MSRI Notes

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July 2022

These are notes on talks given in "Simons Collaboration Meeting" which took place at Simons Foundation in NY. Unfortunately for the reader, these notes are live-texed and so their quality is upper bounded both by my (quite) limited ability to understand the material in real time and by my typing speed. With that in mind, they are doubtlessly missing content/insight present in the talks and certainly contain confusions not present in the talks. Despite all this, I hope that you can still find some use of them. Enjoy and happy mathing.

The website for this seminar is available here.

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1 An analogy 1

1 Day 1 (2/6/23)

1.1 Joseph Silverman (Brown): Moduli Spaces for Dynamical Systems on Projective Space

Today we'll talk about a mix of diophantine geometry, arithmetic geometry, and dynamical systems. We'll start with an analogy. On one side of it is arithmetic geometry, and on the other side is dynamical systems (See table 1).

TODO: Make table look good

Arithmetic Geometry	Dynamical Systems
Abelian varieties of dim ≥ 1	Morphisms $f : \mathbb{P}^N \to \mathbb{P}^N$ of degre
Moduli space $\mathcal{A}_g$ of PPAVs of dimension g (up to isomorphism)	moduli space $\mathcal{M}_d^N$ of degree d endomorphisms of $\mathbb{P}^N$, u
torsion points to add level structure	preperiodic points to add level st

Table 1: An analogy

We'll mainly talk about the RHS of Table 1.

Question 1.1.1 (Audience)**.** *Is there a notion of 'isogeny' on the dynamical side?*

Answer. The short answer is no. Something things have analogies and some don't. In fact, we don't have a notion of 'Hecke correspondence' on the RHS currently, but this would be really amazing to have. ⋆

Let's consider the space of degree d endomorphisms of $\mathbb{P}^N$. How does one specify such a thing? Well, it's given by an $(N+1)$-tupe

$$f = [f_0, \ldots, f_N] : \mathbb{P}^N \to \mathbb{P}^N$$

of homogeneous degree d polynomials (in $X_0, \ldots, X_N$) which have no non-trivial common geometric zeros.

Note that f is determined by the coefficients of $f_0, \ldots, f_N$. One can compute that the number of coefficients is $(N+1)\binom{N+d}{d} =: M(N,d) = M$. These are homogeneous, so these f_i's are really only determined up to scalar multiplication. Hence, the space

$$\mathrm{End}_d^N = \left\{\text{degree } d \text{ morphisms } \mathbb{P}^N \to \mathbb{P}^N\right\} \subset \mathbb{P}^{M-1}.$$

The complement of End_d^N in $\mathbb{P}^{M-1}$ is the set where $\mathrm{Res}(f) \neq 0$ (this is a multi-polynomial version of the resultant). Thus, End_d^N is the complement of a hypersurface, and in particular is affine.

This is a start, but it's sort of like classifying elliptic curves as pairs (A, B), but that's not what you want to do. You want a classification up to isomorphism.

Question 1.1.2. *When do f and g give the same dynamical system?*

The answer is when they differ by a change of variables on $\mathbb{P}^N$. In other words, say f, g are **dynamically equivalent** if $\exists \varphi \in \mathrm{PGL}_{N+1}$ such that

$$\begin{tikzcd} \mathbb{P}^N \arrow[r, "g"] \arrow[d, "\varphi"'] & \mathbb{P}^N \arrow[d, "\varphi"] \\ \mathbb{P}^N \arrow[r, "f"] & \mathbb{P}^N \end{tikzcd}$$

commutes. For technical reasons, we may later take $\varphi \in \mathrm{SL}_{N+1}$ instead (note $\mathrm{SL}_{N+1} \twoheadrightarrow \mathrm{PGL}_{N+1}$ with finite kernel μ_{N+1}). In any case, we denote this by writig $f \sim g$ or $g = f^\varphi = \varphi^{-1} \circ f \circ \varphi$. Note that, in fact, SL_{N+1} actually acts on all of $\mathbb{P}^{M-1}$ (all $(N+1)$-tuples of polynomials of degree d).

At this point, we want to look at $\mathbb{P}^{M-1}/\mathrm{SL}_{N+1}$. To what extend does this have a nice structure? If you take this quotient with all of $\mathbb{P}^{M-1}$, then this doesn't have a nice structure. Instead, one will want to quotient out only a subset of $\mathbb{P}^{M-1}$. To figure out the write subset, one turns to Geometric Invariant Theory (GIT). The utility of GIT is to give subsets (or subschemes) of $\mathbb{P}^{M-1}$ which have nice quotients. Note the plurality; there are different subsets whose quotients have different nice properties. One gets the subset $(\mathbb{P}^{M-1})^s$ of stable points as well as one $(\mathbb{P}^{M-1})^{ss}$ of semistable points. These have nice quotients, which are usually denotes $(\mathbb{P}^{M-1})^s/\!/\mathrm{SL}_{N+1} =: (M_N^d)^s$ and $(\mathbb{P}^{M-1})^{ss}/\!/\mathrm{SL}_{N+1} =: (M_N^d)^{ss}$ with the double /.

Note 1. Missed some stuff

These are algebraic varieties (schemes) satisfying

$$(\mathbb{P}^{M-1})^s(\overline{K})/\mathrm{SL}_{N+1}(\overline{K}) = (M_N^d)^s(\overline{K}) \subset (M_N^d)^{ss}(\overline{K})$$

with the latter proper over $\overline{K}$. That is, the stable locus gives a 'good quotient' while the semistable locus gives a proper quotient.

Theorem 1.1.3 (Milnor, S., Levy, Petsche-Szpiro-Tepper, ...)**.**

(a) $\mathrm{End}_c^N \subset (\mathbb{P}^{M-1})^s$. *Hence,*

$$M_d^N := \mathrm{End}_d^N /\!/ \mathrm{SL}_{N+1}$$

is a scheme over $\mathbb{Z}$.

(b) *(Levy)* M_d^1 *is always a rational variety (of dimension* $2d-2$*?)*

(c) $M_2^1 \cong \mathbb{A}^2$ *over* $\operatorname{Spec}\mathbb{Z}$. *In this case,* $(M_2^1)^s = (M_2^1)^{ss} = \mathbb{P}^2$

Question 1.1.4. *Is* M_d^N *rational?*

Question 1.1.5 (Audience)**.** *Is there a moduli interpretation of the boundary of* M_2^1*?*

Answer. The boundary is a $\mathbb{P}^1 \cong \mathbb{A}^1 \sqcup *$. Most of it looks like linear maps masquerading as quadratic maps, but the single points corresponds to the constant map masquerading as a quadratic map. ⋆

Question 1.1.6 (Audience)**.** *If* M_d^n *known or not known to be rational for any* $n \neq 1$*?*

Answer. Not sure off the top of my head. Maybe the rationality is known for M_2^2? Certainly, it is *not* known that M_d^2 is rational for all $d \gg 0$ or anything like that. ⋆

Question 1.1.7 (Audience)**.** *If* M_d^N *a fine or coarse moduli space?*

Answer. It is a coarse moduli space. There are points with non-trivial automorphisms. One reason to add level structure is to get a fine moduli space. ⋆

Note 2. There were more questions, but I wasn't fast enough to write them all down.

1.1.1 Level Structure

Recall 1.1.8. Torsion points of abelian varieties are analogous to preperiodic points of dynamical systems. ⊙

One might want to classify pairs (f, P) where $f \in \mathrm{End}_d^N$, $P \in \mathbb{P}^N$, and, say, $P = f(P)$. We'd do this up to $(f, p) \sim (f^\varphi, \varphi^{-1}(P))$. Let's set some notation.

Notation 1.1.9. $f^n = \underbrace{f \circ f \circ \cdots \circ f}_{n}$. Given $P \in \mathbb{P}^N$ its **orbit** (or **forward orbit**) is

$$\mathscr{O}_f(P) := \{f^n(P) : n \geq 0\}.$$

Definition 1.1.10. We say P is **preperiodic** if $\#\mathscr{O}_f(P) < \infty$. ◇

Definition 1.1.11. A **(preperiodic) portrait** is a finite directed graph w/ weighted vertices[1] w/ no sinks (this is the preperiodic condition). A **sink** is a vertex with no outgoing arrow (if I heard correctly). ◇

Remark 1.1.12. Portraits do not have to be connected. ○

[Joe drew some pictures]

Let $\mathcal{P}$ be a portrait. Can consider

$$\mathrm{End}_d^N[\mathcal{P}] := \left\{(f, P_1, \dots, P_n) \in \mathrm{End}_d^N \times ((\mathbb{P}^N)^n \setminus \Delta^{\mathrm{big}} \mid f \text{ acting on } P_1, \dots, P_n \text{ looks like } \mathcal{P}\right\}$$

Example. Imagine you have a disconnected portrait $\mathcal{P}$. One component is a self-loop with weight 2. The other is a 2-cycle with a single vertex feeding into it. One point in $\mathrm{End}_d^N[\mathcal{P}]$ is $(x^2 - 1, \infty, 1, 0, -1)$. △

Theorem 1.1.13 (S., Doyle)**.** *$\mathrm{End}_d^N[\mathcal{P}] \subset (\mathbb{P}^{M-1} \times ((\mathbb{P}^N)^n \setminus \Delta))^s$, so one gets a good quotient $M_d^N[\mathcal{P}] = \mathrm{End}_d^N[\mathcal{P}] // \mathrm{SL}_{N+1}$.*

(The implicit line bundle above is $\mathscr{O}(1, 1, \dots, 1)$). The proof is a big calculation using the 'numerical criterion'.

Example. Let $\mathcal{P}_n$ be the portrait consisting of a weight 2 fixed point along with a (disjoint) n-cycle. Then,

$$M_2^1[\mathcal{P}_n] = \left\{(x^2 + c, \alpha) : \alpha \text{ has period } n \text{ for } f.\right\}$$

(Move critical point to ∞ to get degree 2 polynomials). It turns out that this is a curve, analogous to $X_1(n)$. △

Question 1.1.14 (Audience)**.** *Is this curve proper?*

Answer. No. You need to add a bunch of points to complete. The affine curve turns out to be nonsingular, but the natural compactification is highly singular. ⋆

[1] these will correspond to ramification indices of the map f

1.1.2 Uniform Boundedness Conjecture (UBC)

In Arithmetic geometry, the UBC says the following: Say $K/\mathbb{Q}$ is a number field of degree $D = [K : \mathbb{Q}]$ and A/K is a g-dimensional abelian variety. Then, $\#A(K)_{\text{tors}} \le C(D, g)$. This is true for $g = 1$ by work of Mazur, Kaminy (spelling?), Morel.

What's the dynamical analogue? Say $f \in \text{End}_d^N(K)$, then $\text{PrePer}(f, \mathbb{P}^N(K))$ is finite (using a height/Northcott argument[2]). Dynamical UBC predicts that

$$\# \text{PrePer}(f, \mathbb{P}^N(K)) \le C(N, d, D).$$

Fakhruddin showed that dynamical UBC implies arithmetic UBC.

Let's end with one last analogy by looking at the geometry of $M_d^N[\mathcal{P}]$.

Theorem 1.1.15. *$\mathcal{A}_g$ is og general type if $g \gg 1$.*

Question: Is this equivalent to the same statement for M_g?

Conjecture 1.1.16. *If $\mathcal{P}$ is unweighted and $\#(\text{vertices of } \mathcal{P}) \ge C(N, d)$, then $M_d^N[\mathcal{P}]$ is irreducible and of general type.*

This may not get you proofs of very many things, but combined with Bombieri-Lang, it would tell you much.

Theorem 1.1.17 (Blanc, Canci, Elkies)**.** *Let $\mathcal{P}_n$ be an n-cycle. Then, $M_2^2[\mathcal{P}_n]$ is rational for $n \le 5$ and $M_2^2[\mathcal{P}_6]$ is of general type. However, they found a rational curve and two elliptic curves with positive rank sitting inside of it.*

Question 1.1.18 (Audience)**.** *Is it reasonable to conjecture that if the number of vertices is large enough, then you get a fine moduli space?*

Answer. Not quite. It's not just the number of vertices, but whether the graph has vertices. If you have an n-cycle, for example, you may worry that a matrix with rotates the n-points could be an automorphism of a point. $\star$

Question 1.1.19 (Audience, paraphrased)**.** *Is there a moduli interpretation of the boundary?*

Answer. The boundary consists of rational maps of degree d or of maps of lower degree. So there should be some stratification there. I don't think anyone has worked it out in general. Sounds like DeMarco has some work along these lands for $\mathbb{P}^1$. $\star$

Question 1.1.20 (Audience)**.** *Can anything be said about the cohomology of these moduli spaces? Are there analogues of Hecke operators?*

Answer. Sounds like no one has done this. The natural GIT compatification is highly singular, so there would be work to do this. No good analgoues of Hecke operators are known. We mentioned these curves which were analogous to $X_1(n)$. Their Jacobians tend to be simple (as opposed to $J_1(N)$ which breaks apart into something like Hecke eigenspaces). $\star$

Note 3. There was a question I didn't quite get/hear. The takeaway was that a generic f will have an n-cycle though.

[2] Essentially, $h(f(P)) = dh(P) + O_f(1)$

1.2 Thomas Tucker (University of Rochester): A Finite Index Conjecture for Iterated Galois Groups

Conjecture 1.2.1. *Let k be a number field or char 0 function field. Let f be a polynomial in $k[x]$. Suppose that $\deg f > 1$. Write $\mathbb{A}^1 = \operatorname{Spec} k[t]$ and*

$$f^{-n}(t) = \textit{roots of } f^n(x) - t$$

(e.g. in $\overline{K}$). Let $G_n = \operatorname{Gal}(k(f^{-n}(t))/f(t))$ and set $G_\infty = \varprojlim G_\infty$. Take a point $\alpha \in k$. Can similarly consider $G_n(\alpha) = \operatorname{Gal}(k(f^{-n}(\alpha))/k)$ and $G_\infty(\alpha) = \varprojlim G_n(\alpha)$.[3] *One conjectures that $G_\infty(\alpha)$ has finite index in G_∞ unless α is special.*

What do we mean by special above? There are two 'obvious' things that can go wrong.

Definition 1.2.2. We say α is **special** if α is either (or both)

- (**post-critical**) $f^n(\gamma) = \alpha$ for some critical point γ (i.e. $f'(\gamma) = 0$)
- $\exists g \in k(x)$ s.t. $g(\alpha) = \alpha$, $g \neq \mathrm{id}$, and g commutes with some power of f, i.e. $g \circ f^n = f^n \circ g$ for some n.

  (In the simplest case, imagine $g = f$)

$\diamond$

Remark 1.2.3. Can ask this for any finite map $f : X \to X$. In higher dimensional cases, need to tweak the definition of special (e.g. g only required to fix a proper subvariety containing α). $\circ$

The conjecture was originally made for quadratic polynomials over number fields by Boston-Jones. They said they made the conjecture in analogy with the Serre finite index theorem. Thomas doesn't think it's all that similar to Serre finite index. However, it is very similar to a result of BashneKov-Ribet (spelling?) related to multiplication-by-m on semiabelian varieties.

Question 1.2.4 (Audience)**.** *Can you state the theorem?*

Theorem 1.2.5 (Bashnekov-Ribet, paraphrase)**.** *Have $[m] : S \to S$ and $x \in S$. Set*

$$G_\infty := \varprojlim_n \operatorname{Gal}([m]^{-n}(x)/k).$$

Then has finite index in the thing it should have finite index in.

Call the conjecture we started with **FIC** (i.e. that $[G_\infty : G_\infty(\alpha)] < \infty$ unless α is special). When is it known?

- quadratics $x^2 + c$ when 0 is not preperiodic.
  - Sounds like known for function fields in char 0

    (Need to assume not isotrivial, so $c \notin \overline{\mathbb{Q}}$)

[3] Can think of this as a 'decomposition group' of G_∞

- Sounds like known for function fields in char p (need to be careful about inseparability to state things properly) if one assumes the "eventual stability" conjecture.

  (Again, $c \notin \overline{\mathbb{F}}_p$)
- Sounds like known for number fields if one assumes abc + eventual stability.

Question 1.2.6 (Audience)**.** *What do you mean by function field in char* 0*?*

Answer. Function field of a curve (over $\mathbb{Q}$), so f.generated w/ transcendence degree 1. ⋆

Question 1.2.7 (Audience)**.** *In the function field case, do you not need abc or are you using it, but it's just known to hold?*

Answer (paraphrased, I didn't get the answer)**.** Need the Vojta conjecture for algebraic points. Over function fields, this is equivalent to a strong form of abc. ⋆

Question 1.2.8 (Audience)**.** *What does* G_∞ *looks like?*

Answer. One thing that makes this hard is that we don't know what G_∞ looks like in general. When f is quadratic, G_∞ does have a simple description. G_∞ is $\mathrm{Aut}(T_2)$ where T_2 is the tree where every vertex has 2 parents and there's a single leaf (if I'm understanding). ⋆

Note 4. As usual, more questions, but I missed and/or didn't understand them...

Let's take a second to mention some of the people who have worked on this in the quadratic case (no promises on spelling for any of these names): Baedutte, Loopa, Jones, Boston, Gnioca, Doyle, Hindes, ...

