STAGE Fall 2021 Notes

Niven Achenjang

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These are my course notes for "Class name" at School name. Each lecture will get its own "chapter." These notes are live-texed or whatever, so there will likely to be some (but hopefully not too much) content missing from me typing more slowly than one lectures. They also, of course, reflect my understanding (or lack thereof) of the material, so they are far from perfect.[1] Finally, they contain many typos, but ideally not enough to distract from the mathematics. With all that taken care of, enjoy and happy mathing.

The instructor for this class is Prof name, and the course website can be found by clicking this link. Extra extra read all about it

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1 Angus Andrew (Boston Univeristy): Intersection Theory with Divisors, 9/22/2021

Today we're doing classic intersection theory, but later in the seminar, we'll see some Arakelov stuff it sounds.

1.1 Some buildup

Let's begin with a "theorem" Angus was thought at school.

Theorem 1.1. *A quadratic has* $0, 1,$ *or* 2 *roots.*

This is maybe a bit upsetting, so can upgrade to a slightly better 'theorem'.

Theorem 1.2. *Over* $\mathbb{C}$*, a quadratic has* 1 *or* 2 *roots.*

We still have issues like $p(x) = x^2$. To get something even better, we should count roots with multiplicity.

Theorem 1.3. *With multiplicity, a quadratic (over* $\mathbb{C}$*) has* 2 *roots.*

Now we're getting somewhere. Quadratics aren't so special, so let's go ahead and generalize.

Theorem 1.4 (**Fundamental Theorem of Algebra**)**.** *With multiplicity, a degree d polynomial has d roots.*

Still only dealing with single variable polynomials here. We want to do algebraic geometry, with systems of multi-variate polynomials. What can we say about these, is there any analogue in higher dimensions?

A first case to look at is intersections of two curves.

Example. Two lines can have 0, 1, or ∞ intersection points in the plane $\mathbb{A}^2$. △

This is annoying again. The ∞ case is not too bad; this is just the two lines being the same. To get rid of the 0, we compactify $\mathbb{A}^2$ to $\mathbb{P}^2$. Skipping ahead, the punchline is

Theorem 1.5 (**Bezout's Theorem**)**.** *Given projective plane curves $X, Y \subset \mathbb{P}^2$ of degree d, e, respectively. If X, Y have no common irreducible component, then, counting with multiplicity, they have de intersection points.*

What are these intersection multiplicities appearing in the statement of this theorem?

Let's start classically. If $P \in X \cap Y$, one computes the **intersection multiplicity at** P via

$$(X \cdot Y)_p = \text{length}\left(\frac{\mathscr{O}_p}{(f, g)}\right),$$

where $X = \{f = 0\}$ and $Y = \{g = 0\}$. Above $\mathscr{O}_p$ is the stalk at p of the structure sheaf of the ambient space.

Goal. Generalize this story.

1.2 Divisors

Let X be a scheme.

Definition 1.6. A **Weil divisor** D on X is a formal $\mathbb{Z}$-linear combination of closed, integral codimension 1 subschemes. The group of such divisors is denoted $\operatorname{Div} X$.

There are also Cartier divisors. These could be described as (formal differences) of subschemes which are locally principal. Locally principal is saying it's locally cut out by one function, so can say this in a more slick way as follows...

Definition 1.7. A **Cartier divisor** is a global section of $\mathscr{K}_X^\times/\mathscr{O}_X^\times$. Here, $\mathscr{K}_X$ is the sheafification of $U \mapsto \operatorname{Frac}\Gamma(X, \mathscr{O}_X)$, where Frac means total ring of quotients or whatever it's called (invert all non-(zero divisors)).

There is a natural map $\mathrm{H}^0(X, \mathscr{K}_X^\times/\mathscr{O}_X^\times) \to \operatorname{Div} X$ sending $\{(U_i, f_i)\} \mapsto \sum_Y \nu_Y(f_i)Y$.

Remark 1.8. These two notions of divisor coincide if X is sufficiently nice. See Hartshorne (chapter II, section 6 maybe?). ○

Given an effective (Cartier) divisor[2], it determines a subscheme $Y \subset X$. Hence, this gives an exact sequence

$$0 \longrightarrow \mathscr{I}_Y \longrightarrow \mathscr{O}_X \longrightarrow \mathscr{O}_Y \longrightarrow 0,$$

and the ideal sheaf is the line bundle $\mathscr{I}_Y \simeq \mathscr{O}_X(-D) \in \operatorname{Pic} X$.

We will define an intersection theory that will not just allow us to intersect two divisors, but to intersect two line bundles. The approach to intersection theory we'll give was initially due to Snapper, and then further developed by Kleiman and others. It's often called...

1.3 Snapper-Kleiman Intersection Theory

Let $X \to S$ be an S-scheme.

Definition 1.9. Let F be the category of coherent sheaves $\mathscr{F}$ on X such that $\operatorname{supp}\mathscr{F}$ is proper over an Artinian subscheme of S. Let F_r be the subcategory consisting of the $\mathscr{F}$ such that $\dim\operatorname{supp}\mathscr{F} \le r$.

Definition 1.10. Let K be the Grothendieck group of F, i.e. $\mathbb{Z}$-linear combinations of objects in the category, modulo short exact sequences. Similarly, let $K_r \le K$ be the subgroup generated by the elements of F_r.

In this setting, there is a form of induction known as **Grothendieck's dévissage**[3].

Lemma 1.11. *Suppose G_r is a subcategory of F_r such that*

(D1) *If $Z \subset X$ is a closed, integral subscheme proper over an Artininan subscheme of S such that $\mathscr{O}_Z \in F_r$, then $\mathscr{O}_Z \in G_r$.*

Question: Artinian scheme just means spec of an Artinian ring? i.e. 0-dimensional and noetherian

(D2) *If $0 \to \mathscr{F}_1 \to \mathscr{F}_2 \to \mathscr{F}_3 \to 0$ is exact, and $\mathscr{F}_1, \mathscr{F}_3 \in G_r$, then $\mathscr{F}_2 \in G_r$.*

[2] An effective Cartier divisor means the f_i's live in $\Gamma(X, \mathscr{O}_X)$, not just $\Gamma(X, \mathscr{K}_X^\times)$

[3] For some reason, when Angus said this, I initially head "Grothendieck's baby sandwich" which I think is something else entirely

Then, $G_r = F_r$*.*

Sounds like if you want to work with all of F instead of the filtered pieces F_r, you need a third condition which allows you to pass up increasingly large dimensions of the supports.

Let's see an example application of this.

Lemma 1.12. *Consider* $\mathscr{F} \in F_r$*. Let* $Y_1, \dots, Y_s$ *be the irreducible components of* $\operatorname{supp} \mathscr{F}$*. Let* $y_i \in Y_i$ *be the generic points. Then, in* K_r*, we have*

$$\mathscr{F} \equiv \sum_{i=1}^{s} \left(\operatorname{length}_{y_i} \mathscr{F}\right) \cdot \mathscr{O}_{Y_i} \pmod{K_{r-1}}.$$

Proof idea. Let F_r' be the collection of $\mathscr{F}$ satisfying this relation. If $\mathscr{F} = \mathscr{O}_Z$ for a closed integral subscheme, this obviously holds so we get **(D1)** for free. Since length (at generic points?) is additive in exact sequences, **(D2)** is satisfied. By dévissage, $F_r' = F_r$. ■

1.3.1 Line bundles

Definition 1.13. Given a line bundle $\mathscr{L} \in \operatorname{Pic} X$, we have an endomorphism $c_1(\mathscr{L})$ of K given by

$$c_1(\mathscr{L}) \cdot \mathscr{F} = \mathscr{F} - \mathscr{L}^{-1} \otimes \mathscr{F}.$$

Intuition. If $\mathscr{L} = \mathscr{O}(D)$, you want to tensor $\mathscr{F}$ with $\mathscr{O}_D$, but tensoring isn't so good when non-flat things are involved. Hence you instead tensor with $\mathscr{O}_X$ and $\mathscr{L}$ (which are flat since locally free), and then take the difference in the Grothendieck group.

Example. Think about effective divisors. Say $\mathscr{L} = \mathscr{O}_X(D)$ and $\mathscr{F} = \mathscr{O}_X$ (assuming this $\mathscr{F}$ is in our category). Then, the RHS is

$$\mathscr{O}_X - \mathscr{O}_X(-D) = \mathscr{O}_D \in K$$

via the exact sequence $0 \to \mathscr{O}_X(-D) \to \mathscr{O}_X \to \mathscr{O}_D \to 0$. △

Proposition 1.14.

(1) $c_1(\mathscr{L})c_1(\mathscr{M}) = c_1(\mathscr{L}) + c_1(\mathscr{M}) - c_1(\mathscr{L} \otimes \mathscr{M})$*.*

In particular, $c_1(\mathscr{L})c_1(\mathscr{M}) = c_1(\mathscr{M})c_1(\mathscr{L})$

(2) $c_1(\mathscr{O}_X) = 0$

I think at we're maybe secretly assuming $X \to S$ is proper with S Artinian, since we'll most often be interested in a scheme of finite type over a field

(3) If $Z \subset X$ *s.t.* $\mathscr{O}_Z \in F$ *and* $\mathscr{L}|_Z \simeq \mathscr{O}_Z(D)$ *for some effective divisor* D*, then* $c_1(\mathscr{L}) \cdot \mathscr{O}_Z = \mathscr{O}_D$*.*

Definition 1.15. Let $\mathscr{L}_1, \dots, \mathscr{L}_r \in \operatorname{Pic} X$ and $\mathscr{F} \in F_r$. The **intersection number of** $\mathscr{L}_1, \dots, \mathscr{L}_r$ **on** $\mathscr{F}$ is

$$(\mathscr{L}_1 \cdots \mathscr{L}_r \cdot \mathscr{F}) = \chi\left(X, c_1(\mathscr{L}_1) \dots c_1(\mathscr{L}_r)\mathscr{F}\right),$$

where χ is the Euler characteristic

$$\chi(X, \mathscr{F}) = \sum (-1)^i \dim \mathrm{H}^i(X, \mathscr{F})$$

extended linearly to K.

Theorem 1.16. *Let S be Artinian, $X \to S$ an S-scheme, and $\mathscr{F} \in F_r$. Let $\mathscr{L}_i = \mathscr{O}_X(D_i)$ for D_i effective (Cartier) divisors on X. Assume $\mathscr{F}$ is locally free at each $x \in \bigcap D_i$ and that the local equations $\{f_{i,x}\}$ for $\{D_i\}$ at $x \in \bigcap D_i$ form a regular sequence in $\mathscr{O}_{X,x}$. Then,*

$$(\mathscr{L}_1 \cdots \mathscr{L}_r \cdot \mathscr{F}) = \sum_{x \in \bigcap D_i} \operatorname{rank}_x(\mathscr{F}) \cdot \operatorname{length}\left(\frac{\mathscr{O}_{X,x}}{(f_{1,x}, \ldots, f_{r,x})}\right).$$

Remark 1.17. If $\mathscr{F} \in F_r$, then $\dim \operatorname{supp} \mathscr{F} \leq r$ is at most the number of line bundles, the number of equations we're cutting out above. This will ensure that the length computed above is finite. $\circ$

This theorem can be proved with dévissage. If you read Kleiman (one of the references), there's another proof given in corollary B.11.

Notation 1.18. If $\mathscr{F} = \mathscr{O}_X$, we write

$$(\mathscr{L}_1 \cdots \mathscr{L}_r) := (\mathscr{L}_1 \cdots \mathscr{L}_r \cdot \mathscr{O}_X).$$

Furthermore, if $\mathscr{L}_i = \mathscr{O}_X(D_i)$, we write

$$(D_1 \cdots D_r \cdot \mathscr{F}) := (\mathscr{L}_1 \cdots \mathscr{L}_r \cdot \mathscr{F}).$$

Remark 1.19.

- $c_1(\mathscr{L}) \cdot K_r \subset K_{r-1}$

  Hence, if $\mathscr{F} \in F_{r-1}$, then $(\mathscr{L}_1 \cdots \mathscr{L}_r \cdot \mathscr{F}) = 0$.

- Since χ is additive,

$$(\mathscr{L}_1 \cdots \mathscr{L}_r \cdot \mathscr{F}) = \sum_{Y_i \subset \operatorname{supp} \mathscr{F}} \left(\operatorname{length}_{y_i} \mathscr{F}\right) \cdot (\mathscr{L}_1 \cdots \mathscr{L}_r \cdot \mathscr{O}_{Y_i}),$$

  with the sum of irreducible components of $\operatorname{supp} \mathscr{F}$.

$\circ$

We've expressed intersection numbers in terms of Euler products. Let's do the reverse.

Theorem 1.20 (Snapper)**.** *For $\mathscr{F} \in F_r$ and $\mathscr{L}_i \in \operatorname{Pic} X$,*

Not sure if I copied this down right. See the references

$$\chi(X, \mathscr{L}_1^{m_1} \otimes \cdots \otimes \mathscr{L}_n^{m_n} \otimes \mathscr{F}) = \sum_{\substack{i_j \geq 0 \\ \sum i_j \leq r}} (\mathscr{L}_1^{i_1} \cdots \mathscr{L}_n^{i_n} \cdot \mathscr{F}) \binom{m_1 + i_1 - 1}{i_1} \cdots \binom{m_n + i_n - 1}{i_n}$$

Proof idea ($n = 1$). From the binomial expansion, one can derive the formal identity

$$y^m = \sum \binom{m + i - 1}{i} (1 - y^{-1})^i.$$

Apply this to the operation on F_r of tensoring by $\mathscr{L}$. This gives

$$\mathscr{L}^m \otimes \mathscr{F} = \sum_{i=0}^{r} \binom{m + i - 1}{i} c_1(\mathscr{L})^i \mathscr{F}.$$

$\blacksquare$

Example. Consider X a projective curve (over some artinian base. $S = \operatorname{spec} k$ if you want), and set $\mathscr{F} = \mathscr{O}_X$. Also set $m = n = 1$. Then,

$$\chi(X, \mathscr{L}) = \chi(X, c_1(\mathscr{L})\mathscr{O}_X) + \chi(X, \mathscr{O}_X) = \deg(\mathscr{L}) + \chi(X, \mathscr{O}_X).$$

This is Riemann-Roch. $\triangle$

Example. For X a smooth, proper surface, one derives

$$\chi(X, \mathscr{L}) = \frac{1}{2}\left(\mathscr{L} \cdot (\mathscr{L} \otimes \omega_X^{-1})\right) + \chi(X, \mathscr{O}_X).$$

This is Riemann-Roch for surfaces. $\triangle$

Note 1. Got distracted. Missed some comments about growth-rates...

Theorem 1.21 (**Asymptotic Riemann-Roch I**). *Let X be proper of dimension r, $\mathscr{F}$ coherent on X, $\mathscr{L}$ a line bundle. Then,*

$$\chi(X, \mathscr{L}^n \otimes \mathscr{F}) = \frac{(\mathscr{L}^r \cdot \mathscr{F})}{r!} n^r + O\left(n^{r-1}\right).$$

Theorem 1.22 (**Asymptotic Riemann-Roch II**). *Same setup, but now assume $\mathscr{L}$ is **nef**, i.e. $(\mathscr{L} \cdot \mathscr{O}_C) \geq 0$ for all integral curves $C \subset X$.*[4] *Then,*

$$\dim \mathrm{H}^i(X, \mathscr{L}^n \otimes \mathscr{F}) = \begin{cases} O(n^{r-1}) & \text{if } i > 0 \\ \dfrac{(\mathscr{L}^r \cdot \mathscr{F})}{r!} n^r + O(n^{r-1}) & \text{if } i = 0. \end{cases}$$

Prove this using dévissage.

2 Nathan Chen (Harvard): Big and nef line bundles, 9/29

Let's start with some conventions

- $k = \overline{k}$ an algebraically closed field
- $N^1(X) := \{$(cartier) divisors$\}/$ numerical equivalence, where $D_1 \equiv_{num} D_2$ are **numerically equivalent** if $D_1 \cdot C = D_2 \cdot C$ for all curves C.

2.1 Nef line bundles

The story really begins with ampleness.

Recall 2.1. Let X be a projective variety, and let $\mathscr{L}$ be a line bundle on X. Then, $\mathscr{L}$ is **ample** if any of the following holds

(1) $\mathscr{L}^{\otimes m}$ is very ample for some $m \geq 0$

[4] Secretly, this is equivalent to $(\mathscr{L}^{\dim Y} \cdot Y) \geq 0$ for all integral subschemes.

(2) cohomological condition ($\mathrm{H}^i(X, \mathscr{F} \otimes \mathscr{L}^n) = 0$ for all $i > 0$ for $n \gg_{\mathscr{F}} 0$ if $\mathscr{F}$ coherent, assuming I remember correctly)

(3) Global gen of coh. sheaves (?)

(4) $/\mathbb{C}$ exists hermitian metrics

Let's see a numerical condition for ampleness.

Theorem 2.2 (Nakai-Moishezon-Kleiman)**.** *A line bundle $\mathscr{L}$ is ample iff*

$$\int_V c_1(L)^{\dim V} > 0$$

for all subvarieties $V \subset X$.

Now can ask, what happens if you take limits of ample things?

Fact. $N^1(X)_{\mathbb{R}}$ is a finite-dimensional vector space (theorem of Néron-Severi?)

Definition 2.3. Let $\mathscr{L}$ be a line bundle. Then, $\mathscr{L}$ is **nef** if $\mathscr{L} \cdot C \geq 0$ for any curve.

Being nef is closed under addition and scaling, so the nef things form a cone. Some more facts

(1) Say $f : X \to Y$ is proper. Then, $\mathscr{L}$ nef $\implies$ $f^*\mathscr{L}$ is nef (via projection formula).

(2) Say $f : X \twoheadrightarrow Y$ proper + surjective. Then, $f^*\mathscr{L}$ nef $\implies$ $\mathscr{L}$ nef.

Theorem 2.4 (Kleiman)**.** *Let $\mathscr{L}$ be nef. Then,*

$$\int_V c_1(\mathscr{L})^{\dim V} \geq 0$$

for any $V \subset X$.

Corollary 2.5.

(1) *If D is a nef $\mathbb{R}$-divisor and H is an ample $\mathbb{R}$-divisor, then*

formal sum of ample things with positive coefficients

$$D + \varepsilon \cdot H \text{ is ample for any } \varepsilon > 0.$$

(2) *For any $\mathbb{R}$-divisors D, V, if $D + \varepsilon H$ is ample for all $\varepsilon > 0$ sufficiently small, then D is nef*

Proof. **(2)** $(D + \varepsilon H) \cdot C = (D \cdot C) + \varepsilon(H \cdot C) > 0$ for all ε sufficiently small. Sending $\varepsilon \to 0$ implies $D \cdot C \geq 0$.

(1) Suffices to show $(D + H)$ is ample. Let $k = \dim V$. Then,

$$(D + H)^{\dim V} \cdot V = \sum_{s=0}^{k} \binom{k}{s} (D^{k-s} \cdot H^s \cdot V).$$

If H is a $\mathbb{Q}$-divisor, can clear denominators and then bump it up to get something very ample. Then $H^s \cdot V$ is effective (just slicing by a hyperplane[5]), so $D^{k-s} \cdot (H^s \cdot V) \geq 0$ by Kleiman. Furthermore, $H^k \cdot V > 0$. Hence, the sum is strictly positive.

[5] and then scaling back down

If H has $\mathbb{R}$-coefficients, use that ampleness is an open condition to approximate by $\mathbb{Q}$-divisors. ■

Proposition 2.6 (Nefness in families)**.** *Say $f : X \twoheadrightarrow T$ is surjective and proper. Let $\mathscr{L}$ be a line bundle on X. Fix some $0 \in T$ and consider the central fiber $X_0 = f^{-1}(0)$ with $\mathscr{L}_0 = \mathscr{L}|_{X_0}$. If $\mathscr{L}_0$ is nef on X_0, there exists a countable union $B = \bigcup B_i \subset T$ (each B_i a proper subvariety) so that for all $t \in T \setminus B$, the line bundle $\mathscr{L}_t$ is nef on X_t.*

(apparently unknown if the countable union is needed, or if you can always only throw away finitely many, so get a open condition)

Example. Consider $\delta_1, \dots, \delta_n$ nef $\mathbb{R}$-divisors. Note that $(\delta_1 \cdots \delta_n) \geq 0$. One way to see this is to perturb by a small ample class (turning the intersection positive). △

Definition 2.7. Let $\operatorname{Amp}(X) \subset N^1(X)_{\mathbb{R}}$ denote the subset of ample $\mathbb{R}$-divisors (up to numerical equivalence). Similarly define $\operatorname{Nef}(X)$.

We've shown that $\operatorname{Amp}(X)$ is open with closure $\overline{\operatorname{Amp}}(X) = \operatorname{Nef}(X)$.

Example. Let C_1, C_2 be curves which are *unrelated* in the sense that $N^1(C_1 \times C_2)_{\mathbb{R}} = \mathbb{R}\langle f_1, f_2 \rangle$ with f_i the fiber over a point in C_i (i.e. $f_i = \operatorname{pr}_i^*(*)$). Note that f_1, f_2 are nef (surjective proper pullback), but they are not ample e.g. since $C_i^2 = 0$ (move fibers). So get ample locus $\mathbb{R}_{>0} f_1 \times \mathbb{R}_{>0} f_2$ with Nef cone $\mathbb{R}_{\geq 0} f_1 \times \mathbb{R}_{\geq 0} f_2$. △

Apparently there's a Hartshorne exercise showing that $N^1(C_1 \times C_2)$ is always generated by these two classes along with hom's between their Jacobians

Example. Look at $X = C \times C$. Then, $N^1(X)_{\mathbb{R}} \ni f_1, f_2, \delta = [\Delta]$. These are linearly independent by computing intersection numbers. If $g(C) \geq 1$ and C is general in moduli, then they are also spanning.

Let's assume that $g(C) = 1$, i.e. $C = E$ is an elliptic curve. Then, $\delta^2 = 0 = 2 - 2g$. In this case, $X = E \times E$ is an abelian variety.

Fact. If D is effective (on X), then D is nef.

Proof idea. Write $D = \sum a_i D_i$. Use the group law on X to move C around so that it's not equal to any of the D_i's. Then, $D \cdot C \geq 0$ because we're looking at honest-to-God intersections. ■

Lemma 2.8. *On $X = E \times E$, a divisor D is nef $\iff$ $D^2 \geq 0$ and $D \cdot H \geq 0$ for some ample H.*

Proof idea. For backwards direction, Riemann-Roch for surfaces gives that $D^2 > 0$ and $D \cdot H > 0 \implies D$ effective (trick is to take multiples of D and then plug into Riemann-Roch). Taking limits gives backwards direction of this lemma (use effective $\implies$ nef). ■

Consider $D = xf_1 + yf_2 + z\delta$ and $H = f_1 + f_2 + \delta$ (ample e.g. by Nakai-Moishezon). One gets

$$xy + xz + yz \geq 0 \text{ and } x + y + z \geq 0$$

(as criterion for nefness). These equations define the nef cone. △

Open Question 2.9. *On $C \times C$, have class $\delta = [\Delta]$, and $\delta^2 = 2 - 2g$. If C is general, large genus, is δ the only effective irreducible curve with negative self intersection?*

In positive characteristic, this is false it sounds like (do something funny with frobenius).

Theorem 2.10 (**Fujita vanishing**). *Let $\mathscr{F}$ be any coherent sheaf, let D be a nef divisor, and let H be ample. Then, there exists some $m_0 = m(H, \mathscr{F})$ such that*

$$\mathrm{H}^i(X, \mathscr{F}(mH + D)) = 0$$

for $i > 0$ and $m \geq m_0$.

(note that m_0 is *independent* of D above).

Theorem 2.11 (**Asymptotic Riemann-Roch**). *Let X be projective of dimension n, and let D be a nef divisor on X. Let $\mathscr{F}$ be a coherent sheaf. Then,*

$$h^0(X, \mathscr{F}(mD)) = \operatorname{rank}(\mathscr{F}) \cdot \frac{D^n}{n!} m^n + O\left(m^{n-1}\right).$$

2.2 Big line bundles

Let $\mathscr{L}$ be a line bundle on X. Then one gets a rational map

$$X \dashrightarrow \mathbb{P}\,\mathrm{H}^0(X, \mathscr{L}).$$

Bigness is about what happens when you take powers. Assume $\mathrm{H}^0(X, \mathscr{L}^{\otimes m}) \neq 0$, and look at

$$\varphi_m : X \dashrightarrow \mathbb{P}\,\mathrm{H}^0(X, \mathscr{L}^{\otimes m}).$$

Consider the **Iitaka dimension**

$$\max_m \{\dim \varphi_m(X)\} =: \kappa(X, \mathscr{L})$$

of $\mathscr{L}$. If $\mathrm{H}^0(X, \mathscr{L}^{\otimes m}) = 0$, we define $\kappa(X, \mathscr{L}) = -\infty$. Note that, always, $\kappa(X, \mathscr{L}) \in \{-\infty, 0, 1, \ldots, \dim X\}$. We say that $\mathscr{L}$ is **big** if $\kappa(X, \mathscr{L}) = \dim X$.

Remark 2.12. If $\mathscr{L} = \omega_X$ is the canonical bundle, then $\kappa(X, \omega_X)$ is the **Kodaira dimension** of X. Furthermore, X is **of general type** $\iff$ ω_X is big.

this means sections of the canonical bundle eventually give a birational map onto the image, Nathan said this

Here's an alternative definition

Proposition 2.13. *$\mathscr{L}$ is big iff there exists a constant $C > 0$ so that*

$$h^0(X, \mathscr{L}^{\otimes m}) \geq C \cdot m^{\dim X}.$$

By ARR, nef $\implies$ (big $\iff$ $\mathscr{L}^{\dim X} > 0$).

2.2.1 Iitaka fibration

The φ_m stabilize. For $m \gg 0$ and sufficiently divisble, φ_m is birationally equivalently to a map

$$\varphi_\infty : X_\infty \to Y_\infty$$

(this map is not necessarily birational).

