MSRI Notes

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These are notes on talks given in "Degeneracy of Algebraic Points" which took place at MSRI (SLMath). Unfortunately for the reader, these notes are live-texed and so their quality is upper bounded both by my (quite) limited ability to understand the material in real time and by my typing speed. With that in mind, they are doubtlessly missing content/insight present in the talks and certainly contain confusions not present in the talks. Despite all this, I hope that you can still find some use of them. Enjoy and happy mathing.

The website for this seminar is available here.

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1 Day 1 (4/24/23)

1.1 Umberto Zannier (Scuola Normale Superiore): On Effectivity in Some Diophantine Problems

(joint work (in progress) w/ P. Corvaja)

Remark 1.1.1 (Initial musings on 'effectivity'). Roughly speaking, 'effectivity' means something can be computed in a finite number of steps using e.g. a Turing machine. However, there's not always consensus among the experts about a more precise meaning of effectivity. In this talk, we'll think about effectivity fairly naively. Still, beyond being able to carry out an algorithm/run a Turing machine in finite time, we'll be interested in knowing how much time it'll take. something something (iterated) exponentials something something primitive recursive something something. There was more to this in the beginning that I didn't take notes on... ○

Zannier eventually went on to mention

$$e^{\pi\sqrt{163}} = 262537412640769743.9999\ldots,$$

and the `class number problem`: find the $d < 0$ with $h(d) \leq$ given number. Goldfeld (1976) proved that this problem has an effective solution if $L'(1) = 0$ for an appropriate L-function. Actually proving such an equality had to wait until the work of Gross and Zagier. There are similar open problems coming from work of Beilinson, Deligne, Deninger (spelling?), etc.

Zannier ended this preamble by mentioning that a distinguished analytic number theorist once gave a talk, where they started with the following

Theorem 1.1.2. *There exists a number $C > 0$ s.t. if $\zeta(s)$ has no nontrivial zeros w/ $|\operatorname{Im} s| < C$, then RH holds.*

The issue is that this C is completely ineffective. If you wanna prove this theorem, take C whatever if RH holds, and C larger than the imaginary part of the first counterexample if it does not.

1.1.1 Some Previous Results

Question 1.1.3 (`Hilbert's 10th Problem`, 1900)**.** *Find an algorithm to decide whether or not a given equation $P(x_1, \ldots, x_n) = 0$ has an integer solution.*

($P \in \mathbb{Z}[blah]$)

Answer (Matijesevic (spelling) 1970 after work by Davis, Putnam, Robinsom)**.** Can't be done. ⋆

One can ask similar questions over other rings (e.g. $\mathbb{Q}$). These have been worked on e.g. by Denef (spelling), Poonen, Mazur, Rubin, Shopatoleh (spelling?).

Let's quickly look at some examples of Diophantine equations

- Linear systems. Solved, going back e.g. to Euclid.
- (Single) Quadratic equations. There are algorithms in any number of variables, due to work of Lagrange, Gauss, Siegal, Cassels, Dietmann, etc.
- Quadratic systems. No general algorithm. Logicians can reduce all equations to systems of quadratic equations.

Example 1.1.4. An elliptic curve is cut out by two quadratics in $\mathbb{P}^3$. It's unknown if one can find non-trivial solutions to these in general (there is an algorithm known to work is Ш is finite though). $\triangle$

- Cubics

  Example 1.1.5. $x^3 + y^3 + z^3 = n$, $n \not\equiv \pm 4 \pmod 9$. Probabliclit arguments suggest infinitely many solutions, but this is far from known. In fact, even for small n (e.g. 33 and 42) the smallest solution can be very large and hard to find (see Booker for 33 and Booker-Sutherland for 42). The smallest n w/ no known solution is $n = 114$.

  (Audience remark) sounds like Victor Wang has shown there are solutions for almost all n. $\triangle$

- Linear recurrence sequences $u_{n+h} = a_1 u_{n+h-1} + \cdots + a_h u_n$. Does $u_n = 0$ for any n?

- Isogeny estimates: given abelian varieties A, B, decide whether they are isogeneous (and give a priori upper bound for degree of isogeny).

  Sounds like first bounds make from work of Masser-Wüstholz.

  Question 1.1.6 (Audience)**.** *When you say given abelian varieties, how are they given?*

  Answer. Say, for example, given by explicit equations over some number field. $\star$

- Applications to the "modular context." See work of van Koenil (spelling?), Alpöge, Ullmo, etc.

- Effectivity for rational points on "transcendental varieties." See work of Bombieri-Pila, Pila-Wilkie, Binjamini (spelling?), etc.

1.1.2 Curves

Say $\widetilde{X}$ is a smooth projective curve over $\overline{\mathbb{Q}}$. Here, things split between rational points and integral points (in the affine case). Ideally, these two cases would be unified, but the methods to treat them can be a little different.

Let's start with rational points.

- Hilbert-Hurwitz gave an effective solution when $g = 0$.

- For $g = 1$, have work of Mordell and Weil, but it's not yet effective.

- For $g \geq 2$, have Faltings theorem, but it is non-effective. His original proof didn't even give effective bounds for the number of solutions (yet alone anything like an algorithm for computing them). Later methods due to Vojta (and Bombieri) gave bounds for number of solutions which were made uniform by Dimitrov-Gao-Habeggar.

  Remark 1.1.7 (Audience)*.* Apparently Faltings' argument can be used to give bounds on the number of solutions $\circ$

- Mazur determined all the points on certain modular curves

- Wiles solved Fermat

- There's the Chabauty-Kim method, developed by work of Kim, Balakrishnan, Dogra, etc.

What about integral points?

- Have Siegal's theorem: an *affine* curve X may have infinitely many integral points only if it has $g = 0$ and $\left|\widetilde{X} \setminus X\right| \leq 2$.

  Example 1.1.8. $\mathbb{P}^1 \setminus \{0, 1, \infty\}$ has only finitely many solutions, i.e only finitely many $x, y \in \mathscr{O}_S^\times$ s.t. $x + y = 1$. △

  Siegal's theorem is not yet effective, except when $g = 1$ (using Baker's estimates for linear forms in logarithms of algebraic numbers).

- For genus 2, if the affine model is of the form $y^2 = f(x)$, then Baker's method can be used to find the integral points. However, it may be given by other models, e.g. $y^4 + x^3 + ay^2 + bxy + cx^2 = 0$. For other models, there's no known general algorithm for finding all integral points on such a curve.

  Zannier ended by saying, out loud, a bit of what he did with Corvaja. Sounds like by combining Baker w/ a criterion of Bilu (and maybe work of Poonen?), they were able to find 1-parameter families $F_\xi(x, y) = 0$ of this sort such that you get effectivity (over every number field) for a dense set of algebraic points in the family. Their dense set is one of bounded height.

Related to the existence of independent maps to $\mathbb{G}_m$

1.2 Alina Ostafe (University of New South Wales): Integer Matrices with a Given Characteristic Polynomial and Multiplicative Dependence of Matrices

Look at some questions of arithmetic statics of matrices from

$$M_n(\mathbb{Z}) = \left\{A = (a_{ij})_{i,j=1}^n : a_{ij} \in \mathbb{Z}\right\}.$$

Definition 1.2.1. A tuple $(A_1, \ldots, A_s) \in \mathrm{GL}_n(\mathbb{Z})^s$ is multiplicatively dependent if there exists some $\underline{k} = (k_1, \ldots, k_s) \in \mathbb{Z}^s \setminus \{0\}$ such that

$$A_1^{k_1} \ldots A_s^{k_s} = I_n.$$ ◇

Why study this? Say $(a_1, \ldots, a_s) \in \mathbb{C}^s$ is multiplicatively dependent if the same condition holds. This is looking at tuples of complex/algebraic numbers belong to some algebraic subgroup. In work of Poppelondi (spelling?), Sho, Shpanlinski (spelling?), Stewart (2018), they gave an asymptotic formula for the number of tuples of algebraic integers of bounded height and of bounded degree (or in a given number field).

Warning 1.2.2. The property of being multiplicatively dependent of matrices depends on the choice of ordering the matrices. •

Definition 1.2.3. A tuple $(A_1, \ldots, A_s) \in \mathrm{GL}_n(\mathbb{Z})^s$ is non-free if there exists a non-trivial word in these matrices which gives the identity, i.e. $A_{i_1}^{\pm 1} \ldots A_{i_L}^{\pm 1} = I_n$. ◇

What do we count?

- For $H \geq 1$, let

  $$M_n(\mathbb{Z}, H) = \{A = (a_{ij}) : |a_{ij}| \leq H\} \subset M_n(\mathbb{Z}).$$

  Note $\#M_n(\mathbb{Z}, H) \sim (2H)^{n^2}$.

- Let

  $$M_{n,s}(H) := \{(A_1, \ldots, A_s) \in M_n(\mathbb{Z}, H)^s : (A_1, \ldots, A_s) \text{ mult. dep.}\}$$

  Let $M_{n,s}^+(H) \subset M_{n,s}(H)$ be those tuples which are multiplicative dependent of maximal rank, i.e. any subtuple is multiplicative independent.

- Let

$$\mathcal{F}_{n,s}(H) := \{(A_1, \ldots, A_s) \in M_n(\mathbb{Z}, H) : (A_1, \ldots, A_s) \text{ not free}\}.$$

Goal. Both lower and upper bounds for the cardinalities of these sets.

Remark 1.2.4. For $n = 1$, same as questions in P.S.S.S. ('18) ○

Hard to know how to use matrix structure of such tuples, so one cheats by passing to determinants. In general, want something stronger than what this gives (e.g. imagine matrices are in SL_n).

- They count the number of s-tuples of integers in $[-n!H^n, n!H^n]$ which are multiplicatively dependent.
- Finally, they need to estimate the number of matrices $A \in M_n(\mathbb{Z}, H)$ with given determinant. Set

$$\mathcal{D}_n(H, d) := \{A \in M_n(\mathbb{Z}, H) : \det(A) = d\}.$$

1.2.1 Matrices w/ given determinant

A variant of $\mathcal{D}_n(H, d)$ (using L^2-norm instead of L^∞-norm) has been studied by Duke, Rudnick, Sarnak ('93) when $d \neq 0$ and by Katznelson ('93) when $d = 0$. For fixed d, they gave an asymptotic formula for $\#\widetilde{\mathcal{D}}_n(H, d)$ w/ main terms H^{n^2-n} (when $d \neq 0$) or $H^{n^2-n+o(1)}$ (when $d = 0$. The $o(1)$ comes from a logarithmic factor)

In the current work, they needed an upper bound which is uniform w.r.t. d. Hence, they instead use a result of Shparlinski ('10) whose proved that, uniformly over d, $\#\mathcal{D}_n(H, d) \ll_n H^{n^2-n+o(1)}$. I missed something, but sounds like they in fact ended up wanting to not just fix the determinant, but to instead fix the entire characteristic polynomial.

1.2.2 Matrices w/ a given characteristic polynomial

Let

$$\mathcal{K}_n(H, f) = \{A \in M_n(\mathbb{Z}, H) : \mathrm{charpoly}(A) = f\}.$$

Eskin, Mofes, Sha ('96) gave an asymptotic formula for $\#\widetilde{\mathcal{K}}(H, f)$, proving something like

$$\#\widetilde{\mathcal{K}}_n(H, f) = (c(f) + o(1))H^{\frac{n(n-1)}{2}},$$

where $c(f) > 0$ depends on monic, irreducible f.

Question 1.2.5 (Audience)**.** *How did they prove this, using volume estimates?*

Answer. Don't know, but may using some homogenous dynamics and/or some lattice bounds? ⋆

In the current problem, they need an upper bound which

- holds for arbitrary monic $f \in \mathbb{Z}[x]$
- is uniform w.r.t. to the coefficients of f

Conjecture 1.2.6 (O., Shparlinski '22)**.** *Uniformly over f,*

$$\#\mathcal{K}_n(H, f) \leq H^{\frac{n(n-1)}{2}+o(1)} \quad \text{as } H \to \infty.$$

What is the trvial bound. Only care about determinant:

$$\#\mathcal{K}_n(H,f) \le \#\mathcal{D}_n(H,d) \ll H^{n^2-n+o(1)}.$$

Define γ_n to be the largest real number such that, uniformly over polynomials f, we have $\#\mathcal{K}_n(H,f) \le H^{n^2-n-\gamma_n+o(1)}$ as $H \to \infty$. The conjecture says that $\gamma_n = n(n-1)/2$. Shparlinski ('10) shows $\gamma_n \ge 0$.

