Singular Fibers on Elliptic Surfaces

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July 22, 2021

Abstract

Elliptic curves – smooth curves of genus 1 – exhibit many interesting geometric and arithmetic properties, and so have long been studied by geometers. As is common in mathematics, instead of always studying these objects "one curve at a time," it has proven fruitful to study many of them at once. In particular, there exists a well-developed theory of holomorphic families of elliptic curves, given as maps $\pi : X \to C$ from a surface X onto a base curve C such that the generic fiber $\pi^{-1}(c) \subset X$ is smooth of genus 1. A surface X admitting such a map $\pi : X \to C$ is called an elliptic surface, and one generally breaks the study of these surfaces into a local theory – concerned primarily with the possible singular fibers of π, the possible degenerations of a family of elliptic curves – and a global theory – concerned primarily with the geometry of the surface X itself, including determining its numerical invariants and the existence or lack thereof of a section of π. The goal of this paper is to give a detailed account of the local theory of these elliptic surfaces. In particular, we will given an account of Kodaira's [9] classification of singular fibers of elliptic surface, supplemented with many examples.

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1 Introduction

The main subject of this paper is the local study of elliptic surfaces. Here, an elliptic surface X over a base curve C is a surface equipped with a projection map $\pi : X \to C$ such that $\pi^{-1}(c)$ is an elliptic curve – i.e. a smooth curve of genus 1 – for all but finitely many $c \in C$. A natural question one can ask is, "what do the fibers $\pi^{-1}(c)$ look like when they are not smooth?" Put another way, "what are all the ways a holomorphic family of elliptic curves can degenerate?" This question originally answered by Kodaira [9] who gave a complete classification of the possible singular (i.e. non-smooth) fibers of an arbitrary elliptic fibration. After covering the basics of the theory of compact, complex surfaces, we will describe and give a proof of Kodaira's classification followed by a detailed analysis of explicit constructions of various elliptic surfaces.

We have attempted to keep the exposition both readable and satisfactorily detailed. However, the study of complex surfaces is amenable to a wide array of viewpoints and contains many deep theorems, so some concessions have been made. There are a number of theorems which are useful for our goals, but whose proofs are nevertheless outside of the scope of this text. For these theorems, we have omitted proving them ourselves, but have provided references where one may look up their proofs. As far as the treatment of complex surfaces is concerned, we have maintained a largely algebraic viewpoint, but, when doing so provides a simpler proof or offers a useful alternative perspective, occasionally forego an algebraic argument for a more topological/geometric treatment of some results. This is especially apparent in Chapter 3. Finally, while we have attempted to keep of the level of exposition accessible throughout, there are a few instances where we have opted to include comments which are more technical than the surrounding material, but which offer another perspective from which to see things. These comments will be labelled as *Technical Aside*s in the text.

As far as content is concerned, in Chapter 2, we give an overview of the basics of complex surface theory. There is much material which could potentially be covered this chapter, but we have striven to largely restrict it to the gems of the theory – such as the Riemann-Roch Theorem for Surfaces – and to the aspects especially need for our later study of elliptic surfaces – such as the determination of the singularities arising from cyclic quotients, Theorem 2.3.10. We begin it in Section 2.1 by introducing the category of complex spaces which contains smooth, complex manifolds, but also more general objects such as the types of degenerate spaces appearing as singular fibers. After introducing these, in Section 2.2, familiarize ourselves with some of the basic tools of surface theory: divisors and intersection theory. In Section 2.3, we introduce the study of singular surfaces with a focus on the particular types of singularities which will come up when studying elliptic surfaces. The only reason this section precedes Section 2.4 is because that section will briefly need to use the language of blowups; other than that one dependency, the two can be read in any order. In Section 2.4, we extend the previous section by providing a set of definitions and results which allow us to consider all (possibly non-smooth) curves $C \subset X$ embedded in a surface X on equal footing. Finally, in Section 2.5, we study general properties of fibrations which will come up in the next two chapters.

In Chapter 3, before diving into the material on elliptic surfaces, we see how to tools and techniques developed in Chapter 2 can be applied to the study of a simpler class of surfaces: the ruled surfaces. The purpose of this is twofold. On one hand, it gives the reader a chance to better internalize some of the material of the previous chapter, and on the other hand, it provides an example of the global study of a class of surfaces before we initiate our local study of elliptic surfaces in Chapter 4.

Finally, Chapter 4 contains the main content of this paper. We begin in Section 4.1 by providing the basic definitions and context one needs to study elliptic surfaces. We then dive immediately into Kodaira's classification result, giving a complete proof of it in Section 4.2. This will utilize much of the material covered in Chapter 2. Of note, it will crucially involve appealing to Zariski's lemma to relate the fibers of a fibration to a certain family of graphs, the extended Dynkin diagrams. After this is all explained, Section 4.3 constructs and analyzes many examples of elliptic surfaces.

1.1 Common Notation and Conventions

Here is a list of some notational choices made throughout this paper. Most of these are standard and many of them – especially those which are less standard – will be reintroduced when they are first encountered in the text.

Notation	Description
Ab	the category of abelian groups.
$\mathrm{Ab}(X)$	the category of abelian sheaves on a space X.
$\mathrm{Mod}(X)$	the category of $\mathscr{O}_X$-modules on a ringed space X with structure sheaf $\mathscr{O}_X$.
$\Gamma(\mathscr{F}) = \Gamma(X, \mathscr{F})$	the global sections of a sheaf $\mathscr{F}$ on a space X.
$\mathrm{H}^i(\mathscr{F}) = \mathrm{H}^i(X, \mathscr{F})$	the ith sheaf/derived functor cohomology group of a sheaf $\mathscr{F}$ on a space X.
$\check{\mathrm{H}}^i(\mathscr{F}) = \check{\mathrm{H}}^i(X, \mathscr{F})$	the ith Čech cohomology group of a sheaf $\mathscr{F}$ on a space X.
$h^i(\mathscr{F})$	the dimension $\dim_{\mathbb{C}} \mathrm{H}^i(X, \mathscr{F})$ of the ith cohomology group of a sheaf $\mathscr{F}$.
$\mathscr{O}_X$	the structure sheaf of a ringed space X.
$\mathfrak{m}_x$	the maximal ideal $\mathfrak{m}_x \subset \mathscr{O}_{X,x}$ of the stalk $\mathscr{O}_X$ at the point x.
$E(\mathscr{E})$	the total space of the vector bundle corresponding to the locally free sheaf $\mathscr{E}$.
$\mathscr{E}(x)$	the fibre $\mathscr{E}(x) = \mathscr{E}_x \otimes \mathscr{O}_{X,x}/\mathfrak{m}_x$ of a locally free sheaf above a point $x \in X$.
X_s	the fiber $\pi^{-1}(s) \subset X$ above a point $s \in S$ under a map $\pi : X \to S$.
$[C]$	the fundamental homology class $[C] \in \mathrm{H}_{\dim C}(X; \mathbb{Z})$ of a submanifold $C \subset X$.
$[C]^*$	the Poincaré dual $[C]^* \in \mathrm{H}^{\dim X - \dim C}(X; \mathbb{Z})$ of $[C]$ where $C \subset X$ and X compact.
$C \sim D$	denotes that two divisors C, D are linearly equivalent, i.e. $\mathscr{O}_X(C) \simeq \mathscr{O}_X(D)$.
$\mathbf{e}(x)$	the function $\mathbf{e}(x) = \exp(2\pi i x)$

2 Background Material

2.1 Complex Spaces

To begin, we shall define the category in which we will work. In this document, we will be concerned with certain families of complex manifolds. These will be given in the form of a proper, holomorphic map $f : X \to S$ which corresponds to the family $\{f^{-1}(p) : p \in S\}$ of spaces. Most members of this family will be manifolds, but there will be some number singular fibers which can be thought of as different ways for this family to degenerate. In order to study these degenerate members, we need to work with spaces more general than manifolds, and so we introduce the category of complex spaces. While manifolds are spaces that look locally like open balls, we will only require complex spaces to look locally like vanishing locus of a (set of) holomorphic function(s) on a ball.

Recall 2.1.1. A **ringed space** is a pair $X = (|X|, \mathscr{O}_X)$ consisting of a topological space $|X|$ along with a sheaf $\mathscr{O}_X$ of (commutative) rings on it. The space $|X|$ is called the **underlying (topological) space** of X, and $\mathscr{O}_X$ is called its **structure sheaf**

Example. Let $B \subset \mathbb{C}^n$ be an open ball with structure sheaf $\mathscr{O}_B$ of holomorphic functions, and let $\{f_i\}_{i \in I} \subset \Gamma(B, \mathscr{O}_B)$ be a collection of (global) holomorphic functions. Let $\mathscr{I}$ be the subsheaf of $\mathscr{O}_B$ generated, as an $\mathscr{O}_B$-module, by the f_i, so

$$\mathscr{I}(U) = \left\{ \sum_{i \in I} c_i f_i|_U : c_i \in \mathscr{O}_B(U) \text{ and } c_i = 0 \text{ for all but finitely many } i \right\}.$$

Now, let $|Y| = \{z \in B : f_i(z) = 0 \, \forall i \in I\}$ and let $\mathscr{O}_Y = \mathscr{O}_B / \mathscr{I}$, a sheaf supported on $|Y|$. Then, the ringed space $Y = (|Y|, \mathscr{O}_Y)$, denoted $V(f_i)_{i \in I}$, is a prototype for the types of spaces we will study.

Definition 2.1.1. If Y is a ringed space isomorphic to $V(f_i)_{i \in I}$ for some collection $\{f_i\}_{i \in I}$ of holomorphic functions on an open ball in $\mathbb{C}^n$, then we call Y a **closed analytic subspace** of B.

Definition 2.1.2. A **complex space** is a Hausdorff ringed space $X = (|X|, \mathscr{O}_X)$ which is locally isomorphic to closed analytic subspaces. That is, we can over $|X|$ by open sets $U \subset |X|$ such that $(U, \mathscr{O}_U)$ is a closed analytic subspace, where $\mathscr{O}_U = \mathscr{O}_X|_U$ is the restriction of the structure sheaf on X. Note that, if X is complex space, then its structure sheaf $\mathscr{O}_X$ is a sheaf of $\mathbb{C}$-algebras since this is the case for closed analytic subspaces.

In order to completely specify the category of complex spaces in which we will work, we need to know not only the objects, but also the morphisms.

Recall 2.1.2. Given a continuous map $f : X \to Y$, and a sheaf $\mathscr{F}$ on X, one obtains the **pushforward/direct image** sheaf $f_*\mathscr{F}$ on Y. For an open subset $V \subset Y$, one has $f_*\mathscr{F}(V) = \mathscr{F}(f^{-1}(V))$.

Recall 2.1.3. Given two ringed spaces $X = (|X|, \mathscr{O}_X)$ and $Y = (|Y|, \mathscr{O}_Y)$ a **morphism of ringed spaces** $f = (|f|, \widetilde{f})$ is a pair consisting of a continuous map $|f| : |X| \to |Y|$ and a morphism $\widetilde{f} : \mathscr{O}_Y \to |f|_* \mathscr{O}_X$ of sheaves on $|Y|$.

Notation 2.1.3. If $X = (|X|, \mathscr{O}_X)$ is a ringed space, we will often denote its underlying space still by X instead of by $|X|$. Similarly, if $f = (|f|, \widetilde{f})$ is a morphism of ringed spaces, then we will let f also denote the underlying map $|f|$ on topological spaces.

Notation 2.1.4. Given a sheaf (of sets, rings, groups, etc.) $\mathscr{F}$ on a topological space X, its set (ring, group, etc.) of global section may be denoted using any of the following 3 notations

$$\mathscr{F}(X) = \Gamma(X, \mathscr{F}) = \mathrm{H}^0(X, \mathscr{F}).$$

Starting in Section 2.2, we will prefer the latter two.

Definition 2.1.5. Let X, Y be complex spaces. A **morphism of complex spaces** $f : X \to Y$ is a morphism of ringed spaces such that the map $\widetilde{f} : \mathscr{O}_Y \to f_*\mathscr{O}_X$ is a morphism of sheaves of $\mathbb{C}$-algebras. Morphisms of complex spaces are also called **holomorphic** or **analytic** maps. Isomorphisms of complex spaces are similarly called **biholomorphic** maps.

Complex spaces along with holomorphic maps form our main category of interest. In studying a complex space X, we will be interested in a number of sheaves on X, but we will not be interested in just any sheaf on X. We will focus our attention on sheaves of $\mathscr{O}_X$-modules on X, and in particular, those which are "locally of finite presentation" in the following sense.

Definition 2.1.6. Let X be a complex space, and let $\mathscr{F}$ be an $\mathscr{O}_X$-module. We say that $\mathscr{F}$ is **coherent** if there exists an open cover $\{U_i\}_{i\in I}$ of X along with local exact sequences

$$\mathscr{O}_{U_i}^{\oplus r_i} \longrightarrow \mathscr{O}_{U_i}^{\oplus s_i} \longrightarrow \mathscr{F}|_{U_i} \longrightarrow 0$$

of $\mathscr{O}_{U_i}$-modules, where $r_i, s_i \in \mathbb{Z}$. This essentially says that, as an $\mathscr{O}_{U_i}$-module, $\mathscr{F}|_{U_i}$ is generated by s_i generators satisfying r_i relations.

Coherent sheaves are robust in the sense that coherence is preserved under many operations one naturally encounters. For example, one may pull a coherent sheaf on Y back to one on X along a holomorphic map $X \to Y$.

Definition 2.1.7. Let $f : X \to Y$ be a holomorphic map and let $\mathscr{F}$ be an $\mathscr{O}_Y$-module. Its **analytic inverse image** (or **analytic pull-back**) is the sheaf

$$f^*\mathscr{F} := f^{-1}\mathscr{F} \otimes_{f^{-1}\mathscr{O}_Y} \mathscr{O}_X,$$

where f^{-1} denotes formation of the usual inverse image sheaf.

Remark 2.1.1. It is true in general that for a continuous map $f : X \to Y$, the functor $f^{-1} : \mathrm{Ab}(Y) \to \mathrm{Ab}(X)$ (Here, $\mathrm{Ab}(X)$ is the category of abelian sheaves on X) is exact e.g. because $(f^{-1}\mathscr{F})_x = \mathscr{F}_{f(x)}$ for any $\mathscr{F} \in \mathrm{Ab}(Y)$ and $x \in X$, and because exactness can be checked on stalks. Similarly, for a holomorphic map $f : X \to Y$, the functor $- \otimes_{f^{-1}\mathscr{O}_Y} \mathscr{O}_X : \mathrm{Ab}(X) \to \mathrm{Ab}(X)$ of tensoring with $\mathscr{O}_X$ is right exact because this can be checked on stalks where it reduces to the fact that tensoring with a fixed module is right exact.

Combining these observations, for a holomorphic map $f : X \to Y$, the analytic pullback functor $|f| : \mathrm{Mod}(Y) \to \mathrm{Mod}(X)$, where $\mathrm{Mod}(Y), \mathrm{Mod}(X)$ are the categories of $\mathscr{O}_Y$- and $\mathscr{O}_X$-modules, respectively, is right exact. Because coherence is defined by the existence of right exact sequences, and because pulling back commutes with restricting to an open subset, we arrive at the following.

Proposition 2.1.1. *Let $f : X \to Y$ be a holomrophic map, and let $\mathscr{S}$ be a coherent $\mathscr{O}_Y$-module. Then, $f^*\mathscr{S}$ is a coherent $\mathscr{O}_X$-module.*

If X is a complex space, and $U \subset X$ is an open set, then there is a single, natural complex structure on U. In particular, we give U the structure sheaf $\mathscr{O}_U = \mathscr{O}_X|_U$. The story for closed sets is a little more complicated.

Example. Let $\Delta \subset \mathbb{C}$ be the open unit disk, and let $f_1, f_2 : \Delta \to \mathbb{C}$ be the holomorphic functions given by $f_1(z) = z$ and $f_2(z) = z^2$. Then, $Y_1 = V(f_1)$ and $Y_2 = V(f_2)$ are both closed analytic subspaces of Δ supported on the point $0 \in \Delta$. However, they are not isomorphic as complex spaces since, for example, $\Gamma(Y_1, \mathscr{O}_{Y_1}) = \mathbb{C}$, but $\Gamma(Y_2, \mathscr{O}_{Y_2})$ is 2-dimensional over $\mathbb{C}$.

For closed complex subspaces, more important than the underlying closed set is the defining sheaf of ideals. Let X be a complex space, and let $\mathscr{I} \subset \mathscr{O}_X$ be a coherent $\mathscr{O}_X$-sheaf of ideals. Then, $Y = (|Y|, \mathscr{O}_Y)$ is a complex space where $|Y| = \operatorname{supp}(\mathscr{O}_X/\mathscr{I}) = \{x \in X : (\mathscr{O}_X/\mathscr{I})_x \neq 0\}$ and $\mathscr{O}_Y = \mathscr{O}_X/\mathscr{I}$. Such a Y is called a **(closed) complex subspace** of X. Such a subspace naturally comes with an exact sequence

$$0 \longrightarrow \mathscr{I} \longrightarrow \mathscr{O}_X \longrightarrow \mathscr{O}_Y \longrightarrow 0.$$

Definition 2.1.8. A closed embedding $\iota : Y \hookrightarrow X$ of one complex space into another is an isomorphism from Y onto a closed complex subspace of X. In other words, it is a topological embedding whose map $\widetilde{\iota} : \mathscr{O}_X \longrightarrow \iota_* \mathscr{O}_Y$ on sheaves is surjective. The ideal sheaf of the corresponding closed subspace of X is $\mathscr{I} := \ker \widetilde{\iota}$.

Definition 2.1.9. If X is complex space and $A \subset X$ is a closed set, then A is called a **closed analytic subset** if there exists a closed embedding $Y \hookrightarrow X$ with image A.

Now, general complex spaces can have irregular geometric properties. We mentioned in the beginning that complex places were introduced to allow the study of spaces with singularities. In addition to this, the example of the closed analytic space $V(z^2)$ given before showed that general complex spaces can count points, and even closed analytic subspaces, with multiplicity. In order to distinguish spaces which avoid these, and other, examples of potentially bad behavior, we make the following definitions.

Definition 2.1.10. Let $(X, \mathscr{O}_X)$ be a ringed space. We say it is **reduced** if the stalks $\mathscr{O}_{X,x}$ are reduced rings (i.e. have no nilpotents) for all $x \in X$. This is equivalent to requiring that $\mathscr{O}_X(U)$ be reduced for all open $U \subset X$.

Example. The space $V(z) \subset \Delta$ considered before was reduced, but $V(z^2) \subset \Delta$ was not.

There is a natural process by which one takes an arbitrary complex space X and from it produce a reduced complex space X_{red}. Let $\mathscr{R} \subset \mathscr{O}_X$ be the sheafification of the presheaf

$$U \longmapsto \{f \in \mathscr{O}_X(U) : f^n = 0 \,\text{for some}\; n \geq 0\}.$$

Then, $X_{\text{red}} = (|X|, \mathscr{O}_X/\mathscr{R})$ is called the **reduction** of X.

Proposition 2.1.2. *Let X be a complex space. Then, its reduction X_{red} comes with a natural map $X_{\text{red}} \to X$ which is final among maps from reduced complex spaces to X. That is, if $f : Z \to X$ is a holomorphic map with Z reduced, then f factors as $Z \xrightarrow{f_{\text{red}}} X_{\text{red}} \to X$.*

Proof. The canonical map $X_{\text{red}} \to X$ is the identity $X = X$ on topological spaces, and the natural quotient map $\mathscr{O}_X \to \mathscr{O}_X/\mathscr{R}$ on sheaves, where $\mathscr{R}$ is the sheafification of the presheaf

$$\mathscr{F} : U \longmapsto \{f \in \mathscr{O}_X(U) : f^n = 0 \,\text{for some}\; n \geq 0\}.$$

To show that this is final among maps from reduced complex spaces to X, let $f : Z \to X$ be any holomorphic map with Z reduced. For any open $U \subset X$, f has an induced map

$$\mathscr{O}_X(U) \to f_*\mathscr{O}_Z(U) = \mathscr{O}_Z(f^{-1}(U))$$

at the level of sheaves. Since Z is reduced, $\mathscr{O}_Z(f^{-1}(U))$ has no nilpotents, so the map $\mathscr{O}_X(U) \to f_*\mathscr{O}_Z(U)$ vanishes on $\mathscr{F}(U) \subset \mathscr{O}_X(U)$ and hence factors as $\mathscr{O}_X(U) \to \mathscr{O}_X(U)/\mathscr{F}(U) \to f_*\mathscr{O}_Z(U)$. Thus, f gives rise to a map $X/\mathscr{F} \to f_*\mathscr{O}_Z$ of presheaves. By sheafifying, we can turn this into a map $\mathscr{O}_{X_{\text{red}}} = X/\mathscr{R} \to f_*\mathscr{O}_Z$ of sheaves. Combining this sheaf map with f's underlying map on topological spaces gives rise to a holomorphic map $f_{\text{red}} : Z \to X_{\text{red}}$. By construction, we have that f factors as $Z \xrightarrow{f_{\text{red}}} X_{\text{red}} \to X$. ■

Definition 2.1.11. A complex space X is called **irreducible** if for any decomposition $X = E \cup F$ of X as a union of two closed subspaces, we have $X = E$ or $X = F$. The **irreducible components** of X are its maximal (with respect to inclusion) irreducible subspaces.

Definition 2.1.12. Let X be a complex space. A point $x \in X$ is called **regular** or **smooth** if it has an open neighborhood $x \in U \subset X$ such that $(U, \mathscr{O}_U) \simeq (B, \mathscr{O}_B)$ for some open ball $B \subset \mathbb{C}^n$. If $x \in X$ is not smooth, then we call it **singular**.

Remark 2.1.2. A complex space is a complex manifold iff it is smooth at every point.

Example. Complex projective space $\mathbb{P}^n = \mathbb{CP}^n$ is an important example of a complex manifold. It can be covered by the open sets $U_i = \{[z_0 : z_1 : \cdots : z_n] \in \mathbb{P}^n : z_i \neq 0\} \simeq \mathbb{C}^n$ given by the non-vanishing of a single homogeneous coordinates each, so every point has a neighborhood biholomorphic to a ball in $\mathbb{C}^n$.

Definition 2.1.13. Let X be a reduced complex space, and fix a point $x \in X$. The **local dimension** $\dim_x X$ is the Krull dimension of the local ring $\mathscr{O}_{X,x}$. If x is regular with X looking like a ball $B \subset \mathbb{C}^n$ near x, then $\dim_x X = n$. The **dimension** of X, denoted $\dim X$, is defined to be $\max_{x \in X} \dim_x X$ when this exists. If all the local dimensions are equal, say to d, then we say that X is of **pure dimension** d. If X is not reduced, then we define $\dim X = \dim X_{\text{red}}$.

Example. Irreducible complex spaces are of pure dimension.

Definition 2.1.14. A **curve** is a complex space X with X_{red} of pure dimension 1. If X is furthermore smooth, then we call it a **Riemann surface**. Similarly, a (non-Riemann) **surface** is a complex space with X_{red} of pure dimension 2.

Example. A 1-dimensional complex manifold is the same thing as a Riemann surface, and a 2-dimensional complex manifold is a smooth surface.

Definition 2.1.15. A point x in a reduced complex space X is called **normal** if the local ring $\mathscr{O}_{X,x}$ is integrally closed (in its fraction field). We call X normal if all its points are.

Example. Let $X = \mathbb{C}^n$. Because holomorphic functions are analytic, for any point $x \in X$, the stalk $\mathscr{O}_{X,x}$ can be identified with the ring

$$\mathbb{C}\{x_1, x_2, \ldots, x_n\} \subset \mathbb{C}[\![x_1, x_2, \ldots, x_n]\!]$$

of power series convergent on some disk (depending on the power series in question). This ring is integrally closed. This is a consequence of the fact that $\mathbb{C}[\![x_1, x_2, \ldots, x_n]\!]$ is integrally closed along with the holomorphic implicit function theorem. Hence, $\mathbb{C}^n$ is normal.

Remark 2.1.3. If X is a complex space and $x \in X$ is a normal point of local dimension 1 – i.e. $\mathscr{O}_{X,x}$ is integrally closed and has Krull dimension 1 – then $\mathscr{O}_{X,x}$ is a **discrete valuation ring**. In particular, this means that $\mathscr{O}_{X,x}$ is a (local) PID and, letting $\pi \in \mathscr{O}_{X,x}$ be a generator of the maximal ideal, any nonzero element $s \in \mathscr{O}_{X,x}$ has a unique valuation, denoted $\mathrm{ord}_x(s) = \mathrm{ord}_{\mathscr{O}_{X,x}}(s) \in \mathbb{Z}_{\geq 0}$, which is the largest power e of π such that $x \in (\pi^e)$. The generator π of the maximal ideal is called a **uniformizer** or **local parameter**.

Proposition 2.1.3. *Let X be a reduced complex space. Then, it has a **normalization** $\widetilde{X} = X_{\mathrm{norm}}$ which is a complex space with a finite, surjective holomorphic map $\nu_X : X_{\mathrm{norm}} \to X$ which is final among maps from normal spaces to X.*

Proof. [6, Chapter 8] ■

Remark 2.1.4. As a consequence of the above proposition, if X is a reduced complex space with normalization $\nu : \widetilde{X} \to X$, then the map ν is an isomorphism away from X's non-normal points. Indeed, if $S \subset X$ is its set of non-normal points, then $X \setminus S$ is a normal space with an inclusion map $X \setminus S \hookrightarrow X$. Since the normalization is final among such maps, this inclusion factors as $X \setminus S \hookrightarrow \widetilde{X} \twoheadrightarrow X$. The image of the first map misses $\nu^{-1}(S)$ by definition, so we really have a composition

$$X \setminus S \hookrightarrow \widetilde{X} \setminus \nu^{-1}(S) \twoheadrightarrow X \setminus S$$

giving the identity $X \setminus S = X \setminus S$. This shows that $\widetilde{X} \setminus \nu^{-1}(S)$ maps isomorphically onto $X \setminus S$.

The final notion from the general theory of complex spaces which we will need is the notion of "base change." This provides a way to, given a map $C' \to C$, pull back a space over C to one over C' in a universal fashion. We will describe the general phenomenon in more detail, but first we make note of the two cases that will feature most heavily. We always start with a space $X \xrightarrow{\pi} C$ over C, i.e. a holomorphic map with codomain C, and a map $f : C' \to C$ to C. The first common case is when f is the inclusion of a point $f : \{c\} \hookrightarrow C$. In this case, "pulling X back along f" amounts to restricting π to c, and so the base change is simply the fiber $\pi^{-1}(c)$ with a natural complex structure as a subspace of X. In the second common use case, we think of $X \xrightarrow{\pi} C$ as a family of spaces over C, and our goal is to construct a family of spaces over a related base C'. In instances when we take this perspective, almost every space in the family $X \to C$, i.e. almost every fiber $\pi^{-1}(c)$, will be diffeomorphic to the same differentiable manifold Σ. In this case, the basechange X' of X along f is a family $X' \to C'$ of spaces of C', almost all of which are diffeomorphic to the same Σ. In this one, base change allows us to construct new holomorphic families of Σ's (e.g. of elliptic curves when Σ is a torus) from old ones.

Proposition 2.1.4. *Let $\pi : X \to C$ and $f : C' \to C$ be holomorphic maps. Then, there exists a space X', called the **basechange of** $X \to C$ **along** f and denoted $X' = X \times_C C'$, along with projection morphisms $\pi^* f = f' : X' \to X$ and $f^*\pi = \pi' : X' \to C'$ making it final among spaces fitting into the below commutative diagram.*

$$\begin{tikzcd} X \times_C C' \arrow[r, "\pi^* f"] \arrow[d, "f^*\pi"'] & X \arrow[d, "\pi"] \\ C' \arrow[r, "f"] & C \end{tikzcd}$$

That is, for any complex space Y with morphisms $Y \to X$ and $Y \to C'$ making the outer square below *commute, there exists a unique map $Y \to X \times_C C'$ such that the whole diagram below commutes.*

$$\begin{tikzcd}
Y \arrow[dr, dashed] \arrow[drr, bend left] \arrow[ddr, bend right] & & \\
& X \times_C C' \arrow[r] \arrow[d] & X \arrow[d, "\pi"] \\
& C' \arrow[r, "f"] & C
\end{tikzcd}$$

Finally, the underlying topological space of $X \times_C C'$ is

$$X \times_C C' = \{(x, c') \in X \times C' : \pi(x) = f(c')\} .$$

The space $X \times_C C'$, along with its maps to X and C', is sometimes called the ***fiber product of** X **and** C' **above** C* *or the* ***pullback of** $X \to C$ **along** f* *instead.*

Proof. [3, Corollary 0.32] ■

Remark 2.1.5. When $C' = \{*\}$ is a single point with structure sheaf $\mathscr{O}_* = \mathbb{C}$, the fiber product $X \times_* Y$ of any two spaces over C' is the usual direct product $X \times_* Y = X \times Y$ with its natural complex structure.

We will see basechanges in Section 2.5 where we will be interested in fibers (i.e. base changes along the inclusions of a point) of maps $X \to C$. In that section, we will provide a direct construction of the fibers as subspaces of X. In Chapter 4, we will see base changes more substantially; in particular, we will build "more complicated" elliptic surfaces out of "simpler" ones by base changing along suitable maps.

These definitions provide a robust language for talking about the various complex spaces we will encounter. The concepts of reduced and irreducible spaces will be especially useful, and will in particular be convenient for describing various (effective) divisors of a complex surface.

2.1.1 Important Results Concerning Complex Spaces

In addition to having this language of complex spaces, it will be prove useful to have access to some fundamental results in the theory of complex spaces. For this purpose, we record the results below and give references to their proofs. The first result, Stein factorization, will tell us that analytic maps factor as one with connected fibers followed by one with finite fibers.

Definition 2.1.16. A continuous map $f : X \to Y$ is called **proper** if $f^{-1}(K)$ is compact for every compact $K \subset Y$.

Example. If X is compact and Y is Hausdorff, then every continuous map is automatically proper.

Theorem 2.1.5 (Stein factorization). *Let X, Y be complex spaces and let $f : X \to Y$ be a proper analytic map. Then f admits a unique facotrization as*

$$X \xrightarrow{g} Z \xrightarrow{h} Y$$

such that

(i) *$g : X \to Z$ is a proper surjective holomorphic map with connected fibers and $g_*\mathscr{O}_X = \mathscr{O}_Z$.*

(ii) *h is a finite holomorphic map.*

Furthermore, if X is normal, then Z is too.

Proof. [6, Ch. 10, Sect. 6] ■

For the remaining results, we will need the notion of a higher direct image sheaf. Briefly, given a morphism $f : X \to Y$ of complex spaces, the pushforward functor f_* from $\mathscr{O}_X$-modules to $\mathscr{O}_Y$-modules is left exact, and so is the 0th functor in a infinite family f_{*i} of right derived functors from $\mathscr{O}_X$-modules to $\mathscr{O}_Y$-modules, called the higher direct image functors. Intuitively, f_{*i} can be though of as a way of forming a "relative cohomology theory" for the map $f : X \to Y$ (as opposed have cohomology defined on an individual space). These functor are treated more carefully in Appendix A, but for now, we will be content with knowing they exist and observing their role in the below theorems.