Let Y_m be the image of φ_m. You want morphisms, so you'll need a diagram that looks like

$$\begin{tikzcd} X \arrow[r] \arrow[d] & Y \arrow[d] \\ X_\infty \arrow[r] & Y_\infty \end{tikzcd}$$

with vertical maps birational and the top map rational.

Proposition 2.14. *For D a big divisor and F effective, for $m \gg 0$*

$$\mathrm{H}^0(X, \mathscr{O}_X(mD - F)) \neq 0.$$

Proof. Twist ideal sheaf sequence to get

$$0 \longrightarrow \mathscr{O}_X(mD - F) \longrightarrow \mathscr{O}_X(mD) \longrightarrow \mathscr{O}_F(mD) \longrightarrow 0.$$

When you pass to global sections, the middle term grows like $C \cdot m^n$ ($n = \dim X$) while the right terms only grows like $\leq C_2 m^{n-1}$, so there must be a kernel for large m. ■

Corollary 2.15. *TFAE*

(1) D big

(2) For any ample A on X, there exists $m > 0$ and an effective N on X such that

$$mD \equiv_{lin} A + N$$

(large multiples of big is ample + effective)

(3) There exists an ample A on X, a positive integer $m > 0$, and an effective N such that

$$mD \equiv_{lin} A + N.$$

(4) Same as (3) but linear equivalence replace with numerical equivalence.

Without nefness, $L^{\dim X}$ can be arbitrarily negative.

Example. Let X be a surface, H ample, and E a (-1)-curve. Let $D_k = H + k \cdot E$. This will be big but $D_k^2 \ll 0$ for $k \gg 0$. △

Theorem 2.16 (Kawamata-Viehweg)**.** *Let X be a smooth complex projective variety with a big, nef ($\mathbb{Z}$-)divisor D. Then,*

$$\mathrm{H}^i(X, K_X + D) = 0 \text{ for } i > 0.$$

Remark 2.17. There's a version for $\mathbb{Q}$-divisors (involving round-downs. Can only talk about cohomology of integral things) and for multipler (?) ideals. ○

Example. Consider $X = \mathrm{Bl}_0 \mathbb{P}^2 \xrightarrow{\mu} \mathbb{P}^2$, and let $E \subset X$ be the exceptional divisor. Let $H = \mu^* \mathscr{O}(1)$. Then, $N^1(X)_{\mathbb{R}} = \mathbb{R}\langle H, E \rangle$. Furthermore,

$$E^2 = -1, \;\; H^2 = 1, \;\; \text{and} \;\; H \cdot E = 0$$

(for the last one, pull back a line not passing through 0). Note that H is nef (proper, surjective pullback). Furthermore, $H - E$ corresponds to a map $\mathrm{Bl}_0 \mathbb{P}^2 \xrightarrow{f} \mathbb{P}^1$. Note that $f^* \mathscr{O}(1) = H - E$ so $H - E$ is nef (and note ample e.g. since $(H - E)^2 = 0$). The nef cone here sits between H and $H - E$.

Think of it as the linear system of lines in $\mathbb{P}^2$ through 0. This is the resolution of the projection map $\pi_0 : \mathbb{P}^2 \dashrightarrow \mathbb{P}^1$

$H + \alpha E$ will be big for $\alpha > 0$. I missed why. △

3 Katia Bogdanova (Harvard): Height Machine, 10/6

Plan

- Define heights on projective space
- Heights on projective varieties

  The height machine, and applications

Heights measure the 'size' of a point. We'll want two properties in particular

- Only finitely many points of bounded height
- Should capture both arithmetic and geometric information

3.1 Heights on projective space

Example. Say $P \in \mathbb{P}^n(\mathbb{Q})$. We can write $P = [x_0 : \cdots : x_n]$ with $x_i \in \mathbb{Z}$ and $\gcd(x_0, \ldots, x_n) = 1$. Then, the **naive height** of this point is simply

$$H(P) := \max\{|x_0|, \ldots, |x_n|\}.$$

Note that $\#\{P \in \mathbb{P}^n(\mathbb{Q}) : H(P) \leq B\} < \infty$ always, so get finiteness condition. △

We can generalize this readily to other fields.

Definition 3.1. For K a number field, and $P \in \mathbb{P}^n(K)$, write $P = [x_0 : \cdots : x_n]$. We then get the **multiplicative height**

$$H_K(P) := \prod_{v \in M_K} \max\{\|x_0\|_v, \ldots, \|x_n\|_v\}.$$

Above, M_K is the set of all places, and $\|\cdot\|_v$ is normalized absolute value. WE similarly define the **Log height** $h_K(P) := \log H_K(P)$.

The multiplicative height above recovers the naive height when $K = \mathbb{Q}$.

Lemma 3.2. *For $P \in \mathbb{P}^n(K)$*

(a) *$H_K(P)$ is independent of choice of local coordinates* *(b)* $H_K(P) \geq 1$ *for all* $P \in \mathbb{P}^n(K)$

(c) $H_{K'}(P) = H_K(P)^{[K':K]}$

Proof Sketch. **(a)** follows from the product formula. **(b)** follows from scaling so that one coordinate is equal to 1. **(c)** is a direct computation. ■

(c) above gives compatibility between number fields, and so leads into

Definition 3.3. The **Absolute multiplicative height** on $\mathbb{P}^n$ is the function $H : \mathbb{P}^n(\overline{\mathbb{Q}}) \to [1, \infty)$ given by $H(P) = H_K(P)^{\frac{1}{[K:\mathbb{Q}]}}$ where $P \in \mathbb{P}^n(K)$. The **Absolute logarithmic height** is $h(P) = \log H(P)$.

Lemma 3.4. *For* $P \in \mathbb{P}^n(\overline{\mathbb{Q}})$ *and* $\sigma \in G_{\mathbb{Q}}$, *we have* $H(\sigma(P)) = H(P)$.

(σ simply permutes the places)

Definition 3.5. Given a field element $\alpha \in K$, it's **height** will be

height of correspond-ing element of $\mathbb{P}^1$

$$H_K(\alpha) := \prod_{v \in M_K} \max\{\|\alpha\|_v, 1\}$$

Notation 3.6. Given $P \in \mathbb{P}^n(\overline{\mathbb{Q}})$, we'll write $\mathbb{Q}(P)$ for its field of definition (i.e. its residue field).

Theorem 3.7. *For every constants* $B, D \geq 0$,

$$\#\left\{P \in \mathbb{P}^n(\overline{\mathbb{Q}}) : H(P) \leq B \text{ and } [\mathbb{Q}(P) : \mathbb{Q}] \leq D\right\} < \infty.$$

Proof. Write $P = [x_0 : \cdots : x_n]$. WLOG assume $x_0 = 1$. Note that $H(P) \geq H(x_i)$ for all i (using $x_0 = 1$ here), so it suffices to check that for all $d \in \{1, \ldots, D\}$, $\#\left\{x \in \overline{\mathbb{Q}} : H(x) \leq B \text{ and } [\mathbb{Q}(x) : \mathbb{Q}] = d\right\} < \infty$. For these, we'll show that there are only finitely many possibilities for the minimal polynomial of such an x. Since x has degree d, we can write its minimal polynomial as

$$F_x(T) = \prod(T - x_j) = \sum(-1)^r \underbrace{S_r(x)}_{\in\mathbb{Q}} T^{d-r}.$$

Above,

I got lost in the algebra below...

$$|S_r(x)|_v = \left|\sum x_{i_1} \ldots x_{i_r}\right|_v \leq C \max |x_{i_1} \ldots x_{i_r}|_v \leq \ldots$$

After sorting out the above, one should obtain

$$H([S_0(x) : \cdots : S_d(x)]) \leq 2^d \prod H(x_i)^d \leq 2^d B^d.$$

Finally, we know there are only finitely many rational points of bounded height, so we win. ■

Proposition 3.8.

(a) *Consider the Segre embedding* $S_{n,m} : \mathbb{P}^n \times \mathbb{P}^m \to \mathbb{P}^N$ *given by*

$$[x_0 : \cdots : x_n], [y_0 : \cdots : y_m] \mapsto [x_0y_0 : \cdots : x_ny_m].$$

Then, $h(S_{n,m}(x, y)) = h(x) + h(y)$

(b) *Consider the d-uple embedding $\varphi_d : \mathbb{P}^n \to \mathbb{P}^N$. Then, $h(\varphi_d(x)) = dh(x)$.*

Theorem 3.9. *Let $\varphi : \mathbb{P}^n \dashrightarrow \mathbb{P}^m$ is a rational map of degree d (i.e. defined by $m+1$ homogeneous degree d polynomials), and let Z be the locus of indeterminacy or whatever it's called (where the map isn't defined). Then,*[6]

(a) *$h(\varphi(P)) \le dh(P) + O(1)$ for all $P \in \mathbb{P}^n(\overline{\mathbb{Q}}) \setminus Z(\overline{\mathbb{Q}})$*

(b) *If $X \hookrightarrow \mathbb{P}^n$ is a closed embedding with $X \cap Z = \emptyset$, then $h(\varphi(P)) = dh(P) + O(1)$*

Proof. Write $f_i = \sum_{|e|=d} a_{i,e} x^e$. Choose some $P = [x_0 : \cdots : x_n] \in \mathbb{P}^n(\overline{\mathbb{Q}})$. Then,

$$\left|f_i(P)\right|_v = \left|\sum a_{i,e} x^d\right|_v \le \binom{n+d}{n} \underbrace{\max\left\{\left|a_{i,e}\right|_v\right\}}_{|f_i|_v} \cdot \underbrace{\max\left\{\left|x_i\right|_v\right\}}_{|p|_v}.$$

This gives

$$H(\varphi(P)) \le \binom{n+d}{n} \prod \max\left\{|f_0|_v, \ldots, |f_m|_v\right\}^{???} H(P)^d.$$

Figure out the exponent I couldn't read, and then take logs.

(b) Let $I_X = (p_1, \ldots, p_r)$ be the ideal of X. Then, $\sqrt{(I_X, (f_i))} \ni x_i$ (X disjoint from Z and $I_Z = (f_i)_i$).

Note 2. This proof looks like the notation will get messy, so I'm gonna skip it...

■

3.2 Heights on varieties

Let V be a projective variety over $\overline{\mathbb{Q}}$. Let $\varphi : V \to \mathbb{P}^n$ be some map.

Definition 3.10. The **absolute logarithmic height on V relative to φ** is

$$\begin{array}{rccc} h_\varphi : & V(\overline{\mathbb{Q}}) & \longrightarrow & [0,\infty) \\ & h_\varphi(P) & \longmapsto & h(\varphi(P)). \end{array}$$

Theorem 3.11. *If $V/\overline{\mathbb{Q}}$ as above with two maps $\varphi : V \to \mathbb{P}^n$, $\psi : V \to \mathbb{P}^m$ so that $\varphi^*\mathscr{O}(1) \simeq \psi^*\mathscr{O}(1)$, then $h_\varphi(P) = h_\psi(P) + O(1)$ for all $P \in V(\overline{\mathbb{Q}})$.*

Proof idea. The functions defining φ, ψ are both given by linear combinations of a chosen basis $h_0, \ldots, h_N$ for $\mathscr{L}$. In other words, φ, ψ are both projections of the map defined by the corresponding complete linear system. ■

Theorem 3.12 (Height Machine). *For every smooth projective variety V/k, there exists a map*

$$h_V : \operatorname{Pic}(V) \longrightarrow \mathbb{R}^{V(k)}/O(1)$$

with the following properties

Katia actually stated this with $\operatorname{Div}(V)$ where I wrote $\operatorname{Pic}(V)$. This is why the properties are all stated in terms of

[6] I think these are just saying that $h(\varphi(P)) = dh(P) + O(1)$ when both sides are defined, and it's not clear to me why the statement is broken into two parts. Update: the point is that the implied constant in the $O(1)$ depends on X and the map, not on individual points

(a) *(Normalization) if $H \subset \mathbb{P}^n$ is a hyperplane, then*

$$h_{\mathbb{P}^n,H}(P) = h(P) + O(1)$$

(b) *(Functorality) if $\varphi : V \to W$ and $D \in \operatorname{Div}(W)$, then*

$$h_{V,\varphi^* D}(P) = h_{W,D}(\varphi(P)) + O(1)$$

(c) *(additivity) If $D, E \in \operatorname{Div}(V)$, then*

$$h_{V,D+E}(P) = h_{V,D}(P) + h_{V,E}(P) + O(1).$$

(d) *(linear equivalence) If $D \sim E \in \operatorname{Div}(V)$, then $h_{V,D}(P) = h_{V,E}(P) + O(1)$*

(e) *(Positivity) If $D \in \operatorname{Div}(V)$ is effective, and $P \notin \operatorname{supp} D$, then $h_{V,D}(P) \geq O(1)$*

(f) *(Algebraic equivalence) If $D, E \in \operatorname{Div}(V)$ with D ample and $E \equiv 0$ (algebraically equivalent), then*

$$\lim_{\substack{P \in V(k) \\ h_{V,D}(P) \to \infty}} \frac{h_{V,E}(P)}{h_{V,D}(P)} = 0$$

(if you have two algebraically equivalent divisors, their height functions agree in the limit. Something like this)

(g) *(Finiteness) If $D \in \operatorname{Div}(V)$ is ample, then for all finite extensions k'/k and all constants $B \geq 0$,*

$$\#\{P \in V(k') : h_{V,D}(P) \leq B\} < \infty.$$

(h) *(Uniqueness) $h_{V,D}$ is determined uniquely by **(a)**,**(b)**,**(c)***

Proof idea. You construct this the way you have to. Write a divisor as a difference of two very ample (or even just base-point free) divisors, and then take the difference on the associated absolute logarithmic height functions.

(e) Say $D > 0$ effective. Write $D = D_1 - D_2$ with D_i ample. Let $f_0, \ldots, f_n$ be basis of $\mathrm{H}^0(D_2)$. Note that also $f_0, \ldots, f_n \in \mathrm{H}^0(D_1)$ (difference positive), so complete to a basis $f_0, \ldots, f_n, f_{n+1}, \ldots, f_m \in \mathrm{H}^0(D_1)$. Let $\varphi_{D_1} = (f_0 : \cdots : f_n) : V \to \mathbb{P}^n$ and $\varphi_{D_2} = [f_0 : \cdots : f_m] : V \to \mathbb{P}^m$. Then,

$$h_{V,D}(P) = h_{V,D_1}(P) - h_{V,D_2}(P) + O(1) = h(f_0(P), \ldots, f_n(P), f_{n+1}(P), \ldots, f_m(P)) - h(f_0(P), \ldots, f_n(P)) + O(1) \geq O(1)$$

For some reason this works only when $P \notin \operatorname{supp} D_1$ or something, and then replacing with equivalent divisors, we get it for all $P \in V \setminus |D|$?

(f) D ample and $E \equiv 0$. Fact: exists $m > 0$ s.t. for all $n \in \mathbb{Z}$, $mD + nE$ is basepoint free. Then, $mh_{V,D}(P) + nh_{V,E}(P) \geq -c = -c(n, -n)$. From this, we can get

$$\frac{m}{n} + \frac{c}{nh_{v,D}(P)} \geq \frac{h_{V,E}(P)}{h_{V,D}(P)} \geq -\frac{m}{n} - \frac{c}{nh_{V,D}(P)}.$$

Take limit over heights, then limit over n. ■

4 Alice Lin (Harvard): Comparison of Weil height and canonical height, 10/13

Outline

- Facts about abelian varieties
- Canonical heights on abelian varieties w.r.t divisors (first symmetric, then antisymmetric, then general)
- Canonical heights in families

4.1 Abelian varieties, briefly

Definition 4.1. An **abelian variety** is a complete, connected group variety over a field.

Example. A 1-dimensional abelian variety is an elliptic curve. △

Example. Over $\mathbb{C}$, every abelian variety is of the form $\mathbb{C}^g/\Lambda$ with $\Lambda \cong \mathbb{Z}^{2g}$ a (full rank) lattice. △

Given an abelian variety A, and an integer $m \in \mathbb{Z}$, we let $[m] : A \to A$ denote the multiplication-by-m map.

Definition 4.2. Let A be an abelian variety, and let $D \in \mathrm{Div}(A)$ be a divisor. We say that D is **symmetric** if $[-1]^*D \sim D$ and is **anti-symmetric** if $[-1]^*D \sim -D$.

Let's go over some results we'll need. To prove these, one first proves the *theorem of the cube* which we won't state here.

Fact (Mumford's formula). Let $D \in \mathrm{Div}(A)$. Then,

$$[m]^*D \sim \frac{m^2+m}{2}D + \frac{m^2-m}{2}[-1]^*D.$$

In particular, if D is symmetric, then $[M]^*D \sim m^2D$ and if D is anti-symmetric, then $[m]^*D \sim mD$.

Notation 4.3. Let $\sigma, \delta, \pi_1, \pi_2 : A \times A \to A$ be the maps

$$\begin{aligned} \sigma(P,Q) &= P + Q \qquad & \delta(P,Q) &= P - Q \\ \pi_1(P,Q) &= P & \pi_2(P,Q) &= Q \end{aligned}$$

Fact. If D is symmetric, then $\sigma^*D + \delta^*D \sim 2\pi_1^*D + 2\pi_2^*D$.
(think parallelogram law)
If D is anti-symmetric, then $\sigma^*D \sim \pi_1^*D + \pi_2^*D$

4.2 Heights

Recall 4.4 (Weil Height Machine)**.** Let V be a smooth projective variety over a number field k. For all $D \in \mathrm{Div}(V)$, we get a function

$$h_{V,D} : V(\overline{k}) \to \mathbb{R}$$

and the assignment $D \mapsto h_{V,D}$ satisfies

- (additivity) $h_{V,D+E} = h_{V,D} + h_{V,E} + O(1)$
- (functoriality) if $\varphi : V \to W$ and $D \in \mathrm{Div}(W)$, then $h_{V,\varphi^* D} = h_{W,D} \circ \varphi + O(1)$
- (linear equivalence) if $D \sim E$, then $h_{V,D} = h_{V,E} + O(1)$
- (Northcott) If D is *ample*, then

  $$\#\{P \in V(k') : h_{V,D}(P) < B\} < \infty$$

  for any finite k'/k and any $B > 0$.

These $O(1)$'s are annoying, so let's try and get rid of them. This brings us to canonical heights.

Let A/k be an abelian variety over a number field k, and let $D \in \mathrm{Div}(A)$ be symmetric. Recall that this means $[m]^* D \sim m^2 D$. In particular, $[2]^* D \sim 4D$, so

$$h_{A,D} \circ [2] = h_{A,[2]^* D} + O(1) = h_{A,4D} + O(1) = 4h_{A,D} + O(1),$$

i.e. there exists a constant C so that

$$|h_{A,D}([2]P) - 4h_{A,D}(P)| < C \text{ for all } P \in A(\overline{k}).$$

To get a canonical height, consider the limit

$$\widehat{h}_{A,D}(P) := \lim_{n\to\infty} \frac{h_{A,D}([2]^n P)}{4^n}.$$

Does this limit exist? Yes, it's Cauchy; for $n \geq m$, write difference as a telescoping sum, manipulate a bit, and then see that it goes to 0.

This $\widehat{h}_{A,D}$ is our **canonical height** (for symmetric D). Why is this 'canonical'? Didn't we choose to use 2?

Theorem 4.5. *Let A/k be an abelian variety over a number field, and let $D \in \mathrm{Div}(A)$ be a symmetric divisor. Then, $\widehat{h}_{A,D} : A(\overline{k}) \to \mathbb{R}$ defined above satisfies*

(a) $\widehat{h}_{A,D}(P) = h_{A,D}(P) + O(1)$

(b) *for all $m \in \mathbb{Z}$,* $\widehat{h}_{A,D}([m]^* P) = m^2 \widehat{h}_{A,D}(P)$

(c) $\widehat{h}_{A,D}$ *defines a quadratic form on $A(\overline{k})$, and so has an associated bilinear pairing $\langle \cdot, \cdot \rangle_D : A(\overline{k}) \times A(\overline{k}) \to \mathbb{R}$, defined by*

$$\langle P, Q \rangle = \widehat{h}_{A,D}(P+Q) - \widehat{h}_{A,D}(P) - \widehat{h}_{A,D}(Q).$$

Definition 4.6. A **quadratic function** $f : G \to \mathbb{R}$ on an abelian group is one satisfying

I'm not sure if it's supposed to be $+f(0)$ or $-f(0)$ at the end

$$f(P+Q+R) - f(P+Q) - f(Q+R) - f(P+R) + f(P) + f(Q) + f(R) - f(0) = 0$$

always. If $f(0) = 0$ and for all $m \in \mathbb{Z}$, $f(mP) = m^2 f(P)$, then f is a **quadratic form**.

(d) *If $\widehat{h}, \widehat{h}'$ satisfy **(a)** and **(b)** for just one $m \geq 2$, then $\widehat{h} = \widehat{h}'$.*

Proof Sketch. **(a)** follows from the telescoping sum computation with $m = 0$ and $n \to \infty$

(b) $h_{A,D}([2^n m]P) = m^2 h_{A,D}([2]^n P) + O(1)$ for weil heights. Take the relevant limit of both sides gives the desired relation.

(c) This follows from the parallelogram law (last fact of abelian varieties section):

$$\widehat{h}_{A,D}(P+Q) + \widehat{h}_{A,D}(P-Q) = 2\widehat{h}_{A,D}(P) + 2\widehat{h}_{A,D}(Q).$$

(d) Satisfying **(a)** means that $g = \widehat{h} - \widehat{h}'$ is bounded. Satisfying **(b)** then shows that $g([m]Q) = m^2 g(Q)$ so $g([m]^n Q) = m^{2n} g(Q)$, so $g(Q) = 0$ always or else g won't be bounded. ■

Let's take a second to see a neat application of canonical heights.

Remark 4.7. If $D \in \mathrm{Div}(A)$ is symmetric and ample, then for all finite extensions k'/k and all $B > 0$

$$\#P \in A(k') : \widehat{h}_{A,D}(P) < B < \infty.$$

○

Proposition 4.8. *Let A/k with $D \in \mathrm{Div}(A)$ ample and symmetric. Then,*

(i) *For all $P \in A(\overline{k})$, $\widehat{h}_{A,D}(P) \geq 0$*

(ii) *$\widehat{h}_{A,D}(P) = 0 \iff P \in A(\overline{k})_{tors}$*

(iii) *$\widehat{h}_{A,D}$ extends to a positive definite quadratic form on $A(\overline{k}) \otimes \mathbb{R}$*

Proof. **(i)** Since D ample, nD is very ample, so $h_{A,nD} = h_{\mathbb{P}^r,H} \circ \varphi + O(1)$ for an embedding $\varphi : A \hookrightarrow \mathbb{P}^r$ and hyperplane $H \subset \mathbb{P}^r$. Here, $h_{\mathbb{P}^r,H}(\varphi(P)) \geq 0$ always since this is the absolute logarithmic height. Taking the relevant limit gives the claim.

(ii) If P is torsion, then $\{[2]^n P\}_n$ is finite, so $\{h_{A,D}([2]^n P)\}$ is bounded, so $\widehat{h}_{A,D}(P) = 0$. Conversely, if $\widehat{h}_{A,D}(P) = 0$, then $\widehat{h}_{A,D}([2]^n P) = 0$, so $\{[2]^n P\}_n$ has bounded height, so is finite, so P is torsion.

Question: Does every integer divide a difference of powers of 2? I feel like this is saying this must be the case.

(iii) Suffices to check on finite dimensional $V \subset A(\overline{k}) \otimes \mathbb{R}$. Note $\widehat{h}_{A,D}$ is ≥ 0 on $A(\overline{k}) \cap V \cong \mathbb{Z}^n$ (full rank lattice), so is ≥ 0 on $\mathbb{Q}^n$ and so is ≥ 0 on $\mathbb{R}^n$ by continuity. Hence, just need to show that it's non-degenerate on $\mathbb{R}^n$. Suppose not. Then, it factors as

$$\begin{tikzcd} \mathbb{R}^n \arrow[r, "\rho"'] \arrow[rr, bend left, "\widehat{h}_{A,D}"] & \mathbb{R}^{n-1} \arrow[r, "q"'] & \mathbb{R} \end{tikzcd}$$

for some quadratic form on $\mathbb{R}^{n-1}$. Note that $\mathbb{Z}^n \to \rho(\mathbb{Z}^n)$ must be injective since $\widehat{h}_{A,D}$ is non-degenerate on $A(\overline{k})/A(\overline{k})_{tors} \cong \mathbb{Z}^n$. Thus, $\rho(\mathbb{Z}^n)$ is a rank n subgroup of $\mathbb{R}^{n-1}$, and so not discrete. Hence, there's some distinct $x_i \in \mathbb{Z}^n$ with $\rho(x_i) \to 0$. So $\widehat{h}_{A,D}(x_i) \to 0$ which contradicts Northcott. ■

Corollary 4.9 (of **(ii)** above)**.** *There are only finitely many torsion points in $A(k')$ for any finite k'/k.*

This has all been w.r.t. symmetric divisors, but one can tell a very similar story w.r.t. anti-symmetric divisors.

Recall 4.10. A divisor $D \in \mathrm{Div}(A)$ is anti-symmetric iff $[m]^*D \sim mD$ iff $[-1]^*D \sim -D$

Given an anti-symmetric divisor D, one defines

$$\widehat{h}_{A,D}(P) = \lim_{n\to\infty} \frac{h_{A,D}([2]^nP)}{2^n}.$$

Theorem 4.11. *Let D be anti-symmetric. Then,*

(a) $\widehat{h}_{A,D}(P) = h_{A,D}(P) + O(1)$

(b) $\widehat{h}_{A,D}(P+Q) = \widehat{h}_{A,D}(P) + \widehat{h}_{A,D}(Q)$ *for all* $P, Q \in A(\overline{k})$

(c) $\widehat{h}_{A,D}$ *is uniquely determined by **(a)**,**(b)***

Now, let $D \in \mathrm{Div}(A)$ be an arbitrary divisor. We would like a canonical height for it. Note that $D_\pm := D \pm [-1]^*D$ are symmetric/anti-symmetric. Also note that $2D = D_+ + D_-$. We define

$$\widehat{h}_{A,D} := \frac{1}{2}\widehat{h}_{A,D+[-1]^*D} + \frac{1}{2}\widehat{h}_{A,D-[-1]^*D}.$$

Theorem 4.12.

