Outline

- State (without proof) a motivating theorem (related char 0 to char p).
- Describe some categories (equivalent to categories) of char p Galois reps.
- Combine above two to get a description of category of p-adic Galois reps.

Fact (Motivation, Fontaine-Wintenberger)**.** There is a (topological) isomorphism

$$G_{\mathbb{Q}_p(\mu_{p^\infty})} \simeq G_{\mathbb{F}_p(\!(T)\!)}$$

which, moreover, extends to an embedding $G_{\mathbb{Q}_p} \hookrightarrow \mathrm{Aut}(\mathbb{F}_p(\!(t)\!)^s)$.

Takeaway: To understand representations of $\mathbb{Q}_p$, suffices to understand representations of $\mathbb{F}_p(\!(t)\!)$ equipped with an additional action of $\Gamma = \mathrm{Gal}(\mathbb{Q}_p(\mu_{p^\infty})/\mathbb{Q}_p) \simeq \mathbb{Z}_p^\times$.

1 Char p fields

Let E be a characteristic p field. Can we find a non-scary category equivalent to $\mathrm{Rep}_{\mathbb{Z}_p}(G_E)$?

finitely generated $\mathbb{Z}_p$-modules w/ continuous G_E-action

1.1 $\mathbb{F}_p$-linear reps when E perfect

First assume E is perfect, and also look instead of $\mathrm{Rep}_{\mathbb{F}_p}(G_E)$.

Recall 1 (Galois theory à la Grothendieck)**.** There are equivalences of categories

$$\left\{\begin{matrix}\text{étale}\\ E\text{-algs}\end{matrix}\right\}^{\mathrm{op}} \longleftrightarrow \left\{\begin{matrix}\text{fin. étale}\\ E\text{-schemes}\end{matrix}\right\} \longleftrightarrow \left\{\begin{matrix}\text{finite sets w/}\\ \text{continuous } G_E\text{-action}\end{matrix}\right\}$$

given by

$$\begin{array}{ccccc} \mathscr{O}(Z) & \longleftarrow\!\shortmid & Z & \longmapsto & Z(E^s) \\ A & \longleftarrow\!\shortmid & \mathrm{Spec}\, A & \longmapsto & \mathrm{Hom}_{k\text{-alg}}(A, E^s) \end{array}$$

⊙

Corollary 2 (Take commutative group objects)**.** $\mathrm{Rep}_{\mathbb{F}_p}(G_E) \longleftrightarrow \{$*fin. étale group schemes killed by* $p\}$

Corollary 3 (using E perfect)**.** $\mathrm{Rep}_{\mathbb{F}_p}(G_E) \longleftrightarrow \{$*étale Dieudonné modules killed by* $p\}$.

Lemma 4. *Let G/E be a finite commutative group scheme. Then, G is étale $\iff$ $F : G \to G^{(p)}$ is an isomorphism.*

Proof. We first claim F is always an isomorphism on $\overline{E}$ points. Indeed, here we have the map

$$\begin{array}{ccc} \mathrm{Hom}(A, \overline{E}) & \longrightarrow & \mathrm{Hom}(A^{(p)}, \overline{E}) \\ \varphi & \longmapsto & \left[\varphi^{(p)}(a \otimes x) = x\varphi(a)^p\right]. \end{array}$$

This is bijective since A is f.dim and $\overline{E}$ is perfect. Thus, if G is étale then F is an iso. Conversely, $F|_{G^0}$ is nilpotent (because $A^0 = \mathscr{O}(G^0)$ is local Artinian, so its maximal ideal is nilpotent). If $F : G \to G^{(p)}$ is an iso, then $F|_{G^0}$ is nilpotent and an iso, so $G^0 = 0$ and G is étale. ■

Frobenius will map every non-unit of A^{p^r} to 0 if $r \gg 0$

So $\mathbb{F}_p$-linear G_E-representations are equivalent to finite length Dieudonné modules killed by p with F bijective.

1.2 $\mathbb{F}_p$-linear reps for general E of char p

Say E is any field of characteristic p.