Theorem 2.1.6 (**Grauert's direct image theorem**)**.** *Let X, Y be complex spaces, and let $f : X \to Y$ be a proper analytic map. Then, for every coherent sheaf $\mathscr{S}$ on X and integer $i \geq 0$, the (higher) direct image sheaf $f_{*i}\mathscr{S}$ is also coherent.*

Proof. [6, Ch. 10] ■

Corollary 2.1.7 (**Finiteness theorem of Cartan-Serre**)**.** *Let X be a compact complex space, and let $\mathscr{S}$ be a coherent sheaf on X. Then, $\mathrm{H}^i(X, \mathscr{S})$ is finite dimensional for all $i \geq 0$.*

Proof. Fix any $i \geq 0$. We will first show that $\mathrm{H}^i(X, \mathscr{S})$ is finite dimensional. Let $f : X \to \{y\}$ be the map to the one point space whose structure sheaf is $\mathscr{O}_y = \mathbb{C}$. Then, y is a complex space (it is the space $V(z)$ from an earlier example), so Grauert's theorem applies to f. In particular, the sheaf $f_{*i}\mathscr{S} = \mathrm{H}^i(X, \mathscr{S})$ is coherent on y, which means that we have an exact sequence

$$\mathbb{C}^r \longrightarrow \mathbb{C}^s \longrightarrow \mathrm{H}^i(X, \mathscr{S}) \longrightarrow 0$$

for some s, r. In particular, $\mathrm{H}^i(X, \mathscr{S})$ is finite dimensional. ■

Corollary 2.1.8 (**Remmert's Mapping Theorem**)**.** *Let X, Y be reduceced complex spaces with a proper, analytic map $f : X \to Y$. If $A \subset X$ is a closed analytic subset, then so is $f(A) \subset Y$.*

Proof. Let $\iota : Z \hookrightarrow X$ be a closed embedding. We wish to show that $f(\iota(Z)) \subset Y$ is a closed analytic subset. Note that the morphism $f \circ \iota : Z \to Y$ comes equipped with morphism $\widetilde{f \circ \iota} : \mathscr{O}_Y \to f_*\iota_*\mathscr{O}_Z$. Because ι,f are both proper, Grauert's theorem tells us that $f_*\iota_*\mathscr{O}_Z$ is coherent on Y, so the 2-out-of-3 principle then tells us that $\mathscr{I} := \ker \widetilde{f \circ \iota}$ is a coherent sheaf of ideals. It is clear from the construction that $\mathrm{supp}(\mathscr{O}_Y/\mathscr{I}) = f(\iota(Z))$, so the claim holds. ■

Definition 2.1.17. Let $f : X \to Y$ be a proper, holomorphic map between reduced complex space. Then, a coherent sheaf $\mathscr{S}$ on X is called **flat over** Y if $\mathscr{S}_x$ is a flat $\mathscr{O}_{Y,f(x)}$-module for all $x \in X$.

Notation 2.1.18. Let $\mathscr{F}$ be a (locally free) sheaf on a space X. Then, for a point $x \in X$ the fibre of $\mathscr{F}$ above x is denoted $\mathscr{F}(x) := \mathscr{F}_x \otimes_{\mathscr{O}_{X,x}} \mathscr{O}_{X,x}/\mathfrak{m}_x$ where $\mathfrak{m}_x \subset \mathscr{O}_{X,x}$ is the maximal ideal at x.

Theorem 2.1.9. *Let X, Y be reduced complex spaces, and let $f : X \to Y$ be a proper holomorphic map. Let $\mathscr{S}$ be a coherent sheaf on X which is **flat** over Y. Then, if the function*

$$Y \ni y \longmapsto \dim_{\mathbb{C}} \mathrm{H}^q(X_y, \mathscr{S}_y)$$

is constant, we have that $f_{*q}\mathscr{S}$ is locally free on Y. Moreover,

$$(f_{*q}\mathscr{S})(y) \simeq (f_{*q}\mathscr{S})_y \otimes_{\mathscr{O}_{Y,y}} \frac{\mathscr{O}_{Y,y}}{\mathfrak{m}_y} \simeq \frac{(f_{*q}\mathscr{S})_y}{\mathfrak{m}_y \cdot (f_{*q}\mathscr{S})_y} \xrightarrow{\sim} \mathrm{H}^q(X_y, \mathscr{S}_y)$$

where $\mathfrak{m}_y \subset \mathscr{O}_{Y,y}$ is the maximal ideal at $y \in Y$. In particular, the rank of $\mathscr{S}_{*q}$ is $h^q(X_y, \mathscr{S}_y)$.

Proof. [2, Ch. I, Thm. 8.5] ■

Definition 2.1.19. Let $\mathscr{S}$ be a coherent sheaf on a compact complex space X. We can define its **Euler characteristic** to be

$$\chi(\mathscr{S}) = \chi(X, \mathscr{S}) = \sum_{i=0}^{\infty} (-1)^i h^i(X, \mathscr{S}),$$

where $h^i(X, \mathscr{S}) = \dim \mathrm{H}^i(X, \mathscr{S})$, whenver this sum is finite.

2.2 Divisors and Intersection Numbers

2.2.1 Divisors

Setup. Fix a complex manifold X of dimension 2, i.e., a compact **(smooth) surface**. We wish to introduce some basic tools in the study of the geometry of X which we will see throughout the remainder of this text.

We begin by fixing the notation for many of the important sheaves on X.

Notation 2.2.1. Let

- $\mathcal{T}_X$ denote the **(holomorphic) tangent bundle** of X.
- $\mathscr{O}_X^\times$ or $\mathscr{O}_X^*$ denote the **sheaf of nonvanishing holomorphic functions** on X.
- $\Omega_X^i = \bigwedge^i \Omega_X$ denote the **sheaf of holomorphic i-forms** on X. In particular, $\Omega_X^0 \simeq \mathscr{O}_X$ and $\Omega_X^1 \simeq \mathcal{T}_X^\vee$.
- $\omega_X \simeq \Omega_X^2$ denote the **canonical (line) bundle** on X.
- $\mathcal{N}_{Y/X}$ denote the **normal bundle** of a complex submanifold $Y \hookrightarrow X$, which is defined by the following sequence

  $$0 \longrightarrow \mathcal{T}_Y \longrightarrow \mathcal{T}_X|_Y \longrightarrow \mathcal{N}_{Y/X} \longrightarrow 0.$$

  Note that $\mathcal{N}_{Y/X}$ is a sheaf (supported) on Y.

Attached to these sheaves are certain numerical invariants of interest.

Definition 2.2.2. The **arithmetic genus** of X is $p_a(X) = \chi(\mathscr{O}_X) - 1$. Its **geometric genus** is $p_g(X) = \dim \mathrm{H}^0(X, \omega_X)$. For $0 \le p, q \le 2$, the **Hodge numbers** of X are $h^{p,q}(X) = \dim \mathrm{H}^q(X, \Omega_X^p)$.

Now, one often studies a space by studying certain vector bundles on it. The line bundles on a space are particularly amenable to study because they form not only a set, but indeed a group under the operation of tensoring. Hence, we will let $\operatorname{Pic} X$ denote the group of line bundles on X, and we hope to gain some understanding of this object. For studying this group, it will be useful to give it a cohomological interpretation.

Notation 2.2.3. Given a sheaf $\mathscr{F}$ on X, let $\check{\mathrm{H}}^k(\mathscr{F}) = \check{\mathrm{H}}^k(X, \mathscr{F})$ denote its kth Čech cohomology group.

Recall 2.2.1. Let $\mathrm{GL}_n(\mathscr{O}_X)$ denote the sheaf of holomorphic maps to $\mathrm{GL}_n(\mathbb{C})$. Then, there is a bijection between the Čech cohomology group $\check{\mathrm{H}}^1(X, \mathrm{GL}_n(\mathscr{O}_X))$ and the set of isomorphism classes of rank n vector bundles over X. This is shown, for example, in [1, Ch. 7]. In particular, because $\check{\mathrm{H}}^1(X, \mathscr{F}) \simeq \mathrm{H}^1(X, \mathscr{F})$ for any abelian sheaf $\mathscr{F}$ on X, we have that $\operatorname{Pic} X \simeq \mathrm{H}^1(X, \mathscr{O}_X^\times)$.

Notation 2.2.4. Let $\mathscr{L} \in \operatorname{Pic} X$ be a line bundle on X. By default, we think of this as an invertible sheaf on X, but we may sometimes want to reason about $\mathscr{L}$'s underlying space. In such case, we may use $E(\mathscr{L})$ to denote this space, so in particular, $E(\mathscr{L})$ is a complex space equipped with a holormophic map $p_{\mathscr{L}} : E(\mathscr{L}) \to X$ giving it the structure of a line bundle in the geometric sense.

The main utility of this result is that it allowws us to fit $\operatorname{Pic} X$ into exact sequences from which we can extract information about it. This is mainly exploited through use of the **exponential exact sequence**

$$0 \longrightarrow \underline{\mathbb{Z}}_X \longrightarrow \mathscr{O}_X \xrightarrow{\mathbf{e}} \mathscr{O}_X^\times \longrightarrow 0$$

of sheaves on X, where $\underline{\mathbb{Z}}_X$ is the constant sheaf with stalks isomorphic to $\mathbb{Z}$, and $\mathbf{e}$ is the exponential map sending a section $f \in \mathscr{O}_X(U)$ to its exponential $\exp(2\pi i f) \in \mathscr{O}_X^\times(U)$ (here $U \subset X$ is any open set). Exactness of the above sequence can be checked on stalks where one can make use of the fact that on a simply connected domain it is possible to construct holomorphic logarithms. From the existence of this short exact sequence of sheaves, we get a long exact sequence in cohomology

$$\cdots \longrightarrow \mathrm{H}^1(X, \mathbb{Z}) \longrightarrow \mathrm{H}^1(X, \mathscr{O}_X) \longrightarrow \mathrm{H}^1(X, \mathscr{O}_X^\times) \xrightarrow{\delta} \mathrm{H}^2(X, \mathbb{Z}) \longrightarrow \cdots$$

The map $\delta : \mathrm{H}^1(X, \mathscr{O}_X^\times) \longrightarrow \mathrm{H}^2(X, \mathbb{Z})$ above can be identified with the map $c_1 : \operatorname{Pic} X \longrightarrow \mathrm{H}^2(X, \mathbb{Z})$ taking a (holomorphic) line bundle its (underlying topological line bundle's) first Chern class.

Technical Aside 2.2.1. As there are numerous perspectives from which one can understand Chern classes, there are also numerous possible proofs of this fact. It is not hard to show that the map δ above is functorial. Taking advantage of this, if you are familiar with classifying spaces, then one possible proof is to use the classifying map $f : X \to \mathbb{P}^\infty = BU(1)$ for a given line bundle $\mathscr{L} \in \operatorname{Pic} X$, along with the inclusion map $\iota : \mathbb{P}^1 \hookrightarrow \mathbb{P}^\infty$ (which induces an isomorphism on H^2) to reduce to the case that $X = \mathbb{P}^1$ and $\mathscr{L} = \mathscr{O}_{\mathbb{P}^1}(-1)$ is the tautological line bundle. Then, one can simply calculate $\delta(\mathscr{O}_{\mathbb{P}^1}(-1))$ explicitly to see that it agrees with $c_1(\mathscr{O}_{\mathbb{P}^1}(-1))$.

At this point, we still have not given many methods of constructing line bundles. The main source of line bundles lies in their connection to codimension 1 submanifolds.

Definition 2.2.5. A **hypersurface** $H \subset X$ is a (non-empty) closed subset of X where every point $p \in H$ has a connected open neighborhood U in X such that $H \cap U$ is the zero set of a non-constant holomorphic function on U.

Definition 2.2.6. A **divisor** D on X is a formal sum $D = \sum_{i=1}^\infty d_i D_i$ where $d_i \in \mathbb{Z}$ and $\{D_i\}$ is a collection of irreducible hypersurfaces which is **locally finite**, i.e. every $p \in X$ has a neighborhood U meeting only finitely many of the D_i. We say that D is **finite** if in fact $d_i = 0$ for all but finitely many $i \in \mathbb{N}$. We say D is **effective** if $d_i \geq 0$ for all i. We let $\operatorname{Div} X$ denote its group of divisors. We define the **support** of D to be

$$\operatorname{supp} D = \overline{\bigcup_{\substack{i=1 \\ d_i \neq 0}}^{\infty} D_i}.$$

Remark 2.2.1. If X is compact, then any divisor is a finite (formal) sum of irreducible hypersurfaces as opposed to just a locally finite one.

Remark 2.2.2. Let S, T be irreducible hypersurfaces in X. Then, one should think of S as a reduced, irreducible complex space, of $S+T$ as a reduced but not irreducible complex space, and $2S$ as an unreduced complex space.

Definition 2.2.7. Let $f : X \to Y$ be a holomorphic map between connected, complex manifolds. If $D \in \operatorname{Div} Y$ such that $f(X) \not\subset \operatorname{supp} D$, then $f^*(D) \in \operatorname{Div} X$ is defined to be the divisor obtained by lifting the local equations for the irreducible components of D. In particular, if $f : X \to \mathbb{P}^1$ is a non-constant map to the projective line, then the divisor

$$(f) := f^*(0 - \infty)$$

is called the **principal divisor** associated to f.

Remark 2.2.3. Pick some finite $D \in \operatorname{Div} X$. We will show that D defines a line bundle $\mathscr{O}_X(D)$ on X. Write $D = \sum_{i=1}^n d_i D_i$. Our construction will satisfy

$$\mathscr{O}_X(D) \simeq \bigotimes_{i=1}^n \mathscr{O}_X(D_i)^{\otimes d_i}$$

where, for a line bundle $\mathscr{L}$ and integer $d < 0$, we write $\mathscr{L}^{\otimes d}$ to denote $(\mathscr{L}^\vee)^{\otimes |d|}$ where $\mathscr{L}^\vee$ is $\mathscr{L}$'s dual bundle. Put another way, our construction $D \rightsquigarrow \mathscr{O}_X(D)$ will give a group homomorphism $\operatorname{Div} X \to \operatorname{Pic} X$, and so it suffices to define $\mathscr{O}_X(D)$, assuming that D is itself an irreducible hypersurface. Let $\{U_\alpha\}_{\alpha \in A}$ be an open cover of X such that $D \cap U_\alpha = \{f_\alpha = 0\}$ for some $f_\alpha \in \mathscr{O}_X(U_\alpha)$ for all $\alpha \in A$. Then, $\mathscr{O}_X(D)$ is formed by gluing together the trivial line bundles $\{U_\alpha \times \mathbb{C}\}_{\alpha \in A}$ according to the transition functions

$$\tau_{\alpha\beta} : U_\alpha \cap U_\beta \to \mathbb{C}^\times,\ x \mapsto \frac{f_\beta(x)}{f_\alpha(x)}.$$

That is, for $x \in U_\alpha \cap U_\beta$ and $z \in \mathbb{C}$ we identify $(x, z) \in U_\alpha \times \mathbb{C}$ with $(x, \tau_{\alpha\beta}(x) z) \in U_\beta \times \mathbb{C}$.

Remark 2.2.4. If $D \in \operatorname{Div} X$ is the finite divisor $D = \sum d_i D_i$, then, on an open cover $\{U_\alpha\}_{\alpha \in A}$ such that $D_i = \left\{ f_\alpha^{(i)} = 0 \right\}$ for all i and all α, $\mathscr{O}_X(D)$ is determined by the transition functions

$$\tau_{\alpha\beta} : U_\alpha \cap U_\beta \to \mathbb{C}^\times,\ x \mapsto \prod \left(\frac{f_\beta^{(i)}(x)}{f_\alpha^{(i)}(x)} \right)^{n_i}.$$

As such $\mathscr{O}_X(D)$ has a canonical (global) meromorphic section $s : X \to \mathscr{O}_X(D)$ given locally by

$$s|_{U_\alpha}(x) = \prod \left(f_\alpha^{(i)} \right)^{d_i}$$

which is moreover holomorphic if D is effective. When D is effective, we will sometimes denote this section as $1_D \in \mathrm{H}^0(X, \mathscr{O}_X(D))$ since it corresponds to the image of $1 \in \mathrm{H}^0(X, \mathscr{O}_X)$ under the natural inclusion $\mathscr{O}_X \hookrightarrow \mathscr{O}_X(D)$.

Remark 2.2.5. Let $s : X \to \mathscr{L}$ be a meromorphic section of a line bundle $\mathscr{L}$ on X, and let $\{U_\alpha\}$ be an open cover of X which trivialized $\mathscr{L}$. Then, $s_\alpha := s|_{U_\alpha}$ is identifiable with a meromorphic map $U_\alpha \to \mathbb{C}$, i.e. holomorphic map $U_\alpha \to \mathbb{P}^1$. We can glue together the principal divisors $(s_\alpha) \in \operatorname{Div}(U_\alpha)$ in order to form a divisor $(s) \in \operatorname{Div} X$ which satisfies $\mathscr{L} \simeq \mathscr{O}_X((s))$.

Remark 2.2.6. Let $C \subset X$ be an irreducible hypersurface. Then, $\mathscr{O}_X(-C) \simeq \mathscr{I}_C$ is identifiable with the ideal sheaf of C in X. Similarly, if $D = \sum_{i=1}^n d_i D_i$ is an effective divisor, then $\mathscr{O}_X(-D)$ is identifiable with a subsheaf of ideals in $\mathscr{O}_X$, and so D can also be considered as an analytic subspace of X.

Remark 2.2.7. Every line bundle $\mathscr{L}$ on a compact, complex surface X is of the form $\mathscr{L} = \mathscr{O}_X(D)$ for some divisor $D \in \operatorname{Div}(X)$. In particular, there exists a divisor $K_X \in \operatorname{Div}(X)$ such that $\omega_X = \mathscr{O}_X(K_X)$. Any such divisor will be called a **canonical divisor** on X.

We will mainly interact with divisors in relation to their connection to line bundles. Motivated by this, we say two divisors D, E are **linearly equivalent** if $\mathscr{O}_X(D) \simeq \mathscr{O}_X(E)$. This is the case iff there exists a holomorphic function $f : X \to \mathbb{P}^1$ such that $D - E = (f)$, so D and E are related by a "family $f^*(s)$ of divisors above the projective line $\mathbb{P}^1$."

Notation 2.2.8. Given a divisor $D \in \operatorname{Div} X$, we let $|D|$ denote the set of effective divisors linearly equivalent to D. This set is naturally isomorphic to the projective space $\mathbb{P}(\mathrm{H}^0(X, \mathscr{O}_X(D)))$.

Notation 2.2.9. Because of the natural homomorphism $\operatorname{Div} X \to \operatorname{Pic} X$, we will often use divisors directly in notation originally defined only for line bundles. Of note, if $D \in \operatorname{Div} X$ is a divisor, then we set

$$\chi(D) = \chi(\mathscr{O}_X(D)) \qquad h^i(D) = h^i(\mathscr{O}_X(D)) = \dim_{\mathbb{C}} \mathrm{H}^i(\mathscr{O}_X(D))$$

2.2.2 Intersections, Riemann-Roch, and Adjunction

One of the benefits of working in dimension 2 is the presence of a robust intersection theory. Given two distinct irreducible curves $C, D \subset X$ in a compact, smooth surface, their intersection $C \cap D$ is necessarily a finite set of points, and the number $i(C; D)$ of such points, counted with multiplicity, tells us about how to two curves are situated in relation to each other. By Poincaré duality, these curves have corresponding cohomology classes $[C]^*, [D]^* \in \mathrm{H}^{4-2}(X; \mathbb{Z}) = \mathrm{H}^2(X; \mathbb{Z})$, and the number $i(C; D)$ corresponds to their cup product $[C]^* \smile [D]^* \in \mathrm{H}^4(X; \mathbb{Z})$ under the natural isomorphism $\mathrm{H}^4(X; \mathbb{Z}) \simeq \mathbb{Z}$ coming from the complex structure on X. Since the formation of $i(C; D)$ corresponds to taking cup products, we see that we can get a well-defined intersection number for any pair of divisors on X by sending the pair $D = \sum d_i D_i$ and $E = \sum e_i E_i$ to

$$i(D; E) := \sum_{i,j} d_i e_i ([D_i]^* \smile [E_i]^*) \in \mathrm{H}^4(X; \mathbb{Z}) \simeq \mathbb{Z}.$$

Using the relation between divisors and line bundles, we can extend the above to allow for intersections between line bundles or between a line bundle and a curve. From another perspective, given two line bundles $\mathscr{L}, \mathscr{L}'$, we can define their intersection number to be $c_1(\mathscr{L}) \smile c_1(\mathscr{L}') \in \mathrm{H}^4(X; \mathbb{Z}) \simeq \mathbb{Z}$, the cup product of their first Chern classes. These two viewpoints agree with each other. Indeed, we have the following.

Theorem 2.2.1. *Let $D \in \operatorname{Div}(X)$ be a divisor on a compact smooth surface X, and let $[D]^* \in \mathrm{H}^2(X; \mathbb{Z})$ denote its corresponding cohomology class coming from Poincaré duality. Then,*

$$c_1(\mathscr{O}_X(D)) = [D]^*$$

as cohomology classes.

Proof. This is shown in [7, Ch. 3, Sec. 3]. ■

This allows us to unambiguously define intersection numbers of pairs of divisors and/or line bundles.

Definition 2.2.10. Given two divisors $D, E \in \mathrm{Div}(X)$ on a compact smooth surface X, we will denote their **intersection number** by

$$D \cdot E := f\left([D]^* \smile [E]^*\right) = f\left(c_1(\mathscr{O}_X(D)) \smile c_1(\mathscr{O}_X(E))\right) \in \mathbb{Z}$$

where $f : \mathrm{H}^4(X; \mathbb{Z}) \xrightarrow{\sim} \mathbb{Z}$ is the natural isomorphism coming from the complex structure on X. Similarly, if $\mathscr{L}, \mathscr{L}'$ are line bundles, then we define

$$\mathscr{L} \cdot \mathscr{L}' := f\left(c_1(\mathscr{L}) \smile c_1(\mathscr{L}')\right) \in \mathbb{Z},$$

and $D \cdot \mathscr{L} := f\left(c_1(\mathscr{L}) \smile [D]^*\right) \in \mathbb{Z}$.

Notation 2.2.11. In practice, we will often drop the $\cdot$ in the above notation, opting to write e.g. $DE \in \mathbb{Z}$ instead of $D \cdot E \in \mathbb{Z}$. We will similarly often drop the $\smile$ when forming cup products, writing e.g. $c_1(\mathscr{L})c_1(\mathscr{L}') \in \mathrm{H}^4(X; \mathbb{Z})$ instead of $c_1(\mathscr{L}) \smile c_1(\mathscr{L}') \in \mathrm{H}^4(X; \mathbb{Z})$.

The main utility of intersection theory in the context of studying complex surface lies in its connection to cohomology. That is, we will soon see that we can gain bounds on the sizes of the cohomology groups of a line bundle $\mathscr{L}$ on X in terms of $\mathscr{L}$'s intersections with other divisors/line bundles. In particular, we will prove a Riemann-Roch theorem for surfaces mirroring the one for curves.

Recall 2.2.2 (Riemann-Roch for Curves). Let $D \in \mathrm{Div}\, C$ be a line bundle on a smooth curve (i.e. Riemann surface) C. Then,

$$\chi(\mathscr{O}_C(D)) = \chi(\mathscr{O}_C) + \deg D,$$

and $\chi(\mathscr{O}_C) = 1 - g$ where $g = \dim \mathrm{H}^1(C, \mathscr{O}_C)$ is the genus of the curve.

To prove a similar result for surfaces, we would like to make use of the above for curves. Hence, we need a way of relating divisors on a surface to those on a curve. We will have two main techniques for doing so: intersecting with a curve, and adjunction. Before expanding on these techniques, we remark that there is an alternate description of intersection numbers which can be more convenient for calculations. We describe this now in the case of intersections between irreducible curves.

Definition 2.2.12. Let C, C' be distinct irreducible curves on a smooth, compact surface X, and fix a point $x \in C \cap C'$. If $f, g \in \mathscr{O}_{X,x}$ are (germs of) local equations for C, C', respectively, at x, then we define the **local intersection number** of C and C' and x to be

$$i_x(C; D) := \dim_{\mathbb{C}} \mathscr{O}_{X,x}/(f, g).$$

Theorem 2.2.2. *If C, C' are distinct irreducible curves on a smooth, compact surface X, then*

$$C \cdot C' = \sum_{x \in C \cap C'} i_x(C; D).$$

Proof. [2, Ch. II, sect. 10] ■

We will see this point of view in our next result. Recall that we want a "Riemann-Roch for Surfaces," and that we aim to use the usual Riemann-Roch for curves as part of our process of attaining an analogue for surfaces. In order to relate line bundles of surfaces with those on curves, we first prove the following.

Lemma 2.2.3. *Let A be a discrete valuation ring*[1] *with uniformizer $\pi \in A$ and residue field $k = A/(\pi)$. Let $s \in A$ be any nonzero element. Then, $\operatorname{ord}_A(s) = \dim_k A/(s)$.*

Proof. Let $e = \operatorname{ord}_A(s)$, so $(s) = (\pi)^e \subset A$. If $e = 0$, then $s \in A^\times$ is a unit so $A/(s) = A/A = 0$ is 0-dimensional and the claim holds. Hence, assume $e \geq 1$ so that $A/(s) = A/(\pi)^e$. Then, we can filter $A/(\pi)^e$ via

$$0 = (\pi)^e/(\pi)^e \subsetneq (\pi)^{e-1}/(\pi)^e \subsetneq \cdots \subsetneq (\pi)/(\pi)^e \subsetneq A/(\pi)^e,$$

so $\dim_k A/(\pi)^e$ is the sum of the dimensions of the successive quotients in the filtration above, i.e.

$$\dim_k \frac{A}{(\pi)^e} = \sum_{n=0}^{e-1} \dim_k \left(\frac{(\pi)^n/(\pi)^e}{(\pi)^{n+1}/(\pi)^e} \right) = \sum_{n=0}^{e-1} \dim_k \frac{(\pi)^n}{(\pi)^{n+1}}.$$

Finally, $\dim_k (\pi)^n/(\pi)^{n+1} = 1$ by Nakyama's lemma since $(\pi)^n$ is generated as a module (i.e. ideal) over the local ring A by a single element. Since there are e terms in the above sum, we conclude that $\dim_k A/(s) = e = \operatorname{ord}_A(s)$ as claimed. ■

Theorem 2.2.4. *Let $D \in \operatorname{Div} X$ be a divisor on a smooth, compact surface, and let $C \subset X$ be a smooth, irreducible curve. Then,*

$$D \cdot C = \deg(\mathscr{O}_X(D)|_C).$$

Proof. First note that, by linearity, we may assume that D is an irreducible curve as well. Let $s : X \to E(\mathscr{O}_X(D))$ be the canonical holomorphic section constructed in Remark 2.2.3. This restricts to a meromorphic section $s|_C : C \to E(\mathscr{O}_X(D)|_C)$ of $\mathscr{O}_X(D)|_C$. Thus, $\mathscr{O}_X(D)|_C = \mathscr{O}_C((s|_C))$ where $(s|_C)$ is its associated divisor. By construction, $\operatorname{supp}(s|_C) \subset C \cap D$, so we only need to determine the multiplicty at each point. Fix a point $x \in C \cap D$ and let $f, g \in \mathscr{O}_{X,x}$ be local equations for C, D, respectively. Note that, by construction, viewing s as an element $s \in \mathrm{H}^0(X, \mathscr{O}_X(D))$, its stalk x is $s_x = g \in \mathscr{O}_{X,x}$. Thus, the order of vanishing $\operatorname{ord}_x(s|_C)$ of $s|_C$ at $x \in C \cap D$ is given by the valuation of $s_x = g$ in the (local) valuation ring $\mathscr{O}_{C,x} = \mathscr{O}_{X,x}/(g)$. Thus, by the lemma above,

$$\operatorname{ord}_x(s|_C) = \operatorname{ord}_{\mathscr{O}_{C,x}}(s_x) = \dim_{\mathbb{C}} \mathscr{O}_{C,x}/(s) = \dim_{\mathbb{C}} \mathscr{O}_{X,x}/(f,g) = i_x(C; D).$$

Hence,

$$\deg(s|_C) = \sum_{x \in C \cap D} i_x(C; D) = C \cdot D$$

as claimed. ■

Notation 2.2.13. In the situation of the above theorem, we let $\mathscr{O}_C(D) \in \operatorname{Pic} C$ denote the restriction $\mathscr{O}_X(D)|_C$. With this notation, the above theorem can be stated succinctly as $\deg \mathscr{O}_C(D) = CD$.

This gives a way of gaining control over restrictions of line bundles from a surface X to one of its embedded curves C. However, in addition to restricting line bundles on X, there is a second natural way to obtain a

[1] See Remark 2.1.3

line bundle on C. Namely, one can consider its canonical bundle ω_C, and so one can hope to relate this to data on X. This is done by means of the adjunction formula which provides a formula for ω_C in terms of the line bundles ω_X and $\mathscr{O}_X(C)$.

Lemma 2.2.5. *Let $C \subset X$ be a smooth curve in a smooth, compact surface. Then, the normal bundle $\mathcal{N}_{C/X}$ of C in X can be identified with the line bundle $\mathscr{O}_C(C) = \mathscr{O}_X(C)|_C$.*

Proof. Note that we have an exact sequence

$$0 \longrightarrow \mathcal{T}_C \longrightarrow \mathcal{T}_X|_C \longrightarrow \mathcal{N}_{C/X} \longrightarrow 0.$$

Taking duals, we obtain the sequence

$$0 \longrightarrow \mathcal{N}_{C/X}^\vee \longrightarrow \Omega_X|_C \longrightarrow \Omega_C \longrightarrow 0.$$

At the same time, there is an exact sequence $0 \longrightarrow \mathscr{I}_C/\mathscr{I}_C^2 \xrightarrow{\mathrm{d}} \Omega_X|_C \longrightarrow \Omega_C \longrightarrow 0$ where $\mathscr{I}_C$ is the ideal sheaf of $C \subset X$ and d is the differential map. Thus, $\mathcal{N}_{C/X}^\vee \simeq \mathscr{I}_C/\mathscr{I}_C^2$, but also $\mathscr{I}_C = \mathscr{O}_X(-C)$ and

$$\mathscr{O}_X(-C)/\mathscr{O}_X(-2C) = \mathscr{O}_X(-C) \otimes_{\mathscr{O}_X} \mathscr{O}_X/\mathscr{O}_X(-C) = \mathscr{O}_X(-C) \otimes \mathscr{O}_C = \mathscr{O}_C(-C),$$

so $\mathcal{N}_{C/X} \simeq \mathscr{O}_C(C)$ as claimed. ∎

Theorem 2.2.6 (Adjunction Formula). *Let $C \subset X$ be a smooth curve in a smooth, compact surface. Then,*

$$\omega_C \simeq \omega_X \otimes \mathscr{O}_X(C)|_C \simeq \omega_X(C)|_C$$

where $\omega_X(C) := \omega_X \otimes \mathscr{O}_X(C)$.