(a) $\widehat{h}_{A,D} : A(\overline{k}) \to \mathbb{R}$ *is the unique quadratic function satisfying* $\widehat{h}_{A,D} = h_{A,D} + O(1)$ *and* $\widehat{h}_{A,D}(0) = 0$.

(b) $\widehat{h}_{A,D}$ *depends only on the divisor class of D.*

(c) $\widehat{h}_{A,D+E} = \widehat{h}_{A,D} + \widehat{h}_{A,E}$

(d) *If $\varphi : B \to A$ is a morphism (not necessarily homomorphism) of abelian varieties and* $D \in \mathrm{Div}(A)$, *then*

$$\widehat{h}_{B,\varphi^*D} = \widehat{h}_{A,D} \circ \varphi - \widehat{h}_{A,D}(\varphi(0)).$$

4.3 Canonical Heights in Families

Let k be a number field. Let S, A be smooth projective irreducible k-varieties. Say furthermore that we have a flat map $\pi : A \to S$ with generic fiber A_η an abelian variety over $k(S)$. Note that there is a Zariski open $S^0 \subset S$ so that for all closed points $s \in S^0(\overline{k})$, the fiber A_s is a (smooth) abelian variety. Let $U := \pi^{-1}(S_0)$.

Now, pick a divisor $D \in \mathrm{Div}(A)$. By the Weil machinery, we get a height $h_{A,D} : A(\overline{k}) \to \mathbb{R}$.

Question 4.13. *Can we get a canonical height on A (associated to D)?*

From the previous discussion, we know how to do this on (most) fibers. For $s \in S^0(\overline{k})$, restricting D to A_s gives a divisor $D_s \in \mathrm{Div}(A_s)$ for which we can define a canonical height $\widehat{h}_{A_s,D_s} : A_s(\overline{k}) \to \mathbb{R}$. Recall that

$$\left|\widehat{h}_{A_s,D_s}(P) - h_{A_s,D_s}(P)\right| < C = C(s)$$

with C independent of $P \in A_s(\overline{k})$. We define a canonical height

$$\widehat{h}_{A,D} : U(\overline{k}) \to \mathbb{R}$$

defined fiberwise as above. How does the C from before depend on s?

Theorem 4.14 (Silverman-Tate, '82)**.** *Fix an ample divisor $\xi \in \mathrm{Div}(S)$ downstairs. Then, there exists a constant $c = c(\xi, D, \pi)$ such that for all $P \in U(\overline{k})$,*

$$\left|\widehat{h}_{A,D}(P) - h_{A,D}(P)\right| < c \cdot h_{s,\xi}(\pi(P)) + O(1).$$

5 Vijay Srinivasan (MIT): Line bundles on complex tori, 11/3

Complex torus $X = V/U$ with V a $\mathbb{C}$-vector space and U a full rank lattice.

Outline

- Hodge theory
- sheaf cohomology for complex tori
- classifying line bundles on complex tori: $\mathrm{Pic}(X) = \mathrm{H}^1(X, \mathscr{O}_X^\times)$

5.1 Hodge Theory

Theorem 5.1. *Let X be a compact, Kähler manifold. Then,*

$$\mathrm{H}^*(X; \mathbb{C}) \cong \bigoplus_{p,q \geq 0} \mathrm{H}^{p,q}(X)$$

where $\mathrm{H}^{p,q}(X) \cong \mathrm{H}^q(X, \Omega^p) \cong \mathrm{H}^0(X, \Omega^{p,q})$ is the base of smooth $\overline{\partial}$-closed (p,q)-forms. Furthermore, $\mathrm{H}^n(X;\mathbb{C}) \cong \bigoplus_{p+q=n} \mathrm{H}^{p,q}(X)$ and $\mathrm{H}^{p,q}(X) = \overline{\mathrm{H}^{q,p}(X)}$.

Fact. The inclusion of sheaves $\mathbb{C} \hookrightarrow \mathscr{O}_X$ induces $\mathrm{H}^*(X;\mathbb{C}) \to \mathrm{H}^*(X, \mathscr{O}_X)$ which is the projection $\mathrm{H}^*(X;\mathbb{C}) \twoheadrightarrow \mathrm{H}^{0,*}(X)$ from above.

On a complex analytic space[7] X, one has the **exponential exact sequence**

$$0 \longrightarrow \mathbb{Z} \longrightarrow \mathscr{O}_X \longrightarrow \mathscr{O}_X^\times \longrightarrow 0$$

where the map of the right is $f \mapsto e^{2\pi i f}$. This induces a long exact sequence in cohomology:

$$\cdots \to \mathrm{H}^1(X,\mathbb{Z}) \to \mathrm{H}^1(X, \mathscr{O}_X) \to \mathrm{H}^1(X, \mathscr{O}_X^\times) \xrightarrow{c_1} \mathrm{H}^2(X,\mathbb{Z}) \to \mathrm{H}^2(X, \mathscr{O}_X) \to \cdots.$$

Note that the Picard group $\mathrm{Pic}(X) = \mathrm{H}^1(X, \mathscr{O}_X^\times)$ of holomorphic line bundles on X appears above.

Definition 5.2. The coboundary map $c_1 : \mathrm{H}^1(X, \mathscr{O}_X^\times) \to \mathrm{H}^2(X,\mathbb{Z})$ is called the **first Chern class**. The image of this map is called the **Néron-Severi group** $\mathrm{NS}(X)$.

[7] Maybe it actually needs to be a manifold? I can't remember

Theorem 5.3 (Lefschetz (1,1) Theorem). *If X is a compact Kähler manifold, then* $\mathrm{NS}(X) = \mathrm{H}^2(X,\mathbb{Z}) \cap H^{1,1}(X)$ *inside* $\mathrm{H}^2(X,\mathbb{C})$.

Proof. Note that

$$\mathrm{im}(c_1) = \ker(\mathrm{H}^2(X,\mathbb{Z}) \to \mathrm{H}^2(X,\mathscr{O}_X)) \to \mathrm{H}^2(X,\mathbb{C}) \cong \mathrm{H}^{0,2}(X) \oplus \mathrm{H}^{1,1}(X) \oplus \mathrm{H}^{2,0}(X)$$

If you're integral, you have equal parts in $\mathrm{H}^{0,2}$ and $\mathrm{H}^{2,0}$. Anything in this kernel maps to 0 in $\mathrm{H}^{0,2}$ by definition, and so lives in $\mathrm{H}^{1,1}(X)$. ■

Warning 5.4. $\mathrm{NS}(X)$ can have torsion in general, so it's not an intersection so much as a preimage. For tori, we won't have to worry about this.

Proposition 5.5. *If V is a real vector space and U is a full rank lattice, then the cohomology of $X = V/U$ is*

$$\mathrm{H}^*(X,\mathbb{Z}) \cong \bigwedge U^\vee \text{ where } U^\vee = \mathrm{Hom}(U,\mathbb{Z}).$$

Proof. Induct. If $\dim V = 1$, $X \cong S^1$ and we just need $\mathrm{H}^1(X,\mathbb{Z}) \cong U^\vee$. This is because

$$U^\vee \cong \mathrm{Hom}(U,\mathbb{Z}) \cong \mathrm{Hom}(\pi_1(V/U),\mathbb{Z}) \cong \mathrm{H}^1(V/U;\mathbb{Z}).$$

If $\dim V > 1$, decompose $U = U_1 \oplus U_2$ and $V = V_1 \oplus V_2$. Then, $X \cong X_1 \times X_2$ where $X_i = V_i/U_i$. Now we apply the Künneth formula (note $\mathrm{H}^*(X_i;\mathbb{Z})$ has no torsion) which gives

$$\mathrm{H}^*(X;\mathbb{Z}) \cong \mathrm{H}^*(X_1,\mathbb{Z}) \otimes \mathrm{H}^*(X_2;\mathbb{Z}) \cong \bigwedge U_1^\vee \oplus \bigwedge U_2^\vee \cong \bigwedge (U_1^\vee \oplus U_2^\vee) \cong \bigwedge U^\vee.$$

■

Assumption. From now on, V is a complex vector space, so $X = V/U$ is a complex torus.

Proposition 5.6. *When we tensor* $\mathrm{H}^*(X;\mathbb{Z}) \cong \bigwedge U^\vee$ *with* $\mathbb{C}$, *we get*

$$\mathrm{H}^*(X;\mathbb{C}) \cong \bigwedge V^\vee \otimes \bigwedge \overline{V}^\vee,$$

and this gives the Hodge decomposition on X.

Above, $\overline{V}^\vee$ is the space of $\mathbb{C}$-antilinear functional on V, i.e. $f : V \to \mathbb{C}$ so that $f(\lambda v + w) = \overline{\lambda} f(v) + f(w)$ for $v, w \in V$ and $\lambda \in \mathbb{C}$. Above, $V^\vee$ will be the $(1,0)$ space in the decomposition and $\overline{V}^\vee$ will be the $(0,1)$ space.

Explanation. Tensoring with $\mathbb{R}$ gives $\mathrm{H}^*(X;\mathbb{R}) \cong \bigwedge V^\vee$. Tensoring with $\mathbb{C}$ then gives

$$\mathrm{H}^*(X,\mathbb{C}) \cong \bigwedge (V^\vee \otimes_\mathbb{R} \mathbb{C}).$$

The point is that $V^\vee \otimes_\mathbb{R} \mathbb{C} \simeq V^\vee \oplus \overline{V^\vee}$ via $f \otimes \alpha \mapsto (\alpha f, \alpha \overline{f})$, so the above is indeed $\bigwedge V^\vee \otimes \bigwedge \overline{V}^\vee$. To see that this decomposition is the same one coming from the Hodge decomposition, one notes that both sides are generated in degree 1, so suffices to identify the degree 1 pieces. First, $\mathrm{H}^1(X;\mathbb{R}) \cong V^\vee$ consists of real-valued differential forms, identified with the cotangent space at $0 \in X$. After tensoring with $\mathbb{C}$, we get a decomposition into holomorphic differentials and anti-holomorphic differentials; these two pieces exactly correspond to $V^\vee$ and $\overline{V}^\vee$.

Lemma 5.7. *The natural map* $\mathrm{H}^1(X, \mathbb{R}) \to \mathrm{H}^1(X, \mathscr{O}_X)$ *is a (real-linear) isomorphism.*

Proof. This is the composition $V^\vee \to V^\vee \oplus \overline{V}^\vee \to \overline{V}^\vee$ where the first map is $f \mapsto (f, \overline{f})$. ■

(I think the same argument with $\mathrm{H}^1(X, \mathbb{R})$ directly in place of $V^\vee$ works for any Kähler manifold)

5.2 Sheaf cohomology

We'd ultimately like to study the cohomology of the sheaf $\mathscr{O}_X^\times$. For now, let's just consider some general sheaf $\mathscr{F} \in \mathrm{Sh}_{\mathrm{Ab}}(X)$ of abelian groups. Recall $X = V/U$. Let $\pi : V \to X$ be the quotient map (a covering space map). Then, one has

$$\mathrm{H}^0(X, \mathscr{F}) \cong \mathrm{H}^0(V, \pi^*\mathscr{F})^U,$$

the U-invariants of the pullback (above iso true generally for covering maps). Let $\varphi_0 : \mathrm{H}^0(V, \pi^*\mathscr{F})^U \xrightarrow{\sim} \mathrm{H}^0(X, \mathscr{F})$ be this isomorphism.

Theorem 5.8. *There is a unique extension of φ_0 to*

$$\varphi : \mathrm{H}^*(U, \mathrm{H}^0(V, \pi^*\mathscr{F})) \longrightarrow \mathrm{H}^*(X, \mathscr{F})$$

in a way that's compatible with everything you care about. Note this is group cohomology on the left.

(this should be coming from some spectral sequence)

Proof idea. Have global sections functor $\mathrm{H}^0(X, -) : \mathrm{Sh}_{\mathrm{Ab}}(X) \to \mathrm{Ab}$. This factors through the functor $\mathrm{H}^0(V, \pi^*(-)) : \mathrm{Sh}_{\mathrm{Ab}}(X) \to \mathrm{Mod}_U$, i.e. have

$$\begin{tikzcd} \mathrm{Sh}_{\mathrm{Ab}}(X) \arrow[r, "\mathrm{H}^0(V,\pi^*(-))"'] \arrow[rr, bend left, "\mathrm{H}^0(X,-)"] & \mathrm{Mod}_U \arrow[r, "(-)^U"'] & \mathrm{Ab} \end{tikzcd}$$

Now we appeal to

Lemma 5.9 (in Mumford's abelian varieties book)**.** *If $\mathscr{F}$ is an injective sheaf, then $\pi^*\mathscr{F}$ is flasque and* $\mathrm{H}^0(V, \pi^*\mathscr{F})$ *are injective U-modules.*

As a consequence, one gets a spectral sequence

$$E_2^{pq} = \mathrm{H}^p(U, \mathrm{H}^q(V, \pi^*\mathscr{F})) \implies \mathrm{H}^{p+q}(X, \mathscr{F}).$$

In particular, you get an edge map $\varphi : \mathrm{H}^p(U, \mathrm{H}^0(V, \pi^*\mathscr{F})) \to \mathrm{H}^p(X, \mathscr{F})$. If $\pi^*\mathscr{F}$ is acyclic, then φ is an isomorphism (have only one row in the E_2-page). ■

Example. Take $\mathscr{F} = \mathbb{Z}$. This just becomes $\mathrm{H}^*(U, \mathbb{Z}) \xrightarrow{\sim} \mathrm{H}^*(X, \mathbb{Z})$ coming from that the fact that $BU \simeq X$. △

Lemma 5.10. $\mathscr{O}_V$ *and* $\mathscr{O}_V^\times$ *are both acyclic.*

Corollary 5.11. *Get an isomorphism*

$$\mathrm{H}^p(U, \mathscr{O}(V)^\times) \xrightarrow{\sim} \mathrm{H}^p(X, \mathscr{O}_X^\times).$$

We can make this explicit when $p = 1$. Say we're given a 1-cocycle $u \mapsto e_u(z)$ (e_u a global nonvanishing holomorphic function), we get the line bundle $(\mathbb{C} \times V)/U$ where U acts via $u \cdot (t, v) = (e_u(v)t, v + u)$. Get commutative diagram

$$\begin{tikzcd}
\mathrm{H}^1(U, \mathscr{O}(V)^\times) \arrow[r] \arrow[dd] & \mathrm{H}^2(U, \mathbb{Z}) \arrow[dd] \arrow[dr, equal] & \\
& & \bigwedge^2 U^\vee \\
\mathrm{H}^1(X, \mathscr{O}_X^\times) \arrow[r, "c_1"] & \mathrm{H}^2(X, \mathbb{Z}) \arrow[ur, equal] &
\end{tikzcd}$$

Get first Chern class $c_1 : \mathrm{H}^1(U, \mathscr{O}(V)^\times) \to \mathrm{H}^2(U, \mathbb{Z})$. Let $\mathrm{NS}(V, U) \cong \mathrm{NS}(X)$ denote the image of this.

Claim 5.12. *Given $E \in \bigwedge^2 U^\vee$, E is in $\mathrm{NS}(V, U)$ iff*

$$E(ix, iy) = E(x, y)$$

where E is viewed as an (alternating, bilinear) map $V \times V \to \mathbb{R}$.

Proof. By Lefschetz (1,1), $E \in \mathrm{NS}(V, U) \iff E_{\mathbb{C}} \in \bigwedge^2_{\mathbb{C}} V^\vee$ is in the $V^\vee \otimes \overline{V}^\vee$ part. This part exactly consists of the maps $F : V \times V \to \mathbb{C}$ so that $F(ix, iy) = F(x, y)$ ∎

Corollary 5.13. *For $E \in \mathrm{NS}(V, U)$, there is a unique Hermitian form H on V with $\mathrm{Im}(H) = E$.*

Just take $H(x, y) = E(ix, y) + iE(x, y)$.

Corollary 5.14.

$$\mathrm{NS}(V, U) \cong \{H \in \mathrm{Herm}(V) : (\mathrm{Im}\, H)(U \times U) \subset \mathbb{Z}\}.$$

Proposition 5.15. *Say $H \in \mathrm{NS}(V, U)$ is such a Hermitian form, and let $E = \mathrm{Im}\, H$. Then,*

(1) *There exists a map $\alpha : U \to S^1$ so that $\alpha(u_1 + u_2) = e^{\pi i E(u_1, u_2)} \alpha(u_1)\alpha(u_2)$. Such a map is called a* ***semicharacter*** *for H, and it is unique up to multiplication by $\mathrm{Hom}(U, S^1)$.*

(2) *Given such an α, $e_u(z) = \alpha(u) e^{\pi H\left(z + \frac{u}{2}, u\right)}$ is a 1-cocycle in $\mathrm{H}^1(U, \mathscr{O}(V)^\times)$.*

Not sure if I copied this down correctly

Proof. Suffices to find a map $\delta : U \to \mathbb{Z}$ so that

$$\delta(u_1 + u_2) - \delta(u_1) - \delta(u_2) \equiv E(u_1, u_2) \pmod 2,$$

and then set $\alpha(u) = e^{\pi i \delta(u)}$. One constructs δ by induction. If $\mathrm{rank}\, U = 1$, then $E = 0$ ($\bigwedge^2 U^\vee = 0$), so choose $\delta = 0$. If $\mathrm{rank}\, U > 1$, write $U = U' \oplus L$ where $\mathrm{rank}\, L = 1$. Get δ' on U'. For $u' \in U'$ and $\ell \in L$, we get

$$\delta(u' + \ell) = \delta'(u') + E(u', \ell).$$

This gives **(1)**. **(2)** is a computation. ∎

There's a group of pairs $\{(H,\alpha) : H \in \mathrm{NS}(V,U) \text{ and } \alpha \text{ a semichar}\}$ with operation $(H_1,\alpha_1)+(H_2,\alpha_2) := (H_1+H_2, \alpha_1\alpha_2)$. Get exact sequence

$$0 \longrightarrow \mathrm{Hom}(U,S^1) \longrightarrow \{(H,\alpha)\} \xrightarrow{\mathrm{pr}_1} \mathrm{NS}(V,U) \longrightarrow 0$$

(use prop **(1)**). By prop **(2)**, this actually extends to a diagram

$$\begin{tikzcd}
0 \arrow[r] & \mathrm{Hom}(U,S^1) \arrow[r] \arrow[d] & \{(H,\alpha)\} \arrow[r] \arrow[d] & \mathrm{NS}(V,U) \arrow[r] \arrow[d, equal] & 0 \\
0 \arrow[r] & \ker c_1 \arrow[r] & \mathrm{H}^1(U, \mathscr{O}(V)^\times) \arrow[r, "c_1"] \arrow[d, "\wr"] & \mathrm{NS}(V,U) \arrow[r] & 0 \\
 & & \mathrm{H}^1(X, \mathscr{O}_X^\times) & &
\end{tikzcd}$$

We'll prove that $\mathrm{Hom}(U,S^1) \to \ker c_1$ is an isomorphism, and co conclude that line bundles are characterized by Hermitian forms with some integrality property + a semicharacter.

Proof that $\mathrm{Hom}(U,S^1) \xrightarrow{\sim} \ker c_1$. Say $e \in \mathrm{H}^1(U, \mathscr{O}(V)^\times)$ and $c_1(e) = 0$. Consider

$$\begin{tikzcd}
\mathrm{H}^1(U,\mathbb{R}) \arrow[r] \arrow[d] & \mathrm{H}^1(U, \mathscr{O}(V)^\times) \arrow[r] \arrow[d] & \mathrm{H}^2(U,\mathbb{Z}) \arrow[d] \\
\mathrm{H}^1(X,\mathbb{R}) \arrow[r] & \mathrm{H}^1(X, \mathscr{O}_X^\times) \arrow[r] & \mathrm{H}^2(X;\mathbb{Z})
\end{tikzcd}$$

(with all vertical maps isomorphisms). This tells us that e has a representative $e_u(z) = e^{2\pi i t(u)} =: \alpha(u)$ where $t : U \to \mathbb{R}$ is a homomorphism. This shows that the map is surjective (and also writes down what the map is). Say $e_u(z) = \alpha(u)$ where $\alpha : U \to S^1$ is a coboundary. Then, $e_u(z) = g(z+u)/g(z)$ for some $g \in \mathscr{O}(V)^\times$. Then, $|g(z+u)| = |g(u)|$ which, by complex analysis, implies that g is constant, so $\alpha = 1$. ■

Remark 5.16. Given (H,α) the line bundle is $(\mathbb{C} \times V)/U$ where U acts via

$$u \cdot (t,v) = \left(\alpha(u) e^{\pi H(v+u/2,u)} t, v+u\right)$$

○

6 Weixiao Lu (MIT): Algebraization of Complex Tori, 11/10

Note 3. 10ish minutes late

Let V be a g-dim $\mathbb{C}$-vector space, and let $U \subset V$ be a lattice. Let $X = V/U$.

Goal. Determine when V/U is algebraic.

Recall 6.1. $\mathrm{NS}(X) = \{E : V \times V \to \mathbb{R} \text{ alternating with } E(U \times U) \subset \mathbb{Z} \text{ and } E(iu,iv) = E(u,v)\}$.

Specifically, we will prove

Theorem 6.2. *X is algebraic $\iff$ $\exists$ positive $E \in \mathrm{NS}(X)$.*

Above, positive means the associated Hermitian form is positive, i.e. $E(iu, v)$ is positive.
We will give two proofs of this theorem

- Proof by Kodaira embedding theorem
- Proof directly by finding ample line bundles

Note that, as a corollary of our main theorem, we see that any 1-dimensional complex torus is an elliptic curve (there exist easier proofs of this fact[8]

6.1 Proof 1

Theorem 6.3 (Kodaira embedding). *Let X be any compact Kähler manifold. Then, X is projective $\iff$ $\exists \mathscr{L} \in \mathrm{Pic}(X)$ with $\mathscr{L}$ positive (i.e. its first Chern class is a positive $(1,1)$-form)*

Corollary 6.4 (using Lefschetz (1,1))**.** *Let X be compact Kähler. Then, X is projective iff $\exists$ Kähler metric g with fundamental form ω s.t. $[\omega] \in \mathrm{H}^2(X;\mathbb{Z})$ is integral.*

Let X be a compact, Kähler manifold. The **Kähler cone** $\mathcal{K}_X \subset \mathrm{H}^2(X;\mathbb{R})$ consists of all cohomology classes coming from a Kähler manifold.

Corollary 6.5. *X is projective $\iff$ $\mathcal{K}_X \cap \mathrm{H}^2(X;\mathbb{Z}) \neq 0$.*

Lemma 6.6. *Let $X = V/U$ be a complex torus, so $\mathrm{H}^2(X;\mathbb{R}) = \mathrm{Alt}(V \times V, \mathbb{R})$. Then,*

$$\mathcal{K}_X = \{E : E \text{ positive (1,1)-form}\} = \{E : E = \operatorname{Im} H \text{ for } H \text{ positive Hermitian form}\}.$$

$(1,1)$-form here means $E(ix, iy) = E(x, y)$ and positive means $E(ix, x) > 0$ for all nonzero $x \in V$

Note, $\mathrm{H}^2(X;\mathbb{Z})$ just consists of alternating forms with integral values on the lattice U, so this lemma would entail the main theorem.

Proof. ($\supset$) Let $E : V \times V \to \mathbb{R}$ be a positive $(1,1)$-form. Then, $E(ix, y) : V \times V \to \mathbb{R}$ is an inner product (in particular, positive definite), so it induces a metric on V which descends to a metric ω on $X = V/U$. This induced metric is Kähler. Tracing through definitions, one sees that $[\omega] = E$.

($\subset$) Let g be any Kähler metric on $X = V/U$ with fundamental class ω. Note that if g were translation invariant, it would come from a metric on V, and we'd by happy. Hence, we define a new metric

$$\widetilde{g}(v, w) = \int_X g(\tau_{x,*}v, \tau_{x,*}w)\mathrm{d}x.$$

Then, $\widetilde{\omega} = \int_X \omega$ and we only need show that $[\omega] = \left[\int_X \omega\right]$. By the Hodge theorem, we can write $\omega = \eta + \mathrm{d}\alpha$ with η harmonic. Then, $\int \omega = \eta + \mathrm{d}\left(\int \alpha\right)$ ■

Question: Why? Uniqueness of harmonic representative?

[8] e.g. the Weierstrass function $\wp(z) = \sum_{u \in U \setminus 0} \left(\frac{1}{(z-u)^2} - \frac{1}{u^r}\right)$ gives embedding $[1 : \wp : \wp^2] : X \hookrightarrow \mathbb{CP}^2$. Also any compact Riemann surface is algebraic. As a third, if h is any Hermitian form on $\mathbb{C}$ and e_1, e_2 is a basis, can take $h' = h/h(e_1, e_2)$, and then let $E := \operatorname{Im} h'$

6.2 Example of non-algebraic complex torus

There is a matrix interpretation of the previous result. Let $e_1, \dots, e_{2g}$ be a basis for U. Get a $g \times 2g$ matrix $\Pi := (e_1\, e_2\, \dots\, e_{2g})$, the **period matrix**.

