. Say E/F is a cyclic extension of *number fields* and $G = \mathrm{GL}_2$ (or GL_n).

Goal. We want to defines a 'basechange map'

$$\left\{\begin{matrix}\text{cuspidal auto}\\ \text{rep of } G_F\end{matrix}\right\} \xrightarrow{\mathrm{BC}} \left\{\begin{matrix}\text{auto rep}\\ \text{of } G_E\end{matrix}\right\}$$

sending $\pi \mapsto \mathrm{BC}(\pi) = \Pi$ with image containing all Galois invariant cuspidal Π of G_E.

How do we approach this? The idea is to compare the Arthur-Selberg trace formula for G_F to a "twisted" trace formula for G_E. On the E side, we only want to keep Galois invariant automorphic reps. Without this restriction, we cannot hope for a direct comparison between

$$\sum_{\pi \text{ cusp}} \mathrm{tr}(\pi(f)) \text{ and } \sum_{\Pi} \mathrm{tr}(\Pi(f_E)).$$

Recall 24.1. We assume E/F is cyclic, so let $\sigma \in \mathrm{Gal}(E/F)$ be a generator.

We only want $\Pi = {}^{\sigma}\Pi$.

Recall 24.2.

$$\mathrm{tr}(R(f)) = \int_{[G]} K_f(x,x)\mathrm{d}x$$

where $K_f(x,y) = \sum_{\gamma \in G(F)} f(x,y)$ is a function on $[G] \times [G]$.

Consider some π cuspidal automorphic. Choose some orthonormal basis $ON(\pi)$ of π. Can consider

$$K_{f,\pi}(x,y) = \sum_{\varphi \in ON(\pi)} \pi(f)\varphi(x)\overline{\varphi(y)}.$$

Recall that $L^2_0([G]) = L^2_{\mathrm{cusp}}([G]) = \bigoplus_\pi \pi$. We have $R(f)$ acting on this space.

Note 19. I've probably said this before, but reminder that the lecture notes in the dropbox contain extra details not gone over in lecture.

One has

$$K_{f,\mathrm{cups}} = \sum_{\pi \text{ cusp}} K_{f,\pi} \overset{*}{=} K_f$$

with last equality holding of $[G]$ compact (not the case for $G = \mathrm{GL}_n$) or if one restricts only to 'special' choices of f (this is what you do for $G = \mathrm{GL}_n$). Apparently one sees that

$$\mathrm{tr}\,\pi(f) = \int_{[G]} K_{f,\pi}(x,x)\mathrm{d}x.$$

Keep in mind we'd like to incorporate the Galois-invariancy condition ${}^{\sigma}\Pi = \Pi$. We start with

$$L^2_0([G_E]) = \bigoplus \Pi.$$

To test whether two representations are the same or not, one can use inner products (different reps are orthogonal). If $\langle \varphi, \psi \rangle \neq 0$ with $\varphi \in \Pi_1$ and $\psi \in \Pi_2$, then $\Pi_1 \simeq \Pi_2$. Recall this inner product is

$$\langle \varphi, \psi \rangle = \int_{[G_E]} \varphi(g)\overline{\psi}(g)\mathrm{d}g = \int_{[G_E]} (\varphi \otimes \overline{\psi})(g, g)\mathrm{d}g.$$

Let $\widetilde{G}_E := G_E \times G_E$, so $\Pi_1 \boxtimes \Pi_2$ is an automorphic rep on $\widetilde{G}_E$. The above integral over (the diagonal of) $\widetilde{G}_E$ can be used to detect whether two reps are isomorphic.

Definition 24.3. Given algebraic groups $H \subset G$ over F (= number field?) and π a cuspidal auto rep (on G?). For $\varphi \in \pi$, its **automorphic period integral** is

$$\wp_H(\varphi) = \int_{[H]} \varphi(h)\mathrm{d}h.$$

Note this is a map $\wp_H : \pi \to \mathbb{C}$ and is $H(\mathbb{A})$-invariant.