What's the idea in the quadratic case? We have these groups $G_1, G_2, \dots$. It suffices to show that

$$\#\,\mathrm{Gal}(k(f^{-n-1}(\alpha))/k(f^{-n}(\alpha))) = 2^{2^n}$$

for all $n \gg 0$ (to see why it should be this, think about automorphisms of the tree). In the simplest case $n = 0$, just need to know there's a prime p dividing c to an odd power. In general, need p so that

$$v_p(f^{n+1}(0) - \alpha) \text{ is odd and } v_p(f^m(0) - \alpha) = 0 \text{ for all } m \le n.$$

Note that abc tells you that certain things are relatively squarefree. One gets that $f^n(0) - \alpha$ is "reasonably square-free" via

- abc (Granweld, spelling?) over number fields
- Vojta conjecture, something algebraic points (Yamato), char 0
- Char p, (Kim-Szpiro)

Progress has been made, as well, for non-PCF cubics

- Function fields of char 0.

  Need eventual stability assumption.
- Number fields.

  Need eventual stability + abc + Vojta conjecture for blow-ups of $\mathbb{P}^1 \times \mathbb{P}^1$

In quadratic case, wanted primes where critical point hits α but hadn't met it before. There are now two critical points γ_1, γ_2. One needs to control

$$\gcd(f^n(\gamma_1) - \alpha, f^n(\gamma_2) - \alpha)$$

Note that Bigeach-Corvaja-Zannier (spelling?) used Schmidt subspace to prove $\log\gcd(a^n - 1, b^n - 1) \le \varepsilon n$. I missed, but somehow, this is related to a Vojta conjecture for blow-ups (ans Silverman was the one to notice this?). Consider

$$\log\gcd(f^n(\gamma_1) - \alpha, f^n(\gamma_2) - \alpha) < \varepsilon(\log f)^n \text{ for all large } n \text{unless } \gamma_1, \gamma_2, \alpha \text{ are all related by } f \tag{1.1}$$

(I missed what related means). Apparently (1.1) holds for all polynomials if one assumes Vojta's conjecture for blowups of $\mathbb{P}^1 \times \mathbb{P}^1$. It was proved by Favry-Gauthier (spelling?), Gnioca-Ye (spelling?) using the Bogomolov form of dynamical André-Oort.

Definition 1.2.9. Say f is a polynomial of degree $\deg f > 1$, over some field k. We say f is **eventually stable** if the number of factors (irred.) of f^n over k stays bounded as $n \to \infty$. ◇

Notation 1.2.10. Let Δ_n denote the number of factors of f^n/k. Note that $\Delta_n \le \Delta_{n+1} \le \Delta_{n+2} \le \dots$. Stability holds when this chain eventually equalizes.

Conjecture 1.2.11 (Jones-Levy)**.** *Let k be a number field. If 0 is not a root of f^n for any n (i.e. 0 is not periodic under f), then f is eventually stable.*

(Could ask this over any f.generated field)

Note the conjecture holds if f is Eisenstein. Jones-Levy proved it if f is 'almost Eisenstein' (sounds like relaxing condition that constant divisible by p only once). Here are two approaches.

(1) Relate to **dynamical Lahmer conjecture**: There exists c_f s.t. $\widehat{h}_f(\alpha) \ge c_f/\deg\alpha$ for all α (w/ $\widehat{h}_f(\alpha) \ne 0$). This holding for any f would imply eventual stability for that f.[4]

(2) $f^{-n}(0)$ equidistribute w.r.t. lots of measures of maximal entropy.

1.3 Jordan Ellenberg (University of Madison-Wisconsin): Heights of Rational Points on Stacks

(joint w/ Matt Satriano and David Zureick-Brown)

Note 5. A few minutes late

Jordan gave shoutout to AMS-MRC, Explicit computations in stacks (apply by Feb 15)

Let K be a global field (e.g. $\mathbb{Q}$ or $\mathbb{F}_q(t)$). Let's start about talking about what heights mean to us in this talk. Say X is a projective variety. Then, the height of a point of $X(K)$ is a measure of the "arithmetic complexity" on X. It gives us a way of putting points in order from simplest to complex. The most fundamental feature a height should have is that there are only finitely many points of height $< B$ (**Northcott property**).

[4]Note: only need dynamical Lahmer in the field $\bigcup_{n=1}^{\infty} k(f^{-n}(\alpha))$ which is somehow reminiscent of $\mathbb{Q}^{\mathrm{ab}}$

Example. Say $K = \mathbb{Q}$, $X = \mathbb{P}^n$. Then, any $P \in \mathbb{P}^N(\mathbb{Q})$ can be written in the form $p = (a_0 : a_1 : \cdots : a_n)$ s.t. all $a_i \in \mathbb{Z}$ and they have no common factor. Then, $\operatorname{ht}(P) = \max |a_i|$. If you think about a number field with nontrivial class number, then you'd need to be a little more clever with your definition.

If $X \overset{\iota}{\hookrightarrow} \mathbb{P}^n$, we simply define $\operatorname{ht}_X(P) = \operatorname{ht}_{\mathbb{P}^n}(\iota(P))$. A variety with a projective embedding is just a variety with a very ample line bundle. So, if X is endowed w/ a line bundle L, use L to define $i_L : X \hookrightarrow \mathbb{P}^n$ and then define $h_L(P) = h_{\mathbb{P}^n}(i_L(P))$ for $P \in X(K)$. △

Here is a fact that is generally believed: the number of $\mathbb{Q}$-points on the cubic surface $x_1^3+x_2^3+x_3^3+x_4^3 = 0$ not contained in the 27 lines, of height $< B$, should be $\sim cB$ (for some constant c). "Nobody knows this, and if somebody could prove it, it would be a big deal."

(Above is example of Batyrev-Manin heuristics)

Remark 1.3.1. There are like B^4 points on $\mathbb{P}^3$ (of height $< B$). This equation has size about B^3, so maybe you expect the chance it evaluates to 0 is about $1/B^3$, so expect like B points. ◦

Fact (Bhargava)**.** The number of quintic S_3-extensions of $\mathbb{Q}$ with $|\text{discriminant}| < B$ is $\sim cB$ (for some constant c).

(Above is example of Malle conjecture)

More generally,

- (**Batyrev-Manin**) Say X is Fano (so conjecturally is has infinitely many rational points). Then,

$$N_X(B) := \#\{x \in X(K) : \operatorname{ht}(x) < B\} \sim cB^a(\log B)^b.$$

- (**Malle**)

$$N_G(B) := \#\left\{L/K : \operatorname{Gal}(L^{\text{clos}}/K) = G \text{ and } |\operatorname{disc}(L/K)| < B\right\} \sim cB^a(\log B)^b.$$

These look similar? Why's that?

Observation 1.3.2. A G-extension of K is a K-rational point on something, but *not* on a variety. Rather, it is K-point of the stack BG.

"Am I gonna define what a stack is in this talk? I am not. Am I gonna talk about the different between a Deligne-Mumford stack and other kinds of stacks? I am not."

"A stack is like a variety, it was put on this Earth to represent a functor."

Example. $BG(-)$=étale G-torsors for $-$. △

If one checks that the height of a point on BG is the discriminant of a number field, then it really seems like you have reason to think these two conjecture are the same.

Question 1.3.3 (Audience)**.** *Does that mean that I should think of BG as being like a Fano variety?*

Answer. Great question. We don't have a clear criterion for which stacks such an asymptotic to apply to. ⋆

Question 1.3.4 (Audience)**.** *For B-M, need to remove some subvarieties where you have too many points. Is there an analogue for the BG/stack case where you know what you have to remove?*

Answer. Sounds like less is known about exact what needs to be removed, but that some of the original counterexamples of Malle's conjecture are of this form. ⋆

However, when Jordan and David (and Matt?) were thinking about this (at MSRI circa 2006?), they realized there was no standard definition for the height of a point on a stack.

Question 1.3.5. *What is the height of a point on a stack $\mathcal{X}$ (pronounced 'eeks')?*

One runs into the issue that projective stacks are schemes, so can't just choose an embedded $\mathcal{X} \hookrightarrow \mathbb{P}^N$.

What do they do? Let C be either a curve/$\mathbb{F}_q$ or $\operatorname{Spec} \mathscr{O}_K$ for K a number field. Let $\mathcal{X} \to C$ be a good enough stack.[5] Let V be a vector bundle on $\mathcal{X}$. Then, we define a function

$$h_V : \mathcal{X}(K) \longrightarrow \mathbb{R}$$

which they think deserves to be called a height function on K-points of $\mathcal{X}$. We won't give the definition right now, but let's start with some examples.

Example. If $\mathcal{X}$ is actually a scheme, then $h_V(x) = \mathrm{ht}_{\det V}(x)$ for any $x \in \mathcal{X}(K)$. △

Example. If $\mathcal{X}$ is BG, a vector bundle on $\mathcal{X}$ is a representation of G. Let V be a permutation representation of G (i.e. G acting on cosets for some subgroup of G, i.e. $G \curvearrowright G/H$ for some $H \le G$). If $x \in \mathcal{X}(K)$ corresponds to the F/K Galois w/ group G, then

$$\mathrm{ht}_V(x) = \mathrm{disc}(F^H/K). \qquad \triangle$$

Example. Say $\mathcal{X}$ is $\mathbb{P}^1$ w/ 1/2 points at $0, -1, \infty$ (i.e. the residual gerbes at these points are $B\mu_2$). If (a, b) is a point of $\mathbb{P}^1 \setminus \{0, 1, \infty\}$, thought of as a point on the corresponding open patch of $\mathcal{X}$, then

$$\mathrm{ht}_{\mathcal{X}, T\mathcal{X}}(a, b) = [\mathrm{sqf}(a)\, \mathrm{sqf}(b)\, \mathrm{sqf}(a+b) \max(|a|, |b|)]^2$$

(Above sqf is squarefree part). Note that

$$\mathrm{ht}_{\mathbb{P}^1, T\mathbb{P}^1}(a, b) = \max(\left|a^2\right|, |ab|, \left|b^2\right|) = \max(|a|, |b|)^2. \qquad \triangle$$

Example. Say $\mathcal{X}$ is the moduli stack of elliptic curves, i.e. $\mathcal{X} = X(1)$. Let $\mathscr{L}$ be the Hodge bundle on $\mathcal{X}$. Assume $K = \mathbb{Q}$. Then,

$$\mathrm{ht}_{\mathscr{L}}(y^2 = x^3 + Ax + B) = \max\left(|A|^{1/4}, |B|^{1/6}\right).$$

Over number fields, still get something that looks like naive height of a number field. If $\operatorname{char} K = 2, 3$, the height function attached to the Hodge bundle is not so good. In fact, there are infinitely many elliptic curves over K with $\mathrm{ht}_{\mathscr{L}}(E) = 0$. In fact, every line bundle on $X(1)$ in $\operatorname{char} K = 2, 3$ gives a height which is not Northcott. In characteristics $2, 3$, $X(1)$ does have vector bundles which are not sums of line bundles though. △

Question 1.3.6. *In char 2 and 3, is there a vector bundle V on $X(1)$ such that ht_V is Northcott?*

[5] e.g. a proper Artin stack w/ finite diagonal

Note that Aaron Landesman showed that Faltings height is not obtained by any vector bundle on $X(1)$ is characteristics 2 and 3.

Let's comment on some non-properties are these stacky height functions.

- not invariant under field extensions
  (Imagine $\mathbb{Q}(\sqrt{d}) \in B(\mathbb{Z}/2\mathbb{Z})(\mathbb{Q})$ after base change to $\mathbb{Q}(\sqrt{d})$)
- non-additive, i.e. $\mathrm{ht}_{L_1 \otimes L_2}$ is not $\mathrm{ht}_{L_1} + \mathrm{ht}_{L_2}$
  (Imagine taking a line bundle on $B(\mathbb{Z}/2\mathbb{Z})$. All such things are 2-torsion, if I heard correctly)
- No easy local decomposition (like Weil height, but unlike discriminant)

In fact, one finds that

$$\mathrm{ht}_V = \mathrm{ht}_V^{\mathrm{st}} + \sum_p \delta_p.$$

There's a stable part which is additive and behaves well under field extensions, and then there's another part which does have a local decomposition.

Let's end w/ new stuff. Darda-Yasuda (2022) give an alternate definition of heights on stacks which incorporates ours (when both are defined). It sounds like their definition doesn't work so well in low characteristic, but does give a clearer picture of what the asymptotics are supposed to be. It also gives a clearer criterion for which stacks should be considered Fano. Jordan said something like the difference in definitions being something like the difference between $\operatorname{disc} L/K$ and "variant discriminants." For example, if L/K cubic w/ $\mathrm{Gal}(L^{\mathrm{clos}}/K) = S_3$, this discriminant looks like MN^2 with M product of simply ramified primes; could alternatively consider something like $M^\alpha N^\beta$ for any α, β (though this doesn't work so well with wild ramification).

Lots of new work on counting points on stacks which are neither schemes nor BG. Nasserdan-Xiao, Le Budec worked out count for $\mathbb{P}^1$ w/ 3 half points. For $X_1(N)$, count worked out by Harron-Snowden (2014). For $X_0(N)$ for many values of N, Boggess-Sankar worked this out. $X_0(3)$ was worked out by Pizzo-Pommerance-Voight (this example has accumulation phenomenon. 100$ of points correspond to $j = 0$). Stacky $\mathrm{Sym}^2 \mathbb{P}^1$ was done by Yin.

1.4 Anna Cadoret (Sorbonne Université): On the Toric Locus of L-Adic Local Systems (Joint work with J. Stix)

Note 6. Many minutes late

Say k is a number field, X is a smooth, geom. connected variety over k, ℓ is a prime, and V_ℓ is a $\mathbb{Q}_\ell$-local system on X. Think of this as a representation $\rho_\ell : \pi_1(X, \overline{x}) \to V_{\ell,\overline{x}}$ through which $\mathrm{Gal}(\kappa(x)) = \pi_1(x, \overline{x}) \xrightarrow{\rho_{\ell,x}} V_{l,\overline{x}}$ factors, for $x : \operatorname{Spec} \kappa(x) \to X$.

Example. Say $V_\ell = R^i f_* \mathbb{Q}_\ell$ for $f : Y \to X$ smooth and proper. $V_{\ell,\overline{x}} = \mathrm{H}^i(Y_{\overline{x}}, \mathbb{Q}_\ell) \curvearrowleft \pi_1(X)$. △

1.4.1 Exceptional Locus

$\mathrm{GL}_{V_{\ell,\overline{x}}} \supset \overline{\operatorname{Im} \rho_\ell} =: G_\ell \supset \overline{\operatorname{Im} \rho_{\ell,x}} =: G_{\ell,x}$ (both bars are Zariksi closures). The **exceptional locus** is

$$X_{V_\ell}^{\mathrm{ex}} = \left\{ x \in X : G_{\ell,x}^\circ \subsetneq G_\ell^\circ \right\}.$$

Note 7. There was an example I missed.

The **Toric locus** is

$$X_{V_\ell}^{\mathrm{tor}} = \left\{x \in X : G_{\ell,x}^{\circ} \text{ is a torus}\right\}$$

which is contained in the analogously defined **solvable locus**.

Note 8. I'm low-key already lost, so I'm gonna stop taking notes, and probably work on something else...

2 Day 2

2.1 Emmanuel Ullmo (IHES): Algebraicity of the Hodge Locus

(joint w/ G. Baldi and B. Klingler)

Let's start with some motivation. Say $f : X \to S$ is a smooth, projective morphism over $\mathbb{C}$. We are interested in

$$\{s \in S : X_s \text{ has more "symmetries" (or more structures) than a general } s \in S\}.$$

Example (Level 1)**.** Consider a family of Abelian varieties. △

Example (Level d)**.** Family of degree d hypersurfaces in $\mathbb{P}^{n+1}$. △

We will see that there is different behavior depending on whether you're in level 1, level 2, or level ≥ 3. We'll say what we mean by extra structures.

2.1.1 Hodge structures and Mumford-Tate Groups

Definition 2.1.1. A **$\mathbb{Z}$-Hodge structure of weight** w is a free $\mathbb{Z}$-module $V_{\mathbb{Z}}$ of finite rank equipped with a decomposition

$$V_{\mathbb{Z}} \otimes \mathbb{C} = \bigoplus_{p+q=w} V_{\mathbb{C}}^{p,q} \text{ where } V_{\mathbb{C}}^{q,p} = \overline{V_{\mathbb{C}}^{p,q}}.$$

Given such a thing, one gets a decreasing filtration

$$F^p V_{\mathbb{C}} = \bigoplus_{n \geq p} V_{\mathbb{C}}^{n,w-n}.$$

Note that the decomposition can be recovered from the filtration. ⋄

Another say of writing this decomposition is to say that there exists a morphism (of algebraic groups)

$$\alpha : \mathbb{S} \longrightarrow \mathrm{GL}(V_{\mathbb{R}}),$$

where $\mathbb{S} = \mathrm{Res}_{\mathbb{C}/\mathbb{R}} \mathbb{G}_m$ is the **Deligne Torus**. From this perspective,

$$V_{\mathbb{C}}^{p,q} = \left\{v : \alpha(z)v = z^{-p}\overline{z}^{-q}v \text{ for all } z \in \mathbb{C}^\times = \mathbb{S}(\mathbb{R}).\right\}$$

Definition 2.1.2. The **Mumford-Tate Group** $\mathrm{MT}(\mathbb{V}) = \mathrm{MT}(\alpha)$ is the smallest $\mathbb{Q}$-algebraic group H of $\mathrm{GL}(V_{\mathbb{Q}})$ such that α factorizes as

$$\alpha : \mathbb{S} \longrightarrow H_{\mathbb{R}} \hookrightarrow \mathrm{GL}(V_{\mathbb{R}}). \qquad \diamond$$

There is a notion of polarization of Hodge Structure. We won't get into to this, but all the Hodge structures in this talk will be polarized. One thing this nets us is that $\mathrm{MT}(\alpha)$ will always be a reductive group.

Principal Example. Let X is projective + smooth/$\mathbb{C}$. Then, its Betti cohomology $H^w_B(X, \mathbb{Z})$ (mod torsion) naturally carries a polarized Hodge structure.