Theorem 6.7. *X algebraic iff there exists skew-symmetric $A \in M_{2g}(\mathbb{Z}) \cap \mathrm{GL}_{2g}(\mathbb{Q})$ so that $\Pi A^{-1} \Pi^t = 0$ and $i\Pi A^{-1}\overline{\Pi^t} > 0$. These two are called the **Riemann relations***

Non-example. Say $V = \mathbb{C}^2$ and U has basis

$$\begin{pmatrix}1\\0\end{pmatrix}, \begin{pmatrix}0\\1\end{pmatrix}, \begin{pmatrix}\sqrt{-2}\\\sqrt{-3}\end{pmatrix}, \begin{pmatrix}\sqrt{-5}\\\sqrt{-7}\end{pmatrix}.$$

Then V/U is not algebraic. Can check that first Riemann relation can never be satisfies (use that various square roots are $\mathbb{Q}$-linearly independent).

6.3 Proof 2

Let's first study $\mathrm{H}^0(X, L)$ with $L \in \mathrm{Pic}(X)$.

Recall 6.8. There is an exact sequence

$$0 \longrightarrow \mathrm{Pic}^0(X) \longrightarrow \mathrm{Pic}(X) \longrightarrow \mathrm{NS}(X) \longrightarrow 0$$

which can be identified with

$$0 \longrightarrow \mathrm{Hom}(U, S^1) \longrightarrow \{(H, \alpha)\} \longrightarrow \{H\} \longrightarrow 0$$

Say $L \in \mathrm{Pic}(X)$ corresponds to (H, α). Then,

$$\mathrm{H}^0(X, L) = \left\{\theta : V \to C \text{ holomorphic} \mid \theta(z+u) = \alpha(u) e^{\pi H(z,u) + \frac{\pi}{2} H(u,u)} \theta(z) \text{ for any } z \in V, u \in U\right\}$$

Just gotta solve the functional equation appearing above.

Lemma 6.9. *Let $N = \mathrm{rad}(H)$. Then, $N = \mathrm{rad}(E)$ where $E = \mathrm{Im}\, H$. Also, N is a $\mathbb{C}$-subspace and $N \cap U$ is a (full rank) lattice of N.*

Proof. $H(z, w) = E(iz, w) + iE(z, w)$ so $\mathrm{rad}(E) = \mathrm{rad}(H)$. Since H is $\mathbb{C}$-linear in first component, N is a $\mathbb{C}$-subspace. For the last claim, we use

Fact. If B ia an alternating form on a free abelian group, then $\mathrm{rad}(B) \otimes_{\mathbb{Z}} \mathbb{R} = \mathrm{rad}(B_{\mathbb{R}})$.

Apply this to $B = E|_U$; the LHS is $(N \cap U) \otimes \mathbb{R}$ and the RHS is N. ■

Lemma 6.10. *Let $N = \mathrm{rad}(E)$. Then, Θ is constant on each coset $z + N$ of N.*

Proof. Fix z. If $u \in N$, then $\theta(z + u) = \alpha(u)\theta(z)$, so $|\theta(z + u)| = |\theta(z)|$. Thus, $\theta|_{z+N}$ is bounded, so θ is constant by Liouville. ■

Corollary 6.11. $\mathrm{H}^0(X, L) = \mathrm{H}^0(\overline{X}, \overline{L})$ *where $\overline{X} = X/(N/(N \cap U))$ and $\overline{L}$ is the descent of L.*

Thus, we may assume that H is non-degenerate.

Lemma 6.12. *If H is not positive, then $\mathrm{H}^0(X, L) = 0$.*

Proof. Let $W \subset V$ be a maximal subspace with $H|_W$ negative definite. Convince yourself that $W \cap U$ is a (full rank) lattice of W. For fixed z, we'll show $\Theta|_{z+W} = 0$. The functional equation with $u \in W \cap U$ (and $u \to \infty$) shows that θ is bounded on each such coset, tending to 0 as $u \to \infty$ (the $e^{\pi H(u,u)/2}$ term dominates and behaves liek $e^{-|x|^2}$). Hence, it's constant by Liouville and so must be 0. ■

Theorem 6.13. *If H is positive, then $h^0(X, L) = \sqrt{\det E}$.*

Lemma 6.14. *There exists a basis of U s.t. the matrix of E is of the form*

$$\begin{pmatrix} 0 & D \\ -D & 0 \end{pmatrix} \text{ with } D = \operatorname{diag}(d_1, d_2, \ldots, d_n) \text{ and } d_1 \mid d_2 \mid \cdots \mid d_n.$$

Proof idea. Can assume E non-degenerate. Pick a symplectic basis $e_1, f_1, \ldots, e_n, f_n$ of $U \otimes \mathbb{Q}$, and use that $U \subset \mathbb{Z}e_1 \oplus \ldots \mathbb{Z}e_n$. ■

Example. Say $V = \mathbb{C}$ and $U = \mathbb{Z} \oplus \mathbb{Z}i$. Take $H(z_1, z_2) = z_1\overline{z}_2$, $E(z_1, z_2) = \operatorname{Im} H(z_1, z_2)$. Let $\alpha : U \to S^1$ be $\alpha(a + bi) = e^{i\pi ab}$. We want to find θ s.t.

$$\theta(z + u) = \alpha(u)e^{\pi z\overline{u} + \frac{\pi}{2}|u|^2}\theta(z).$$

Plug in $u = 1$ and $u = i$. Consider $\theta^* = \theta \cdot e^{-\frac{z^2}{2}}$. Get functional equation

$$\theta^*(z + 1) = \theta^*(z) \text{ and } \theta^*(z + i) = e^{-\pi}e^{2\pi iz}\theta^*(z).$$

Consider Fourier expansion of $\theta^* = \sum_{n\in\mathbb{Z}} c_n e^{2\pi inz}$ and conclude that Fourier coefficients satisfy $c_n/c_{n-1} = e^{\pi(2n-1)}$. Thus, θ^* uniquely determined by value of c_0, so get 1-dimensional space of solutions, spanned by $\theta^* = \sum_{n\in\mathbb{Z}} e^{-\pi n^2} q^n$. △

The general proof is the same, but much more annoying notationally.

Theorem 6.15 (Lefschetz)**.** *Let $L \in \operatorname{Pic}(X)$ correspond to (H, α).*

(1) *If H is not positive, then L is not ample*

(2) *If H is positive, then $L^{\otimes 3}$ is very ample.*

Proof. **(1)** If H is not positive, no powers of L with have global sections?

(2) Have $\theta \in \mathrm{H}^0(X, L)$. For any $a, b \in V$, $\theta(z - a)\theta(z - b)\theta(z + a + b) \in \mathrm{H}^0(X, L^{\otimes 3})$ (theorem of square). Do some calculations to show that $L^{\otimes 3}$ globally generated and separated tangent vectors and whatnot. ■

7 Ryan Chen (MIT): Moduli Space of Curves and Abelian Varieties, 11/17

We ultimately want to understand uniform Mordell, rational points on all curves. For this, it's natural to turn to the moduli space of all curves and also look at its associated Jacobian.

7.1 Moduli of Abelian Varieties

Fix a field $k = \overline{k}$.

Definition 7.1. A **polarization** of an abelian variety X/k is any of the following equivalent data

(1) An ample class in $\mathrm{NS}(X) = \mathrm{Pic}(X)/\mathrm{Pic}^0(X)$.

Note ampleness of a line bundle depends only on its class in the Néron-Severi group, e.g. by **Nakai-Moishezon** $\mathscr{L}$ ample $\iff c_1(\mathscr{L})^{\dim Y} \cdot Y > 0$ for all $Y \hookrightarrow X$.

(2) An isogeny $\varphi : X \to X^\vee$ which is symmetric ($\varphi = \varphi^\vee$) and furthermore: if $\wp$ is the Poincaré bundle on $X \times X^\vee$, the pullback $(1 \times \varphi)^* \wp$ is ample.

Remark 7.2 (Passing from one to the other). Starting with **(1)**, we have an ample $\mathscr{L} \in \mathrm{Pic}(X)$. Associated to this is the map

$$\begin{array}{rccc} \varphi_{\mathscr{L}} : & X & \longrightarrow & X^\vee \\ & a & \longmapsto & t_a^*\mathscr{L} \otimes \mathscr{L}^{-1} \end{array}$$

where t_a is translation by a. ○

Example (when $K = \mathbb{C}$)**.** Say $X = V/\Lambda$. Recall that

$$\mathrm{NS}(X) \simeq \{H : V \times V \to \mathbb{C} : H \text{ Hermitian and } \mathrm{Im}(H)(\Lambda) \subset \mathbb{Z}\} \simeq \left\{ \begin{array}{r} \psi \text{ skew-symmetric pairing, } \mathbb{Z}\text{-valued on } \Lambda \text{ satisfying} \\ \psi(ix, y) = \psi(x, y) \end{array} \right\}$$

Above, $H(x, y) = \psi(ix, y) + i\psi(x, y)$ and $\psi(x, y) = \mathrm{Im}\, H(x, y)$. Furthermore, $H \in \mathrm{NS}(X)$ represents an ample class $\iff$ H is positive definite ($\iff$ $\psi(ix, x) > 0$ for all nonzero x). What's the corresponding isogeny to the dual abelian variety? Here, $X^\vee = \overline{V}^\vee/\Lambda^\vee$ where

$$\overline{V}^\vee = \{\ell : V \to \mathbb{C} \mid \ell(\alpha v) = \overline{\alpha}\ell(v) \text{ and } \ell(v + w) = \ell(v) + \ell(w)\}.$$

Furthermore, $\Lambda^\vee = \left\{\ell \in \overline{V}^\vee \mid \mathrm{Im}(\ell)(\Lambda) \subset \mathbb{Z}\right\}$. The Hermitian pairing $H : V \times V \to \mathbb{C}$ gives a map $V \to \overline{V}^\vee$ which descends to $V/\Lambda \to \overline{V}^\vee/\Lambda^\vee$. △

Definition 7.3. A polarization $\varphi : X \to X^\vee$ is **principle** if φ is an isomorphism.

Example ($k = \mathbb{C}$)**.** Say $X = V/\Lambda$ with polarization ψ (a skew-symmetric pairing). This is principle iff Λ is *self-dual* in the sense that

$$\Lambda = \{v \in V : \psi(v, w) \in \mathbb{Z} \text{ for all } w \in \Lambda\}.$$

That is, in some basis for Λ, the pairing ψ is given by the standard symplectic matrix

$$\begin{pmatrix} & I_g \\ -I_g & \end{pmatrix}$$

where I_g means the $g \times g$ identity matrix. △

Example. Any Elliptic curve X/k admits a unique principle polarization. This is because $\deg : \mathrm{NS}(X) \xrightarrow{\sim} \mathbb{Z}$ with ample classes given by the positive integers $\mathbb{Z}_{>0}$. For any $d \in \mathbb{Z}_{>0}$, the corresponding isogeny $X \to X^\vee$ has degree d^2, and so is an isomorphism $\iff d = 1$. △

Example. Let C/k be a (smooth) curve of genus $g \geq 2$. Then, its Jacobian $\mathrm{Jac}(C)$ is canonically principally polarized by its Theta divisor $\Theta = \underbrace{C + \cdots + C}_{g-1} \subset \mathrm{Jac}(C)$. Here, we're embedding $C \hookrightarrow \mathrm{Jac}(C)$ via some choice of basepoint (the theta divisor is independent of choice of Abel-Jacobi map $C \hookrightarrow \mathrm{Jac}(C)$) △

Question: The divisor itself or just its class in $\mathrm{NS}(\mathrm{Jac}(C))$?

Definition 7.4. An **abelian scheme** $X \to S$ is a smooth, proper group scheme w/ geometrically integral fibers. A **polarization of an abelian scheme** X/S is an isogeny $X \xrightarrow{\varphi} X^\vee := \mathrm{Pic}^0_{X/S}$ s.t. on geometric fibers, φ is a polarization in the sense defined earlier. We call φ a **principal polarization** if it is an isomorphism.

In order to get a moduli space represented by a variety instead of a stack, we impose some level structure.

Fix $N \geq 2$ and a base scheme S over $\operatorname{spec} \mathbb{Z}[1/N, \zeta_N]$.

Fact. Let X be an abelian variety. Then, $X[N]$ and $X^\vee[N]$ are Cartier dual, so there is a pairing

$$X[N] \times X^\vee[N] \longrightarrow \mu_N,$$

called the **Weil pairing**. If $X \to X^\vee$ is a polarization, this gives an alternating pairing

$$X[N] \times X[N] \longrightarrow \mu_N.$$

Example. If $S = \operatorname{spec} \mathbb{C}$, say $X = V/\Lambda$ with polarization given by ψ. Then this pairing is explicitly given by[9]

$$\begin{array}{ccc} X[N] \times X[N] & \longrightarrow & \mu_N \\ (x, y) & \longmapsto & \exp(-2\pi i N \psi(x, y)). \end{array}$$

Fix $\zeta_N = \exp(-2\pi i/N)$ some primitive N root of unity. Get identification $\mu_N \cong \mathbb{Z}/N\mathbb{Z}$. △

Definition 7.5. A **level** N **structure** on a polarized abelian scheme X/S of relative dimension g is a symplectic isomorphism

$$\left(\frac{\mathbb{Z}}{N\mathbb{Z}}\right)^{2g} \xrightarrow{\sim} X[N]$$

with $(\mathbb{Z}/N\mathbb{Z})^{2g}$ given the standard $\mathbb{Z}/N\mathbb{Z}$-valued symplectic pairing $\begin{pmatrix} & I_g \\ -I_g & \end{pmatrix}$. The RHS has the symplectic pairing given by the Weil pairing.

Theorem 7.6. *Fix $N \geq 3$, $g \geq 1$. Consider the moduli functor*

$$\begin{array}{rccl} \mathcal{A}_{g,N} : & \mathrm{Sch}/\mathbb{Q}(\zeta_N) & \longrightarrow & \mathrm{Set} \\ & S & \longmapsto & \{\text{iso classes of ppav's rel dim } g \text{ over } S \text{ w/ level } N \text{ structure.}\} \end{array}$$

This is represented by a smooth, quasi-projective geom. irreducible variety (over $\mathbb{Q}(\zeta_N)$) of dimension $g(g+1)/2$.

What are the complex points of this variety?

[9] Note $X[N] = \frac{1}{N}\Lambda/\Lambda$ so $\psi : X[N] \times X[N] \to \frac{1}{N^2}\mathbb{Z}$ so N is exponent gets us to $\frac{1}{N}\mathbb{Z}$

Goal. Describe $\mathcal{A}_{g,N}(\mathbb{C}) = \{\text{isom. classes of ppavs / } \mathbb{C}, \text{ dim } g \text{ w/ level } N \text{ structure}\}$. Also describe $\mathcal{A}_g(\mathbb{C})$ which is same thing w/o level structure.

Define

$$\widetilde{\mathcal{A}}_g(\mathbb{C}) = \left\{ \begin{matrix} \text{isom. classes of ppavs, dim } g/\mathbb{C} \\ \text{w/ choice of symplectic basis for } \mathrm{H}_1(X,\mathbb{Z}) \end{matrix} \right\} = \left\{ (\Lambda \subset V, \psi, (e_i)_{i=1}^{2g}) \,\middle|\, \begin{matrix} \psi(ix,iy) = \psi(x,y) \\ \psi(ix,x) > 0 \end{matrix} \right\}.$$

(above, (e_i) symplectic basis for Λ, ψ). There's a forgetful map $\widetilde{\mathcal{A}}_g(\mathbb{C}) \to \mathcal{A}_g(\mathbb{C})$ realizing $\mathcal{A}_g(\mathbb{C}) \cong \mathrm{Sp}_{2g}(\mathbb{Z})\backslash\widehat{\mathcal{A}}_g(\mathbb{C})$. There is also a map $\widetilde{\mathcal{A}}_g(\mathbb{C}) \to \mathcal{A}_{g,N}(\mathbb{C})$ sending

$$(\Lambda \subset V, \psi, (e_i)) \longmapsto (\Lambda \subset V, \psi, (e_i/N)).$$

Let $\Gamma(N) := \ker\left(\mathrm{Sp}_{2g}(\mathbb{Z}) \twoheadrightarrow \mathrm{Sp}_{2g}(\mathbb{Z}/N\mathbb{Z})\right)$. Then, $\mathcal{A}_{g,N}(\mathbb{C}) \cong \Gamma(N)\backslash\widetilde{\mathcal{A}}_g(\mathbb{C})$.

Fact.

$$\widetilde{\mathcal{A}}_g \cong \mathcal{H}_g := \{z \in M_{g\times g}(\mathbb{C}) \text{ symmetric with } \mathrm{Im}(z) > 0\}.$$

Given $g \in \mathrm{Sp}_{2g}(\mathbb{R})$ and $z \in \mathcal{H}_g$, the corresponding action on $g = \begin{pmatrix} a & b \\ c & d \end{pmatrix}$ is

$$g \cdot z = (az+b)(cz+d)^{-1}.$$

Example ($g=1$)**.** $\mathrm{Sp}_2(\mathbb{R}) \cong \mathrm{SL}_2(\mathbb{R})$ and we get $\mathcal{A}_1(\mathbb{C}) \cong \mathrm{SL}_2(\mathbb{Z})\backslash\mathcal{H}$ and $\mathcal{A}_{1,N}(\mathbb{C}) = \Gamma(N)\backslash\mathcal{H}$. △

Instead of fixing a complex space and varying the lattice, we can fix a lattice and vary the complex structure...

Fix a choice of lattice $\Lambda_0 = \mathbb{Z}e_1 \oplus \cdots \oplus e_{2g}$ with choice of symplectic basis for the standard symplectic pairing $\psi_0 := \begin{pmatrix} & I_g \\ -I_g & \end{pmatrix}$. Let $V_0 = \Lambda_0 \otimes \mathbb{R}$. Then,

$$\widetilde{\mathcal{A}}_g(\mathbb{C}) \cong \left\{ J : V_0 \to V_0 \text{ complex structure} \,\middle|\, \begin{matrix} \psi_0(Jx,Jy) = \psi_0(x,y) \\ \psi_0(Jx,x) > 0 \text{ for all } x \in V_0 \setminus 0 \end{matrix} \right\}$$

(above J is $\mathbb{R}$-linear and $J^2 = -1$, i.e. J is multiplication by i).

Example. Take $J_0 = \begin{pmatrix} & I_g \\ -I_g & \end{pmatrix} \in \widetilde{\mathcal{A}}_g(\mathbb{C})$. △

Remark 7.7. $g \in \mathrm{Sp}_{2g}(\mathbb{R})$ acts on complex structure via conjugation $J \mapsto gJg^{-1}$. Some linear algebra shows that this action is transitive, so if $K \le \mathrm{Sp}_{2g}(\mathbb{R})$ is the stabilizer of J_0, then

$$\widehat{\mathcal{A}}_g(\mathbb{C}) \cong \mathrm{Sp}_{2g}(\mathbb{R})/K.$$

Thus, we get a double coset description

$$\mathcal{A}_g(\mathbb{C}) \cong \mathrm{Sp}_{2g}(\mathbb{Z})\backslash\mathrm{Sp}_{2g}(\mathbb{R})/K.$$

○

Giving a complex structure is the same as giving the associated Hodge decomposition, i.e. given such a J, we can decompose $V_0 \otimes \mathbb{C} = W \oplus \overline{W}$ with $W, \overline{W}$ the $(\pm i)$-eigenspaces of J. Conversely, given such a decomposition, get complex structure by letting J act by i on W and by $-i$ on $\overline{W}$. This Hodge decomposition is determined by either factor, so equivalent to give a g-dimensional $W \subset V_0 \otimes \mathbb{C}$ so that $W \cap \overline{W} = 0$. We can translate this into saying

$$\widetilde{\mathcal{A}}_g(\mathbb{C}) \cong \left\{ \text{dim } g \text{ subspaces } \overline{W} \subset V_0 \otimes \mathbb{C} \,\middle|\, \begin{matrix} \textbf{(1)}\psi_0(x,y) = 0 \text{ when } x, y \in \overline{W} \text{ and } \textbf{(2)} - i\psi_0(x,\overline{x}) > 0 \text{ for all } x \in \overline{W} \setminus 0 \\ W \cap \overline{W} = 0 \end{matrix} \right\}$$

Remark 7.8. The last condition $W \cap \overline{W}$ is redundant. If $x \in W \cap \overline{W}$, then $\psi_0(x,x) = 0$ but also $-i\psi_0(x,\overline{x}) > 0$, a contradiction. ◦

Get map $\widetilde{\mathcal{A}}_g(\mathbb{C}) \hookrightarrow \mathrm{Gr}_g(V_0 \otimes \mathbb{C})$ given by $\overline{W}$. Condition **(1)** defines Lagrangian Grassmanian $\mathrm{LGr}(V_0 \otimes \mathbb{C})$ (which has dimension $g(g+1)/2$). Condition **(2)** is an open condition. We still need to say how to turn $\widetilde{\mathcal{A}}_g(\mathbb{C})$ into the Siegal (spelling?) upper half space.

Example. The point $J_0 = \begin{pmatrix} & I_g \\ -I_g & \end{pmatrix} \in \widetilde{\mathcal{A}}_g(\mathbb{C})$ corresponds to the column span of $\begin{pmatrix} iI_g \\ I_g \end{pmatrix}$ in $\mathrm{LGr}(V_0 \otimes \mathbb{C})$, i.e. the $(-i)$-eigenspace of J_0 is spanned by $ie_1 + e_{1+g}, ie_2 + e_{2+g}, \ldots, ie_g + e_{2g}$. △

Note that $\mathrm{Gr}_g(V_0 \otimes \mathbb{C}) = \{\text{full rank } N \in M_{2g\times g}(\mathbb{C})\} / \mathrm{GL}_g(\mathbb{C})$. Given a matrix on the RHS, send it to its column span. With this perspective, $\widehat{\mathcal{A}}_g(\mathbb{C}) \subset \mathrm{Gr}_g(V_0 \otimes \mathbb{C})$ is precisely

$$\left\{ \begin{pmatrix} z \\ 1 \end{pmatrix} : z \in \mathcal{H}_g \right\}.$$

This falls out of some linear algebra. z being symmetric comes from **(1)** while $\mathrm{Im}(z) > 0$ comes from **(2)**, the two conditions on $\overline{W}$ from earlier.

7.2 A little about moduli of curves

Definition 7.9. A **relative genus** g curve $C \to S$ is a smooth, proper map whose geometric fibers are genus g curves. Such a map has a **relative Jacobian** $\mathrm{Jac}(C/S) := \mathrm{Pic}^0_{C/S}$ which is an abelian scheme.

Fact. The relative Jacobian $\mathrm{Jac}(C/S)$ carries a canonical principal polarization, fiberwise recovering the theta divisor polarization we mentioned earlier.

Remark 7.10. Say C/k. If $i : C \hookrightarrow \mathrm{Jac}(C)$ is the Abel-Jacobi map, we can take Pic^0's to get a map $i^\vee : \mathrm{Jac}(C)^\vee \to \mathrm{Jac}(C)$. Furthermore, $-i^\vee = \varphi^{-1}$ where $\varphi : \mathrm{Jac}(C) \to \mathrm{Jac}(C)^\vee$ is the polarization coming from the theta divisor. ◦

Definition 7.11. A **level** N **structure** on a relative curve $C \to S$ is a level N structure on $\mathrm{Jac}(C/S)$.

Theorem 7.12. *Fix $N \geq 3$ and $g \geq 2$. Consider the functor*

$$\begin{array}{rccc} \mathcal{M}_{g,N} : & \mathrm{Sch}/\mathbb{Q}(\zeta_N) & \longrightarrow & \mathrm{Set} \\ & S & \longmapsto & \left\{ \begin{matrix} \textit{isom. classes of rel. genus } g \textit{ curves over } S \\ \textit{w/ level } N \textit{ structure} \end{matrix} \right\}. \end{array}$$

This is representable by a smooth, quasi-projective geometrically irreducibel variety over $\mathbb{Q}(\zeta_N)$, of dimension $3g-3$.

One has a **Torelli morphism** $\mathcal{M}_{g,N} \xrightarrow{\tau} \mathcal{A}_{g,N}$ sending a curve to its Jacobian.

Fact. τ above has finite fibers.

Fact. In the stacky version of things, one has $\tau : \mathcal{M}_g \to \mathcal{A}_g$ which is an injection on $\mathbb{C}$-points.

8 Tony Feng (MIT): Uniform Mordell: review and preview 1, 2/23/2022

Note 4. Roughly 7 minutes late

Theorem 8.1 (Mordell/Faltings)**.** *Let X be a smooth projective curve over a number field K. If $g(X) \geq 2$, then $X(K)$ is finite.*

Goal. Control $\#X(K)$ uniformly in X.

Conjecture 8.2. *There exists $c = c(g, K)$ so that $\#X(K) \leq c$*

Theorem 8.3 (Dimitrov-Gao-Harbegger)**.** *There is some $c = c(g, [K : \mathbb{Q}])$ so that $\#X(K) \leq c^{\operatorname{rank} \operatorname{Jac}_X(K)}$*

Proof is via an "amplification argument." We will try to illustrate it in a simpler example (due to Caporaso-Harris-Mazur, I think).

Let's start with the following question: how do we generalize Mordell to $\dim X > 1$?

- How to generalize "$X(K)$ finite"?

  Could still ask $X(K)$ finite, but this is too restrictive (e.g. a higher dimensional variety may contain a line or elliptic curve or what have you as a subvariety). Instead, we'll ask that the points are sparse. That is, we'll want "$X(K)$ is not Zariski-dense in X"

- How to generalize "$g \geq 2$"?

  For a curve, this is equivalent to the condition that K_X be ample.

  Recall 8.4. In general, a line bundle L is **big** if

  $$h^0(X, L^{\otimes n}) \sim cn^{\dim X}$$

  for some constant c.

  Definition 8.5. We say X is of **general type** if K_X is big.

  Slogan. Big is a birational analoge of ample

  For example, if $f : X \to Y$ is birational and L on Y is big, then f^*L on X is big. This is false if 'big' is replaced by 'ample'.