Example. Take $G = \mathrm{GL}_{n,E} \times \mathrm{GL}_{n,E}$ and $H = \Delta(\mathrm{GL}_{n,E})$ the diagonal (or use ${}^{\sigma}H$ = graph of σ, i.e. the image of $h \mapsto (h, \sigma(h))$)

Warning 24.4. As often, we're ignoring issues caused by having a nontrivial center.

Example. Can take $G = \mathrm{GL}_{2,F}$ and $H = A = \left\{ \begin{pmatrix} * & \\ & * \end{pmatrix} \right\} \subset G$. Hecke-Jacquet-Langlands says

$$\wp_A(\varphi) = \int_{[A]} \varphi(a)\mathrm{d}a \overset{s=1/2}{\sim} L(\pi, 1/2)$$

(there should be some local factors on the right, and imagine a factor of $|a|^{s-1/2}$ in the integrand on the left)

Definition 24.5. We say π is ***H*-distinguished** if $\wp_H(\varphi) \neq 0$ for some $\varphi \in \pi$.

Remark 24.6. π is H-distinguished $\implies \mathrm{Hom}_{H(\mathbb{A})}(\pi, \mathbb{C}) \neq 0$.

Recall we want a nice way to detect whether or not Π is Galois invariant. This led us to considering representations of the product $\mathrm{GL}_{n,E} \times \mathrm{GL}_{n,E}$. Consider the diagram

$$H_1 := \Delta(\mathrm{GL}_{n,E}) \hookrightarrow \mathrm{GL}_{n,E} \times \mathrm{GL}_{n,E}.$$

Let $\Sigma = \Pi_1 \boxtimes \Pi_2$ be some product rep. Then,

Fact. Σ is H_1-distinguished $\iff \Pi_1 \simeq \Pi_2^{\vee}$.

(the contragradient comes from taking complex conjugation at some point)

Let $H_2 = \Gamma_\sigma$ be the graph of $\sigma \in \mathrm{Gal}(E/F)$. Then,

Fact. Σ is H_2-distinguished $\iff \Pi_1 \simeq {}^\sigma\Pi_2^\vee$.

Corollary 24.7. *Σ is both H_1 and H_2-distinguished iff it is of the form $\Sigma \simeq \Pi \boxtimes \Pi^\vee$ where $\Pi \cong {}^\sigma\Pi$.*

Now we want to design a trace formula.

24.1 (Relative) Trace formula stuff, I think

Say G some algebraic group (e.g. $G = \mathrm{GL}_{n,E} \times \mathrm{GL}_{n,E}$) with chosen subgroups $H_1, H_2 \subset G$. Can consider the integral

$$I(f) := \int_{[H_1]} \int_{[H_2]} K_f(x,y)\mathrm{d}x\mathrm{d}y.$$

Warning 24.8. We're going to ignore non-cuspidal contributions.

Spectral side We get something like

$$I_\pi(f) := \int_{[H_1 \times H_2]} K_{f,\pi} = \sum_{\varphi \in ON(\pi)} \wp_{H_1}(\pi(f)\varphi)\overline{\wp_{H_2}(\varphi)}.$$

Then, $I_\pi(f) \neq 0$ for some $f \in C_c^\infty(G(\mathbb{A})) \iff \pi$ is distinguished by both H_1 and H_2.

Example. We're interested in the case $G = \mathrm{GL}_{n,E} \times \mathrm{GL}_{n,E}$, $H_1 = \Delta = \Gamma_{\mathrm{id}}$, and $H_2 = \Gamma_\sigma$.

The trace formula one gets like this is called a **relative trace formula** or **twisted trace formula**.