Proof. Once again consider the exact sequence

$$0 \longrightarrow \mathcal{N}_{C/X}^\vee \longrightarrow \Omega_X|_C \longrightarrow \Omega_C \longrightarrow 0.$$

Taking determinants of each vector bundle above, and noting that most of them are rank 1, we see that

$$\omega_X|_C \simeq \mathcal{N}_{C/X}^\vee \otimes \omega_C.$$

Since $\mathcal{N}_{C/X}^\vee \simeq \mathscr{O}_X(-C)|_C$, the claim follows from tensoring both sides with $\mathscr{O}_X(C)|_C$. ∎

Corollary 2.2.7 (Genus Formula). *Let $C \subset X$ be a smooth curve in a smooth, compact surface, and let $g(C)$ denote the genus of C. Then,*

$$2g(C) - 2 = (K_X + C)C = K_X C + C^2,$$

where $K_X \in \operatorname{Div} X$ is a canonical divisor.

Proof. The adjunction formula shows that $\omega_C = \omega_X \otimes \mathscr{O}_X(C)|_C$. Taking degrees of both sides and applying Theorem 2.2.4 gives the desired result. ∎

This gives our second way of relating line bundles on an embedded curve to data on the ambient surface. Combining these will allow us to prove our Riemann-Roch analogue for surfaces.

Theorem 2.2.8 (**Riemann-Roch for Surfaces**). *Let $D = \sum d_i D_i$ be a divisor on a compact, smooth surface X. Then,*

$$\chi(\mathscr{O}_X(D)) = \frac{1}{2}D(D - K_X) + \chi(\mathscr{O}_X).$$

Proof. We must make one concession in our proof. So far, we only have Riemann-Roch for smooth curves, but the D_i are only assumed irreducible. In a later section on embedded curves, we will show that Riemann-Roch for curves actually holds verbatim for non-smooth, irreducible curves as well. Granting this for now, we proceed with the proof.

We proceed by induction on $\sum |d_i|$. If this sum is 0, then $\mathscr{O}_X(D) = \mathscr{O}_X$ and the claim is apparent. Hence, assume the claim holds for D', and let C be an irreducible curve. We need to show that it then holds for $D' \pm C$. We start with the case $D = D' - C$. Tensoring the exact sequence $0 \to \mathscr{O}_X(-C) \to \mathscr{O}_X \to \mathscr{O}_C \to 0$ by $\mathscr{O}_X(D')$ gives

$$0 \longrightarrow \mathscr{O}_X(D) \longrightarrow \mathscr{O}_X(D') \longrightarrow \mathscr{O}_C(D') \longrightarrow 0 \ .$$

Now, appealing to induction, adjunction, and Riemann-Roch on C, we get

$$\begin{aligned}
\chi(\mathscr{O}_X(D)) &= \chi(\mathscr{O}_X(D')) - \chi(\mathscr{O}_C(D')) \\
&= \left[\frac{1}{2}D' \cdot (D' - K_X) + \chi(\mathscr{O}_X)\right] - [\deg(D' \mid_C) + (1 - g(C))] \\
&= \left[\frac{1}{2}D' \cdot (D' - K_X) + \chi(\mathscr{O}_X)\right] + \left[\frac{1}{2}(K_X + C) \cdot C - C \cdot D'\right] \\
&= \frac{1}{2}D(D - K) + \chi(\mathscr{O}_X)
\end{aligned}$$

which completes the $D = D' - C$ case. The $D = D' + C$ case is handled complete analogously. ■

The Riemann-Roch theorem is the main method to gain control over the sizes $h^i(\mathscr{O}_X(D))$ of cohomology groups of line bundles. In practice, one is usually mostly interested in the number $h^0(\mathscr{O}_X(D))$ of holomorphic sections of a line bundle. With this quantity in mind, Riemann-Roch by itself is often insufficient. In order to gain finer control on sizes of these cohomology groups, we will accept without proof both Noether's formula and Serre duality which, respectively, give a formula for $\chi(\mathscr{O}_X)$ and make the group $\mathrm{H}^2(\mathscr{O}_X(D))$ less mysterious.

Theorem 2.2.9 (**Noether's formula**). *Let X be a compact, smooth surface. Then,*

$$\chi(\mathscr{O}_X) = \frac{1}{12}\left(K_X^2 + \chi_{\mathrm{top}}(X)\right)$$

where $\chi_{\mathrm{top}}(X) = \sum_{k=0}^{4}(-1)^k \dim \mathrm{H}^k(X; \mathbb{R})$ is X's topological Euler characteristic.

Proof. [7, Ch. 4, Sect. 6] ■

Theorem 2.2.10 (**Serre Duality**). *Let X be a smooth, compact manifold of dimension n, and let $\mathscr{E}$ be a vector bundle on X. Then, for all i, we have*

$$h^i(\mathscr{E}) = h^{n-i}(\mathscr{E}^\vee \otimes \omega_X).$$

Consequently, $h^i(\mathscr{E}) = 0$ whenever $i > n$.

Proof. [12] ■

Corollary 2.2.11. *Let D be a divisor on smooth, compact surface X. Then,*

$$h^0(D) + h^0(K_X - D) \geq \frac{1}{2}D(K_X - D) + \chi(\mathscr{O}_X).$$

Proof. Riemann-Roch for surfaces gives

$$h^0(D) + h^2(D) \geq h^0(D) - h^1(D) + h^2(D) = \chi(\mathscr{O}_X(D)) = \frac{1}{2}D(D - K_X) + \chi(\mathscr{O}_X).$$

Now apply Serre duality to see that $h^2(D) = h^0(K_X - D)$. ■

Remark 2.2.8. The inequality form of Riemann-Roch given in the above corollary is how the theorem is often applied in practice. It is commonly the case that one of $h^0(D), h^0(K_X - D)$ above will vanish, and this vanishing can commonly be shown by finding an irreducible curve $C \subset X$ such that $DC < 0$ (or $(K_X - D)C < 0$). Indeed, if $h^0(D) > 0$, then there must exist an effective divisor E linearly equivalent to D such that $EC = DC < 0$. Because pairings between distinct irreducible curves are always non-negative this is possible only if $C \subset \operatorname{supp} E$ and $C^2 < 0$. Hence, if we know that $C^2 \geq 0$, we can conclude that $h^0(D) = 0$.

There is another immediate application of Riemann-Roch which is sometimes useful and which will close out this section. We can easily show that the intersection pairing $\mathscr{L} \cdot \mathscr{L}'$ between two line bundles can be calculated completely in terms of dimensions of various cohomology groups. This viewpoint is sometimes simpler to work with than thinking in terms of cup products or local intersection numbers.

Lemma 2.2.12. *Let $\mathscr{L}, \mathscr{L}'$ be line bundles on a smooth, compact surface X. Then,*

$$\mathscr{L} \cdot \mathscr{L}' = \chi(\mathscr{O}_X) - \chi(\mathscr{L}) - \chi(\mathscr{L}') + \chi(\mathscr{L} \otimes \mathscr{L}').$$

Proof. For notational convenience, write $\mathscr{L} = \mathscr{O}_X(D)$ and $\mathscr{L}' = \mathscr{O}_X(D')$. Now, expand the right hand side of the desired equality and apply Riemann-Roch for surfaces to each summand to get

$$\begin{aligned}
\chi(\mathscr{O}_X) - \chi(\mathscr{L}) - \chi(\mathscr{L}') + \chi(\mathscr{L} \otimes \mathscr{L}') &= \chi(\mathscr{O}_X) - \left[\frac{1}{2}D(D - K_X) + \chi(\mathscr{O}_X)\right] - \left[\frac{1}{2}D'(D' - K_X) + \chi(\mathscr{O}_X)\right] \\
&\quad + \left[\frac{1}{2}(D + D')(D + D' - K_X) + \chi(\mathscr{O}_X)\right] \\
&= \frac{1}{2}DD' + \frac{1}{2}D'D \\
&= DD' \\
&= \mathscr{L} \cdot \mathscr{L}',
\end{aligned}$$

where, above, there was much cancellation in the second equality. ■

The above lemma will play an important role in the section on Ruled surfaces where it will be used to understand the canonical divisor for a large class of surfaces. This covers all the basic results one needs from intersection theory. In order to avoid future issues with dealing with the existence of non-smooth spaces (like the one that came up when proving Riemann-Roch), we devote the next two sections to studying singular surfaces and (possibly singular) embedded curves.

2.3 Singularities

While complex manifolds constitute the nicest class of complex spaces, it is not possible to avoid singular spaces altogether. In cases where a singular space X arises, one often wishes to find a related space $\widetilde{X}$ which is a "resolution of singularities" or "desingularization" of X. This means $\widetilde{X}$ is smooth and comes equipped with a morphism $\pi : \widetilde{X} \to X$ which is an isomorphism away from the singularities of X, i.e. if $S \subset X$ is its singular set, then $\widetilde{X} \setminus \pi^{-1}(S) \xrightarrow{\pi} X \setminus S$ is an isomorphism.

Within the context of surfaces, i.e. 2 dimensional complex spaces, the basic tool in the formation of such desingularizations is the process of blowing up a surface at a point, the subject of the below subsection. After introducing this process, we will study a class of singularities which feature prominently in our later local study of elliptic fibrations, namely the Hirzebruch-Jung strings.

2.3.1 Blowups

One way to motivate the blowups is their application to resolving singularities of curves embedded in surfaces. Consider, for example, the cuspidal cubic $E : y^2 = x^2(x+1)$ whose real points are pictured in Figure 1. This

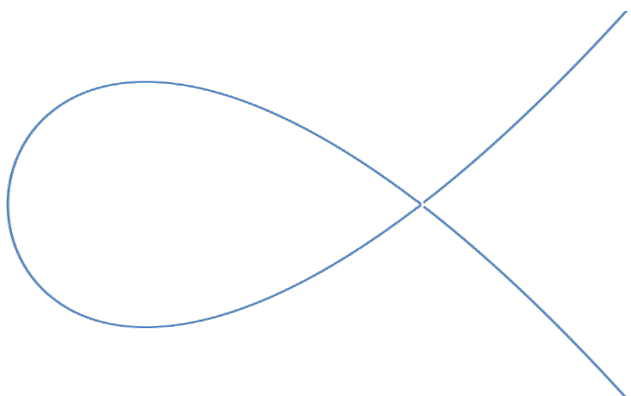

Figure 1: The real points of the cuspidal cubic $y^2 = x^2(x+1)$

curve has a singularity at the origin $(0,0)$ evidenced by its two separate tangent directions at this point. In order to resolve this singularity, we would like to replace the origin by (at least) 2 points, one for each of the direction from which the curve passes through it. This is exactly the idea of blowing up a surface (in this case, the complex plane $\mathbb{C}^2$). You pick a point and replace it with a copy of $\mathbb{P}^1$ representing all the possible tangent directions of a curve passing through that point. This description characterizes the blow up.

Proposition 2.3.1. *Let X be a (possibly singular or even non-reduced) complex space, and let $p \in X$ be a smooth point of local dimension 2, so p has a neighborhood isomorphic to an open ball $B \subset \mathbb{C}^2$. Then, there exists a complex space $\operatorname{Bl}_p X$, called the **blowup of X at p**, with a canonical map*

$$\pi : \operatorname{Bl}_p X \to X$$

*such that $E := \pi^{-1}(p)$, which we call the **exceptional curve** of the blowup, is isomorphic to $\mathbb{P}^1$ and π restricts to an isomorphism $\operatorname{Bl}_p X \setminus E \xrightarrow{\sim} X \setminus \{p\}$.*

Because the characterization of the blowup is entirely local, in order to prove this proposition, it suffices to construct $\operatorname{Bl}_0 \mathbb{C}^2$, the blowup of the plane at the origin. To construct $\operatorname{Bl}_p X$ is general, we simply take a coordinate neighborhood $U \subset X$ centered at p – i.e. we fix an isomorphism $\phi : U \xrightarrow{\sim} B$ taking U to the unit ball $B \subset \mathbb{C}^2$ such that $\phi(p) = (0,0)$ – and then let

$$\operatorname{Bl}_p X = (X \setminus \{p\}) \underset{U\setminus\{p\}}{\cup} \operatorname{Bl}_0 B,$$

the result of gluing $\operatorname{Bl}_0 B$ to $X \setminus \{p\}$ along $\operatorname{Bl}_0 B \setminus \pi^{-1}(0) \simeq U \setminus \{p\}$ where $\pi : \operatorname{Bl}_0 B \to B$ is the canonical map. In other words, $\operatorname{Bl}_p X$ fits into the following push out diagram

$$\begin{array}{ccc} U \setminus \{p\} & \xrightarrow{\pi^{-1}\circ\phi} & \operatorname{Bl}_0 B \\ \downarrow & & \downarrow \\ X \setminus \{p\} & \longrightarrow & \operatorname{Bl}_p X \end{array}$$

Hence, we turn to giving this construction. In fact, we will give two constructions. Either is fine individually for the construction of general blowups (and really, they are one in the same), but seeing both helps one more easily recognize blow ups in the wild.

Construction 2.3.1. Let x, y be coordinates on $\mathbb{C}^2$, and let X, Y be homogeneous coordinates on $\mathbb{P}^1$. Then, $\operatorname{Bl}_0 \mathbb{C}^2 \subset \mathbb{C}^2 \times \mathbb{P}^1$ is given as the the zero set of the equation

$$\operatorname{Bl}_0 \mathbb{C}^2 : xY - yX = 0.$$

Indeed, by restricting the projection map $\operatorname{pr}_1 : \mathbb{C}^2 \times \mathbb{P}^1 \to \mathbb{C}^2$, we obtain a natural projection map $\pi : \operatorname{Bl}_0 \mathbb{C}^2 \to \mathbb{C}^2, ((x,y),[X:Y]) \mapsto (x,y)$ with

$$\pi^{-1}(0,0) = \left\{((x,y),[X:Y]) \in \mathbb{C}^2 \times \mathbb{P}^1 : (x,y) = (0,0) \text{ and } xY = yX\right\} = 0 \times \mathbb{P}^1 \simeq \mathbb{P}^1.$$

At the same time, away from the origin, π has an inverse $s : \mathbb{C}^2 \setminus 0 \to \operatorname{Bl}_0 \mathbb{C}^2 \setminus (0 \times \mathbb{P}^1)$ given by $s(x,y) = ((x,y),[x:y])$.

Construction 2.3.2. Let $p : L \to \mathbb{P}^1$ be the **tautological line bundle** on $\mathbb{P}^1$,[2] that is

$$L = \left\{((x,y),\ell) \in \mathbb{C}^2 \times \mathbb{P}^1 : (x,y) \in \ell\right\} \subset \mathbb{C}^2 \times \mathbb{P}^1,$$

and p is simply the restriction of the projection map $\mathbb{C}^2 \times \mathbb{P}^1 \to \mathbb{P}^1$. Let $\pi : L \to \mathbb{C}^2$ be the restriction of the projection map $\mathbb{C}^2 \times \mathbb{P}^1 \to \mathbb{C}^2$. Then, $L \simeq \operatorname{Bl}_0 \mathbb{C}^2$. Indeed, $\pi^{-1}(0,0) \simeq \mathbb{P}^1$ since every element of $\mathbb{P}^1$ passes through the origin by definition, and we have an isomorphism $s : \mathbb{C}^2 \setminus 0 \to L$ given by $s(x,y) = ((x,y),[x:y])$ much like last time.

Combining either construction with the discussion preceding them proves Proposition 2.3.1 in general. Now that we have blowups, we will prove some key theorems regarding their intersection theory. This will, unsurprisingly, be strongly related to the intersection theory of the base space, so we begin by understanding how divisors on the base pull back to divisors on the blowup.

Let X be a smooth surface, let $\widehat{X} = \operatorname{Bl}_p X \xrightarrow{\pi} X$ be its blowup at a fixed point $p \in X$, and let $E = \pi^{-1}(p)$ be the exceptional curve.

Definition 2.3.1. Let C be an irreducible curve passing through p with multiplicity m. Then, the closure

[2] We're calling this L and not $\mathscr{O}_{\mathbb{P}^1}(-1)$ because one usually thinks of the latter as a sheaf, not a topological space

$\overline{\pi^{-1}(C \setminus \{fp\})}$ of the preimage of $C \setminus \{p\}$ is an irreducible curve $\widehat{C} \subset \mathrm{Bl}_p X$ called the **strict transform** of C.

Proposition 2.3.2. *As divisors, $\pi^* C = \widehat{C} + mE$.*

Proof. Because the blowup map is an isomorphism away from p, we must have $\mathrm{supp}(\pi^* C) \setminus E = \pi^{-1}(C \setminus \{p\}) = \widehat{C} \setminus E$ from which we conclude that $\pi^* C = \widehat{C} + kE$ for some $k \in \mathbb{Z}$. Now, the fact that C passes through p with multiplicity m means exactly that we can find local coordinates x, y near p, say on a neighborhood U, such that, $C \cap U$ is given as the zero set of a function

$$f(x,y) = \sum_{d \geq m} f_d(x,y)$$

with f_d homogeneous of degree d, and $f_m \neq 0$. Now, let $\widehat{U} \subset U \times \mathbb{P}^1$ be the vanishing set of $xY - yX = 0$ where X, Y are homogeneous coordinates on $\mathbb{P}^1$. Hence, $\widehat{U}$ can be viewed as an open subset of $\widehat{X}$ containing E. Near the point $q = (p, [1:0]) \in \widehat{U}$ we can take the functions x and $t = Y/X$ as local coordinates since x is a coordinate on U near p, t is one on $\mathbb{P}^1$ near $[1:0]$, and $y = xt$ is determined by these two. Hence, near q, $\pi^* f$ has the form

$$(\pi^* f)(x,t) = f(x,xt) = \sum_{d \geq m} f_d(x,xt) = \sum_{d \geq m} x^d f_d(1,t) = x^m \left(f_m(1,t) + \sum_{n \geq 1} x^n f_{m+n}(1,t) \right). \qquad (1)$$

Since E, in these coordinates, is given as the vanishing set of x and x divides the above equation with degree exactly m, we see that $k = m$. ■

Since the pullback of any curve $C \subset X$ not containing p is necessarily isomorphic to C, we understand the entire pullback map $\pi^* : \mathrm{Pic}\, X \to \mathrm{Pic}\, \widehat{X}$. We now prove finer results on the intersection theory in $\widehat{X}$.

Proposition 2.3.3. *With the same setup as before,*

(1) $E^2 = -1$.

(2) *For $D, D' \in \mathrm{Pic}\, X$, we have*

$$(\pi^* D) \cdot (\pi^* D') = DD', \qquad E \cdot (\pi^* D) = 0.$$

As a consequence, if $\widehat{C}$ is the proper transform of C, then also

$$(\pi^* D) \cdot \widehat{C} = (\pi^* D) \cdot (\pi^* C - mE) = (\pi^* D) \cdot (\pi^* C) = D \cdot C.$$

(3) *There is an isomorphism* $\mathrm{Pic}\, X \oplus \mathbb{Z} \xrightarrow{\sim} \mathrm{Pic}\, \widehat{X}$ *given by* $(D, n) \mapsto \pi^* D + nE$.

(4) $K_{\widehat{X}} = \pi^* K_X + E$

Proof. We will prove these one at time.

(1) We will start with the claim that $E^2 = -1$ which, because $E \simeq \mathbb{P}^1$, is equivalent to the claim that the normal bundle $\mathscr{O}_{\widehat{X}}(E)|_E =: \mathscr{O}_E(E)$ corresponds to the tautological line bundle $\mathscr{O}_{\mathbb{P}^1}(-1)$ under the identification $E \simeq \mathbb{P}^1$. Since this claim is local, we may assume for the time being that $X = \mathbb{C}^2$, so $\widehat{X} = \left\{(v,\ell) \in \mathbb{C}^2 \times \mathbb{P}^1 : v \in \ell\right\} \subset \mathbb{C}^2 \times \mathbb{P}^1$ is the total space of $\mathscr{O}_{\mathbb{P}^1}(-1)$. Let $q : \widehat{X} \to \mathbb{P}^1$ be the projection map, and let $\mathcal{L} = q^*\mathscr{O}_{\mathbb{P}^1}(-1)$ be the line bundle on $\widehat{X}$ whose total space is

$$q^*\widehat{X} = \left\{((v,\ell),(v',\ell')) \in \widehat{X} \times \widehat{X} : \ell = \ell'\right\},$$

and whose bundle map $\mathscr{L} \to \widehat{X}$ is projection onto the first factor. In other words, we have a pullback square

$$\begin{tikzcd} q^*\widehat{X} \arrow[r] \arrow[d] & \widehat{X} \arrow[d, "q"] \\ \widehat{X} \arrow[r, "q"] & \mathbb{P}^1 \end{tikzcd}$$

Then, $\mathscr{L}$ has a (holomorphic) section $t \in \mathrm{H}^0(\mathscr{L})$ which, when viewed as a map $t : \widehat{X} \to q^*\widehat{X}$, is given by $t(v,\ell) = ((v,\ell),(v,\ell))$. The zero set of this section is $\left\{(v,\ell) \in \widehat{X} : v = 0 \in \ell\right\} = E$, and t visibly vanishes with multiplicity one, so $\mathscr{O}_{\widehat{X}}(E) \simeq \mathscr{L} \simeq q^*\mathscr{O}_{\mathbb{P}^1}(-1)$. Since the map $q|_E : E \to \mathbb{P}^1$ is an isomorphism, this shows that $\mathscr{O}_E(E)$ is identified with $\mathscr{O}_{\mathbb{P}^1}(-1)$. Thus, $E^2 = \deg(\mathscr{O}_E(E)) = -1$ as claimed.

(2) By linearity, it suffices to check these formulas for irreducible divisors, so let $C, C' \subset X$ be irreducible curves. We will first show that $E \cdot (\pi^*C) = 0$. Write $\pi^*C = \widehat{C} + mE$ where $m \in \mathbb{Z}$ is the (possibly zero) multiplicity with which C passes through p. Looking back at equation (1), we see that, in the notation of that equation, $\widehat{C}$ was given by

$$\widehat{C} : 0 = f_m(1,t) + \sum_{n\geq 1} x^n f_{m+n}(1,t),$$

where E is line $E : 0 = x$. Hence, determining their intersection corresponds solving the equation defining $\widehat{C}$ when $x = 0$. Since all terms of degree greater than m vanish, we see that this equation has $m = \deg f_m(1,t)$ many zeroes along the line $x = 0$, i.e. $\widehat{C}$ meets E in m points, so $E\widehat{C} = 0$. Thus,

$$E \cdot (\pi^*C) = E\widehat{C} + mE^2 = m - m = 0.$$

Finally, $(\pi^*C) \cdot (\pi^*C') = CC'$ simply because π is a degree 1 map.

(3) We first show surjectivity. Let $C \subset \widehat{X}$ be an irreducible curve on the blowup. We have three cases. (1) $C \cap E = \emptyset$. In this case, $C = \pi^*\pi(C)$ is in the image of this map. (2) $C \cap E = E$. In this case, $C = E$ is in the image of this map. (3) $C \cap E$ is a finite set of points. In this case, C is the strict transform of $\pi(C)$, so $C = \pi^*(C) - mE$ for some $m \in \mathbb{Z}$ and hence is in the image of this map. This shows surjectivity. For injectivity, suppose $\pi^*D + nE$ is trivial. Intersecting with E, we get $-n = 0$, so π^*D is trivial. Now restricting to $\widehat{X} \setminus E$ and noting that $\mathrm{Pic}(\widehat{X} \setminus E) \simeq \mathrm{Pic}(X \setminus \{p\}) \simeq \mathrm{Pic}\, X$ (this last map sends an irreducible curve on $X \setminus \{p\}$ to its topological closure in X), we see that D must itself be trivial, so the map under investigation is injective as well.

(4) We know from **(3)** that $K_{\widehat{X}} = \pi^*D + nE$ for some $D \in \mathrm{Div}\, X$ and $n \in \mathbb{Z}$. Restricting to $\widetilde{X} \setminus E$ and from there to $X \setminus \{p\}$ like before, we see that $D = K_X$ is the canonical divisor on X. We now apply the adjunction formal to $E \subset \widehat{X}$ to get that

$$\mathscr{O}_{\mathbb{P}^1}(-2) \simeq \omega_E \simeq \omega_{\widehat{X}} \otimes \mathscr{O}_{\widehat{X}}(E)|_E \simeq \mathscr{O}_E((n+1)E) \simeq \mathscr{O}_{\mathbb{P}^1}(-(n+1)),$$

and so $n = 1$ as claimed.

$\blacksquare$

Corollary 2.3.4. *Let $C \subset X$ be a curve passing through p with multiplicity m, and let $\widehat{C} \subset \widehat{X}$ be its strict transform. Then $\widehat{C}^2 = C^2 - m^2$. In particular, if $m = 1$, we have $C^2 = \widehat{C}^2 + 1$.*

Proof. Write $\pi^* C = \widehat{C} + mE$. Then,

$$C^2 = (\pi^* C)^2 = (\widehat{C})^2 + 2m\widehat{C}E + m^2E^2 = (\widehat{C})^2 + 2m^2 - m^2 = (\widehat{C})^2 + m^2.$$

$\blacksquare$

We motivated blowups by claiming they assist in the resolution of singularities of curves embedded in (smooth) surfaces. This is indeed the case, but blowups also play a role resolving surfaces which are themselves singular. We shall give an example of this by resolving the (cyclic) quotient singularities $\mathbb{C}^2/(\mathbb{Z}/n\mathbb{Z})$.

Example ($\mathbb{C}^2/(\mathbb{Z}/2\mathbb{Z})$, an A_1 singularity)**.** Let $i : \mathbb{C}^2 \to \mathbb{C}^2$ denote the reflection $(x, y) \mapsto (-x, -y)$, and let $G = \langle i \rangle \simeq \mathbb{Z}/2\mathbb{Z}$ be the group of automorphisms of $\mathbb{C}^2$ generated by i. Then, the surface $X = \mathbb{C}^2/G$ has a single singularity at the point $p = [(0, 0)]$, the image of the origin under the natural projection $\mathbb{C}^2 \to X$. This is because this is the only point fixed by i. In order to resolve this singularity, we extend i to the blowup $\mathrm{Bl}_0\,\mathbb{C}^2 = \big\{(x, y; X : Y) \in \mathbb{C} \times \mathbb{P}^1 : xY - yX = 0\big\}$, and then show that $(\mathrm{Bl}_0\,\mathbb{C}^2)/G$ is smooth. This extension is given by

$$i : (x, y; X : Y) \mapsto (-x, -y : -X; -Y) = (-x, -y : X : Y)\,.$$

Thinking of $\mathrm{Bl}_0\,\mathbb{C}^2$ as the tautological line bundle on $\mathbb{P}^1$, we see immediately that i fixes the fibers of the natural map $\mathrm{Bl}_0\,\mathbb{C}^2 \to \mathbb{P}^1$, so the quotient $(\mathrm{Bl}_0\,\mathbb{C}^2)/G$ will be a fibre bundle over $\mathbb{P}^1$ with fibers $\mathbb{C}/G \simeq \mathbb{C}$, i.e. it will again be a line bundle. More concretely, the map

$$\begin{array}{cccc} f: & \mathrm{Bl}_0\,\mathbb{C}^2 & \longrightarrow & \mathbb{C} \times \mathbb{P}^1 \\ & (x, y; X : Y) & \longmapsto & (x^2, y^2; X : Y) \end{array}$$

descends to an isomorphism

$$(\mathrm{Bl}_0\,\mathbb{C}^2)/G \xrightarrow{\sim} L := \big\{(x, y; X : Y) \in \mathbb{C} \times \mathbb{P}^1 : xY^2 - yX^2 = 0\big\}\,.$$

Now, the projection $p : L \to \mathbb{P}^1$ makes L a line bundle over $\mathbb{P}^1$ and one can check that its sheaf of sections is $\mathscr{O}_{\mathbb{P}^1}(-2)$. Indeed, this has a meromorphic section

$$s : \mathbb{P}^1 \to L,\ [X : Y] \mapsto \big(X^2/Y^2, 1; X : Y\big)$$

which has a double pole at at $[0 : 1]$ and no other poles or zeros, so L's sheaf of sections is $\mathscr{O}_{\mathbb{P}^1}((s)) = \mathscr{O}_{\mathbb{P}^1}(-2)$. Given this, much in the same way we showed the zero section/exceptional divisor of $\mathscr{O}_{\mathbb{P}^1}(-1) = \mathrm{Bl}_0\,\mathbb{C}^2$ has self-intersection -1, one shows that the $F := \{(x, y; X : Y) \in L : x = y = 0\} \simeq \mathbb{P}^1$ has self-intersection -2.

All in all, we've formed a square of surfaces

$$\begin{tikzcd}
\mathrm{Bl}_0\,\mathbb{C}^2 \arrow[r] \arrow[d, "\pi"'] & L \arrow[d, "g"] \\
\mathbb{C}^2 \arrow[r] & \mathbb{C}^2/G
\end{tikzcd}$$

where the vertical maps are isomorphisms away from a single point on the base, $\mathbb{C}^2/G$ is singular, and L is smooth. Hence, L is a resolution of singularities for $\mathbb{C}^2/G$, and since $F^2 = -2$ (and $F \simeq \mathbb{P}^1$) where $F = g^{-1}([(0,0)])$, we say that the singularity in $\mathbb{C}^2/G$ is resolved by a (-2)-curve.

Remark 2.3.3. The example above is fairly representative of how one calculates the resolution of $\mathbb{C}^2/(\mathbb{Z}/n\mathbb{Z})$ in general, when $\mathbb{Z}/n\mathbb{Z}$ acts by component-wise multiplication with (ζ, ζ^{-1}) for a primitive n root of unity ζ. To simplify things, you first blow up the plane and lift the action there. Then, when $n > 2$, you observe that the quotient of this blowup still has some singularities remaining, but these are somehow "milder." You can then resolve these using the same technique of blowing up the space before quotienting, and you eventually end up with a nonsingular surface resolving $\mathbb{C}^2/(\mathbb{Z}/n\mathbb{Z})$ via some chain of blowups. To keep the resolution as simple as possible, you end by "blowing down" any curves in your resolution that look like the exceptional divisor of a blowup (this being possible will be justified by Theorem 2.3.8), and you will find that you end up resolving the singularity by a chain (of length $n-1$) of $\mathbb{P}^1$'s, each with self-intersection -2. For this reason, the original singularity (the image of the origin in $\mathbb{C}^2/(\mathbb{Z}/n\mathbb{Z})$) is called an A_{n-1} singularity.

We will encounter these A_n singularities briefly when constructing some specific elliptic surfaces towards the end of these notes. Since the main difficulties in resolving these singularities in general is present already in the cases of small n, we will contend ourselves with only going through the details for the cases $n = 2$ above and $n = 3$ below.