Remark 8.6. There are other ways of generalizing Mordell. For example, Lang has a conjecture that if $X_{\mathbb{C}}$ is hyperbolic (in some sense), then $X(K)$ should actually be finite. $\circ$

Conjecture 8.7 (Weak Lang Conjecture)**.** *Let X be smooth, projective over K (and $\dim X > 0$) + general type. Then, $X(K)$ is not Zariski dense in X.*

Theorem 8.8 (Caparaso-Harris-Mazur)**.** *Weak Lang Conjecture $\implies$ (strong) uniform version, i.e. $\#X(K) < c(g, K)$*

Unclear if the strong uniform version is widely believed by experts (at least, Bjorn believes it). It sounds like Caparso-Harris-Mazur wanted to use this as evidence against weak Lang.

Remark 8.9. If you believe that the Mordell Weil rank (say of Jacobians of genus g curves over a fixed number field) is bounded, then the Dimitrov-Gao-Harbegger theorem implies the stronger version of uniformity. $\circ$

Remark 8.10. If X dominates Y of general type with $\dim Y > 0$ (call this condition **(*)** on X), then $\exists U \overset{\text{open}}{\subset} X$ s.t. $U(K) = \emptyset$, assuming 'weak Lang'.

There's an open subset of Y with no rational points (since rational points are not zariski dense). It's preimage U in X will also have no rational points (assuming map defined over K), so we get above statement. $\circ$

Example. If X is a $\mathbb{P}^1$-bundle over a genus ≥ 2 curve, it will not be of general type, but will dominate something of general type, e.g $X = \mathbb{P}^1 \times Y$ with Y a curve. $\triangle$

Consider a family of curves $\mathcal{C} \xrightarrow{f} S$ (with $\mathcal{C}$ irreducible). Suppose $\mathcal{C}$ satisfies condition **(*)**. Then, there exists $U' \overset{\text{open}}{\subset} \mathcal{C}$ w/ no K-points. Thus, $\mathcal{C} \setminus U \to S$ is generically finite of some degree d. So we can assume it's finite of degree d over $V \subset S$. Then, $\#\mathcal{C}_s(K) \leq d$ for $s \in V$. We now want to try to keep going by noetherian induction. This will give a uniform bound for the fibers of this family. Now, apply to something like universal family $\mathcal{C}_g \to \mathcal{M}_g$.

Question: Why?

Answer: $\dim \mathcal{C} = \dim S + 1$ (family of curves), so $\dim(\mathcal{C} \setminus U) = \dim \mathcal{C} - 1 = \dim S$. Have dominant map of 'nice' things of the same dimension

Remark 8.11. There are subtleties ignored above. For example, to induct, need **(*)** for all the schemes appearing along the way. $\circ$

Theorem 8.12 (Correlation Theorem)**.** *Let $\mathcal{C} \xrightarrow{f} S$ be an "integral, proper, flat" family of genus $g \geq 2$ curves, and let $\mathcal{C}^{(n)} := \mathcal{C} \times_S \times \mathcal{C} \times_S \ldots \times_S \mathcal{C}$ (n factors). Then, $\mathcal{C}^{(n)}$ will satisfy **(*)** for some $n \gg 0$.*

Heuristic: canonical bundle of fibers are big and taking powers is "increasing along fiber direction," so hope bigness of fibers overcomes any non-bigness of the base. "amplifying fibers increases total positivity"

Claim 8.13. *Consider family $\mathcal{C} \xrightarrow{f} S$ of curves (with $\mathcal{C}$ irreducible). Then, $\exists U \overset{open}{\subset} S$ s.t. $\#\mathcal{C}_y(KK_y) < d(\mathcal{C}/S)$ for all $y \in U(\overline{K})$ (where K_y is the residue field at y).*

proper, flat, generic fiber genus ≥ 2, blah, blah, blah

This is the sort of statement one can actually by Noetherian induction to.

Proof. By correlation, there's some n s.t. $\mathcal{C}^{(n)}$ satisfies **(*)**. If $n = 1$, we saw earlier how to prove this statement. A similar argument will work for larger n.

Example (Say $n = 2$)**.** Let $S' = \mathcal{C}$ so we have

$$\begin{tikzcd} \mathcal{C}^{(2)} \arrow[r] \arrow[d, "f'"'] & \mathcal{C} \arrow[d, "f"] \\ S' \arrow[r, "f"] & S \end{tikzcd}$$

By earlier argument applied to f', we conclude there exists some $V \subset S' = \mathcal{C}$ s.t. $\mathcal{C}_y^{(2)}(K) < d$ for all $y \in V$. Finally, $f(V)$ contains some open set U in S, and so we win using this U, i.e. fibers of f over U are the same as fibers of f' over V, so we get the desired conclusion. △

■

The two main ingredients of an "amplification argument" are

- Some property giving generic control of the rational points, e.g. condition **(*)**
- A correlation theorem saying the property holds after taking iterated fiber products

8.1 Gap principles for rational points

Let's start w/ a summary: Vojta bounds rational points of large height. One of the innovations of [DGH] is that they give a bound on points of small height as well.

Let's say X is a (smooth, projective) curve of genus $g \geq 2$. Assume it has a rational point, so we get an embedding $X \hookrightarrow \mathrm{Jac}(X)$ defined over K. Consider a Néron-Tate height $\mathrm{Jac}(X)(\overline{K}) \to \mathbb{R}$.

Theorem 8.14.

TODO: Improve formatting

- *There exists a constant $R > 0$ s.t. for any $P \neq Q \in X(\overline{\mathbb{Q}})$ with $|Q| > |P| > R$, one has*

  (Mumford's Gap Principle) *If $\langle P, Q\rangle \geq \frac{3}{4}|P|\,|Q|$, then $|Q| \geq 2\,|P|$*

  Think of this as a 'repulsion principle,' i.e. "the rational points repel each other." If the angle between them is too small, then their magnitudes differ by a lot.

  (Vojta's inequality) *If $\langle P, Q\rangle \geq \frac{3}{4}|P|\,|Q|$, then $|Q| \leq \kappa\,|P|$ for some $\kappa = \kappa(g)$.*

  Taken together, this means that in some small angular sector, there can't be that many rational points of large height. There are some remaining rational points of small height, but for a fixed curve X, there are only finitely many left by Northcott.

Remark 8.15 (Vojta in Families). Start with a family of curves $\mathcal{C} \to S$. The height bound R can be made to work in families. One can compactify $S \hookrightarrow \overline{S}$ and choose an ample line bundle M on $\overline{S}$ so that Vojta holds with $R_s \ll \max(1, h_M(s))$. ○

Theorem 8.16 (Vojta in Families)**.** *Consider a family of curves $\mathcal{C} \to S$ embedded (over S) in some abelian scheme $\mathcal{A}/S$. Let $\mathscr{L}$ be a relatively ample line bundle on $\mathcal{A}$. Compactify $S \hookrightarrow \overline{S}$ and choose an ample line bundle $\mathscr{M}$ on $\overline{S}$. We get Néron-Tate $\widehat{h}_{\mathscr{L}}$ on $\mathscr{A}$ and Weil $h_{\mathscr{M}}$ on S. Fix a finite rank subgroup $\Gamma \subset \mathscr{A}_s(\overline{\mathbb{Q}})$. Now,*

$$\#\left\{P \in C_s(\overline{\mathbb{Q}}) \cap \Gamma : \widehat{h}_{\mathscr{L}}(P) \gg \max(1, h_{\mathscr{M}}(s))\right\} < c^{\operatorname{rank}\Gamma} \text{ for some } c = c(\mathcal{A}/S, \mathcal{C}/S, \mathscr{L}, \mathscr{M}).$$

In particular, c does not depend on $s \in S$.

Upshot: we have a family version of a bound on the number of large height points.

This says nothing about the number of small height points in families. We just now each fiber has finitely many, but this finite amount might grow wildly with s. To deal with that, we'll need the 'new gap principle'.

Theorem 8.17 (**New Gap Principle**)**.** *There exists a constant $c = c(\mathcal{C}/S, \mathcal{A}/S, \mathscr{L}, \mathscr{M})$ such that given $P \in \mathcal{C}_s(\overline{\mathbb{Q}})$, one has*

$$\#\left\{Q \in \mathcal{C}_s(\overline{\mathbb{Q}}) : \widehat{h}_{\mathscr{L}}(Q-P) \ll \max(1, h_{\mathscr{M}}(s))\right\} < c$$

Think of this as saying that even points of small height also repel each other, there aren't that many points Q close to a given point P. This let's you bound the number of small points in terms of the volume of the sphere defining 'small' (in some finite rank subgroup, have some sphere of radius R in which Vojta doesn't apply. However, the new gap principle will apply and give a bound of the form $c^{\operatorname{rank}\Gamma}$, essentially something like the volume of a sphere of radius R in $\operatorname{rank}\Gamma$-dimensional space).

The proof of the New Gap Principle is based on amplification

- The geometric property is 'non-degeneracy' for subvarieties $X \subset \mathcal{A}$ of abelian schemes

  This will give generic control on the number of rational points

- There will be a similar correlation type result

Claim 8.18 (generic control)**.** *If X is non-degenerate inside $\mathcal{A}/S$, then there is a Zariski open $U \subset X$ s.t.*

$$\widehat{h}_{\mathscr{L}}(P) \gg h_{\mathscr{M}}(\pi(P)) + C$$

for all $P \in U(\overline{\mathbb{Q}})$ ("height lower bound for non-degenerate varieties")

Claim 8.19 (correlation)**.** *Given $X \subset \mathcal{A}/S$ satisfying certain hypotheses, then $X^{(n)} \subset \mathcal{A}^{(n)}$ will be non-degenerate for $n \gg 0$.*

We will apply this sort of thing to $X = \mathcal{C}_g \times_{\mathcal{M}_g} \mathcal{C}_g \to U_g$ over $\mathcal{M}_g \to \mathcal{A}_g$ where above map sends $(P, Q) \mapsto \mathscr{O}(P-Q)$.

9 Tony Feng (MIT): Ovewview 2, 3/2

Goal. Understand *non-degeneracy*, a property of subschemes $X \subset \mathcal{A}$ of an abelian scheme $\mathcal{A} \to S$ over S.

9.1 Betti map

Recall 9.1. If A is an abelian variety over $\mathbb{C}$, then $A(\mathbb{C}) = \mathbb{C}^g/\mathbb{Z}^{2g} = \mathbb{R}^{2g}/\mathrm{H}_1(A, \mathbb{Z})$

For an abelian scheme $\mathcal{A}/S$, the Betti map keeps track of the lattice $\mathrm{H}_1(A_s, \mathbb{Z}) \subset \mathbb{R}^{2g}$ as it varies with s.

Construction 9.2. Let $\mathcal{A} \xrightarrow{\pi} S$ be an abelian scheme with fibers $\mathcal{A}_s = A_s = \pi^{-1}(s)$. Suppose we have $\Delta \subset S(\mathbb{C})$ along with a trivialization of $R^1\pi_*\mathbb{Z}$ (as a $\mathrm{Sp}_{2g}(\mathbb{Z})$-local system) over Δ. Concretely, this means that for all $s \in S$, we have a "choice" of a symplectic base $\omega_1(s), \dots, \omega_{2g}(s)$ for $\mathrm{H}_1(A_s(\mathbb{C}), \mathbb{Z})$. This lets us write $A_s(\mathbb{C}) = \mathbb{R}^{2g}/\mathrm{H}_1(A_s(\mathbb{C}), \mathbb{Z})$ and write $x \in A_s(\mathbb{C})$ in coordinates $x = (b_1(x), b_2(x), \dots, b_{2g}(x))$ with $b_i(x) \in \mathbb{R}/\mathbb{Z}$. The trivialization will let us view things as mapping to a fixed $\mathbb{R}^{2g}/\mathbb{Z}^{2g}$, and so we obtain

$$b_\Delta : A(\mathbb{C})_\Delta \to \mathbb{R}^{2g}/\mathbb{Z}^{2g}.$$

In the above discussion, note that such a Δ will exist locally on the base in the analytic topology.

The **Betti map** is – given $\Delta \subset S$ + a trivialization of $R^1\pi_*\mathbb{Z}|_\Delta$ – is a map $b_\Delta : A(\mathbb{C})_\Delta \to \mathbb{R}^{2g}/\mathbb{Z}^{2g}$. To make this less dependent on choices, can bake the choice into the base. For $\widetilde{S} \to S(\mathbb{C})$, the $\mathrm{Sp}_{2g}(\mathbb{Z})$-torsor of trivializations of $R^1\pi_*\mathbb{Z}$, we similarly get $A(\mathbb{C})_{\widetilde{S}} \to \mathbb{R}^{2g}/\mathbb{Z}^{2g}$.

All these Betti maps will ultimately come from the universal case. Let $U_g \to A_g$ be the universal (principally polarized) abelian scheme of relative dimension g. Thus, given $\mathcal{A} \xrightarrow{\pi} S$, we get a diagram

$$\begin{tikzcd} \mathcal{A} \arrow[r] \arrow[d, "\pi"'] & U_g \arrow[d, "\pi^{univ}"] \\ S \arrow[r] & A_g. \end{tikzcd}$$

Let's take the second perspective on Betti maps. Note that $\mathcal{A}_g(\mathbb{C}) = [\mathbb{H}_g/\mathrm{Sp}_{2g}(\mathbb{Z})]$ and so $\widetilde{S}_{univ} = \mathbb{H}_g$. A point $\tau \in \mathbb{H}_g$ really is the same data as a tuple $(A_\tau, \omega_1, \dots, \omega_{2g})$ with A_τ an abelian variety and $\omega_1, \dots, \omega_{2g}$ a basis for $\mathrm{H}_1(A_\tau, \mathbb{Z})$. This gives the universal Betti map

Brackets hint that we secretly are thinking about stack quotients

$$b^{univ} : U_g(\mathbb{C}) \times_{A_g(\mathbb{C})} \mathbb{H}_g \longrightarrow \mathbb{R}^{2g}/\mathbb{Z}^{2g}.$$

Remark 9.3. Here's a third perspective on Betti maps. If you don't like passing to covers or to small opens, you can define a map

$$\mathcal{A}(\mathbb{C}) \xrightarrow{b} \left[\left(\mathbb{R}^{2g}/\mathbb{Z}^{2g}\right)/\mathrm{Sp}_{2g}(\mathbb{Z})\right].$$

○

We'll be interested in local properties of the Betti map (e.g. if the derivative surjective on tangent space), and these won't be sensitive to the perspective one adopts.

In some sense these Betti maps have been studied for a while, see e.g.

9.2 André-Corvaja-Zannier

Setup. We're given an abelian scheme $\mathcal{A} \to S$ (of relative dimension g) with a section $\xi : S \to \mathcal{A}$. Let $\mathcal{A}_{tors} \subset \mathcal{A}$ be the torsion subscheme. Note $\mathcal{A}_{tors} = \bigcup_n \mathcal{A}[n]$.

Question 9.4. *What does does $\xi^{-1}(\mathcal{A}_{tors})$ look like in S?*

(How does the image of ξ intersect the torsion in $\mathcal{A}$?)

These sorts of questions belong/relate to the theory of unlikely intersections.

Remark 9.5. The "expected codimension" of $\xi^{-1}(\mathcal{A}[n])$ is g since $\mathcal{A}[n]$ has codimension g in $\mathcal{A}$. This maybe suggests the following heuristic dichotomy

- If $\dim S < g$, $\xi^{-1}(A_{tors})$ is not Zariski dense in S
- If $\dim S \geq g$, $\xi^{-1}(A_{tors})$ is Zariski dense in S

$\circ$

They proposed to address this sort of question using the Betti map. Locally over $\Delta \subset S(\mathbb{C})$ we have $b : A(\mathbb{C})_\Delta \longrightarrow \mathbb{R}^{2g}/\mathbb{Z}^{2g}$ and $b^{-1}(\mathbb{Q}^{2g}/\mathbb{Z}^{2g}) = A_{tors}(\mathbb{C})_\Delta$. They then considered $\operatorname{rank}(b \circ \xi)$. If rank $= 2g$ is maximal, then the composite

$$\Delta \xrightarrow{\xi} A(\mathbb{C})_\Delta \xrightarrow{b} \mathbb{R}^{2g}/\mathbb{Z}^{2g}$$

is a submersion, so its image will contain an open set which will then contain a lot of torsion points; in fact, $\xi^{-1}(A_{tors})$ will be analytically dense in some $\Delta \subset S(\mathbb{C})$.

Remark 9.6. The rank is bounded above by $2\dim S = \dim_{\mathbb{R}}(S)$ $\circ$

Their work highlights the utility of studying the rank of the Betti map and in particular the cases where it has maximal possible rank.

9.2.1 towards "no fixed part"

Still have $\pi : \mathcal{A} \to S$ an abelian scheme.

Definition 9.7. Let K/k be a field extension, and let A/K be an abelian variety.[10] The (K/k)**-trace** of A is the universal B/k equipped with a map $B_K \to A$.

Want to capture the idea of having a subabelian scheme of $\mathcal{A}$ which is constant.

Warning 9.8. The map $B_K \to A$ is not necessarily injective.

We make this definition to be able to rule out the sort of behavior which would futz with our heuristic dichotomy. Imagine, for example, that $\mathcal{A} = A \times S$ is constant and you take the section $\xi : S \to \mathcal{A}$ given by a fixed torsion point. Then, $\xi^{-1}(\mathcal{A}_{tors}) = S$ independent of $\dim S$.

Definition 9.9. Say S/k is irreducible with function field K. We say $\mathcal{A}/S$ has **no fixed point** if the K/k-trace of $A = \mathcal{A}_K$ is trivial.

Note 5. There was some discussion about whether it's equivalent to ask that the image of the K/k trace be trivial int he above discussion. Sounds like yes? something something image parameterizes abelian quotients instead of abelian subschemes something something + Poincaré complete reducibility something something.

Non-example. Let $E/\mathbb{C}$ be an elliptic curve, and let $\mathcal{A}'/S/\mathbb{C}$ be an abelian scheme over S. Then, $\mathcal{A} = \mathcal{A}' \times E_K$ has a fixed part e.g. since there's a natural non-trivial map $E_K \to \mathcal{A}$.

9.2.2 Trivial bound on $\operatorname{rank}(b \circ \xi)$

Of course $\operatorname{rank}(b \circ \xi) \leq 2g$ since the target is $(2g)$-dimensional. Similarly $\operatorname{rank}(b \circ \xi) \leq 2\dim S$. More creatively, the Betti map factors through the universal case

$$S \to \mathcal{A} \to U_g \overset{b_{univ}}{\dashrightarrow} \mathbb{R}^{2g}/\mathbb{Z}^{2g}$$

[10] Imagine S irreducible and $K = k(S)$ the function field

(dashed arrow to indicate that really you need to pass to a small analytic open or to a cover or take a stack quotient of the target or whatever you want to make this make sense), so in fact

$$\operatorname{rank}(b \circ \xi) \leq \min\left\{2g, 2\dim \operatorname{Im}(S \to U_g)\right\}.$$

Conjecture 9.10 (ACZ). *If $\mathbb{Z}\xi$ is Zariski dense in $\mathcal{A}$ and $\mathcal{A} \to S$ has no fixed part, then* $\operatorname{rank}(b \circ \xi)$ *achieves the trivial bound.*

Theorem 9.11 (Gao). *ACZ Conjecture holds if $\mathcal{A} \to S$ has simple geometric generic fiber, but is false in general.*

9.3 Non-degeneracy

Setup. Say $X \subset \mathcal{A}/S$ is an (irreducible) S-subscheme of an abelian variety $\mathcal{A} \xrightarrow{\pi} S$.

Definition 9.12. We say X is **non-degenerate** if $\operatorname{rank} b_{\mathcal{A}}|_X = 2\dim X$ for some $x \in X(\mathbb{C})$, i.e. the Betti map has maximal rank at some point (and so has maximal rank on an open).

Non-example. If $\dim X > g$, then X is degenerate. This is simply because the target $\mathbb{R}^{2g}/\mathbb{Z}^{2g}$ of the Betti map only has dimension $2g$.

Example. If $S = *$ is a point, then any $X \subset A$ is non-degenerate. In this case, the Betti map is an isomorphism $b_A : A(\mathbb{C}) \xrightarrow{\sim} \mathbb{R}^{2g}/\mathbb{Z}^{2g}$ of real manifolds. △

Example. If $X = A_s$ is a single fiber, then it will be non-degenerate. △

We will mostly be interested in X's which dominate S.

Example. Say we have $A \to S$ with S irreducible and $\eta = \operatorname{spec} \mathbb{C}(S)$ its generic point. If X_η is in the $(\mathbb{C}(s)/\mathbb{C})$-trace of A_η and X dominates S, then X is degenerate. Similarly, if X is a (finite union of) torsion-translate(s) of such a thing ("generically special"), then it will again be degenerate. △

Example. Say $S = A$ is an abelian variety. Take $\mathcal{A} = A \times A$ and $X = \Delta$ be the diagonal in $\mathcal{A}$. Is X degenerate? Seems like it shouldn't be since the Betti map is essentially $\mathrm{pr}_2 : \mathcal{A} \to A$, so its restriction to X should be an isomorphism. However, the previous example would suggest that it is degenerate. What's the resolution? I don't know △

Example. Say we have $X, Y \subset \mathcal{A}/S$ with X non-generate. Then, $X \times_S Y \subset \mathcal{A} \times_S \mathcal{A}$ will also be non-degenerate. The idea is that $\dim(X \times_S Y) = \dim X +$"rel dimension of $Y \to S$". Need to justify that as you move around in the Y direction, you account for $\operatorname{reldim}(Y/S)$ amount of extra freedom, and this comes from the Betti map being an iso in the fiber direction or something? I don't know, see the survey paper by Gao for a proof I think △

We want non-degeneracy to be the geometric property used in our amplification argument. For this, it will have the following key properties

- A height lower bound

- Correlation

  Let $X \subset \mathcal{A}/S$ be some closed subvariety dominating S. Assume further that the geometric generic fiber of $X \to S$ is irreducible. Also assume

  (1) $\dim X > \dim S$

  Proposition 9.13. *If X is non-degenerate, then there's an open $U \overset{open}{\subset} X$ and constants c_1, c_2 (independent of $s \in S$) so that*

  $$\left|\widehat{h}(P)\right| \geq c_1 h(\pi(P)) - c_2 \text{ for all } P \in U(\overline{\mathbb{Q}}).$$

  The proof here makes use of the Betti form defined last semester. It shows non-degeneracy of X relates to $\mathscr{L}|_X$ being big ($\mathscr{L}$ coming from principal polarization of $\mathcal{A}/S$)

  (2) $\mathbb{Z}X_s$ is Zariski dense in $\mathcal{A}_s$ for all $s \in S(\mathbb{C})$

  Rules out e.g. torsion translates of fixed part

  (3) $\mathrm{Stab}_{A_{\overline{\eta}}}(X_{\overline{\eta}})$ is finite

  Rules out e.g. $X = \mathcal{A}$

  Then, $X^{(n)} \subset \mathcal{A}^{(n)}$ will be non-degenerate for $n \geq \dim S$.

Remark 9.14. Above, $\dim X \leq \dim A - 1$. Trivial bound for non-degeneracy requires $\dim X^{(n)} \leq ng$. Note $\dim X^{(n)} = \dim S + n\dim(X \to S) \leq \dim S + n(g-1)$ and $\dim S + n(g-1) \leq ng \iff \dim S \leq n$. Thus, the bound in the correlation result is the least we could expect to always work. ○

10 Alice Lin (Harvard): Height Bounds for nondegenerate varieties, 4/6

Outline

- Recall Néron-Tate height
- Silverman-Tate theorem
- Height inequality for NT height on nondeg. subvars. of abelian schemes

10.1 Néron-Tate Height

(Reference: Section B.5 of Hindry-Silverman)

Let A be an abelian variety defined over a number field K. Let $\mathscr{L}$ be a symmetric line bundle on A (so $[-1]^*\mathscr{L} \simeq \mathscr{L}$). The **Néron-Tate height** w.r.t. $\mathscr{L}$ is defined as

$$\begin{array}{rccc} \widehat{h}_{A,\mathscr{L}}: & A(\overline{\mathbb{Q}}) & \longrightarrow & \mathbb{R}_{\geq 0} \\ & P & \longmapsto & \lim_{n\to\infty} \dfrac{h_{A,\mathscr{L}}([2^n]P)}{4^n}. \end{array}$$

Above, $h_{A,\mathscr{L}}$ is any representative of the Weil height associated to $\mathscr{L}$.

Let's generalize this to abelian schemes.

Setup (for rest of talk).

- Let S be a regular, irreducible quasi-projective variety over $\overline{\mathbb{Q}}$.
- Let $\pi : \mathcal{A} \to S$ be an **abelian scheme** (smooth, proper group scheme with geometrically integral fibers).
- Let η be the (unique) generic point of S, and let A_η be the generic fiber of π, an abelian variety over $\overline{\mathbb{Q}}(\eta) = \overline{\mathbb{Q}}(S)$.
- Assume we're given a closed immersion $\mathcal{A} \hookrightarrow \mathbb{P}^n_{\overline{\mathbb{Q}}} \times S$
- Let $\overline{S} \subset \mathbb{P}^m$ be the closure of $S \subset \mathbb{P}^m$ (recall S quasi-projective)
- Let $\overline{\mathcal{A}}$ be the Zariski closure of $\mathcal{A}$ in $\mathbb{P}^n \times \mathbb{P}^m$
- Let $\overline{\mathscr{L}} := \mathscr{O}_{\mathbb{P}^n \times \mathbb{P}^m}(1,1)|_{\overline{\mathcal{A}}}$ and $\mathscr{L} := \overline{\mathscr{L}}|_{\mathcal{A}}$
- We may and do assume that $\mathscr{L}_\eta$ is symmetric, so $[2]^*\mathscr{L}_\eta \cong \mathscr{L}_\eta^{\otimes 4}$

Notation 10.1. We let $h : \overline{\mathcal{A}}(\overline{\mathbb{Q}}) \to \mathbb{R}_{\geq 0}$ denote the absolute logarithmic height on $\mathbb{P}^n \times \mathbb{P}^m \supset \overline{\mathcal{A}}$, so h is a representative for the Weil height $h_{\overline{\mathcal{A}},\overline{\mathscr{L}}}$.