Example. When $G = H \times H$ and $H_1 = H_2 = \Delta(H)$, one recovers the Arthur-Selberg trace formula for H. One has something like $\Sigma = \pi \boxtimes \pi^\vee$ and

$$\int_{[H]} \int_{[H]} K_f(blah) \rightsquigarrow \int_{[H]} K_{f_1 * f_2}(x,x)\mathrm{d}x \text{ where } f = f_1 \otimes f_2 \text{ on } (H \times H)(\mathbb{A}).$$

Geometric side Let's (formally) consider a general case, so still have G with subgroups H_1, H_2. What's the geometric side of the relative trace formula attached to (G, H_1, H_2)? Recall $K_f = \sum_{\gamma \in G(F)} f(x^{-1}\gamma y)$. One gets (w/o worrying about convergence)

$$\begin{aligned} I(f) &= \int_{[H_1]} \int_{[H_2]} K_f(x,y)\mathrm{d}x\mathrm{d}y \\ &= \sum_{\gamma \in H_1(F) \backslash G(F) / H_2(F)} \mathrm{vol}(H_\gamma)\mathrm{Orb}(\gamma, f) \end{aligned}$$

Note $H := H_1 \times H_2$ acts on G via $(h_1, h_2) \cdot g := h_1^{-1} g h_2$. Above,

$$\mathrm{Orb}(\gamma, f) := \int_{H_\gamma(\mathbb{A}) \backslash H(\mathbb{A})} f(h_1^{-1}\gamma h_2)\mathrm{d}h_1\mathrm{d}h_2,$$

where $H_\gamma = \mathrm{Stab}_H(\gamma)$ (for $\gamma \in G$).

Note the parameter space fo the sum on the geometric side is the double coset space $H_1(F) \backslash G(F) / H_2(F)$.

24.2 Back to base change

What is the double coset space

$$\Delta(\mathrm{GL}_n(\mathbb{E}))\backslash \mathrm{GL}_n(E) \times \mathrm{GL}_n(E)/\Gamma_\sigma?$$

Can first consider the quotient on the left to see this is

$$\mathrm{GL}_n(E)//^\sigma \mathrm{GL}_n(E),$$

where $//^\sigma$ means take quotient by *σ-conjugation*:

$$h \cdot \gamma = {}^\sigma h^{-1}\gamma h \text{ for } h, \gamma \in \mathrm{GL}_n(E).$$

Definition 24.9. We call an element of $\mathrm{GL}_n(E)/\sigma$-conjugation a *σ-conjugacy class.*

We want to relate this to $\mathrm{GL}_n(F)/\mathrm{conj}$.

Recall 24.10. Recall the norm map $\mathrm{Nm} = \mathrm{Nm}_{E/F} : E^\times \to F^\times$. This sends $x \mapsto \mathrm{Nm}(x) = \prod_{i=0}^{d-1} \sigma^i(x)$, where $d = [E : F]$.

We want to generalize this from GL_1 to GL_n. For any $\gamma \in \mathrm{GL}_n(E)$, we define

$$\mathrm{Nm}\,\gamma = \prod_{i=0}^{d-1} \sigma^i(\gamma) \in \mathrm{GL}_n(E)$$

(note that the order matters). Say γ' is σ-conjugate to γ, i.e. $\gamma' = g\gamma\sigma(g)^{-1}$. Then,

$$\begin{aligned}
\mathrm{Nm}\,\gamma' &= \gamma'\sigma(\gamma')\ldots\sigma^{d-1}(\gamma') \\
&= g\gamma\sigma\,(g)^{-1}\,\sigma(g)\sigma(\gamma)\sigma^2(g^{-1})\ldots \\
&= g\,\mathrm{Nm}(\gamma)\sigma^d(g^{-1}) \\
&= g\,\mathrm{Nm}(\gamma)g^{-1}
\end{aligned}$$

That is, Nm "untwists" conjugacy. Note that the image of Nm is contained in $\mathrm{GL}_n(E)$, not $\mathrm{GL}_n(F)$. However, one can check that the characteristic polynomial of $\mathrm{Nm}\,\gamma$ has F-coefficients (even in $\mathrm{Nm}\,\gamma$ itself does not). Hence, there will always be some $\delta \in \mathrm{GL}_n(F)$ such that $\delta \underset{\mathrm{GL}_n(E)}{\sim} \mathrm{Nm}\,\gamma$, i.e. δ is $\mathrm{GL}_n(E)$-conjugate to $\mathrm{Nm}\,\gamma$. Thus, Nm does in fact induce a map

$$\mathrm{Nm} : \frac{\mathrm{GL}_n(E)}{\sigma\text{-conj}} \longrightarrow \frac{\mathrm{GL}_n(F)}{\mathrm{conj}}$$

(where, as usual, we restrict attention to regular semisimple elements only).