Definition 2.1.3. If $W = V^{\otimes m} \otimes (V^{\vee})^{\otimes m}$ and V has a Hodge structure, then W inherits one as well. In this context, a **Hodge tensor** is an element of $W_{\mathbb{Q}} \cap W^{0,0}_{\mathbb{C}}$ in some tensorial construction. θ is a Hodge tensor $\iff$ θ is invariant under $\mathrm{MT}(\alpha)$. $\diamond$

Question: Is α coming from V or W?

2.1.2 Polarized Variation of Hodge Structure

Let S be a quasi-projective smooth variety over $\mathbb{C}$.

Definition 2.1.4. A **$\mathbb{Z}$-Variation of Hodge Structure** on S is

Not sure I copied this down correctly. In particular, I think that Θ_S might actually be an $\mathscr{O}_S$.

$$\mathbb{V} = (\mathbb{V}_{\mathbb{Z}}, \nu = \mathbb{V}_{\mathbb{Z}} \otimes \Theta_S, F, \nabla)\,.$$

Without getting into the details, informally, for all $s \in S$, $\mathbb{V}_{\mathbb{Z},s}$ is a polarized Hodge structure of weight w, and $s \mapsto F^{\bullet}(\mathbb{V}_{\mathbb{Z},s} \otimes \mathbb{C})$ varies holomorphically.

- $\mathbb{V}_{\mathbb{Z}}$ is a finite local system $\iff$ $\rho : \pi_1(S) \to \mathrm{GL}(V_{\mathbb{Z}})$.
- ν is a vector bundle
- ∇ is a connection $\diamond$

Principal Example. Say $f : X \to S$ is a smooth, projective morphism. Then, $R^w f_* \mathbb{Z}$ is a $\mathbb{Z}$-VHS, $(R^w f_* \mathbb{Z})_s = \mathrm{H}^w(X_s, \mathbb{Z})$.

2.1.3 Hodge Locus and Mumford-Tate domain

Assume the base S is always connected.

Definition 2.1.5. Say $s \in S(\mathbb{C})$ is **Hodge generic** if $\mathrm{MT}(\mathbb{V}_s)$ has maximal dimension (among $\mathrm{MT}(\mathbb{V}_t)$ for $t \in S(\mathbb{C})$). $\diamond$

Fact. Hodge generic points all have isomorphic MT-groups.

This groups is call the "**generic Mumford-Tate group**," and we will usually denote it by $G_S = \mathrm{MT}(\mathbb{V})$.

Definition 2.1.6. The Hodge locus $\mathrm{HL}(S, V^{\otimes})$ is the set of $s \in S$ which are not Hodge generic, i.e. $\{s \in S : \mathbb{V}_{\mathbb{Z},s}$ has more Hodge tensors than general$\}$. $\diamond$

(Recall Hodge tensors are elements of $W_{\mathbb{Q}}$ which are invariant under $\mathrm{MT}(\alpha)$, I think)

Theorem 2.1.7 (Cattami-Deligne-Kaplan)**.** *The Hodge Locus is a countable union of algebraic subvarieties of S.*

Klingler-Otwinanska (spelling?) proved that $\mathrm{HL}(S, V^{\otimes})$ is either analytically dense or algebraic. This prompts of the question of how one decides which of these two cases they are in.

2.1.4 Period Maps and Hodge locus

Let $G = G_S$ be the generic Mumford-Tate group of $\mathbb{V}$. Let $\widetilde{S} \to S$ be the universal cover. Let s be a Hodge generic point, and choose some lift $\widetilde{s} \in \widetilde{S}$. Associated to $\widetilde{s}$ is a Hodge structure

Question: Is this the same Hodge structure associated to s? i.e. is the family on $\widetilde{S}$ the pullback of the family on S?

$$h_{\widetilde{s}} : \mathbb{S} \longrightarrow \mathrm{GL}(V_{\widetilde{s}} \otimes \mathbb{R})$$

Let $D = G(\mathbb{R})^+$ conjugacy classes of $h_{\widetilde{s}} = G(\mathbb{R})^+/M$, where M is some (not necessarily maximal) compact subgroup of $G(\mathbb{R})^+$. This D will be a complex analytic variety (in fact, an open in some flag variety). We get a **period map**

$$\begin{tikzcd} \widetilde{S} \arrow[r, "\widetilde{\varphi}"] \arrow[d] & D \arrow[d] \\ S \arrow[r, "\varphi"] & \Gamma\backslash D, \end{tikzcd}$$

where $\Gamma \subset G(\mathbb{Z})$ is a finite-index subgroup. Informally, you should think

Slogan. "$\Gamma\backslash D$ is the moduli space of Hodge structures sharing the same symmetries as those of a general point of $\mathbb{V}$."

Definition 2.1.8. Let $\widetilde{s}' \in \widetilde{S}$ be any point, define $G' = \mathrm{MT}(\widetilde{s}')$ and $D' = \mathrm{MT}(\widetilde{s}')(\mathbb{R})^+ \cdot \widetilde{s}' = G'/M'$. Then, $\Gamma'\backslash D' \mapsto \Gamma\backslash D$ (the image of this map) is called a **special subvariety** of $\Gamma\backslash D$ ($\Gamma' = G' \cap \Gamma$). $\diamond$

Fact (Main)**.** Let Y be a component of $\mathrm{HL}(S, V^{\otimes})$. Let G_Y be the generic Mumford-Tate group of $\mathbb{V}|_Y$. Let $\Gamma_Y\backslash D_Y \hookrightarrow \Gamma\backslash D$ be the associated period sub-domain, $\Gamma_Y = \Gamma \cap D_Y$. Then, Y is a component of $\varphi^{-1}(\varphi(S) \cap \Gamma_Y\backslash D_Y)$.

Definition 2.1.9 (Main)**.** A component Y of $\mathrm{HL}(S, V^{\otimes})$ is **atypical** if $\mathrm{codim}_{\Gamma\backslash D}\, \varphi(Y) < \mathrm{codim}_{\Gamma\backslash D}\, \varphi(S) + \mathrm{codim}_{\Gamma\backslash D}(\Gamma_Y\backslash D_Y)$. Otherwise, Y is said to be typical. Furthermore, we say Y is **of positive period dimension** if $\dim \varphi(Y) > 0$. $\diamond$

(Note: 'typical' means that $\varphi(S)$ and $\Gamma_Y\backslash D_Y$ properly intersect at Y)

Can decompose the Hodge locus as

There's some subtlety I didn't understand, where the CM points should really be defined as the union of components of the Hodge locus satisfying the

$$\begin{aligned} \mathrm{HL}(S, V^{\otimes})_{\mathrm{pos}} &= \bigcup \text{components of positive period dimension} \\ \mathrm{HL}(S, V^{\otimes})_0 &= \bigcup \text{components of period dimension } 0 \\ \text{CM points} &= \{s \in S : \mathrm{MT}(\widetilde{s}) \text{ is a torus}\} \end{aligned}$$

Sounds like the CM points are contained in $\mathrm{HL}(S, V^{\otimes})_0$. Think of $\mathrm{HL}(S, V^{\otimes})_0$ as being more 'arithmetic' and of $\mathrm{HL}(S, V^{\otimes})_{\mathrm{pos}}$ as being more 'geometric'.

2.1.5 Main Results

Assume for simplicity that $G = \mathrm{MT}(\mathbb{V})$ is $\mathbb{Q}$-simple and that $G = \mathrm{ad}G$.

Theorem 2.1.10 (BKU, **Geometric Zilber-Pink**)**.** *$\mathrm{HL}(S, V^{\otimes})_{pos,atyp}$ is algebraic.*

What is expected?

Conjecture 2.1.11 (Zilber-Pink-Klingler)**.** *$\mathrm{HL}(S, V^{\otimes})_{atyp}$ is algebraic.*

Theorem 2.1.12 (BKU)**.** *If $\mathrm{HL}(S, V^{\otimes})_{typ} \neq \emptyset$, then $\mathrm{HL}(S, V^{\otimes})$ is analytically dense in S.*

Theorem 2.1.13 (BKU)**.** *If the "Level" of $\mathbb{V}$ is at least 3, then $\mathrm{HL}(S, V^{\otimes})_{typ} = \emptyset$. Consequently, $\mathrm{HL}(S, V^{\otimes})_{pos}$ is algebraic.*

Example. Let $\mathbb{P}_{\mathbb{C}}^{N(n,d)}$ be the projective space parameterizing degree d hypersurfaces in $\mathbb{P}^{n+1}$. Let $V_{n,d}$ be the Zariski open set parameterizing smooth hypersurfaces. Let $\mathbb{V}$ be the $\mathbb{Z}$-VHS corresponding to $\mathrm{H}^3(X_s, \mathbb{Z})_{\mathrm{prim}}$. If $n \geq 3$ and $d \geq 5$ (and $(n, d) \neq (4, 5)$), then $\mathrm{HL}(V_{n,d}, V^{\otimes})_{\mathrm{pos}} \subset V_{n,d}$ is algebraic. △

Question 2.1.14 (Audience)**.** *What's the definition of level?*

Answer. Any Hodge generic point induces a HS on $\mathrm{Lie}(G) = \mathfrak{g}$ of weight 0:

$$\mathfrak{g}_{\mathbb{C}} = \bigoplus_{k=-w}^{2} \mathfrak{g}_{\mathbb{C}}^{-k,k} \text{ with } g_{\mathbb{C}}^{-w,w} \neq 0.$$

The **level** is this w. Can think of Level 1 as corresponding to Shimura varieties. ⋆

2.2 Evelina Viada (Universität Göttingen): Some Effective Methods in the Context of the Mordell-Lang Conjecture

Note 9. This is a slide talk, so there's no chance I can take notes fast enough to keep up... I'm not gonna take notes

2.3 Ziyang Gao (Leibniz Universität Hannover): Sparsity of rational and algebraic points

Note 10. Another slide talk...

2.4

Summer of 1984: Koblitz' book mentions some analogy between Γ-functions and Gauss sums. I missed what's going on, but somehow there's some analogy between $2\pi i$ and p? I'm already confused...

Somehow speaker thought there should be some product formula, something like

$$\log 2\pi - \sum_p v_p(2\pi i) \log p = 0?$$

Or something like $v_p(2\pi i) = 1$? Consider

$$\frac{\zeta'(s)}{\zeta(s)} = -\sum_p \frac{\log p}{p^s - 1} \text{ where } p^s = e^{s \log p}.$$

Compare with

$$\frac{1}{e^x - 1} = \frac{1}{x} - \frac{1}{2} + \frac{x}{12} + \dots.$$

At 0, have $\zeta'(0)/\zeta(0) = \frac{1}{2} \sum \log p$. Somehow from all this, one gets

$$\log 2\pi - 2\frac{\zeta'(0)}{\zeta(0)} = -\log 2\pi \neq 0.$$

This is not quite what was originally wanted, so speaker didn't pursue this at the time. But then in the Spring of 1985, Fontaine was introducing period rings which involved a 'genuine p-adic $2\pi i$'. In the Fall of 1986, Boxcall had a paper on p-adic L-functions of CM elliptic curves. This paper included the following formula. Say $K = \mathbb{Q}(\sqrt{-p})$, $h(K) = 1$, $E/\mathbb{Q}$ CM by $\mathscr{O}_K$. Then,

$$v_p(\Omega_K) = \begin{cases} 0 & \text{if } p \text{ split} \\ \frac{1}{p-1} - \frac{1}{p^2-1} & \text{if } p \text{ inert} \\ \frac{1}{2(p-1)} & \text{if } p \text{ ramified} \end{cases}$$

Morally, $\Omega_p = 2\pi i/\Omega_\infty$. Somehow this suggests that $v_p(2\pi i) = 1/(p-1)$ and $v_p(\Omega_\infty) = 1/(p-1)$ or $1/(p^2-1)$ or $1/(2(p-1))$. Speaker asked Fontaine if $v_p(2\pi i) = 1/(p-1)$ makes sense. Fontaine said yes e.g. because $2\pi i$" $=$ " $\lim_{n\to\infty} p^n \mathrm{d}\zeta_{p^n}/\zeta_{p^n}$ and $v_p(\mathrm{d}\zeta_{p^n}/\zeta_{p^n}) = \frac{1}{p-1} - n$. Redoing earlier stuff with this in mind gives

$$\log(2\pi) - \sum_p \frac{\log p}{p-1} = \log 2\pi + \frac{\zeta'(1)}{\zeta(1)} := \log 2\pi - \frac{\zeta'(0)}{\zeta(0)} = 0.$$

This is what we were after in the beginning.

Note 11. I am so lost, gonna stop taking notes...

3 Day 3

3.1 Mark Kisin (Harvard University): Heights in the Isogeny Class of an Abelian Variety

(joint w/ Mocz)

Let A be an abelian variety over a number field K. Fix an algebraic closure $\overline{K}$ of K. Attached to A is its Faltings height $h(A)$. Despite this being an intro workshop, we haven't yet seen a definition of the Faltings height (though its appeared in some talks). We won't see one in this talk either, but we won't need it. We'll only need to know how it changes under isogeny, and this we will explain.

Let $\mathscr{I}(A)$ denote the isogeny class of $A/\overline{K}$. Note that we're allowing isogenier defined over any extension of K.

Conjecture 3.1.1 (Mocz/folklore)**.** *If $c > 0$, then*

$$\#\left\{A' \in \mathscr{I}(A) : h(A') < c\right\}/isom. < \infty$$

(Isomorphism over $\overline{K}$).

(Made by Mocz in her thesis, but there were hints of it in the folklore)

Remark 3.1.2. Morally, if you have a sequence of isogenies, the heights of the abelian varieties involved must go to ∞. You need to worry about silly things like considering $\cdots \to A \xrightarrow{[2]} A \xrightarrow{[2]} A \xrightarrow{[2]} A$ if you want this to be literal. ○

The CM case of this conjecture was handled in Mocz' thesis.

Theorem 3.1.3 (K.-Mocz)**.** *Suppose that the Mumford-Tate conjecture holds for A (call this statement* $\mathrm{MT}(A)$*). Then, Mocz's conjecture for A holds.*

Let $G_{\mathrm{MT}} = G_{\mathrm{MT}}(A)$ denote the Mumford-Tate group of A.

Theorem 3.1.4 (Deligne)**.** $\rho_\ell : \mathrm{Gal}(\overline{K}/K) \to \mathrm{GL}_2(\mathbb{Z}_\ell)$ *factors through the Mumford-Tate group.*

Conjecture 3.1.5 ($\mathrm{MT}(A)$)**.** $\operatorname{Im} \rho_A$ *is Zariski dense in* $G_{\mathrm{MT}}(A)$*.*

Kisin: "I'm not a fan of theorems of the form unknown conjecture A implies unknown conjecture B, but you shouldn't think of this as that, because the Mumford-Tate conjecture is known in many cases." (paraphrase)

Example. If $G_{\mathrm{MT}} = \mathrm{GSp}_{2g}$ and $g = \dim A$, then $\mathrm{MT}(A)$ holds for all but 82 values of $g < 10^6$. The smallest g where this holds if $g = 4$. △

One should compare this result of K.-Mocz to the following theorem of Faltings (which somehow says the opposite):

Theorem 3.1.6 (Faltings)**.** *If $A \to A'$ is defined over K, then $|h(A) - h(A')|$ is bounded. Thus,*

$$\#\left\{h(A') : A \to A'/K\right\} < \infty.$$

In order to study the conjecture, they need a way to measure the change in height. For this, they use

Theorem 3.1.7 (**Faltings' isogeny lemma**)**.** *Suppose A extends to an abelian scheme $\mathcal{A}/\mathscr{O}_K$. Extend $A \to A'$ to a morphism $\mathcal{A} \xrightarrow{\varphi} \mathcal{A}'/\mathscr{O}_K$. Then, $\mathcal{G} := \ker\varphi$ is a finite flat group scheme over $\mathscr{O}_K$. It's generic fiber is étale, so $\Omega^1_{\mathcal{G}/\mathscr{O}_K}$ is a torsion $\mathscr{O}_K$-module, as is $s^*\Omega^1_{\mathcal{G}/\mathscr{O}_K} =: \omega_{\mathcal{G}/\mathscr{O}_K}$, where $s : \operatorname{Spec}\mathscr{O}_K \to \mathcal{A}$ is the identity section. Given all of this,*

$$h(A') - h(A) = \frac{1}{2}\log\deg(\varphi) - \frac{1}{[K:\mathbb{Q}]}\log\#\omega_{\mathcal{G}/\mathscr{O}_K}.$$

(Can always assume A extends to a semiabelian scheme by extending the base field. We assume everywhere good reduction above just to simplify the exposition)

Note that the primes dividing $\#\omega_{\mathcal{G}/\mathscr{O}_K}$ will also divide $\deg\varphi$. Hence, the right term is a sum of local contributions.

Suppose $\deg\varphi = p^n$ for some prime p, and let $w \mid p$ be a prime of K over p. Note that

$$\omega_{\mathcal{G}/\mathscr{O}_K} \otimes \mathscr{O}_{K_w} = \bigoplus_i \mathscr{O}_{K_w}/a_i\mathscr{O}_{K_w}$$

(for some a_i's). Then,

$$\deg\mathcal{G} = \sum_i v_p(a_i) \text{ and } \operatorname{ht}\mathcal{G} = \log_p \#\mathcal{G}(\overline{K}).$$

Define

$$\mu_w(\mathcal{G}) := \deg\mathcal{G}/\operatorname{ht}(\mathcal{G}) \in [0,1],$$

and think of this as a '*local slope*'. The initial utility of this is that it lets us restate Faltings' isogeny lemma as

$$h(A') - h(A) = \log\deg(\varphi)\left(\frac{1}{2} - \underbrace{\sum_{w\mid p}\frac{[K_w:\mathbb{A}_p]}{[K:\mathbb{Q}]}\mu_w(\mathcal{G})}_{\nu(\mathcal{G})}\right)$$

(if $\deg\varphi$ is a p-power). This $\nu(\mathcal{G})$ should be thought of as a '*global slope*'. Note it's a weighted average of local invariants. Note the height increases/decreases depending only on whether $\nu(\mathcal{G})$ is $> 1/2$ or $< 1/2$.