Theorem 1.2.7 (O.-Shparlinski '22)**.** *$\gamma_2 = 1$, $\gamma_3 \ge 1$, and*

$$\gamma_n \ge \frac{1}{(n-3)^2} \text{ for } n \ge 1.$$

Only $\gamma_2 = 1$ corresponds to the conjecture. For $n \ge 4$, the proof of this result is only using that the determinant and trace are fixed, not all coefficients of the char poly. Let

$$\mathcal{S}_n(H,d,t) = \{A \in M_n(\mathbb{Z},H) : \det A = d \text{ and } \operatorname{Tr} A = t\}.$$

They prove that

$$\#\mathcal{S}_n(H,d,t) \ll H^{n^2-n-\sigma_n}, \text{ where } \sigma_n = \frac{1}{(n-3)^2}$$

if $n \ge 4$. Idea of the proof

- reduce the problem to estimating the cardinality $\#\mathcal{U}_n(H)$, where

  $$\mathcal{U}_n(H) := \left\{ A = \begin{pmatrix} R^* & a^* \\ b^* & 0 \end{pmatrix} \in M_n(\mathbb{Z},H) : \det(A) = 0, a^*, b^* \ne 0, a_{nn} = 0 \right\}$$

  (a^*, b^* are column,row vectors).

- Use ideas of Katznelson ('93). There is a set of "good" λ's (primitive integer vectors) where $|\lambda| \ll H^{n-1}$, $A\lambda = 0$, and dual lattices are almost cubic.

- Sounds like they count matrices one row at a time (each row belongs to dual lattice of λ) using some result of Schmidt ('68).

- There's additional difficulty w/ the last row since $(\lambda_1, \dots, \lambda_{n-1})$ might not be good.

1.2.3 Coming back to multiplicative dependence

The prove

$$H^{5n^2-\lceil\frac{5}{2}\rceil n+o(1)} \ge \#M^*_{n,s}(H) \ge \begin{cases} H^{(s-1)\frac{n^2}{2}+\frac{n}{2}+o(1)} & \text{if } s \text{ even} \\ H^{(s-1)\frac{n^2}{2}+o(1)} & \text{if } s \text{ odd} \end{cases}$$

(lower bound from construction, upper bound from taking determinants).

Remark 1.2.8. Unclear what the truth should be. ○

Other results as well that I didn't bother writing down...

1.3 Harry Schmidt (Universität Basel): Effective Counting and Families of Elliptic Curves

(joint w/ Bingamini, Jones, Thomas)

The first part will be related to counting rational points on transcendental varieties. Then, we'll give some applications, related e.g. to Manin-Mumford and André-Oort.

Consider the Legenedre family

$$\mathcal{E} : y^2 = x(x-1)(x-\lambda) \text{ with } \mathscr{O} = \text{"point at } \infty\text{"}$$

($\lambda \neq 0, 1$, i.e. $\lambda \in Y(2) = \mathbb{P}^1 \setminus \{0, 1, \infty\}$).

We say that $\lambda_0 \in Y(2)$ is `special` if $\operatorname{rank} \operatorname{End}(\mathcal{E}_{\lambda_0}) = 2$ (i.e. if λ_0 is a CM point).

Example 1.3.1. $\mathcal{E}_\lambda : y^2 = x^3 - x$ has $[i](x, y) = (-x, iy)$ and $\operatorname{End}(\mathcal{E}_\lambda) = \mathbb{Z}[i]$. Here, $\lambda = -1$ is special. △

Fact:

$$\#\{\lambda : [\mathbb{Q}(\lambda) : \mathbb{Q}] = D, H(\lambda) \leq T, \operatorname{rank} \operatorname{End}(\mathcal{E}_\lambda) = 1\} \gg_\varepsilon T^{1+D-\varepsilon}$$

(assuming I copied this down correctly).

Special subvarieties of $Y(2)^g$ will be of the form $S_0 \times S_1 \times Y(2)^s$ (up to permutation of the coordinates), where S_0 is a product of special points, and S_1 is a product of modular curves.

Let's introduce a measure of complexity of these special subvarieties. We want to know that if we bound this complexity, then we can list all special subvarieties that are of bounded complexity. This will be

$$\max\{|\operatorname{Disc} \operatorname{End}(\mathcal{E}_{\lambda_i})|\} + \deg(S_1)$$

(discriminant of endomorphism rings along w/ degree of modular curves)

Question 1.3.2 (Audience)**.** *What is meant by degree of S?*

Answer. Literally degree as a projective subvariety of $(\mathbb{P}^1)^g$, nothing fancy. ⋆

Fact. Special subvarities contain a Zariski dense set of special points.

(isogeneous ellipitic curves are either both CM or neither CM)

Consider $\pi : \mathcal{E}^g \to Y(1)^g$. We say $S \subset \mathcal{E}^g$ is `special` if it is a component of a subgroup scheme $\mathcal{G} \subset \mathcal{E}^g$ such that $\pi(\mathcal{G}) \subset Y(2)^g$ is special.

Example 1.3.3. $\mathcal{E}^{(g)} = \mathcal{E} \times_\pi \ldots \times_\pi \mathcal{E}$ above the diagonal downstairs. △

A `special point` $p \in \mathcal{E}^g(\overline{\mathbb{Q}})$ is one such that p is torsion in $\mathcal{E}^g_{\pi(p)}$ and $\mathcal{E}^g_{\pi(p)}$ is a product of CM elliptic curves.

Theorem 1.3.4 (`André-Oort for` $\mathcal{E}^g$, Gao, Pila)**.** *$V \subset \mathcal{E}^g$ contains a Zariski dense set of special points if and only if it is special.*

This apparently implies Manin-Mumford for products of elliptic curves w/ CM.

Theorem 1.3.5 (`Manin-Mumford`, Raynaud)**.** *Let $V \subset A$ be a subvariety of an abelian variety A. Then,*

$$\overline{V \cap A_{tors}}^{Zar} = \bigcup_{i=1}^{N} (T_i + H_i)$$

is a finite union of translates of connected algebraic subgroups by torsion points.

This theorem is effective. Given V, A (e.g. in terms of explicit equations), there is an algorithm which can return the T_i's and H_i's. André-Oort, on the other hand, is not effective.

This talk is about reducing effectiveness (of AO) for this family $\mathcal{E}^g$ to effectiveness on the base.

1.3.1 Effective methods

The proof of AO for $\mathcal{E}^g$ uses Pila-Wilkie.

Theorem 1.3.6 (Pila-Wilkie). *Let $X \subset \mathbb{R}^n$ be definable in an o-minimal structure. Then,*

$$X(\mathbb{Q}, T) := \{\underline{x} \in X \cap \mathbb{Q}^n : H(\underline{x}) \leq T\}$$

satisfies $\#X^{trans}(\mathbb{Q}, T) \ll_\varepsilon T^\varepsilon$.

It's great that this theorem is very general. However, it's not effective. Can't even easily compute the implicit constant.

Let $f_1, \ldots, f_r$ be analytic functions on $U \subset \mathbb{R}^n$, a product of open integrals, such that

$$\partial_j f_i = P_{ij}(\underline{x}, f_1, \ldots, f_i) \text{ for } i, j = 1, \ldots, r \text{ and } P_{ij} \in \mathbb{R}[x_1, \ldots, x_n, x_{n+1}, \ldots, x_{n+r}].$$

For any polynomial F, a function of the form $f = F(x, f_1, \ldots, f_r)$ is called a Pffaffian of format $n+r$ and degree $\sum \deg(P_{ij}) + \deg F$.

Example 1.3.7 (I think, I'm confused)**.** Take $f_1 = \exp(x)$ (so $P_{11} = X$). Then, $f = f_1$ is a Pffaffian. △

Definition 1.3.8. Say $X \subset \mathbb{R}^n$ is semi-Pfaffian if it is defined by equalities and inequalities of Pfaffian functions $h_1, \ldots, h_\ell$. The format and degree of X are

$$\text{format}(X) = \sum \text{format}(h_i) \text{ and } \deg(X) = \sum \deg(h_i).$$

Say Y is subpfaffian if it is a projection of a semipfaffian set (format/degree of Y are the same as of X). ⋄

Theorem 1.3.9 (Bingamini-Jones-S.-Thomas)**.** *Let $X \subset \mathbb{R}^n$ be subpfaffian of format r and degree d, and choose $\varepsilon > 0$. Then,*

$$\#X^{trans}(\mathbb{Q}, T) \leq c d^\gamma T^\varepsilon,$$

for some effectively computable $c = c(\varepsilon, r)$ and $\gamma = \gamma(r)$.

Should not expect to get a bound that's better than polynomial in d.

For an abelian variety A of dimension g, have uniformization map

$$\exp_A : \mathbb{C}^g \longrightarrow A$$

(exponential at identity). Would not expect graph of this to be algebraic, but can write it in terms of Pfaffians.

Theorem 1.3.10. *Let A be a product of elliptic curves. The graph of $\exp_A$ ([1]) is a sub-pfaffian set whose format and degree only depend on $g = \dim A$ and are effectively computable.*

Theorem 1.3.11 (Gao)**.** *Let A be an abelian variety w/ CM, and let $P \in A(\overline{\mathbb{Q}})_{tors}$ of order N. Then,*

$$[\mathbb{Q}(A, P) : \mathbb{Q}(A)] \gg_\varepsilon N^{1-\varepsilon}.$$

(Gives Galois lower bound for torsion points in CM abelian varieties)

This a uniform arithmetic result. Can combine this w/ the uniform transcendental result above it.

[1] restricted to a fundamental domain

Let A be a product of g CM elliptic curves. Let $V \subset A$ be an algebraic subvariety defined over a number field K.

Theorem 1.3.12 (BJST)**.** *Choose $\varepsilon > 0$. Then,*

$$\overline{V \cap A_{tors}} = \bigcup_{i=1}^{N} (T_i + H_i),$$

where $\operatorname{ord}(T_i) \leq c[K:\mathbb{Q}]^{2g+\varepsilon} \deg(V)^{\gamma}$*,* $\deg(H_i) \leq c \deg(V)^{\gamma}$*, and* $N \leq c[K:\mathbb{Q}]^{2g+\varepsilon} \deg(V)^{\gamma}$ *for effectively computable $c = c(\gamma, \varepsilon)$ and $\gamma = \gamma(g)$.*

Dill proved an analogous result for E^g (though didn't include a uniform bound for the number of torsion cosets).

Theorem 1.3.13. *Let $V \subset \mathcal{E}^{(g)}$, defined over a number field K. Then, any maximal special subvariety $S \subset V$ has complexity at most* $\exp(c[K:\mathbb{Q}]\deg(V)^{\gamma})$ *for some effectively computable $\gamma = \gamma(g)$ and $c = c(g)$.*

Habbeggar proved this using different methods w/o the effectiveness (or uniformity?).

How does one get this from the previous theorem? For the proof idea, say $g = 2$, we have $V \subset \mathcal{E}^{(2)}$, and $\pi : V \twoheadrightarrow Y(2)$. Look at some $V_{\lambda_0} = \pi^{-1}(\lambda_0)$ (λ_0 CM). By previous theorem, for $P \in V_{\lambda_0}$ torsion, either

(1) $\operatorname{ord}(P) \leq c[K:\mathbb{Q}]^{4+\varepsilon} \deg(V)^{\gamma}$; or

(2) $T + H \subset V_{\lambda_0}$ (and we have bounds on $\deg(H), \operatorname{ord}(T)$).

I didn't follow how to conclude...

$S_{\lambda_0} \subset V_{\lambda_0}$ with S_{λ_0} a torsion point or translate of a connected algebraic subgroup.

(i) S_{λ_0} =specialization of a special subvariety $S \twoheadrightarrow Y(2)$. Either $S \subset V$ or $\dim(S \cap V) < \dim(V)$. Induct.

(ii) S cannot be spread out

$$S_{\lambda_0} = \{(P_1, P_2) \in \mathcal{E}_{\lambda_0} \times \mathcal{E}_{\lambda_0} : \alpha_1 P_1 + \alpha_2 P_2 = 0\} \text{ for some } \alpha_1, \alpha_2 \in \operatorname{End}(\mathcal{E}_{\lambda_0}).$$

Sounds like can assume $(\alpha_1, \alpha_2) \notin \mathbb{Z}^2$ or else in case **(i)**? Can prove $\deg S_{\lambda_0} \gg \max\{|\alpha_1|, |\alpha_2|\} \gg \operatorname{disc}\operatorname{End}(\mathcal{E}_{\lambda_0})^{\gamma}$.

1.4 Levent Alpöge (Harvard University): Modularity and Effective Mordell

Conjecture 1.4.1 (Mordell, 1922)**.** *Let K be a number field. Let C/K be a smooth, projective, hyperbolic ($g > 1$) curve. Then, $C(K)$ is finite.*

(Proved by Faltings in 1983)

History. Mordell was an American mathematician born in Philadelphia who spent most of hist life in Cambridge. As a kid, he tutored to save up enough money for one trip across the Atlantic + a one word telegram. He took Cambridge entrance exams, passed, and sent his father, "Hazzah!"

Levent then told more history, but talked faster than I can comfortably type... $\ominus$

We'd like to find all of $C(K)$.

Question 1.4.2 (Diophantus, Book VI of Arithmetica)**.** *Solve* $y^2 = x^6 + x^2 + 1$ *over* $\mathbb{Q}$

Diophantus finds $(\pm 1/2, \pm 9/8)$ as well as $(0, \pm 1)$ (and, implicitly, $\pm\infty$).

Theorem 1.4.3 (Wetherell, 1998)**.** *That's all*

(in Berkeley thesis)
Here's the point of the talk.