Example ($\mathbb{C}^2/(\mathbb{Z}/3\mathbb{Z})$, an A_2 singularity)**.** Let $\zeta \in \mathbb{C}^\times$ be a primitive 3rd root of unity, and let $i : \mathbb{C}^2 \to \mathbb{C}^2$ denote the map $(x, y) \mapsto (\zeta x, \zeta^{-1} y)$. Then, $G := \langle i \rangle \simeq \mathbb{Z}/n\mathbb{Z}$ is a group of automorphisms of $\mathbb{C}^2$ with no fixed points except the origin $(0,0)$. Hence, the quotient $X = \mathbb{C}^2/G$ is a surface with a single singularity which we wish to resolve. First note that i lifts to a map on the blowup $\mathrm{Bl}_0\,\mathbb{C}^2 = \left\{(x, y; X : Y) \in \mathbb{C} \times \mathbb{P}^1 : xY - yX = 0\right\}$ which is given by

$$i : (x, y; X : Y) \mapsto (\zeta x, \zeta^{-1} y;\, \zeta X : \zeta^{-1} Y).$$

This map has two isolated fixed points at $p = (0, 0; 1 : 0)$ and $q = (0, 0; 0 : 1)$, so the quotient $\mathrm{Bl}_0\,\mathbb{C}^2/G$ is still singular. Let us investigate its singularities. Near p (i.e. on the open where $X \neq 0$), the functions $x, t := Y/X$ give coordinates on $\mathrm{Bl}_0\,\mathbb{C}^2$. In these coordinates, the map i is given by

$$(x, t) \mapsto \left(\zeta x, \frac{\zeta^{-1} Y}{\zeta X}\right) = (\zeta x, \zeta^{-2} t) = (\zeta x, \zeta t).$$

Hence, near p, i acts by multiplication by ζ in both coordinates. Since we have the same factor in both coordinates, repeating the same type of argument as in the previous example (i.e. resolving the singularity p produces on $\mathrm{Bl}_0\,\mathbb{C}^2/G$ by first blowing up $\mathrm{Bl}_0\,\mathbb{C}^2$ at p and then taking the quotient of that space under its lifted action of G), we see that the singularity p produces in $\mathrm{Bl}_0\,\mathbb{C}^2/G$ is resolved by attaching an $\mathscr{O}_{\mathbb{P}^1}(-3) \simeq \mathscr{O}_{\mathbb{P}^1}(-1)/G$ near p whose zero section will have self-intersection -3. Similarly, the action near q looks like multiplication by ζ^{-1} in both coordinates and so it also leads to a singularity resolved by a copy of $\mathbb{P}^1$ with self-intersection -3. Finally, since p, q lived on the exceptional divisor of $\mathrm{Bl}_0\,\mathbb{C}^2$, we have shown that the singularity of $\mathbb{C}^2/G$ can be resolved by a surface $f : Y \to X$ such that $f^{-1}([(0,0)])$ consists of 3 copies of $\mathbb{P}^1$ meeting with the dual graph in Figure 2. In this figure, each node represents a copy of $\mathbb{P}^1$, two

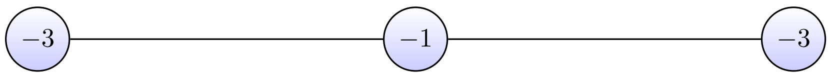

Figure 2: The dual graph of a non-minimal resolution of the singularity in $\mathbb{C}^2/(\mathbb{Z}/3\mathbb{Z})$

nodes share an edge if the corresponding $\mathbb{P}^1$'s intersect (necessarily in exactly 1 point by construction), and each node is labelled by its self-intersection number. The middle $\mathbb{P}^1$ has self-intersection -1, and so looks like the exceptional curve of a blowup. It is a fact that we will soon see that in such a situation, there in fact exists a smooth surface Z equipped with a blowup map $g : Y \to Z$. The resolution map $f : Y \to X$ clearly factors through $g : Y \to Z$ since f collapses the $\mathbb{P}^1$ that g collapses, so $Y \to Z$ represents another, smaller, resolution of X's singularity. The spaces we have considered fit in the below diagram.

$$\begin{tikzcd}
\mathrm{Bl}_{p,q}(\mathrm{Bl}_0\,\mathbb{C}^2) \arrow[r] \arrow[d] & Y \arrow[d, "g"'] \arrow[dd, bend left, "f"] \\
\mathrm{Bl}_0\,\mathbb{C}^2 \arrow[d] & Z \arrow[d, "h"'] \\
\mathbb{C}^2 \arrow[r] & X
\end{tikzcd}$$

Now, $h^{-1}([(0,0)])$ consists of 2 copies of $\mathbb{P}^1$, since it comes from collapsing one of the $\mathbb{P}^1$'s in $f^{-1}([(0,0)])$, and, as a consequence of Corollary 2.3.4, each of these $\mathbb{P}^1$'s has self-intersection -2. Thus, $h^{-1}([(0,0)])$ has the dual graph shown in Figure 3.

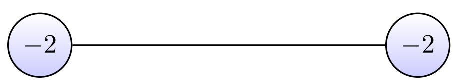

Figure 3: The dual graph of a minimal resolution of an A_2 singularity.

2.3.2 More on Singularities

In this section, we introduce the concept of bimeromorphic maps, and study some of the types of surface singularities that will come up in these notes. The point of bimeromorphic maps is that they present a more lenient notion of equivalence of complex spaces than biholomorphism; in particular, the concept of a bimeromorphic map captures the relationship between a singular space and any space resolving its singularities.

Definition 2.3.2. Let X, Y be irreducible surfaces. A proper, holomorphic, surjective map $\pi : X \to Y$ is called **bimeromorphic** if there are discrete subsets $T \subset X$ and $S \subset Y$ such that $\pi : X \setminus T \to Y \setminus S$ is biholomorphic. If X, Y are furthermore normal, a **bimeromorphic correspondence** between X, Y is a triple (Z, π_1, π_2) where Z is an irreducible, normal surface and $\pi_1 : Z \to X, \pi_2 : Z \to Y$ are bimeromorphic maps.

Example. The blowup $\mathrm{Bl}_p\, X \to X$ of a surface at a point is the quintessential example of a bimeromorphic map.

Remark 2.3.4. If $X_1 \leftarrow Z_1 \rightarrow X_2 \leftarrow Z_2 \rightarrow X_3$ are two bimeromorphic correspondences, then there exists a surface Z fitting into the below diagram

$$\begin{tikzcd}
 & & Z \arrow[dl] \arrow[dr] & & \\
 & Z_1 \arrow[dl] \arrow[dr] & & Z_2 \arrow[dl] \arrow[dr] & \\
X_1 & & X_2 & & X_3
\end{tikzcd}$$

We can take Z to be a well-chosen component of the normalization of the fiber product $Z_1 \times_{X_2} Z_2$.

Definition 2.3.3. We call two surfaces **bimeromorphically equivalent** if there is a bimeromorphic correspondence between them. The above remark shows that this is an equivalence relation.

Singularities on surfaces often arise from bimeromorphic maps which contract curves to points. Consider, for instance, a bimeromorphic map $\pi : X \to Y$ with X, Y normal. In this case [5], there exists a discrete subset $S \subset Y$ such that $\pi|_{\pi^{-1}(Y \setminus S)}$ is biholomorphic and $\pi^{-1}(y)$ is a curve in X for every $y \in S$. In this case, the points $y \in S$ are called **fundamental points for** π, and the curves $\pi^{-1}(y)$ are called **exceptional curves for** π. In general, a compact, reduced, connected curve C on a smooth surface X is called **exceptional** if there is some bimeromorphic map $\pi : X \to Y$ for which it is exceptional. This may seem like a hard property to detect intrinsically, but it is possible by the following theorem of Grauert.

Theorem 2.3.5 (Grauert's criterion). *A reduced, compact connected curve C with irreducible components C_i on a smooth surface is exceptional if and only if the intersection matrix (C_iC_j) is negative definite.*

Proof. [4] ∎

Inspired by Grauert's criterion, as well as our earlier discussion of blow ups, we remark that the simplest examples of exceptional curves are the following.

Definition 2.3.4. A **(-1)-curve** is a smooth rational curve with self-intersection -1.

Proposition 2.3.6. *An irreducible curve $C \subset X$ in a smooth surface is a (-1)-curve iff*

$$C^2 < 0 \ \text{ and } \ K_X C < 0.$$

Proof. $(\to)$ Let $C \subset X$ be a (-1)-curve so $C^2 - 1$. By the genus formula, we have

$$-2 = 2g(C) - 2 = K_X C + C^2 = K_X C - 1,$$

and so $K_X C = -1 < 0$.

$(\leftarrow)$ Assume instead that $C^2 < 0$ and $K_X C < 0$. Appealing to the genus formula again, we see that $2g(C) - 2 = K_X C + C^2 < 0$. Since $g(C) \geq 0$, we see that this is possible if and only if $g(C) = 0$, so C is smooth rational. Since $K_X C$ and C^2 are both strictly negative and their sum is $K_X C + C^2 = 2g(C) - 2 = -2$, we conclude furthermore that $K_X C = C^2 = -1$, so C is a (-1)-curve. ∎

The canonical source of (-1)-curves is given by blowups of points on surfaces. In fact, every (-1)-curve arises in this fashion as is made precise by the following two results.

Theorem 2.3.7. *Let X be a nonsingular surface, $E \subset X$ a (-1)-curve and $\pi : X \to Y$ the map contracting E. Then, $y = \pi(E)$ is nonsingular on Y.*

Proof. [2, Ch. III, Theorem 4.1] ■

Theorem 2.3.8. *Let X, Y be smooth surfaces and let $\pi : X \to Y$ be a bimeromorphic map such $E := \pi^{-1}(y) \subset X$ is an irreducible curve for some $y \in Y$. Then, near E, π is equivalent to blowing up a neighborhood of y.*

Proof. Since the contraction of E is local, we may assume that $Y \simeq \mathbb{C}^2$ and so has global coordinates. Let u, v be coordinates on Y centered at y. Let $U = \{u = 0\} \subset Y$ and $V = \{v = 0\} \subset Y$. Furthermore, let $\overline{U} = \overline{\pi^{-1}(U \setminus \{y\})}$ and $\overline{V} = \overline{\pi^{-1}(V \setminus \{y\})}$ be the proper transforms of U and V, respectively, in X. Note that $\{y\} = U \cap V = \{u = 0 \text{ and } v = 0\}$ so $\overline{U} \cap \overline{V} \subset E$. Consider now any $x \in \overline{U} \cap E$. The local intersection number between $\overline{U}$ and $\pi^*(V) = \overline{V} + E$ is

$$i_x(\overline{U}, \pi^*(V)) = \text{ord}_x(\pi^*(v)|_{\overline{U}}) = \text{ord}_y(v|_U) = 1$$

where the first equality comes from applying Lemma 2.2.3 to the dvr $\mathscr{O}_{\overline{U},x} = \mathscr{O}_{X,x}/(\pi^*(U))$ which shows that

$$\text{ord}_x(\pi^*(v)|_{\overline{U}}) = \dim_{\mathbb{C}} \mathscr{O}_{U,x}/(\pi^*(v)|_{\overline{U}}) = \dim_{\mathbb{C}} \mathscr{O}_{X,x}/(\overline{u}, \pi^*(v)) = i_x(\overline{U}, \pi^*(v))$$

where $\overline{u}$ is a local equation for $\overline{U} \subset X$. Since $\pi^*(V) = \overline{V} + kE$ for some k and $\overline{U} \cdot E > 0$ by construction, we conclude that $\overline{U} \cdot \overline{V} = 0$, i.e. that they are disjoint. Furthermore, $U = \{u = 0\}$ passes through y with multiplicity one, so $\overline{U} \cap E$ is a single point and $\overline{U} \cdot \pi^*(V) = \overline{U} \cdot E = 1$. Thus, $k = 1$. We can reason similarly with $\overline{V}$ in place of $\overline{U}$ to see that

$$\overline{U} \cdot E = 1 \qquad \overline{U} \cdot \overline{V} = 0 \qquad \overline{V} \cdot E = 1,$$

Recall that $\overline{u}$ is a local equation for $\overline{U}$, and let $\overline{v}$ be a local equation for $\overline{V}$. Shrinking X is necessary to these local equations become global, we have a meromorphic map $\phi = \overline{u}/\overline{v} : x \mapsto [\overline{u}(x) : \overline{v}(x)]$ from $X \to \mathbb{P}^1$. This map is defined everywhere since $\overline{U} \cdot \overline{V} = 0$ and the restriction $\phi|_E : E \to \mathbb{P}^1$ is an isomorphism since it is degree 1 as $\overline{U} \cdot E = 1 = E \cdot \overline{V}$. Thus, we see that the product map $\pi \times \phi : X \to Y \times \mathbb{P}^1$ maps X isomorphically onto

$$\overline{Y} = \{(y, [X : Y]) : u(y)Y - v(y)X = 0\} \subset Y \times \mathbb{P}^1,$$

the result of blowing up Y at y. ■

Remark 2.3.5. Inspired by the above theorem, we call the result of contracting a (-1)-curve a **blowdown**.

Taking on the mindset that bimeromorphic correspondences are intended to model desingularizations of surfaces, we hope to find, in each bimeromorphic equivalence class, a "best" or "simplest" smooth surface. By the preceding theorems, any (smooth) surface containing a (-1)-curve is not a good candidate for the simplest smooth surface in its bimeromorphic equivalence class since it is blown up from an even "simpler" smooth surface. This motivates the following definition.

Definition 2.3.5. A smooth surface is called a **minimal** if it does note contain any (-1)-curves. If a smooth, minimal surface X is bimeromorphically equivalent to a singular surface Y, then we call X the **minimal desingularization** of Y.

Remark 2.3.6. By successively contracting any (-1)-curves, you can show than any compact, smooth surface is bimeromorphically equivalent to a minimal one. It is a fact [2, Ch. I, Theorem 9.1(iv)] that each blowdown reduces the second Betti number by 1, and so there can only be finitely many (-1)-curves to start with.

Now that we have gained an understanding of (-1)-curves, we move on to another type of singularity: the Hirzebruch-Jung strings. These will feature more heavily towards to end of this document during the construction of some examples of elliptic surfaces.

Definition 2.3.6. A **Hirzebruch-Jung string** is a union $C = \bigcup_{i=1}^{r} C_i$ of smooth rational curves C_i such that

- $C_i^2 \leq -2$ for all i
- $C_i C_j = 1$ if $|i-j| = 1$
- $C_i C_j = 0$ if $|i-j| \geq 2$

We visualize this as a path of length r with vertices labeled by the self-intersections $e_i = C_i^2$. See, for example, Figure 4. If $r = 1$ and $e_1 = -2$, then we call C a (-2)**-curve**.

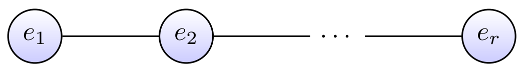

Figure 4: A visualization of a Hirzebruch-Jung string

Definition 2.3.7. A Hirzebruch-Jung string of length r with self-intersections $e_i = -2$ for all i is called a **singularity of type** A_r. In particular, a singularity of type A_1 is a (-2)-curve.

In the case of Hirzebruch-Jung strings with self-intersections other than -2, there is a way of notating them that is more concise than listing out the various numbers e_i. This alternate description comes from gathering the e_i into a continued fraction. For the sake of brevity, we will not cover the story here in detail, but an account of it can be found in [2, Ch. III, Sect. 5].

Notation 2.3.8. We will denote the continued fraction

$$b_0 + \cfrac{b_1}{a_1 + \cfrac{b_2}{a_2 + \cfrac{b_3}{\ddots\, a_{n-1} + \cfrac{b_n}{a_n}}}}$$

simply by

$$b_0 + \frac{b_1 \rvert}{\lvert a_1} + \frac{b_2 \rvert}{\lvert a_2} + \frac{b_3 \rvert}{\lvert a_3} + \cdots + \frac{b_n \rvert}{\lvert a_n}.$$

Theorem 2.3.9. *Let $C = \bigcup_{i=1}^{r} C_i$ be a Hirzebruch-Jung string and let $e_i = C_i^2$. Write*

$$\frac{n}{q} = |e_1| - \frac{1 \rvert}{\lvert |e_2|} - \cdots - \frac{1 \rvert}{\lvert |e_r|}$$

where $0 < q < n$ are coprime and the RHS denotes a continued fraction. Then, the singularity obtained by contracting C is isomorphic to the unique singularity lying over $0 \in \mathbb{C}^3$ in the normalization of the surface

$$W = \left\{(w, z_1, z_2) \in \mathbb{C}^3 : w^n = z_1 z_2^{n-q}\right\}$$

Proof. [2, Ch. III, Theorem 5.2]. ∎

Definition 2.3.9. As a consequence of the theorem above, the singularity only depends on n, q, and so is called the **singularity of type** $A_{n,q}$.

For our purposes, it is most important to know that these singularities appear when taking quotients of surfaces by cyclic groups. The local picture is as follows.

Any linear action $\mathbb{Z}/n\mathbb{Z} \curvearrowright \mathbb{C}^2$ can be put in the form

$$k \cdot \begin{pmatrix} u_1 \\ u_2 \end{pmatrix} = \begin{pmatrix} \mathbf{e}(q_1 k/n) u_1 \\ \mathbf{e}(q_2 k/n) u_2 \end{pmatrix}$$

with $0 \leq q_1, q_2 < n$. These numbers $\{q_i\}_{i=1}^2$ are called the **weights** of the operation, and they are unique up to reordering. The quotient $\mathbb{C}^2/(\mathbb{Z}/n\mathbb{Z})$ will contain a singularity of type $A_{n,q}$ for some $n, q \in \mathbb{Z}$ which are determined by the following theorem.

Theorem 2.3.10. *Let $\mathbb{Z}/n\mathbb{Z}$ act on $\mathbb{C}^2$ as above. Assume that $q_1 \neq 0 \neq q_2$ and $\gcd(n, q_1, q_2) = 1$. For $i = 1, 2$, let*

$$\begin{aligned} d_i &= \gcd(n, q_i) & n &= n_i d_i & q_i &= p_i d_i \\ m &= \gcd(n_1, n_2) \\ p_i' &= \text{ unique integer with } p_i p_i' \equiv 1 \pmod{m} \text{ and } 0 < p_i' < m \\ q &= \text{ unique integer with } q \equiv p_1 p_2' \pmod{m} \text{ and } 0 < q < m \end{aligned}$$

Then, the image of $(0,0) \in \mathbb{C}^2$ in the quotient $\mathbb{C}^2/(\mathbb{Z}/n\mathbb{Z})$ is a singularity of type $A_{m,q}$.

Proof. [2, Ch. III, Proposition 5.3]. ∎

Corollary 2.3.11. *If G is a finite cyclic group acting on a smooth surface X, then the quotient X/G has only singularities of Hirzebruch-Jung type.*

Remark 2.3.7. If $\mathbb{Z}/n\mathbb{Z}$ acts on $\mathbb{C}^2$ with weights $q_1 = q_2 = 1$, then the resulting singularity is of type $A_{n,1}$ which is to say that it is resolved a single smooth rational curve $C \simeq \mathbb{P}^1$ with self-intersection $C^2 = -n$.

Remark 2.3.8. On the opposite end of the spectrum, if the $\mathbb{Z}/n\mathbb{Z}$ acts on $\mathbb{C}^2$ with weights $(q_1, q_2) = (1, -1)$, then the resulting singularity is of type A_{n-1}, which is to say that it is resolved by a string of $(n-1)$ smooth rationals $C_i \simeq \mathbb{P}^1$, each of self-intersection -2. We saw this explicitly for $n = 2$ and for $n = 3$ at the end of the Section 2.3.1.

2.4 Curves Embedded in Surfaces

This came up already in the section on divisors – most notably in the proof of Riemann-Roch – but for studying curves in surfaces, it will be useful to be able to simultaneously treat smooth curves and merely irreducible (or just reduced, or even just embedded) curves on an equal playing field. This will come up in our study of fibrations where, for example, by appealing to certain numerical invariants shared by all fibers, we will be able to use knowledge of the general, smooth fiber to gain knowledge of the singluar fibers. This will result from the tension between two types of genera – the arithmetic and geometric genera – which are equal exactly when a curve is smooth and which we will introduce in this section.

2.4.1 The Arithmetic Genus and Other Invariants

Setup. Let X be a smooth, compact surface.

Definition 2.4.1. An **embedded curve** $C \subset X$ is a 1-dimensional analytic subspace locally defined a single equation. As such, there is a natural 1-1 correspondence between embedded curves and effective divisors.

Remark 2.4.1. Because we are assuming X is compact in this section, any embedded curve $C \subset X$ is automatically also compact since it is closed in a compact set.

Our goal in this section is to provide a language/toolset for all embedded curves which behaves as one expects when C is smooth (i.e. a Riemann surface). To start with, for $C \subset X$ a smooth curve, we have the notions of a canonical bundle ω_C and a normal bundle $\mathcal{N}_{C/X}$, both on C, which we previously computed in terms of data on X. The end results of these computations were formulas which did not depend on C being smooth, and so we adopt them for general embedded curves.

Notation 2.4.2. As usual, for $\iota : C \hookrightarrow X$ an embedded curve, and $D \in \operatorname{Div} X$ a divisor on X, we let $\mathscr{O}_C(D) := \mathscr{O}_X(D)|_C = \iota^*\mathscr{O}_X(D)$ denote the restriction of $\mathscr{O}_X(D)$ to C.

Definition 2.4.3. Let $C \subset X$ be an embedded curve. We define its **normal bundle** to be $\mathscr{O}_C(C)$ and its **canonical bundle** to be $\omega_C := \omega_X \otimes \mathscr{O}_X(C)|_C$.

The above definitions may seem somewhat artificial, but these objects on arbitrary embedded curves actually maintain many of the nice properties they have for smooth curves. Of note, we will see soon that, essentially by definition, the Genus Fomrula contains to hold for an appropatiely defined notion of genus, and more surprisingly, embedded curves satisfy Serre duality.

Theorem 2.4.1 (**Serre Duality for Embedded Curves**)**.** *Let $C \subset X$ be an embedded curve. Then, there is a perfect pairing*

$$\mathrm{H}^1(\mathscr{E}) \otimes \mathrm{H}^0(\mathscr{E}^\vee \otimes \omega_C) \longrightarrow \mathbb{C}$$

whenever $\mathscr{E}$ is locally free. In particular, $h^0(\mathscr{E}) = h^1(\mathscr{E}^\vee \otimes \omega_C)$. Furthermore, $h^i(\mathscr{E}) = 0$ for all $i > 1$.

Proof. [2, Ch. II, Thm. 6.1]. ■

Remark 2.4.2. Because of its role in Serre duality, the sheaf ω_C on an embedded curve $C \subset X$ is sometimes also called the **dualizing sheaf**.

In order to really have a unified account of all embedded curves, in addition to Serre duality, we will need Riemann-Roch to still hold as well. We will prove that it does by reducing to the case of smooth curves, but before we do that, to even state Riemann-Roch in general, we need a notion of degree of line bundles on embedded curves.

Recall 2.4.1. Any reduced complex space S has a unique normalization $\widetilde{S}$ which is a normal space with a map $\nu : \widetilde{S} \to S$ that is final among all maps from normal spaces to X. In particular, if S is a curve, then its normalization $\widetilde{S}$ is smooth, and so a Riemann surface.

Definition 2.4.4. Let $C \subset X$ be an embedded curve, and write $C = \sum_{i=1}^{r} n_i C_i \in \mathrm{Div}(X)$ with $n_i > 0$ and C_i irreducible. Let $\nu_i : \widetilde{C}_i \to C_i$ be the normalization of the ith irreducible component of C, and let $\mathscr{L} \in \operatorname{Pic} C$ be a line bundle on C. Then, we defined the **degree** of $\mathscr{L}$ to be

$$\deg \mathscr{L} = \sum_{i=1}^{r} n_i \cdot \deg\left(\nu_i^*(\mathscr{L}|_{C_i})\right).$$

Remark 2.4.3. Let $C \subset X$ be reduced with normalization $\nu : \widetilde{C} \to C$. Let $\mathscr{L}$ be any line bundle, and let $\widetilde{\mathscr{L}} = \nu^*\mathscr{L}$. Then, from [2, Ch. II, Sect. 3], we see that

$$\deg(\widetilde{\mathscr{L}}) = \deg(\mathscr{L}) \qquad \chi(\widetilde{\mathscr{L}}) = \chi(\nu_*\widetilde{\mathscr{L}}) \qquad \chi(\mathscr{O}_{\widetilde{C}}) = \chi(\nu_*\mathscr{O}_{\widetilde{C}})$$

We will take these equalities for granted without proof.

Our strategy for proving Riemann-Roch will be twofold: we will first prove it for reduced curves by reduction to the smooth case, and then we will prove it for general curves by reduction to the reduced case (using induction). The input needed for reducing to the smooth case, aside from the equalities in Remark 2.4.3, is the fact that the cokernel $\mathscr{O}_C/\nu_*\mathscr{O}_{\widetilde{C}}$ is supported on a discrete subset of C. For reducing from the general case to the reduced case, the main input we will need is a way to relate line bundles on a(n) (effective) divisor $C = A + B$ to those on A, B respectively. These two inputs are collected in the following two remarks.

Remark 2.4.4. Let $C \subset X$ be a reduced embedded curve with normalization $\nu : \widetilde{C} \to C$. Recall that ν is an isomorphism away from C's (finitely many) non-normal points. With this in mind, consider the **normalization sequence**

$$0 \longrightarrow \mathscr{O}_C \longrightarrow \nu_*\mathscr{O}_{\widetilde{C}} \longrightarrow \mathscr{S} \longrightarrow 0.$$

Because $\mathscr{O}_C \to \nu_*\mathscr{O}_{\widetilde{C}}$ is an isomorphism almost everywhere, the cokernel $\mathscr{S} := \operatorname{coker}(\mathscr{O}_C \to \nu_*\mathscr{O}_{\widetilde{C}})$ is concentrated[3] at the singular points of C. In particular, since $\mathscr{S}$ is supported on a finite set of points, $h^1(\mathscr{S}) = 0$ while

$$h^0(\mathscr{S}) = \sum_{x \in C} \dim_{\mathbb{C}} \mathscr{S}_x = \sum_{x \in C} \dim_{\mathbb{C}} (\nu_*\mathscr{O}_{\widetilde{C}}/\mathscr{O}_C)_x.$$

Remark 2.4.5. Let $C = A + B$ be a sum of two effective divisors on X. Then, we get an exact sequence

$$0 \longrightarrow \mathscr{O}_A(-B) \longrightarrow \mathscr{O}_C \longrightarrow \mathscr{O}_B \longrightarrow 0$$

called the **decomposition sequence** for $C = A + B$. To see this, note that we have a homomorphism

$$\begin{tikzcd}
0 \arrow[r] & \mathscr{O}_X(-C) \arrow[r] \arrow[d] & \mathscr{O}_X \arrow[r] \arrow[d, equal] & \mathscr{O}_C \arrow[r] \arrow[d] & 0 \\
0 \arrow[r] & \mathscr{O}_X(-B) \arrow[r] & \mathscr{O}_X \arrow[r] & \mathscr{O}_B \arrow[r] & 0
\end{tikzcd}$$

of short exact sequences. Applying the snake lemma to it then shows that

$$\ker(\mathscr{O}_C \to \mathscr{O}_B) \simeq \operatorname{coker}(\mathscr{O}_X(-A-B) \to \mathscr{O}_X(-B)) = \mathscr{O}_A(-B),$$

[3] has nonzero stalk only at

from which we get the decomposition sequence.

Theorem 2.4.2 (**Riemann-Roch for Embedded Curves**). *Let $C \subset X$ be an embedded curve, and let $\mathscr{L} \in \operatorname{Pic} C$ be a line bundle on C. Then,*

$$\chi(\mathscr{L}) = \chi(\mathscr{O}_C) + \deg \mathscr{L}.$$

Proof. First suppose that C is reduced with normalization $\nu : \widetilde{C} \to C$. Let $\widetilde{\mathscr{L}} = \nu^* \mathscr{L}$, a line bundle on the smooth curve $\widetilde{C}$. Hence, Riemann-Roch for smooth curves combined with Remark 2.4.3 gives

$$\deg \mathscr{L} = \deg(\widetilde{\mathscr{L}}) = \chi(\widetilde{\mathscr{L}}) - \chi(\mathscr{O}_{\widetilde{C}}) = \chi(\nu_* \widetilde{\mathscr{L}}) - \chi(\nu_* \mathscr{O}_{\widetilde{C}}).$$

Now, tensoring the normalization sequence with $\mathscr{L}$ gives rise to

$$0 \longrightarrow \mathscr{L} \longrightarrow \nu_* \mathscr{L} \longrightarrow \nu_* \mathscr{O}_{\widetilde{C}} / \mathscr{O}_C \longrightarrow 0$$

with the last term unchanged because $\nu_* \mathscr{O}_{\widetilde{C}} / \mathscr{O}_C, \nu_* \widetilde{\mathscr{L}} / \mathscr{L}$ are both line bundles on the discrete set of singular points of C, and so isomorphic. Hence,

$$\chi(\mathscr{L}) = \chi(\nu_* \mathscr{L}) - \chi(\nu_* \mathscr{O}_{\widetilde{C}} / \mathscr{O}_C) = \deg \mathscr{L} + \chi(\nu_* \mathscr{O}_{\widetilde{C}}) - \chi(\nu_* \mathscr{O}_{\widetilde{C}} / \mathscr{O}_C) = \deg \mathscr{L} + \chi(\mathscr{O}_C),$$

as desired.

We now proceed inductively, so write $C = A + B$ with A, B effective and Riemann-Roch holding for them. Tensor the decomposition sequence for $C = A + B$ with $\mathscr{L}$ to get

$$0 \longrightarrow \mathscr{L}|_A \otimes \mathscr{O}_A(-B) \longrightarrow \mathscr{L} \longrightarrow \mathscr{L}|_B \longrightarrow 0.$$

Hence, $\chi(\mathscr{L}) = \chi(\mathscr{L}|_B) + \chi(\mathscr{L}|_A \otimes \mathscr{O}_A(-B))$ and similarly $\chi(\mathscr{O}_C) = \chi(\mathscr{O}_B) + \chi(\mathscr{O}_A(-B))$. By induction, i.e. Riemann-Roch on A, B, we see that

$$\begin{aligned}
\chi(\mathscr{L}|_B) &= \deg(\mathscr{L}|_B) + \chi(\mathscr{O}_B) \\
\chi(\mathscr{L}|_A \otimes \mathscr{O}_A(-B)) &= \deg(\mathscr{L}|_A \otimes \mathscr{O}_A(-B)) + \chi(\mathscr{O}_A) \\
&= \deg(\mathscr{L}|_A) + \deg(\mathscr{O}_A(-B)) + \chi(\mathscr{O}_A) \\
&= \deg(\mathscr{L}|_A) + \chi(\mathscr{O}_A(-B))
\end{aligned}$$

Finally, it is clear by definition that $\deg(\mathscr{L}) = \deg(\mathscr{L}|_A) + \deg(\mathscr{L}_B)$, so we finally get that

$$\begin{aligned}
\chi(\mathscr{L}) &= \chi(\mathscr{L}|_B) + \chi(\mathscr{L}|_A \otimes \mathscr{O}_A(-B)) \\
&= (\deg(\mathscr{L}|_B) + \chi(\mathscr{O}_B)) + (\deg(\mathscr{L}| + A) + \chi(\mathscr{O}_A(-B))) \\
&= (\deg(\mathscr{L}|_A) + \deg(\mathscr{L}|_B)) + (\chi(\mathscr{O}_B) + \chi(\mathscr{O}_A(-B))) \\
&= \deg \mathscr{L} + \chi(\mathscr{O}_C),
\end{aligned}$$

as claimed. ∎

Remark 2.4.6. This almost completes the proof of Riemann-Roch for surfaces. In addition to Riemann-Roch for embedded curves, we also need a version of Theorem 2.2.4 for embedded curves, i.e. we need that $CD = \deg(\mathscr{O}_X(D)|_C)$ when C is not assumed smooth. We will obtain this below after introducing the arithmetic genus of a curve.

