Outline

- Definition of r-(Re)parameterizations
- "Proof by Example" of existence of r-(re)parameterizations
- Uniform parameterizations via compactness

1 Parameterizations

Let R be an ordered field with some given O-minimal structure. We begin with several definitions.

Ultimately interested in $R = \mathbb{R}$, but need to work in greater generality e.g. to apply compactness at the end

Definition 1.1.

(1) A set $X \subset R^m$ is **strongly bounded** if there is some $N \in \mathbb{N}$ such that $X \subset [-N, N]^m$. A map $f : X \to Y$ is **strongly bounded** if its graph Γ_f is. Equivalently, both X and $f(X)$ are strongly bounded.

I think

(2) Let $X \subset R^m$ be definable and set $d = \dim(X)$. A finite set S of definable maps $\varphi :]0,1[^d \to X$ is called an **parameterization** of X if

$$\bigcup_{\varphi \in S} \mathrm{Im}(\varphi) = X.$$

(3) A parameterization S of a definable set X is called an **r-parametrization** if every $\varphi \in S$ is of class $C^{(r)}$ and has the property that $\varphi^{(\alpha)}$ is strongly bounded for every $\alpha \in \mathbb{N}^k$ with $|\alpha| = \alpha_1 + \cdots + \alpha_k \leq r$

(4) An r-parametrization is called a **strong r-parameterization** if for every $\varphi \in S$, $\varphi^{(\alpha)}$ is bounded by 1 for every $\alpha \in \mathbb{N}^k$ with $|\alpha| \leq r$

Remark 1.2. A definable set admits an r-parameterization iff it admits a strong r-parameterization.

Example. If $\varphi :]0,1[\to X$ satisfies $\left|\varphi^{(\alpha)}\right| \leq c$ for all $\alpha \in \mathbb{N}$, then consider

$$\psi_i :]0,1[\to X, \quad \psi_i(t) = \varphi\left(\frac{i+t}{c}\right) \quad \text{for} \quad i = 0, \ldots, c-1.$$

Then, $\psi_i'(t) = \frac{1}{c}\varphi\left(\frac{i+t}{c}\right)$ and so on... $\triangle$

In higher dimension, do the same thing but break domain up into cubes of size $1/c$. $\circ$

(5) Let S be an r-parameterization of a definable set $X \subset R^m$ and $F : X \to R^n$ a definable map. Then, S is an **r-reparameterization** of F if $F \circ S := \{F \circ \varphi : \varphi \in S\}$ is an r-parameterization of $f(X)$.

$F \circ \varphi$ of class $C^{(r)}$ and $(F \circ \varphi)^{(\alpha)}$ strongly bounded for all $\alpha \in \mathbb{N}^{\dim X}$ with $|\alpha| \leq r$

Theorem 1.3 (Goal)**.** *Fix any $r \in \mathbb{N}$.*

(1) *Any strongly bounded, definable set X admits an r-parametrization.*

(2) *Any strongly bounded, definable map F admits an r-reparametrization.*

We won't go over the proof of this theorem in detail, but here's an outline. The proof is inductive. First consider the statements

$B(r)$ Every strongly bounded, definable $F :]0,1[\to R$ admits an r-reparameterization S such that for each $\varphi \in S$, either φ or $F \circ \varphi$ is a polynomial with strongly bounded coefficients.

$R(m,n,r)$ Every strongly bounded, definable $F : X \to R^m$ (w/ $X \subset]0,1[^n$) admits an r-reparameterization.

Note $R(1,1,r) \implies P(1,r)$. More generally, $R(n,n,r) \implies P(n,r)$ (since X strongly bounded. Consider $X \hookrightarrow R^n \xrightarrow[\cdot 1/N]{\sim} R^n$)

$P(n,r)$ Every strongly bounded, definable set $X \subset R^n$ admits an r-parameterization.