Theorem 1.4.4 (A.)**.** *Let $K/\mathbb{Q}$ be an odd-degree totally real number field. Let S be a finite set of places of K. Let $g \in \mathbb{Z}^+$. Then, the finite set*

$$\{A/K \text{ ab var.} : \dim A = g, \text{ good reduction outside } S, \text{ of } \mathrm{GL}_2\text{-type}/K\}$$

can be computed.

Definition 1.4.5. Say A/K is a simple g-dimensional abelian variety. Then, it is `of` GL_2`-type/`K if and only if there is a number field $E/\mathbb{Q}$ with $[E : \mathbb{Q}] = g$, and order $\mathscr{O} \subset E$, and an embedding $\mathscr{O} \hookrightarrow \mathrm{End}_K(A)$. ◇

Theorem gives effective determination of some integer points of certain Hilbert modular varieties.

Corollary 1.4.6. *Let $K/\mathbb{Q}$ be an odd-degree totally real field. As $a \in K^\times$ varies, consider*

$$C_a : x^6 + 4y^3 = a^2.$$

Then, $C_a(K)$ can be computed.

You can deduce analogous statements for other families of curves if you want.

Remark 1.4.7. The content of this talk is related, at least somewhat, to Brian's talk on Friday. ○

Why the odd degree, totally real hypothesis? The proof will use (parallel weight 2?) Hilbert modular forms, and we need to know these are associated to abelian varieties, or something like this? Basically, wanna use modularity, so need these assumptions w/ current technology.

When $g = 1, K = \mathbb{Q}$, the theorem is due to Murty-Pasten. When $K = \mathbb{Q}$ and g arbitrary, the theorem is due to von Känel. When $K = \mathbb{Q}$, modularity is known, which simplifies things. In general, only potential modularity is known, so need to "make this effective."

Another technique for effectivity: find $C \to A/K$ with $\operatorname{rank} A(K) = 0$ (compute height differences of torsion points).

Theorem 1.4.8 (Demjanenko)**.** *If* $\operatorname{rank}_{\mathbb{Z}} \operatorname{Hom}(\operatorname{Jac} C, A) > \operatorname{rank} A(K)$*, then also done.*

p-adic techniques (e.g. Lawrence-Venkatesh, Chabauty-Kim) allow one to compute a finite set containing $C(K)$. The question then becomes one of recognizing/determining whether a zero of a p-adic analytic function is transcendental or algebraic. This are generally practical when they apply.

Recall 1.4.9 (Faltings' argument)**.** First find a map $C \to \mathcal{A}_{\widetilde{g}}$ ($\widetilde{g} \neq g(C)$ in general). Kodaira-Parshin gives a finite-to-one such map (think: non-isotrivial family of abelian varieties on your curve). Can extend K, expand S to get nice integral model $\mathcal{C} \to \mathcal{A}_{\widetilde{g}}/\mathscr{O}_{K,S}$ of this. To ease notation, let's drop the tilde of $\widetilde{g} =: g$.

Choose prime ℓ away from S. Second,

$$\{A/K : \dim A = g, \text{good outside } S\}/_{\sim_K} \hookrightarrow \left\{\rho : \mathrm{Gal}(\overline{\mathbb{Q}}/K) \to \mathrm{GL}_{2g}(\mathbb{Q}_\ell) \,\middle|\, \begin{matrix} \text{unramified outside } S \text{ and } \ell \\ \text{pure of weight 1, etc.} \end{matrix} \right\}$$

Faltings bounds the RHS, so only finitely many isogeny classes on the LHS. Third,

$$\{B/K : B \sim_K A\} \text{ finite}$$

(e.g. by Raynaud/Masser-Wüstholz) $\odot$

Where did we lose? Why is this non-effective? Step 2. Characterizing which representations come from abelian varieties is an issue.[2]

Remark 1.4.10. In Faltings argument, who forced us to use the Kodaira-Parshin family? For example, if you start w/ a canonical model of a Shimura curve, then it comes equipped w/ a natural non-isotrivial family of abelian varieties over it. However, Faltings' argument would have you forget this and use Kodaira-Parshin? $\circ$

Claim 1.4.11 (slash motivation, Bogomolov-Tschinkel, Poonen). *Every solvable cover C of $\mathbb{P}^1$ admits a diagram of the form*

$$\begin{tikzcd} & \widetilde{C} \arrow[dl, "\acute{e}t"'] \arrow[dr] & \\ C & & \{y^2 = x^6 + 1\} \end{tikzcd}$$

over $\mathbb{P}^1$.

Above, $y^2 = x^6 + 1$ is a Shimura curve over $\overline{\mathbb{Q}}$ (associated to discriminant 6 quaternion algebra $B_6/\mathbb{Q}$), and admits a canonical family $\{y^2 = x^6 + 1\} \to A_2$ of abelian surfaces.

Recall we earlier had mentioned $x^6 + 4y^3 = 1$ ($a = 1$). Consider the Prym of the desingularization of

$$Y^6 = X^4(1 - X)^3(\lambda - X),$$

and call this A_λ. Then $A_\lambda = 2$ (above curve of genus 3 and maps to an elliptic curve). Evidently, $\mathbb{Z}[\zeta_3] \hookrightarrow \mathrm{End}_{\mathbb{Q}(\lambda,\zeta_3)}(A_\lambda)$. It also has quaternionic multiplication defined over $\mathbb{Q}(\lambda, \zeta_3, (\lambda[(1-\lambda)/4]^2)^{1/6})$. This assignment gives a map

$$C(K) \ni P \longmapsto A_P/K \text{ w/ quat mult over } K(\zeta_3)$$

(so GL_2-type$/K$). The quaternion algebra here is again B_6.

1.4.1 Sketch of Proof of Theorem

If we had an L/K (w/ L odd degree, totally real) such that all relevant abelian varieties were modular over L, we'd be done. Then, every such abelian variety would be a quotient of Jac(explicit Shimura curve) over L. To get this Shimura curve, look at the quaternion algebra ramified at all but one infinite place of L, and show that A is a quotient by comparison of L-functions (something something Jaccuet-Langlangs something something). Thus, this Jacobian is L-isogenous to $A \times B$ and

$$h(A) \le -h(B) + \log(\text{M.W.}) + h(\text{Jac}),$$

and every term on the right can be bounded explicitly.

Question 1.4.12 (Audience). *What happened to the level?*

[2] Vaguely similar to deciding which zeros of a p-adic analytic function come from rational points, or which elements of n-Selmer come from $E(K)/nE(K)$.

Answer. This is hidden in the "explicit Shimura curve." All relevant A/K have good reduction outside S. Can use work of Brumer-Kramer ("The conductor of an abelian variety") to bound the level in terms of S. ⋆

Now, the whole fight is to produce such an L. If we had an a priori λ such that all relevant A/K had $\overline{\rho}_{A,\lambda}$ w/ large image, then we'd be done. One follows Taylor's proof of potential modularity. Construct an X (depending on $\overline{\rho}_{A,\lambda}$). Find a totally real (Galois over $\mathbb{Q}$) point on X (Taylor used work of Moret-Bailly/Rumely). Then, L is basically a field of definition of that point (really, $L = \widetilde{L}^{\text{2-Sylow}}$ if $\widetilde{L}$ is the fixed field. Using solvable descent for GL_2, whatever that is).

Proposition 1.4.13. *Let K, E be number fields. Let S be a finite set of places of K. Then, for* $\mathrm{Nm}(\lambda) \gg_{[E:\mathbb{Q}],K,S} 1$ *effectively, for all non-CM, $\overline{\mathbb{Q}}$-simple, g-dimensional ($g := [K : \mathbb{Q}]$) abelian varieties A/K which are $\mathrm{GL}_2(\mathscr{O}_E)$-type over K, $\overline{\rho}_{A,\lambda}$ has large image. It'll contain a conjugate of $\mathrm{SL}_2(\mathbb{F}_\ell)$ (where $\lambda \mid (\ell)$).*

2 Day 2

2.1 Xinyi Yuan (Peking University): Bigness of the Admissible Canonical Bundle

Note 1. This talk is over zoom (using notability). The audience asked the speaker to write bigger, and then the (projector operator?) just increased the zoom on the projection instead.

2.1.1 Uniform Bogomolov

For us, a curve is a smooth, projective, geometrically integral 1-dimensional scheme over a field.

Theorem 2.1.1 (Ullmo (1998?), Bogomolov Conjecture)**.** *Let $C/\overline{\mathbb{Q}}$ be a curve of genus $g > 1$. Fix $\alpha \in \mathrm{Pic}^1(C)$, so we get an embedding*

$$i_\alpha : C \hookrightarrow J, \;\; x \mapsto x - \alpha.$$

Then, there exists a constant $\varepsilon > 0$ such that

$$\left\{x \in C(\overline{\mathbb{Q}}) \mid \widehat{h}(x-\alpha) = \widehat{h}(i_\alpha(x)) < \varepsilon\right\} < \infty.$$

I guess, as a consequence, there's an absolute lower bound on the heights of "non-torsion" points.

Sounds like this was proved using an equidistribution argument.

Nowadays, we have a uniform version of this statement.

Theorem 2.1.2 (Dimitrov-Gao-Habegger, Kühne)**.** *Fix $g > 1$. There exists constants $c_1, c_2 > 0$, depending only on g, such that for all $C/\overline{\mathbb{Q}}$ of genus g and all $x_0 \in C(\overline{\mathbb{Q}})$, we have*

$$\#\left\{x \in C(\overline{\mathbb{Q}}) \mid \widehat{h}(x - x_0) < c_1 \max\left\{h_{Fal}(C), 1\right\}\right\} < c_2.$$

Note that "uniform Bogomolov" could mean the above with $c_1 \max\{blah\}$ replaced by c_1. The stronger version above is needed for their application to uniform Mordell.

Remark 2.1.3. Above, the Faltings' height of C is the Faltings' height of $\mathrm{Jac}(C)$. ○

Remark 2.1.4. DGH proved this theorem when $h_{\mathrm{Fal}}(C)$ is large. Kühne strengthened it to the above form (did it for small Faltings height). ○

Theorem 2.1.5 (Uniform Mordell)**.** *For any $X/\mathbb{C}$ of genus $g > 1$ and any finite rank $\Gamma \subset J(\mathbb{C})$, we have*

Not necessarily finitely generated

$$\#(X(\mathbb{C}) \cap \Gamma) \leq c(g)^{\operatorname{rank}\Gamma+1}.$$

(Sounds like Mazur conjectured this)

Remark 2.1.6. This is implied by

(1) (DGH, K) $\#\{\text{small points}\} <?$

(2) (Vojta's proof) $\#\{\text{large points}\} <?$ ◦

The main theorem of this talk generalizes uniform Bogomolov Theorem 2.1.2.

Theorem 2.1.7 (Y.)**.** *Fix $g > 1$. There exists constants $c_1, c_2 > 0$, depending only on g, such that for any K which is either a number field K or a function field of 1 variable over k, for any genus g curve $X/\overline{K}$, for any $\alpha \in \operatorname{Pic}^1(X)$ (as long as (X, α) is non-isotrivial in the function field case), then*

$$\#\left\{x \in X(\overline{K}) \mid \widehat{(}x-\alpha) < c_1\Big(\max\{h_{Fal}(X), 1\} + \widehat{h}((2g-2)\alpha - \omega_X)\Big)\right\} < c_2.$$

Remark 2.1.8. Compared to Theorem 2.1.2, above theorem allows K to be a function field, allows the basepoint to be a divisor $\alpha \in \operatorname{Pic}^1(X)$, and adds an extra height term to the set being bounded. Somehow this extra term is like a "uniform Mumford inequality"? Also, c_1, c_2 don't depend on the field K at all. ◦

Remark 2.1.9. Looper-Silverman-Wilms proved the result for function fields independently, but still w/o the extra term. However, they have surprisingly explicit constants c_1, c_2 in terms of g (e.g. c_2 can be a quadratic polynomial in g). Their proof using admissible pairing (introduced by Shou-Wu) on single curve X. In constrast, DGH+K and Y. worked in families, on the moduli space. ◦

Remark 2.1.10 (Response to audience question)*.* For small points, no finite rank assumption is needed for finiteness. However, for large points (e.g. in Vojta's argument), you do need to work in something of finite rank. ◦

Let's mention some of the ideas in some of these arguments

(1) (DGH, K) used

- non-degeneracy (model theory)
- height inequality
- equidistribution

(2) (Y.) used

- adelic line bundles by Y.-Zhang
- bigness of admissible canonical sheaf

2.1.2 Admissible Canonical Sheaf

Recall 2.1.11. If K is a number field, and X/K is quasi-projective, then an adelic line bundle is a pair $(L, \{\|\cdot\|_v\}_{v\in M_K})$ where

(1) L is a line bundle on X

(2) $\|\cdot\|_v$ is a metric of L on the (Berkovich/complex) space $X^{\mathrm{an}}_{K_v}$ (or on $X(\overline{K}_v)$)

(3) the metric is a limit of metrics induced by projective models $(\mathcal{X}, \overline{\mathscr{L}})$ on (X, L) over $\mathscr{O}_K$

$\mathcal{X}$ is projective over $\mathscr{O}_K$ and $X \hookrightarrow \mathcal{X}_K$ is an open immersion. The limit is taken w.r.t. to a "boundary topology" which they define. $\odot$

Assumption. Go ahead an always assume K is a number field.