Example. If $\mathcal{G}$ is étale over $\mathscr{O}_{K_w}$, then $\mu_w(\mathcal{G}) = 0$ (recall: étale means relative differentials are 0). If it's multiplicative, then $\mu_w(\mathcal{G}) = 1$. These are the two extremes. △

Note that Faltings theorem as stated is only for isogenies defined over K. What if it's only defined over an extension of K?

Suppose $A \to A'$ is defined over K', where K'/K is finite, Galois. You get a similar formula as before, but now the global slope takes the form

$$\nu(\mathcal{G}) = \sum_{w'\mid p}\frac{[K'_{w'}:\mathbb{Q}_p]}{[K':\mathbb{Q}]}\mu_{w'}(\mathcal{G}) = \sum_{w\mid p}\frac{[K_w:\mathbb{Q}_p]}{[K:\mathbb{Q}]}\sum_{\sigma\in\operatorname{Gal}(K'/K)}\mu_w(\sigma(\mathcal{G})),$$

where $\sigma(\mathcal{G}) \leq \mathcal{A}$ is the subgroup scheme with

$$\sigma(\mathcal{G})(\overline{K}) = \sigma(\mathcal{G}(\overline{K})) \subset A(\overline{K}).$$

So above, we got something like a (weighted) average of $\mu_w(\sigma(\mathcal{G}))$'s.

Remark 3.1.8. To make since of $\mu_w(\sigma(\mathcal{G}))$ above, since $\mathcal{G}$ (and $\sigma(\mathcal{G})$) is only defined over K'), choose any $w' \mid w$, and set $\mu_w(\sigma(\mathcal{G})) := \mu_{w'}(\sigma(\mathcal{G}))$. ○

Example. Suppose $A = E$ is an elliptic curve over $\mathbb{Q}$ which is ordinary at p, $\mathcal{G} \subset E[p]$, and $\overline{\rho}_E : \operatorname{Gal}(\overline{\mathbb{Q}}/\mathbb{Q}) \to \operatorname{GL}_2(\mathbb{F}_p)$ is surjective. Under these assumptions, the height will always increase when you move by an isogeny. Will be $\nu(\mathcal{G})$ be more or less than 1/2? Note that $\sigma(\mathcal{G}) \not\subset E[p]^{\text{mult}}$ for some σ (think of $E[p]^{\text{mult}}$ as a line in $E[p]$. Galois group moves around this line, using surjectivity of mod p representation). When this holds, $\sigma(\mathcal{G})$ will be étale and so satisfy $\mu_w(\sigma(\mathcal{G})) = 0$. Hence, most terms in the sum will be 0, not 1, so one gets that $\nu(\mathcal{G}) < 1/2$, and hence $h(A') > h(A)$. △

(Above, I think, $E[p]^{\text{mult}}$ is fixed essentially since it's defined over $K = \mathbb{Q}$)

To prove the theorem, consider some sequence $\mathcal{G}_n \subset \mathcal{A}$ of finite, flat group schemes (ffgs.). You want some statement like $\lim \nu(\mathcal{G}_n) < 1/2$. Because of the local natures of this global slope, the key cases are

(1) $\mathcal{G}_n \subset \mathcal{A}[p_n]$ for some distinct primes $p_1, p_2, \ldots$.

(2) $\#\mathcal{G}_n = p^{r_n}$ for some fixed p.

Today: we talk about **(2)** in a special case. However, the general case of **(2)** can be deduced from this special case.

Proposition 3.1.9. *Let $H \subset A[p^\infty](\overline{K})$ be a p-divisible subgroup. Let $\overline{H}[p^n]$ denote the closure of $H[p^n]$ in $\mathcal{A}$. Then,*

(i) *$\{\mu_w(\overline{H}[p^n])\}$ is non-increasing (so same is true for global invariants)*

(ii)

$$\lim_n \mu_w(\overline{H}[p^n]) = \frac{\dim(\mathrm{Lie}\, A \otimes \mathbb{C}_p \cap T_pH \otimes \mathbb{C}_p(-1))}{\mathrm{ht}\, H}$$

To explain this intersection, note that

$$T_pH \otimes \mathbb{C}_p \subset T_pA \otimes \mathbb{C} \simeq (\mathrm{Lie}\, A \otimes \mathbb{C}_p(1)) \oplus (blah)$$

(Hodge-Tate decomp on RHS)

(iii) *$\lim \nu(\overline{H}[p^n]) \leq 1/2$. Furthermore, if equality holds, then H is defined over a finite extension of K.*

Corollary 3.1.10 (of **(iii)**)**.**

$$\#\left\{A/\overline{H}[p^n] : h(A/\overline{H}[p^n]) < c\right\} / \simeq < \infty.$$

If the limit is less than a half, then the heights eventually decrease. If the limit is equal to $1/2$, then everything above is defined over a finite extension of K, so apply Faltings.

How might one prove that **(ii)** implies **(iii)**? Recall this global slope is some weighted average of local slopes of Galois translates, i.e.

$$\lim_n \nu(\overline{H}[p^n]) = \lim_n \sum_{w|p} \text{avg of } \mu_w(\sigma(\overline{H}[p^n])).$$

($\sigma \in \mathrm{Gal}(\overline{K}/K)$) Apply **(ii)** yields

$$\lim_n \mu_w(\sigma(\mathcal{G})) = \frac{\dim(\mathrm{Lie}\, A \otimes \mathbb{C}_p \cap \sigma(T_pH) \otimes \mathbb{C}_p(-1))}{\mathrm{ht}\, H}.$$

You imagine that if the Galois action is not too small, this average will be something like the dimension of a generic intersection. Something like this + some bookkeeping gives **(iii)**. It's in this implication that one uses the Mumford-Tate conjecture.

How might one prove **(ii)**? You might imagine that you want to make some kind of limiting object out of these $\overline{H}[p^n]$'s.

Warning 3.1.11. In general, $\{\overline{H}[p^n]\}$ is not a p-divisible group over $\mathscr{O}_{\mathbb{C}_p}$. •

(Sounds like, after some fiddling, it would be if all the H's were defined over a fixed number field?)

Proposition 3.1.12. *Consider p-divisible groups $\mathscr{I}/\mathscr{O}_{\mathbb{C}_p}$ equipped with a map $\chi : \mathscr{I} \longrightarrow \mathcal{A}[p^\infty]$ s.t. $\chi|_{\mathbb{C}_p}$ factors through H. Then, the category of such things has a final object $\mathcal{H}$. Furthermore, $\mathcal{H}|_{\mathbb{C}_p} = H$.*

Corollary 3.1.13. $\lim \mu_w(\overline{H}[p^n]) = \dim \mathcal{H}/\operatorname{ht} H$.

(ii) follows from this + the construction of $\mathcal{H}$. Sounds like they give two constructions of $\mathcal{H}$. One gives $\dim \mathcal{H}/\operatorname{ht} H$ = (RHS of **(ii)**), while the other gives the above corollary. To show that these two constructions are the same, they show they share a universal property.

Proof of Prop. Use Scholze-Weinstein classification of p-divisible groups over $\mathscr{O}_{\mathbb{C}_p}$.

$$\left\{p\text{-divislbe groups}/\mathscr{O}_{\mathbb{C}_p}\right\} \longleftrightarrow \{(T, W, f) : f : W(1) \hookrightarrow T \otimes \mathbb{C}_p\}$$

where T is a finite, free $\mathbb{Z}_p$-module and W is a $\mathbb{C}_p$-vector space. One direction of this takes

$$\mathcal{G} \longmapsto (T_p\mathcal{G}, \operatorname{Lie}\mathcal{G}, f_{\mathcal{G}}).$$

They simply take $\mathcal{H} = (T_pH, \operatorname{Lie}\mathcal{A} \otimes \mathbb{C}_p \cap T_pH \otimes \mathbb{C}_p(-1), f_{\mathcal{H}})$, with $f_{\mathcal{H}}$ induced by the corresponding thing for $\mathcal{A}$. One then checks this satisfies the universal property. ■

To prove formula, one needs to now construct $\mathcal{H}$ in a different way as a limit of these $\overline{H}$'s.

Question 3.1.14 (Audience)**.** *How far is $\overline{H}[p^n]$ away from being $\mathcal{H}[p^n]$?*

Answer. There's always a map $\mathcal{H}[p^n] \to \overline{H}[p^n]$. I didn't get the way to think of this, but something about applying some construction of Raynaud (for getting p-divisble groups out of sequences of ffgs's over dvrs) in this setting and taking a limit? I don't know. ⋆

3.2 Ananth Shankar (University of Wisconsin-Madison): Tate-Semisimplicity Over Finite Fields

(joint w/ B. Bakker & J. Tsimerman)

3.2.1 Intro

Let $S/\mathbb{F}_q$ be a smooth, connected variety. Let $\mathbb{L}$ be an étale ℓ-adic local system on S (where $\ell \neq p = \operatorname{char}\mathbb{F}_q$). $\mathbb{L}$ is equivalently a representation

$$\rho : \pi_1^{\text{ét}}(S) \longrightarrow \mathrm{GL}_n(\mathbb{Z}_\ell).$$

Assumption. Assume throughout that

(1) $\rho|_{\pi_1(S_{\overline{\mathbb{F}}_q})}$ is irreducible.

(2) $\det\rho$ is a finite order character

Given this setup, Laffargue proved the following result

Theorem 3.2.1 (Lafforgue)**.** *Given $x \in S(\mathbb{F}_{q^i})$,* $\mathrm{Frob}_x \curvearrowright \mathbb{L}_{\overline{x}}$ *w/ algebraic eigenvalues. Furthermore, they are integral over $\mathbb{Z}[1/p]$ and have complex absolute value 1.*

(Frob_x is the image of a generator under $\mathrm{Gal}_{\mathbb{F}_{q^i}} \simeq \pi_1^{\text{ét}}(x) \to \pi_1^{\text{ét}}(S) \xrightarrow{\rho} \mathrm{GL}_n(\mathbb{Z}_\ell)$)

This is related to Deligne's work on the Weil conjectures. Say $S = \operatorname{Spec} \mathbb{F}_q$ and X/S is a smooth projective variety. Then, $\mathrm{H}_{i,\text{ét}}(X_{\overline{\mathbb{F}}_p}, \mathbb{Q}_\ell)$ has Frob_q action. Deligne proved the eigenvalues are algebraic, integral over $\mathbb{Z}[1/p]$, and that they have complex absolute value $q^{i/2}$.

Question 3.2.2. *In either case, is the Frobenius endomorphism semisimple, i.e. diagonalizable over $\overline{\mathbb{Q}}_\ell$?*

Example. Say $X = A$ is an abelian variety. Then semisimplicity is known. In this case, have a Frobenius morphism $\mathrm{Frob} : A \to A$ inducing the Frobenius action on étale homology, i.e. on the Tate module $T_\ell(A)$. One also knows there's an embedding

$$\mathrm{End}(A) \otimes \mathbb{Q}_\ell \hookrightarrow \mathrm{End}_{\mathrm{Frob}}(T_\ell(A) \otimes \mathbb{Q}_\ell).$$

From this, one sees that Frob is in the center of $\mathrm{End}(A) \otimes \mathbb{Q}_\ell$, which is a semisimple algebra. Any element in the center of a semisimple algebra is semisimple. $\triangle$

(This argument appears in a paper of Tate's)

Sounds like one knows the Frobenius endomorphism is semisimple if X is a curve or K3 surface (see work of Deligne, Z. Yhang, Ho-Ho-Koshikawa).

Theorem 3.2.3 (Bakker-S.-Tsimerman)**.** *Let S be a compact exceptional Shimura variety. Let $p \gg_S 1$. Then,* Frob*-semisimplicity holds for $\mathbb{L}/S$ mod p, i.e. have semisimplicity at the stalks $\mathbb{L}_x$ for all $x \in S(\mathbb{F}_q)$.*

Question 3.2.4 (Audience)**.** *Do you expect semisimplicity for local systems coming from geometry, but not in general?*

Answer. Sounds like you expect semisimplicity in the Lafforgue setting, and expect that those local systems come from geometry. $\star$

Question 3.2.5 (Audience)**.** *Is the emphasis on exceptional Shimura varieties to avoid having an easy proof using (some result of Tate, I didn't hear which)?*

Answer. Yes. $\star$

Let's explain some of the words in the theorem.

3.2.2 Shimura Varieties

The first examples are modular curves $\mathcal{Y}$. The complex points of the connected component of such a thing look like $\mathbb{H}/\Gamma$, where $\mathbb{H} = \{z \in \mathbb{C} : \operatorname{Im} z > 0\}$ and $\Gamma \subset \mathrm{SL}_2(\mathbb{Z})$ is a (torsion-free) congruence subgroup. Such a $\mathcal{Y}$ is not just a complex manifold, but is in fact an algebraic curve. Furthermore, it carries a family $\mathcal{E} \to \mathcal{Y}$ defined over $\mathbb{Z}[1/N]$, with N explained defined in terms of Γ.

Question: If $\mathcal{Y}$ is connected, do you not need $\mathbb{Z}[1/N, \zeta_N]$ sometimes?

In more generality, Shimura varieties are associated to appropriate reductive groups $G/\mathbb{Q}$ as well as some homomorphism $h : \mathbb{S} \to G_{\mathbb{R}}$ ($\mathbb{S}$ = Deligne torus) satisfying the "Shimura axioms." This data gives rise to a generalization of the upper half-plane (hermitian symmetric domain) X. The components of the resulting Shimura varieties will be of the form X/Γ for suitable (torsion-free) congruence subgroups $\Gamma \subset G(\mathbb{Q})$. As stated, these sound like complex manifolds. However, Bailey-Borel proved that S is algebraic. Work of Deligne, Milne et al shows that S can be defined over an explicitly defined number field E.

Remark 3.2.6. How to get local systems. Should think that X is the universal cover of X/Γ, so $\pi_1(X/\Gamma) = \Gamma$ and local systems are Γ-reps. It's a theorem that, often, reps of Γ come from reps of G. ○

Let V be an algebraic representation of G defined over $\mathbb{Q}$. This gives a complex local system on Γ. Deligne, Milne, and friends tell you that this will actually descend to an étale local system over E.

G^{der} will be either of type A or B or C or D or E(xceptional groups). In type A, the associated Shimura variety will more-or-less parameterize abelian varieties (will parameterize 'abelian motives'[6]). In types B, C, also parameterize abelian motives. Some groups of type D do as well, though not all. We'll broadly term all these cases as being of "abelian type." In these cases, Kisin has constructed good integral models, so get Shimura variety S defined over $\mathscr{O}_E[1/N]$ (Mark's theorem says exactly which primes to invert). The local systems in these cases will be related to cohomology of abelian varieties[7], so Tate's argument gives Frobenius semisimplicity in these cases.

Other SVs (*mixed of type D* or associated to exceptional groups) will be called **exceptional Shimura Varieties**. This have not good theory of integral models or good moduli interpretation (sounds like it's conjecture they should be some moduli interpretation, but no one knows what sort of geometric objects they should parameterize). Let S be a compact exceptional Shimura variety. By formally spreading out, one gets a model of S over $\mathscr{O}_E[1/N]$ for some multiplicatively large N. Choose N large enough so that the model $\mathcal{S}/\mathscr{O}_E[1/N]$ satisfying

- $\mathcal{S}$ is smooth + projective over $\mathscr{O}_E[1/N]$
- For all $p \nmid N$, also $p \nmid \# \mathrm{GL}(\mathbb{V}/3\mathbb{V})$ ($\mathrm{GL}(\mathbb{F}_{3^n})$ where $n = \dim \mathbb{V}$?)
- Canonical bundle is ample mod v for all $v \nmid N$.

The first two conditions imply that $\mathbb{V}_3$ extends to $\mathcal{S} \otimes \mathscr{O}_{E,v}$. To deduce theorem for all ℓ (not just $\ell = 3$), one uses Kedlaya's,Abe-Esnault's theory of companions.

(aside Kledval + Patriks: $\mathbb{V}_\ell, \mathbb{V}_{\ell'}$ mod p are companions)

The third bullet point lets you extend (prime-to-N) Hecke correspondences to the integral model.

Note 12. Got distracted and missed some stuff

3.2.3 Different proof of semisimplicity for elliptic curves

Let $E/\mathbb{F}_q$ be an elliptic curve, with Tate module T_ℓ. If the eigenvalues of Frobenius are distinct, it's automatically semisimple, so suppose they are the same and that Frobenius is not diagonalizable (over $\overline{\mathbb{F}}_q$?). Then, get something like

$$\mathrm{Frob}_E = \begin{pmatrix} q^{1/2} & * \\ 0 & q^{1/2} \end{pmatrix}$$

[6]something something Tannakian category generated by abelian varieties something something

[7]something something $\mathbb{V}_\ell$ built from cohomology of abelian varieties something something

with $*$ some nonzero number. Call the basis which gives this matrix e_1, e_2. Also assume $* = 1$ because why not? Get a ℓ^n isogeny $E \to E_n$ defined over $\mathbb{F}_q$ (I missed how, but not e_1 spans a line defined over $\mathbb{F}_q$). The Tate module of E_n will be spanned by $\frac{e_1}{\ell^n}, e_2$. Thus, Frob $\curvearrowright T_\ell(E_n)$ via

$$\begin{pmatrix} q^{1/2} & \ell^n \\ & q^{1/2} \end{pmatrix}$$

Now, finiteness (of isom classes within an isogeny class) tells us there will be some $1 \le n < m$ with $E_n \xrightarrow{\sim} E_m$. Hence, the Frobenius actions will be conjugate integrally, i.e. over $\mathrm{GL}_2(\mathbb{Z}_\ell)$. This is impossible, e.g. because if you mod out by $\ell^m\mathbb{Z}_\ell$, then one of these matrices will be in the center, but the other won't be.