Definition 5. An **étale φ-module** (= 'étale Dieudonné module killed by p') is a f.dim E-vector space M equipped with an additive bijection $\varphi : M \to M$ so that $\varphi(\alpha m) = \alpha^p \varphi(m)$ for all $\alpha \in E$ and $m \in M$. $\diamond$

Theorem 6. *The category of étale φ-modules is equivalent to the category $\mathrm{Rep}_{\mathbb{F}_p}(G_E)$. The functors are given by*

Compatible with tensor products and duals

$$M \mapsto (M \otimes_E E^s)^{\varphi=1} =: V_E(M) \text{ and } V \mapsto \left(V \otimes_{\mathbb{F}_p} E^s\right)^{G_E} =: D_E(V)$$

(M an étale φ-module, and V an $\mathbb{F}_p$-linear G_E-rep).

Remark 7. $\overline{E}$ has two structures, both a G_E-action and a Frobenius semilinear automorphism. By 'using up' one of these, we are able to pass between the two corresponding categories. $\circ$

Proof Sketch. To check properties of D_E, use the natural (Galois- and Frobenius-equivariant) isomorphism

$$D_E(V) \otimes_E E^s \xrightarrow{\sim} V \otimes_{\mathbb{F}_p} E^s.$$

This shows $\dim_E D_E(V) = \dim_{\mathbb{F}_p}(V)$ is finite, and that Frobenius on $D_E(M)$ is bijective since this holds after field extension (frobenius on RHS obviously bijective). Taking φ invariants, induces iso $V_E(D_E(V)) \xrightarrow{\sim} V$.

Checking that V_E lands in the right category and has D_E as a quasi-inverse is a bit more involved. Via similar base change trick, we're interested in showing that

$$V_E(M) \otimes_{\mathbb{F}_p} E^s \longrightarrow M \otimes_E E^s$$

is an isomorphism. For injectivity, suppose $v_1, \ldots, v_r \in V_E(M)$ satisfy a linear relation

$$\sum_i a_i v_i = 0 \text{ with } a_i \in E^s$$

with r minimal. Assume wlog that $a_r = 1$. Then (using $v_i = \varphi(v_i)$)

$$0 = 0 - \varphi(0) = \sum_{i=1}^{r} (a_i - \varphi(a_i)) v_i = \sum_{i=1}^{r-1} (a_i - \varphi(a_i)) v_i,$$

so $a_i = \varphi(a_i)$ by minimality of r. Thus, $a_i \in \mathbb{F}_p$, so the v_i's satisfies a linear relation over $\mathbb{F}_p$ already.

For surjectivity, we use the following neat trick. Let $m_1, \ldots, m_n \in M$ ($n = \dim_E M$) be a basis, and write

$$\varphi(m_j) = \sum_{i=1}^{n} C_{ij} m_i.$$

So, if $m \in M \otimes E^s$ corresponds to the vector $v \in (E^s)^{\dim M}$, then $\varphi(m)$ corresponds to the vector $C \cdot v^p$. Hence, m is frob-fixed $\iff v = C \cdot v^p \iff v^p = C^{-1} \cdot v$. That is, $V_E(M)$ is identified with the E^s points of

$$X = \operatorname{Spec} A \text{ where } A := \frac{E[x_1, \ldots, x_n]}{\left(x_j^p - \sum_{i=1}^{n} (C^{-1})_{ij} x_i\right)}.$$

This is étale ($\Omega_{X/E} = 0$ since the Jacobian of this system is the invertible matrix C^{-1}), so $\#V_E(M) = \#X(E^s) = p^{\dim M}$, and we win. ■

1.3 $\mathbb{Z}_p$-linear reps for general E of char p

To understand $\mathbb{Z}_p$-linear reps, we would want étale φ-modules over a nice char. 0 lift $\mathscr{O}_{\mathscr{E}}$ of E, equipped with a Frobenius action.