Remark 24.11. Sounds like this Nm map is injective (at least, on regular semisimple elements).

Notation 24.12. We now understand $\mathrm{Nm}\,\gamma$ to refer to the associated conjugacy class of $\mathrm{GL}_n(F)$.

To compare the trace formulas for $\mathrm{GL}_{n,F}$ and $\mathrm{GL}_{n,E}$, we need to obtain identities between orbital integrals.

Recall 24.13. For $f \in C_c^\infty(G(E))$,

$$\mathrm{Orb}(\gamma, f) = \int_{\mathrm{GL}_n(E)/\,\mathrm{Stab}} f(g\gamma\sigma(g)^{-1})\mathrm{d}g.$$

For $f' \in C_c^\infty(\mathrm{GL}_n(F))$,

$$\mathrm{Orb}(\delta, f')$$

This comparison is a local question, so can assume E/F local. One should expect

$$\mathrm{Orb}(\gamma, f) = \mathrm{Orb}(\mathrm{Nm}\,\gamma, f')$$

when f, f' are appropriately related. Recall the Hecke algebra $\mathcal{H}_E = C_c^\infty(\mathrm{GL}_n(E)//K_E)$. It turns out that there is a map (really, $\mathbb{C}$-algebra homomorphism)

$$\mathrm{BC} : \mathcal{H}_E \to \mathcal{H}_F$$

'dual' to basechange of representations so that $\mathrm{Orb}(\gamma, f) = \mathrm{Orb}(\mathrm{Nm}\,\gamma, \mathrm{BC}(f))$ (**Fundamental Lemma**).

Recall 24.14. There's

$$\mathrm{BC} : \mathrm{Rep}^{\mathrm{unr}}(\mathrm{GL}_n(F)) \longrightarrow \mathrm{Rep}^{\mathrm{unr}}(\mathrm{GL}_n(E))$$

for unramified reps. It sends $\pi = \mathrm{Ind}_{B_F}^{G_F}\chi$ to $\mathrm{BC}(\pi) = \mathrm{Ind}_{B_E}^{G_E}(\chi \circ \mathrm{Nm})$.

For $f \in \mathcal{H}_E$, $\mathrm{BC}(f)$ is characterized by

$$\mathrm{Tr}(\mathrm{BC}(\pi)(f)) = \mathrm{Tr}(\pi(\mathrm{BC}(f))).$$

In other symbols, $\langle \mathrm{BC}(\pi), f\rangle = \langle \pi, \mathrm{BC}(f)\rangle$. The *fundamental lemma*

$$\mathrm{Orb}(\gamma, f) = \mathrm{Orb}(\mathrm{Nm}\,\gamma, \mathrm{BC}(f))$$

is one of the harder parts of carrying out this comparison.

Note 20. Got distracted and missed some stuff. I think Wei said something about who proved fundamental lemmas for various groups and how base change leads to something called 'endoscopy functoriality'. Apparently Ngô won a fields medal in part for proving one (or many?) fundamental lemma(s?).

This is all we'll say about basechange.

24.3 Waldspurger Theorem

Let $G = \mathrm{PGL}_2$ and let $A \subset G$ be the diagonal torus as usual. Let E/F be a quadratic field. Choosing a basis of E/F gives rise to an embedding $E \hookrightarrow M_2(F)$ inducing $E^\times/F^\times \hookrightarrow \mathrm{GL}_2(F)$ (this embedding is unique up to conjugation). Let T be the image of this embedding, a non-split (since E a field, not $F \times F$) maximal torus.

Let π be a cuspidal automorphic rep of $G(\mathbb{A})$. Recall the period integral

$$\wp_T(\varphi) = \int_{[T]} \varphi(t)\mathrm{d}t.$$

Remark 24.15. $[T]$ is compact. Note that

$$[T] = T(F)\backslash T(\mathbb{A}) \simeq E^\times\backslash\mathbb{A}_E^\times/\mathbb{A}_F^\times.$$

Recall that $E^\times\backslash\mathbb{A}_E^1$ is compact (where $\mathbb{A}_E^1$ is the norm 1 ideles).