Remark 3.2.7. Tate uses an idea like this in his proof of the Tate conjecture. $\circ$

Somehow the above idea is bootstrapped to get hte main theorem.

3.2.4 Proof Sketch of Main Theorem

Say $\mathrm{Frob}_x \curvearrowright \mathbb{V}_{\ell,x}$. Write $\mathrm{Frob}_x = \varphi_s\varphi_u = \varphi_u\varphi_s$ with some semisimple and the other unipotent. These will both lie in $G(\mathbb{Z}_\ell)$. The role of isogenies is replaced by that of Hecke operators. Define $H := Z_G(\varphi_s)$. Note that $\varphi_u \in H$. Jacobson-Morozon Theorem tells you there exists some $h \in H(\mathbb{Q})$ such that $h^{-1}\varphi_u h = \varphi_u^{\ell^2}$ (in particular, is closer to the identity). Let's build a Hecke corrspondence

$$T_h : \mathcal{S}\text{``} \longrightarrow \text{''}\mathcal{S}$$

(quotes b/c multi-valued). There exists some $x_1 \in T_h(X) \subset \mathcal{S}(\mathbb{F}_q)$ where $\mathbb{V}_{\ell,x_1} \subset \mathbb{V}_{\ell,x} \otimes \mathbb{Q}_\ell$ (and $V_{\ell,x_1} = h \cdot \mathbb{V}_{\ell,x}$). Somehow this tells you this lattice will be Frobenius-stable and that x_1 will be defined over $\mathbb{F}_q$. Keep doing this to get unipotent part as close to the identity as possible. Get an infinite sequence of points all defined over $\mathbb{F}_q$, a contradiction.

4 Day 4

4.1 Min Ru (University of Houston): The Filtration Method in Diophantine Geometry

4.1.1 G.C.D., (and) a method proposed by Silverman

Let's begin by stating Vojta's conjecture.

Notation 4.1.1. Let k be a number field, and M_k be its set of places. We will use S to denote a finite set of places of M_k, and always assume S contains all archimedean places.

Conjecture 4.1.2. *Let X/k be a smooth, projective variety. Let $D \subset X$ be a simple normal crossings divisor, and let A be an ample divisor. Choose any $\varepsilon > 0$. Then, there will be a Zariski closed subset $Z = Z(X, D, A, \varepsilon) \subset X$ as well as a constant $c = c(X, D, A, \varepsilon)$ such that*

$$m_{D,S}(x) + h_{K_X}(x) \le \varepsilon h_A(x) + c \text{ for all } x \notin Z.$$

Above, $m_{D,S}(x) = \sum_{v\in S} \lambda_{D,v}(x)$*. One can also write this as*

$$h_{D+K_X}(x) \leq_{exc} \varepsilon h_A(x) + N_{D,S}(x) + O(1).$$

Above,

$$h_D(x) = \underbrace{\sum_{v\in S} \lambda_{D,v}(x)}_{m_{D,S}(x)} + \underbrace{\sum_{v\notin S} \lambda_{D,v}(x)}_{N_{D,S}(x)}.$$

Remark 4.1.3 (Audience, paraphrased). Think that m measures the "proximitiy of the point to the divisor D". The conjecture says that, if you take into account the canonical divisor, then the proximity is not too big. So, somehow, the divisor repels rational points.. ◦

Notation 4.1.4. $\leq_{\text{exc}}$ means inequality holds outside of an (exceptional) Zariski closed.

Theorem 4.1.5 (Bugeaud-Corvaja-Zannier)**.** *Let a, b be multiplicatively independent integers ≥ 2. Then, for any $\varepsilon > 0$, there is a constant $N = N(a, b, \varepsilon)$ such that for all $n > N$,*

$$\gcd(a^n - 1, b^n - 1) < 2^{\varepsilon n}.$$

Motivation of this result comes from Piscot's conjecture.

Conjecture 4.1.6 (Piscot)**.** *If $(a^n - 1) \mid (b^n - 1)$ for $n \gg 0$, then b is a power of a.*

(If I heard correctly this became a theorem before the BCZ result?)

What's the general G.C.D problem? Say $Y \subset X$ is a subvariety w/ $\operatorname{codim}_X(Y) \geq 2$. Then, $h_Y(x)$ is "small." Maybe we'll spell out this statement later?

Method by Silverman Assume Vojta's conjecture holds. Let $\pi : \widetilde{X} \to X$ be the blowup along Y, and let $D = \pi^*(-K_X)$ (assume properties on Y which guarantee that this is simple normal crossing). Note that $K_{\widetilde{X}} = \pi^* K_X + (r-1)E$ (where $r = \operatorname{codim} Y$?). Choose ample $\widetilde{A} = \pi^* A - \frac{1}{\ell} E$ with A ample on X. Vojta then says that

$$\left(r - 1 + \frac{1}{\ell}\right) h_E(\widetilde{x}) \leq_{\text{exc}} \varepsilon h_A(x) + N_{-K_X,S}(x) + O(1).$$

What to bound the N above.

Remark 4.1.7. (S, D)-integral points are those where the N-part is bounded. ◦

I didn't really follow how we conclude (I also don't know what sort of statement we're after...)

Theorem 4.1.8 (Wang-Yasokutu (spelling?), 2021)**.** *Let X/k be a smooth projective variety. Let $D_1, \ldots, D_{n+1}$ be effective divisors in general position. Assume there's an ample divisor A and integers d_i such that $D_i \equiv d_i A$ for all i (numerical equivalence). Let $Y \subset X$ be a subavariety of* codim ≥ 2 *which does not contain any point of the set $\bigcup_{i=1}^{n+1} \left(\bigcap_{1\leq j\neq i\leq n+1} D_j\right)$. For any $\varepsilon > 0$, one has*

$$h_Y(p) \leq_{exc} \varepsilon h_A(p)$$

for all (S, D)-integral points p ($D = D_1 + \cdots + D_{n+1}$).

(This was proved w/o using Vojta conjecture)

If I understand/heard correctly, above, $h_Y(p)$ means $h_E(p)$ where E is the exceptional divisor of the blowup of Y.

Example. $X = \mathbb{P}^n$ and $-K_X = H_0 + \cdots + H_n$ with $H_i = \{Z_i = 0\}$. Then condition is Y does not contain $(1 : 0 : \cdots : 0), \ldots, (0 : \cdots : 0, 1)$. △

Above theorem was proved using.

Theorem 4.1.9 (Ru-Vojta, 2020)**.** *Assume X is smooth/k. Say $D_1, \ldots, D_q$ are divisors in general position.*[8] *Let L be a big line bundle. Then,*

$$\sum_{j=1}^{q} \beta(L, D_j) m_{D_j,S}(x) \leq_{exc} (1+\varepsilon) h_L(x),$$

where

$$\beta(L, D) = \lim_{N\to\infty} \sum_{m\geq 1} \frac{h^0(L^N(-mD))}{N h^0(L^N)}.$$

Say $X = \mathbb{P}^n$ and take $L = \ell(n+1)\pi^* H_0 - E$ ($\pi : \widetilde{X} \to X$ blowup of Y?). One can prove

$$1/\beta(L, \pi^* H_i) \leq \frac{1}{\ell}\left(1 + \frac{1}{\ell\sqrt{\ell}}\right).$$

One needs $\pi^* H_i$ (for $i = 0, \ldots, n$) in general position. This depends on Y. Ru-Vojta gives

$$h_E(\widetilde{x}) \leq_{\text{exc}} \frac{1}{\sqrt{\ell}} h_A(x) + \sum_{i=0}^{n} N_{H_i,S}(x)$$

(or something like this, who knows?). Apparently this is how you prove the Wang-Yasokutu result.

4.1.2 Filtration Method

Say we have X (smooth, projective) and a divisor D, both defined over k. There are local heights $\lambda_{D,v}(x)$. Think of this as the negative of the log-distance from x to D.

Example (Complex case)**.** Have X and section $s_D : X \to \mathscr{O}(D)$. Then, $\lambda_D(x) = -\log \|s_D(x)\|$. △

Definition 4.1.10. One says $R \subset X(k) \setminus D$ is a set of (D, S)**-integral points** if there exists $C \geq 0$ so that

$$N_{D,S}(x) \leq C$$

for all $x \in R$. ◇

In qualitative terms, one its interested in finiteness or degeneracy (= non-Zariski closedness) of sets of integral points. One likes statements of the form

Conjecture 4.1.11. *All integral sets in $X(K) \setminus D$ are degenerate $\iff$ $K_X + D$ is big.*

[8] Each point has at most $n := \dim X$ divisors passing through it

(follows from Vojta's conjecture).

Remark 4.1.12. If K_X is big, can take $D = 0$ and recover Bombieri-Lang. $\circ$

Note that the conjecture holds when X is a curve by Siegel's theorem.

In quantitative terms, one wants "Schmidt-Vojta type" inequalities, i.e. bounds on $m_{D,S}(x)$ in terms of heights of x.

Sounds like the Filtration method originated in a 1994 paper of Faltings-Wüstholz (spelling?). It's ultimately based on the Schmidt subspace theorem. Additional progress started with the 2002 paper of Corvaja-Zannier reprove Siegel using Schmidt. They did the surface case in 2004 and projective spaces in 2004. Further results by Levin and Autissier.

Remark 4.1.13. Sounds like Schmidt subspace is Ru-Vojta when $X = \mathbb{P}^n$ and $D_i = H_i$ are hyperplanes (the β-constants go away apparently). Sounds like Roth's theorem – that

$$\left|\frac{p}{q} - \alpha\right| >_{\text{exc}} \frac{1}{q^{2+\varepsilon}}$$

for any algebraic number α of degree ≥ 2 – is Ru-Vojta/Schmidt for $X = \mathbb{P}^1$. $\circ$

Theorem 4.1.14 (Schmidt)**.** *Let $L_{1,v}, \ldots, L_{n+1,v}$ be linearly independent linear forms in $(n+1)$-variables. For all $\varepsilon > 0$,*

$$\sum_{i=1}^{n+1} \sum_{v \in S} \lambda_{L_i,v}(x) \leq_{exc} (n+1+\varepsilon)h(x).$$

Note 13. There was a bit more stuff after this, but oh well...

4.2 Hector Pasten (Pontificia Universidad Católica de Chile): Modularity and Height Bounds

This should be more of a survey lecture.

4.2.1 S-unit equation

Let S be a finite set of primes. Consider the equation

$$x + y = 1 \text{ with } x, y \in \left(S^{-1}\mathbb{Z}\right)^{\times}.$$

Example. If $S = \{2\}$, have solutions like $2 + (-1) = 1$ and $\frac{1}{2} + \frac{1}{2} = 1$. $\triangle$

Theorem 4.2.1 (Mahler)**.** *The S-**unit equation** $x+y = 1$ has only finitely many solutions in $\left(S^{-1}\mathbb{Z}\right)^{\times}$.*

Question 4.2.2. *How many solutions?*

By work of Evertse, one knows that the number of solutions is $\leq 21 \cdot 49^{\#S}$.

Remark 4.2.3. Compare to Uniform Mordell: $\#C(K) \leq (1+c)^{\operatorname{rank} J(K)}$. Note that $\#S$ is the rank of the group $\left(S^{-1}\mathbb{Z}\right)^{\times}$. $\circ$

Can this finiteness result be made effective? Linear functions in logarithms (LFL) gives

$$h(x,y) \ll_\varepsilon \left(\prod_{p\in S} p\right)^{\frac{1}{3}+\varepsilon}$$

(by work of Stewart and Yu). The implicit constant depends explicitly on ε.

We won't go into this LFL stuff today. Here is an alternative point of view. Consider $a+b=c$ w/ coprime positive integers. Want upper bound for c (think: height of solution) in terms of $\mathrm{rad}(abc) := \prod_{p\mid abc} p$. In this language, Stewart and Yu. say that

$$\log c \ll_\varepsilon \mathrm{rad}(abc)^{\frac{1}{3}+\varepsilon}.$$

What is the expectation?

Conjecture 4.2.4 (**ABC Conjecture**)**.**

$$c \ll \mathrm{rad}(abc)^\kappa$$

for some κ (possibly $\kappa = 1+\varepsilon$).

An approach using modularity can give a result of a similar flavor

$$\log c \ll_\varepsilon R^{1+\varepsilon}.$$

This has a worse exponent than the LFL one, but the implicit constant here is must better than there, so it can work better in practice. In fact, one can prove

$$\log c \ll R \log R$$

(Murty-P.) from which he see where the mysterious ε comes from. At roughly the same time, it was independently proved that

$$\log c \ll R(\log R)^2$$

(von Känel) also using modular forms.

4.2.2 Modular Approach

Let $E/\mathbb{Q}$ be an elliptic curve, N its conductor, and $h(E)$ its Faltings height "which has been defined a couple of times in this workshop, and which has not been defined so far. I would like to say that I am not gonna do it, but actually I will need to" (paraphrase).

Conjecture 4.2.5 (Frey)**.** *$h(E) \ll \log N$ (w/ an explicit constant)*

Even easier than this, bound $h(E)$ in terms of N. There are many expressions for the Faltings height, including one as a sum of local contributions. However, N forgets multiplicities (it's a product of primes of bad reduction w/ bounded exponents). So the question is somehow to bound the complexity of E using only coarse information about its bad reduction.

connection to S-units Associated to $a+b=c$ is the **Frey elliptic curve**

$$E : y^2 = x(x-a)(x-b).$$

One can compute that

$$\Delta_E = (abc)^2 \text{ and } N = \operatorname{rad}(abc).$$

(up to constants which we will ignore in this talk). Keep in mind that a, b, c are coprime. E has semistable reduction everywhere away from $2, 3$. Sounds like one can show[9]

$$\log(abc) \leq 6h(E) + 10.$$

If you can prove Frey's conjecture, then you get abc: $\log(abc) \ll 6 \log N$ (so $c \leq abc \ll N^6 = \operatorname{rad}(abc)^6$, a polynomial bound on c in terms of the radical).

Corollary 4.2.6. *Frey's conjecture $\implies$ abc.*

In fact, any inequality for the Faltings height in terms of the conductor will give some abc-like information, which then tells you something about the S-unit equation.

How does the modular approach work? By modularity, there is a modular parameterization

$$\varphi : X_0(N) \longrightarrow E$$

(normalized so that $\infty \mapsto 0$). Assume φ is of minimal degree. Let ω be a Néron differential on $E/\mathbb{Z}$ (this is unique up to scaling by $\mathbb{Z}^\times = \{\pm 1\}$). Then, the **Faltings' height** of E is

$$h(E) = -\frac{1}{2} \log\left(\frac{i}{2} \int_{E(\mathbb{C})} \omega \wedge \overline{\omega}\right).$$

This is *not* always positive, but still satisfies Northcott.

One has

$$\varphi^* \omega = 2\pi i c \cdot f(z) \mathrm{d}z$$

on $X_0(N) = \Gamma_0(N) \backslash \mathfrak{H}^*$ w/ $c \in \mathbb{Z} \setminus \{0\}$ the Manin constant (whatever this is), $f(z)$ is newform in $S_2(\Gamma_0(N)) =: S_2(N)$, and z the coordinate on the upper half plane. One gets that

$$\log \deg \varphi = 2 \log(2\pi |c|) + 2 \log \|f\|_2 + 2h(E)$$

(compute integral of $(\varphi^* \omega) \wedge (\varphi^* \overline{\omega})$). From this, one can deduce that[10]

$$h(E) \leq \frac{1}{2} \log \deg \varphi + 10.$$

We're in good shape because φ comes from the modular curve site, so you can think of this as only a function of N.

[9]Apparently this isn't too bad and comes from throwing at the archmidean contributions to the Faltings' height

[10]To bound $\|f\|_2$, use that f is normalized and its further Fourier coefficients decrease exponentially, so can give a lower bound (that's close to 1?)

Theorem 4.2.7 (Ribet)**.** $\deg\varphi \mid n_f$*, where* n_f *is a congruence number (it measures congruences of* f *and* $f^\perp \subset S_2(N,\mathbb{Z})$*).*

So, how do you bound the congruence number?

Theorem 4.2.8 (Morty-P.)**.** $\log n_f \le \frac{1}{5}N\log N$

Climbing back up the tower of implications, one gets

$$h(E) \ll N\log N$$

w/ explicit constant.

Question 4.2.9 (Audience)**.** *Is there an expectation for how big* n_f *should be?*

Answer. We expect that n_f should be at most polynomial in the level N. ⋆

4.2.3 Applications

Work of von Känel: other diophantine equations, e.g. $y^2 = x^3 + k$. Just need equations whose points parameterize elliptic curves (here, k is discriminant and x, y are the A, B parameters).

von Känel-Matschke: tables of solutions for S-unit equations.

Alpöge: extended the idea to GL_2-type abelian varieties, leading to height bounds for rational points on certain curves (can find compact curves inside the moduli space of GL_2-type abelian varieties, so their integral points are rational points).