The proof proceeds in several steps

(1) Use induction to prove $B(r)$ ($\implies R(1,1,r) \implies P(1,r)$) for all r

(2) Another induction then proves, for fixed n, r,[1]

$$R(1,d,r) \text{ for all } d \le n \implies R(m,n,r) \text{ for all } m.$$

In particular, this + **(1)** gives $R(m,1,r)$ for all m, r.

(3) Finally, prove that, for fixed r,[2]

$$P(n,k) \text{ and } R(m,n,k) \text{ for all } k \le r, \text{ all } m \implies P(n+1,r) \text{ and } R(1,n+1,r).$$

(this let's us increase n)

Now, one proves $\forall (m,r) : R(m,n,r)$ via strong induction on n.

2 "Proof" by Example

Proposition 2.1 ($B(r)$ for all r)**.** *Every definable map $F :]0,1[\to]0,1[$ admits a (strong) r-reparameterization for any r.*

Proof. Induct on r. If $r = 1$, a strengthened version of the monotonicity theorem from Katia's talk let's us partition

$$0 = a_0 < a_1 < \cdots < a_n = 1$$

so that $F|_{]a_i,a_{i+1}[}$ is continuously differentiable and monotone for all i and furthermore $|F'| \le 1$ or $|F'| > 1$ on each subinterval. Two cases

- Say $|F'| \le 1$ on $]a,b[\subset]0,1[$. Then, $\varphi(x) = a + (b-a)x$ is a parameterization of $]0,1[$ so that $|(F \circ \varphi)'| \le 1$.

  $\varphi :]0,1[\to]0,1[$

- Say $|F'| > 1$ on $]a,b[\subset]0,1[$. F is strictly monotone (so invertible) and strongly bounded[3], so we can write $F(]a,b[) =]m,M[$ with m, M strongly bounded. Now, $\varphi(x) = F^{-1}(m + (M-m)x)$ gives a 1-reparameterization of F on $]a,b[$ (Note $F \circ \varphi = m + (M-m)x$ has derivative $(M-m)$).

We still need to cover the remaining finitely many points $a_1, \ldots, a_{n-1}$. For these, we use constant functions $\varphi_i :]0,1[\mapsto a_i$.

[1] Specifically, $R(m-1,n,r)$ and $R(1,d,r)$ for all $d \le n$ together imply $R(m,n,r)$

[2] Possibly $m = 1, 2$ suffices. Didn't read closely enough to be sure. Certainly, $m = 1$ alone does not suffice

[3] $m = \lim_{x \to a^+} F(x)$

Now fix $r > 1$ and assume $B(k)$ holds for any $k < r$. Let S be an $(r-1)$-reparametrization of F with the extra property. For $\varphi \in S$, write $\{\varphi, F \circ \varphi\} = \{g, h\}$ where g is a polynomial with strongly bounded coefficients. Then, $g^{(k)}$ exists and is strongly bounded for every k, so we focus on h. We know $h^{(k)}$ exists, is continuous, and is strongly bounded for $k \leq r-1$ by hypothesis. We can partition $]0,[1$ into f.many intervals $]a,b[\subset]0,1[$ such that h is of class $C^{(r)}$ with $\left|h^{(r)}\right|$ monotonic on each subinterval. Given one such $]a,b[$, define

$$\psi(x) = \begin{cases} a + (b-a)x & \text{if } \left|h^{(r)}\right| \text{ is decreasing} \\ b + (a-b)x & \text{if } \left|h^{(r)}\right| \text{ is increasing} \end{cases}$$

Now, $h \circ \psi :]0,1[\to R$ is of class $C^{(r)}$, $(h\circ\psi)^{(k)}$ is strongly bounded for $k < r$ and $\left|(h\circ\psi)^{(k)}\right|$ is decreasing. An analytic argument using the chain rule and Rolle's theorem then shows that

This analytic input is the heart of things

$$x \longmapsto h(\psi(x^2))$$

has all derivatives up to order r (not just $r-1$) strongly bounded. Similarly $x \mapsto g(\psi(x^2))$ is still a polynomial with strongly bounded coefficients. Varying the $]a,b[$, this functions $x \mapsto \varphi(\psi(x^2))$ with ranges covering $\mathrm{Im}(\varphi)$ except maybe finitely many points. For the missing points, we add constant maps and so obtain an r-reparameterization of F. ∎