Theorem 24.16 (Waldspurger's Theorem). *TFAE*

(i) *π is T-distinguished ($\wp_T(\varphi) \neq 0$ for some $\varphi \in \pi$)*

(ii) $L(\pi, \frac{1}{2})L(\pi \otimes \eta_{E/F}, \frac{1}{2}) \neq 0$
(and $\mathrm{Hom}_{T(\mathbb{A})}(\pi, \mathbb{C}) \neq 0$*)*

This is a local condition: $\mathrm{Hom}_{T(F_v)}(\pi_v, \mathbb{C}$ 0 for all v

Above, $\eta_{E/F} : F^\times\backslash\mathbb{A}_F^\times \to \{\pm 1\}$ *is* $F^\times\backslash\mathbb{A}_F^\times \twoheadrightarrow F^\times\backslash\mathbb{A}_F^\times/\mathbb{A}_E^\times \xrightarrow{\sim} \{\pm 1\}$ *(recall E/F quadratic).*

Note that if we let $\pi_E = \mathrm{BC}(\pi)$ (exists since E/F quadratic so cyclic), then

$$L(\pi_E, s) = L(\pi, s)L(\pi \otimes \eta, s)$$

(unclear to me if this a theorem or follows quickly from definitions).

This result is related to B-SD. Let $A/\mathbb{Q}$ be an elliptic curve. B-SD concerns $L(A, s) = L(\pi, s+\frac{1}{2})$ where $A \rightsquigarrow \pi$ for $\mathrm{GL}_{2,\mathbb{Q}}$ (by modularity). Leet $E/\mathbb{Q}$ be a quadratic extension, so $L(A/E, s) = L(\pi_E, s + \frac{1}{2})$. Then B-SD predicts

$$L(A/E, 1) \neq 0 \implies \operatorname{rank} A(E) = 0 \text{ and } \#\text{Ш}(A/E) < \infty.$$

This has been proved using Waldpurger's theorem (along with other stuff).

In the last lecture, we will try to indicate how to design a trace formula needed to prove Waldspurger.

25 Lecture 25 (5/19): Last lecture (AKA Waldspurger's theorem)

Last lecture, so let's be more relaxed.

Let's state a more general version of Waldspurger's result than last time.

Let F be a global field, and let E/F be a quadratic extension. Let B be some quaternion algebra of F with an embedding $E \hookrightarrow B$ of F-algebras ($\iff$ all places in Σ_B are non-split in E). Let $G = B^\times/F^\times$ and $G' = \mathrm{PGL}_2$. Given (cuspidal automorphic) π on G, let $\pi' = JL(\pi)$ on G' correspond to it by Jacquet-Langlands.

For $v \in \Sigma_B$, B_v is a division algebra. If v is split in E, $E_v = F_1 \times F_2$ will locally be a product, and so cannot embed into a division algebra.

For $H = E^\times/F^\times \subset G$ (a nonsplit torus), recall the period integral

$$\wp_H(\varphi) = \int_{[H]} \varphi(h)\mathrm{d}h$$

(so $\wp_H \in \mathrm{Hom}_{H(\mathbb{A})}(\pi, \mathbb{C})$). Recall we say that π is (globally[50]) *H-distinguished* $\iff$ $\wp_H(\varphi) \neq 0$ for some φ.

[50]Note this notion depends upon the embedding $\pi \hookrightarrow \mathcal{A}_0([G])$

Theorem 25.1 (**Waldspurger's Theorem**). *TFAE: for all $B \supset E$*

(a) *π is H-distinguished*

(b) *$L(\pi'_E, \frac{1}{2}) \neq 0$ (with $_E$ denoting base change to E)*

(and $\mathrm{Hom}_{H(\mathbb{A})}(\pi, \mathbb{C}) \neq 0$, i.e. $\pi = \otimes'_v \pi_v$ locally distinguished[51])

Remark 25.2. Say we're given $\pi' = \otimes'_v \pi'_v$ cuspidal auto rep of $G' = \mathrm{PGL}_2$.