Recall 2.1.12. Say X/K is a curve of genus $g > 1$. Then, there is an *admissible* adelic metric $\|\cdot\|_v$ on $\omega_{X/K}$ for any place $v \in M_K$.

(1) $v \mid \infty$, use Arakelov metric

(2) $v \nmid \infty$, use Zhang metric

Hence, get adelic line bundle $\overline{\omega}_{X/K}$ on X/K. $\odot$

Fact (Zhang)**.** $\overline{\omega}_{X/K}^2 > 0 \iff$ Bogomolov conjecture

This path to prove Bogomolov was finished by Zhang, de Jong, Cinkir.

Xinyi's approach to uniform Bogomolov is like a family version of this path (w/o using equidistribution or model theory).

Question 2.1.13 (Audience)**.** *Can you explain what admissible means?*

Answer. It's a bit complicated, but here's one way to see it. Consider $C \xhookrightarrow{i} J$ via $x \mapsto (2g-2)x - \omega_{X/K}$. Have canonical metric on theta divisor $\overline{\Theta}$ (satisfying e.g. $[2]^*\overline{\Theta} = 4\overline{\Theta}$). Recall that $i^*\Theta \cong \omega_{X/K}^e$ for some power e. Admissibility means that $i^*\overline{\Theta} \cong \overline{\omega}_{X/K}^e$ (so metric compatible w/ canonical metric on theta divisor). $\star$

2.1.3 Family version

Assume S/K is a quasi-projective variety, and assume $\pi : X \to S$ is a family of curves, i.e. smooth, projective w/ fibers genus $g > 1$ curves (e.g. take $S = M_{g,N}$ and X the universal family). Then, $\omega_{X/S}$ extends to a canonical adelic line bundle $\overline{\omega}_{X/S}$.

Let J be the relative Jacobian of X/S. Map $X \xrightarrow{i} J$ over S via $x \mapsto (2g-2)x - \omega_{X/S}$:

$$\begin{tikzcd} X \arrow[rr, "i"] \arrow[dr, "\pi"'] & & J \arrow[dl] \\ & S. & \end{tikzcd}$$

Have symmetric, relatively ample line bundle Θ on J along w/ canonical metric $\overline{\Theta}$ (so $[2]^*\Theta = 4\Theta$). Let $\overline{L} := i^*\overline{\Theta}$.

(1) $\overline{L} = 4g(g-1)\overline{\omega}_{X/S} - \pi^* \left\langle \overline{\omega}_{X/S}, \overline{\omega}_{X/S} \right\rangle$

(2) $\left\langle \overline{L}, \overline{L} \right\rangle = 16g(g-1)^3 \left\langle \overline{\omega}, \overline{\omega} \right\rangle$

Here, $\left\langle \overline{L}, \overline{L} \right\rangle$ is the Deligne pairing (line bundle on S).

One consequence of this is that $\overline{\omega}_{X/S}$ is nef, because $\overline{L}$ is nef.

Theorem 2.1.14. *Assume S has* `maximal variation`*, i.e. $S \to M_g$ is generically finite. Then, $\overline{\omega}_{X/S}$ is big.*[3]

[3]Because it is already known to be nef, this just means that its top self intersection is positive

2.1.4 Proof of Unif. Bog.

Keep in mind the diagram

$$\begin{tikzcd} X \arrow[rr, "i"] \arrow[dr, "\pi"'] & & J \arrow[dl, "\pi'"] \\ & S, & \end{tikzcd}$$

with $i : x \mapsto (2g-2)x - \omega_{X/S}$. Also, S/K is quasi-projective.

Note 2. For reasons I missed, the above is a "simplified" setting compared to what's really need to prove the theorem.

We want $h_{\overline{L}}(x) = h_{\overline{\Theta}}(i(x))$ to be "big" for most $x \in X(\overline{K})$. If $\overline{L}$ is big, this is the case by the height inequality: for any Weil height $h_M : S(\overline{K}) \to \mathbb{R}$, there is some $\varepsilon > 0$ such that

$$\left\{x \in X(\overline{K}) \mid h_{\overline{L}}(x) < \varepsilon h_M(\pi(x))\right\}$$

is not Zariski dense in X.

Warning 2.1.15. $\overline{L}$ is never big for our purposes. Somehow this is related to $\dim M_g = 3g-3 > g = \dim J$ and it being a pullback from something on J? •

The solution is that $\left\langle \overline{L}, \overline{L} \right\rangle = c \left\langle \overline{\omega}, \overline{\omega} \right\rangle$ is big on S (consequence of bigness of $\overline{\omega}$). This is weaker, but turns out to still imply that $\overline{L}$ is `potentially big`, i.e. for $m > \dim S$, on $X^m_{/S} = X \times_S \ldots \times_S X$, the adlic line bundle $\overline{L}^{\boxtimes m}$ is big. The proof of this proceeds by compute top intersection numbers. Once you have this, apply the height inequality for $\overline{L}^{\boxtimes m}$ on $X^m_{/S}$ to finish the proof.

2.2 Niki Myrto Mavraki (Harvard University): Preperiodic Points in Families of Rational Maps

We'll talk about some dynamical problems inspired by the relative Manin-Mumford conjecture.

Setup 2.2.1. Let S be an irreducible, quasi-projective variety over $\mathbb{C}$. Let $\pi : \mathcal{A} \to S$ be an abelian scheme of relative dimension $g \geq 1$. Also set

$$\mathcal{A}_{\text{tors}} = \bigcup_{s \in S(\mathbb{C})} (\mathcal{A}_s)_{\text{tors}}.$$

Relative Manin-Mumford is about how these torsion points distribute in subvarieties.

Theorem 2.2.2 (Gao-Habegger '23, Relative Manin-Mumford Conjecture (proposed by Pink, Zhang, Zannier))**.** *Let $\mathfrak{X} \subset \mathcal{A}$ be an irreducible subvariety which is flat over $\pi(\mathcal{X}) = S$ and which satisfies* $\operatorname{codim}_{\mathcal{A}} \mathfrak{X} > \dim S$.[4] *If $\mathfrak{X} \cap \mathcal{A}_{tors}$ is Zariski dense in $\mathfrak{X}$, then $\mathfrak{X}$ is contained in a proper subgroup scheme.*

Remark 2.2.3.

- When S is a point, this is Manin-Mumford (theorem due to Raynaud).
- When $\dim \mathfrak{X} = 1$, theorem establish by works of Masser, Zannier, Corvaja
- Kühne proved stronger version for $\mathcal{A}$ product of families of elliptic curves

[4]So unlikely for $\mathfrak{X}$ to intersect torsion points

- some cases for surfaces, e.g. by Habegger and Zannier-Tsimermann-Corvaja ◦

Example 2.2.4 (Masser-Zannier)**.** Say we have an elliptic surface $\mathcal{E} \to S$ ($\dim S = 1$), assumed non-trivial, along with a fixed section $p : S \to \mathcal{E}$. Its image defines a curve $\Gamma_p \subset \mathcal{E}$. Then, $\Gamma_p \cap \mathcal{E}_{\text{tors}}$ is Zariski dense, regardless of whether the section is torsion or not.[5] Say, we have a second section $Q : S \to \mathcal{E}$. Now, can look at $\Gamma_P \times_S \Gamma_Q \subset \mathcal{E} \times_S \mathcal{E}$. In this case,

$$(\Gamma_P \times_S \Gamma_Q) \cap (\mathcal{E} \times_S \mathcal{E})_{\text{tors}}$$

is finite unless $[n]P_t = [m]Q_t$ for all t. △

2.2.1 The ultimate Manin-Mumford problem

We have a family $\pi : \mathcal{A} \to S$ of abelian varieties, and we're intersecting it w/ points which are special w.r.t. the group law on fibers.

Imagine, instead, we have a (flat) family $\pi : \mathcal{A} \to S$ of schemes (not necessarily group schemes), but also have an endomorphism $\varphi : \mathcal{A} \to \mathcal{A}$ over S (think: $\varphi = [2]$). Can replace torsion points by

$$\operatorname{Prep}(\varphi) = \bigcup_{s \in S(\mathbb{C})} \operatorname{Prep}(\varphi_s)$$

(a preperiodic point is one w/ finite forward orbit under iteration).

Problem 2.1 (Dynamical Relative Manin-Mumford Problem)**.** *Characterize the irreducible subvarieties $\mathcal{X} \subset \mathcal{A}$ with $\pi(\mathcal{X}) = S$, $\operatorname{codim}_{\mathcal{A}} \mathcal{X} > \dim S$, and $\mathcal{X} \cap \operatorname{Prep}(\varphi)$ is Zariski dense in $\mathcal{X}$.*

This contains the regular relative Manin-Mumford as a special case.

Remark 2.2.5. This is open even when $\dim S = 0$. This is the dynamical Manin-Mumford conjecture (Zhang). ◦

What sort of answer would you expect? As a first guess, the subvariety should be preperiodic, i.e. $\varphi^n(\mathcal{X}) = \varphi^m(\mathcal{X})$ as sets/subvarieties. This turns out to fail (counterexamples coming from abelian varieties w/ CM, eg. by Ghioca-Tucker-Zhang). There's a newer proposed answer by Ghioba-Tucker.

Much of the progress towards this problem has focussed on split maps.

2.2.2 Split maps

Consider maps

$$\begin{array}{cccc} \Phi : & S \times (\mathbb{P}^1)^n & \longrightarrow & S \times (\mathbb{P}^1)^n \\ & (t, x_1, \dots, x_n) & \longmapsto & \left(t, f_t^{(1)}(x_1), \dots, f_t^{(n)}(x_n)\right), \end{array}$$

where $f_t^{(i)} \in \mathbb{C}(z)$ of degree $d \geq 2$. These are called `split maps`

Example 2.2.6. $f_t(z) = z^2 + t$ or $f_{t_1,t_2}(z) = z^3 + t_1 z + t_2$. △

- $\dim S = 0$

  The answer is known in this case, by work of Ghioca-Nguyen-Ye + M.-Schmidt-Wilms. These were the final step, following work/special case of many people: Mimer, Baker, De Marco, Yuan, Zhang, Hsia, Tucker, etc.

[5] Here, neither conclusion nor hypotheses (codim inequality) of theorem are satisfied

Example 2.2.7. Say $\Phi = (f,g)(x,y) = (f(x), g(y))$ and take $\mathcal{X} = \Delta = \{x,y\}$. When does the diagonal contain a Zariski dense subset of preperiodic points? When is $\#\operatorname{Prep}(f) \cap \operatorname{Prep}(g) = \infty$? The theorem (due to many people) is that this is the case if and only if $\operatorname{Prep}(f) = \operatorname{Prep}(g)$, which is furthermore the case if and only if Δ is (f,g)-`special` (= preperiodic or group related). $\triangle$

Let's say a bit about the proof ingredients (for the above example?). Sounds like it was partially inspired by Ullmo's proof of the Bogomolov conjecture. At the very least, it relies on an equidistribution result. This result needs heights to be formulated, so it's first done for maps over $\overline{\mathbb{Q}}$.

Theorem 2.2.8 (ca. '13)**.** *Say $f(z) \in \mathbb{Q}(z)$ and that you have an infinite, non-repeating sequence $\{a_n\} \subset \operatorname{Prep}(f)$. Then,*

$$\frac{1}{\deg \alpha_n} \sum_{\sigma} \delta_{\sigma(\alpha_n)} \longrightarrow \mu_f.$$

The measure μ_f appearing above was constructed ca. '80 by complex dynamisists (Brolin, Lynbich, etc.). It is the only measure w/ no point masses which is canonical for f in the sense that $f^*\mu_f = \deg(f)\mu_f$ (pullback in the sense of currents).

This lets you conclude that $\mu_f = \mu_g$ since same sequence equidistributes w.r.t. both of them. Sounds like then you conclude by works of Levin-Przytychos (spelling?), Levin, Baker, Beardon.

- What if $\dim S = 1$?

  Say we're look at a split endomorphism Φ of $S \times \mathbb{P}^1 \times \mathbb{P}^1$, and we have two sections $a, b : S \rightrightarrows \mathbb{P}^1$ whose graphs Γ_a, Γ_b define curves on $S \times (\mathbb{P}^1)^2$. When is

$$\#(\Gamma_a \times_S \Gamma_b) \cap \operatorname{Prep}(\Phi) = \infty?$$

Does this force a, b to be dynamically related? E.g. $f_t(a_t) = f_t^2(b_t)$ for all t, or something like this. Is $\Gamma_a \times_S \Gamma_b$ contained in a proper Φ-invariant subvariiety $\mathcal{Y}$ (i.e. $\Phi(\mathcal{Y}) \subset \mathcal{Y}$)? This is a case of conjecture by Baker-DeMarco.

Sounds like first important case was done by Baker-DeMarco, and that the answer in general is not yet known. Sounds like it's known that the Baker-DeMarco conjecture holds if $f^{(1)}, f^{(2)}$ are polynomials (Favre-Gauthier). There are also some sporadic cases known for rational maps, by work of many people.