Proposition 2.2 ($R(1,1,r) \implies R(2,1,r)$)**.** *Every definable map $F : X \to]0,1[^2$ (with $X \subset]0,1[$) admits a (strong) r-reparameterization for any r.*

Proof. Fix r. Let $F, f : X \to]0,1[$ be strongly bounded definable maps. We will show that $(F,f) : X \to]0,1[^2$ admits an r-reparameterization. $R(1,1,r)$ given an r-reparameterization S of F. For any $\varphi \in S$, $\varphi :]0,1[^d \to X$ ($d = \dim X$), $R(1,d,r)$ gives an r-reparam. S_φ of $f \circ \varphi :]0,1[^d \to]0,1[$. The chain rule then shows that, for any $\psi \in S_\varphi$, all order $\leq r$ derivatives of $\varphi \circ \psi$ are strongly bounded. Hence,

$$\left\{\varphi \circ \psi :]0,1[^d \to]0,1[: \varphi \in S, \psi \in S_\varphi\right\}$$

is an r-reparameterization of (F,f). ∎

Proposition 2.3 ($P(2,r)$)**.** *Every open cell $X \subset]0,1[^2$ admits a (strong) r-parameterization for any r.*

Proof. Write

$$X =]f,g[:= \{(x,y) : x \in D \text{ and } f(x) < y < g(x)\}$$

where $D \subset]0,1[$ is a 1-cell and $f < g : D \to]0,1[$ are definable, continuous maps. Note that D is strongly bounded, so $P(1,r)$ gives an r-parameterization S of D. For each $\varphi \in S$, $\varphi :]0,1[\to D$, the map

$$(f,g) \circ \varphi :]0,1[\longrightarrow]0,1[^2$$

has an r-parameterization S_φ by Lemma 2.2 ($R(2,1,r)$). For each $\psi \in S_\varphi$, we define $\theta_{\varphi,\psi} :]0,1[^2 \to X$ via

$$\theta_{\varphi,\psi}(x_1,x_2) = \big(\varphi\circ\psi(x_1)\,,\,(1-x_2)f(\varphi(\psi(x_1))) + x_2 g(\varphi(\psi(x_1)))\big).$$

The collection of theses forms an r-parameterization of X. ∎

3 Uniformity

Recall 3.1 (**Compactness Theorem**)**.** Fix a structure M. Let V be a definable set, and let $(V_i)_{i\in I}$ be a collection of definable subsets of V. Suppose that for every elementary extension $M \subset M^*$, we have

$$V^* = \bigcup_{i\in I} V_i^*.$$

Then, there exists a finite subset $I_0 \subset I$ such that

$$V = \bigcup_{i\in I_0} V_I.$$

Lemma 3.2. *Let R be an o-minimal structure, and let $R \subset R^*$ be an elementary extension. Then, R^* is again o-minimal (only sketch a proof).*

Proof. To keep life simple, suppose all singletons in R are basic definable, so definable = basic definable for R. Let $V^* \subset R^*$ be a definable subset; we want to prove that V^* is a finite union of points and open intervals. First write $V^* = (W^*)_{y^*}$ for some basic definable $W^* \subset (R^*)^{n+1}$ and point $y^* \in (R^*)^n$. The set W^* is the extension of some (basic) definable $W \subset R^{n+1}$. By the cell decomposition theorem, W is a finite disjoint union of cells C, so it suffices to prove that each $(C^*)_{y^*}$ is a finite union of points and open intervals. Suppose

$$C =]f, g[= \{(x, y) : x \in D \text{ and } f(x) < y < g(x)\}$$

where $D \subset R^n$ is a cell and $f < g : D \to R$ are definable, continuous functions. Then, $C^* =]f^*, g^*[$, so the fiber above y^* is either an interval $]f^*(y^*), g^*(y^*)[$ if $y^* \in D^*$ or is empty otherwise. The other sorts of cells can be handled similarly. ∎