Remark 4.2.10. S-unit equation is $\mathbb{P}^1 \setminus \{0,1,\infty\}$ is moduli for elliptic curves w/ full 2-torsion. This is Frey's construction. ○

Somehow this sort of idea works whenever you have an equation which parameterizes a moduli space for abelian varieties satisfying modularity.

4.2.4 Product of valuations

Bounds previously discussed were of the form $\log c \ll N^\kappa$. This is basically the same as saying

$$\max_{p\mid abc} v_p(abc) \ll_\kappa N^{\kappa+\varepsilon}$$

(it's literally the same as $\log c \ll_\varepsilon N^{\kappa+\varepsilon}$). Something stronger would be to prove

$$\prod_{p\mid abc} v_p(abc) \ll N^\kappa.$$

Why want this?

- $d(abc) \ll R^\kappa$, where $d(-)$ is number of divisors.

  (also $N = R = \mathrm{rad}(abc)$)

- Say $E/\mathbb{Q}$ an elliptic curve. Let $\mathrm{Tam}(E)$ be the product of the Tamagawa numbers. One may conjecture
$$\mathrm{Tam}(E) \ll_\varepsilon N^\varepsilon.$$
This would prove a subexponential version of abc (best known is exponential). One can prove a polynomial bound $\mathrm{Tam}(E) \ll N^\kappa$

- $$\#\{\text{level lowering primes for } f\} \ll \frac{\log N}{\log\log N}.$$

Now, how do you prove this? Make use of Shimura curves $X_0^D(M)$ with $N = DM$ **admissible** (D is squarefree w/ evenly many prime factors). By Jacquet-Langlands, the usual modular paramerterization $\varphi : X_0(N) \to E$ will induce maps
$$\varphi_{D,M} : X_0^D(M) \longrightarrow E,$$
say of degree $\delta_{D,M}$. Let $\delta_{1,N} = \deg \varphi$.

Warning 4.2.11. $X_0^D(M)$ may not have cusps (so no Fourier expansions) or even rational points. •

One can think
$$\frac{\delta_{1,N}}{\delta_{D,M}} \text{ “ = ” } \frac{\text{cong of } f \text{ in } S_2(N)}{D\text{-new congruences}} \stackrel{?}{=} D\text{-old congruences} \stackrel{?}{=} \prod_{p|D} v_p(\Delta_E).$$

Theorem 4.2.12 (P. after Ribet-Takahashi (spelling?))**.** *Let S be a finite set of primes. For every $E/\mathbb{Q}$ semistable away from S, we have*
$$\frac{\delta_{1,N}}{\delta_{D,M}} = \gamma_{E,D,M} \cdot \prod_{p|D} v_p(\Delta_E),$$
where $H(\gamma_{E,D,M}) \ll_\varepsilon N^\varepsilon$.

(γ above is a rational number).

Recall 4.2.13. $\log \delta_{1,N} = (\mathrm{const}) + 2\log\|f\|_2 + 2h(E)$ ⊙

In the Shimura curve case:
$$\log \delta_{D,M} = 2\log\|f'\|_2 + 2h(E)$$
with f' a quaternionic modular form which is integral over $\mathbb{Z}$ (pulling back a Néron differential). One ends up with
$$\sum_{p|D} \log v_p(\Delta_E) = 2\log\|f\|_2 - \log\|f'\| + o(\log N) = \log N - 2\log\|f'\|_2 + o(\log N).$$

Need an upper bound for $-2\log\|f'\|_2$. This is where things get complicated, and so is the part of the story we won't talk about in this intro workshop.

Theorem 4.2.14 (P.)**.** *Let $\varepsilon > 0$. For all nonzero $g \in S_2^D(M,\mathbb{Z})$, we have*
$$\|g\|_2 \gg_\varepsilon \frac{1}{N^{5/6+\varepsilon}\sqrt{D}}.$$

What goes into this? Need arakelov geometry, average Colmez conjecture, and more analytitc number theory.

Question 4.2.15 (Audience). *Does this generalize to number fields?*

Answer. Works out case of totally real fields in the same paper. Just need good enough notion of modularity. ⋆

4.3 Jennifer Balakrishnan (Boston University): Quadratic Chabauty for Modular Curves

Note 14. Sadly, this is a slide talk...

Goal. Quadratic Chabauty for modular curves.

4.3.1 Chabauty for curves

Proposition 4.3.1 (Diophantus, 3rd Century AD). *Solve the equation* $y^2 = x^6 + x^2 + 1$ *in rational numbers* x, y.

Diophantus new about $(x, y) = (1/2, 9/8)$. Are there more?

Similarly, consider the curve

$$X : y^2 = -x^6 - 9x^4 - 11x^2 + 37.$$

What we determine $X(\mathbb{Q})$?

Faltings theorem tells us $X(\mathbb{Q})$ is finite if X is a curve of genus $g \geq 2$. Can you actually compute this set?

For certain curves, by associating other geometric objects to X, we can explicitly compute a slightly larger (but still finite) set of p-adic points containing $X(\mathbb{Q})$.

- This story starts w/ Chabauty-Coleman
- There are variants (covering collections, elliptic Chabauaty, restriction of scalars, symmetric power Chabauty, ...)
- There is a vast generalization of this called nonabelian Chabauty

Theorem 4.3.2 (Chabauty, '41). *Let X be a curve of genus $g \geq 2$ over $\mathbb{Q}$. If* $\operatorname{rank} \operatorname{Jac}(X)(\mathbb{Q}) < g$, *then* $\#X(\mathbb{Q}) < \infty$.

- Coleman ('85) made this effective by re-interpreting in terms of p-adic line integrals of regular 1-forms. In fact, we gave the bound

  $$\#X(\mathbb{Q}) \leq \#X(\mathbb{F}_p) + 2g - 2$$

  when $p > 2g$ is a prime of good reduction.

Say $p > 2$ is a prime of good reduction for X. Fix $b \in X(\mathbb{Q})$ and embed it into its Jacobian via Abel-Jacobi $\iota : X \hookrightarrow J$, $P \mapsto [(P) - (b)]$. This induces an iso $\mathrm{H}^0(J_{\mathbb{Q}_p}, \Omega^1) \xrightarrow{\sim} \mathrm{H}^0(X_{\mathbb{Q}_p}, \Omega^1)$. Suppose ω_J restricts to ω. Define

$$\int_Q^{Q'} \omega := \int_0^{[(Q')-(Q)]} \omega_J.$$

If $r < g$, there exists $\omega \in \mathrm{H}^0(X_{\mathbb{Q}_p}, \Omega^1)$ s.t.

$$\int_b^p \omega = 0$$

for all $P \in X(\mathbb{Q})$. So we want to understand zeros of $\int_b \omega$. Let

$$X(\mathbb{Q}) \subset X(\mathbb{Q}_p)_1 := \left\{ z \in X(\mathbb{Q}_p) : \int_b^z \omega = 0 \right\}.$$

Example. $X_0(37)$ is the X from the start of the talk. This has rank $J_0(37)(\mathbb{Q}) = 1$ and is genus 2. One has $\{(\pm 1, \pm 4)\} \subset X(\mathbb{Q})$, and we set $b = (-1, 4) \in X(\mathbb{Q})$.

- Have $\mathrm{H}^0(X_{\mathbb{Q}_p}, \Omega^1) = \left\langle \frac{\mathrm{d}x}{y}, \frac{x\mathrm{d}x}{y} \right\rangle$
- Take $p = 3$
- The point $P := [(1, -4) - (-1, 4)] \in J_0(37)(\mathbb{Q})$ is non-torsion, e.g. since

  $$\int_p \frac{x\mathrm{d}x}{y} = 3^2 + 2 \cdot 3^3 + O(3^4).$$

  Moreover $\int_P \mathrm{d}x/y = O(3^9)$, so we may take $\mathrm{d}x/y$ as our annihilating differential.
- We solve

  $$\int_b^z \frac{\mathrm{d}x}{y} = 0$$

  for $z \in X(\mathbb{Q}_3)$. △

Note 15. I'm already missing stuff because I'm lagging behind tryint to type notes, so I'll stop here...

4.4 Minhyong Kim (International Centre for Mathematical Sciences): Path Integrals and p-adic L-functions

Note 16. Slide talk again...

Kim began with an advertisement for the 'Mathematics for Humanity' program at the ICMS.
Plan

- Preliminary remarks on QFT
- Arithmetic topology
- Arithmetic fields
- many more things...

4.4.1 Prelim remarks on QFT

Typical ingredients of (quantum) field theory of dimension d:

(1) Manifold M of dimension d, the model for spacetime.

(2) Fiber bundle $F \to M$, e.g. $M \times N$, tensor bundles, principal bundles (F could be stack in general, e.g. $M \times BG$)

(3) $\mathcal{F}_M = \Gamma(M, F)$, spece of fields/sections (e.g. vector fields, tensor fields, connections, map to some other manifold, bundles themselves, etc.)

(4) A **theory** consists of a function

$$S : \mathcal{F}_M \longrightarrow \mathbb{C}$$

called the **action**, typically expressed as a scary integral

poly in values and derivatives of φ

$$S(\varphi) = \int_M L(\varphi(x), \nabla\varphi(x), \nabla^2\varphi(x), \dots) \mathrm{d}\,\mathrm{vol}_M$$

The function L is usually of the form $\langle D\varphi(x), D\varphi(x)\rangle$ + higher order terms for some linear differential operator D.

(5) The space of classical states $\mathcal{C}_M \subset \mathcal{F}_M$ consisting of fields satisfying the Euler-Langrange equation for S describing the extrema of the function.

That's field theory. What's quantum?

(6) In quantum field theory, one consiers integrals like

$$\int_{\mathcal{F}_M} \exp(-\pi S(\varphi)) \mathrm{d}\,\mathrm{vol}_{\mathcal{F}}$$

or

$$\int_{\mathcal{F}_M} g_1(\varphi) \dots g_k(\varphi) \exp(-\pi S(\varphi)) \mathrm{d}\,\mathrm{vol}_{\mathcal{F}},$$

where the $g_i(\varphi)$ are usually local functions of φ, e.g. $\varphi \mapsto \varphi(x), \partial_t\varphi(x)$. Integrals like the first are often viewed as invariants of the manifold M, once the theory is fixed and makes sense on any manifold.

Example. For electromagnetism on a compact Riemannian manifold w/ $\mathrm{H}^1(M) = 0$, one might get

$$\int_{\mathcal{F}_M} \exp(-\pi S(\varphi)) \mathrm{d}\,\mathrm{vol}_{\mathcal{F}} = \frac{1}{\sqrt{\det \Delta_1}},$$

where Δ_1 is the Laplacian on 1-forms $\triangle$

When N is a manifold of dimension $d-1$, since one can consider the theory on $M = N \times [0,1]$, there is also a vector space of initial conditions $H(N)$ attached to N, approximately thought of as $H(N) = L^2(\mathcal{F}_N, \mathbb{C})$. If M is a cobordism from N_1 to N_2, one should also get a linear transformation

$$H(M) : H(N_1) \longrightarrow H(N_2),$$

thought of as an integral operator w/ kernel

$$K(\varphi_1, \varphi_2) := \int ...$$

There is a monoidal property

$$\begin{aligned} H(\varphi) &= \mathbb{C} \\ H(N \sqcup N') &= H(N) \otimes H(N') \\ H(-N) &= H(N)^\vee \end{aligned}$$

The operator associated to a cobordism can be compactly expressed as $H(M) \in H(N)$ when $\partial M = N$. In the case of topological quantum field theory, all these data only depend on the differential topology of M.

4.4.2 Arithmetic Topology

Let $\mathscr{O}_F$ be the ring of algebraic integers in a number field F, and let $X := \operatorname{Spec} \mathscr{O}_F$. It has many properties of a compact, closed 3-manifold. If v is a maximal ideal in $\mathscr{O}_F$, then κ_v is a finite field and the inclusion $\operatorname{Spec} \kappa_v \hookrightarrow X$ is analogous to the inclusion of a knot. The completion $\operatorname{Spec} \mathscr{O}_{F,v}$ is like the tubular neighborhood of a knot. The completion F_v of F is $\operatorname{Frac} \mathscr{O}_{F,v}$, so $\operatorname{Spec}(F_v) = \operatorname{Spec}_{\mathscr{O}_{F,v}} \setminus v$ is like the tubular neighborhood w/ central knot deleted, which should be homotopic to a torus.

If B is a finite set of primes, then $X_B := \operatorname{Spec} \mathscr{O}_{F,B} = \operatorname{Spec} \mathscr{O}_F \setminus B$ is like the 3-manifold w/ boundary, the boundary having one torus component $\operatorname{Spec} F_v$ for each prime in B:

$$\partial X_B = \bigsqcup_{v \in B} \operatorname{Spec} F_v \to X_B \hookrightarrow X.$$

Mazur's Analogy

- p-adic L-function $\leftrightarrow$ Alexander polynomial of a know K

  Analogoy between $\operatorname{Spec} \mathbb{Z}[\mu_{p^\infty}][1/p] \to \operatorname{Spec} \mathbb{Z}[\zeta_p][1/p]$ w/ Galois group

  $$\Gamma := \operatorname{Gal}(\mathbb{Q}(\mu_{p^\infty})/\mathbb{Q}(\mu_p)) \simeq \mathbb{Z}_p$$

  and the maximal abelian covering $D_K \longrightarrow S^3 \setminus K$ (missed stuff)

  The p-adic L-function is a determinant of the torsion $\mathbb{Z}_p[\![\Gamma]\!]$-module $\mathrm{H}_1(\operatorname{Spec} \mathbb{Z}[\mu_{p^\infty}], \mathbb{Z}_p)$ (stated carefully, need to take various isotypic components).

  The Alexander polynomial is a determinant of the torsion $\mathbb{Z}_p[\mathbb{Z}]$-module $\mathrm{H}_1(D_K, \mathbb{Z})$.

Quantum field theory interpretation of this analogy?

Jones polynomial and path integrals Witten: construction of the Jones polynomial of a know using the methods of quantum field theory, made rigorous by Reshetikhin and Turaev.

$\mathcal{A}$: SU(2) connectsion on S^3 acted upon by a group $\mathcal{G}$ of gauge transformations. A knot $K \subset S^3$ defined a Wilson loop function

$$\begin{aligned} W_K : \quad \mathcal{A} &\longrightarrow \quad \mathbb{C} \\ A &\longmapsto \operatorname{Tr}(\rho(\mathrm{Hol}_K(A))), \end{aligned}$$

the trace of the holonomy of the connection around K evaluated in the standard representation ρ of SU(2).

Remark 4.4.1. Compare to taking traces of Frobenius under some Galois rep. Suggests viewing knots/primes as functions (instead of $K(\mathbb{Z}, 1)$'s) and comparing them this way. ◦

Global Chern-Simons function given by

$$CS(A) = \frac{1}{8\pi} \int_M \operatorname{Tr}\left(A \wedge \mathrm{d}A + \frac{2}{3} A \wedge A \wedge A \right)$$

which is only gauge-invariant up to integers (i.e. is in $\mathbb{R}/\mathbb{Z}$). Witten's formula:

$$\int_{\mathcal{A}/\mathcal{G}} W_K(A) \exp(2\pi i k CS(A)) \mathrm{d}A = J_K\left(\exp\left(\frac{2\pi i}{k+2} \right) \right),$$

equating a path integeral w/ the value of the Jones polynoimal J_K of K at a root of unity.

Question 4.4.2. *Is there a path integral formula for L-functions?*

4.4.3 Arithmetic fields

"One thing you learn from physicists is that they call everything a field."

Most of the time, **arithmetic fields** will be sections of a sheaf or stack over an arithmetic scheme X_B.

Example.

$$\mathcal{M}(X_B, R) = \mathrm{H}^1(X_B, R) = \mathrm{H}^1(\pi_1(X_B), R),$$

a moduli space of principal R-bundles, where R is a sheaf of topological groups. The group R is often a p-adic Lie group like $\mathrm{GL}_n(\mathbb{Z}_p)$. But it might be a sheaf of finite groups like $A[p]$ or Lie groups T_pA for an algebraic group A. △

Other possible spaces of arithmetic fields

- families $\{P_v\}_v$ where P_v is a principal R-bundle over $\operatorname{Spec} F_v$
- Families ...