In some cases, things can be easier when $\dim \mathcal{X}$ is huge.

Theorem 2.2.9 (Schmidt-M.)**.** *Say $\Phi_1, \Phi_2 : S \times \mathbb{P}^1 \times \mathbb{P}^1 \rightrightarrows S \times \mathbb{P}^1 \times \mathbb{P}^1$ are two split maps. Fix flat surfaces $\mathcal{C}_1, \mathcal{C}_2 \subset \mathbb{P}^1 \times \mathbb{P}^1 \times S$. Assume that $\mathcal{C}_1$ is not weakly Φ_1-special, and $\mathcal{C}_2$ is not Φ_2-special. Then,*

$$(\mathcal{C}_1 \times_S \mathcal{C}_2) \cap \operatorname{Prep}(\Phi_1 \times_S \Phi_2)$$

is not Zariski dense in $\mathcal{C}_1 \times_S \mathcal{C}_2$.

Think 'special' means 'preperiodic or coming from a group'.

Remark 2.2.10. If they did not avoid weakly special, they could have $\mathcal{C}_1 = \mathbb{P}^1 \times \Gamma_a$ and $\mathcal{C}_2 = \mathbb{P}^1 \times \Gamma_b$, and so their theorem would be solving Baker-DeMarco. This is why they chose to exclude this case. $\circ$

Corollary 2.2.11. *Say $\Phi : S \times \mathbb{P}^1 \times \mathbb{P}^1 \to S \times \mathbb{P}^1 \times \mathbb{P}^1$ split endomorphism and $\mathcal{C} \subset S \times \mathbb{P}^1 \times \mathbb{P}^1$ is a flat surface, not weakly special. Then, there exists a constant $M = M(\Phi, \mathcal{C}) > 0$ such that*

$$\#\mathcal{C}_t(\overline{\mathbb{Q}}) \cap \mathrm{Prep}(\Phi_t) \le M$$

for all but finitely many $t \in S(\mathbb{C})$.

Let's end by saying a bit about what the proof is like

(1) Establish a geometric Bogomolov-type statement
Allows one to use Yuan-Zhang's recent equidistribution results

(2) Use (slicings of) currents to compare the limiting measures by

(3) reducing to answering when do invariant measures satisfy certain linear relations.

2.3 Salim Tayou (Harvard University): Reduction of Brauer Classes on K3 Surfaces

(joint w/ D. Malik)

Let X be a `K3 surface` ($\omega_X \simeq \mathscr{O}_X$ and $\mathrm{H}^1(X, \mathscr{O}_X) = 0$) over a number field K. Let $\alpha \in \mathrm{Br}(X)$. Spread these out to some

$$\mathfrak{X} \longrightarrow \mathrm{Spec}\, \mathscr{O}_{K,S}$$

(e.g. $S = \{\text{bad red}\} \cup \{\alpha \text{ ramified}\}$). For any $\rho \in T$, can form the reduction $\alpha_\rho \in \mathrm{Br}(\mathfrak{X}_\rho)$.

Question 2.3.1 (Frei-Hassett-Várilly-Alvarado)**.** *What can be said about the vanishing set $V(\alpha) := \{\rho : \alpha_\rho = 0\}$.*

Theorem 2.3.2 (TM, '23)**.** *The set $V(\alpha)$ is infinite*.*

Prior work

- F.-H.-VA. paper

  Fix embedding $K \hookrightarrow \mathbb{C}$, so $X(\mathbb{C})$ is a complex K3 surface. Consider the transcendental lattice $L = \mathrm{Pic}(X(\mathbb{C}))^\perp \subset \mathrm{H}^2(X(\mathbb{C}), \mathbb{Z})$. This has a quadratic form Q coming from the pairing on H^2. Furthermore, it is a (polarized) Hodge structure of weight 2 with Hodge numbers $(1, b, 1)$. The signature of Q is $(b, 2)$. Consider $E := \mathrm{End}_{\mathrm{HS}}(L)$. This turns out to be either a totally real field (say RM, for real multiplication) or a CM field.

  Theorem 2.3.3 (F-H-VA)**.** *If E is RM and $\dim_E(L)$ is odd, then $V(\alpha)$ contains a set w/ positive natural density, i.e.*

  $$\liminf \frac{V(\alpha) \cap \{\rho : \|\rho\| < H\}}{\{\rho : \|\rho\| < H\}} > 0.$$

  In fact, if E is CM or $\dim_E L$ is even, and α is nonzero in $\mathrm{Br}(X_{\overline{K}})$, then $V(\alpha)$ has zero density (Charles). Consider the diagram (use Kummer sequence)

$$\begin{tikzcd}
0 \arrow[r] & \mathrm{Pic}(X_{\overline{K}}) \otimes \mathbb{Z}/r\mathbb{Z} \arrow[r] & \mathrm{H}^2_{\text{ét}}(X_{\overline{K}}, \mu_r) \arrow[r] \arrow[d, "\wr"] & \mathrm{Br}(X_{\overline{K}})[r] \arrow[r] & 0 \\
0 \arrow[r] & \mathrm{Pic}(X_{\overline{\rho}}) \otimes \mathbb{Z}/r\mathbb{Z} \arrow[r] & \mathrm{H}^2_{\text{ét}}(X_{\overline{\rho}}, \mu_r) \arrow[r] & \mathrm{Br}(X_{\overline{\rho}})[r] \arrow[r] & 0
\end{tikzcd}$$

Note $\mathrm{Br}(X_{\overline{K}})$ is torsion, so let r be the torsion order of $\alpha \in \mathrm{Br}(X_{\overline{K}})$. Above relates vanishing of α to jumps in Picard rank (if α is nonzero in $\mathrm{Br}(X_{\overline{K}})$ but 0 in $\mathrm{Br}(X_{\overline{\rho}})$, then get a class in Pic downstairs which doesn't lift).

Note that $V(\alpha)$ can have zero density, but still be infinite.

The mysterious condition * appearing in the theorem is

($\star$) $\gcd(r, \mathrm{NDisc}(L)) = 1$

Question 2.3.4. *effectivity? estimate number of primes?*

After putting their paper on arxiv, got email from Daniel Loughran who mentioned a paper of Serre studying similar questions.

Question 2.3.5 (Serre '90s)**.** *Say $\alpha \in \mathrm{Br}(K(V))$, where V/K is a quasi-projective algebraic variety. Choose height function H. Let $D \hookrightarrow V$ be divisor s.t. α extends away from D. How big is*

$$\{x \in V(K) \setminus D(K) : \alpha_x \neq 0 \text{ and } H(x) \leq H\}.$$

Theorem 2.3.6 (Serre)**.** *When $V = \mathbb{P}^n$ and α is 2-torsion, there is an upper bound of*

$$\frac{H^{n+1}}{(\log H)^{\frac{d}{2}}},$$

where d is the number of irreducible components of D.

Serre asks about a lower bound, but doesn't formulate a conjecture.

Note 3. I've been writing ρ, where Salim has been writing p...

2.3.1 Proof

X/K our K3 and $\alpha \in \mathrm{Br}(X)$ our Brauer class. Let r be the torsion order of α in $\mathrm{Br}(X_{\overline{K}})$.

The first case is $r = 1$ (i.e. α vanishes in $\mathrm{Br}(X_{\overline{K}})$). Pick Galois extension L/K s.t. $\alpha_L = 0$ and $\mathrm{Pic}(X_{\overline{K}}) = \mathrm{Pic}(X_L)$. The set of primes which are totally split in L has positive density. Choosing such a prime ρ, get diagram

$$\begin{tikzcd} \mathrm{Br}(X)[r] \arrow[r] \arrow[d] & \mathrm{Br}(X_\rho)[r] \arrow[d, "\wr"] \\ \mathrm{Br}(X_L)[r] \arrow[r] & \mathrm{Br}(X_\rho)[r] \end{tikzcd}$$

Thus, $V(\alpha)$ contains all totally split prime.

The second case is $r \geq 2$. In this case α is a transcendental Brauer class. Consier

$$\begin{tikzcd} & \mathrm{H}^2(X(\mathbb{C}), \mathbb{Z}) \arrow[r] & \mathrm{H}^2(X(\mathbb{C}), \mathscr{O}_X) \arrow[r] & \mathrm{Br}^{\mathrm{an}}(X(\mathbb{C})) \arrow[r] & 0 \\ 0 \arrow[r] & \mathrm{H}^2(X(\mathbb{C}), \mathbb{Z}) + \mathrm{Pic}(X)_{\mathbb{Q}} \arrow[r] & \mathrm{H}^2(X(\mathbb{C}), \mathbb{Q}) \arrow[r] \arrow[u] & \mathrm{Br}(X(\mathbb{C})) \arrow[r] \arrow[u] & 0 \end{tikzcd}$$

(top row from exponential exact sequence)

Somehow, maybe using condition ($\star$), one can use this to show that $\mathrm{Br}(X(\mathbb{C}))[r] \simeq \frac{1}{u}L^\vee/L^\vee \simeq \frac{1}{r}L/L$, so get parameterization of Brauer classes using transcendental lattice.

The strategy, now, is to construct $\lambda \in \mathrm{Pic}(X_{\overline{\rho}})$ with "residue" β for ∞-many ρ. For this, consider

$$\begin{tikzcd} \mathrm{Pic}(X_{\overline{\rho}}) \arrow[r] & \mathrm{H}^2_{\text{ét}}(X_{\overline{\rho}}, \mathbb{Z}_\ell(1)) \arrow[d, "\wr"] \\ L_{\mathbb{Z}_\ell} \arrow[r, hook] & \mathrm{H}^2_{\text{ét}}(X_{\overline{K}}, \mathbb{Z}_\ell(1)) \end{tikzcd}$$

Want to construct $\lambda \in \mathrm{Pic}(X_{\overline{\rho}})$ with image landing in $L_{\mathbb{Z}_\ell}$ reducing to $\beta \in L_\ell/rL_\ell \bmod r$.

Somehow if you do this right, you get $\alpha_\rho = 0$ in $\mathrm{Br}(X_{\overline{\rho}})$ and then H-Serre spectral sequence tells you that $\alpha_\rho = 0$ in $\mathrm{Br}(X_\rho)$.

Note 4. I'm kinda lost as to what we're doing...

Consider period domain $D = \{x \in \mathbb{P}(L_{\mathbb{C}}) : (x \cdot \overline{x}) < 0 \text{ and } (x \cdot x) = 0\}$. Let $\Gamma_r := \ker\big(O(L) \to O\big(\frac{1}{u}L^\vee/rL\big)\big)$. Now, $\Gamma_r \backslash D$ is (the $\mathbb{C}$-points of) a Shimura variety. For any $\beta \in L/rL$ and integer $m \in \mathbb{Z}$, define

$$Z(\beta, m) = \left\{x \in D : \exists \gamma \in L \cap L^{1,1} \text{ such that } Q(\gamma) = m \text{ and } \gamma = \beta \in L/rL\right\}.$$

Alternatively, $Z(\beta, m) = \bigcup_{\substack{\gamma \in \beta + rL \\ \varphi(\gamma) = m}} \{x \in D : (x, \gamma) = 0\}$.

Note 5. I'm too confused to pretend like I know what's happening. I'm gonna stop taking notes...

3 Day 4

3.1 Ziyang Gao (Leibniz Universität Hannover): Degeneracy Loci in Families of Abelian Varieties and their Applications

3.1.1 Motivation/Application (Why?)

Degeneracy loci are used crucially in recent works of uniform Mordell-Lang (Dimitrov-G.-Habeggar, Kühne, G.-Ge-Kühne). One important phenomenon is to understand/construct so called *non-degenerate subvarieties* (defined by Habegger).

Setup 3.1.1. Let S be a quasi-projective variety over $\mathbb{C}$. Let $\mathcal{A} \to S$ be an abelian scheme of relative dimension $g \geq 1$. Assume this carries a principal polarization (+ level structure), so we have Carteseian diagram

$$\begin{tikzcd} \mathcal{A} \arrow[r] \arrow[d, "\pi"'] & U_g \arrow[d] \\ S \arrow[r, "i_S"] & \mathbb{A}_g. \end{tikzcd}$$

Furthermore, assume i_S is quasi-finite. Also, choose a subvariety $X \subset \mathcal{A}$.

To define non-degeneracy, need Betti map. Consider uniformization

$$\mathbb{C}^g \times \mathfrak{H}_g \longrightarrow \mathfrak{U}_g$$

I missed the definition because I was busy messing with Latex, but the upshot is that the Betti map is, morally,

$$B : \mathfrak{U}_g \longrightarrow \mathbb{T}^{2g}$$

realizing all abelian varieties as some fixed real torus. This was introduced by Masser-Zannier.