TL;DR WTS: $V^* \subset R^*$ is a finite union of points and open intervals. This is a fiber of some basic definable $W^* \subset (R^*)^{n+1}$. By cell decomposition, can assume that $W \subset R^{n+1}$ is a cell, and then just directly check that $(W^*)_{y^*}$ is a point, empty, or an interval for each cell type

Theorem 3.3. *Let $X \subset]0,1[^n \times Y$ be a definable family with fiber dimension k over Y. Then, there is a finite set I and definable maps*

$$\left\{\varphi_{i,y} :]0,1[^{\dim X_y} \to X_y \subset]0,1[^n\right\}_{i\in I, y\in Y}$$

such that for each $y \in Y$, there is a subset $I_0 \subset I$ so that $\{\varphi_{i,y}\}_{i\in I_0}$ gives a strong r-parameterization of X_y.

(Similarly statement for families of strongly bounded definable maps)

Each fiber has a strong r-parameterization of uniformly bounded size

Proof. We may assume X, Y are basic definable. For any $N \in \mathbb{N}$, any basic definable $Z \subset Y$, and any N-tuple of basic definable functions $f_1, \ldots, f_N :]0,1[^k \times Z \to X$ over Z, we write

$$Y_{Z,f_1,\ldots,f_N} := \left\{y \in Z \mid f_{1,y}, \ldots, f_{N,y} :]0,1[^k \to X_y \text{ form a strong } r\text{-parameterization}\right\}.$$

We claim all such sets taken together cover Y.

Fix some $y_0 \in Y$, and let $g_1, \ldots, g_N :]0,1[^k \to X_{y_0}$ be a strong r-param. Since each g_i is definable w/ basic definable (co)domain, there exists some basic definable set $V \ni v$ along with basic definable functions $h_1, \dots, h_N :]0,1[^k \times V \to X \times V$ over V s.t. $g_i = h_{i,v}$ for all i. Set

$$W := \{(y,v) \in Y \times V \mid h_{1,z}, \dots, h_{N,z} \text{ form a strong } r\text{-param. of } X_y\} \quad \text{and} \quad Z := \mathrm{pr}_1(W) \subset Y.$$

Note that $y_0 \in Z$ by construction. Basic definable choice gives a splitting $s : Z \to W$ which we use to form the compositions

$$f_i :]0,1[^k \times Z \xrightarrow{(\mathrm{id},s)}]0,1[^k \times W \xrightarrow{(\mathrm{id},\mathrm{pr}_2)}]0,1[^k \times V \xrightarrow{h_i} X \times V \xrightarrow{\mathrm{pr}_1} X.$$

By construction, $f_{1,y}, \dots, f_{N,y} :]0,1[^k \to X_y$ form a strong r-param. for all $y \in Z$. Thus, $y_0 \in Y_{Z,f_1,\dots,f_N}$.

The above argument holds not only in R, but in any elementary extension R^*. Hence, by compactness, $Y = \bigcup Y_{Z,f_1,\dots,f_N}$ is in fact covered by only finitely many such sets; whence the claim. ∎

Lemma 3.4. *Let $g : X \to Y$ be a definable function with basic definable (co)domain X, Y. Then, there exists a basic definable set $V \ni v$ and a basic definable function $h : X \times V \to Y \times V$ over V so that $g = h_s$.*

Definable functions are fibers of basic definable functions

Proof. Write $\Gamma_g = W_s$ for some basic definable $W \subset R^m \times X \times Y$ and $s \in R^m$. Let $p : W \to R^m \times X$ denote the natural projection, and note that

$$B := \big\{(t,x) \in R^m \times X : \#p^{-1}(t,x) = 1\big\} \supset \{s\} \times X$$

is basic definable. Furthermore, $p^{-1}(B) \xrightarrow{p} B$ is an isomorphism. Let

$$V := \{t \in R^m \mid \{t\} \times X \subset B)\} \ni s.$$

Then, $p^{-1}(V)$ is the graph of our desired function h. ∎