4.4.4 Arithmetic actions (two examples)

For technically reasons, we will assume throughout that F it totally complex. Would like to define

$$S : \mathcal{M}(X_B, R) = \mathrm{H}^1(\pi_1(X_B), R) \longrightarrow K$$

as well as path integrals

$$\int_{\rho \in \mathcal{M}(X_B, R)} \ldots$$

Let μ_n be the nth roots of 1. Then,

$$\mathrm{H}^3(X, \mu_n) = \mathrm{H}^3(\operatorname{Spec} \mathscr{O}_F, \mu_n) \simeq \frac{1}{n}\mathbb{Z}/\mathbb{Z}$$

(consequence of class field theory, see Mazur "étale cohomology of ..." (1970's)) Assume $\mu_n \subset F$, so $\mathrm{H}^3(X, \mathbb{Z}/n\mathbb{Z}) \simeq \frac{1}{n}\mathbb{Z}/\mathbb{Z}$, so we get a map

$$\mathrm{inv} : \mathrm{H}^3(\pi_1(X), \mathbb{Z}/n\mathbb{Z}) \longrightarrow \mathrm{H}^3(X, \mu_n) \simeq \frac{1}{n}\mathbb{Z}/\mathbb{Z}$$

Example. Let R have trivial $\pi_1(X)$-action. On the moduli space

$$\mathcal{M}(X, R) = \mathrm{Hom}(\pi_1(X), R)//R,$$

of continuous reps of $\pi_1(X)$, a Chern-Simons functional is defined as follows:

The function will depend on the choice of a cohomology class (a level) $c \in \mathrm{H}^3(R, \mathbb{Z}/n\mathbb{Z})$. Then,

$$\begin{array}{rccc} CS_c : & \mathcal{M}(X, R) & \longrightarrow & \frac{1}{n}\mathbb{Z}/\mathbb{Z} \\ & c & \longmapsto & \mathrm{inv}(\rho^*(c)) \end{array}$$

△

If $R = \mathbb{Z}/n\mathbb{Z}$, can define $c = a \cup \delta a$ where $a \in \mathrm{H}^1(R, \mathbb{Z}/n\mathbb{Z}) = \mathrm{Hom}(\mathbb{Z}/n\mathbb{Z}, \mathbb{Z}/n\mathbb{Z})$ is the identity, and $\delta : \mathrm{H}^1(R, \mathbb{Z}/n\mathbb{Z}) \to \mathrm{H}^2(R, \mathbb{Z}/n\mathbb{Z})$ is the Bockstein operator (connecting map from $0 \to \mathbb{Z}/n\mathbb{Z} \to \mathbb{Z}/n^2\mathbb{Z} \to \mathbb{Z}/n\mathbb{Z} \to 0$)

Example (BF-theory)**.** Let V be a finite n-torsion group scheme admitting a suitable Bockstein $\mathrm{d} : \mathrm{H}^1(X, V) \to \mathrm{H}^2(X, V)$. Have function

$$\begin{array}{ccc} \mathrm{H}^1(X, V) \times \mathrm{H}^1(X, D(V)) & \longrightarrow & \frac{1}{n}\mathbb{Z}/\mathbb{Z} \\ (a, b) & \longmapsto & \mathrm{inv}(\mathrm{d}a \cup b) \end{array}$$

where $D(-)$ is Cartier duality. △

4.4.5 Arithmetic Path Integrals

[Joint work w/ H. Chung, D. Kim, G. Pappas, J. Park, H. Yoo]

Let $n = p$ be a prime and assume the Bockstein map $\mathrm{d} : \mathrm{H}^1(X, \mathbb{Z}/p\mathbb{Z}) \to \mathrm{H}^2(X, \mathbb{Z}/p\mathbb{Z})$ is an isomorphism. Then,

$$\sum_{\rho \in \mathrm{H}^1(X, \mathbb{Z}/p\mathbb{Z})} \exp(2\pi i CS(\rho)) = \sqrt{\# \operatorname{Cl}_X[p]} \left(\frac{\det d}{p}\right) i^{\frac{(p-1)^2 \dim(\operatorname{Cl}_X[p])}{4}}.$$

Arithmetic BF-theory (joint w/ Magnus Carlson): Start w/

$$\begin{array}{rccc} BF : & \mathrm{H}^1(X, \mu_n) & \longrightarrow & \mathrm{H}^1(X, \mathbb{Z}/n\mathbb{Z}) \quad \mathrm{inv}(\mathrm{d}a \cup b) \\ & \frac{1}{n}\mathbb{Z}/\mathbb{Z} & \longmapsto & (a, b) \end{array}$$

Proposition 4.4.3. *for $n \gg 0$,*

$$\sum_{(a,b)\in \mathrm{H}^1(X,\mu_n)\times \mathrm{H}^1(X,\mathbb{Z}/n\mathbb{Z})} \exp(2\pi i BF(a,b)) = \#\,\mathrm{Cl}_X[n]\cdot \#\frac{\mathscr{O}_X^\times}{(\mathscr{O}_X^\times)^n}.$$

Compare w/ (some L-function stuff I missed)

If E is an elliptic curve w/ Néron model $\mathscr{E}$, then we have

$$0 \longrightarrow \mathcal{E}[n] \longrightarrow \mathcal{E}[n^2] \longrightarrow \mathcal{E}[n] \longrightarrow 0,$$

for n coprime to the conductor and the orders of component groups. This gives us a $BF : \mathrm{H}^1(X, \mathscr{E}[n]) \times \mathrm{H}^1(X, \mathcal{E}[n]) \to \frac{1}{n}\mathbb{Z}/\mathbb{Z}$.

Proposition 4.4.4. *Assuming $\#\text{Ш}(E) < \infty$, for $n \gg 0$, have*

$$\sum_{(a,b)\in \mathrm{H}^1(X\mathcal{E}[n])\times \mathrm{H}^1(X,\mathcal{E}[n])} \exp(2\pi i BF(a,b)) = \#\text{Ш}(E)[n]\cdot \#(E(F)/n)^2.$$

Compare with

$$\frac{L^{(r)}(T_pE,0)}{r!} = \left(\prod_v c_v\right)\#\text{Ш}(E)...$$

(leading term from BSD)

4.4.6 p-adic L-functions as path integrals?

(Spoiler: they haven't succeeded yet)

Recall Wittens path integral determinantion of the Jones polynomial.

For each $j \in \{1, 2, \dots, p-1\}$ odd, there is a unique power series $Z_j(T) \in \mathbb{Z}_p[\![T]\!]$ such that

$$Z_j((1+p)^n - 1) = (1 - p^{-n})\zeta(n),$$

for all $n < 0$, $n \equiv j \pmod{p-1}$. Allows the interpolation of the negative...

(Joint work w/ Carlson, Chung, Kim, Park, Yoo)

Let $X_B = \operatorname{Spec}\mathbb{Z}[\mu_{p^n}][1/(\zeta_{p^n} - 1)]$ and define the space of fields as

$$\mathcal{F}^m := \mathrm{H}^1(X_B, \mu_{p^m}) \times \mathrm{H}^1_c(X_B, \mathbb{Z}/p^m\mathbb{Z}).$$

There is a natural action of $G = \mathrm{Gal}(\mathbb{Q}(\mu_{p^n})/\mathbb{Q})$ on the space of eilds $\mathcal{F}^m$. Let $G' \subset G$ be unique subgroup of order $p-1$...

(missed definition of $\mathcal{F}_k$)

$$\mathcal{F}_k = \mathrm{H}^1(X_B, \mathbb{Z}_p(1))_{\omega^k} \times \mathrm{H}^1_c(X_B, \mathbb{Q}_p/\mathbb{Z}_p)_{\omega^{-k}}$$

and

$$\int_{\mathcal{F}_k} \exp(2\pi i BF(a,b))\mathrm{d}a\mathrm{d}b := \lim_{m\to\infty} \sum_{(a,b)\in\mathcal{F}_k^m} \exp(2\pi i BF(a,b))$$

(Think of Riemann sums)

Theorem 4.4.5. *Let $k \neq 1$ be odd. We have*

$$\int_{\mathcal{F}_k} \exp(2\pi i BF(a,b)) \mathrm{d}a\mathrm{d}b = \left| \prod_{j=0}^{p^n-1} Z_{1-k}(\exp(2\pi ij/p^n) - 1)^{-1} \right|_p .$$

(Note complex numbers on the left, not p-adic ones, so can't hope for direct equality)

Essentially just a repackaging of the main conjecture of Mazur and Wiles together w/ some generalities on arithmetic duality.

Remark 4.4.6. It would be far more interesting to get

$$\left| \prod_{j=0}^{p^n-1} Z_{1-k}(\exp(2\pi ij/p^n) - 1)^{-1} \right|_p$$

expressed as an integral. ○

4.4.7 Future work

(Some thoughts coming out of discussion w/ David Jordan, Jon Pridham, and Pavel Safranov)

Remark 4.4.7. "If turns into real results, it's joint work. If turns out to be nonsense, those people aren't responsible." ○

Galois reps

$$G_p := \mathrm{Gal}(\mathbb{Q}_{(p)}/\mathbb{Q}) \longrightarrow \mathrm{GL}(V),$$

with $\mathbb{Q}_{(p)}$ the maximal extension of $\mathbb{Q}$ unramified outside p and V is a $\mathbb{Q}_p$-vector space. Assume p is odd and V is crystalline at p.

Proposition 4.4.8 (joint work)**.**

$$\mathcal{M}(V) := \mathrm{H}^1_f(G_p, V \times V^\vee(1))$$

suitable interpreted, has the structure of a Lagrangian intersection, and hence, a (-1)-shifted symplectic structure

Construct $\mathcal{H}(M(V))$, a quantization of $\mathcal{M}(V)$, a module over a suitable completion R of $\mathbb{Z}_p[\![T]\!] \otimes \mathbb{Q}_p$.

Conjecture 4.4.9 (Owen Gwilliam $+\varepsilon$)**.** $\mathrm{rank}_R\, \mathcal{H}(M(V)) = 1$

Conjecture 4.4.10. *There is a canonical isomorphism*

$$\mathcal{H}(M(V)) \simeq \det R\Gamma_f(X, V \otimes R)$$

(both of these are "essentially theorems" once one nails then definitions and irons out some details)

Conjecture 4.4.11. *There is a (QFT construction of a) canonical vector*

$$\mathcal{L}_p(V) \in \mathcal{H}(M(V))$$

that can be identified with the p-adic L-function of V.

(This really is a conjecture)

5 Day 5

5.1 Jean-Benoit Bost (Faculté des Sciences d'Orsay): Formal-Analytic Surfaces: Nori's Bound in Arakelov Geometry and Archimedean Overflow

(joint work w/ Francois Charles, see `https://arxiv.org/abs/2206.14242`)

Note 17. Slide talk (w/ slides in cursive)...

Want to talk about formal-analytic arithmetic surfaces. Think of analogues of germs of complex formal analytic surfaces along a projective curve C.

Say $\psi \in \mathbb{R}[x]$ w/ $\psi(0)$ and $\psi'(0) \neq 0$. Consider $\mathbb{Z}$-algebra (+ normal domain)

$$\widetilde{\mathscr{O}}(\psi) := \{\widehat{\alpha} \in \mathbb{Z}[\![T]\!] : \text{radius of convergence of } \widehat{\alpha} \circ \psi > 1\}.$$

Theorem 5.1.1 (CDT21). *If $|\psi'(0)| > 1$, and if $\widehat{p} \in \widetilde{\mathscr{O}}(\psi) \setminus \mathbb{Z}$, then*

$$[\operatorname{Frac} \widetilde{\mathscr{O}}(\psi) : \mathbb{Q}(\widehat{p})] \leq \textit{some explicit bound}$$

Note 18. I can't read these slides well enough to take notes and pay attention....

There theory can also be used to prove finiteness of fundamental groups of some modular curves. For $N \geq 3$, consider $\mathcal{Y}(N) \to \operatorname{Spec} \mathbb{Z}$ smooth, affine geometrically irred. fibers classifying $\mathcal{E}/S$ with $i : \mathcal{E}[N] \xrightarrow{\sim} (\mu_N \times \mathbb{Z}/N\mathbb{Z})_S$. This has finite fundamental group (even after base change to any ring of integers).

5.1.1 Geometric motivations: germs of complex surfaces along projective curves

Say D is a projective (not necessarily irreducible) curve in a surface V.

Fact. The geometric/analytic properties of V are controlled by the properties of the intersection matrix

$$\Gamma := (D_i \cdot D_j) \text{ where } D = \bigcup_{i \in I} D_i$$

as well as $\lambda := \lambda_{\max}(\Gamma)$, the largest (simple) eigenvalue (its eigenvector has > 0 coefficients).

Trichotomy

(1) $\lambda < 0$

Basic example: resolution of a germ of normal surface w/ isolated singularity

many holomorphic functions on V and D may be contracted

(2) $\lambda > 0$

Basic example $\mathbb{P}^1 \hookrightarrow V := \mathbb{P}^2 \setminus$ large ball in $\mathbb{C}^2$.

few meromorphic functions $\operatorname{trdeg}_{\mathbb{C}} \mathcal{M}(V) \leq 2$. algebraicity: $\varphi : V \to \mathbb{P}^N$ analytic, then $\varphi(V)$ will be included in an algebraic surface.

(3) $\lambda = 0$

The $\lambda > 0$ case above are generally not algebraic.

Fact. If there exists an open immersion $i : V \hookrightarrow X$ w/ X an algebraic surface, then $\Gamma := (D_i \cdot D_j)$ has at most one > 0 eigenvalue.

This follows from **Hodge Index Theorem**: If X is a smooth...

5.1.2 Nori's degree formula and fundamental groups of surfaces

Note 19. I'm gonna go ahead and stop trying to take notes...

Proposition 5.1.2 (Nori)**.** *If $f : Y \to X$ is a dominant morphism b/w integral, normal projectuve surfaces (over $k = \overline{k}$). Let $B \hookrightarrow Y$ be a divisor and $A := f_* B$ (a divisor on X). If $B \cdot B > 0$, then $\deg f \leq (A \cdot A)/(B \cdot B)$. In particular, $A \cdot A > 0$.*

Proof. Use Hodge index. ∎

5.2 François Charles (École Normale Supérieure): Formal-Analytic Surfaces: Finiteness Theorems, and Applications to Arithmetic Fundamental Groups – Didn't Go

5.3 Laura Capuano (Università degli Studi Roma Tre): Unlikely Intersections and Applications to Diophantine Problems

Let's start w/ the **Mordell Conjecture** (**Faltings' Theorem**): Let C be an irreducible curve of genus 2 defined over a number field K; then, $\#C(K) < \infty$.

Here's a perspective of this. If $C(K) \neq \emptyset$, can define an embedding $C \hookrightarrow \operatorname{Jac}(C)$. Note that $C(K) = C \cap J(K)$ ($J = \operatorname{Jac}(C)$), i.e. the set $C(K)$ we want to study is the intersection of some variety C w/ points $J(K)$ in an abelian variety (also, $J(C)$ is f.g. group by Mordell-Weil). Note that $\dim J = g > 1 = \dim C$, so believable it's rare for curve to fit f.g. group $J(K)$. One can generalize this in many ways

- Replace J w/
  - $\mathbb{G}_m^n$
  - (semi)abelian variety
  - Shimura variety
- Replace C w/
  - subvariety of higher dimension
- Replace $J(K)$ w/

– f.g. subgroup, or subgroup of finite rank (e.g. $\dim(\Gamma \otimes_{\mathbb{Z}} \mathbb{Q}) < \infty$)

This gives rise to statements like Manin-Mumford, Mordell-Lang, etc.

Say G is an ambient, say, semiablien, variety. Let $X \subset G$ be an (irreducible) subvariety. If H is an algebraic subgroup such that

$$\dim H + \dim X < \dim G,$$

then one generally expects that $X \cap H = \emptyset$. What if instead of taking one algebraic subgroup, one takes a union of algebraic subgroups satisfying this dimension inequality. Then, one still expects the union of intersections to be small compared to X.

Conjecture 5.3.1 (**Zilber-Pink** for semiabelian varieties)**.** *Let $X \subset G$ with G/K a semiabelian variety (and $K \subset \mathbb{C}$). Let*

$$G^{[m]} = \bigcup_{\substack{H \text{ alg. subgrp.} \\ \operatorname{codim}_G H \geq m}} H.$$

If X is not contained in any proper algebraic subgroup, then

$$X \cap G^{[\dim X + 1]}$$

is not Zariski dense in X.

Remark 5.3.2.

- Zilber-Pink $\implies$ **Manin-Mumford**, i.e. that the set of torsion points on such an X is not Zariski dense.

  Indeed, $G^{\text{tors}} = \bigcup_{\operatorname{codim}_G H = n} H$ (where $n = \dim G$?)

- Say $G = \mathbb{G}_m^n$ is a torus and X is a curve over $\overline{\mathbb{Q}}$. This case was handled by BMZ '99, Maurin '08, and BMZ '08 (for $X/\mathbb{C}$ now). If $\operatorname{codim} X = 2$, result of BMZ in '07. Habegger also has some results along these lines.

- G an abelian variety

  – X curve$/\overline{\mathbb{Q}}$ settled by Habegger-Pila '16
  – X curve$/\mathbb{C}$ Barroero-Dill '20 (using Habegger-Pila)
  – $\operatorname{codim} X = 2$ for $G = E^n$ by Checholi-Viada-Veneziano

- G semiabelian variety

  – $X/\overline{\mathbb{Q}}$ curve by Barroero-Kühne-Schmidt
  – $X/\mathbb{C}$ curve by Barroero-Dill ○

Question 5.3.3. *What if $K \subset \mathbb{C}$ is replaced by a field w/ positive characteristic?*

Say $G = \mathbb{G}_m^n$ is a split torus. Let $C \subset G$ is a curve, parameterized by rational functions $C \rightsquigarrow (f_1(a), \ldots, f_n(a))$ ($f_i \in \mathbb{F}_p(C)$). Something about multiplicative relations, but every element of $\overline{\mathbb{F}}_p$ has finite multiplicative order, so can't expect Zilber-Pink to hold w/ the same statement here. Masser '14, under more restrictive hypotheses on C, proved a Zilber-Pink type result when $n = 3$ ($n = \dim G$?). It's open in general.

Work of F. Berroero-C.-Mesai-Ostafe-Sha Take $f_1(x), \dots, f_n(x) \in \mathbb{Q}(x)$ and $p \gg 0$ so that the reductions $\overline{f}_1, \dots, \overline{f}_n \in \mathbb{F}_p(x)$ make sense. Consider the set

$$\mathcal{A}(p, L) = \left\{ \alpha \in \overline{\mathbb{F}}_p \;\middle|\; \begin{array}{c} \exists \underline{\ell}, \underline{k} \in \mathbb{Z}^n \text{ lin. indep w/} \\ \prod f_i^{\ell_i}(\alpha) = 1 = \prod f_i^{k_i}(\alpha) \text{ and } \max\{|\ell_i|, |k_i|\} \leq L \end{array} \right\}.$$

Want to explore relation between mod p intersections and global intersections.