Note we still have a doubling map $[2] : \mathcal{A} \to \mathcal{A}$ defined over S, and we can use this to define a fiberwise Néron-Tate height, i.e. for $P \in \mathcal{A}(\overline{\mathbb{Q}})$ s.t. $\pi(P) = s \in S(\overline{\mathbb{Q}})$, we can define

$$\widehat{h}_{\mathcal{A},\mathscr{L}}(P) := \widehat{h}_{\mathcal{A}_s,\mathscr{L}_s}(P) = \lim_{n\to\infty} \frac{h_{\mathcal{A}_s,\mathscr{L}_s}([2^n]P)}{4^n}.$$

One then gets that

$$\left|\widehat{h}_{\mathcal{A},\mathscr{L}}(P) - h_{\mathcal{A},\mathscr{L}}(P)\right| < C_s$$

for some constant C_s depending only on s. The variation of this constant is controlled by

Theorem 10.2 (Silverman-Tate Theorem). *With the setup as above, there exists a constant $c > 0$ s.t. for all $P \in \mathcal{A}(\overline{\mathbb{Q}})$, we have*

$$\left|\widehat{h}_{\mathcal{A}}(P) - h(P)\right| \leq c \max\{1, h_{\overline{S}}(\pi(P))\}$$

This c depends on the abelian scheme $\mathcal{A}/S$ and on the choice of immersions, but not on the point P.

We will obtain this as a corollary of the following proposition

Proposition 10.3. *With setup as above, there exists some constant $c_1 > 0$ s.t. for all $P \in \mathcal{A}(\overline{\mathbb{Q}})$, the following holds*

$$|h([2]P) - 4h(P)| \leq c_1 \max\{1, h_{\overline{S}}(\pi(P))\}.$$

Proof of Theorem 10.2, assuming Proposition 10.3. We will use a telescoping sum along with the triangle inequality. Observe

$$\left|\frac{h([2^\ell]P)}{4^\ell} - h(P)\right| = \left|\sum_{m=0} \frac{h([2^{m+1}]P)}{4^{m+1}} - \frac{h([2^m]P)}{4^m}\right|$$

$$
\begin{aligned}
&\leq \sum_{m=0}^{\ell-1} \left| \frac{h([2^{m+1}]P)}{4^{m+1}} - \frac{h([2^m]P)}{4^m} \right| \\
&= \sum_{m=0}^{\ell-1} 4^{-m-1} \left| h([2][2^m]P) - 4h([2^m]P) \right| \\
&= \sum 4^{-m-1} c_1 \max\{1, h_{\overline{S}}(\pi(P))\} \\
&\leq \frac{c_1}{3} \max\{1, h(\pi(P))\}
\end{aligned}
$$

This holds for all ℓ, so we win by taking the limit as $\ell \to \infty$ and setting $c = c_1/3$. ■

Now let's prove the proposition itself.

Proof of Proposition 10.3.

(Step 1) Define $\overline{\mathcal{A}}'$ and the line bundle $\mathscr{F}'$ on $\overline{\mathcal{A}}'$

Let $[2] : \mathcal{A} \to \mathcal{A}$ be multiplication by two, and consider its graph $\Gamma_{[2]} \subset \mathcal{A} \times_S \mathcal{A}$. Let $\overline{\mathcal{A}}'$ be the closure of $\Gamma_{[2]}$ inside $\overline{\mathcal{A}} \times_S \overline{\mathcal{A}}$. The two projection maps $\overline{\mathcal{A}} \times_S \overline{\mathcal{A}} \rightrightarrows \overline{\mathcal{A}}$ give two maps $\overline{\mathcal{A}}' \rightrightarrows \overline{\mathcal{A}}$, which we'll call $\rho : \overline{\mathcal{A}}' \to \overline{\mathcal{A}}$ (projection onto first factor) and $[2] : \overline{\mathcal{A}}' \to \overline{\mathcal{A}}$ (projection onto second factor). Identify $\mathcal{A} \cong \Gamma_{[2]}$ (via the inverse of projection onto the first factor), we have $\rho|_{\mathcal{A}} = \mathrm{id}_{\mathcal{A}}$ and $[2]|_{\mathcal{A}} = [2]$. Finally, we set

$$\mathscr{F}' := [2]^*\overline{\mathscr{L}} \otimes \rho^*\overline{\mathscr{L}}^{\otimes(-4)} \in \operatorname{Pic}(\overline{\mathcal{A}}').$$

Remark 10.4. $\mathscr{F}'|_{A_\eta} = [2]^*\mathscr{L}_\eta \otimes \mathscr{L}_\eta^{\otimes -4} \simeq \mathscr{O}_{A_\eta}$ since $\mathscr{L}_\eta$ is symmetric. ∘

Fact (EGA IV-4 Cor 21.4.13, p. 361)**.** Let $f : X \to Y$ be a proper, flat morphism of regular irreducible schemes w/ regular fibers. Let $\mathscr{L}$ be a line bundle on X s.t. $\mathscr{L}|_{X_\eta}$ is trivial. Then, there exists a line bundle $\mathscr{M}$ on Y such that $\mathscr{L} \cong f^*\mathscr{M}$.

In present circumstances, this tells us that there is a line bundle $\mathscr{M}$ on S so that $\pi^*\mathscr{M} \cong \mathscr{F}'|_{\mathcal{A}}$.

(Step 2) Regularizing $\overline{S}$ to $\overline{S}' \to \overline{S}$

Since we are in characteristic 0, Hironaka tells us that there exists a proper birational morphism $b : \overline{S}' \to \overline{S}$ s.t. $\overline{S}'$ is regular. Since $S \subset \overline{S}$ was already regular, this b will be an isomorphism above S, so S is open in $\overline{S}'$.

(Step 3) Base change to $\overline{S}'$

Upshot. On $\overline{S}'$, Cartier divisor, Weil divisors, and line bundles are all the same.

Note that

$$\mathcal{A} = \overline{\mathcal{A}}' \times_{\overline{S}} S \overset{\text{open}}{\subset} \overline{\mathcal{A}}' \times_{\overline{S}} \overline{S}'$$

Let $\overline{\overline{\mathcal{A}}} \subset \mathcal{A}' \times_{\overline{S}} \overline{S}'$ be the irreducible component containing $\mathcal{A}$. We have a Cartesian square

$$
\begin{array}{ccc}
\mathcal{A} & \longrightarrow & \overline{\overline{\mathcal{A}}} \\
{\scriptstyle\pi}\big\downarrow & & \big\downarrow{\scriptstyle\overline{\pi}} \\
S & \xrightarrow{\text{open}} & \overline{S}.
\end{array}
$$

Now, $\overline{\overline{\mathcal{A}}}$ may not be regular, so we apply Hironaka once more to obtain a regular $\widetilde{\mathcal{A}}$ with a birational proper map $\widetilde{\mathcal{A}} \to \overline{\overline{\mathcal{A}}}$. Now, we still have a Cartesian square

$$\begin{tikzcd} \mathcal{A} \arrow[r] \arrow[d, "\pi"'] & \overline{\widetilde{\mathcal{A}}} \arrow[d, "\widetilde{\pi}"] \\ S \arrow[r, "\text{open}"] & \overline{S}. \end{tikzcd}$$

Everything above is regular.

(Step 4) Convert to Weil divisors.

Pull $\mathscr{F}'$ back to a line bundle $\mathscr{F}$ on $\widetilde{\mathscr{A}}$. Extend $\mathscr{M}$ from a line bundle on S to one on $\overline{S}'$ (take closures of prime divisors defining $\mathscr{M}$). Since $\widetilde{\mathscr{A}}$ is regular, $\mathscr{F} \otimes \widetilde{\pi}^* \mathscr{M}^{-1} \cong \mathscr{O}(D)$ for some Weil divisor

$$D = \sum n_i Z_i.$$

Remark 10.5. One can choose Z_i to be supported in $\widetilde{\mathcal{A}} \setminus \mathcal{A}$. This is because $D|_{\mathcal{A}}$ is principle. Now use the exact sequence

$$\bigoplus_{Z_i \text{ codim 1 in } X \setminus U} \mathbb{Z}[Z_i] \longrightarrow \mathrm{Cl}(X) \longrightarrow \mathrm{Cl}(U) \longrightarrow 0.$$

○

(Step 5) Create an effective divisor

For each Z_i, $\widetilde{\pi}(Z_i) =: Y_i \subset \overline{S}' \setminus S$. Note there will exist a prime Weil divisor E_i on $\overline{S}'$ so that $\mathrm{supp}(E_i) \supset Y_i$. Hence, $\widetilde{\pi}^* E_i$ is a divisor on $\widetilde{\mathcal{A}}$ whose support contains Z_i. Define divisors

$$D_{\pm} := \widetilde{\pi}^* \left(\sum |n_i| \, E_i \right) \pm \sum n_i Z_i$$

which will both be effective by construction. They are also both supported on $\bigcup_i \mathrm{supp}\,(\widetilde{\pi}^* E_i)$.

(Step 6) Pass back to line bundles & get height inequality

Recall 10.6. The Weil height machinery is additive and functorial (both up to $O(1)$). Furthermore, if D is effective, then the height $h_{X,D}(P) \geq O(1)$ for $P \notin \mathrm{supp}(D)$. Finally, for D ample and E an arbitrary divisor, there exists a constant c s.t. $h_{X,E}(P) \leq c h_{X,D}(P) + O(1)$.

Consider the line bundle $\mathscr{E} := \mathscr{O}\left(\sum |n_i| \, E_i\right)$ on $\overline{S}'$, and note that

$$\mathscr{O}(D_{\pm}) \cong \widetilde{\pi}^* \mathscr{E} \otimes \left(\mathscr{F} \otimes \widetilde{\pi}^* \mathscr{M}\right)^{\otimes \pm 1}.$$

Thus,

$$O(1) \leq h_{\widetilde{\mathcal{A}}, \mathscr{O}(D_{\pm})}(\widetilde{P}) = h_{\overline{S}', \mathscr{E}}(\widetilde{\pi}(P)) \pm \left(h_{\widetilde{\mathcal{A}}, \mathscr{F}}(\widetilde{P}) - h_{\overline{S}', \mathscr{M}}(\widetilde{\pi}(P)) \right)$$

(with lower bound holding for $\widetilde{P}$ outside the support of $D_{\pm}$). If one rearranges this, they get

$$\left| h_{\overline{\mathcal{A}}', \mathscr{F}'}(P) \right| \leq h_{\overline{S}', \mathscr{E}}(\pi(P)) + h_{\overline{S}', \mathscr{M}}(\pi(P)) + O(1).$$

Let's consider each side separately

(LHS) Consider $P \in \mathcal{A}(\overline{\mathbb{Q}})$. Then, $\mathscr{F}'|_{\mathcal{A}} = [2]^*\mathscr{L} \otimes \mathscr{L}^{\otimes -4}$ so the LHS is

$$|h_{\mathcal{A},\mathscr{L}}([2]P) - 4h_{A,\mathscr{L}}(P)|.$$

Note that $h_{\mathcal{A},\mathscr{L}}$ is the height we called h earlier.

(RHS) Let $i : \overline{S} \hookrightarrow \mathbb{P}^m$ be our closed immersion and set $T := i^*\mathscr{O}_{\mathbb{P}^m}(1)$. There are constants $c, c' > 0$ s.t.

$$h_{\overline{S},\mathscr{M}} < ch_{\overline{S},T} + O(1) \text{ and } h_{\overline{S},\mathscr{E}} < c'h_{\overline{S},T} + O(1).$$

Note that $h_{\overline{S},T}$ is the naive absolute logarithmic height on $\mathbb{P}^m$.

Putting these together, we obtain

$$|h([2]P) - 4h(P)| \leq (c + c')h_{\overline{S},T}(\pi(P)) + O(1) \leq C \max\{1, h_{\overline{S}}(\pi(P))\},$$

where C is the max of $c + c'$ and $O(1)$. Note this holds for $P \notin \bigcup \text{supp}(\pi^* E_i)$. To win, consider the abelian scheme $\mathcal{A}|_{E_i} \to E_i$ and apply noetherian induction. ■

10.2 Application to Uniform Mordell

Theorem 10.7 (Theorem 1.6 in [DGH])**.** *Let $\pi : \mathcal{A} \to S$ as above. Let X be a non-degenerate closed, irreducible subvariety of $\mathcal{A}$, defined over $\overline{\mathbb{Q}}$, that dominates S. Then, there exists constants $c_1, c_2 > 0$ and a Zariski dense open subset U of X such that for all $P \in U(\overline{\mathbb{Q}})$, one has*

$$\widetilde{h}_{\mathcal{A}}(P) \geq c_1 h(\pi(P)) - c_2.$$

Recall 10.8 (Proposition 4.1 in [DGH])**.** Let X be as above. There exists a constant $c_3 > 0$ satisfying the following. For $N \in 2^{\mathbb{N}}$, there is a Zariski open $U_N \subset X$ defined over $\overline{\mathbb{Q}}$ and a constant $c_4(N)$ s.t. for all $P \in U_N(\overline{\mathbb{Q}})$, one has

$$h([N]P) \geq c_3 N^2 h(\pi(P)) - c_4(N).$$

Proof of Theorem 10.7. Choose N later. Take $P \in U_N(\overline{\mathbb{Q}})$. Apply Theorem 10.2 and then the recall to get

$$\widehat{h}_A([N]P) \geq h([N]P) - c_0(1 + h_{\overline{S}}(\pi(P))) \geq c_3 N^2 h_{\overline{S}}(\pi(P)) - c_4(N) - c_0(1 + h_{\overline{S}}(\pi(P))).$$

The line bundle $\mathscr{L}$ used to define our heights is symmetric, so $[N]^*\mathscr{L} \cong \mathscr{L}^{\otimes N^2}$ and we know $\widehat{h}_{\mathcal{A}}([N]P) = N^2 \widehat{h}_{\mathcal{A}}(P)$. Using this above gives

$$\widehat{h}_{\mathcal{A}}(P) \geq \left(c_3 - \frac{c_0}{N^2}\right) h(\pi(P)) - \left(\frac{c_4(N) + c_0}{N^2}\right).$$

This is almost what we'd like, except there's a dependency on N in the constants which we don't want. To fix this, pick k smallest so that $N = 2^k \geq 2c_0/c_3$. Then, $c_3 - c_0/N^2 > c_3/2$. Now that N is fixed, we get the above equality for all $P \in U_N(\overline{\mathbb{Q}})$ and this U_N is the U is the theorem statement. ■

11 Niven Achenjang (MIT): Proof of the New Gap Principle I, 4/13 – notes here

12 Sam Marks (Harvard): ???, 10/4

Talks here and below are from Kisin Seminar

Let $C/\overline{\mathbb{Q}}$ be a curve of genus $g \geq 2$, and fix a basepoint $P_0 \in C(\overline{\mathbb{Q}})$ giving an embedding $j : C \hookrightarrow J := \operatorname{Jac}(C)$.

Recall 12.1. The **theta divisor** $\Theta := \{j(P_1) + \cdots + j(P_{g-1}) : P_i \in C\} \in \operatorname{Div}(J)$ is irreducible and ample. We write $\Theta^- = [-1]^*\Theta$.

Recall 12.2 (height machine)**.** If $A/\overline{\mathbb{Q}}$ is an abelian variety, and $D \in \operatorname{Div}(A)$, we get a normalized height $\widehat{h}_D : A(\overline{\mathbb{Q}}) \to \mathbb{R}$. Some properties:

- If D symmetric, then $\widehat{h}_D$ is a quadratic form.
- If D is ample, then $\widehat{h}_D$ is positive definite.

Notation 12.3. We write $\widehat{h} = \widetilde{h}_{\Theta+\Theta^-} : J(\overline{\mathbb{Q}}) \longrightarrow \mathbb{R}_{\geq 0}$, and let $\langle -, - \rangle$, $|\cdot|$ denote the associate bilinear form and norm.

Theorem 12.4. *Let $\varepsilon > 0$. There exists $R = R(C, P_0, \varepsilon) > 0$ and $\kappa = \kappa(g)$ such that for all distinct $P \neq Q \in C(\overline{\mathbb{Q}})$ satisfying $|q| \geq |p| \geq R$ and $\frac{\langle p,q \rangle}{|p||q|} \geq \frac{3}{4} + \varepsilon$ (where $p = j(P), q = j(Q)$), then*

> ***(Mumford's Gap principle)*** $|q| \geq 2\,|p|$
>
> ***(Vojta's inequality)*** $|q| \leq \kappa\,|p|$

Remark 12.5. For uniform results, need both of the above. For Falting's theorem, only need Vojta since there are only finitely many points of bounded height. ◦

Corollary 12.6.

(1) *If $\Gamma \subset J(\overline{\mathbb{Q}})$ is finitely generated, then*

$$\#\left\{P \in C(\overline{\mathbb{Q}}) : j(P) \in \Gamma, |j(P)| \geq R\right\} \leq (\log_2 \kappa + 1)7^{\operatorname{rank}\Gamma}.$$

(2) ***(Mordell's Conjecture)*** *If C is defined over a number field K, then $\#C(K) < \infty$.*

Proof. **(1)** Use following fact

Fact. $(\Gamma \otimes_{\mathbb{Z}} \mathbb{R}, \langle \cdot, \cdot \rangle)$ can be covered by $\leq 7^{\operatorname{rank}\Gamma}$ cones of vertex angle $< \cos^{-1}\left(\frac{3}{4} + \varepsilon\right)$.

If $P_1, \ldots, P_n \in \left\{P \in C(\overline{\mathbb{Q}}) : j(P) \in \Gamma, |j(P)| \geq R\right\}$ and we write $p_i = j(P_i) \in \Gamma$ with increasing heights $|p_1| \leq \cdots \leq |p_n|$. The gap principle + Vojta then tells us that

$$2^{n-1}\,|p_1| \leq |p_n| \leq \kappa\,|p_1| \implies n \leq \log_2 \kappa + 1.$$

This + the fact gives the bound on the number of large points.

(2) Take $\Gamma = J(K)$ in **(1)**. Then, $C(K) \simeq j(C(\overline{\mathbb{Q}})) \cap \Gamma$. Split into big points and small points. Finitely many big points by **(1)**, and finitely many small points by Northcott.[11] ∎

If you can make a uniform choice of R, get uniform bound on large points. However, bound on small points is not uniform. Need to make this uniform in order to get a uniform version of Faltings' theorem. We won't do that today.

We will prove the Gap principle though. To do so, we'll need a bit of setup.

Proposition 12.7.

(1) *There exists $\alpha \in J(\overline{\mathbb{Q}})$ such that $t_\alpha^*\Theta = \Theta^{-1}$*

(2) *For $c \in J(\overline{\mathbb{Q}})$, write*

$$\begin{array}{rccc} j_c: & C & \longrightarrow & J \\ & P & \longmapsto & j(P) + c. \end{array}$$

Then,

(a) $j_c^*\Theta^- \sim g[P_0] - c \in \operatorname{Pic}(C)$

(b) $j_c^*\Theta \sim g[P_0] - c + \alpha$.

Proof. **(1)** Let $\alpha = j(K_C)$. Since Θ, Θ^- are irreducible, suffices to show $t_\alpha^*\Theta \subset \Theta^-$. Let $x \in t_\alpha^*\Theta$, so $x = t - \alpha$ for some $t \in \Theta$. Write $t = j(D)$ for some effective degree $g-1$ divisors D on C. By Riemann-Roch,

$$h^0(K_C - D) = \deg D - g + 1 + h^0(D) = h^0(D) \geq 1,$$

so $K_C - D \sim E$ for some effective E of degree $g-1$. Thus, $x = t - \alpha = j(D) - j(K_C) = -j(E) \in \Theta^-$

(2) We'll use a lemma

Lemma 12.8. *There exists $U \subset \operatorname{Sym}^g C$ open, nonempty such that $\pi : \operatorname{Sym}^g C \to J$ is injective on U.*

Proof. By semi-continuity, if $D \in \operatorname{Div}(C)$ is effective, there's an open $U \subset C$ such that $h^1(D+Q) = h^1(D) - 1$ for all $Q \in U$. Iterate the lemma, starting with $D = 0$, $h^1(D) = g$ to get nonempty open $U \subset \operatorname{Sym}^g C$ such that $h^0(P_1 + \cdots + P_g) = 0 + g - g + 1 = 1$ – where the 0 is an h^1 – for all $(P_1, \ldots, P_g) \in U \subset \operatorname{Sym}^g C$. ∎

For any $D \in U$ above, $\pi^{-1}(\pi(D)) = \{E \sim D : E \text{ effective}\} \cong \mathbb{P}\,\mathrm{H}^0(D) \cong \mathbb{P}^0$ (so the lemma says there's only one).

Shrink U so that if $(P_1, \ldots, P_g) \in U$, then the P_i are distinct (remove diagonals). Let $V = \pi(U) \subset J$ a nonempty open. Let $c \in -V$, and write $-c = j(P_1) + \cdots + j(P_g)$ (unique description by lemma) with P_i distinct. Let $t \in \Theta^-$. Then,

$$j_c^{-1}(\{t\}) = \{P \in C : j(P) - t = -c = j(P_1) + \cdots + j(P_g)\} = \begin{cases} \{P_{i'}\} & \text{if } -t = \sum_{i \neq i'} j(P_i) \\ \emptyset & \text{otherwise.} \end{cases}$$

[11] Maybe not this simple. Maybe worry about torsion points or something. Actually, sounds like you can show $\Gamma \cap C(\overline{\mathbb{Q}})$ is finite. Look at torsion points + points of bounded height in lattice $\Gamma \to \Gamma \otimes \mathbb{R}$

From this, we see that $j_c^*\Theta^- = (P_1) + \cdots + (P_g) \sim g(P_0) - c$.[12] This proves **(2a)** for $c \in -V$.

By theorem of the square, $t^*_{a+b-c} = t^*_a + t^*_b - t^*_c$. This is therefore also true with the t's replaced by j's. To finish, we note that

$$(-V) \times (-V) \times (-V) \twoheadrightarrow J, \quad (a, b, c) \mapsto a + b - c$$

is surjective.

(2b) $j_c^*\Theta = g(P_0) - c + \alpha$ follows from **(2a)** + **(1)** since

$$j_c^*\Theta = j_c^* t^*_{-\alpha}\Theta^- = j^*_{c-\alpha}\Theta^- = g(P_0) - c + \alpha.$$

∎

Proposition 12.9. *Let $\delta = s^*\Theta - p_1^*\Theta - p_2^*\Theta \in \mathrm{Div}(J \times J)$.*

This is secretly the Poincaré bundle, i.e. $\delta = (\mathrm{id} \times \varphi_\Theta)^* \wp$

(1) $\Delta^*\delta = \Theta + \Theta^-$

(2) $(j \times j)^*\delta = \{P_0\} \times C + C \times \{P_0\} - \Delta$

Proof. **(1)** Use functoriality of pull backs:

$$\begin{aligned}\Delta^*\delta &= (s \circ \Delta)^*\Theta - (p_1 \circ \Delta)^*\Theta - (p_2 \circ \Delta)^*\Theta \\ &= [2]^*\Theta - 2\Theta \\ &= \Theta + \Theta^-\end{aligned}$$

(2) Use seesaw. It suffices to show that LHS $\sim$ RHS, when pulled back to $\{P\} \times C$ for $P \in U \subset C$ for some nonempty open U. Let $\iota_P : \{P\} \times C \hookrightarrow C \times C$. For $P \neq P_0$,

Unclear how correct this is

$$\iota_P^*(RHS) = 0 - (P_0) - (P)$$

and

$$\begin{aligned}\iota_P^*(LHS) &= (s \circ (j \times j) \circ \iota_P)^*\Theta - (p_1 \circ (j \times j) \circ \iota_P)^*\Theta - (p_2 \circ (j \times j) \circ \iota_P)^*\Theta \\ &= g(P_0) - j(P) + \alpha - 0 - (g(P_0) + \alpha) \\ &= -j(P) = P_0 - P\end{aligned}$$

First map above sends $Q \mapsto j(P) + j(Q)$, so is $j_{j(P)}$. Second map is $Q \mapsto j(P)$. Thirs map is j. ∎

Corollary 12.10 (Mumford's Formula). *For $P, Q \in C(\overline{\mathbb{Q}})$, $p = j(P), q = j(Q)$, one has*

$$\langle p, q\rangle = \frac{1}{2g}\left(|p|^2 + |q|^2\right) - h_\Delta(P, Q) + O(|p| + |q| + 1).$$

Proof.

$$\langle p, q\rangle = \frac{1}{2}\left(\widehat{h}(p+q) = \widehat{h}(p) - \widehat{h}(q)\right)$$

[12] Only computed the support of the pullback. Need to know it's actually degree g. We'll ignore this. Need intersection theory to actually prove it.

$$
\begin{aligned}
&= \frac{1}{2}\left(\widehat{h}_\delta(\Delta_J(p+q)) - \widehat{h}_\delta(\Delta_J(p)) - \widehat{h}_\delta(\Delta_J(q))\right)\\
&= \widehat{h}_\delta(p,q)\\
&= \widehat{h}_\delta(j(P), j(Q))\\
&= h_{(j\times j)^*\delta}(P,Q) + O(1)\\
&= h_{(P_0)}(P) + h_{(P_0)}(Q) - h_\Delta(P,Q) + O(1)\\
&= \frac{1}{g}\left(h_{j^*\Theta}(P) + h_{j^*\Theta^-}(Q)\right) - h_\Delta(P,Q) + O(1)\\
&= \frac{1}{g}(h_{\Theta^-}(p) + h_{\Theta^-}(q)) - h_\Delta(P,Q) + O(1)\\
&= \frac{1}{2g}\left(|p|^2 + |q|^2 - \widehat{h}_{\Theta-\Theta^-}(p) - h_{\Theta-\Theta^-}(q)\right) - h_\Delta(P,Q) + O(1)
\end{aligned}
$$

(get third equality by expanding things out). Now we want to show $\widehat{h}_{\Theta-\Theta^-}(p) + \widehat{h}_{\Theta-\Theta^-}(q) = O(|p| + |q| + 1)$.