For each point $x \in X^{\mathrm{sm}}(\mathbb{C})$, consider the rank

$$\mathrm{rank}_{\mathrm{Betti}}(X, x) := \mathrm{rank}_{\mathbb{R}}(\mathrm{d}B|_X)_x\,.$$

This has trivial bounds

$$\mathrm{rank}(X, x) \le 2\min\{g, \dim X\}\,.$$

Definition 3.1.2 (Habegger, GH, DGH)**.** X is called `non-degenerate` if $\mathrm{rank}_{\mathrm{Betti}}(X, x) = 2\dim X$ for all $x \in U(\mathbb{C})$ for some non-empty open $U \subset X$. ⋄

Recall $[(A, \lambda)] \in \mathbb{A}_g$ parameterizes abelian varieties w/ polarization. In the universal picture, this polarization is captured by the tautological line bundle $\mathscr{L}_g$ over $\mathfrak{U}_g$.

Remark 3.1.3. Yuan-Zhang extends $\mathscr{L}_g$ to an adelic line bundle $\widetilde{\mathscr{L}_g}$ on $\mathfrak{U}_g$. ∘

Fact. non-degeneracy $\iff$ $\widetilde{\mathscr{L}_g}|_X$ is big.

Non-degeneracy is a real geometry way of capturing this algebraic phenomenon (bigness of $\widetilde{\mathscr{L}_g}$).

Theorem 3.1.4. *Assume X maps onto S and that X is not contained in a (proper) subgroup scheme of $\mathcal{A} \to S$.*[6] *Then, X is degenerate (i.e. $\widetilde{\mathscr{L}_g}|_X$ is not big) if and only if there exists an abelian subscheme $\mathcal{B}$ of $\mathcal{A}/S$ such that* $\dim X - \dim \iota(X) > g - g'$*, with notation as below:*

$$\begin{tikzcd}
\mathcal{A} \arrow[r] \arrow[d] \arrow[rr, bend left, "\iota"] & \mathcal{A}/\mathcal{B} \arrow[r] \arrow[d] & \mathfrak{U}_{g'} \arrow[d] \\
S \arrow[r, equal] & S \arrow[r] & \mathbb{A}_{g'}
\end{tikzcd}$$

Remark 3.1.5.

(1) X is degenerate if $\dim X > g$ for "naive" reasons. The Betti rank will be $\le 2g < 2\dim X$.

(2) In general, theorem above says any degenerate X is "built up" from naively degenerate subvarietes.

In the condition "$\dim X - \dim \iota(X) > g - g'$," the LHS is the dimension of a generic fiber of $\iota|_X$. The RHS is the dimension of the abelian direction of ι. ∘

Corollary 3.1.6. *Can construct non-degenerate subvarieties via fibered copies.*

Example 3.1.7. Say have family of curves $\mathcal{C} \hookrightarrow \mathrm{Jac}(\mathcal{C}/S)$. This is usually naively degenerate, but will be non-degenerate after sufficient fiber powers. (Second proof by Yuan, Tuesday). Led to uniform Mordell-Lang for curves. △

Example 3.1.8. Hilbert schemes (Ge, G.-Ge-Kühne). Led to full uniform Mordell-Lang. △

There are two steps in the proof of Theorem 3.1.4, and they both use degeneracy loci.

3.1.2 Degeneracy loci and some properties (What?)

Recall 3.1.9. Say A is an abelian variety w/ irreducible subvariety $Y \subset A$. We say Y is `weakly special` if $Y = a + B$ is *any* translate of an abelian subvariety. ⊙

How do we get an analogue for families?

[6] Really mean a torsion coset over S

Definition 3.1.10. Given $Y \subset \mathcal{A}/B$ dominating the base, we say Y is `weakly special` if it is the translate of an abelian subscheme $\mathcal{B}$ by a torsion section or an `iso-constant section`[7]. $\diamond$

Remark 3.1.11. All torsion sections are iso-constant. $\circ$

In a family, the Betti map will map torsion sections to torsion points, but arbitrary sections may have large image. However, iso-constant sections will have a single point as the its image. In general, a section has a single point as its image under the Betti map if it is torsion or iso-constant.

Definition 3.1.12. For any $t \in \mathbb{Z}$, the tth `degeneracy locus` is defined to be

$$X^{\deg}(t) = \bigcup_{\substack{Y \subset X, \dim Y > 0 \\ \dim \langle Y \rangle_{\mathrm{ws}} - \dim \pi(Y) < \dim Y + t}} Y,$$

where $\langle Y \rangle_{\mathrm{ws}}$ is the smallest weakly special subvariety of $\mathcal{A}|_{\pi(Y))} \to \pi(Y)$ which contains Y. $\diamond$

Remark 3.1.13. If Y is weakly special, then $\langle Y \rangle_{\mathrm{ws}} - \dim \pi(Y) = \dim Y - \dim \pi(Y)$ is the relative dimension of $\mathcal{B}$. $\circ$

Why is this definition not so horrible?

Example 3.1.14. Say $S = \{*\}$ is a point, so $\pi(Y) = *$ always.

- When $t \leq 0$, $X^{\deg}(t) = \emptyset$.

  So, for negative t, the degeneracy locus is really something about families.

- When $t = 1$, each Y must be weakly special, so

  $$X^{\deg}(1) = \bigcup_{a + B \subset X, \dim B > 0} (a + B)$$

  is the `Ullmo locus` of X. $\triangle$

Theorem 3.1.15. *$X^{deg}(t)$ is Zariski closed.*

Key idea. The maximal relevant Y's come from finitely many families. In the classical caseo of Ullmo locus, this is a theorem of Bogomolov. He showed for $X \subset A$, the Ullmo locus $\bigcup_{a+B \subset X, \dim B > 0}(a + B)$ is closed. He should that the number of B such that

- $\dim B > 0$
- $a + B \subset X$ for some $a \in A(\mathbb{C})$, and B is maximal

is finite. $\blacksquare$

Theorem 3.1.16. *Assume $X \twoheadrightarrow S$, and that X is not contained in a proper subgroup scheme of $\mathcal{A}/S$. Then, $X^{deg}(t) = X$ iff and only if there exists an abelian subgroup scheme $\mathcal{B} \subset \mathcal{A}/S$ such that $g' < g$ and $\dim \iota(X) < \dim X - (g - g') + t$, with notation as below:*

$$\begin{tikzcd}
\mathcal{A} \arrow[r] \arrow[d] \arrow[rr, bend left, "\iota"] & \mathcal{A}/\mathcal{B} \arrow[r] \arrow[d] & \mathfrak{A}_{g'} \arrow[d] \\
S \arrow[r, equal] & S \arrow[r] & \mathbb{A}_{g'}
\end{tikzcd}$$

[7]Say $\mathcal{C} \subset \mathcal{A}$ is the largest iso-constant/isotrivial part. An iso-constant section is one that's constant in this part after finite base change.

(Proved by G. when $t \le 1$, and by GH for general t)
Compare this to Theorem 3.1.4.

3.1.3 Applications (How?)

(1) $t = 0$

As an application of mixed Ax-Schanuel (for $\mathbb{A}_g$), one can show that X is degenerate $\iff$ $X^{\deg}(0) = X$. This is why Theorem 3.1.16 is relevant to the proof of Theorem 3.1.4.

If X is degenerate, then $\mathrm{rank}_{\mathrm{Betti}}(X, x) < 2\dim X$ for all $x \in X^{\mathrm{sm}}(\mathbb{C})$. Somehow, Gao went from this to X degenerate $\iff$ "$\forall x \in X^{\mathrm{sm}}(\mathbb{C})$" there's a complex analytic curve $C \subset \mathcal{A}$ s.t. $B(C) = *$ and $x \in C \subset X$. This is the case iff

$$X = \bigcup_{\text{all such } C} \overline{C}^{\mathrm{Zar}}.$$

Mixed Ax-Schanuel will tell you that each $\overline{C}^{\mathrm{Zar}}$ above satisfies the condition in the definition of 0th degeneracy. This computation is what motivated the general definition.

(2) $t = 1$ and $t = 0$

Theorem 3.1.17 (Relative Manin-Mumford conjecture, G.-Habegger '23)**.** *Say $X \subset \mathcal{A}/S$ dominating the base is not contained in any proper subgroup scheme of $\mathcal{A}/S$. Set*

$$\mathcal{A}_{tors} := \bigcup_{s \in S(\mathbb{C})} \mathcal{A}_s(\mathbb{C})_{tor}.$$

If $\overline{X \cap \mathcal{A}_{\mathrm{tors}}}^{Zar} = X$, then $\dim X \ge g$.

Remark 3.1.18 (Response to Audience Question). Sounds like maybe you can't strengthen the conclusion here (except to say this holds $\iff$ Betti rank is $2g$)? ○

Corollary 3.1.19 (Uniform Manin-Mumford Conjecture for curves, Kühne)**.** *For any $g \ge 2$, let C be a smooth projective curve of genus g. For any point $P \in C(\mathbb{C})$, get uniform bound on $\#(C - P)(\mathbb{C}) \cap J_{\mathrm{tors}}$, depending only on g.*

Remark 3.1.20. $\dim X = 1$ case done by Corvaja, Masser, Zannier. Some surfaces done by Habegger, Corvaja-Tsimerman-Zannier. Some other case done by Kühne. ○

In UML, always start by constructing some non-degenerate subvariety. In RMM, need to use another philosophy (related to starting with a family). Separate into cases, $X^{\deg}(0) = X$ or $\neq X$. If $\neq$, use Pila-Zannier method to show that $X^{\deg}(1) = X$. Thus, in either case $X^{\deg}(1) = X$. Now, apply Theorem 3.1.16 and do induction on g.

3.2 Vesselin Dimitrov (Institute for Advanced Study): A Twisting-Free Converse Theorem for GL(2)

The setting is 'Integral converse theorems.'

These involve some nice L-function $L(s, \pi) = L(k - s, \pi)$. Somehow we want to use nice integrality properties of its coefficients (to say something?).

Example 3.2.1. Say $E/\mathbb{Q}$ is an elliptic curve. Can consider Hasse-Weil L-function

$$L(s,E) = \sum_{n\geq 1} a_n n^{-s} = \prod_{p\nmid N_E} \left(1 - a_p p^{-s} + p^{1-2s}\right)^{-1} \prod_{p\mid N_E} \left(1 - a_p p^{-s}\right)^{-1},$$

where $a_p = (p+1) - \#E^{\mathrm{ns}}(\mathbb{F}_p)$. △

Hasse gave Humbert a thesis problem related to this. He asked him to think about the function

$$\Lambda(s,E) := \left(\frac{\sqrt{N_E}}{2\pi}\right)^s \Gamma(s) L(s,E) \tag{3.1}$$

These days, one knows

$$\Lambda(s,E) = \pm\Lambda(2-s,E)$$

with sign (conjecturally?) equal to $(-1)^{\operatorname{rank} E}$.

Example 3.2.2. Hasse conjecture $\iff$ Modularity: $L(s,E) = L(s,f)$ for some (Hecke newform) $f \in S_2(\Gamma_0(N_E))$. △

Say we're given $f \in S_k(\Gamma_0(N))$ holomorphic cusp form. One can attach to this

$$\Lambda(s,f) := A^{s/2}\Gamma(s) \sum_{n=1}^{\infty} a_n n^{-s}, \text{ where } A = \left(\frac{\sqrt{N}}{2\pi}\right)^s.$$

By taking Mellin transofrms, one can show

$$\Lambda(s,f) = \int_0^\infty f(iy) N^{s/2} y^s \,\mathrm{d}\log y = \int_0^1 (blah) + \int_1^\infty (blah).$$

Substituting $y \mapsto 1/y$ in the first summand, one can show

$$\Lambda(s,f) = i^k \Lambda(k-s, \widetilde{f}), \text{ where } \widetilde{f} = f|_{w_N} \text{ for } w_N = \begin{pmatrix} 0 & -1 \\ N & 0 \end{pmatrix}.$$

Note that

$$\begin{pmatrix} 1 & 0 \\ N & 1 \end{pmatrix} = w_N^{-1} \begin{pmatrix} 1 & -1 \\ 0 & 1 \end{pmatrix} w_N.$$

For some reason, this makes us care about $\Gamma_N^0 = \left\langle \begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix}, \begin{pmatrix} 1 & 0 \\ N & 1 \end{pmatrix} \right\rangle \leq \Gamma_0(N)$. Somehow, "this" (don't ask me what "this" is) proves Hecke's converse theorem?

Theorem 3.2.3 (`Hecke's converse theorem`)**.** *Suppose $N \leq 4$, $A = \left(\frac{\sqrt{N}}{2\pi}\right)^s$. Let $L(s), \widetilde{L}(s)$ are Dirichlet series which extend to entire functions on $\mathbb{C}$ and whose complete L-functions satisfy $\Lambda(s) = i^k\widetilde{\Lambda}(k-s)$, then there exists some $f \in S_k(\Gamma_0(N))$ such that $\Lambda(s) = \Lambda(s,f)$ and $\widetilde{\Lambda}(s) = \Lambda(s, f|_{w_N})$.*

Above, 'Dirichlet series' means $L(s) = \sum_{n\geq 1} a_n n^{-s}$ w/ $a_n \in \mathbb{C}$. Above theorem does not consider integrality properties of the a_n's at all.

The above fails (very badly) for any $N > 4$. Vesselin sketched a picture of a fundamental domain (for Γ_N^0? This is a congruence subgroup for $N \leq 4$, but always has congruence closure $\Gamma_0(N)$.) when $N \geq 5$.