Fact. There is at most one quaternion algebra B s.t $\mathrm{Hom}_{H(\mathbb{A})}(\pi^B, \mathbb{C}) \neq 0$, i.e. where π^B locally distinguished

(Above, $\pi^B \leftrightarrow \pi$ via Jacquet-Langlands). In fact,

$$\dim \mathrm{Hom}_{H(F_v)}(\pi'_v, \mathbb{C}) + \dim \mathrm{Hom}_H(\pi_v^{B_v}, \mathbb{C}) = 1,$$

so there's some dichotomy here.

How does one prove Walspurger? Let's ignore the 'locally distinguished' stuff. The idea is to compare two local trace formulas. We rewrite

(a) π is H-distinguished

(b) $\pi' = JL(\pi)$ is A-distinguished and (A, η)-distinguished

(Note $L(\pi'_E, \frac{1}{2}) = L(\pi, \frac{1}{2}) L(\pi \otimes \eta_{E/F}, \frac{1}{2})$ where $\eta = \eta_{E/F}$ is the character attached to the quadratic extension).

For seeing that this **(b)** is equivalent to the earlier **(b)**, recall that

$$L(\pi', \frac{1}{2}) \sim \int_{[A]} \varphi(a) \mathrm{d}a \ \text{ and } \ L(\pi' \otimes \eta, \frac{1}{2}) \sim \int_{[A]} \varphi(a) \eta(a) \mathrm{d}a,$$

for suitable choice of automorphic form φ.

Relative trace formula For $f \in C_c^\infty(G(\mathbb{A}))$, we have our old friend the kernel function $K_f(x,y)$. We can attach to this the integral

$$I(f) = \int_{[H]} \int_{[H]} K_f(x,y) \mathrm{d}x \mathrm{d}y.$$

- On the spectral side (cuspidal part), one has

$$\sum_\pi I_\pi(f) \ \text{ where } \ I_\pi(f) = \sum_{\varphi \in ON(\pi)} \wp_H(\pi(f)\varphi) \overline{\wp_H(\varphi)}.$$

  One has $I_\pi(f) \neq 0$ for some $f \in C_c^\infty(G(\mathbb{A})) \iff \wp_H \neq 0 \in \mathrm{Hom}(\pi, \mathbb{C})$. One calls $I_\pi(f)$ a "relative trace" of $\pi(f)$.

  Note 21. Wei said more on this, but I was distracted.

  This relative trace detects distinction by this subgroup.

[51] This is almost like just requiring $\mathrm{Hom}_{H(F_v)}(\pi_v, \mathbb{C}) \neq 0$ for all v

We can do the same thing for $G' = \mathrm{PGL}_2$. For $f' \in C_c^\infty(G'(\mathbb{A}))$, we set

$$J(f') = \int_{[A]}\int_{[A]} K_f(x,y)\eta(y)\mathrm{d}x\mathrm{d}y.$$

(Note the quadratic twist. Without it, the J side has no information on the quadratic fields). On the spectral side, one similarly gets $J_{\pi'}(f')$.

Hence, Waldspurger can be written as $I_\pi \neq 0 \iff J_{\pi'} \neq 0$. Thus, to prove it, suffices to relate $J_{\pi'}$ and I_π.

Comparing orbits The minimal thing one can do is match "orbits," the double cosets

$$H(F)\backslash G(F)/H(F) \longleftrightarrow A(F)\backslash G'(F)/A(F),$$

i.e. we'd like a bijection between 'good' (think: regular semisimple) orbits. Once one can matchup orbits like these, you want transfers and fundamental lemmas, i.e. you want identities between orbit integrals.