So far, we have seen how facts about smooth curves extend to possibly singular curves embedded in surfaces. However, we have introduced no techniques or invariants differentiating a singular irreducible curve from a smooth one, or even better, for measuring how far from being smooth a given singular curve is. To remedy situation, we introduce the geometric and arithmetic genera of an embedded curve.

Definition 2.4.5. Let $C \subset X$ be a reduced, embedded curve. Then, the genus $g(\widetilde{C})$ of its normalization is called its **geometric genus** while its **arithmetic genus**, denoted $g(C)$, is given by the formula

$$g(C) = 1 - \chi(\mathscr{O}_C) = 1 + \chi(\omega_C).$$

These coincide when C is smooth.

Remark 2.4.7. Let $C \subset X$ be a reduced embedded curve with normalization $\nu : \widetilde{C} \to C$. We wish to compare $g(\widetilde{C})$ and $g(C)$. Recall the normalization sequence

$$0 \longrightarrow \mathscr{O}_C \longrightarrow \nu_* \mathscr{O}_{\widetilde{C}} \longrightarrow \mathscr{S} \longrightarrow 0$$

whose cokernel $\mathscr{S} := \operatorname{coker}(\mathscr{O}_C \to \nu_* \mathscr{O}_{\widetilde{C}})$ is concentrated at the singular points of C. It gives rise to the equality

$$\chi(\nu_* \mathscr{O}_{\widetilde{C}}) = \chi(\mathscr{O}_C) + \chi(\mathscr{S}) = \chi(\mathscr{O}_C) + h^0(\mathscr{S}).$$

Because $\chi(\nu_* \mathscr{O}_{\widetilde{C}}) = \chi(\mathscr{O}_{\widetilde{C}})$, we conclude from this that

$$g(C) = g(\widetilde{C}) + h^0(\mathscr{S}),$$

and since $\mathscr{S}$ is concentrated at a finite number of points, we have

$$h^0(\mathscr{S}) = \sum_{x \in C} \dim_{\mathbb{C}} \mathscr{S}_x = \sum_{x \in C} \dim_{\mathbb{C}} (\nu_* \mathscr{O}_{\widetilde{C}} / \mathscr{O}_C)_x.$$

Hence, $g(C) \geq g(\widetilde{C})$ with equality if and only if $\mathscr{S} = 0$, i.e. if and only if C is smooth. Furthermore, the difference $g(C) - g(\widetilde{C}) = h^0(\mathscr{S})$ gives us a measure for how singular C is.

Corollary 2.4.3. *If $C \subset X$ is an embedded curve with arithmetic genus $g(C) = 0$, then C is necessarily smooth and $C \simeq \mathbb{P}^1$.*

Proof. This follows immediately from Remark 2.4.7 since $0 = g(C) \geq g(\widetilde{C}) \geq 0$, so $g(C) = g(\widetilde{C}) = 0$ which implies $C = \widetilde{C} \simeq \mathbb{P}^1$. ■

Now that we have a measure for the singularity of an embedded curve, we will prove a generalization of Theorem 2.2.4, as promised earlier. Our strategy will be to reduce the general case to the smooth case already handled earlier. In order to do so, we will show that normalization $\widetilde{C}$ of an embedded cure $C \subset X$ can itself be embedded into a surface $\widetilde{X}$ with a well-understood map $\widetilde{X} \to X$, restricting to the normalization map $\nu : \widetilde{C} \to C$. In fact, we will show that $\widetilde{X}$ is given a finite sequence of blow-ups of X. Pulling a line bundle $\mathscr{O}_X(D)$ back along this map will give the reduction to Theorem 2.2.4. Now that we know the strategy, let us implement it.

Theorem 2.4.4. *Let $C \subset X$ be a singular, reduced, embedded curve. Then, by blowing X up a finite number of times, we can obtain a map $\widetilde{X} \to X$ such that the proper transform of C is smooth, and so is necessarily C's normalization.*

Proof. The strategy behind the proof is simple: we will show that each blow up of X at a singular point of C gives rise to a proper transform $\widehat{C}$ which is less singular. With that said, let $x \in C \subset X$ be a singular point of C, and let $\widehat{X} = \mathrm{Bl}_x X$ with projection map $\pi : \widehat{X} \to X$. Let m be the multiplicity with which C passes through x, and note that, since x is a singular point of C, we necessarily have $m \geq 2$. Let $\widehat{C} \subset \widehat{X}$ be the proper transform of C, so C and $\widehat{C}$ have the same normalization, $\widetilde{C}$. Let $\nu : \widetilde{C} \to C$ and $\widehat{\nu} : \widetilde{C} \to \widehat{C}$ be their respective normalization maps. Then, we have a sequence

$$\mathscr{O}_C \hookrightarrow \pi_*\mathscr{O}_{\widehat{C}} \hookrightarrow \pi_*\widehat{\nu}_*\mathscr{O}_{\widetilde{C}} = \nu_*\mathscr{O}_{\widetilde{C}}$$

of inclusions of sheaves on C. Now, recall that we measure how singular C is using the non-negative integer

$$h^0(\nu_*\mathscr{O}_{\widetilde{C}}/\mathscr{O}_C) = \sum_{p \in C} \dim_{\mathbb{C}} \left(\nu_*\mathscr{O}_{\widetilde{C}}/\mathscr{O}_C\right)_p.$$

Since $\widehat{C} \xrightarrow{\pi} C$ is an isomorphism away from $x \in C$, we see that

$$\dim_{\mathbb{C}} \left(\nu_*\mathscr{O}_{\widetilde{C}}/\mathscr{O}_C\right)_p = \dim_{\mathbb{C}} \left(\nu_*\mathscr{O}_{\widetilde{C}}/\pi_*\mathscr{O}_{\widehat{C}}\right)_p$$

for all $p \neq x$ as $(\pi_*\mathscr{O}_{\widehat{C}})_p = \mathscr{O}_{C,p}$. However,

$$\dim_{\mathbb{C}} \left(\nu_*\mathscr{O}_{\widetilde{C}}/\mathscr{O}_C\right)_x = \dim_{\mathbb{C}} \left(\nu_*\mathscr{O}_{\widetilde{C}}/\pi_*\mathscr{O}_{\widehat{C}}\right)_x + \dim_{\mathbb{C}} \left(\pi_*\mathscr{O}_{\widehat{C}}/\mathscr{O}_C\right)_x,$$

so

$$h^0(\widehat{\nu}_*\mathscr{O}_{\widetilde{C}}/\mathscr{O}_{\widehat{C}}) = h^0(\nu_*\mathscr{O}_{\widetilde{C}}/\pi_*\mathscr{O}_{\widehat{C}}) = h^0(\nu_*\mathscr{O}_{\widetilde{C}}/\mathscr{O}_C) - \dim_{\mathbb{C}} \left(\pi_*\mathscr{O}_{\widehat{C}}/\mathscr{O}_C\right)_x.$$

Thus, if $\dim_{\mathbb{C}}(\pi_*\mathscr{O}_{\widehat{C}}/\mathscr{O}_C)_x > 0$, then $\widehat{C}$ is less singular than C. Since our measure of singularity is a non-negative integer, it can only decrease finitely many times, and so we would indeed obtain a smooth proper transform after finitely many blow-ups. Since $m > 1$, it is easy to see that $\pi_*\mathscr{O}_{\widehat{C}}/\mathscr{O}_C \neq 0$ and so has nonzero dimension. For example, letting $U \subset X$ be a neighborhood of x small enough that $\pi^{-1}(U)$ is disconnected – i.e. there are two points of $\widehat{C}$ above x lying in two different components of $\pi^{-1}(U)$ – we have $\pi_*\mathscr{O}_{\widehat{C}}(U) = \mathscr{O}_{\widehat{C}}(\pi^{-1}(U)) = \mathscr{O}_{\widehat{C}}(V_1) \oplus \mathscr{O}_{\widehat{C}}(V_2)$ where $V_1, V_2 \subset \pi^{-1}(U)$ are disjoint opens covering $\pi^{-1}(U)$. Thus, $(0,1) \in \mathscr{O}_{\widehat{C}}(V_1) \oplus \mathscr{O}_{\widehat{C}}(V_2) = \pi_*\mathscr{O}_{\widehat{C}}(U)$ is an element whose germ at x does not lie in $\mathscr{O}_{C,x}$, and so which gives a nonzero element of $(\pi_*\mathscr{O}_{\widehat{C}}/\mathscr{O}_C)_x$. Therefore, the claim holds. ■

Corollary 2.4.5. *Theorem 2.2.4 holds for non-smooth embedded curves as well. That is, if $C \subset X$ is an embedded curve and $D \in \operatorname{Div} X$ is a divisor, then*

$$C \cdot D = \deg(\mathscr{O}_X(D)|_C).$$

Proof. By Theorem 2.4.4, there exists a exists a morphism $\pi : \widetilde{X} \to X$, obtained as a finite sequence of blowups, such that the proper transform $\widetilde{C}$ of C under this morphism is smooth. First note that $\pi^*\mathscr{O}_X(D) = \mathscr{O}_{\widetilde{X}}(\pi^*D)$, essentially by definition and, since $\pi|_{\widetilde{C}} : \widetilde{C} \to C$ is C's normalization map, also $\deg \mathscr{O}_X(D)|_C = \deg \mathscr{O}_{\widetilde{X}}(\pi^*D)|_{\widetilde{C}}$ by definition. Since $\widetilde{X}$ is obtained by repeated blowups, repeated application of part **(2)** of

Proposition 2.3.3 shows that $\widetilde{C} \cdot (\pi^* D) = C \cdot D$. Combining this all with Theorem 2.2.4 applied to $\widetilde{C} \subset \widetilde{X}$, we see that

$$\deg \mathscr{O}_X(D)|_C = \deg \mathscr{O}_{\widetilde{X}}(\pi^* D)|_{\widetilde{C}} = \widetilde{C} \cdot (\pi^* D) = C \cdot D$$

as desired. ■

Remark 2.4.8. At this point, we have now finished the proof of Riemann-Roch for surfaces.

Returning to our discussion of arithmetic genera, while linearly equivalent curves may have different geometric genera, the arithmetic genus is more "stable," depending only on the homology class of the curve $C \subset X$. This is a consequence of the genus formula still holding for general embedded curves.

Theorem 2.4.6. *Let $C \subset X$ be a reduced embedded curve. Then,*

$$2g(C) - 2 = \deg(\omega_X \otimes \mathscr{O}_X(C)|_C) = (K_X + C)C$$

where $g(C)$ denotes its arithmetic genus.

Proof. By Riemann-Roch, $\deg(\omega_C) = \chi(\omega_C) - \chi(\mathscr{O}_C)$, but by Serre duality, $\chi(\mathscr{O}_C) = -\chi(\omega_C)$, so $\deg(\omega_C) = 2\chi(\omega_C) = 2g(C) - 2$. Finally, $\deg(\omega_C) = \deg(\omega_X \otimes \mathscr{O}_X(C)|_C)$ by definition. ■

Remark 2.4.9. The genus formula above allows us to define the arithmetic genus of an arbitrary divisor $D \in \mathrm{Div}(X)$ via

$$2g(D) - 2 = \deg(\omega_X \otimes \mathscr{O}_X(D)|_D) = (K_X + D)D.$$

With this definition, we see that we can relate the genus of a reducible (i.e. not irreducible) divisor $D = A + B$ to those of its components. Indeed,

$$2g(D) - 2 = K_X(A + B) + (A + B)^2 = (K_X A + A^2) + (K_X B + B^2) + 2AB = 2g(A) + 2g(B) + 2AB - 4,$$

and so

$$g(D) = g(A + B) = g(A) + g(B) + AB - 1. \tag{2}$$

2.4.2 Ordinary Double Points

There is one particular type of curve singularity we will encounter later on: the ordinary double point which is also known as a node. It will arise in Section 4.3.2 where we will analyze a particular family of elliptic curves degenerating to a nodal curve, and then use this family to construct other families of elliptic curves with different types of degenerations. When carrying this out, it will be helpful to have a local normal form for a curve with such a singularity. Therefore, in this short section, we produce such a normal form.

Consider an irreducible curve $C \subset X$ embedded in a smooth, compact surface X. Let $p \in C$ be a fixed point, and let $f \in \mathscr{O}_{X,p}$ be a local equation for C near p. Since X is smooth, we can find a neighborhood $U \subset X$ of p isomorphic to the unit ball in $\mathbb{C}^2$. In particular, on U we can find coordinates $x, y \in \Gamma(U, \mathscr{O}_X)$ centered at p (i.e. $\{p\} = \{x = 0 \text{ and } y = 0\} \subset U$). Hence, $\mathscr{O}_{X,p} \simeq \mathscr{O}_{U,p} \simeq \mathscr{O}_{\mathbb{C}^2,(0,0)}$ and f can be identified with a (convergent) power series in the two variables x, y centered at $(0, 0)$.

Recall 2.4.2. Write $f(x, y) = \sum_{d \geq 0} f_d(x, y)$ as a power series with $f_d(x, y)$ a homogeneous polynomial of degree d. The least d such that $f_d \neq 0$ is called the **multiplicity of p in C**. If p is of multiplicity 2, then we call it a **double point**.

Assume now that p is a double point of C so $f(x,y) = \sum_{d\geq 2} f_d(x,y)$. The degree 2 part $f_2(x,y) = ax^2 + bxy + cy^2$ is a nonzero homogeneous polynomial in two variables. All such polynomials factor as a product of degree 1 homogeneous polynomials in two variables, so we can write

$$f_2(x,y) = (\alpha_1 x + \beta_1 y)(\alpha_2 x + \beta_2 y)$$

with $(\alpha_i, \beta_i) \neq (0,0) \in \mathbb{C}^2$ for $i = 1, 2$. If $[\alpha_1 : \beta_1] \neq [\alpha_2 : \beta_2] \in \mathbb{P}^1$, then we call p an **ordinary double point** or **node**. Geometrically, this means that C has two distinct tangent vectors at p. Algebraically, this means we can perform the following change of variables

$$u = \frac{(\alpha_1 x + \beta_1 y) + (\alpha_2 x + \beta_2 y)}{2} \qquad v = \frac{(\alpha_1 x + \beta_1 y) - (\alpha_2 x + \beta_2 y)}{2i}$$

in order to rewrite f_2 as

$$f_2(u,v) = u^2 + v^2.$$

This also gives us our normal form. We would like to promote the above equality from simply holding for f_2 to one holding for all of f. Since every monomial in u, v of degree at least 3 contains x^2 or y^2, we may write $f(u,v) = u^2\phi_1(u,v) + v^2\phi_2(u,v)$ where $\phi_1(0,0), \phi_2(0,0) \neq 0$. Finally, we change variables once more by letting $s = u\sqrt{\phi_1}$ and $t = v\sqrt{\phi_2}$, so in these coordinates we have

$$f(s,t) = s^2 + t^2.$$

This is our local normal form. In the end, we have proven the following.

Theorem 2.4.7. *Let $C \subset X$ be an irreducible curve in a smooth surface X, and let $p \in C$ be an ordinary double point (i.e. a node). Then, there exists coordinates x, y on X near p such that, near p, C is given as the vanishing set of the function*

$$f(x,y) = x^2 + y^2.$$

2.5 Fibrations

Setup. Let X be a connected, smooth surface (not necessarily compact), S a smooth connected curve (again, not necessarily compact), and $f : X \to S$ a proper, surjective holomorphic map. Assume furthermore that f is **connected**, i.e. $f^{-1}(p)$ is connected for all $p \in S$.

Remark 2.5.1. Our reason for not requiring compactness in this section is to prepare for later giving an account of the local theory of elliptic surfaces. In particular, the final chapter of this book will involve many fibrations of the form $X \to \Delta$ where the base $\Delta = \{z \in \mathbb{C} : |z| < 1\}$ is not compact.

Upon seeing this setup, one may reasonably ask why this is enough for a section titled "fibrations." That is, one may wonder why the map $f : X \to S$ is not explicitly assumed to be a topological fibration. The reason is that this is automatic from Ehresmann's lemma.

Lemma 2.5.1 (Ehresmann). *Let $f : M \to N$ be a smooth map between smooth manifolds. If f is proper, surjective, submersion, then f is a locally trivial fibration, i.e. a fibre bundle. In particular, all the fibers of f are diffeomorphic.*

Proof. [11, Theorem 4.1] ∎

Recall 2.5.1. Given a holomorphic map $f : X \to Y$, a point $x \in X$ is a **critical point** if $\mathrm{d}f = 0$ at x, and a point $p \in S$ is a **critical value** if $p = f(x)$ for x a critical point. Here $\mathrm{d}f : \mathcal{T}_X \to \mathcal{T}_Y$ denotes the induced map between tangent spaces.

In order to apply the above in our current setting, note that by Theorem 2.1.8, the critical values of f form an analytic subset of S since, after choosing local coordinates, the condition that $\mathrm{d}f = 0$ is given by the vanishing of a holomorphic function. Because $\dim S = 1$ and not every point is critical, the critical points must form a 0-dimensional analytic subset, i.e. a discrete set.

Corollary 2.5.2. *Away from a finite set of points $\Sigma \subset S$, the map $f : X \to S$ considered in the setup gives X the structure of a smooth fibre bundle over S. That is, there exists a finite set $\Sigma \subset S$ such that the map $f : X \setminus f^{-1}(\Sigma) \to S \setminus \Sigma$ is a smooth fiber bundle.*

Proof. The map f is holomorphic and hence smooth when X, S are viewed as real manifolds. Since we explicitly assumed that it was proper and surjective, we only need to show that it is a submersion. Since $\dim S = 1$, $\mathrm{d}f$ is surjective at a point $x \in X$ iff it is nonzero there. By the remark above the claim, the set $\Sigma \subset S$ of images of points $x \in X$ such that $\mathrm{d}f_x = 0$ is discrete and closed. Since S is compact, this makes it finite. ∎

If $s \in S$ with $\mathscr{I}_s \subset \mathscr{O}_S$ its ideal sheaf, the **fibre** X_s **above** s is the curve $f^{-1}(s)$ on X with sheaf of ideals $f^*(\mathscr{I}_s)$ (i.e. $\mathscr{I}_{X_s} \simeq f^*(\mathscr{I}_s)$ so $\mathscr{O}_{X_s} = \mathscr{O}_X / f^*(\mathscr{I}_s)$). Put another way, X_s is the fibre product

$$\begin{tikzcd} X_s \arrow[r] \arrow[d] & X \arrow[d, "f"] \\ \{s\} \arrow[r] & S \end{tikzcd}$$

where the structure sheaf on $\{s\}$ is simply the residue field $\mathscr{O}_s = \mathscr{O}_{S,s}/\mathfrak{m}_s$. This fibre X_s is singular if and only if s is a critical value, so almost all fibres are smooth. Furthermore, since all smooth fibres are diffeomorphic (by Ehresmann), they all have the same genus.

We will occasionally encounter fibrations where the base space can be embedded back in the total space. To make discussing these linguistically easier when they arise, we recall the following terminology.

Recall 2.5.2. A **section** of the map $f : X \to S$ is a map $s : S \to X$ such that $f \circ s = \mathrm{Id}_S$. We may sometimes also refer to just the image $C := s(S) \subset X$ of s in X as a section of f.

Non-example. Write $X_s = \sum n_i C_i$ with each C_i irreducible. If $n = \gcd\{n_i\} > 1$, then X_s is a multiple fibre and no section $s : S \to X$ of f can exist. Indeed, if one did exist, we would have $n \mid X_s C$ where $C = s(S)$, so $X_s C > 1$. However, $X_t C = 1$ for a generic fiber, so we also get $X_s C = 1$ since all fibers are homologous, but this is a contradiction.

2.5.1 Invariants of a Fibrations

Our goal in this section is to prove some general facts about fibrations which will be useful in our later study of elliptic fibrations. Of note are Zariski's lemma and the existence of relatively minimal models. We begin with a simple observation. Given a fiber $X_s \subset X$, we can consider its fundamental class $[X_s]^* \in \mathrm{H}^2(X;\mathbb{Z})$ in cohomology, where we have $[X_s]^* = f^*[s]^*$ ($[s]^* \in \mathrm{H}^2(S;\mathbb{Z})$), so

$$X_s^2 = [X_s]^* \smile [X_s]^* = f^*[s]^* \smile f^*[s]^* = f^*([s]^* \smile [s]^*) = 0$$

since $[s]^* \smile [s]^* \in \mathrm{H}^4(C;\mathbb{Z}) = 0$. Hence, every fiber has self-intersection 0.

Remark 2.5.2. The fact that the cohomological fundamental class $[X_s]^* \in \mathrm{H}^2(X;\mathbb{Z})$ is the pullback of the cohomological fundamental class $[s]^* \in \mathrm{H}^2(S;\mathbb{Z})$ of the point s above which X_s is a fiber shows that $[X_s]^* = [X_t]^*$, as cohomology classes, for all $s,t \in S$. This is simply because any two points $s,t \in S$ are homologous and so their Poincaré duals $[s]^*, [t]^* \in \mathrm{H}^2(S;\mathbb{Z})$ are equal. In particular, this shows that $c_1(\mathscr{O}_X(X_s)) \in \mathrm{H}^2(X;\mathbb{Z})$ is independent of s, so all fibers behave the same way under the intersection pairing.

We can in fact say something slightly stronger, and in the process obtain an alternate proof that fibers have zero self-intersection.

Lemma 2.5.3. *The normal bundle $\mathscr{O}_{X_s}(X_s)$ of each fibre is trivial.*

Proof. For notational convenience, let $F = X_s$ be the fiber above s. Start with the exact sequence

$$0 \longrightarrow \mathcal{T}_{X/S} \longrightarrow \mathcal{T}_X \longrightarrow f^*\mathcal{T}_S \longrightarrow 0$$

where $\mathcal{T}_{X/S}$ is defined as the above kernel. Then, $\mathcal{T}_{X/S}|_F \cong \mathcal{T}_F$ since $f^*\mathcal{T}_S|_F = f|_F^*\mathcal{T}_s$ is trivial. Hence, letting $i : F \hookrightarrow X$ be the inclusion, the normal bundle sequence $0 \longrightarrow \mathcal{T}_F \longrightarrow \mathcal{T}_X|_F \longrightarrow \mathcal{N}_{F/X} \longrightarrow 0$ becomes

$$0 \longrightarrow i^*\mathcal{T}_{X/S} \longrightarrow i^*\mathcal{T}_X \longrightarrow i^*f^*\mathcal{T}_S \longrightarrow 0$$

where, necessarily, $i^*f^*\mathcal{T}_S \cong \mathcal{N}_{F/X}$ since they are cokernels of the same map. Now, $i^*f^*\mathcal{T}_S = f|_F^*\mathcal{T}_s$ is trivial, so we win. ■

Corollary 2.5.4. *$X_s^2 = 0$ for any fiber of X.*

Proof. $X_s^2 = \deg(\mathscr{O}_{X_s}(X_s)) = 0$. ■

Corollary 2.5.5. *The canonical/dualizing sheaf of a fiber X_s is $\omega_{X_s} \simeq \omega_X|_{X_s}$, the restriction of the canonical bundle on X.*

Proof. Depending on whether the curve X_s is smooth or not, by adjunction or by definition, we have

$$\omega_{X_s} \simeq \omega_X|_{X_s} \otimes \mathscr{O}_{X_s}(X_s) \simeq \omega_X|_{X_s}$$

since we have just shown that the normal bundle $\mathscr{O}_{X_s}(X_s)$ is trivial. ■

Remark 2.5.3. As a consequence of Theorem 2.4.6, the arithmetic genus of a fiber is independent of the point it lies over. Indeed, by that theorem, for any $s \in S$, we have

$$2g(X_s) - 2 = K_X X_s + X_s^2 = K_X X_s,$$

but $K_X X_s$ is independent of s because all fibers are homologous. Hence, $g(X_s) = \frac{1}{2}(K_X X_s + 2)$ is independent of s as well.

Example. Let $\Delta \subset \mathbb{C}$ be the complex unit disk, and let

$$X = \left\{([z_0 : z_1 : z_2], s) \in \mathbb{P}^2 \times \Delta \mid z_2 z_1^2 = (z_0 - 1)(z_0^2 - s z_2^2)\right\}$$

with projection map $\pi : X \to \Delta$. Then, π is a fibration whose generic fiber is smooth elliptic, but $X_0 = \pi^{-1}(0)$ is (isomorphic to) the nodal rational given by the projective closure of $\{y^2 = x^2(x-1)\} \subset \mathbb{C}^2$. Thus, the geometric genus of X_s is 1 if $s \neq 0$, but it is 0 if $s = 0$.

2.5.2 Intersection Theory on Fibers

We will next state and prove Zariski's lemma. The brunt of the proof relies on general facts concerning bilinear forms, so after stating the lemma, we will take a detour to prove these fact before proving Zariski itself.

Lemma 2.5.6 (**Zariski's lemma**). *Let $X_s = \sum n_i C_i$, $n_i > 0$, $C_i \subset X$ irreducible, be a fibre of the fibration $X \to S$. Then, we have*

- ***(a)*** *$C_i X_s = 0$ for all i.*
- ***(b)*** *If $D = \sum m_i C_i$, $m \in \mathbb{Z}$, then $D^2 \leq 0$.*
- ***(c)*** *$D^2 = 0$ holds in **(b)** iff $D = rX_s$, $r \in \mathbb{Q}$.*

Zariski's lemma gives quite strong restrictions on the intersection theory of the components of a fiber. In particular, it tells us that the intersection matrix $(C_iC_j)_{i,j}$ associated to a fiber is negative semi-definite with 1-dimensional kernel spanned by the fiber itself. It is these restrictions which will be the main force allowing us to classify singular fibers of elliptic surfaces later on. For now, we aim to prove Zariski, and as alluded to before, will do so by briefly studying general symmetric bilinear forms.

With that said, we now briefly turn to the theory of symmetric bilinear forms. In the preceding discussion, the fiber $X_s = \sum n_i C_i$ gives rise to a vector space with basis $\{C_i\}$ and bilinear form given by $Q(C_i, C_j) = C_iC_j$. Part **(a)** of Zariski's lemma shows that this form has nontrivial kernel (it contains X_s), and the fact that X_s is connected (i.e. $C_iC_j \neq 0$ when $i \neq j$) show that this form is "simple" in the sense that it is not the direct sum of two other bilinear forms. These facts restrict the structure of this form, and in particular, imply that it is negative semi-definite with a 1-dimensional kernel, i.e. that **(b)** and **(c)** above hold. We will show this in 3 steps.

Lemma 2.5.7. *Let Q be a symmetric, bilinear form on the (real or rational) vector space V, and let $q(v) = Q(v,v)$. Fix $a_1, \ldots, a_n \in V$ s.t. $Q(a_i, a_j) \leq 0$ for all $i \neq j$. Then,*

- ***(i)*** *If $Q(v,v) = 0$ for $v = \sum_i c_i a_i$, we have $Q(w,w) = 0$ as well with $w = \sum_i |c_i| a_i$.*
- ***(ii)*** *If Q is non-degenerate and there exists a linear $T : V \to V$ with $T(a_i) > 0$ for all i, then $\{a_1, \ldots, a_n\}$ are linearly independent.*

Proof. The relation $Q(a_i, a_j) \leq 0$ for $i \neq j$ implies that

$$q\left(\sum |c_i| a_i\right) \leq q\left(\sum c_i a_i\right) \text{ since } q\left(\sum c_i a_i\right) = \sum_{i,j} c_i c_j Q(a_i, a_j),$$

so **(i)** is clear. Now, assume q non-degenerate. Then, if we write $\sum_i c_i a_i = 0$, we get by **(i)** that $q\left(\sum_i |c_i| a_i\right) = 0 \implies \sum_i |c_i| a_i = 0$. Thus, $\sum_i |c_i| T(a_i) = 0$, but this implies that $|c_i| = 0$ since $T(a_i) > 0$. ■

Lemma 2.5.8. *Let Q be a symmetric bilinear form on V given by the matrix $\{q_{ij}\}$ such that* *(a)* $q_{ij} \le 0$ *for* $i \neq j$

(b) there is no partition $\{1, \dots, n\} = I \sqcup J$ *with* $I \neq \emptyset \neq J$ *and* $i \in I, j \in J \implies q_{ij} = 0$.

(c) $Q \ge 0$

Then, the kernel K *of* Q *has dimension 0 or 1. If* $\dim K = 1$*, then* K *is generated by a vector with strictly positive coordinates.*

Proof. Let $a_1, \dots, a_n$ be a basis for V. Suppose that $v = \sum_i c_i a_i \in K$.[4] By the previous lemma, we see that $\sum_i |c_i| a_i \in K$ as well, so $Q\left(\sum_i |c_i| a_i, a_j\right) = \sum_i q_{ji} |c_i| = 0$ for all j. Now, let $I = \{i : c_i \neq 0\}$. If $j \notin I$, we have $q_{ij} |c_i| \le 0$ for $i \in I$ and $q_{ij} |c_i| = 0$ for $i \notin I$, so $q_{ij} = 0$ for $j \notin I$ and $i \in I$. By **(b)**, this means that $I = \emptyset$ of $I = \{1, \dots, n\}$. Thus, for any $0 \neq v \in K$, it must be the case that all of V's coefficients are nonzero. Thus, we have $0 = \dim(K \cap \{x_i = 0\}) \ge \dim K - 1$ so $\dim K \le 1$, so we win. ■

Lemma 2.5.9. *Let* Q *be a symmetric bilinear form on* V *given by the matrix* $\{q_{ij}\}$ *such that*

(a) $q_{ij} \le 0$ *for* $i \neq j$.

(b) there is no partition $\{1, \dots, n\} = I \sqcup J$ *with* $I \neq \emptyset \neq J$ *and* $i \in I, j \in J \implies q_{ij} = 0$.

(c) The annihilator N *of* Q *contains some* $v = (v_1, \dots, v_n)$ *with* $v_i > 0$ *for all* i.