Theorem 3.2.4 (Weil's converse theorem, 1967)**.** *Say $L(s) = \widetilde{L}(s)$. Then, $L(s) = L(s, f)$ for some $f \in S_k(\Gamma_0(N))$ if for all $(r, N) = 1$ (suffices to only consider $r < N^2$?) and all primitive Dirichlet characters $\chi \mod r$,*

$$\left(\frac{\sqrt{r^2 N}}{2\pi}\right)^s \Gamma(s) \sum_{n\geq 1} a_n \chi(n) n^{-s} =: \Lambda(s, \chi)$$

satisfies $\Lambda(s, \chi) = w_\chi \Lambda(k - s, \overline{\chi})$, where $w_\chi = i^k \chi(N) \tau(\chi)^2 / r$, and

$$\tau(\chi) = \sum_r \chi(n) e^{2\pi i n / r}$$

is the usual Gauss sum.

Theorem 3.2.5 (Integral converse theorem)**.** *Let $N \in \mathbb{N}$ arbitrary.Fix $k \in \mathbb{N}_0$ and $w \in \mathbb{C}^\times$. Let $A = \sqrt{N}/(2\pi)$. Let $L(s) = \sum_{n=1}^\infty a_n n^{-s}$ be a finite $\overline{\mathbb{Q}}$-linear combination of $\mathbb{Z}[p^{-s} : p \text{ primes}]$. Assume we also have some $\widetilde{L}(s) = \sum_{n\geq 1} b_n n^{-s}$ w/ $b_n \in \mathbb{C}$. Normalizing by assuming $a_0 = b_0 = 1$. Assume $L, \widetilde{L}$ are entire on $\mathbb{C}$ and either*

- *bounded in vertical strips*
- *entire of finite order.*

Furthermore, setting $\Lambda(s) = A^s \Gamma(s) L(s)$ as usual, assume

$$\Lambda(s) = w \widetilde{\Lambda}(k - s)$$

for some $k \in \mathbb{Z}$. Then, $\exists f \in S_k(\Gamma_0(N))$ such that $L(s) = L(s, f)$, $\widetilde{L}(s) = L(s, f|_{w_N})$, and $w = i^k$.

The integrality assumption allows one to do some diophantine approximation argument.

Assume wlog that N is even. Let $F(q) = \sum_{n\geq 1} a_n q^n$. Assume, for simplicity, that $F(q) \in \mathbb{Z}[\![q]\!]$. By Hecke's lemma, already know that $F|_k \gamma = F$ for all $\gamma \in \Gamma_N^0 = \left\langle \begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix}, \begin{pmatrix} 1 & 0 \\ N & 1 \end{pmatrix} \right\rangle$. Let

$$f(q) := \frac{F(q) \lambda(q)^k}{\eta^{2k}(q)} \in \mathbb{Z}[\![q]\!] \,.$$

(λ modular function for $\Gamma(2)$? I guess it must be the map to $\mathbb{P}^1 \setminus \{0, 1, \infty\}$)

We want to show that $f \in K(X_0(N))$. Consider

$$x = \frac{\lambda(\tau)}{16} + \lambda(Nt) = q + \cdots \in \mathbb{Z}[\![q]\!] \,, \text{ where } q = e^{\pi i \tau},$$

a modular function on $Y_0(N)$. This is a local parameter around $i\infty$ and 'index N' at 0? One uses that $f\mathbb{Z}[\![q]\!] = \mathbb{Z}[\![x]\!]$. We want to show that $f(x) \in \mathbb{Z}[\![q]\!]$ is actually an algebraic function in $\mathbb{Z}[\![x]\!]$. This will give finite index $\Gamma \subset \Gamma(2)$ such that $F \in S_k(\Gamma)$. Finally need to show can take $\Gamma = \Gamma_0(N)$ (this follows by some previous paper by CDT?).

Note $(\mathbb{H}/\left\langle \begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix} \right\rangle) \cup \{i\infty\}$ is the unit disk $\mathbb{D} = \{|q| < 1\}$, $q = e^{\pi i \tau}$. One cares about the composition

$$(\mathbb{H}/\left\langle \begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix} \right\rangle) \cup \{i\infty\} \longrightarrow (\mathbb{H}/\Gamma_N^0) \cup \{i\infty\} \longrightarrow Y_0(N) \cup \{i\infty\} \xrightarrow{x} \mathbb{C}.$$

Lemma 3.2.6 (Algebraization lemma). *Suppose we have*

$$\begin{tikzcd} z \in & (\mathbb{D},0) \arrow[r, "\varphi"] \arrow[dr] & (U,u_0) \arrow[d, dashed, "t"] \arrow[dr, "x"] & \\ & & (\mathbb{C},0) & (\mathbb{C},0) \end{tikzcd}$$

Suppose all solid arrows are holomorphic and that $f \in \mathbb{Z}[\![x]\!]$. Suppose

$$|\varphi'(0)| := \left| D\varphi|_0\left(\frac{\partial}{\partial x}\right)\right| > 1.$$

Then, $f \in \overline{\mathbb{Q}(x)}$.

(André, *G*-functions and Geometry)

A priori, we have $x'(0) = 1$, so are on the borderline of this lemma. To overcome this, one use the modularity relation $f(-1/(N\tau)) = f(\tau)$.

Apparently, the universal covering map

$$\varphi : (\mathbb{D}, 0) \longrightarrow \left((\mathbb{H}/\Gamma_N^0) \cup \{i\infty, 0\}, i\infty\right)$$

has $|\varphi'(0)| > 1$, so can apply Lemma.

The Lemma is proved using Diophantine approximation.

4 Day 5

4.1 Brian Lawrence (University of Wisconsin-Madison): Conditional Algorithmic Mordell

(in progress w/ Levent Alpöge)

“Today, I'm gonna talk about a result that sounds very nice at first, but then it begins to disappoint in every possible way."

Theorem 4.1.1 (Flatings 1983, Mordell Conjecture). *Let K be a number field, and let X/K be a curve of genus ≥ 2. Then, $X(K)$ is finite.*

This is very concrete. A curve is something like an equation in two variables. In practice, we may want to know what are all the solutions. This theorem tells you there are only finitely many. However, you may also want to know what they are, so, beyond this, we'd really like some deterministic procedure to compute all the points on a given curve.

Question 4.1.2. *Given K, X, how can one determine $X(K)$?*

Levent gave a talk on this sort of problem earlier this week. His result was stronger in that it is unconditional. Today's result will require assuming many conjecture. However, today's result is stronger in that it applies to any curve.

Theorem 4.1.3 (In Progress). *There is a Turing machine that takes K, X as input, and*

- *terminates, assuming the Fontaine-Mazur, Hodge, and Tate conjectures.*

  “Then you know the program will eventually stop. ‘Eventually' is a very long long time. It won't actually stop, but eventually it stops.” (paraphrase)

- *If it stops, its output is unconditionally $X(K)$*

("85% written")

Question 4.1.4 (Audience)**.** *Why don't you assume more conjectures? For example, would it help to assume BSD for abelian varieties?*

Answer. I mean, sure, we can also assume BSD. However, sounds like assuming more wouldn't really help. One reason it is hopelessly slow is that it involves a brute force search for Galois representations. If you accept the form of Fontain-Mazur that says all of these things are automorhpic, and maybe you could compute the associated automorphic representations, then maybe that could help speed things up? The approach they use is through the Shafarevich conjecture (not via an Abel-Jacobi embedding), so unclear how something like BSD would help. ⋆

4.1.1 Faltings' Proof

Slogan. Shafarevich $\implies$ Mordell

Write X as the base of of some nonisotrivial family of abelian varieties $\mathcal{A}/X$. For every $x \in X(K)$, the fiber A_x is an abelian variety over K. There are lots of abelian varieties over K, but A_x can't just be any of them.

So the map $X \to \mathbb{A}_g$ is quasi-finite (because $\dim X = 1$ is small)!

Fact. A_x has good reduction outside a fixed set S of primes of $\mathscr{O}_K$

Shafarevich says there are only finitely many such abelian varieties.

Note 6. Brian gave an out loud description of the algorithm, but I missed the details. I think, eseentially, compute a family, compute S and all AVs w/ good reduction outside S, then check which ones appear in the family.

Note 7. There some discussion about finding integral points on (affine) curves, espeically in the wake of Hilbert's 12th problem. I didn't bother taking notes.

Theorem 4.1.5 (Faltings, `Shafarevich Conjecture`)**.** *There are only finitely many principally polarized g-dimensional abelian varieties defined over K which have good reduction outside of S.*

Thus Theorem 4.1.3 will follow from an "equally terrible algorithm for the Shafarevich conjecture."

Theorem 4.1.6 (In Progress)**.** *There is a Turing machine that*

- *takes K, g, S as input*
- *terminates, assuming Fontaine-Mazur, Hodge, Tate*
- *If it terminates, it outputs all principally polarized abelian varieties defined over K of dimension g w/ good reduction outside S.*

Faltings' proof of Shafarevich was ineffective in a few places. Let's say a bit about this. Somehow, the basic idea is to study abelian varieties using Galois reps, and to study those using Frobenius traces.

$$(\text{ab. vars}) \xrightarrow{\alpha} \begin{pmatrix}\text{Galois}\\ \text{reps}\end{pmatrix} \longrightarrow \begin{pmatrix}\text{Frob}\\ \text{traces}\end{pmatrix}$$

$$A \longmapsto \mathrm{H}^1_{\text{ét}}(A_{\overline{K}}, \mathbb{Q}_\ell) \longmapsto (t_1, \dots, t_r)$$

$\mathrm{H}^1_{\text{ét}}(A, \mathbb{Q}_\ell)$ is known to be semisimple, so it's determined by Frobenius traces. By some Chebatarev density type argument, finitely many traces will suffice.

Theorem 4.1.7 (Faltings)**.** *Given K, g, S, can compute a set of primes $p_1, \dots, p_r$ of $\mathscr{O}_K$ such that if two semisimple representations*

$$\rho_1, \rho_2 : \mathrm{Gal}_{K,S'} \longrightarrow \mathrm{GSp}_{2g}(\mathbb{Q}_\ell)$$

satisfy $\operatorname{Tr} \rho_1(\mathrm{Frob}_{p_i}) = \operatorname{Tr}(\mathrm{Frob}_{p_i} \mid \rho_2)$ *for all i, then $\rho_1 \cong \rho_2$.*

Above, $S' = S \cup \{\mathfrak{p} \mid \ell\}$.

Remark 4.1.8. Something like, consider all extensions of K of degree ℓ^{g^2} which are unramified outside of S. Then apply Chebotarev density to that. Now, check all the primes one by one until you exhaust conjugacy classes. ◦

Question 4.1.9 (Audience)**.** *Would assuming GRH help?*

Answer. It would tell you how far you need to look to find enough p_i's, so would help get bounds on runtime, but unclear if it would help make things faster (in practice). ⋆

By the Weil conjectures, there are only finitely many r-tuples $(t_1, \dots, t_r)$ you can get. Note that the fibers of the map α are isogeny classes. Thus, the second thing Faltings' shows is that isogeny classes are finite. His proof of that is unfortunately ineffective.

Remark 4.1.10 (key ineffectivities when Faltings' finished his work).

(1) Don't know how to find all abelian varieties in an isogeny class.

(2) Which tuples of Frobenius traces (satisfying the Weil conjectures) come from Galois representations, and which Galois representations come from abelian varieties? ◦

Question 4.1.11 (Audience)**.** *If $(t_1, \dots, t_r)$ comes from an abelian variety, how do you find it?*

Answer. The algorithm is already terribly slow, so we don't do anything clever. Just search by brute force. ⋆

Remark 4.1.12. Someone suggested writing down equations for moduli space of abelian varieties and searching in there. Brian suggested the following alternative (just to prove a point): an abelian variety is a projective variety equipped w/ some structure morphisms (e.g. multiplication). These can all be can described using a finite bitstring. Conversely, given a finite bitstring, can check if it describes an abelian variety. So just brute force search over all bitstrings, ordered by length. Once you find one abelian variety that works, you win by the below. ◦

Of the two obstructions above, **(1)** is already solved.

Theorem 4.1.13 (Masser-Wüstholz, Bost-David, Gaudron-Rémond)**.** *Solve **(1)**.*

Given some A/K can find all abelian varieties which are K-isogenous to it. If two abelian varieties are isogeneous, this theorem tells you that there's some explicitly computable bound on the degree of an isogeny between them. Thus, you can (in finite time) brute force all abelian varieties which are isogenous to a given A.

Remark 4.1.14. We haven't used any of Fontaine-Mazur, Hodge, or Tate yet. ◦

Now, we need to know, which tuples lie in the image.

Theorem 4.1.15 (Patrikis-Voloch-Zarhin ∼2014, Alpoöge-L.). *Assume FM,H,T, and suppose you're given* $\rho : G_{K,S'} \to \mathrm{GSp}_{2g}(\mathbb{Z}_\ell)$*. Assume*

- *for all* $p \notin S'$*, the characteristic polynomial of* Frob_p *has coefficients in* $\mathbb{Z}$ *and all roots are Weil numbers (of the right type); and*
- *at every prime above* ℓ*,* ρ *is de Rham w/ Hodge-Tate weights* $0, \ldots, 0, 1, \ldots, 1$ *(g of each)*

Then, ρ *is a direct summand in some* $\mathrm{H}^1(A)$*.*

Hence, when they do their brute force search of abelian varieties, they're not searching for abelian varieties of dimension g, they search for abelian varieties of dimension $\geq g$.