Example. If $E = \mathbb{F}_p$, can take $\mathscr{O}_{\mathscr{E}} = \mathbb{Z}_p$. If E is perfect, can take $\mathscr{O}_{\mathscr{E}} = W(E)$. Apparently, if $E = \mathbb{F}_p((T))$, can take

$$\mathscr{O}_{\mathscr{E}} = \left\{ \sum_{n \geq -\infty} a_n T^n \,\middle|\, a_n \in \mathbb{Z}_p \text{ and } \lim_{n \to -\infty} a_n = 0 \right\}.$$

I think this is $\mathbb{Z}_p[\![T]\!]\widehat{[T^{-1}]}$

Above, Frobenius acts e.g. via $T \mapsto (T+1)^p - 1$. △

Definition 8. Let E be a char p field. We'll say $\mathscr{O}_{\mathscr{E}}$ is a **Cohen ring** for E if it is a complete dvr with uniformizer p such that $\mathscr{O}_{\mathscr{E}}/(p) \simeq E$. ⋄

Theorem 9. *Let E be a char p field. A Cohen ring $\mathscr{O}_{\mathscr{E}}$ exists.*

Proof Sketch. We claim that exists a flat, local $\mathbb{Z}_p$-algebra $\mathbb{Z}_p \to R$ with maximal ideal $\mathfrak{m}_R = pR$ and residue field $R/pR \simeq E$. Given this, one can take $\mathscr{O}_{\mathscr{E}} := \widehat{R}$.

$\widehat{R}$ noetherian (since R/p noetherian and pR f.gen) local, then $\dim \widehat{R} \leq \dim \widehat{R}/x\widehat{R} + 1$ for any $x \in \widehat{R}$

To get R, order the set[1]

$$S := \{(R, \mathbb{Z}_p \to R, R/p \hookrightarrow E) : \mathfrak{m}_R = p\mathbb{Z}_p\}$$

by saying $(R, \mathbb{Z}_p \to R, R/p \hookrightarrow E) \leq (R', \mathbb{Z}_p \to R', R'/p \hookrightarrow E')$ iff $\exists R \to R'$ making obvious diagram commute. Zorn[2] gives a maximal element $(R, \mathbb{Z}_p \to R, R/p \hookrightarrow E)$, and we claim $R/p \hookrightarrow E$ is an iso. If not, let $F := R/p$, and choose some $\alpha \in E$ not in the image. If α is transcendental, can form $R' := R[x]_{pR[x]}$ which is local with residue field $F(\alpha)$, contradicting maximality. If α has minimal polynomial

$$\overline{f}(T) = T^n + \overline{a}_{n-1}T^{n-1} + \cdots + \overline{a}_1 T + \overline{a}_0 \text{ with } a_i \in R,$$

then can form $R' = R[T]/(f(T))$ which is local with residue field $F(\alpha)$, contradicting maximality. ■

Fact. $\mathscr{O}_{\mathscr{E}}$ is unique (up to non-unique isomorphism) and supports a lift φ of Frobenius.

Theorem 10. *The category of étale φ-modules over $\mathscr{O}_{\mathscr{E}}$ is equivalent to the category* $\mathrm{Rep}_{\mathbb{Z}_p}(G_E)$ *via*

f.g. $\mathscr{O}_{\mathscr{E}}$-module with φ-semilinear additive bijection

$$M \mapsto (M \otimes_{\mathscr{O}_{\mathscr{E}}} \mathscr{O}^s_{\mathscr{E}})^{\varphi=1} =: V_E(M) \text{ and } V \mapsto \left(V \otimes_{\mathbb{Z}_p} \mathscr{O}^s_{\mathscr{E}}\right)^{G_E} =: D_E(V),$$

where $\mathscr{O}^s_{\mathscr{E}} := \widehat{\mathscr{O}^{sh}_{\mathscr{E}}}$ is the completion of the strict hensilization of $\mathscr{O}_{\mathscr{E}}$.

Argue for finite length modules via induction to reduce to statements involve E in place of $\mathscr{O}_{\mathscr{E}}$. Then, argue for general

Corollary 11. $\mathrm{Rep}_{\mathbb{Q}_p}(G_E)$ *is equivalent to étale φ-modules over $\mathscr{E} = \mathrm{Frac}\,\mathscr{O}_{\mathscr{E}} = \mathscr{O}_{\mathscr{E}}[1/p]$.*

[1] ignore set-theoretic issues, see tag 03C3. Alternative: apply transfinite induction to the size of a set of generators for E over R/p