Then, $Q \ge 0$ *with 1-dimensional annihilator (necessarily generated by* v*).*

Proof. Let $e_1, \dots, e_n$ be the basis for V, so $e_i e_j = q_{ij}$ and $v = \sum_{i=1}^n v_i e_i$. Note that

$$0 = v \cdot e_i = v_i q_{ii} + \sum_{i \neq j} v_j q_{ij} \le v_i q_{ii} \implies q_{ii} \ge 0.$$

Now, let $f_i = v_i e_i$, so $q'_{ii} := f_i^2 = v_i^2 q_{ii} \ge 0$ and $q'_{ij} := f_i \cdot f_j = v_i v_j q_{ij} \le 0$ for $i \neq j$. Furthermore, for i fixed,

$$0 = v \cdot f_i = \left(\sum_j f_j\right) \cdot f_i = \sum_j q'_{ij}.$$

Now, consider some $x = \sum_{i=1}^n x_i f_i \in V$ with $x_i \ge 0$ for all i. Then,

$$\begin{aligned} x \cdot x &= \sum_{i,j} x_i x_j q'_{ij} \\ &= \sum_i x_i \left(\sum_j x_j q'_{ij}\right) \\ &= \sum_i x_i \left(x_i \sum_j q'_{ij} + \sum_{j \neq i} (x_j - x_i) q'_{ij}\right) \\ &= \sum_i x_i \left(\sum_{j \neq i} (x_j - x_i) q'_{ij}\right) \end{aligned}$$

[4] Since $Q \ge 0$, this is equivalent to $q(v) = 0$. This is because, for $\lambda \in \mathbb{Q}$ and $w \in V$, we have $0 \le q(v - w\lambda) = q(w) - 2\lambda Q(v, w)$ but λ can be really big, so we must have $Q(v, w) = 0$.

$$= \sum_{i,j} x_i(x_j - x_i)q'_{ij}$$

The term q'_{ij} appears twice above, once with coefficient $x_i(x_j - x_i)$ and once with coefficient $x_j(x_i - x_j)$, so

$$x \cdot x = \sum_{i<j} \left(2x_ix_j - x_i^2 - x_j^2\right) q'_{ij} = -\sum_{i<j}(x_i - x_j)^2 q'_{ij} \geq 0.$$

By the same logic, if $x_i \leq 0$ for all i, then we get that $x \cdot x \geq 0$. Finally, for general x, we can write $x = p - n$ where, after relabeling the f_i's if necessary, $p = \sum_{i=1}^m p_i f_i$ and $n = \sum_{j=m+1}^n n_j f_j$ with $p_i, n_j \geq 0$. In this case, we have $x^2 = x \cdot x = p^2 + n^2 - 2p \cdot n$, but it's clear from assumption **(a)** that $p \cdot n \leq 0$, so $x^2 \geq 0$, making Q positive semi-definite. Once we know this, it follows from the previous lemma that $\dim N = 1$ generated by v. ■

With our 3 steps complete, we end our foray into symmetric bilinear forms, and return to complex geometry. In particular, we are now ready to prove Zariski's lemma. For the reader's convenience, we restate the theorem.

Lemma 2.5.10 (**Zariski's Lemma**). *Let $X_s = \sum n_i C_i$, $n_i > 0$, $C_i \subset X$ irreducible, be a fibre of the fibration $X \to S$. Then, we have*

- ***(a)*** *$C_i X_s = 0$ for all i.*
- ***(b)*** *If $D = \sum m_i C_i$, $m \in \mathbb{Z}$, then $D^2 \leq 0$.*
- ***(c)*** *$D^2 = 0$ holds in **(b)** iff $D = rX_s$, $r \in \mathbb{Q}$.*

Proof. Part **(a)** follows from Lemma 2.5.3 since $C_i X_s = \deg(\mathscr{O}_{C_i}(X_s))$, but $\mathscr{O}_{C_i}(X_s)$ is the restriction of the trivial bundle $\mathscr{O}_{X_s}(X_s)$. Given this, the rest of the claims now follow from Lemma 2.5.9 applied to $-Q$ where Q is the bilinear form on $\bigoplus \mathbb{Q}C_i$ induced by the intersection form on $\mathrm{H}^2(X;\mathbb{Z})$. Indeed, **(b)** says exactly that $-Q$ is positive semi-definition while **(c)** says that it has a 1-dimensional annihilator. ■

It occasionally occurs that the fibration $\pi : X \to S$ will have some non-reduced fibers. In these cases, we would like to be able to reason about the fiber X_s by considering instead its reduction $(X_s)_{\mathrm{red}}$. When doing so, the following lemma showing that $(X_s)_{\mathrm{red}}$ corresponds to a torsion line bundle will come in handy.

Definition 2.5.1. A singular fiber $X_s = \sum n_i C_i$ is called a **multiple fibre** (**of multiplicity** n) if $n = \gcd\{n_i\} > 1$. In this case, $X_s = nF$ with F another effective divisor on X.

Lemma 2.5.11. *Let $S = \Delta \subset \mathbb{C}$ be the unit disk, and let $X_0 = nF$ be a multiple fiber with multiplicity n. Then, $\mathscr{O}_X(F)$ and $\mathscr{O}_F(F)$ are both torsion of order n.*

Proof. First note that X is not compact since it surjects onto the non-compact space Δ. By looking at the exponential exact sequence, we see that the Picard group $\operatorname{Pic}\Delta$ fits into an exact sequence

$$\mathrm{H}^1(\Delta, \mathscr{O}_\Delta) \longrightarrow \operatorname{Pic}\Delta \longrightarrow \mathrm{H}^2(\Delta;\mathbb{Z}) = 0,$$

but $\mathrm{H}^1(\Delta, \mathscr{O}_\Delta) = 0$ as well, so $\operatorname{Pic}\Delta = 0$. Hence, $\mathscr{O}_\Delta(0) \simeq \mathscr{O}_\Delta$ is trivial, and so is $\mathscr{O}_X(X_0) = \pi^*\mathscr{O}_\Delta(0)$.

This shows that $\mathscr{O}_X(F)$ is torsion of order dividing n as $\mathscr{O}_X(F)^{\otimes n} = \mathscr{O}_X(nF) = \mathscr{O}_X(X_0)$ is trivial, so let m be the order of $\mathscr{O}_X(F)$ in $\operatorname{Pic} X$. Then, $h^0(\mathscr{O}_X(-mF)) \geq 1$, so there is a holomorphic function on X

vanishing along F to order $m \leq n$. We will show that n is the least nonzero order to which a holomorphic function on X can vanish, and so $m = n$. Let $f \in \mathrm{H}^0(\mathscr{O}_X)$ be any holomorphic function vanishing along F. Note that, by Theorem 2.1.9, the push forward $\pi_*\mathscr{O}_X$ is a locally free sheaf on Δ of rank $h^0(X_s, \mathscr{O}_{X_s}) = 1$, i.e. a line bundle. Since $\operatorname{Pic}\Delta = 0$, this means that $\pi_*\mathscr{O}_X \simeq \mathscr{O}_\Delta$ and that the natural map $\mathscr{O}_\Delta \to \pi_*\mathscr{O}_X$ is an isomorphism. Thus, every holomorphic function on X is pulled back from one on Δ, so $f = g \circ \pi$ for some $g \in \mathrm{H}^0(\mathscr{O}_\Delta)$. Consequently, f vanishes to order $n \operatorname{ord}_0(g) \geq n$ along F. Therefore, we must have $m = n$, so $\mathscr{O}_X(F)$ is torsion of order n as claimed.

We now consider $\mathscr{O}_F(F)$. Since $\mathscr{O}_F(nF) = \mathscr{O}_F(X_0) = \mathscr{O}_X(X_0)|_F$, we see that $\mathscr{O}_F(F)$ is torsion of order dividing n. To see that its order actually is n, we will need the fact that, possibly after shrinking the base Δ, the restriction maps $\mathrm{H}^i(X, \mathbb{Z}) \to \mathrm{H}^i(F, \mathbb{Z})$ are bijective. This is a consequence of [2, Ch. I, Theorem 8.8] which says that F is a deformation retraction of some open around it. Combining this with the exponential exact sequences for X and F, we arrive at the following commutative diagram

$$\begin{tikzcd}
\mathrm{H}^1(X;\mathbb{Z}) \arrow[r] \arrow[d, "\wr"] & \mathrm{H}^1(X,\mathscr{O}_X) \arrow[r] \arrow[d] & \operatorname{Pic} X \arrow[r, "c_1"] \arrow[d] & \mathrm{H}^2(X;\mathbb{Z}) \arrow[d, "\wr"] \\
\mathrm{H}^1(F;\mathbb{Z}) \arrow[r] & \mathrm{H}^1(F,\mathscr{O}_F) \arrow[r] & \operatorname{Pic} F \arrow[r, "c_1"] & \mathrm{H}^2(F;\mathbb{Z}) \arrow[d, equal] \\
 & & & \mathbb{Z}^N
\end{tikzcd}$$

We now diagram chase. First note that $\mathscr{O}_X(F) \in \operatorname{Pic} X$ is torsion while $\mathrm{H}^2(X;\mathbb{Z})$ is torsion-free, so $c_1(\mathscr{O}_X(F)) = 0$ and hence $\mathscr{O}_X(F)$ is the image of some $\alpha \in \mathrm{H}^1(X, \mathscr{O}_X)$. Let m be the order of $\mathscr{O}_F(F)$, so $m\alpha|_F \in \ker\left(\mathrm{H}^1(F, \mathscr{O}_F) \to \operatorname{Pic} F\right)$. Hence, there's some $c_F \in \mathrm{H}^1(F;\mathbb{Z})$ mapping onto $m\alpha|_F$ which we can pull back to a $c \in \mathrm{H}^1(X;\mathbb{Z})$. Now, by [2, Ch. II, Proposition 2.1], the map $\mathrm{H}^1(F;\mathbb{Z}) \to \mathrm{H}^1(F, \mathscr{O}_F)$ is injective, so the composition $\mathrm{H}^1(X;\mathbb{Z}) \to \mathrm{H}^1(X, \mathscr{O}_X) \to \mathrm{H}^1(F, \mathscr{O}_F)$ is injective as well. Now, since $n\alpha = 0 \in \operatorname{Pic} X$, its the image of some $d \in \mathrm{H}^1(X;\mathbb{Z})$, but d and $(n/m)c$ (recall that $m \mid n$) both have the same image in $\mathrm{H}^1(F, \mathscr{O}_F)$, so $d = (n/m)c$. Hence, $m\alpha$ must be the image of c in $\mathrm{H}^1(X, \mathscr{O}_X)$. By exactness, this means that $m\alpha$ has trivial image in $\operatorname{Pic} X$, i.e. that $\mathscr{O}_X(mF)$ is trivial. Since $\mathscr{O}_X(F)$ has order n, we conclude that $m = n$. ■

Remark 2.5.4. Lemma 2.5.11 will play a key role in classifying the possible multiple fibers in elliptic surfaces.

2.5.3 Minimal Fibrations

Definition 2.5.2. Let $f : X \to S, g : Y \to S$ be two (connected) fibrations. We call them **bimeromorphically equivalent** if there exists a **bimeromorphic correspondence** between then respecting the fibrations, i.e. if there is a surface Z bitting into a commutative diagram

$$\begin{tikzcd}
 & Z \arrow[dl, dashed] \arrow[dr, dashed] & \\
X \arrow[dr, "f"] & & Y \arrow[dl, "g"'] \\
 & S &
\end{tikzcd}$$

where dashed lines are meant to indicate that the maps are (bi)meromorphic.

Definition 2.5.3. A fibration which has no (-1)-curves in any of its fibers is called **relatively minimal**.

Proposition 2.5.12. *If the genus of the general fiber is strictly positive, then* $X \xrightarrow{f} S$ *factors through a unique nonsingular surface* Y *with* $Y \to S$ *relatively minimal.*

Proof. We blow down all (-1)-curves contained in fibres X_s, and repeat this until they're all gone (this requires finitely many blow downs in each fiber). This Y is uniquely determined unless some fibre X_s contains two intersecting (-1)-curves C_1, C_2. Then, $(C_1+C_2)^2 \geq 0$ with $(C_1+C_2)^2 = 0$ only if $(C_1, C_2) = 1$. By Zariski, $(C_1+C_2)^2 \leq 0$ so indeed $C_1C_2 = 1$ and $(C_1+C_2)^2 = 0$. Again, appealing to Zariski, this further implies $(C_1 + C_2)^2 = qX_s$ for some $q \in \mathbb{Q}$, so C_1, C_2 are the only irreducible curves in X_s and actually $X_s = n(C_1 + C_2)$ for some $n \in \mathbb{Z}$. Hence, for the (connected) general fibre X_t, we have

$$K_X X_t = nK_X(C_1 + C_2) = -2n.$$

Hence, $n = 1$ and the general fibre is rational. This is because the genus formula gives $g(X_t) = 1+\frac{1}{2}X_t(K_X + X_t) = 1 + \frac{1}{2}X_tK_X$, so $X_tK_X = 2g(X_t) - 2 \geq -2$. ■

Hence, in later discussions, we may safely assume that all fibers under consideration are relatively minimal.

3 Ruled Surfaces

3.1 Ruled Surfaces and Projective Bundles

Our main objects of interest in this document will be elliptic surfaces, which are, briefly, surfaces $\pi : X \to C$ fibered over a curve such that the generic fiber is smooth of genus 1. Before studying this, it will prove instructive to look at the case of fibered surfaces whose generic fiber is smooth of genus 0 instead; these are the so-called ruled surfaces which we study in this section. The material here is not logically necessary for the later material of elliptic surfaces, but it is a good warm up and offers a view of a different set of complex geometric techniques than those we will see when studying elliptic surfaces. Briefly put, our later study of elliptic surfaces will be "local" in nature, focusing on the behavior of individual (singular) fibers. Here, we will be more concerned with the "global structure" of ruled surfaces.

Definition 3.1.1. A **ruled surface** $\pi : X \to C$ is a smooth surface X equipped with a fibration π over a smooth curve C such that the generic fiber $X_c = \pi^{-1}(c)$ is (smooth) of genus 0.

3.1.1 Ruled Surfaces are Projectivizations of Vector Bundles: First Proof

The first fundamental result is that ruled surfaces always arise as the projectivization of a vector bundle on the base curve. We will give two proofs of this fact, but will first remind the reader how to projectivize a vector bundle. Let C be a smooth curve, and let $p : E \to C$ be a (holomorphic) vector bundle of rank n. By restricting p to a trivializing open cover $\{U_\alpha\}$ of C, we see that p is determined by its set of transition functions

$$\tau_{\alpha\beta} : U_\alpha \cap U_\beta \to \mathrm{GL}_n(\mathbb{C})$$

which describe how to form E by gluing trivial bundles. Given this data, we form the projectivization $\mathbb{P}(E)$ of E by letting it be the $\mathbb{P}(\mathbb{C}^n) = \mathbb{P}^{n-1}$–bundle $\mathbb{P}(E) \to C$ whose transition functions are the reductions $\overline{\tau}_{\alpha\beta} : U_\alpha \cap U_\beta \to \mathrm{PGL}_n(\mathbb{C})$ of those of E. When $\operatorname{rank} E = 2$, $\mathbb{P}(E)$ is a $\mathbb{P}^1$ bundle over C and hence a ruled surface. The converse holds as well.

Theorem 3.1.1. *Let $\pi : X \to C$ be a ruled surface. Then, there exists a rank 2 vector bundle E on C such that $\mathbb{P}(E) \simeq X$, and the following diagram commutes*

$$\begin{tikzcd}
\mathbb{P}(E) \arrow[rr, "\sim"] \arrow[dr] & & X \arrow[dl] \\
& C &
\end{tikzcd}$$

As mentioned above, we will give two proofs of this theorem. The first will be cohomological in nature, deriving the result from something akin to the exponential exact sequence. Because this proof is potentially somewhat opaque, we will also give a second, more topological proof of this fact. The topological proof will be noticeably more involved, but will elucidate more of the structure of ruled surfaces than the algebraic one does.

Cohomological proof of Theorem 3.1.1. We will actually show the stronger claim that any $\mathbb{P}^n$-bundle is the projectivization of some rank $(n+1)$ vector bundle. Let $\mathrm{GL}_n(\mathscr{O}_C), \mathrm{PGL}_n(\mathscr{O}_C)$ denote the sheaves of holomorphic maps from C to $\mathrm{GL}_n(\mathbb{C})$ or $\mathrm{PGL}_n(\mathbb{C})$, respectively. While these sheaves are, in general, not abelian, the fact that they are sheaves of groups is alone enough to define $\check{\mathrm{H}}^1$, Čech cohomology in degree 1. Because they are only sheaves of groups, they do not have any higher cohomology and their $\check{\mathrm{H}}^1$'s are merely pointed sets, not groups. Mimicking the proof that $\mathrm{H}^1(C, \mathscr{O}_C^\times)$ classifies line bundles, one can show that $\mathrm{H}^1(C, \mathrm{GL}_n(\mathscr{O}_C))$ classifies vector bundles of rank n and $\mathrm{H}^1(C, \mathrm{PGL}_n(\mathscr{O}_C))$ classifies $\mathbb{P}^{n-1}$-bundles. The sheaves fit a sequence

$$0 \longrightarrow \mathscr{O}_C^\times \longrightarrow \mathrm{GL}_n(\mathscr{O}_C) \longrightarrow \mathrm{PGL}_n(\mathscr{O}_C) \longrightarrow 0$$

whose exactness is readily checked on stalks. This induces the exact sequence

$$\mathrm{H}^1(C, \mathscr{O}_C^\times) \longrightarrow \check{\mathrm{H}}^1(C, \mathrm{GL}_n(\mathscr{O}_C)) \longrightarrow \check{\mathrm{H}}^1(C, \mathrm{PGL}_n(\mathscr{O}_C)) \longrightarrow \mathrm{H}^2(C, \mathscr{O}_C^\times)$$

of (pointed) cohomology sets. At the same time, the usual exponential exact sequence in cohomology includes

$$0 = \mathrm{H}^2(C, \mathscr{O}_C) \longrightarrow \mathrm{H}^2(C, \mathscr{O}_C^\times) \longrightarrow \mathrm{H}^3(C, \mathbb{Z}) = 0,$$

where $\mathrm{H}^2(C, \mathscr{O}_C), \mathrm{H}^3(C, \mathbb{Z}) = 0$ because C is only 1 (complex) dimensional. By exactness, this shows that $\mathrm{H}^2(C, \mathscr{O}_C^\times) = 0$ as well. As such, the map $\mathrm{H}^1(C, \mathrm{GL}_n(\mathscr{O}_C)) \to \mathrm{H}^1(C, \mathrm{PGL}_n(\mathscr{O}_C))$, which coincides with the projectivization map $E \mapsto \mathbb{P}(E)$, is surjective. ■

This gives one proof of Theorem 3.1.1. The strategy for the other proof, which is more direct and topological, is to explicitly construct the vector bundle E. It will end up being given as the push-forward of a line bundle on the total space associated to a (holomorphic) section of the ruling. The argument will proceed in several steps, the first couple of which are aimed at showing that a section even exists.

3.1.2 Ruled Surfaces are Projectivizations of Vector Bundles: Second Proof

Let $\pi : X \to C$ be a ruled surface. We aim to construct a vector bundle E such that $X \simeq \mathbb{P}(E)$, and we claim that E is the pushforward of a section of π. Hence, we first show the existence of such a section. Before beginning, note that since any given fiber of π has arithmetic genus 0, all of the fibers of π are smooth, and so π is a topological fibration.

Remark 3.1.1. Since $\mathbb{P}^1 \to X \to C$ is a fibration, we get a long exact sequence

$$\cdots \longrightarrow \pi_{n+1}(C) \longrightarrow \pi_n(\mathbb{P}^1) \longrightarrow \pi_n(X) \longrightarrow \pi_n(C) \longrightarrow \pi_{n-1}(\mathbb{P}^1) \longrightarrow \cdots$$

in homotopy. This shows that $\pi_1(X) \simeq \pi_1(C)$, so $\mathrm{H}_1(X) \simeq \mathrm{H}_1(C) \simeq \mathbb{Z}^{2g(C)}$. By [2, Ch. IV, Thm. 3.1], a compact surface is Kähler iff its first Betti number is even. Hence, all ruled surfaces are Kähler.

Lemma 3.1.2. *There exists a topological (i.e. continuous) map $s : C \to X$ such that $\pi \circ s = \mathrm{Id}_C$.*

Proof. We will provide a rather topological proof of this fact, using that C can be given the structure of a CW-complex and then constructing this topological section $s : C \to X$ one cell at a time. For the duration of this proof, let $X \approx Y$ denote that X is homeomorphic to Y. Also, since C will be viewed as a CW-complex, let C_k denote its k-skeleton. Note that the morphism $X \xrightarrow{\pi} C$ gives X the structure of a $\mathbb{P}^1$-bundle over C, and $\mathbb{P}^1 \approx S^2$, a sphere.

Since C is a smooth curve, it is a topological surface and so can be given the structure of a 2-dimensional CW-complex with a single 0-cell $e^0 \in C$. Let $s_0 : \{e^0\} \to X$ be any map such that $\pi(s_0(e^0)) = e^0$. This gives a section for $C_0 = \{e^0\}$, C's 0-skeleton.

We now move onto C's 1-skeleton. Let $e^1 \subset C$ be any 1-cell with attaching map $\phi : \partial e^1 \to C_0$ where $\partial e^1 \approx S^0$ is the boundary of e^1. Consider the restriction $\pi|_{e^1} : X|_{e^1} \to e^1$ of π where $X|_{e^1} := \pi^{-1}(e^1)$. Since e^1 is contractible and $X|_{e^1} \to e^1$ is an S^2-bundle, there must be a commutative diagram

$$\begin{tikzcd} X|_{e^1} \arrow[rr, "\sim"] \arrow[dr, "\pi|_{e^1}"'] & & S^2 \times e^1 \arrow[dl, "\mathrm{pr}_2"] \\ & e^1 & \end{tikzcd}$$

so, after identifying $X|_{e^1} \approx S^2 \times e^1$, sections can be identified with maps to S^2 and extending the section $s_0 : C_0 \to X$ to a section $C_0 \cup \{e^1\} \to X|_{e^1}$ amounts to extending the composition $\partial e^1 \xrightarrow{\phi} C_0 \xrightarrow{s_0} S^2$ to a map $e^1 \to S^2$. This is possible if and only if $\partial e^1 \xrightarrow{\phi} C_0 \xrightarrow{s_0} S^2$ is nullhomotopic, but this is obviously the case since the image of this map is already a single point. Thus, we can extend s_0 to a section over e^1. By simultaneously carrying this process out for all 1-cells, we obtain a section $s_1 : C_1 \to X$ of π over C's 1-skeleton.

Finally, we need to extend s_1 to a section over C's 2-skeleton (i.e. over all of C). As before, let $e^2 \subset C$ be any 2-cell with attaching map $\phi : \partial e^2 \to C_1$. By considering the restriction $\pi : |_{e^2} : X|_{e^2} \to e^2$ of π and again noting that e^2 is contractible, we can once again identify $X|_{e^2} \approx S^2 \times e^2$ so sections of π over e^2 again correspond to maps $e^2 \to S^2$. In particular, extending the section $s_1 : C_1 \to X$ to one over $C_1 \cup \{e^2\}$ amounts to extending the composition $\partial e^2 \xrightarrow{\phi} C_1 \xrightarrow{s_1} S^2$ to a map $e^2 \to S^2$, and this is again possible if and only if $\partial e^2 \xrightarrow{\phi} C_1 \xrightarrow{s_1} S^2$ is nullhomotopic. The image of this map is no longer a single point, but $\partial e^2 \approx S^1$, so this composition represents an element in $\pi_1(S^2)$, the fundamental group of S^2. Since S^2 is simply connected ($\pi_1(S^2) = 0$), this composition is necessarily nullhomotopic, and so we may extend s_1 to a section over e^2. Performing this simultaneously for all 2-cells of C, we obtain our desired section $s : C = C_2 \to X$ of π. ■

Technical Aside 3.1.1. Readers familiar with homotopy theory may find the following observation interesting. The main point of the proof of the above lemma is the following: since C is a CW-complex, constructing a section $s : C \to X$ only involves taking a map $S^k \to \mathbb{P}^1$ – from a sphere to a fiber of $\pi : X \to C$ – and extending it to a map $e^{k+1} \to \mathbb{P}^1$ from the $(k+1)$-cell the original S^k bounded. Furthermore, since C is 2-dimensional as a CW-complex, every sphere encountered had dimension strictly less than 2, so we can always form the extension as $\mathbb{P}^1$ is path-connected and simply connected (you can form the extension exactly when the original map $S^k \to \mathbb{P}^1$ is nullhomotopic). More generally, if $p : E \to B$ is any topological fibration such that B is an n-dimensional CW-complex, and the (homotopy type of the) fiber F is $(n-1)$-connected (i.e. $\pi_i(F) = 0$ for all $i < n$), then one can construct a section $s : B \to E$ of p. We will not need this fact beyond the single case handled explicitly in Lemma 3.1.2.

Corollary 3.1.3. *There exists a holomorphic line bundle $\mathscr{L}$ on X such that $\mathscr{L}|_F \simeq \mathscr{O}_F(1)$ for any fiber $F \simeq \mathbb{P}^1$ of π.*

Proof. Part of the exponential exact sequence in cohomology looks like

$$\mathrm{H}^1(X, \mathscr{O}_X^\times) \xrightarrow{c_1} \mathrm{H}^2(X; \mathbb{Z}) \longrightarrow \mathrm{H}^2(X; \mathscr{O}_X).$$

We claim that $\mathrm{H}^2(X; \mathscr{O}_X) = 0$, so the map c_1 above is surjective. To see this, let K_X be a canonical divisor, and let $F \simeq \mathbb{P}^1$ be any fiber. The adjunction formula tells us that

$$\mathscr{O}_F(-2) = \omega_F \simeq \omega_X \otimes \mathscr{O}_X(F)|_F = \mathscr{O}_X(K_X + F)|_F,$$

so $-2 = (K_X + F)F = K_X F$. Suppose $\omega_X = \mathscr{O}_X(K_X)$ has a nonzero holomorphic section σ, and write $(\sigma) = D + nF$ where $\operatorname{supp} D$ does not contain F as a subspace and D is effective. Then, $-2 = FK_X = F(D + nF) = FD$ contradicts effectivity of D, so σ cannot exist. As a consequence $h^0(\omega_X) = 0$. By Serre duality, we then get $h^2(\mathscr{O}_X) = h^0(\omega_X) = 0$ as well, so the first Chern map $c_1 : \operatorname{Pic} X \to \mathrm{H}^2(X;\mathbb{Z})$ is surjective. The preceding lemma produces a topological section $S \subset X$ of π, so we can take $\mathscr{L}$ to be any holomorphic line bundle with $c_1(\mathscr{L})$ equal to S's fundamental class $[S] \in \mathrm{H}^2(X;\mathbb{Z})$ in cohomology. ■

With this, we have almost everything we need in order to show that every ruled surface has a holomorphic section. Since we have produced a line bundle $\mathscr{L}$ such that $\deg \mathscr{L}|_F = 1$ for any fiber F, one wishes to simply take a divisor $D \in \operatorname{Div} X$ such that $\mathscr{O}_X(D) \simeq \mathscr{L}$ and conclude that D must be a section since $DF = 1$ for all fibers F. However, there is an issue with this. We have no guarantee that this divisor D is effective; equivalently, we do not know if $\mathscr{L}$ has any holomorphic sections. To overcome this, we will show that if we twist $\mathscr{L}$ by a sufficiently high power of $\mathscr{O}_X(F)$, then we do get a line bundle coming from a (holomorphic) section of π. This technique of producing a section of a line bundle by twisting it by high powers of a simple divisor will be used repeatedly in this chapter.

Lemma 3.1.4. *Let $D \in \operatorname{Div}(X)$ be any divisor such that $DF = 1$ for a fiber F. Then, $D \sim S + nF$ for some section $S \subset X$ and integer n where $\sim$ denotes linear equivalence.*

Proof. For each $n \in \mathbb{Z}$, let $D_n = D + nF$. As before, let K_X be a canonical divisor, and note that, since F is a fiber, we have $F^2 = 0$ and $K_X F = -2$, the latter a consequence of adjunction. We then get the equalities

$$D_n^2 = D^2 + 2n \qquad D_n F = 1 \qquad D_n K_X = DK_X - 2n$$

Hence, by Riemann-Roch for surfaces,

$$h^0(D_n) + h^0(K_X - D_n) \geq \chi(\mathscr{O}_X(D_n)) = \frac{1}{2} D_n(D_n - K_X) + \chi(\mathscr{O}_X) = \chi(\mathscr{O}_X(D)) + 2n.$$

Furthermore, $(K_X - D_n)F = -3 < 0$ so $h^0(K_X - D_n) = 0$ as no effective divisor can have negative intersection with a fiber. Thus, for n sufficiently large, we have $h^0(D_n) > 0$, i.e., there exists some effective divisor E with $E \sim D_n$. In particular, $EF = 1$ which, because E is effective, forces E to be of the form $E = S + mF$ for some section S and integer $n \in \mathbb{Z}$. Hence, $D \sim S + (m - n)F$ as desired. ■

Corollary 3.1.5. *$\pi : X \to C$ has a holomorphic section.*

Proof. Apply the theorem above to the line bundle $\mathscr{L}$ guaranteed by Corollary 3.1.3. ■

This gives the existence of a holomorphic section $S \subset X$. Note that $\mathscr{O}_X(S)$ is flat over C in the sense that $\mathscr{O}_X(S)_x \simeq \mathscr{O}_{X,x}$ is a flat $\mathscr{O}_{C,f(x)}$-module for all $x \in X$. This, combined with the fact that $X_c \simeq \mathbb{P}^1$ for all $c \in C$, shows that the hypotheses of Theorem 2.1.9 apply to π, and so we may conclude that $\mathscr{E} := \pi_* \mathscr{O}_X(S)$ is a locally free sheaf on C. By the same theorem, the rank of $\mathscr{E}$ is

$$\operatorname{rank} \mathscr{E} = h^0(F, \mathscr{O}_X(S)|_F) = h^0(\mathbb{P}^1, \mathscr{O}_{\mathbb{P}^1}(1)) = \chi(\mathscr{O}_{\mathbb{P}^1}(1)) = 1 + \chi(\mathscr{O}_{\mathbb{P}^1}) = 2$$

where F above is any fiber, the last equality is Riemann-Roch for curves, and the second to last equality is a consequence of Serre duality since $h^1(\mathscr{O}_{\mathbb{P}^1}(1)) = h^0(\mathscr{O}_{\mathbb{P}^1}(-3))$. Let $p : E \to C$ be the vector space associated to the sheaf $\mathscr{E}$. We claim that $\mathbb{P}(E) \simeq X$ as complex spaces over C.

To show this it will be good to know a little about the structure of $\mathbb{P}(E)$.