What's the basic idea of the proof? Fontaine-Mazur tells you the representation comes from geometry. So it shows up in the étale cohomology of some variety. Look a the Hodge structure of this variety. The weights are just right, so it has a piece that looks like an abelian variety. That piece will be an abelian variety (using Hodge and Tate conjectures), and now pass back to étale cohomology to say that abelian variety will have the same cohomology.

Question 4.1.16 (Audience). *How do you know the complex torus coming from the Hodge structure will actually be an abelian variety?*

Answer (Audience). It will be polarized and have the right weights. This is enough to conclude that it's an abelian variety. ⋆

4.1.2 Algorithm

Have finitely many possible tuples of frobenius traces. Need to figure out which are in the image of this map.

- Make a list of "candidates" $(t_1, \ldots, t_r)$
- By day, search for abelian varieties.

  Compute its endomorphism ring, find all subrepresenations of its étale cohomology, and check which come from abelian varieties. If you find $(t_1, \ldots, t_r)$, then it comes from an abelian variety.
- By night, search for (mod ℓ^n) Galois representations. This is a finite search (essentially looking for number fields w/ bounded ramification)

  For each of them, check to see if Frobenius at primes (up to some bound N which increases w/ time) away from S' satisfies Weil conjectures.

  If you find no Galois rep consistent with $(t_1, \ldots, t_r)$ (mod ℓ^n), then can cross it off the lift.

4.2 Congling Qiu (Yale University): Joint Unlikely Almost Intersections on Ordinary Siegel Spaces

Note 8. A slide talk, so little chance I keep up. Expect me to stop taking notes before too long.

Outline

- motivation
- unlikely almost intersections
- Ax-Lindemann principle
- Perfectoid approach to unlikely almost intersections

4.2.1 Motivation

Let S be a Shimura Variety, $V \subset S$ closed subvariety.

Conjecture 4.2.1 (André-Oort)**.** *Let* CM *denote the set of CM points. If* $V \cap \mathrm{CM}$ *is Zariski dense in* V*, then* V *is a "Shimura subvariety"*

Conjecture 4.2.2 (André-Pink)**.** *Let* $O \subset S$ *be a Hecke orbit. If* $V \cap O$ *is Zariski dense in* V*, then* V *is weakly special.*

For abelian vareities, "Mordell-Lang + Bogomolov" was proved by Poonen, and indepdnently by S. Zhang. Ge proved a uniform version.

ε-neighborhoods of divison points of a lattice in the height topology.

Question 4.2.3. *How to include distance on Shimura varieties using heights?*

No easy direct way, but for any variety over a valued field, $\mathbb{R}$-valued distance from points to a subvariety is defined. E.g., local heights.

Example 4.2.4. Take equations defining subvariety. Evaluate them at the point. Take (max of?) absolute value of valuations. △

Note 9. Stopped taking notes here

4.3 Paul Vojta (University of California, Berkeley): Roth's Theorem over Adelic Curves

Theorem 4.3.1 (Thue-Segal-Gel'fond-Dyson-Roth, over number fields)**.** *Let* K *be a number field, let* S *be a finite set of places of* K*, each extended in* $\overline{K}$ *in some way, and for all* $v \in S$*, let* $\alpha_v \in \overline{K}$*. Let* $\varepsilon > 0$*. Then, the set of all* $x \in K$ *satisfying the condition*

$$\prod_{v \in S} \min\{1, \|x - \alpha_v\|_v\} \leq \frac{1}{H_K(x)^{2+\varepsilon}}$$

is finite.

4.3.1 Adelic Curves

Definition 4.3.2 (H. Chen, A. Moriwaki)**.** An `adelic curve` is a tuple $(K, (\Omega, \mathcal{A}, \mu), \varphi)$, where

(1) K is a field

(2) $(\Omega, \mathcal{A}, \mu)$ is a measure space (i.e. Ω a set, $\mathcal{A}$ a σ-algebra, μ a measure)

(3) $\varphi : \Omega \to \{\text{all absolute values on } K\}$ is a function, denoted $\omega \mapsto |\cdot|_\omega$, such that, for all $a \in K^\times$, the function $\omega \mapsto \log |a|_\omega$ is A-measurable and integral w.r.t. μ (i.e. in $L^1(\mu)$).

It is `proper` if it satisfies a product formula

$$\int_\Omega \log |a|_\omega \, \mathrm{d}\mu(\omega) = 0 \text{ for all } a \in K^\times.$$

It has a `height function` $h(x) = \int_\Omega \log^+ |x|_\omega \, \mathrm{d}\mu(\omega)$ for all $x \in K$. Here, $\log^+ x = \log \max\{1, x\}$, and $\log^- x = \log \min\{1, x\}$ for $x \in \mathbb{R}_{\geq 0}$. ⋄

Remark 4.3.3. For function fields, the conclusion would be that the considered set consists of elements of bounded height. ○

Assumption. In this talk, all adelic curves considered are assumed to be proper, and have $\operatorname{char} K = 0$.

Example 4.3.4.

(1) Number field K, with $\Omega = M_K^0 \sqcup M_K^\infty$, where $M_K^\infty = \operatorname{Hom}(K, \mathbb{C})$ (so each complex place appears twice), and $M_K^0 = \operatorname{Spec} \mathscr{O}_K$. Use the counting measure for μ.

Roth theorem's is known here, due to Roth.

(2) (Geometric) function fields over a constant field F. Let $K = F(X)$ for some smooth, projective F-variety X of dimension $d = \dim X = \operatorname{trdeg}(K/F) \geq 1$, equipped with a polzarization (ample line bundle). Take $\Omega =$ set of prime divisors on X.[8]

Lang (and separately Corvaja?) proved Roth here.

(3) (Moriwaki) Arithmetic function fields

$K = K(X)$, where X is a projective, flat, integral scheme over $\operatorname{Spec} \mathbb{Z}$, of relative dimension $d \geq 0$ (hence, number fields included as special case), equipped w/ a polarization given by a metrized line sheaf $\mathscr{L}$ on X. X should have smooth generic fiber (and normal total space?). One takes $\Omega = M_K^0 \sqcup M_K^\infty$, where $M_K^\infty = X(\mathbb{C}) \setminus \bigcup_{Z \in X^{(1)}} Z(\mathbb{C}) = \operatorname{Hom}(K, \mathbb{C})$ and M_K^0 is the set of prime divisors on X.

Example 4.3.5. If $X = \mathbb{A}^1_{\mathbb{Z}}$, then $\pi = 3.14\ldots$ is not in this union of divisors △

Take μ to be the counting measure on M_K^0 and the measure corresponding to $\bigwedge^d c_1(\mathscr{L})$ on M_K^∞.

Roth is proved (V. 2021)

(4) $K = \overline{\mathbb{Q}}$, $\Omega = M_K^0 \sqcup M_K^\infty$, where $M_K^\infty = \operatorname{Hom}(\overline{\mathbb{Q}}, \mathbb{C})$ and $M_K^0 = \bigsqcup_{p>0} \operatorname{Hom}(\overline{\mathbb{Q}}, \overline{\mathbb{Q}}_p)$.

Here, Roth's theorem is false.

Remark 4.3.6. Roth implies that Thue's equation, e.g. $x^3 - 2y^3 = 1$, has finitely many solutions (or solutions of bounded height). Here, can take anything for y, and so get solutions w/ arbitrary large height. ○

△

If you want a version of Roth for adelic curves, need to impose some further condition(s).

4.3.2 Statement of Theorem

Definition 4.3.7 (Dolce-Zucconi)**.** An adlic curve (K, Ω, μ)

(a) Satisfies the `μ-equicontinuity condition` if for any measurable $S \subset \Omega$ with $\mu(S) < \infty$ and any $\delta, \varepsilon > 0$, there exists a finite cover $C_1, \ldots, C_m$ of S by measurable sets such that: for all $x \in K^\times$, there is a measurable $U_x \subset S$ such that $\mu(U_x) < \delta$ and

$$\left| -\log^- |x|_\omega + \log^- |x|_{\omega'} \right| < \varepsilon h(x)$$

for all $\omega, \omega' \in C_j \setminus U_x$ for all j.

[8] To get product formula to work, take a projective embedding and use degrees of divisors in this embedding

(b) Satisfies the `uniform integrability condition` if for all $\varepsilon > 0$ there is a $\delta > 0$ such that

$$\int_T -\log^- |x|_\omega \, \mathrm{d}\mu(\omega) < \varepsilon h(x).$$

for all $x \in K^\times$ and all measurable $T \subset \Omega^\infty = \{\omega \in \Omega : |\cdot|_\omega \text{ is archimedean}\}$ with $\mu(T) < \delta$. ◇

Exercise. Check these conditions for number fields and geometric function fields.

Theorem 4.3.8 (Roth)**.** *Let (K, Ω, μ) be an adelic curve satisfying the above two properties; fix distinct $\alpha_1, \ldots, \alpha_q \in K$; let $S \subset \Omega$ be a measurable subset of finite measure; let $\varepsilon > 0$; and let $c \in \mathbb{R}$. Then, the inequality,*

$$\int_S \max_{1 \le i \le q} \left(-\log^- |x - \alpha_i|_\omega\right) \mathrm{d}\mu(\omega) < (2 + \varepsilon)h(x) + c \tag{4.1}$$

holds for all $x \in K$ outside of a set of bounded height.

Remark 4.3.9. Above, have $\alpha_i \in K$, not $\overline{K}$. Can always pass to Galois extension though (See [V '21] prop 6.8 or SLN 1239,[9] 2.2.3) ○

4.3.3 Reduction to Simultaneous Approximation

Classically, if

$$\sum_{v \in S} \left(-\log^- |x - \alpha_v|_v\right) < (2 + \varepsilon)h(x) + c$$

fails to hold for many $x \in K$, then there exists constants $c_v \in \mathbb{R}$ for all $v \in S$ such that

$$\sum c_v > 2 + \frac{\varepsilon}{2} \text{ and } -\log^- |x - \alpha|_v \ge c_v h(x) \text{ for all } x \in S \text{ for many } x \in K$$

(pigeonhole argument).

Remark 4.3.10 (Adelic curves case). If (4.1) fails to hold for `many` $x \in K$ (i.e. w/ $h(x)$ unbounded), then for any given $m > 0$, there is a partition $S = S_1 \sqcup \cdots \sqcup S_q$ of S by measurable subsets, and $\varepsilon' > 0$ such that

$$\sum_{h=1}^{q} \int_{S_h} \min_{1 \le i \le m} \left(\frac{-\log^- |x_i - \alpha_h|_\omega}{h(x_i)} \right) \mathrm{d}\mu(\omega) \le 2 + \varepsilon'$$

also fails to hold for sequences $(x_1, \ldots, x_m) \in K^M$ with $h(x_1)$ and $h(x_i)/h(x_{i-1})$ $(i = 2, \ldots, m)$ arbitrarily large. ○

Note 10. Vojta went on to sketch the proof of this remark, but I didn't bother taking notes. In fact, I stopped taking notes altogether.

4.4 Shou-Wu Zhang (Princeton University): Diophantine Geometry: All Our Yesterdays

"Thanks to my co-organizers for forcing me to give this talk."

Yesterdays = (1901-2000)

4.4.1 Diophantine Geometry

Lang's book 1962 popularized (coined?) the name.

[9] (Vojta's) Spring Lecture Notes

History (1901, Poincaré)**.** People had studied many equations, and had realized that even some which look different are really the same, e.g. since they're related by a rational transformation. Poincaré suggested studying these equations via using geometry. For the most part, he was considered with (plane) curves $f(x, y) = 0$.

If $g = 0$, he knew curves with linear or quadratic. That is most of what was known in the time from Diophantus to Poincaré.

$g = 1$. Work of Jacobi, Euler product formula. Definite group law, and he proposed Poincaré Hypothesis that the group law is finitely generated. This is now the Mordell-Weil theorem (proved, over $\mathbb{Q}$, by 1922 by Mordell).

"I think this is a good thing in mathematics. Somebody proposes a conjecture, and 20 years later, somebody proves it. This should be tradition." (paraphrase) $\ominus$

History. In 1928, Mordell proposed his famous conjecture. Sounds like proving this was suggested as a PhD problem to Weil. He tried very hard, but was unable to prove it. However, he did manage to prove that the Mordell theorem can be extended to $\mathrm{Jac}(C)$ if $g(C) \geq 1$. His idea was then to show that $\#C(K) \cap \mathrm{Jac}(C)(K) < \infty$. Something like this was only realized much later, by Paul Vojta. In working on this, Weil introduced the theory of heights and Galois descent (in his PhD thesis). $\ominus$

Note 11. I'm gonna stop taking notes, and just listen. This doesn't feel like a "take notes" kinda talk...

5 List of Marginal Comments

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