Theorem 5.3.4 (BCMOS)**.** *Assume $f_1, \dots, f_n \in \mathbb{Q}(x)$ are multiplicatively independent. Then,*

$$\#\mathcal{C} \cap G^{[2]} < \infty \quad (\textit{Maurin}).$$

There exists an effectively computable constant $c = c(f_1, \dots, f_n) > 0$ such that for all $p \geq e^{cL^2}$, then L is fixed

$$\#\mathcal{A}(p, L) \leq \#\mathcal{C} \cap G^{[2]},$$

and all the solutions in $\mathcal{A}(p, L)$ come from reductions of global solutions.

This is just for rational curves. There is in progress work by Campagna-Dill dealing with more general cases. Sounds like BCMOS have also results in $E^n \times \mathbb{G}_m^r$.

Remark 5.3.5 (Response to Audience question)*.* $\mathcal{A}(p, \infty)$ is always infinite. ○

5.3.1 What about families?

Let S be an irreducible smooth curve, defined over a number field K. Let A/S be an abelian scheme of relative dimension $g \geq 2$. Let $C \subset A$ be an irreducible curve dominating S. For every $s \in S(\mathbb{C})$, set

$$A_s^{[2]} = \bigcup_{\substack{H \text{ alg. subgrp} \\ \operatorname{codim}_{A_s} H \geq 2}} H \quad \text{and} \quad A^{[2]} = \bigcup_{s \in S(\mathbb{C})} A_s^{[2]}.$$

Conjecture 5.3.6 (Pink)**.** *Assume that $C \not\subset$ any proper subgroup scheme. Then, $C \cap A^{[2]}$ is not Zariski dense (so finite over S).*

Remark 5.3.7*.* Say $g = 2$. Then, $\mathcal{A}^{[2]} = \bigcup$ torsion sections. In this case, Pink's conjecture give relative Manin-Mumford. ○

Masser-Zannier (+ Corvaja/$\mathbb{C}$) proved this case (if I heard correctly). Habegger-Gao proved relative Manin-Mumford for higher dimensional bases and subvarieties (recently announced). Pink's conjecture is known for $\mathcal{A} = \mathcal{E}^n$ fiber product of elliptic schemes via work of Barroero-C. + Barroero (for number fields) + Barroero-Dill (for $\mathbb{C}$).

The general case is known only partially.

Theorem 5.3.8 (Barroero-C.)**.** *Assume $C \not\subset$ any proper subgroup scheme. Then,* This is for any abelian scheme over a curve

$$C \cap \left(\bigcup_{\text{codim} \geq 2} \textit{flat subgroup schemes} \right) < \infty.$$

(for S a curve, flat $\iff$ each irreducible component dominates the base. Something like this is true; flatness should be easy/mild over Dedekind domains.)

What goes into the proof?

- Pila-Zannier method
  - o-minimality & point-counting
  - large Galois orbit bound $/\overline{\mathbb{Q}}$

  Silverman gives $h(P)$ is bounded. Then want to bound their degrees, and this is where Pila-Zannier method comes in.

Binyamini effectivized above sort of point-counting result, so should give an effective version of their theorem.

5.3.2 Application to GCD results for divisibility sequences

Theorem 5.3.9 (Bugeaud-Corvaja-Zannier '03). *Let $a, b \in \mathbb{Z}_{\geq 1}$ multiplicatively independent. Then, for all $\varepsilon > 0$, there is some $c = c(a, b, \varepsilon)$ such that*

$$\log \gcd(a^n - 1, b^n - 1) \leq \varepsilon n + c.$$

Conjecture 5.3.10 (Ailon-Rudnick). *There are infinitely many $n \geq 1$ such that*

$$\gcd(a^n - 1, b^n - 1) = \gcd(a - 1, b - 1).$$

(Completely open)

Consider this conjecture for polynomials, so $a, b \in k[t]$ (multiplicative independent) w/ $k \subset \mathbb{C}$. In this case, the GCD conjecture is a theorem of Ailon-Rudnick.

Theorem 5.3.11 (Ailon-Rudnick). *a, b mult. independent polynomials. Then, $\exists c_{a,b} \in k[t]$ such that for all $n \geq 1$,*

$$\gcd(a^n - 1, b^n - 1) \mid c_{a,b}$$

Say $E/k(t)$ an elliptic curve and choose $P \in E(k(t))$. Write $x_P = A_P/D_P^2$.

Fact. For all $m \mid n$, $D_{mP} \mid D_{nP}$.

Conjecture 5.3.12 (Silverman, now theorem of Silverman '05 and Ghioca-Hsia-Tucker '18). *Assume $P, Q \in E(k(t))$ s.t. there are no isogenies $f, g : E \to E$ w/ $f(P) = g(Q)$. Then, $\exists c_{P,Q} \in k[t]$ such that for all $n \geq 1$,*

$$\gcd(D_{nP}, D_{nQ}) \mid c_{P,Q}$$

Given $E(k(t))$ can get $\mathscr{E} \to \mathbb{P}^1$ w/ generic fiber E. Any $P \in E(k(t))$ will give a section $\sigma_P : \mathbb{P}^1 \to \mathscr{E}$. The roots of the denominator give places where point becomes torsion. Hence this is studying the intersection of $\sigma_P(\mathbb{P}^1)$ with the zero section. Relative Manin-Mumford (applied in $\mathcal{E} \times \mathcal{E}$?) gives that $\operatorname{supp} \operatorname{div}_0 \gcd(D_{nP}, D_{nQ})$ is finite. I'm a little confused, but she's writing

$$\operatorname{div}_0(\gcd(D_{nP}, D_{nQ})) = \sum_{\gamma \in \mathbb{P}^1} \min\left\{\operatorname{ord}_\gamma(\sigma_{nP}^*(\overline{O}), \sigma_{nQ}^*(\overline{O}))\right\} \cdot \gamma.$$

To generalize Silverman's conjecture, replace $\mathbb{P}^1$ w/ any curve S, and give a formulation for every group scheme/S. Can also replace zero section $\overline{O}$ w/ any other section.

Sounds like Barroero-C.-Turchet proved generalization of Silverman's conjecture to $\mathcal{G} \to S$ s.t. the generic fiber is a split semiabelian variety.

5.4 Arul Shankar (University of Toronto): Polynomials with Squarefree Discriminant

(Joint w/ M. Bhargava & X. Wang)

We have a polynomial $F(x_1, \ldots, x_n) \in \mathbb{Z}[x_1, \ldots, x_n]$. A classical question in analytic number theory is to understand the values that F takes, e.g. how often does it take prime values (very difficult). Today, we'll ask

Question 5.4.1. *What is the "probablitiy" that $F(x)$ is squarefree?*

By "probabability" we mean

$$\lim_{X \to \infty} \frac{\#\{x : |x_i| < X, F(x) \text{ is squarefree}\}}{(2X+1)^n}.$$

Conjecture 5.4.2 (Folklore)**.** *The above probability is*

$$\prod_p \Pr(p^2 \nmid F(x)) = \prod_p \frac{\#\{x \in (\mathbb{Z}/p^2\mathbb{Z})^n : p^2 \nmid F(x)\}}{p^{2n}}.$$

When $n = 1, \deg F = 1$, basically determining density of squarefree numbers. This is classical and gives something like $1/\zeta(2)$. If $\deg F = 2$, a similar argument works and the answer is classically known. When $\deg F = 3$, this is quite hard, but was done by Heath-Brown. When $\deg F \geq 4$, wide open (sounds like no known cases, even e.g. $x^4 + 1$).

Let $V_n = \mathrm{Sym}^n(2) = \left\{a_n x^n + a_{n-1}x^{n-1}y + \cdots + a_0 y^n\right\}$ (binary n-ic forms). Associated to this is the discriminant polynomial $\Delta_n = F$ (vanishes when form has a double root in $\mathbb{P}^1$). Note $\deg F = 2n - 2$.

Theorem 5.4.3 (Bhargava-S.-Wang)**.** *Conjecture 5.4.2 is true for Δ_n.*

($n = 3$ case due to Davenport-Heilbronn)

Given a binary n-ic form $f(x, y) \in V_n(\mathbb{Z})$, one can associate a rank n ring R_f (construction due to Nakagawa. Melanie Wood showed it's the ring of global functions associated to the scheme $\{f = 0\} \subset \mathbb{P}^1$). What is the probability that R_f is maximal?

Theorem 5.4.4 (Bhargava-S.-Wang)**.** $\Pr(R_f$ *is maximal in* $R_f \otimes \mathbb{Q}) = 1/(\zeta(2)\zeta(3))$

(something something Arithmetic Bertini for $\mathbb{P}^1$ something something)

Corollary 5.4.5.

$$\#\{\deg n \textit{ fields } K \textit{ with } |\Delta(K)| < X\} \gg X^{\frac{n+1}{2n+2}}$$

(slight improvement of Ellenberg-Venkatesh bound $X^{1/2+1/n^2}$. The correct answer should be X)

This is still far from the truth, essentially because you do not expect most rings to show up as R_f.

What sort of things are needed to prove this result in the general case? In the simplest possible sieve, throw away all x s.t. $4 \mid f(x), 9 \mid f(x), \dots$, add in the ones s.t. $36 \mid f(x), \dots$, throw away.... This will net you

$$\#\{x : |x| < X, F(x) \text{ is squarefree}\} = \sum_{q \geq 1} \mu(q) \# \left\{x : |x| < X, q^2 \mid F(x)\right\}$$

(Easy if $q^2 < X$. If q is large, this is difficult to count exactly, so look for upper bounds.)

What is a good upper bound for

$$\#\{x : |x| < X, m \mid F(x)\}?$$

You might guess that there are X^n points in total, so might guess S^n/m. However m could be larger than X^n, so try $X^n/m + 1$. To be safe, say

$$\left(\frac{X^n}{m} + 1\right) X^{\varepsilon}.$$

This sort of thing is hard to prove (about as hard as counting integral points on some varieties). Even if $F = \Delta_4$, because binary quartic forms parameterize 2-Selmer elements of elliptic curves, a bound of this sort would produce great estimates on integral points of elliptic curves. It'd give

$$|\mathrm{Sel}_2(E)| \ll_{\varepsilon} |\Delta(E)|^{\varepsilon} \implies \#\text{integer points on } E \ll |\Delta(E)|^{\varepsilon},$$

but nothing so strong is known. So a bound like this in full generality should be difficult. However, we don't need such a thing. Only need m which are squares. Also, don't need bound for individual q^2's, but only need it on average.

Need: for some $M > 1, X > 1$

Probably want this for $M = \sqrt{X}$ since $q^2 < X$ is easy.

$$\sum_{q \geq M} \# \left\{x : |x| < X, q^2 \mid F(x)\right\} \ll \frac{X^n}{M} + X^{n-\delta} \text{ for some } \delta > 0.$$

The $1/M$ is coming from $\sum_{q \geq M} 1/q^2$. Essentially, we want to say that it is rare for $F(x)$ to be divisible by a large square.

Things are hard because you have a very sparse set of lattice points. To get around this, find some reductive group acting on them which brings them closer together. Then, you can bound the number by something like the volume of a region.

Case $n = 3$ Look a binary cubics

$$V_3 = \left\{ax^3 + bx^2y + cxy^2 + dy^3\right\}$$

Note that $\mathrm{GL}_2 \curvearrowright V_3$ via $\gamma f(x, y) = \frac{1}{\det \gamma} f((x, y) \cdot \gamma)$.

Davenport-Heilbron tail estimate in words: binary cubic forms give cubic rings. The $\mathrm{GL}_2(\mathbb{Z})$ orbit doesn't change the ring. In fact, in this case, this is a bijection (b/w cubic rings and integral orbits of binary cubic forms). Suppose you have a binary cubic form w/ discriminant divisible by p^2. Then, discriminant is divisible by p, so it must have a double (or triple) root mod p. Use GL_2-action to move that root to 0. Hence, form has coefficients looking like (a, b, pc, pd). We know more: $p^2 \mid \Delta_3$. For that to happen, either $p \mid b$ (have triple root) and/or $p \mid d$. Call these Type 1 points ($p \mid b$) and Type 2 points ($p \mid d$). There's a sieve originally developed by Ekhadahl (w/ refinements by Sarnak, Poonen-Stoll, Bhargava) which gives good enough tail estimates on type 1 points. Type 2 points here will have corresponding rings which are not maximal (they have index at least p in the maximal ring), so can count them by couting their overrings instead. This last sentence does not generalize well to other cases.

Remark 5.4.6. Somehow, in general, type 1 means $p^2 \mid \Delta$ can be detected mod p, but type 2 means $p^2 \mid \Delta$ can only be detected mod p^2. $\circ$

Here's an alternate approach to dealing w/ type 2 points.

$$f(x,y) \overset{\gamma}{\rightsquigarrow} (a, b, pc, p^2 d) \overset{\begin{pmatrix} 1 & \\ & 1/p \end{pmatrix}}{\rightsquigarrow} (pa, b, c, d).$$

This latter operation increases size of coefficient, so bad if ordering points by height. However, if you order orbits of forms by discriminant, this is useful because it lowers the discriminant. Now, can use geometry of numbers of get a tail estimate.

Ternary cubic forms Take action of $\mathrm{GL}_3(\mathbb{Z}) \curvearrowright \mathrm{Sym}^3(\mathbb{Z}^3)$. Want tail estimates on points in which p^2 divides the discriminant. Suppose $p^2 \mid \Delta(f(x,y,z))$. It must have a singularity. If it's type 2, this must be a nodal singularity (anything and it's a type I form). Move nodal singularity to $(0:0:1)$, so coefficients of z^3, xz^2, yz^2 are all 0 mod p. Since p^2 divides discriminant, in fact $p^2 \mid$ coeff of z^3. Now consider

$$f(x,z,y) \longmapsto pf \overset{z \mapsto z/p}{\longmapsto} g.$$

In new form, $z \mid g \mod p$ (this transformation keeps discriminant the same), so g is a type 1 form. Use Ekhadahl.

This argument is due to Bhargava-S.

Binary n-ic forms, $n = 2g+1$ odd Have $\mathrm{GL}_2 \curvearrowright V_n = \mathrm{Sym}^n(2)$. This is "not big enough." Consider also the space $W_n = 2 \otimes \mathrm{Sym}_2(n)$, pairs (A, B) of symmetric $n \times n$ matrices. $\mathrm{GL}_2 \times \mathrm{SL}_n \curvearrowright W_n$. This is an "invariant map"

$$\begin{array}{rccc} \mathrm{inv}: & W_n & \longrightarrow & V_n \\ & (A,B) & \longmapsto & \det(Ax + By). \end{array}$$

If I heard correctly, this is invariant under the SL_n-action and equivariant for the GL_2-action. Suppose $f \in V_n(\mathbb{Z})$ is integral binary form. Work of Melanie Wood shows that

$$\frac{\mathrm{inv}^{-1}(f)}{\mathrm{SL}_n(\mathbb{Z})} \longleftrightarrow \text{``}\,\mathrm{Cl}(R_f)[2]\text{''}$$

which proves, for example, that $\mathrm{inv}^{-1}(f)$ is nonempty (have identity element on RHS). One n is even, only parameterize a torsor for this group, which may be empty.

They construct a lift

$$\left\{f \in V_n(\mathbb{Z}) : p^2 \mid \Delta(f) \text{ for mod } p^2 \text{ reasons}\right\} \longhookrightarrow W_n(\mathbb{Z})$$

(making a choice of element of $\mathrm{SL}_n(\mathbb{Z})$-orbit corresponding to trivial element of the class group)

Instead of counting binary n-ic forms, can instead count $\mathrm{SL}_n(\mathbb{Z})$-orbits of $W_n(\mathbb{Z})$. Turns out lift sits inside subspace $W_n^0(\mathbb{Z}) \subset W_n(\mathbb{Z})$ where the top-left $g \times g$ block is all 0's. $\mathrm{SL}_n(\mathbb{Z})$ does not act on this subset, but there is a non-reductive subgroup $G_0(\mathbb{Z})$ which does. Note that the discriminant Δ_n on V_n lifts to a $(\mathrm{GL}_2 \times \mathrm{SL}_n)$-invariant on W_n. However, it's not irreducible as a $G_0(\mathbb{Z})$-invariant on W_n^0. In $\mathbb{Z}[W_0]$, there is some Q such that $Q^2 \mid \Delta$. One gets that

$$\#\left\{f(x,y) : H(f) < X, m^2 \mid \Delta(f), m > M\right\} \leq \#\left\{(A,B) \in G_0(\mathbb{Z})\backslash W_n^0(\mathbb{Z}) : H(\mathrm{inv}(A,B)) < X, Q(A,B) > M\right\}$$

($H(-)$ is max of abs. val. of coeffs.). Since $G_0(\mathbb{Z})$ is not reductive, there are no nice fundamental domains. To get around this, they replace $G_0(\mathbb{Z})$ with $\mathrm{SL}_n(\mathbb{Z})$ as much as possible.

Have to worry about losing control of the Q-invariant in making this switch; they show it's rare for it to change too much, I think. Arul drew imagines of the fundamental domains for $G_0(\mathbb{Z})$ vs $\mathrm{SL}_n(\mathbb{Z})$ and how one handles the cusp(s) vs. the main bodies.

Arul then ended by saying a bit about what happens when n is even. Essentially, they consider

$$\begin{array}{ccc} V_n(\mathbb{Z}) & \longrightarrow & V_{n+1}(\mathbb{Z}) \\ f(x,y) & \longmapsto & yf(x,y) \end{array}$$

and keep track of the resulting points of $W_{n+1}(\mathbb{Z})$ which lie in the image of this (want (A,B) with $\det A = 0$).

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