Lemma 12.11. *If $E \in \mathrm{Pic}^0(J)$ and $D \in \mathrm{Div}(J)$ is ample and symmetric, then $\widehat{h}_E(p) = O(\widehat{h}_D(p)^{1/2})$.*

Proof. If D ample, then $\varphi_D : J \to \mathrm{Pic}^0(J)$, $a \mapsto t_a^*D - D$ is an isogeny. There exists $a \in J$ such that $E \sim t_a^*D - D$ (isogenies are surjective). Hence,

$$\widehat{h}_E(p) = \widehat{h}_{t_a^*D-D}(p) = \widehat{h}_D(p+q) - \widehat{h}_D(a) - \widehat{h}_D(p) = 2\langle p, a\rangle_D \leq 2\,|p|_D\,|a|_D\,.$$

by Cauchy-Schwarz in the last inequality. ■

■

This just leaves the deduction of the gap principle.

Proof of Mumford's Gap Principle. By corollary, have $c > 0$ such that for $R \gg 0$, for all $P \neq Q \in C(\overline{\mathbb{Q}})$ with $|q| \geq |p| \geq R$:

$$\langle p, q\rangle \leq \frac{1}{2g}\left(|p|^2 + |q|^2\right) - h_\Delta(P,Q) + c(|p| + |q|)$$

(This is just what big-O means). Since P, Q are distinct and Δ is effective, we know $h_\Delta(P,Q) \geq 0$. Thus,

$$\frac{3}{4} + \varepsilon \leq \frac{\langle p, q\rangle}{|p|\,|q|} \leq \frac{1}{2g}\left(\frac{|q|}{|p|} + 1\right) + c\left(\frac{1}{|p|} + \frac{1}{|q|}\right).$$

Taking R larger, get

$$\frac{3}{4} \leq \frac{1}{2g}\left(\frac{|q|}{|p|} + 1\right)$$

and rearrange to see $2 \leq \frac{3}{2}g - 1 \leq |q|\,/\,|p|$. ■

Next semester we'll prove a relative version of our main theorem today.

Let $S/\overline{\mathbb{Q}}$ be an irreducible variety, and let $\pi : \mathcal{A} \to S$ be an abelian scheme of relative dimension ≥ 1. Let $\mathscr{L}$ be a relatively ample, symmetric line bundle on $\mathcal{A}/S$. This choice induces a height function $\widehat{h}_{\mathscr{L}} : \mathcal{A}(\overline{\mathbb{Q}}) \to \mathbb{R}_{\geq 0}$. Let $\mathcal{M}$ be an ample line bundle on a compactification $\overline{S}$ of S. This induces a height $h_{\mathcal{M}} : \overline{S}(\overline{\mathbb{Q}}) \to \mathbb{R}$.

Theorem 12.12. *Let $\mathcal{C} \subset \mathcal{A}$ be an irreducible closed subvariety dominating S such that $\mathcal{C}$ is a flat family of genus $g \geq 2$ curves. Then, there is a constant $c = c(\pi, \mathscr{L}, \mathcal{M}, \mathcal{C}) \geq 1$ such that for all $s \in S(\overline{\mathbb{Q}})$, $\Gamma \subset \mathcal{A}_s(\overline{\mathbb{Q}})$ subgroup or rank $\rho < \infty$:*

$$\#\left\{P \in \mathcal{C}_s(\overline{\mathbb{Q}}) \cap \Gamma : \widehat{h}_{\mathscr{L}}(P) > c\max\{1, h_{\mathcal{M}}(S)\}\right\} < c^{\rho}.$$

Update: we won't do this next semester.

13 Sasha? (Harvard): Roth

We want to prove Roth's theorem. This will depend on Roth's Lemma, and that Lemma will be used to give Vojta's inequality.

Theorem 13.1 (Roth)**.** *Let $\alpha \in \overline{\mathbb{Q}}$. For all $\varepsilon > 0$, there are at most finitely many $p/q \in \mathbb{Q}$ such that*

$$\left|\alpha - \frac{p}{q}\right| \leq \frac{1}{q^{2+\varepsilon}}$$

Remark 13.2. If α is irrational, then infinitely many p/q such that

$$\left|\alpha - \frac{p}{q}\right| \leq \frac{1}{q^{2}}$$

so Roth's theorem is optimal. ○

Let's start with a weaker result.

Theorem 13.3 (Liouville)**.** *For $d = [\mathbb{Q}(\alpha) : \mathbb{Q}]$, for all $\varepsilon > 0$, there are only finitely many p/q such that*

$$\left|\alpha - \frac{p}{q}\right| \leq \frac{1}{q^{d+\varepsilon}}$$

Proof. Let $f(x) = a_d x^d + \cdots + a_1 x + a_0$ be the minimal polynomial of α, so $a_i \in \mathbb{Z}$. Observe

(a) If $|\alpha - p/q| < 1$, then

$$|f(p/q)| \leq \left|\alpha - \frac{p}{q}\right| \cdot C(\alpha).$$

Why? Use Taylor

$$|f(p/q)| = \left|f(\alpha) + \left(\frac{p}{q} - \alpha\right) f'(\alpha) + \left(\frac{p}{q} - \alpha\right)^2 \frac{f''(\alpha)}{2!} + \dots\right| \leq \left|\alpha - \frac{p}{q}\right| C(\alpha)$$

using p/q and α close to each other.

(b) $f(p/q) = 0$ or $|f(p/q)| \geq 1/q^d$. This is because

$$f\left(\frac{p}{q}\right) = \frac{a_d p^d + a_{d-1} q^{d-1} p + \cdots + a_0 q^d}{q^d} \in \frac{1}{q^d}\mathbb{Z}.$$

Together, these give infinitely many p/q with $|p/q - \alpha| < 1/q^d$ forces $f(p/q) = 0$ as

$$\frac{1}{q^d} \le \left| f\left(\frac{p}{q}\right) \right| \le \left| \alpha - \frac{p}{q} \right| \cdot C(\alpha) \le \frac{C(\alpha)}{q^{d+\varepsilon}}$$

for all such p/q and then **(b)** gives $f(p/q) = 0$. This would force $f = 0$, a contradiction (I think). ∎

In previous proof, could replace f by g such that $g^{(i)}(\alpha) = 0$ for $i < N$. This by itself does not work, but will work if we use polynomials in several variables.

Definition 13.4. Say $P \in \mathbb{C}[x_1, \dots, x_m]$ along with a point $(\beta_1, \dots, \beta_m) \in \mathbb{C}^m$. Choose also integers $r_1, \dots, r_m \in \mathbb{Z}$. We define the **index**

$$\operatorname{Ind}_{(\beta,r)}(P) := \min \left\{ \frac{i_1}{r_1} + \cdots + \frac{i_m}{r_m} \,\middle|\, (\partial_{i_1,\dots,i_m} P)(\beta) \neq 0 \right\}.$$

This is some sort of weighted measure of a singularity of a hypersurface? Can think of it as getting "the leading term." In one variable, this is the multiplicity of P at β, divided by r.

Definition 13.5. If $P \in \overline{\mathbb{Q}}[x_1, \dots, x_m]$, $P = \sum a_I x^I$, its height is

$$h(P) = \max h(a_I),$$

the maximum (logarithmic) height of its coefficients.

Proposition 13.6. *Let α be an algebraic integer. There exists $B(\alpha)$ so that for all $\delta > 0$ and any m such that $e^{\frac{\delta^2 m}{4}} > 2d$ (where $d = [\mathbb{Q}(\alpha) : \mathbb{Q}]$), for any $r_1, \dots, r_m \in \mathbb{Z}_{\ge 0}$ there exists $P \in \mathbb{Z}[x_1, \dots, x_m]$ such that*

(1) $\deg_{x_i} P \le r_i$ *for all i; and*

(2) $\operatorname{Ind}_{(\alpha,\dots,\alpha;r_1,\dots,r_m)} P \ge \frac{m}{2}(1 - \delta)$*; and*
(point on the diagonal)

(3) $h(P) \le B(\alpha)(r_1 + \cdots + r_m)$

Maybe $B(\alpha)$ also depends on δ

How do you prove something like this. Do some (integral) linear algebra? Have P with bounded coefficients giving $(r_1 + 1) \dots (r_m + 1)$ degrees of freedom. **(2)** essentially gives a linear condition and one ends up with something like $\le N/2$ conditions. Then Siegal's lemma gives a solution of sufficiently small height.

Probably proof of this in chapter 3 of Levent's thesis

Non-example. Consider $P = f(x_1)^{N_1} \dots f(x_m)^{N_m}$ with f the minimal polynomial of α. f is of degree d, so first condition says $dN_i \le r_i$. Indices of products are easy to compute (Ind gives a valuation), so $\operatorname{Ind} = \frac{N_1}{r_1} + \cdots + \frac{N_m}{r_m} \le \frac{m}{d}$. We want at least $m/2$, so this doesn't quite work. This 2 vs d is the 2 vs d in the final results of Liouville vs. Roth.

Remark 13.7. The proposition is easy to prove if $r_1 = \cdots = r_m$. We need a polynomial vanishing on diagonal point. How about we take $P = \left(\prod_{i<j}(x_i - x_j)\right)^N$? This works. ◦

We won't prove this proposition, but the polynomial it gives is the analogue of the minimal poly in Liouville's proof. We need analogues of **(a)**,**(b)**.

Remark 13.8. **(a)** + **(b)** of that proof give that if $\left|\alpha - \frac{p}{q}\right| < \frac{1}{q^{d+\varepsilon}}$ and q can be large, then $f(p/q) = 0$. ∘

Proposition 13.9 (Analogue of **(a)**+**(b)**)**.** *For any $\varepsilon > 0$, there is a $\delta = \delta(\varepsilon)$ so that for all $m, r_1, \ldots, r_m$,P as in Proposition 13.6: if $\beta_1, \ldots, \beta_m \in \mathbb{Q}$ so that*

- *(1)* $|\alpha - \beta_i| < \frac{1}{H(\beta_i)^{2+\varepsilon}}$
- *(2)* $H(\beta_i) \geq C(\varepsilon)$
- *(3)* $\max(r_i h(\beta_i)) \leq \min(r_i h(\beta_i)) \cdot (1+\varepsilon)$*, so the β's are 'balanced'*

Then

$$\operatorname{Ind}_{(\beta_1,\ldots,\beta_m;r_1,\ldots,r_m)} P > m\varepsilon.$$

Think: want to evaluate $P(\beta_1, \ldots, \beta_m)$ by Taylor expanding at α:

$$P(\beta_1, \ldots, \beta_m) = \sum \partial_I P(\alpha, \ldots, \alpha) \cdot \frac{(\beta_1 - \alpha)^{i_1}}{i_1!} \cdots \frac{(\beta_m - \alpha)^{i_m}}{i_m!}$$

(something like this). The hypotheses tell us that many of the early terms vanish, and the remaining terms are small, so $|P(\beta_1, \ldots, \beta_m)|$ must be small. Because that β's are rational, if $P(\beta_1, \ldots, \beta_m)$ is nonzero, it's bounded away from zero, so if it's sufficiently small, then it must vanish. Something like this.

Proposition 13.10 (**Roth's Lemma**, not most general case)**.** *There exits $\eta(\varepsilon), C(\varepsilon)$ so that for any $\varepsilon > 0$, any $m > 0$, any $r_1, \ldots, r_m \in \mathbb{Z}$, any $P \in \mathbb{Q}[x_1, \ldots, x_m]$, and any $(\beta_1, \ldots, \beta_m) \in \mathbb{Q}$ satisfying*

- *(1)* $r_{i+1} \leq r_i \eta(\varepsilon)$
- *(2)* $\deg_{x_i} P \leq r_i$
- *(3) For all i, $r_i h(\beta_i) \geq C(\varepsilon)(h(P) + mr_1)$* r_1 largest by **(1)**

Then,

$$\operatorname{Ind}_{(\beta_1,\ldots,\beta_m,r_1,\ldots,r_m)} P \leq m\varepsilon.$$

Let's first use this to prove Roth.

Proof of Roth's Theorem. Assume wlog that $\alpha \in \overline{\mathbb{Z}}$. Assume there are infinitely many $\beta \in \mathbb{Q}$ such that $|\alpha - \beta| \leq \frac{1}{H(\beta)^{2+\varepsilon}}$. Use Proposition 13.9 to go from ε to $\delta(\varepsilon)$. Choose n such that $e^{\delta^2 m/4} > 2d$. Now let's choose β's. Say we have $h(\beta_1) \geq D(\alpha, \varepsilon)$ (some constant), and then choose $\beta_2, \ldots, \beta_m$ so that $h(\beta_{i+1}) > h(\beta_i)/\eta(\varepsilon)$. Choose $r_1, \ldots, r_m$ now so that the hypotheses of Proposition 13.9**(3)** are satisfied. Choose P by Proposition 13.6. Now, we conclude with opposing inequalities (from Propositions 13.10 and 13.6), which is bullshit. ■

I'm not sure what went on in that proof, but hopefully I wrote down enough to reconstruct the argument...

Proof of Proposition 13.10. Homogenize P and consider $X = \{P = 0\} \subset \mathbb{P}^1_{[x_1:y_1]} \times \ldots \times \mathbb{P}^1_{[x_m:y_m]}$. For $\sigma \in R$, let

$$Z_\sigma = V\left(\partial_{i_1,\ldots,i_m} P : \frac{i_1}{r_1} + \cdots + \frac{i_m}{r_m} < \sigma\right)$$

(a subscheme, likely non-reduced). Note that $Z_\sigma(\overline{\mathbb{Q}})$ is points β with $\mathrm{Ind}_\beta P > \sigma$. In particular, $Z_0 = X$. We want $(\beta_1, \ldots, \beta_m) \in Z_{m\varepsilon}(\mathbb{Q}) \implies h(\beta_i)$ small. This will allow us to arrive at a contradiction (of hypothesis **(3)**).

We need something special about $Z_{m\varepsilon}$ to obtain a result like this. Note that

$$X = Z_0 \supset Z_\varepsilon \supset Z_{2\varepsilon} \supset \cdots \supset Z_{m\varepsilon} \ni \beta$$

Since $\dim X = m-1$ (hypersurface) and $\dim Z_{m\varepsilon} \geq 0$ (nonempty), there must be some i so that $\dim Z_{i\varepsilon} = \dim Z_{(i+1)\varepsilon}$. Thus $Z_{i\varepsilon}^{red}$ and $Z_{(i+1)\varepsilon}^{red}$ must share some irreducible component Z. Now we appeal to

Theorem 13.11 (**Falting's product theorem**). *For any $\varepsilon > 0$, there exists $\eta(\varepsilon)$ such that if $r_{i+1} \leq \eta(\varepsilon) r_i$, $\deg_{x_i} P \leq r_i$, and Z is a common irreducible component of $Z^{red}_{\sigma,\overline{\mathbb{Q}}}, Z^{red}_{\sigma+\varepsilon,\overline{\mathbb{Q}}}$, then*

(1) *Z is a product*

$$Z = X_1 \times \ldots \times X_m$$

with each X_i a point or all of $\mathbb{P}^1_{\overline{\mathbb{Q}}}$.

(2)

$$\sum_{\dim X_i = 0} r_i h(x_i) \leq C(\varepsilon)(h(P) + m r_1).$$

This will finish Roth's lemma by taking $\sigma = i\varepsilon$. Get $\beta \in X_1 \times \ldots \times X_m$ so there exists i so that $\{\beta_i\} = X_i$ ($\dim Z < m$). Then,

$$r_i h(\beta_i) \leq \sum r_i h(x_i) \leq C(\varepsilon)(h(P) + m r_1),$$

a contradiction to assumption **(3)**.

Why's Falting's product theorem true? Say $m = 2$, so we're looking at a curve $Z \subset \mathbb{P}^1 \times \mathbb{P}^1$. We'd like to show $Z = x_1 \times \mathbb{P}^1$ or $\mathbb{P}^1 \times x_2$. Look at tangent bundles (over smooth locus of Z or even just at generic point?) $T_Z \subset T_{\mathbb{P}^1 \times \mathbb{P}^1}|_Z \supset p_1^* T_{\mathbb{P}^1}$ (recall $r_2 \leq \eta(\varepsilon) r_1$ is smaller). If generically, $T_Z = \mathrm{pr}_1^* T_{\mathbb{P}^1}$, then we're done, so assume $T_Z \cap \mathrm{pr}_1^* T_{\mathbb{P}^1} = 0$ generically (horizontal vectors generically not tangent). Let $I_\sigma \subset \mathscr{O}_{\mathbb{P}^1 \times \mathbb{P}^1}$ be the ideal sheaf of $\mathbb{Z}_\sigma$. Note $I_\sigma \subset I_Z$. For all i_1, i_2 with $i_1/r_1 + i_2/r_2 \leq \varepsilon$, we have

$$\partial_{i_1, i_2}(I_\sigma) \subset I_{\sigma+\varepsilon} \subset I_Z.$$

Example. $I_{Z_\sigma} = (x^a) \subset I_Z = (x) \subset k[x,y]$ quite non-reduced. △

Let $\mathrm{mult}_\eta Z_\sigma := \mathrm{length}_{\mathscr{O}_{Z,\eta}} \mathscr{O}_{Z_\sigma, \eta}$ with $\eta \in Z$ generic point, and we know this is at least

$$\dim \frac{\{D \text{ diff op on } \mathbb{P}^1 \times \mathbb{P}^1 \mid D(I_\sigma) \subset I_Z\}}{\{D : D(I_Z) \subset I_Z\}} \geq \varepsilon r_1$$

(last bound coming from taking $\partial_{i_1,0}$ with $i_1/r_1 + 0/r_2 \leq \varepsilon$). To show $Z = x_1 \times \mathbb{P}^1$, suffices to show $Z \cdot p_1^* \mathscr{O}(1) = 0$. Let $H_1 = p_1^* \mathscr{O}(1)$. Then,

$$\mathbb{Z} \ni Z \cdot H_1 \leq \frac{1}{\mathrm{mult}_Z Z_\sigma} Z_\sigma|_Z \cdot H_1 \leq \frac{1}{\mathrm{mult}_Z Z_\sigma}(r_1 H_1 + r_2 H_2) \cdot H_1 = \frac{r_2}{\mathrm{mult}_Z Z_\sigma} \leq \frac{1}{\varepsilon} \frac{r_2}{r_1}$$

(note $Z \subset \mathbb{P}^1 \times \mathbb{P}^1$ cut out by one equation with degree small than that of P. The equation will be some derivative). We have a small integer, so better have $Z \cdot H_1 = 0$. For $m > 2$, do some induction? ■

14 Si Ying (Harvard): ?? (11/8)

Let $C \to S$ be a relative curve of genus $g \geq 2$. Consider the Jacobian $\mathrm{Jac}(C) \to S$, an abelian scheme.

Goal. We want a bound on # of points $P \in C_s(\overline{\mathbb{Q}})$ which have small height, relative to the base. We would like this bound to be uniform (independent of s).

We'll actually work in a slightly more general setup.

Setup. $A \to S$ abelian scheme, defined over $\overline{\mathbb{Q}} =: k$. Assume

(1) S is regular, irreducible, and quasi-projective. Also assume there is a compactification $\overline{S} \hookrightarrow \mathbb{P}^m_k$ of S.

(2) There is a closed immersion $A \hookrightarrow \mathbb{P}^n_k \times S$ over S. Note

$$A \hookrightarrow \mathbb{P}^n_K \times S \subset \mathbb{P}^n_k \times \overline{S} \hookrightarrow \mathbb{P}^n_k \times \mathbb{P}^m_k.$$

(3) There is a symmetric, ample line bundle $\mathscr{L}$ on A with embedding on generic fiber induced by $\mathscr{L}|_{K(A)}$.

(4) Let's also assume that $A \to S$ is principally polarized and has a given level ℓ-structure (with $\ell \geq 3$). This is not too important[13]

To get above, start with symmetric ample L on generic fiber $A_{k(S)}$. Spread out to $\mathscr{L}$ on all of A. Get closed immersion $A \hookrightarrow \mathbb{P}^n_K \times S$ using $\mathscr{L} \otimes \pi^*\mathscr{M}$ for some $\mathscr{M} \in \mathrm{Pic}(S)$.

Question: Why tensor w/ a pull-back from something on the base?

Theorem 14.1. *Let $X \subset A$ be a closed, irreducible, non-degenerated subvariety defined over $\overline{\mathbb{Q}}$ which dominants S. Then, there exists $c_1 > 0$, $c_2 \geq 0$, and a dense Zariski open $U \subset X$ such that*

$$\widehat{h}_{A,\mathscr{L}}(P) \geq c_1 h(\pi(P)) - c_2 \text{ for all } P \in U(\overline{\mathbb{Q}}).$$

Remark 14.2. non-degeneracy aboves implies there's a smooth point where $\omega^{\wedge \dim X} \neq 0$. ○

First step in proof to get remove the Néron-Tate height. We will replace it with height $h(P)$ defined using $\mathscr{O}_A(1,1) \simeq \mathscr{L} \otimes \pi^*\mathscr{M}$. Consider the graph $X_{[N]} \subset A \times_S A$ of the multiplication by N map restricted to X (i.e. $[N] : X \to A$), so[14]

$$X_{[N]} \subset A \times_S A \subset \mathbb{P}^n_k \times \mathbb{P}^n_k \times \mathbb{P}^m_k.$$

We let $\overline{X_N}$ be the Zariski closure of $X_N = X_{[N]}$. Get two lines bundles on $\overline{X}_N$: $\mathscr{F} = \mathscr{O}(0,1,1)$ (this is the (pullback of the) $\mathscr{L} \otimes \pi^*\mathscr{M}$ from before) and $\mathscr{M} := \mathscr{O}(0,0,1)$. Given $P' = (P, [N]P, s) \in X_N$, one has

$$h_{\mathscr{F}}(P') = h([N]P) + h(S) + O(1) \text{ and } h_{\mathscr{M}}(P') = h(S) + O(1).$$

[13] To get around this, use that after base change by some quasi-finite étale dominant $S' \to S$, one gets A' isogenous to a principally polarized abelian scheme

[14] The $\mathbb{P}^m$ is the base

Proposition 14.3 (Key)**.** *There exists $c_1 > 0$ s.t. for all N, $\exists U_N \subset X$ Zariski dense open, as well as a constant $c_2 = c_2(N) \geq 0$ so that*

$$h([N]P) \geq c_1 N^2 h(\pi(P)) - c_2(N) \ \textit{ for all } \ P \in U_N(\overline{\mathbb{Q}}).$$

This proposition implies the theorem. One applies a result of Silverman-Tate which gives an inequality

$$\left|\widehat{h}_A(P) - h_{A,F}(P)\right| \leq c_0 \max\left(1, h(\pi(P))\right)$$

(note that N'eron-Tate height is defined fiberwise, so same w.r.t. to $\mathscr{F} = \mathscr{L} \otimes \pi^*\mathscr{M}$ as w.r.t $\mathscr{L}$). Hence,

$$N^2\widehat{h}_A(P) = \widehat{h}_A([N]P) \geq h([N]P) - c_0(1 + h(\pi(P))) \geq c_1 N^2 h(\pi(P)) - c_2(N) - c_0(1 + h(\pi(P))).$$

Divide through by N^2 to deduce theorem from the proposition:

$$\widehat{h}_A(P) \geq \left(c_1 - \frac{c_0}{N^2}\right) h(\pi(P)) - \underbrace{\frac{c_2(N) + c_0}{N^2}}_{c_2}$$

since c_0, c_1 are independent of N, can choose N so that $N^2 \geq 2c_0/c_1$ (want coefficient above to be strictly positive) then we take $c_1/2$ for the constant in the theorem.[15]

This leaves showing the key proposition. For that, we want positive integers p, q so that $\mathscr{F}^{\otimes q} \otimes \mathscr{M}^{\otimes -pN^2}$ is a big line bundle (on $\overline{X}_N$) for all N.

Definition 14.4. A line bundle $\mathscr{L}$ on a projective variety X is **big** if

$$h^0(X, \mathscr{L}^{\otimes j}) \sim O(j^{\dim X})$$

So if $\mathscr{L}$ is big, then a sufficiently large multiple has a global section, so $h_{\mathscr{L}}(P) \geq O(1)$ outside base locus.