Definition 3.1.2. Let $q : E \to C$ be a rank 2 vector bundle on a smooth curve C, and let $p : \mathbb{P}(E) \to C$ be the corresponding natural map. Consider the line bundle p^*E on $\mathbb{P}(E)$. It has a sub-line bundle, called the **tautological subbundle**, $\mathscr{O}_{\mathbb{P}(E)}(-1)$ given by

$$\mathscr{O}_{\mathbb{P}(E)}(-1) = \{(e, \ell) \in E \times \mathbb{P}(E) : q(e) = p(\ell) \text{ and } e \in \ell\} \subset p^*E,$$

which fits into the diagram

$$\begin{tikzcd} \mathscr{O}_{\mathbb{P}(E)}(-1) \arrow[r, phantom, "\subset"] & p^*E \arrow[r] \arrow[d] & E \arrow[d, "q"] \\ & \mathbb{P}(E) \arrow[r, "p"] & C \end{tikzcd}$$

Dually, the **tautological quotient bundle** $\mathscr{O}_{\mathbb{P}(E)}(1)$ is defined via the short exact sequence

$$0 \longrightarrow \mathscr{O}_{\mathbb{P}(E)}(-1) \longrightarrow p^*E \longrightarrow \mathscr{O}_{\mathbb{P}(E)}(1) \longrightarrow 0.$$

Note that $\operatorname{rank} \mathscr{O}_{\mathbb{P}(E)}(1) = \operatorname{rank} p^*E - \operatorname{rank} \mathscr{O}_{\mathbb{P}(E)}(-1) = 1$.

Remark 3.1.2. The above construction works for vector bundles of arbitrary rank on any base space. In the case that $E = \mathbb{C}^{n+1} \times \{*\}$ is the rank $(n+1)$ vector bundle on the 1-point space, we naturally have $\mathbb{P}(E) \simeq \mathbb{P}^n$ with $\mathscr{O}_{\mathbb{P}(E)}(-1)$ corresponding to $\mathscr{O}_{\mathbb{P}^n}(-1)$. Furthermore, in this case, also $\mathscr{O}_{\mathbb{P}(E)}(1)$ corresponds to $\mathscr{O}_{\mathbb{P}^n}(1)$. To see this latter point, note that, in this case, $\mathscr{O}_{\mathbb{P}(E)}(1)$ is defined by

$$0 \longrightarrow \mathscr{O}_{\mathbb{P}(E)}(-1) \longrightarrow \mathbb{C}^{n+1} \times \mathbb{P}^n \longrightarrow \mathscr{O}_{\mathbb{P}(E)}(1) \longrightarrow 0,$$

so taking determinants shows that $\mathscr{O}_{\mathbb{P}(E)}(1) \otimes \mathscr{O}_{\mathbb{P}(E)}(-1) \simeq \mathscr{O}_{\mathbb{P}(E)}$, so $\mathscr{O}_{\mathbb{P}(E)}(1)$ and $\mathscr{O}_{\mathbb{P}(E)}(-1)$ are dual to each other. Since $\mathscr{O}_{\mathbb{P}(E)}(-1)$ corresponds to $\mathscr{O}_{\mathbb{P}^n}(-1)$, we conclude that $\mathscr{O}_{\mathbb{P}(E)}(1)$ corresponds the dual bundle $\mathscr{O}_{\mathbb{P}^n}(1)$.

The utility of these tautological (line) bundles is that the role they play in defining maps to $\mathbb{P}(E)$. The space $\mathbb{P}(E)$ satisfies two universal properties. According to one, given a map $f : X \to C$, factoring it through $\mathbb{P}(E)$, i.e. writing it as a composition $X \to \mathbb{P}(E) \xrightarrow{p} C$, is equivalent to specifying a line subbundle of f^*E. According to the other, such a factorization corresponds to a quotient bundle of f^*E (whose rank is one less than $\operatorname{rank} E$). In our application of interest – constructing an isomorphism $X \xrightarrow{\sim} \mathbb{P}(\pi_*\mathscr{O}_X(S))$ for a ruled surface $\pi : X \to C$ with section S – we will be able to find a natural quotient bundle of $\pi^*\pi_*\mathscr{O}_X(S)$, and so we only state and prove the universal property which is given in terms of quotients.

Theorem 3.1.6 (**Universal Property of Projectivized Vector Bundles**). *Let E be a rank 2 vector bundle on a curve C, and let $\pi : X \to C$ be a complex space over C. A morphism $f : X \to \mathbb{P}(E)$, fitting into the below commutative diagram, is the same thing as a quotient line bundle of π^*E.*

$$\begin{tikzcd} X \arrow[rr, "f"] \arrow[dr, "\pi"'] & & \mathbb{P}(E) \arrow[dl, "p"] \\ & C & \end{tikzcd}$$

Furthermore, if $\mathscr{Q}$ is the quotient line bundle of π^*E giving rise to f, then $f^*\mathscr{O}_{\mathbb{P}(E}(1) \simeq \mathscr{Q}$.

Proof. First consider a morphism $f : X \to \mathbb{P}(E)$. From this, one obtains the line bundle $f^*\mathscr{O}_{\mathbb{P}(E)}(1)$. Because f^* is right exact (i.e. preserves surjections), we can pull back the natural map $p^*E \twoheadrightarrow \mathscr{O}_{\mathbb{P}(E)}(1)$ along f to get a map $\pi^* E = f^* p^* E \twoheadrightarrow f^* \mathscr{O}_{\mathbb{P}(E)}(1)$ which shows that $f^* \mathscr{O}_{\mathbb{P}(E)}(1)$ is indeed a line quotient bundle of $\pi^* E$.

In the other direction, given a surjection $\pi^* E \twoheadrightarrow \mathscr{L}$ with $\mathscr{L} \in \operatorname{Pic} X$ a line bundle, we get the map $f : X \to \mathbb{P}(E)$ given by

$$f : x \mapsto \ker\left(E(\pi(x)) \twoheadrightarrow \mathscr{L}(x)\right)$$

where $\mathscr{L}(x)$ denotes the fiber above x (and similarly for $(\pi^* E)(x) = E(\pi(x))$), and $\mathrm{pr}_2 : \pi^* E \to E$ is the natural map. This map lands in $\mathbb{P}(E)$ because $\ker(\pi^* E \twoheadrightarrow \mathscr{L})$ is a line bundle, so $\mathrm{pr}_2 \ker((\pi^* E)(x) \to \mathscr{L}(x))$ is a 1-dimensional linear subspace of $E(x)$.

All that remains is to show that these two constructions are inverse to each other. To do this, we will show that if $\mathscr{L}$ is a quotient of $\pi^* E$ giving rise to the morphism $f : X \to \mathbb{P}(E)$, then $f^* \mathscr{O}_{\mathbb{P}(E)}(1) \simeq \mathscr{L}$. Indeed, we have a homomorphism of exact sequences (recall $f^* p^* E = \pi^* E$)

$$\begin{tikzcd}
 & f^* \mathscr{O}_{\mathbb{P}(E)}(-1) \arrow[r] \arrow[d, "\alpha"] & \pi^* E \arrow[r] \arrow[d, equal] & f^* \mathscr{O}_{\mathbb{P}(E)}(1) \arrow[r] \arrow[d, dashed, "\gamma"] & 0 \\
0 \arrow[r] & \mathscr{K} \arrow[r] & \pi^* E \arrow[r] & \mathscr{L} \arrow[r] & 0
\end{tikzcd}$$

where $f^* \mathscr{O}_{\mathbb{P}(E)}(-1) = \{(x, e, \ell) \in X \times E \times \mathbb{P}(E) : f(x) = \ell \text{ and } e \in \ell\}$ and the map α exists because the map $f^* \mathscr{O}_{\mathbb{P}(E)}(-1) \to \pi^* E$ given by $(x, e, \ell) \mapsto (x, e)$ lands in $\ker(\pi^* E \to \mathscr{L})$ since, by definition of f, we have

$$(x, e, \ell) \in f^* \mathscr{O}_{\mathbb{P}(E)}(-1) \implies e \in \ell = f(x) = \ker\left(E(\pi(x)) \twoheadrightarrow \mathscr{L}(x)\right),$$

and this implies that $(x, e) \in \ker((\pi^* E)(x) \to \mathscr{L}(x))$. Once we know α exists, we get γ by a simple diagram chase. Then, the snake lemma gives us an exact sequence

$$\ker \alpha \longrightarrow 0 \longrightarrow \ker \gamma \longrightarrow \operatorname{coker} \alpha \longrightarrow 0 \longrightarrow \operatorname{coker} \gamma \longrightarrow 0.$$

This immediately shows that $\operatorname{coker} \gamma = 0$. Furthermore, α is surjective by definition, so $\operatorname{coker} \alpha = 0$ which gives $\ker \gamma = 0$ as well. Thus, γ is an isomorphism.

Finally, start with a morphism $f : X \longrightarrow \mathbb{P}(E)$ and let $\mathscr{L} = f^* \mathscr{O}_{\mathbb{P}(E)}(1)$. Then, $\mathscr{L}$ gives rise to a morphism $f_{\mathscr{L}} : X \longrightarrow \mathbb{P}(E)$ and, by the previous paragraph, $f_{\mathscr{L}}^* \mathscr{O}_{\mathbb{P}(E)}(1) \simeq \mathscr{L} \simeq f^* \mathscr{O}_{\mathbb{P}(E)}(1)$. From this, we may conclude that $f = f_{\mathscr{L}}$. ■

Corollary 3.1.7. *Let E be a rank 2 vector bundle on a curve C. A section $s : C \to \mathbb{P}(E)$ of $\mathbb{P}(E) \to C$ is the same thing as a quotient line bundle of E.*

With our foray into projectivized vector bundles complete, we return to our earlier goal of showing that every ruled surface is such a space. Let $\pi : X \to C$ be a ruled surface with holomorphic section $S \subset X$. Recall that $\mathscr{E} := \pi_* \mathscr{O}_X(S)$ is a rank 2 locally free sheaf on C, and that we have claimed that $\mathbb{P}(E) \simeq X$ where E is the vector bundle corresponding to $\mathscr{E}$. Note that we have a natural morphism $\pi^* \mathscr{E} = \pi^* \pi_* \mathscr{O}_X(S) \to \mathscr{O}_X(S)$ which we claim is surjective (and hence gives a map $X \to \mathbb{P}(E)$). We can check surjectivity on the fibers $(\pi^* \pi_* \mathscr{O}_X(S))(x) = (\pi_* \mathscr{O}_X(S))(\pi(x)) \to (\mathscr{O}_X(S))(x)$ where, by Theorem 2.1.9, we have

$$\begin{tikzcd}
(\pi_* \mathscr{O}_X(S))(\pi(x)) \arrow[r, "\sim"] \arrow[d] & \mathrm{H}^0(X_{\pi(x)}, \mathscr{O}_X(S)|_{X_{\pi(x)}}) \arrow[r, "\sim"] \arrow[d] & \mathrm{H}^0(\mathbb{P}^1, \mathscr{O}_{\mathbb{P}^1}(1)) \arrow[d] \\
(\mathscr{O}_X(S))(x) \arrow[r, "\sim"] & \mathrm{H}^0(x, \mathscr{O}_X(S)|_x) \arrow[r, "\sim"] & \mathrm{H}^0(x, \underline{\mathbb{C}}_x)
\end{tikzcd}$$

from which we see that $\pi^*\pi_*\mathscr{O}_X(S) \to \mathscr{O}_X(S)$ corresponds, on fibers, to the above restriction map which is certainly surjective. Hence, it gives rise to a morphism $f : X \to \mathbb{P}(E)$, commuting with their projections onto C, which we claim is an isomorphism. Again, it suffices to check this on each fiber, so let $p : \mathbb{P}(E) \to C$ be the projection map, and fix any $c \in C$. Note that, by Remark 3.1.2, $\mathscr{O}_{\mathbb{P}(E)}(1)|_{p^{-1}(c)}$ corresponds to $\mathscr{O}_{\mathbb{P}^1}(1)$ under any identification $p^{-1}(c) \simeq \mathbb{P}^1$, and, since S is a section, $\mathscr{O}_X(S)|_{\pi^{-1}(c)}$ also corresponds to $\mathscr{O}_{\mathbb{P}^1}(1)$ under any identification $\pi^{-1}(c) \simeq \mathbb{P}^1$. By Theorem 3.1.6, we have $f^*\mathscr{O}_{\mathbb{P}(E)}(1) \simeq \mathscr{O}_X(S)$, so restricting to the fibers over c, $f^*\mathscr{O}_{\mathbb{P}(E)}(1)|_{p^{-1}(c)} \simeq \mathscr{O}_X(S)|_{\pi^{-1}(c)}$, i.e. f carries $\mathscr{O}_{\mathbb{P}^1}(1)$ to $\mathscr{O}_{\mathbb{P}^1}(1)$ after identifying the fibers with $\mathbb{P}^1$. This says exactly that f is a degree one map $\mathbb{P}^1 \to \mathbb{P}^1$, so f is an isomorphism fiberwise and hence one globally. This completes our second argument that every ruled surface is of the form $\mathbb{P}(E)$ for some rank 2 vector bundle on C.

3.2 The Geometry of General Ruled Surfaces

Setup. Let C be a smooth curve. Fix a rank 2 vector bundle E over C, let $X = \mathbb{P}(E)$, and let $\pi : X \to C$ be the natural projection. Furthermore, let $\mathscr{O}_X(1)$ denote the tautological quotient bundle and let $H \in \operatorname{Div} X$ be a divisor with $\mathscr{O}_X(H) \simeq \mathscr{O}_X(1)$.

Goal. The goal of this section is to calculate many of the fundamental invariants of X. In particular, we aim to determine its Hodge numbers, the structure of its Picard group, and a representative for its canonical divisor class.

We begin by recalling the various details of the geometry of X described in the previous subsection.

Recall 3.2.1.

(1) By Remark 3.1.1, X is Kähler with first Betti number $b_1(X) = 2g(C)$ twice the genus of the base curve. Hence, $h^{0,1}(X) = g(C) = h^{1,0}(X)$.

(2) In the proof of Corollary 3.1.3, we showed that $h^{0,2}(X) = 0 = h^{2,0}(X)$.

(3) Combining **(1)** and **(2)** above, we get that the Hodge diamond for X is

$$\begin{array}{ccccc} & & 1 & & \\ & g(C) & & g(C) & \\ 0 & & b_2(X) & & 0 \\ & g(C) & & g(C) & \\ & & 1 & & \end{array}$$

We determine $b_2(X)$ later in this section.

(4) We used adjunction in our proof of Corollary 3.1.3 to show that $K_X F = -2$ for any fiber F and canonical divisor K_X.

(5) The definition of the tautological quotient bundle given in Definition 3.1.2 entails that $HF = 1$ for any fiber F. Indeed, by definition, the restriction $\mathscr{O}_X(1)|_F$ coincides with the construction of the tautological quotient bundle arising from the trivial fibration $\mathbb{C}^2 \times \{*\} \to \{*\}$ which, as we saw in Remark 3.1.2, is simply $\mathscr{O}_{\mathbb{P}^1}(1)$.

(6) Combining **(5)** above with Lemma 3.1.4 shows that $H \sim S + nF$ for some section S of $\pi : X \to C$ and some $n \in \mathbb{Z}$.

It is straightforward from all of this to determine the structure of the Picard group of X. Much like when we calculated a canonical divisor on blowups, having a description of $\operatorname{Pic} X$ will be tremendously useful in determining a divisor for ω_X.

Theorem 3.2.1. *Let $F \in \operatorname{Div} X$ be any fiber. Let $h = c_1(\mathscr{O}_X(1))$ and $f = c_1(\mathscr{O}_X(F)) \in \mathrm{H}^2(X;\mathbb{Z})$. Then,*

(1) $\operatorname{Pic} X \simeq \mathbb{Z}[H] \oplus \pi^* \operatorname{Pic} C$.

(2) $\mathrm{H}^2(X;\mathbb{Z}) \simeq \mathbb{Z}h \oplus \mathbb{Z}f$.

Proof.

(1) Let $D' \in \operatorname{Div} X$ be any divisor, and let $D = D' - mH$ where $m = D'F \in \mathbb{Z}$. We claim that D is the pullback of a divisor on C. To show this, we will twist D by fibers until it becomes (linearly equivalent to something) effective and the claim becomes clear. With that said, let $D_n = D + nF$ for any $n \in \mathbb{Z}$. Then, recalling that $F^2 = 0$, $HF = 1$, and $K_X F = -2$,

$$D_n^2 = D^2 + 2n \qquad D_n F = 0 \qquad D_n K_X = DK_X - 2n$$

where $K_X \in \operatorname{Div} X$ is a (fixed) canonical divisor. Hence, Riemann-Roch for surfaces gives

$$h^0(D_n) + h^0(K_X - D_n) \geq \frac{1}{2} D_n(D_n - K) + \chi(\mathscr{O}_X) = \frac{1}{2}(D^2 - DK) + \chi(\mathscr{O}_X) + n = \chi(\mathscr{O}_X(D)) + n.$$

At the same time, $(K_X - D_n)F = -2 < 0$, so $K_X - D_n$ cannot be linearly equivalent to any effective divisor, i.e. $h^0(K_x - F_n) = 0$. This means that $h^0(D_n) \geq \chi(\mathscr{O}_X(D)) + n$ is strictly positive for n sufficiently large. Hence, for a fixed large n, we may write $D_n \sim E$ where E is an effective divisor. Since $EF = D_n F = 0$, we must have that E is a sum of fibers of π (any non-fiber irreducible component in E would intersect F with positive multiplicity), but this exactly says that E is the pullback of a divisor on C. Thus the same is true for $D_n \sim E$ and so also for $D = D_n - nF$. Hence, $D' = D + mH \in \pi^* \operatorname{Pic} C \oplus \mathbb{Z}H$ as desired. Since the latter is a subgroup of $\operatorname{Pic} X$, this gives $\operatorname{Pic} X \simeq \mathbb{Z}H \oplus \pi^* \operatorname{Pic} X$.

(2) The exponential exact sequence gives

$$\operatorname{Pic} X \xrightarrow{c_1} \mathrm{H}^2(X;\mathbb{Z}) \longrightarrow 0,$$

so $\mathrm{H}^2(X;\mathbb{Z})$ is a quotient of $\operatorname{Pic} X$. Since any two points of C have the same cohomology class in $\mathrm{H}^2(C;\mathbb{Z})$, we see by **(1)** that $\mathrm{H}^2(X;\mathbb{Z})$ is generated by h, f. To finish, we only need to show that these cohomology classes are linearly independent. Pick $a, b \in \mathbb{Z}$ such that $ah + bf = 0$. Multiplying both sides by f, we get that $a = 0$, so $bf = 0$. Now, multiplying by h, we see that $b = 0$ as well, so h, f are indeed linearly independent. ■

Corollary 3.2.2. *The Hodge diamond of X is*

$$\begin{array}{ccccc} & & 1 & & \\ & g(C) & & g(C) & \\ 0 & & 2 & & 0 \\ & g(C) & & g(C) & \\ & & 1 & & \end{array}$$

and its topological Euler characteristic is $\chi_{\mathrm{top}}(X) = 4 - 4g(C) = 4(1 - g(C))$.

Before we can use this to actually calculate K_X, we will need a couple lemmas about vector bundles on curves. By Theorem 3.2.1 above, we know $K_X \sim aH + \pi^* D$ for some $a \in \mathbb{Z}$ and $D \in \mathrm{Div}(C)$. We will determine a and D by applying adjunction to a fiber of X and a section of X. In the process of doing this, we will want to know $H^2 \in \mathbb{Z}$ which will in turn be calculated using the exact sequence

$$0 \longrightarrow \mathscr{O}_X(-1) \longrightarrow \pi^* E \longrightarrow \mathscr{O}_X(1) \longrightarrow 0.$$

However, performing this calculation will require the below lemmas on rank 2 vector bundles on curves.

Lemma 3.2.3. *Let $\mathscr{E}$ be a vector bundle on a smooth curve C. Then, there exists a divisor $D \in \mathrm{Div}\, C$ such that $\mathscr{E}(D) := \mathscr{E} \otimes_{\mathscr{O}_C} \mathscr{O}_C(D)$ has a holomorphic section. In fact, we can take D to be of the form $D = np$ for some $n \in \mathbb{Z}_{\geq 0}$ and $p \in C$.*

Proof. Fix any point $p \in C$, and note that we have an exact sequence $0 \to \mathscr{O}_C(-p) \longrightarrow \mathscr{O}_C \longrightarrow \mathbb{C}_p \longrightarrow 0$ where $\mathbb{C}_p$ is the skyscraper sheaf supported at p (equivalently, $\mathbb{C}_p = \mathscr{O}_p$ is the structure sheaf of the point p). Tensoring this exact sequence with $\mathscr{E}(p)$ – and noting that $\mathbb{C}_p \otimes_{\mathscr{O}_C} \mathscr{E}(p)$ is a rank $r := \mathrm{rank}\, E$ vector bundle on a point and so trivial – we get the sequence

$$0 \longrightarrow \mathscr{E} \longrightarrow \mathscr{E}(p) \longrightarrow \mathbb{C}_p^{\oplus r} \longrightarrow 0.$$

Hence, $h^0(\mathscr{E}(p)) - h^1(\mathscr{E}(p)) = \chi(\mathscr{E}(p)) = \chi(\mathscr{E}) + \chi(\mathbb{C}_p^{\oplus r}) = \chi(\mathscr{E}) + r$ where $h^1(\mathbb{C}_p^{\oplus r}) = 0$ either because it is supported on a 0-dimensional subspace or because it is flasque. As such, an easy induction argument shows that

$$h^0(\mathscr{E}(np)) \geq h^0(\mathscr{E}(nP)) - h^1(\mathscr{E}(np)) = \chi(\mathscr{E}(np)) = \chi(\mathscr{E}) + nr$$

for every $n \in \mathbb{Z}_{\geq 0}$. Hence, the claim holds when we take $D = np$ for n large enough. ■

Corollary 3.2.4. *Let $\mathscr{E}$ be a rank 2 vector bundle on a smooth curve C. Then, there exists line bundle $\mathscr{L}, \mathscr{M}$ on C fitting into a short exact sequence*

$$0 \longrightarrow \mathscr{L} \longrightarrow \mathscr{E} \longrightarrow \mathscr{M} \longrightarrow 0,$$

i.e. $\mathscr{E}$ is an extension of $\mathscr{M}$ by $\mathscr{L}$.

Proof. By the previous lemma, we can find some divisor $D \in \mathrm{Div}\, C$ such that $h^0(\mathscr{E}(D)) > 0$. Fix a nontrivial holomorphic section $s \in \mathrm{H}^0(\mathscr{E}(D))$. Then, s defines a map $\mathscr{O}_C \to \mathscr{E}(D)$ which we can think of equivalently as a map $\mathscr{E}^\vee(-D) \to \mathscr{O}_C$ from $\mathscr{E}(D)$'s dual bundle. The image of this map is a torsion-free submodule of $\mathscr{O}_C$, i.e. an ideal, so of the form $\mathscr{O}_C(-E)$ for a divisor $E \in \mathrm{Div}\, C$. Let $\mathscr{L} = \ker(\mathscr{E}^\vee(-D) \to \mathscr{O}_C(-E))$, so we have an exact sequence $0 \to \mathscr{L} \to \mathscr{E}^\vee(-D) \to \mathscr{O}_C(-E) \to 0$. Dualizing this sequence and then tensoring with $\mathscr{O}_C(-D)$ gives

$$0 \longrightarrow \mathscr{O}_C(E - D) \longrightarrow \mathscr{E} \longrightarrow \mathscr{L}^\vee(-D) \longrightarrow 0,$$

so the claim holds. ■

Corollary 3.2.5 (Riemann-Roch for Rank 2 Vector Bundles on Curves). *Let $\mathscr{E}$ be a rank 2 vector bundle on a smooth curve C. Note that $\det \mathscr{E} = \bigwedge^2 \mathscr{E}$ is a line bundle, and define the* ***degree*** *of $\mathscr{E}$ to be*

$\deg(\mathscr{E}) := \deg(\det \mathscr{E})$. *Then,*

$$\chi(\mathscr{E}) = \deg(\mathscr{E}) + 2\chi(\mathscr{O}_C).$$

Proof. Fix line bundles $\mathscr{L}, \mathscr{M} \in \operatorname{Pic} C$ such that we have an exact sequence

$$0 \longrightarrow \mathscr{L} \longrightarrow \mathscr{E} \longrightarrow \mathscr{M} \longrightarrow 0.$$

Hence, $\det \mathscr{E} \simeq \mathscr{M} \otimes \mathscr{L}$ so $\deg(\mathscr{E}) = \deg \mathscr{L} + \deg \mathscr{M}$. Furthermore, applying Riemann-Roch for (line bundles on) curves, we have

$$\chi(\mathscr{E}) = \chi(\mathscr{L}) + \chi(\mathscr{M}) = (\deg \mathscr{L} + \chi(\mathscr{O}_C)) + (\deg \mathscr{M} + \chi(\mathscr{O}_C)) = \deg \mathscr{E} + 2\chi(\mathscr{O}_C).$$

Hence, the claim holds. ■

We may now return to our goal of understanding the geometry of the ruled surface $X = \mathbb{P}(E) \xrightarrow{\pi} C$ where E is the vector bundle corresponding to a locally free sheaf $\mathscr{E}$ of rank 2 on C. Our only remaining task is to determine $\omega_X \in \operatorname{Pic} X$. Recall that (Theorem 3.2.1), we can write $\omega X \simeq \mathscr{O}_X(1)^{\otimes a} \otimes \pi^* \mathscr{O}_C(D)$ for some $a \in \mathbb{Z}$ and $D \in \operatorname{Pic} C$, so our only job is to determine a and D. To do so, we will first make one simplifying remark.

Remark 3.2.1. In Section 3.1.2, we showed that there exists a section $S \subset X$ of $X \to C$ such that $X \simeq \mathbb{P}(\pi_* \mathscr{O}_X(S))$. Hence, we may safely assume that $\mathscr{E}$ in the preceding paragraph is $\mathscr{E} = \pi_* \mathscr{O}_X(S)$. This has the benefit that we know from the discussion at the end of Section 3.1.2 that $\mathscr{O}_X(1) \simeq \mathscr{O}_X(S)$. Furthermore, with this choice of $\mathscr{E}$, $\mathrm{H}^0(C, \mathscr{E}) = \mathrm{H}^0(X, \mathscr{O}_X(S)) \neq 0$, so contains a section $\sigma \in \mathrm{H}^0(C, \mathscr{E})$ corresponding to canonical section of $1_S \in \mathrm{H}^0(X, \mathscr{O}_X(S))$ (constructed in Remark 2.2.4). Thinking of σ as a holomorphic map $C \to E$, we can compose with the natural map $E \to \mathbb{P}(E) = X$ to see that σ induces a section $C \to X$ of $\pi : X \to C$. The image of this section is exactly S. At the same time, $\sigma \in \mathrm{H}^0(C, \mathscr{E})$ gives rise to a map $\mathscr{O}_C \to \mathscr{E}$ and the cokernel $\mathscr{E}/\sigma\mathscr{O}_C$ of this map is the quotient line bundle of $\mathscr{E}$ corresponding to $C \xrightarrow{\sigma} E \to \mathbb{P}(E)$ (Recall Corollary 3.1.7), so we have an exact sequence

$$0 \longrightarrow \mathscr{O}_C \xrightarrow{\sigma} \mathscr{E} \longrightarrow \mathscr{E}/\sigma\mathscr{O}_C \longrightarrow 0,$$

and, again by Corollary 3.1.7, since σ (viewed as a section $C \to \mathbb{P}(E) = X$ of π) corresponds to $\mathscr{E}/\sigma\mathscr{O}_C$, we have $\sigma^* \mathscr{O}_X(1) \simeq \mathscr{E}/\sigma\mathscr{O}_C$. Taking determinants of the exact sequence above, we also conclude that $\sigma^* \mathscr{O}_X(1) \simeq \det \mathscr{E}$. This fact will be very important below.

Theorem 3.2.6. *With notation as in the preceding paragraph, we have $h^2 = \deg(\mathscr{E})$ and*

$$\omega_X \simeq \mathscr{O}_X(-2S) \otimes \pi^* (\omega_C \otimes \det \mathscr{E}).$$

Hence, $c_1(\omega_X) = -2h + (\deg \mathscr{E} + \deg \omega_C) f = -2h + (\deg \mathscr{E} + 2g(C) - 2) f$.

Proof. Let $\mathscr{L}, \mathscr{M}$ be line bundles on C giving rise to an exact sequence $0 \to \mathscr{L} \to \mathscr{E} \to \mathscr{M} \to 0$. Pulling this back along π, we get an exact sequence $0 \to \pi^* \mathscr{L} \to \pi^* \mathscr{E} \to \pi^* \mathscr{M} \to 0$. Note that, by Lemma 2.2.12,

$$\begin{aligned} 0 = (\pi^* \mathscr{L}) \cdot (\pi^* \mathscr{M}) &= \chi(\mathscr{O}_X) - \chi(\pi^* \mathscr{L}) - \chi(\pi^* \mathscr{M}) + \chi(\pi^* \mathscr{L} \otimes \pi^* \mathscr{M}) \\ &= \chi(\mathscr{O}_X) - \chi(\pi^* \mathscr{E}) + \chi(\pi^* \det \mathscr{E}). \end{aligned} \tag{3}$$

Now, consider the sequence

$$0 \longrightarrow \mathscr{O}_X(-1) \longrightarrow \pi^* \mathscr{E} \longrightarrow \mathscr{O}_X(1) \longrightarrow 0.$$

Because the RHS of (3) only depends on $\pi^* \mathscr{E}$, we immediately see that $\mathscr{O}_X(-1) \cdot \mathscr{O}_X(1) = 0$. Letting $e = c_1(\det \mathscr{E}) \in \mathrm{H}^2(C; \mathbb{Z})$, the fact that $\mathscr{O}_X(1) \otimes \mathscr{O}_X(-1) \simeq \det \pi^* \mathscr{E} = \pi^* \det \mathscr{E}$ shows us that $c_1(\mathscr{O}_X(-1)) = \pi^* e - h$. Hence,

$$0 = c_1(\mathscr{O}_X(-1))c_1(\mathscr{O}_X(1)) = (\pi^* e - h)h = h \cdot \pi^* e - h^2.$$

Since, by our simplifying remark, $\mathscr{O}_X(1) \simeq \mathscr{O}_X(S)$ for some section S, we also have

$$h \cdot \pi^* e = \deg(\pi^* \det \mathscr{E}|_S) = \deg(\det \mathscr{E}) = \deg(\mathscr{E}),$$

since $\pi|_S : S \to C$ is an isomorphism sending $\pi^* \mathscr{E}$ to $\mathscr{E}$. Thus,

$$h^2 = h \cdot \pi^* e = \deg \mathscr{E}$$

as claimed.