[2] take direct limit of chains

2 p-adic and perfectoid fields

Recall 12 (Fontaine-Wintenberger)**.** The absolute Galois groups of $\mathbb{Q}_p(\mu_{p^\infty})$ and $\mathbb{F}_p(\!(t)\!)$ are (topologically) isomorphic. Moreover, this isomorphism extends into an embedding $G_{\mathbb{Q}_p} \hookrightarrow \mathrm{Aut}(\mathbb{F}_p(\!(t)\!)^s)$. ⊙

Definition 13. Let $\Gamma := \mathrm{Gal}(\mathbb{Q}_p(\mu_{p^\infty})/\mathbb{Q}_p) \simeq \mathbb{Z}_p^\times$. Let $E = \mathbb{F}_p(\!(T)\!)$, and let

$G_{\mathbb{Q}_p} \curvearrowright \mathscr{O}_{\mathscr{E}}$ since $G_{\mathbb{Q}_p} \hookrightarrow \mathrm{Aut}(\mathbb{F}_p(\!(t)\!)^s)$

$$\mathscr{O}_{\mathscr{E}} = \left\{ \sum_{n \geq -\infty} a_n T^n \,\middle|\, a_n \in \mathbb{Z}_p \text{ and } \lim_{n \to -\infty} a_n = 0 \right\}.$$

Note that $\Gamma \curvearrowright \mathscr{O}_{\mathscr{E}}$ via $\gamma \cdot T = (T+1)^\gamma - 1$. An **étale** (φ, Γ)**-module** is a f.g. $\mathscr{O}_{\mathscr{E}}$-module M equipped an additive bijection $\varphi_M : M \to M$ and a continuous Γ-action, denoted $m \mapsto \gamma(m)$, satisfying

$$\varphi_M(\alpha m) = \varphi(\alpha)\varphi_M(m) \text{ and } \gamma(\alpha m) = (\gamma \cdot \alpha)\gamma(m) \text{ and } \gamma(\varphi_M(m)) = \varphi_M(\gamma(m))$$

for all $\alpha \in \mathscr{O}_{\mathscr{E}}$, $\gamma \in \Gamma$, and $m \in M$. ◇

Theorem 14. $\mathrm{Rep}_{\mathbb{Z}_p}(G_{\mathbb{Q}_p})$ *is equivalent to the category of étale* (φ, Γ)*-modules via*

$$M \mapsto (M \otimes_{\mathscr{O}_{\mathscr{E}}} \mathscr{O}_{\mathscr{E}}^s)^{\varphi=1} \text{ and } V \mapsto \left(V \otimes_{\mathbb{Z}_p} \mathscr{O}_{\mathscr{E}}^s\right)^{G_E}.$$

If there's extra time, we can fit the Fontaine-Wintenberger into a larger context...

Definition 15. Let K be a nonarchimedean field of residue characteristic p. It is **perfectoid** if its value group is non-discrete and the Frobenius map

$$\varphi : \mathscr{O}_K/p \longrightarrow \mathscr{O}_K/p$$

is surjective. ◇

(If $\operatorname{char} K = p$, then perfectoid = perfect + non-discrete valuation)

Definition 16. Let K be a perfectoid field. The **tilt** of K, denoted $K^\flat$ is the multiplicative monoid

$$K^\flat := \varprojlim_{z \mapsto z^p} K = \left\{ (a_n)_{n \geq 0} : a_{n+1}^p = a_n \right\}$$

with addition law $(a_n) + (b_n) = (c_n)$, where

$$c_n = \lim_{m \to \infty} (a_{m+n} + b_{m+n})^{p^m}.$$

This is a field of characteristic p. Equipped with the absolute value $|(a_n)| := |a_0|$, it becomes perfectoid. ◇

Theorem 17. *Let K be a perfectoid field. There is an equivalence of categories between finite extensions of K and finite extensions of $K^\flat$. Thus,*

$$\mathrm{Gal}(\overline{K}/K) \cong \mathrm{Gal}(\overline{K^\flat}/K^\flat).$$

Example. $\widehat{\mathbb{Q}_p(\mu_{p^\infty})}$ is perfectoid with tilt $\mathbb{F}_p\left(\!\left(t^{p^{-\infty}}\right)\!\right)$. Since Galois groups are preserved under completions (by Krasner) and perfection, this recovers Fontain-Wintenberger's theorem. △