For all f, you want some f' such that $\mathrm{Orb}(\gamma, f) = \mathrm{Orb}(\delta, f')$ for any corresponding orbits $\gamma \leftrightarrow \delta$. This will let you compare the geometric sides of the trace formulas. Also want $\mathcal{H}_K \ni f = f' \in \mathcal{H}'_{K'}$ (with $\mathcal{H}_K \cong \mathcal{H}'_{K'}$) to imply that the orbit integrals match; this is not a tautology (even though the groups are the same in this case, the subgroups are not the same). You want

$$\int_{(H\times H)(F_v)} f(h_1^{-1}\gamma h_2)\mathrm{d}h_1\mathrm{d}h_2 = \int_{(A\times A)(F_v)} f'(h_1^{-1}\delta h_2)\eta(h_2)\mathrm{d}h_1\mathrm{d}h_2.$$

How do we understand orbits? Say for GL_2, so we're looking at the double coset space $A\backslash \mathrm{GL}_2/A$ ($A =$diagonal torus). Note there's a map

$$\begin{array}{ccc} A\backslash \mathrm{GL}_2/A & \longrightarrow & \mathbb{P}^1 \\ \delta = \begin{pmatrix} a & b \\ c & d \end{pmatrix} & \longmapsto & \dfrac{bc}{ad}. \end{array}$$

This really lands in $\mathbb{P}^1 \setminus \{1\} \cong \mathbb{A}^1$ since $\det\delta = ad - bc \neq 0$. What are the "good" orbits? We have $A \times A \curvearrowright G$, and we like the orbits which are *closed* subschemes of $G = \mathrm{GL}_2$. Also, we like finite stabilizers.

These conditions inspired from GIT if I heard correctly

Definition 25.3. We'll say δ is **regular semisimple** iff $\mathrm{Inv}(\delta) \notin \{0, \infty\}$.

Hence, we can classify the bad δ, e.g.

- $\mathrm{Inv}(\delta) = 0 \iff bc = 0$, giving thee bad orbits $(b = 0, c \neq 0), (b \neq 0, c = 0), (b = c = 0)$. Note that the orbits of $\begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix}$ and $\begin{pmatrix} 1 & 0 \\ 1 & 1 \end{pmatrix}$ are not closed (e.g. the closures of their orbits contain $\begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}$).

  That stabilizer of $\begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}$ is too big, I think (I didn't quite hear).

To summarize, $A\backslash G/G \simeq \mathbb{P}^1 \setminus \{1\}$ is a variety with good orbits $A\backslash G^{\text{reg, s.s.}}/A \xrightarrow{\sim} \mathbb{P}^1 \setminus \{0, 1, \infty\}$.

Remark 25.4. Really understand $A\backslash G/A$ as $G/A \times A$. Can view this as a GIT quotient, but here they're identified with familiar schemes.

Warning 25.5. There's no guarantee that one has e.g.

$$A(F)\backslash G(F)/A(F) \simeq \mathbb{P}^1(F) \setminus \{1\}.$$

This turns out to actually hold in our case though (everything we said makes sense on the level of points).

Back to the non-split torus, so $G = B^\times$.

Lemma 25.6. *Any quaternion algebra $B \supset E$ is of the form*

$$B_\varepsilon := \left\{ \begin{pmatrix} \alpha & \beta \\ \varepsilon\overline{\beta} & \overline{\alpha} \end{pmatrix} \middle| \alpha, \beta \in E \right\} \subset M_{2\times 2}(E).$$

Note $E \hookrightarrow B_\varepsilon$ 'diagonally', $\alpha \mapsto \begin{pmatrix} \alpha & 0 \\ 0 & \overline{\alpha} \end{pmatrix}$. Above, B_ε is unique for choice of $\varepsilon \in F^\times/NE^\times$.

One has an invariant map

$$\begin{array}{cccc} \mathrm{Inv}: & G = B_\varepsilon^\times & \longrightarrow & \mathbb{P}^1 \setminus \{1\} \\ & \gamma = \begin{pmatrix} \alpha & \beta \\ \varepsilon\overline{\beta} & \overline{\alpha} \end{pmatrix} & \longmapsto & \dfrac{\varepsilon\beta\overline{\beta}}{\alpha\overline{\alpha}} = \dfrac{\varepsilon \operatorname{Nm}\beta}{\operatorname{Nm}\alpha} \end{array}$$

This is $H \times H$-invariant, and identifies $H\backslash G/H \simeq \mathbb{P}^1 \setminus \{1\}$.