Theorem 14.5 (Siu)**.** *Let Y be a projective variety of dimension d. Let D, E be nef divisors. If $D^d > d(D^{d-1} \cdot E)$, then $D - E$ is big.*

To apply theorem (with $D = F^{\otimes q}$ and $E = M^{\otimes pN^2}$), we'll wanna choose q so that $qF^d > dN^2 p(F^{(d-1)} \cdot M)$. For this, we want a lower bound on F^d and an upper bound on $(F^{d-1} \cdot M)$. Looks like one gets

$$\mathscr{F}^{\cdot d} \geq \kappa N^{2d} \ \text{ and } \ \mathscr{F}^{\cdot (d-1)} \cdot \mathscr{M} \leq a_1 N^{2(d-1)} \ \text{ for some } \ \kappa, a_1 > 0.$$

Proof of lower bound. Let α be the Chern class of $\mathscr{O}(1,1)$, restricted to $\overline{A}$. Then, α is strictly positive. Since X non-degenerate, there's a point $x_0 \in X$ with $\omega^{\wedge d}|_{X_0} \neq 0$. Choose open neighborhood Δ of $\pi(x_0)$ which is relatively compact[16]. Then, $\exists C > 0$ so that $C\alpha - \omega \geq 0$ over A_Δ, where ω is the Betti form. Choose a smooth bump function $\widetilde{\Theta} : \Delta \to \mathbb{R}$ supported on compact $K \subset \Delta$, and pullback to $\Theta = \widetilde{\Theta} \circ \pi : A \to \mathbb{R}$. Now, $C\Theta\alpha - \Theta\omega \geq 0$ has compact support. Let $\beta := [N]^*(C\Theta\alpha - \Theta\omega) = c[N]^*(\Theta\alpha) - \underbrace{N^2\Theta\omega}_{\delta}$

TODO: Recall and add discussion about positivity. key words: volume form, cross terms (can exist

[15] $c_1 - c_0/N^2 \geq c_1/2$
[16] compact closure in analytic topology

(this is where we use that the Betti forms pulls back to N^2 times itself). Thus,

$$\int_{\overline{X}} (\beta+\delta)^{\wedge d} \geq \int_{\overline{X}} \delta^{\wedge d}$$

by binomial + $\beta, \delta \geq 0$ (i.e. $\sum_{i+j=d} \binom{d}{i} \int_{\overline{X}} \beta^{\wedge i} \delta^{\wedge j}$ has only semi-positive terms). On the left,[17]

Question: Not totally clear why (recall: cross term debacle)

$$\int_X [N]^*(C\Theta\alpha)^{\wedge d} = C^d \int_X \left(p_2 \circ p_1^{-1}\right)^* (\Theta\alpha)^{\wedge d} = C^d \int_{X_N} p_2^*(\Theta\alpha)^{\wedge d} \leq C^d \int_{\overline{X}_N} (p_2^*\alpha)^{\wedge d} = C^d \mathscr{F}^{\cdot d}$$

(recall $p_2^*\alpha$ is first Chern class of $\mathscr{F}$). On the right,

$$\int_{\overline{X}} \delta^{\wedge d} = N^{2d} \int_X (\Theta\omega)^{\wedge d} =: N^{2d}\kappa.$$

The point is that $\kappa > 0$ since ω non-vanishing near x_0. This is the lower bound we wanted. ■

14.1 Part II (11/15)

Setup. $A \to S$ is an abelian scheme, and we have a compactification $\overline{S} \hookrightarrow \mathbb{P}^m_k$ of S ($k = \overline{\mathbb{Q}}$). We also have $A \subset \mathbb{P}^n_k \times S \subset \mathbb{P}^n_k \times \mathbb{P}^m_k$. We fix non-degenerate $X \subset A$. For an integer N, we let

$$X_N \subset X \times A \subset \mathbb{P}^n \times \mathbb{P}^n \times \mathbb{P}^k$$

be the graph of multiplication by $[N]$. We let $\overline{X}_N$ be its closure. We have two line bundles $\mathscr{F} = \mathscr{O}(0,1,1)|_{\overline{X}_N}$ and $\mathscr{M} = \mathscr{O}(0,0,1)|_{\overline{X}_N}$ on $\overline{X}_N$.

Proposition 14.6 (Key, Proposition 14.3)**.** *There is a constant $c_1 > 0$ such that for all $N \in \mathbb{N}$, there is some $U_N \subset X$ Zariski open, dense and a constant $c_2(N) \geq 0$ so that*

$$h([N]P) \geq c_1 N^2 h(\pi(P)) - c_2(N) \text{ for all } P \in U_N(\overline{\mathbb{Q}}).$$

Above, the height on the LHS is induced by $\mathscr{O}(1,1)$; the right on the RHS is induced by $\mathscr{O}(1)$.

In order to prove this, we want to show that there exists p, q such that $L := F^{\otimes q} \otimes M^{\otimes -pN^2}$ is a big line bundle. Then, $h_L(P) \geq O(1)$ on U_N, the complement of base locus. Note $h_L = qh_F - pN^2 h_M$. Rearranging shows that

$$h_F(p) \geq \frac{p}{q} N^2 h_M(p) - \frac{c_2'(N)}{p}$$

(where $c_2'(N)$ is the $O(1)$ from before). Note $h_F(P) = h([N]P)$ and $h_M(P) = h(\pi(P))$ by definition of F, M. Hence, the above gives the key prop.

By Siu's theorem, this being big boiled down to showing the inequality

$$qF^d > dN^2 p\left(F^{(d-1)} \cdot M\right).$$

[17]Have $X \xleftarrow{p_1} X_N \xrightarrow{p_2} A$

For this, we last time proved a lower bound $F^{\cdot d} \geq \kappa N^{2d}$ for some $\kappa > 0$. We still need an upper bound

$$F^{\cdot(d-1)} \cdot M \leq a_1 N^{2(d-1)} \text{ for some } a_1 > 0.$$

Let $\vec{x}$ be the projective coordinates on $\mathbb{P}^n$, and $\vec{s}$ be those on $\mathbb{P}^m$.

Proof of upper bound when $N = 2^\ell$. Invoke a result of Serre: there exists polynomials $f_0, \ldots, f_n \in k[\vec{x}, \vec{s}]$ bihomogeneous of degree $(4, d')$ defined on some $A \setminus Z$ for some proper, Zariski closed $Z \hookrightarrow A$ which satisfy

$$[2](p, s) = ([f_0(p, s) : f_1(p, s) : \cdots : f_n(p, s)], s).$$

Taking $N = 2^\ell$, we can iterate this result to get $f_0^{(\ell)}, \ldots, f_n^{(\ell)}$ so that

$$[2^\ell](p, s) = \left(\left[f_0^{(\ell)}(p, s) : \cdots : f_n^{(\ell)}(p, s)\right], s\right).$$

Note that $\deg f_j^{(\ell)} = (d_\ell, d'_\ell)$, then $d_{\ell+1} = 4d_\ell$ and $d'_{\ell+1} = d' + 4d'_\ell$, so

$$(d_\ell, d'_\ell) = \left(4^\ell, \frac{4^\ell - 1}{3} d'\right).$$

In particular, both degrees are of the order $O(4^\ell) = O(N^2)$. For $i = 1, \ldots, n$, let $g_i = Y_i f_0^{(\ell)} - Y_0 f_i^{(\ell)}$, and note that

$$\overline{X}_N \subset (\overline{X} \times \mathbb{P}_k^n) \cap V(g_1, \ldots, g_n).$$

Note if $\dim X = d$, then we expect this intersection to have dimension $(d + n) - n = d$ as well. Now we use

Proposition 14.7 (Fulton 12.2(a))**.**

$$[\overline{X} \times \mathbb{P}_k^n] \cdot V(g_1) \cdot V(g_2) \cdot \ldots \cdot V(g_n) \in \mathrm{Ch}_d(\mathbb{P}^n \times \mathbb{P}^n \times \mathbb{P}^m)$$

will equal $[\overline{X}_N] + D$ for some positive cycles D.

Potentially $[\overline{X}_N] \leq D$?

We have $[\overline{X} \times \mathbb{P}_k^n] \cdot V(g_1) \cdot V(g_2) \cdot \ldots \cdot V(g_n) \geq [\overline{X}_N]$ so intersecting with $\mathscr{O}(0,1,1)^{\cdot(d-1)} \cdot \mathscr{O}(0,0,1)$ gives

$$F^{\cdot(d-1)} \cdot M \leq \mathscr{O}(0,1,1)^{\cdot(d-1)} \cdot \mathscr{O}(0,0,1) \cdot \mathscr{O}(d_\ell, 1, d'_\ell)^{\cdot n} \cdot [\overline{X} \times \mathbb{P}_k^n].$$

We need to simplify the RHS. Write $[\overline{X} \times \mathbb{P}_k^n] = \sum_{i+p=n+m-d} a_{i,p} H_1^i H_2^p$ with H_1, H_2 the pullback of hyperplanes: $H_2 = \mathscr{O}(0,0,1)$ and $H_1 = \mathscr{O}(1,0,0)$. The RHS can thus be written as

There are almost certainly typos in this displayed equation

$$\sum_{\substack{i+p=n+m-d \\ j'+p'=d-1 \\ i''+j''+p''=n}} a_{ip} \binom{d-1}{j', p'} \binom{n}{i'', j'', p''} d_\ell^{i''} d_\ell'^{p''} \cdot \left(\mathscr{O}(1,0,0)^{i+i''} \cdot \mathscr{O}(0,1,0)^{j'+j''} \cdot \mathscr{O}(0,0,1)^{1+p+p'+p''}\right).$$

The nonzero terms have $i + i'' \leq n, j + j' \leq n, 1 + p + p' + p'' \leq m$. All terms satisfying $i + i'' + j + j' + 1 + p + p' + p'' = 2n + m$, so the nonzero terms always have $i + i'' = n = j' + j''$ and $1 + p + p' + p'' = m$. Hence, the intersection product above is just 1 in the nonzero terms. The only things left which depend on N are the d_ℓ, d'_ℓ. Thinking about the bound we can get, we see we want $i'' + p'' \leq d - 1$. This holds since $i'' + p'' = n - j'' = j' \leq j' + p' = d - 1$. Thus, the RHS from before is

$$\leq C[N^2]^{d-1} = CN^{2(d-1)} \text{ for some } C > 0.$$

■

This proves the key proposition.

Let's iron out the positivity confusion from last time.

Let V be a complex n-dimensional vector space with dual $V^\vee$.

Definition 14.8. A (p,p)-form $u \in \bigwedge^{p,p} V$ is **positive** if for all $\alpha \in V^\vee$ and $1 \leq j \leq q := n - p$:

$$u \wedge i\alpha \wedge \overline{\alpha}_1 \wedge \cdots \wedge i\alpha_q \wedge \overline{\alpha}_q$$

is positive (a non-negative multiple of volume form). This is what Gao calls semi-positive?

This implies something stronger for $(1,1)$-forms. Say a $(1,1)$-form u is positive, and write

$$u = i \sum_{j,k} u_{j,k} \mathrm{d}z_j \wedge \mathrm{d}\overline{z}_k.$$

The matrix $U := (u_{j,k})$ will by symmetric and semipositive definite, i.e. $x^t U x \geq 0$ for all $x \in \mathbb{R}^n$.

Remark 14.9 (Audience). For $(1,1)$-forms, you can restrict to any local holomorphic curve, integrate there, and then ask the answer to be positive. This should get you something about wedging with a form of complementary definition giving you a non-negative integral by locally thinking of your space as a fibration of curves or something? I didn't follow... ○

Let's end with a proof of Siu.

Theorem 14.10 (Siu)**.** *Let X be a projective variety of dimension d. Let D, E be nef divisors. If $D^d > d(D^{d-1} \cdot E)$, then $D - E$ is big.*

Proof. First, we may assume that D, E are ample (add εH for some ample H and small $\varepsilon > 0$). Then, we can take large enough multiples to assume that D, E are very ample. For positive integer m, choose $E_1 \sim \cdots \sim E_m \sim E$, and then consider exact sequence

$$0 \to \mathrm{H}^0(X, \mathscr{O}_X(mD - \sum E_i)) \to \mathrm{H}^0(X, \mathscr{O}_X(mD)) \to \bigoplus_{i=1}^m \mathrm{H}^0(E_i, \mathscr{O}_{E_i}(mD)).$$

Asymptotic Riemann-Roch gives

$$h^0(mD) = \frac{D^d}{d!} m^d + O(m^{d-1}),$$

so

$$h^0(m(D-E)) \geq \frac{D^d}{d!} m^d - \sum_{i=1}^m \frac{(D^{d-1} \cdot E_i)}{(d-1)!} m^{d-1} + O(m^{d-1}) = \frac{D^d - d(D^{d-1} \cdot E)}{d!} m^d + O(m^{d-1}).$$

Now stare at this and compare to the definition of being big. ■

15 Daniel (Harvard): Uniform Mordell-Lang (11/22)

Theorem 15.1 (DGH + K)**.** *Fix $g \geq 2$. There exists some $c > 0$ such that for all char 0 fields F, smooth proper curves C/F of genus g, subgroups $\Gamma \subset \mathrm{Jac}(C)(F)$ of rank rank ρ, and $P_0 \in C(F)$, we have*

$$\#\left((C(F) - P_0) \cap \Gamma\right) \leq c^{1+\rho}.$$

It seems all that matters is that we have a bound where the exponent is linear in ρ

Proof. Passing to $\overline{F}$, spreading out, and then appealing to work of Masser let's us reduce to the case $F = \overline{\mathbb{Q}}$. Consider the universal family $\mathcal{C}_g \to \mathcal{M}_g$. Fix ample M on $\mathcal{M}_g$, and a relatively ample symmetric L on $\mathrm{Jac}(\mathcal{C}_g/\mathcal{M}_g)$. Hensel's + spreading out $\implies \exists$ surjective quasi-finite étale map $S \twoheadrightarrow \mathcal{M}_g$ so that $\mathcal{C}_{g,S} \to S$ has a section $\sigma : S \to C_g$. Write C for the image of

Here, $\mathcal{M}_g$ moduli of curves with some level structure, so it's a variety, not a stack

$$\mathcal{C}_g \times_{\mathcal{M}_g} S \xrightarrow{(\mathrm{id},\sigma)} \mathcal{C}_g \times_{\mathcal{M}_g} \mathcal{C}_g \longrightarrow \mathrm{Jac}(\mathcal{C}_g/\mathcal{M}_g).$$

Recall 15.2 (Sam's talk)**.** There exists a constant $c_0 > 0$ such that for all $s \in \mathcal{M}_g(\overline{\mathbb{Q}})$ and $\Gamma \subset \mathrm{Jac}(C_{g,s})(\overline{\mathbb{Q}})$ of rank ρ, we have

$$\#\left\{P \in C_s(\overline{\mathbb{Q}}) \cap \Gamma : \widehat{h}_L(P) \geq c_0 \max\left\{1, h_{\overline{\mathcal{M}}_g, M}(s)\right\}\right\} < c_0^{\rho}.$$

(note that C_s is a finite union of $C - P_i$ for some $P_i \in C_{g,s}(\overline{\mathbb{Q}})$)

Since C_s is a finite union of those things, we can just pick one of them, so

$$\#\left\{P \in (C_{g,s}(\overline{\mathbb{Q}}) - P_i) \cap \Gamma : \widehat{h}_L(P) \geq c_0 \max\left\{1, h_{\overline{\mathcal{M}}_g, M}(s)\right\}\right\} < c_0^{\rho}.$$

Now, fix s so that C (from the theorem statement) is $\cong C_{g,s}$. First assume $P_0 = P_i$. Then define

$$Y_s := \left\{P - P_i \in \left(C(\overline{\mathbb{Q}}) - P_i\right) \cap \Gamma \mid \widehat{H}_L(P) \leq \underbrace{c_0 \max\left\{1, h_{\overline{\mathcal{M}}_g, M}(s)\right\}}_{R^2}\right\}.$$

Fact. Subsets Y of a closed ball of radius R in $\left(\Gamma \otimes_{\mathbb{Z}} \mathbb{R}, \widehat{h}_L^{1/2}\right)$ can be covered by $\leq \left(1 + 2\frac{R}{r}\right)^{\rho}$ closed balls of radius r whose centers lie in Y.

We need a candidate for little r.

Proposition 15.3. *There exists constants $c_1, c_2 > 0$ so that for all $s \in \mathcal{M}_g(\overline{\mathbb{Q}})$ and any $Q \in C_{g,s}(\overline{\mathbb{Q}})$, we have*

$$\#\left\{P \in C_{g,s}(\overline{\mathbb{Q}}) : \widehat{h}_L(P - Q) \leq \underbrace{c_1 \max\left\{1, h_{\overline{\mathcal{M}}_g, M}(s)\right\}}_{r^2}\right\} < c_2$$

(there's a bounded number of points in balls of the above radius).

Appealing to the fact, this gives us $\left(1+2\sqrt{c_0/c_1}\right)^{\rho}$ balls, each with at most c_2 points, and so there are

$$\leq \left(1+2\sqrt{c_0/c_1}\right)^{\rho} c_2$$

elements in Y_s. Then we take something like $c = \max\left\{c_2, 1+2\sqrt{c_0/c_1}\right\} + c_0$. Hence, we would be done if the point P_0 we started with happened to be the P_i we got later.

For general P_0, set $\Gamma' := \langle \Gamma, P_0 - P_i\rangle$ which has rank ≤ 1 higher than original Γ. Now, addition by $P_0 - P_i$ yields an injection

$$(C - P_0)(\overline{\mathbb{Q}}) \cap \Gamma \hookrightarrow (C - P_i)(\overline{\mathbb{Q}}) \cap \Gamma'.$$

We can now apply what we proved earlier, so this now has size

$$\leq \max\left\{1+2\sqrt{\frac{c_0}{c_1}} + c_0, c_2 + c_0\right\}^{2+\rho} \leq \left(\max\{\dots\}^2\right)^{1+\rho},$$

and we win. ■

We still owe ourselves the proof of Proposition 15.3.

Proof of Proposition 15.3. Write $\mathcal{B}$ for the image of $\mathcal{C}_g \times_{\mathcal{M}_g} \mathcal{C}_g \to \operatorname{Jac}(\mathcal{C}_g/\mathcal{M}_g) \times_{\mathcal{M}_g} \mathcal{C}_g$ (this map is $(x,y) \mapsto (x-y, y)$). Note that for all $s \in \mathcal{M}_g(\overline{\mathbb{Q}})$ and $Q \in C_{g,s}(\overline{\mathbb{Q}})$, the fiber of $\mathcal{B}$ over (s,Q) is simply $C_{g,s} - Q \subset \operatorname{Jac}(C_{g,s})$. Let's punt some more.

Proposition 15.4 (DGH)**.** *There are constants $c_1', c_2', c_3' > 0$ s.t. for all (s,Q), we have*

$$\#\left\{x \in \mathcal{B}_{(s,Q)}(\overline{\mathbb{Q}}) \mid \widehat{h}_L(x) \leq c_1' \max\left\{1, h_{\overline{\mathcal{M}}_g, M}(s)\right\} - c_3'\right\} < c_2'.$$

Proposition 15.5 (K, blackbox for us)**.** *There are constants $c_2'', c_3'' > 0$ s.t. for all (s,Q) we have*

$$\#\left\{x \in \mathcal{B}_{(s,Q)}(\overline{\mathbb{Q}}) \mid \widehat{h}_L(x) \leq c_3''\right\} < c_2''.$$

To prove proposition 15.3, we take

$$c_1 := \min\left\{\frac{c_1'}{2}, \frac{c_1'c_3''}{2c_3'}\right\} \text{ and } c_2 := c_2' + c_2''.$$

I'm gonna leave it as an exercise to show these choices work... (Hint: break into two cases depending on whether $\max\{1, h_{\overline{\mathcal{M}}_g, M}(s)\} > 2c_3'/c_1'$) ■

Now we still have to prove Proposition 15.4.

Proposition 15.6 (Proposition 15.4, generally)**.** *Let $S/\overline{\mathbb{Q}}$ be an irreducible, quasi-projective variety. Let A/S be a principally polarize abelian scheme, and let $\mathscr{L}$ be a symmetric relatively ample line bundle on A. Let $\mathcal{M}$ be an ample line bundle on a compactification $\overline{S}$ of S. Let $\mathcal{C} \subset A$ be an irreducible subvariety. Assume condition **(*)**.*

- ***(*)*** *The fibers $\mathcal{C}_s$ are irreducible curves generate A_s which are not translates of elliptic curves, and for all subvarities $S' \subset S$, the modular map $A_{S'} \to \underline{A}$ is generically finite.*

(for us, we take A/S to be $\mathrm{Jac}(\mathcal{C}_g/\mathcal{M}_g)\times_{\mathcal{M}_g}\mathcal{C}_g$ *and $\mathcal{C}=\mathcal{B}$). Then, there exists constants $c_1', c_2', c_3' > 0$ so that for all $s \in S$*

$$\#\left\{x \in \mathcal{C}_s(\overline{\mathbb{Q}}) : \widehat{h}_{\mathscr{L}}(x) \le c_1' \max\left\{1, h_{\overline{S},\mathcal{M}}(s) - c_3'\right\} < c_2'\right\}$$

Proof. Fix some $N \ge \dim S$. By Yujie's talk (blackboxed), $\mathcal{C}^{[N]} = \mathcal{C}\times_S \ldots \times_S \mathcal{C}$ is a non-degenerate subvariety of $A^{[N]}$. By Si Ying's talk, there exists a Zariski dense open $U \subset \mathcal{C}^{[N]}$ along with constants $a, b > 0$ so that for all $s \in S(\overline{\mathbb{Q}})$ and $(x_1, \ldots, x_N) \in U_s(\overline{\mathbb{Q}})$:

$$ah_{\overline{S},\mathcal{M}}(s) - b \le \widehat{h}_{\mathscr{L}^{\boxtimes N}}(x_1, \ldots, x_N) = \widehat{h}_{\mathscr{L}}(x_1) + \cdots + \widehat{h}_{\mathscr{L}}(x_N).$$

Since $\mathcal{C}$ dominates S, the image of U in S is dense. Now we induct; the stuff not in the image is lower dimensional, so we can induct on $\dim S$. This let's us assume that U surjects onto S. Hence, for all $s \in S$, U_s is nonempty so $Z_s := \mathcal{C}_s^{[N]} \setminus U_s$ is a proper subset.

Lemma 15.7 ($k = \overline{k}$)**.** *Let V/k be an irreducible, projective variety with ample line bundle L. Let C be an irreducible curve on V, and let $Z \subset V^N$ be a Zariski closed subset not containing C^N. Then, there exists some $c = c(N, \deg_L(C), \deg_L(V), \deg_L(Z)) > 0$ s.t. if $\Sigma \subset C(k)$ has $\#\Sigma \ge c_1$, then $Z \not\supset \Sigma^N$.*

To apply this lemma, take $V = A_s$, $L = \mathscr{L}_s$, $C = \mathcal{C}_s$, and $Z = Z_s$. There will be a stratification of your base where your family is flat over each piece (so the degrees in the lemma are constant, and the c's are the same for all s in each piece); there are only finitely many pieces of such a stratification, so we take the biggest c showing up. That is, we conclude that there exists a $c_2' > 0$ so that for all s and $\Sigma \subset \mathcal{C}_s(\overline{\mathbb{Q}})$ w/ $\#\Sigma > c_2'$ then $Z_s \not\supset \Sigma^N$. Set

$$c_1' := \frac{a}{2N} \text{ and } c_3' := \frac{a+b}{N}.$$

Take $\Sigma := \left\{x \in \mathcal{C}_s(\overline{\mathbb{Q}}) : \widehat{h}_{\mathscr{L}}(x) \le c_1' \max\left\{1, h_{\overline{S},\mathcal{M}}(x)\right\} - c_3'\right\}$. If $\#\Sigma \ge c_2'$, then $\exists (x_1, \ldots, x_N) \in \Sigma^N \cap U_s(\overline{\mathbb{Q}})$ which implies that

$$\begin{aligned}
a\max\{1, h(s)\} - a - b &\le a(h(s)+1) - a - b \\
&\le ah(s) - b \\
&\le \widehat{h}(x_1) + \cdots + \widehat{h}(x_N) \\
&\le N\left(c_1' \max\{1, h(s)\} - c_3'\right) \\
&= \frac{a}{2}\max\{1, h(s)\} - a - b,
\end{aligned}$$

which is bullshit. ■

16 List of Marginal Comments

Talks here and below are from Kisin Seminar . . . ii
Question: Artinian scheme just means spec of an Artinian ring? i.e. 0-dimensional and noetherian 2
I think at we're maybe secretly assuming $X \to S$ is proper with S Artinian, since we'll most often be interested in a scheme of finite type over a field . . . 3
Not sure if I copied this down right. See the references . . . 4
formal sum of ample things with positive coefficients . . . 6
Apparently there's a Hartshorne exercise showing that $N^1(C_1 \times C_2)$ is always generated by these two classes along with hom's between their Jacobians . . . 7
this means sections of the canonical bundle eventually give a birational map onto the image, Nathan said this . . . 8
Think of it as the linear system of lines in $\mathbb{P}^2$ through 0. This is the resolution of the projection map $\pi_0 : \mathbb{P}^2 \dashrightarrow \mathbb{P}^1$ . . . 10
height of corresponding element of $\mathbb{P}^1$ . . . 11
I got lost in the algebra below... . . . 11
Katia actually stated this with $\mathrm{Div}(V)$ where I wrote $\mathrm{Pic}(V)$. This is why the properties are all stated in terms of divisors . . . 12
I'm not sure if it's supposed to be $+f(0)$ or $-f(0)$ at the end . . . 16
Question: Does every integer divide a difference of powers of 2? I feel like this is saying this must be the case. . . . 16
Not sure if I copied this down correctly . . . 21
$(1,1)$-form here means $E(ix, iy) = E(x, y)$ and positive means $E(ix, x) > 0$ for all nonzero $x \in V$ 23
Question: Why? Uniqueness of harmonic representative? . . . 23
Question: The divisor itself or just its class in $\mathrm{NS}(\mathrm{Jac}(C))$? . . . 27
Question: Why? . . . 31
Answer: $\dim \mathcal{C} = \dim S + 1$ (family of curves), so $\dim(\mathcal{C} \setminus U) = \dim \mathcal{C} - 1 = \dim S$. Have dominant map of 'nice' things of the same dimension . . . 31
proper, flat, generic fiber genus ≥ 2, blah, blah, blah . . . 31
TODO: Improve formatting . . . 32
Brackets hint that we secretly are thinking about stack quotients . . . 34
Talks here and below are from Kisin Seminar . . . 42
This is secretly the Poincaré bundle, i.e. $\delta = (\mathrm{id} \times \varphi_\Theta)^* \wp$ . . . 44
Unclear how correct this is . . . 44
Maybe $B(\alpha)$ also depends on δ . . . 47
Probably proof of this in chapter 3 of Levent's thesis . . . 47
r_1 largest by **(1)** . . . 48
Question: Why tensor w/ a pullback from something on the base? . . . 50
TODO: Recall and add discussion about positivity. key words: volume form, cross terms (can exist but no condition), positive function, ω_0, $\sum_i \mathrm{d}z_i \wedge \mathrm{d}\bar{z}_i$, etc. . . . 51

Question: Not totally clear why (recall: cross term debacle) 52
Potentially $[\overline{X}_N] \leq D$? 53
There are almost certainly typos in this displayed equation 53
It seems all that matters is that we have a bound where the exponent is linear in ρ 55
Here, $\mathcal{M}_g$ moduli of curves with some level structure, so it's a variety, not a stack 55

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