Warning 25.7. On F-points,

$$G(F) \to (\mathbb{P}^1 \setminus \{1\})(F)$$

is *not* surjective. Not everything is of the form $\varepsilon \operatorname{Nm}\beta / \operatorname{Nm}\alpha$. The image of this map is only $\varepsilon \operatorname{Nm}(E) \cup \{\infty\}$.

As before, our good orbits will be those with invariant $\neq 0, \infty$. Hence, the good orbits have invariants in $\varepsilon \operatorname{Nm}(E^\times)$.

Lemma 25.8. *There is a bijection of good orbits*

$$\bigsqcup_{B \supset E} E^\times \backslash B^\times / E^\times \xrightarrow{\sim} A\backslash G'/A.$$

Recall 25.9. This lemma (after taking invariants) is really just saying

$$\bigsqcup_{\varepsilon \in F^\times/\operatorname{Nm}(E^\times)} \varepsilon \operatorname{Nm}(E^\times) = F^\times,$$

which is true by definition.

This is why, in the our first statement of Waldspurger, we had to use all the divison algebras. This concludes our discussion of Waldspurger.

25.1 Some Remarks

Remark 25.10. The argument via relative trace formulas that we outlined is due to Jacquet. Waldspurger's original proof was more ad hoc.

Remark 25.11. The Arthur-Selberg trace formula (for H) is the relative trace formula attached to $G = H \times H$ with chosen subgroups $H_1 = H_2 = \Delta(H)$.

Hence, the Arthur-Selberg trace formula is about $\mathrm{GL}_2 \times \mathrm{GL}_2$ while the ones coming up in Waldspurger really only involve GL_2 (or $B^\times$) and so our simpler in some sense.

Remark 25.12. When talking about base change, we also saw a twisted trace formula for GL_n. We have a (cyclic) Galois extension E/F and a (generator) $\sigma \in \mathrm{Gal}(E/F)$; we used $G = \mathrm{GL}_{n,E} \times \mathrm{GL}_{n,E}$, $H_1 = \Delta(\mathrm{GL}_{n,E})$, and $H_2 = \Gamma_\sigma$.

We see from these examples that the choice of triple is not something 'purely mechanic.' We also see some limitations in this last example. If the Galois group is not cyclic, one would like to consider more than two subgroups, but then you run into the issue that the kernel function $K_f(x, y)$ only takes two parameters.[52]

Open Question 25.13. *non-solvable base change, even for* GL_2*?*

In another direction, one application we didn't have time to get to is the Gross-Zagier formula. Waldspurger says that

$$\left| \int_{[H]} \varphi(t)\mathrm{d}t \right|^2 \sim L(\pi_E, \frac{1}{2})$$

with E/F quadratic. This can be used to prove B-SD when $\mathrm{ord}_{s=\frac{1}{2}} L = 0$. Gross-Zagier extended things by producing a formula for the first derivative $L'(\pi_E, \frac{1}{2})$, at least when π_∞ is a discrete series of $\mathrm{GL}_2(\mathbb{R})$ (so assuming $F = \mathbb{Q}$ or some other totally real field). Their formula took the form

$$\langle y_H, y_H \rangle_{NT} \sim L'\left(\pi_E, \frac{1}{2}\right).$$

They used this formula to prove B-SD when $\mathrm{ord}_{s=1/2} L = 1$.

The automorphic period integral for H is replaced by a shimura variety; $H \subset G$ becomes $y_H \mathrm{Sh}_H \subset \mathrm{Sh}_G$ for G some Shimura curve$/F$ (and Sh_H corresponding to CM points). The integral itself is replaced by the height pairing $\langle y_H, y_H \rangle$.

Open Question 25.14. *Higher derivatives* $L^{(r)}\left(\pi_E, \frac{1}{2}\right)$*?.*

[52] If the Galois extension is solvable, you can get around this by repeating the cyclic case over and over

26 List of Marginal Comments

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