We now determine ω_X. By Theorem 3.2.1, we may write $\omega_X \simeq \mathscr{O}_X(aS) \otimes \pi^* \mathscr{O}_C(D)$ for some $a \in \mathbb{Z}$ and $D \in \operatorname{Div} C$. Applying the genus formula to a fiber $F \subset X$, we see that

$$-2 = 2g(F) - 2 = K_X F + F^2 = K_X F = (aH + \pi^* D)F = a.$$

We next apply adjunction to S to see that

$$\omega_S \simeq \omega_X \otimes \mathscr{O}_X(S)|_S \simeq \mathscr{O}_X(-S + \pi^* D)|_S.$$

Let $\sigma : C \xrightarrow{\sim} S$ be the inverse of $\pi|_S : S \xrightarrow{\sim} C$. Pulling back the above along σ, and noting that $\sigma^* \mathscr{O}_X(S)|_S \simeq \det \mathscr{E}$ by the end of Remark 3.2.1, we see that

$$\omega_C \simeq (\det \mathscr{E})^\vee \otimes \mathscr{O}_C(D).$$

Thus, $\mathscr{O}_C(D) \simeq \omega_C \otimes \det \mathscr{E}$. Combining this with our earlier calculation that $a = -2$ shows that

$$\omega_X \simeq \mathscr{O}_X(-2S) \otimes \pi^*(\omega_C \otimes \det \mathscr{E})$$

as claimed. This finishes the proof. ■

Technical Aside 3.2.1. The calculation that $c_1(\mathscr{O}_X(-1))c_1(\mathscr{O}_X(1)) = 0$ performed above can be reformulated as the vanishing of the second Chern class $c_2(\pi^* \mathscr{E})$ of $\pi^* \mathscr{E}$. Indeed $c_2(\pi^* \mathscr{E}) = \pi^* c_2(\mathscr{E}) = 0$ since $\mathscr{E}$ is a bundle on a 1-dimensional space, and the existence of the short exact sequence $0 \to \mathscr{O}_X(-1) \to \pi^* \mathscr{E} \to \mathscr{O}_X(1) \to 0$ shows that $0 = c_2(\pi^* \mathscr{E}) = c_1(\mathscr{O}_X(-1))c_1(\mathscr{O}_X(1))$.

3.3 Ruled Surfaces over $\mathbb{P}^1$

Now that we have spent a considerable amount of time studying the structure of general ruled surfaces, we will shift focus slightly by taking a look at some specific ruled surfaces. In this section, we will briefly look at the Hirzebruch surfaces, the surfaces ruled over $\mathbb{P}^1$. We will first show that every ruled surface over $\mathbb{P}^1$ is of the form $\Sigma_n := \mathbb{P}(\mathscr{O}_{\mathbb{P}^1} \oplus \mathscr{O}_{\mathbb{P}^1}(n))$ for some $n \in \mathbb{Z}$ which we will then show is unique (i.e. we will show that $\Sigma_n \not\simeq \Sigma_m$ if $n \neq m$).

Remark 3.3.1. If E is a vector bundle on a curve and L is a line bundle on it, then $\mathbb{P}(E) \simeq \mathbb{P}(E \otimes L)$. Thinking of bundles in terms of their transition functions, it is clear that $\mathbb{P}(E)$ and $\mathbb{P}(E \otimes L)$ have the same transition functions since they differ by an element of $\mathrm{PGL}_1(\mathbb{C}) = 1$.

In light of this remark, to show that every ruled surface over $\mathbb{P}^1$ is of the form $\mathbb{P}(\mathscr{O}_{\mathbb{P}^1} \oplus \mathscr{O}_{\mathbb{P}^1}(n))$, it suffices to show that every rank 2 vector bundle over $\mathbb{P}^1$ splits as a sum of line bundles.

Proposition 3.3.1. *Let $0 \to \mathscr{L} \to \mathscr{E} \to \mathscr{M} \to 0$ be an extension of line bundles $\mathscr{M}, \mathscr{L}$ on a smooth curve C. Then, there exists a cohomology class $\alpha \in \mathrm{H}^1(C, \mathscr{L} \otimes \mathcal{M}^\vee)$ with the property that this sequence splits – i.e. $\mathscr{E} = \mathscr{L} \oplus \mathscr{M}$ – if and only if $\alpha = 0$.*

Proof. Tensor the sequence with $\mathscr{M}^\vee$ to get an exact sequence

$$0 \longrightarrow \mathscr{L} \otimes \mathscr{M}^\vee \longrightarrow \mathscr{E} \otimes \mathscr{M}^\vee \xrightarrow{\phi} \mathscr{O}_C \longrightarrow 0$$

which splits iff the original sequence does. Note that a splitting map $s : \mathscr{O}_C \to \mathscr{E} \otimes \mathscr{M}^\vee$ is identified with a global section of $\mathscr{E} \otimes \mathscr{M}^\vee$ whose image under ϕ is the unit section $1_C \in \mathrm{H}^0(\mathscr{O}_C)$. With this in mind, consider the cohomology sequence

$$\mathrm{H}^0(\mathscr{E} \otimes \mathscr{M}^\vee) \xrightarrow{\phi} \mathrm{H}^0(\mathscr{O}_C) \xrightarrow{\partial} \mathrm{H}^1(\mathscr{L} \otimes \mathscr{M}^\vee).$$

Our original sequence splits iff 1_C is in the image of ϕ above, but $\operatorname{im} \phi = \ker \partial$, so it splits iff $\alpha := \partial(1_C) \in \mathrm{H}^1(\mathscr{L} \otimes \mathscr{M}^\vee)$ vanishes. This proves the claim. ■

Technical Aside 3.3.1. Readers familiar with homological algebra may find find the following explanation of this proposition more conceptual or illuminating. Mirroring the construction of the Ext^i_R functors on R-modules, one can derive the $\mathrm{Hom}(\mathscr{F}, -)$ functor (here $\mathscr{F}$ is any $\mathscr{O}_X$-module) to get $\mathrm{Ext}^i(\mathscr{F}, -) = \mathrm{Ext}^i_{\mathscr{O}_X}(\mathscr{F}, -)$ functors of $\mathscr{O}_X$-modules. Just as with modules, isomorphism classes of extensions of $\mathscr{M}$ by $\mathscr{L}$ turn out to be in bijection with elements of $\mathrm{Ext}^1(\mathscr{M}, \mathscr{L})$. Furthermore, when $\mathscr{M}$ is locally free we have $\mathrm{Ext}^1(\mathscr{M}, \mathscr{L}) \simeq \mathrm{Ext}^1(\mathscr{O}_X, \mathscr{L} \otimes \mathscr{M}^\vee)$, but since $\mathrm{Hom}(\mathscr{O}_X, -) = \Gamma(-)$ is naturally isomorphic to the global sections functor, their right derived functors coincide, i.e. $\mathrm{Ext}^1(\mathscr{O}_X, \mathscr{L} \otimes \mathscr{M}^\vee) \simeq \mathrm{H}^1(X, \mathscr{L} \otimes \mathscr{M}^\vee)$, so this latter group also classifies extensions of $\mathscr{M}$ by $\mathscr{L}$.

Corollary 3.3.2. *Every rank 2 vector bundle on $\mathbb{P}^1$ splits as a sum of line bundles.*

Proof. Let $\mathscr{E}$ be a rank 2 locally free sheaf on $\mathbb{P}^1$. Note that if $\mathscr{L}$ is a line bundle on $\mathbb{P}^1$, then $\deg(\mathscr{E} \otimes \mathscr{L}) = \deg \mathscr{E} + \deg \mathscr{L}$, so by tensoring $\mathscr{E}$ with an appropriate line bundle, we may assume that $\deg \mathscr{E} \in \{0, -1\}$. By Riemann-Roch, we have $h^0(\mathscr{E}) \geq \chi(\mathscr{E}) = \deg \mathscr{E} + 2 \geq 1$, so $\mathscr{E}$ has some nonzero section $s \in \mathrm{H}^0(\mathscr{E})$. Hence, there exists an exact sequence

$$0 \longrightarrow \mathscr{O}_{\mathbb{P}^1}(k_1) \longrightarrow \mathscr{E} \longrightarrow \mathscr{O}_{\mathbb{P}^1}(k_2) \longrightarrow 0$$

with $k_1 \geq 0$.[5] Furthermore, $k_1 + k_2 = \deg \mathscr{E}$ since $\mathscr{O}_{\mathbb{P}^1}(k_1) \otimes \mathscr{O}_{\mathbb{P}^1}(k_2) \simeq \det \mathscr{E}$. By the theorem above, the splitting of this sequence is controlled by a cohomology class in $\mathrm{H}^1(\mathscr{O}_{\mathbb{P}^1}(k_1 - k_2)) = \mathrm{H}^1(\mathscr{O}_{\mathbb{P}^1}(2k_1 - d))$ where $d = \deg \mathscr{E}$. Since $k_1 \geq 0$ and $d \leq 0$, we see $2k_1 - d \geq 0$, so $h^1(\mathscr{O}_{\mathbb{P}^1}(2k_1 - d)) = h^0(\mathscr{O}_{\mathbb{P}^1}(-2 - (2k_1 - d)) = 0$. Thus, $\mathscr{E} \simeq \mathscr{O}_{\mathbb{P}^1}(k_1) \oplus \mathscr{O}_{\mathbb{P}^1}(k_2)$. ■

[5] The proof of Corollary 3.2.4 shows that the line bundle $\mathscr{L}$ in its statement has a section when $\mathscr{E}$ does since $\mathscr{L}$ is the dual of an ideal sheaf (and ideal sheaves correspond to the negatives of effective divisors

This completes the argument that all ruled surfaces over $\mathbb{P}^1$ are of the following form.

Definition 3.3.1. The ruled surface $\Sigma_n := \mathbb{P}(\mathscr{O}_{\mathbb{P}^1} \oplus \mathscr{O}_{\mathbb{P}^1}(n)) \to \mathbb{P}^1$ is called the n**th Hirzebruch surface**.

Theorem 3.3.3. *The Hirzebruch surfaces Σ_n are minimal if $n \neq 1$, and $\Sigma_n \simeq \Sigma_m \implies n = m$.*

Proof. We will prove both parts of this theorem by showing that Σ_n contains a unique curve B with negative self-intersection and that $B^2 = -n$. Thus, $\Sigma_n \not\simeq \Sigma_m$ for $n \neq m$ and Σ_n is minimal when $n \neq 1$ since it contains no (-1)-curves.

Reusing notation from the previous subsection, let $F, H \in \operatorname{Pic}\Sigma_n$ be the classes of a fiber and of the tautological quotient bundle, respectively. By Theorems 3.2.1 and 3.2.6, we know $\operatorname{Pic}\Sigma_n = \mathbb{Z}F \oplus \mathbb{Z}H$ with $F^2 = 0$, $FH = 1$, and $H^2 = \deg(\mathscr{O}_{\mathbb{P}^1} \oplus \mathscr{O}_{\mathbb{P}^1}(n)) = n$. Let $s : \mathbb{P}^1 \to \Sigma_n$ be the section corresponding to the quotient $\mathscr{O}_{\mathbb{P}^1} \oplus \mathscr{O}_{\mathbb{P}^1} \to \mathscr{O}_{\mathbb{P}^1}$, and let $B = s(C) \subset \Sigma_n$. Then, $B \sim H + rF$ for some $r \in \mathbb{Z}$. We have

$$HB = \deg(\mathscr{O}_{\Sigma_n}(1)|_B) = \deg(s^*\mathscr{O}_{\Sigma_n}(1)) = \deg(\mathscr{O}_{\mathbb{P}^1}),$$

so $0 = HB = H(H + rF) = H^2 + rHF = n + r$. Hence, $r = -n$ and $B^2 = (H - nF)^2 = n - 2n = -n$. Therefore, we just need to show B is the only curve with this property.

Let $C \subset \Sigma_n$ be any other curve (i.e. $C \neq B$). Then, $C \sim aH + bF$ for some $a, b \in \mathbb{Z}$. We have $a = CF \geq 0$. Since C, B are distinct curves, we also have

$$0 \leq CB = (aH + bF)B = bFB = b.$$

Thus, $C^2 = (aH + bF)^2 = a^2n + 2ab \geq 0$, so B really is the unique curve with negative self-intersection. ∎

Since Σ_1 is not minimal, one may wonder what surface serves as its minimal model. Determining this will be the last thing we do this section.

Example (An alternate construction of Σ_1)**.** Pick any point $p \in \mathbb{P}^2$ and let $S = \operatorname{Bl}_p \mathbb{P}^2$. Note that we have a rational map $\mathbb{P}^2 \dashrightarrow \mathbb{P}^1$, "projection away from p", which sends any point $q \neq p$ of $\mathbb{P}^2$ to the unique line passing through p and q. This map is not defined at p, but it naturally extends to a morphism $S \to \mathbb{P}^1$ which now gives S the structure of a ruled surface over $\mathbb{P}^1$. Since S contains a (-1)-curve by construction, we conclude by the theorem above that $S \simeq \Sigma_1$. Hence, $\mathbb{P}^2$ is a minimal model for Σ_1.

4 Elliptic Surfaces

In this chapter, we finally fulfill our promise of carrying out the local study of elliptic surfaces. Recall that in the case of ruled surfaces, the fact that the generic fiber had (arithmetic) genus 0 automatically forced every fiber to be smooth. In the case of elliptic surfaces, the generic fiber has (arithmetic) genus 1, and so particular, or "special", fibers may lower geometric genus, i.e. some fibers may be singular. The main goal of this chapter is to completely classify all the possible singular fibers that may arise on elliptic surfaces. The classification we provide will be geometric/combinatorial, stated in terms of the possible layouts of the irreducible components of the fibers. There also exists a more algebraic classification, given in terms of a certain "monodromy" map which measures how (smooth) fibers "twist" as they are "transported around a given (singular) fiber." Towards the end of this chapter, we will determine this monodromy for some types of singular fibers in detail, and then give a brief overview of how one calculates the monodromy for the remaining cases.

4.1 Definitions

Definition 4.1.1. An **elliptic surface** is an analytic space X equipped with a holomorphic map $\pi : X \longrightarrow C$ from a complex surface X to a smooth curve C such that the general fiber of π is a smooth connected curve of genus one. The map π is also called an **elliptic fibration**, and sometimes called the **structure map** of X.

In this definition, we refrained from saying that the general fiber of $\pi : X \to C$ is an elliptic curve, because "elliptic curves" come with a specified choice of a basepoint; however, not all elliptic surfaces have a section $s : C \to X$. Furthermore, we will often call the total space X itself an "elliptic surface" with the implicit understanding that it comes with an auxiliary map $\pi : X \to C$.

Definition 4.1.2. Given an elliptic surface $\pi : X \to C$, we say a curve in X is **vertical for** π if it is contained within a fiber of π. Otherwise, we call it **horizontal**.

Remark 4.1.1. An elliptic surface is relatively minimal if and only if it has no vertical (-1)-curves. Furthermore, relatively minimal elliptic surfaces are not necessarily minimal; they may contain horizontal (-1)-curves.

It will be useful to know that elliptic fibrations are preserved under basechange. This is because performing suitable basechanges[6] can sometimes let you deduce properties of complicated elliptic surfaces from those of simpler ones.

Lemma 4.1.1. *Let $\pi : X \to C$ be an elliptic surface, and let $f : C' \to C$ be a surjective map of (not necessarily compact) curves. Then, the pullback $f^*\pi : X \times_C C' \to C'$ is an elliptic surface as well.*

Proof. For notational convenience, let $X' = X \times_C C'$ and let $\pi' : X' \to C'$ denote the natural projection map. We need to show that the generic fiber $X'_{c'} = (\pi')^{-1}(c')$ $(c' \in C')$ is smooth of genus one. We will do this by showing that $X'_{c'} \simeq X_{f(c')}$ for all $c' \in C'$. Fix any $c' \in C'$, and let $c = f(c') \in C$. In order to show $X'_{c'} \simeq X_c$, it suffices to show that X_c satisfies the universal property of $X'_{c'}$, i.e. that X_c is the base change of $X' \to C'$ along the inclusion $\{c'\} \hookrightarrow C'$.

[6]Recall Proposition 2.1.4

It is worth noting that proofs like this are best oneself since it can be hard to clearly describe the logic textually, but we will attempt to do so nevertheless. With that said, consider any space Y with a map $Y \to C'$ making the below commute

$$\begin{tikzcd}
Y \arrow[r] \arrow[d] & X' \arrow[d, "\pi'"] \\
\{c'\} \arrow[r] & C'
\end{tikzcd}$$

where there is a unique map $Y \to \{c'\}$ since $\{c'\}$ is given the structure sheaf $\mathscr{O}_{c'} = \mathbb{C}$. Since $f(c') = c$, we can extend this diagram into the one below

$$\begin{tikzcd}
Y \arrow[r] \arrow[d] & X' \arrow[r] \arrow[d, "\pi'"] & X \arrow[dd, "\pi"] \\
\{c'\} \arrow[r] \arrow[d] & C' \arrow[dr, "f"] & \\
\{c\} \arrow[rr] & & C
\end{tikzcd}$$

and so we obtain a unique map $Y \to X_c$ respecting their maps to X and $\{c\}$. In order to conclude that $X_c \simeq X'_{c'}$ we need to show X_c has maps $X_c \to X'$ and $X_c \to \{c'\}$ respected by this map $Y \to X_c$. There is only one map $X_c \to \{c'\}$, so there is nothing to show here. We get a map $X_c \to X'$ using X''s universal property since we have natural maps $X_c \to \{c'\} \to C'$ and $X_c \to X$ which both become the constant map to $c \in C$ when pushed to C. This gives rise to the desired map $X_c \to X'$. By construction (i.e. the uniqueness/naturality of everything), the maps $Y \to \{c'\}$ and $Y \to X_c \to \{c'\}$ agree (this is unsurprising since any space has a unique map to $\{c'\}$) as do the maps $Y \to X'$ and $Y \to X_c \to X'$, so X_c really is the base change $X_c \simeq X' \times_{c'} C' =: X'_{c'}$. Pictorially, the proof is encapsulated in the following commutative diagram (the dashed maps exist by universal properties).

$$\begin{tikzcd}
Y \arrow[dr, dashed] \arrow[drr, bend left] \arrow[ddr, bend right] & & & \\
& X_c \arrow[r, dashed] \arrow[d] \arrow[rr, bend left] & X' \arrow[r] \arrow[d, "\pi'"] & X \arrow[dd, "\pi"] \\
& \{c'\} \arrow[r] \arrow[d] & C' \arrow[dr, "f"] & \\
& \{c\} \arrow[rr] & & C
\end{tikzcd}$$

$\blacksquare$

Before diving into the classification of singular fibers of elliptic surfaces, we will give a concrete example, exhibiting that singular fibers may in fact exist.

Example (A smooth elliptic surface with singular fiber)**.** Let $\Delta = \{z \in \mathbb{C} : |z| < 1\}$ be the complex unit disk, and let

$$X = \left\{ \left([z_0 : z_1 : z_2], s\right) \in \mathbb{P}^2 \times \Delta \,\middle|\, z_2 z_1^2 = z_0^3 - s z_0 z_2^2 \right\},$$

with structure map $\pi : X \to \Delta$ given by the natural projection. Then, X is visibly elliptic, and its smoothness follows from the fact that there is no point $p \in X$ at which all 4 derivatives

$$\frac{\partial f}{\partial z_0} = 3z_0^2 - sz_2^2 \qquad \frac{\partial f}{\partial z_1} = -2z_2 \qquad \frac{\partial f}{\partial z_2} = -2sz_0z_2 - z_1^2 \qquad \frac{\partial f}{\partial s} = -z_0z_2^2$$

vanish, where $f(z_0, z_1, z_2, s) = z_0^3 - sz_0z_2^2 - z_2z_1^2$. However, the fiber $X_0 = \pi^{-1}(0)$ above 0 is not smooth as it is isomorphic to the cuspidal cubic given by the projective closure of $\{y^2 = x^3\} \subset \mathbb{C}^2$. When we classify singular fibers, this will be an example of a singular fiber of type II.

While many types of singular fibers can occur, they are constrained by the facts that they must have arithmetic genus 1 and, by Zariski's Lemma, must come from a negative semi-definite bilinear form. These will allow us to give a combinatorial description of all possible singular fibers in terms of so-called extended Dynkin diagrams. Afterwards, one can verify that each possible type does indeed occur by constructing various examples.

From here on out, unless otherwise stated, assume any map denoted $\pi : X \to C$ or $\pi : X \to \Delta$ is an elliptic fibration, and in the latter case, that $\Delta \subset \mathbb{C}$ is the open unit disk. Assume furthermore that all elliptic fibrations under consideration are smooth and relatively minimal.

4.2 Classification of Fibers

The classification on singular fibers rests on showing a connection between (fibers of) elliptic fibrations and a certain explicit family of graphs known as the extended Dynkin diagrams. We have already done half the work in determining this classification. The point is the following: given a fiber $F = \sum n_i C_i$ ($n_i > 0$ and C_i irreducible) of an elliptic fibration $\pi : X \to C$, the intersection form on X induces a symmetric bilinear form B on the $\mathbb{Q}$-vector space $V = \bigoplus \mathbb{Q}C_i$ with basis $\{C_i\}$. To this form, we will associate a graph G encoding information about B. Now, recalling Zariski's lemma, this form is necessarily negative semi-definite with one-dimensional kernel. In the reverse direction, to any graph G, it is possible to associate a vector space V_G equipped with a bilinear form, and the graphs whose associated forms are semi-definite with one-dimensional kernel are exactly the extended Dynkin diagrams. Once we show this, we will have our classification rather quickly.

This classification was first carried out by Kunihiko Kodaira [9], but our treatment of it most closely follows the account given in [10] by Rick Miranda.

The first half of the classification – showing that fibers give rise to negative semi-definite symmetric bilinear forms with one-dimensional kernel – was already completed when we proved Zariski's lemma, so we begin now with the second half: relating fibers to graphs and determining which graphs give rise to negative semi-definite symmetric bilinear forms with one-dimensional kernel.

Definition 4.2.1. Let $F = \sum n_i C_i$ (C_i irreducible) be a fiber of an elliptic surface $X \to C$. Its **dual graph** (or **incidence graph** or the **graph associated to the fiber** F) is the graph $G = G_F$ with one vertex v_i for each irreducible component C_i of F, C_iC_j edges joining v_i to v_j (when $i \neq j$), $\frac{1}{2}(C_i^2 + 2)$ self-loops on the vertex v_i.

Remark 4.2.1. Using adjunction, we will see later than, when F is the fiber of a relatively minimal elliptic fibration, then either $C_i^2 = 0$ (when $F = C_i$ is irreducible) or $C_i^2 = -2$ (when $F \neq C_i$ is reducible). With this in mind, the formula for the number of self-loops is simply a convenient normalization of C_i^2, leading to a nonnegative integer.

This provides a link between fibers and graphs. We are interested in knowing precisely which graphs can arise from this process. Since our main constraints on fibers are summarised by Zariski's lemma and stated in terms of intersection products, we make the following dual definition.

Definition 4.2.2. Let G be an undirected graph, possibly with self-loops and/or multi-edges. Let V_G be the $\mathbb{Q}$-vector space whose basis is $V(G)$, the vertices of G. We give V_G the symmetric bilinear form with

$$v^2 = -2 + 2\#\{\text{loops at } v\} \qquad\qquad vw = \#\{\text{edges joining } v \text{ to } w\}$$

where $v \neq w$ are vertices. The space V_G, together with this form, is called the **bilinear form associated to** G. By analogy with considerations of fibers, we will sometimes also call this G's **intersection form**.

Remark 4.2.2. If F is a fiber of an elliptic surface $X \to C$, then V_{G_F}, the bilinear form associated to F's dual graph, is precisely the intersection form on $\bigoplus \mathbb{Q}C_i$, by construction.

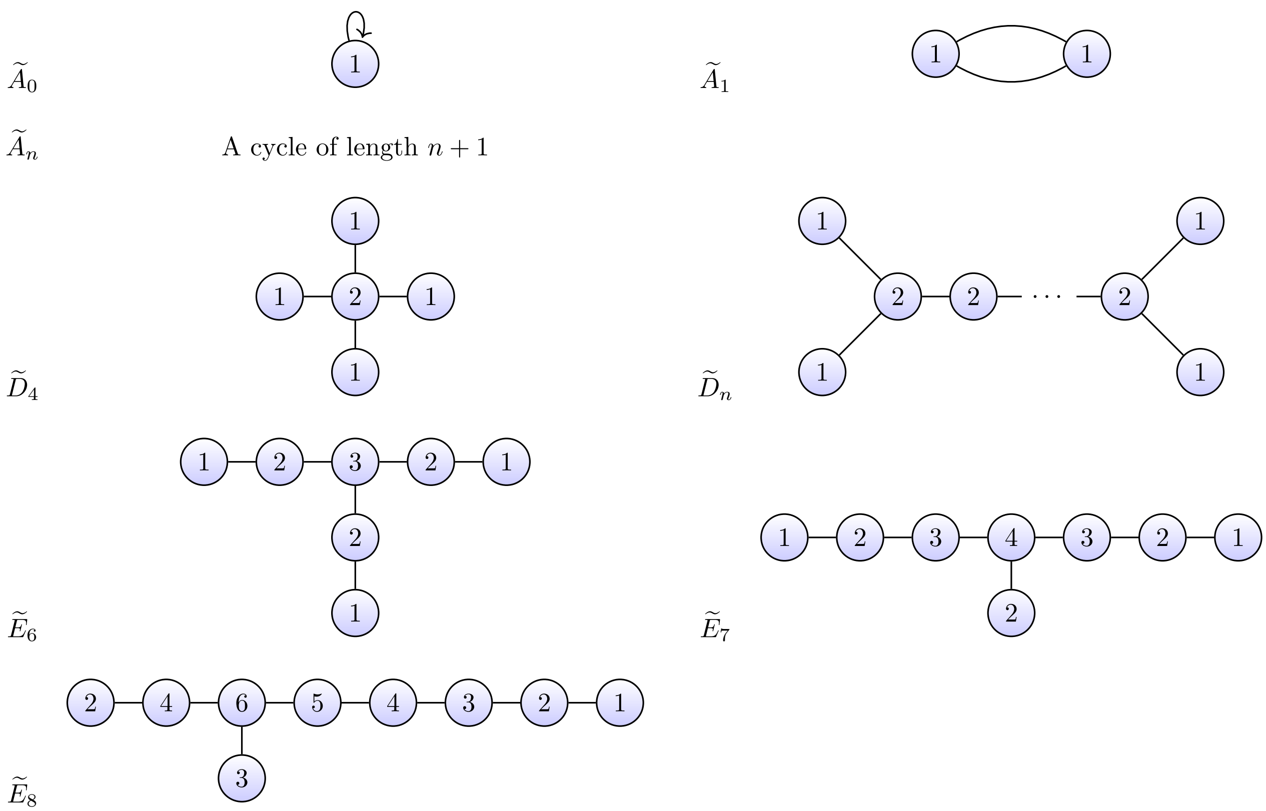

Figure 5: The extended Dynkin diagrams. The subscript on the name denotes one less than the number of vertices (e.g. $\widetilde{D}_n$ has $n+1$ vertices). Each vertex is labelled with its multiplicity in a generator of the kernel of its graph's associated bilinear form (Every vertex of $\widetilde{A}_n$ has multiplicity 1).

In the overview of our strategy for classifying singular fibers, we claimed that the graphs whose associated forms are negative semi-definite with one-dimensional kernel – i.e. the graphs which could possibly be dual graphs of singular elliptic fibers – are exactly the extended Dynkin diagrams. In order to prove this, it is necessary to first define this class of graphs, so we do so now. An **extended Dynkin diagram** is any graph belonging to the list of graphs given in Figure 5.

Proposition 4.2.1. *Let G be an extended Dynkin diagram. Then G's associated bilinear form V_G is negative semi-definite with one-dimensional kernel.*

Proof. Much like in the proof of Zariski, this is seen by applying Lemma 2.5.9 to $-V_G$ (the same vector space but with negated bilinear form). Indeed, **(a)** of Lemma 2.5.9 is satisfied by $-V_G$ since $vw = -\#\{\text{edges joining } v \text{ to } w\} \leq 0$ for any vertices $v \neq w$. Hypothesis **(b)** is also satisfied since G is connected and so in any partition $I \sqcup J = V(G)$ of the vertices (into two nonempty sets), there's some $v \in I$ and $w \in J$ with an edge between them. Finally, hypothesis **(c)** of Lemma 2.5.9 is satisfied by $-V_G$ by verifying that the vector $x_0 \in V_G$ indicated in Figure 5 (e.g. when $G = \widetilde{D}_4$, x_0 is twice the central vertex plus the sum of the outer vertices) squares to 0. This is a simple computation for each family of extended Dynkin digrams. Hence, Lemma 2.5.9 applies and shows that $-V_G$ is positive semi-definite with one-dimensional kernel, so V_G is negative semi-definite with one-dimensional kernel. ■

In order to take the above proposition a step further – to show that the extended Dynkin diagrams are the only graphs with this property – we implement a two step strategy. The first step will be to somehow relate any graph with this property to an extended Dynkin diagram. In fact, we will relate any (connected) graph to an extended Dynkin diagram by showing that any connected graph contains or is contained in some extended Dynkin diagram. Once we have shown this, we will have a connection between the intersection form of a given graph with this desired property and that of some extended Dynkin diagram, so some numerology with bilinear form – in particular, making use of the fact that the kernel of the form associated to an extended Dynkin digram has strictly positive coefficients on each vertex – will force graphs with this desired property to themselves be extended Dynkin diagrams.

We begin with the first step. The proof will require a bit of new notation.

Notation 4.2.3. Let $p, q, r \geq 0$ be non-negative integers. Then, we let $T_{p,q,r}$ denote the **3-spider** with legs of length p, q, and r. That is, $T_{p,q,r}$ consists of 3 paths of lengths p, q, r, respectively, which are disjoint except for all sharing the same starting vertex v. An image of $T_{1,2,3}$ can be found in Figure 6.

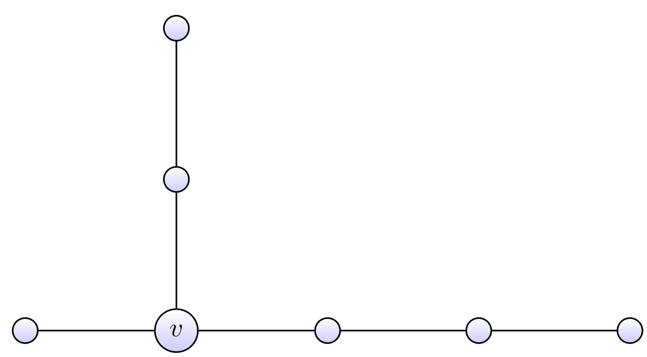

Figure 6: The graph $T_{1,2,3}$

Remark 4.2.3. When we say a graph H is contained in another graph G, we mean that H can be obtained from G by removing some number of vertices, edges, and/or self-loops.

Lemma 4.2.2. *Every connected graph is contained in or contains an extended Dynkin diagram. Furthermore, every connected graph without loops or multiple edges either is contained in or contains an extended Dynkin diagram without loops or multiple edges.*

Proof. Let G be a connected graph. If G has a loop, then G contains $\widetilde{A}_0$, and if G has a multi-edge, then G contains $\tilde{A}_1$. Hence, we may assume that G is simple, and so are reduced to proving the part of the lemma after the word "Furthermore."

If G contains a cycle, then it contains $\tilde{A}_N$ for some N, so assume G is a tree. If G has a vertex of degree 4, then it contains $\tilde{D}_4$, so assume every vertex has degree at most 3. If G has two vertices of degree 3, it will contain $\tilde{D}_N$ for some N. If G has no degree 3 vertex, then every vertex of G has degree at most 2, making G a path and hence contained in $\tilde{A}_N$ for some N. Thus, we may and do assume that G has exactly one vertex of degree 3, i.e. that G is a $T_{p,q,r}$ graph for some $p, q, r \geq 1$. Note that $\tilde{E}_6 = T_{2,2,2}$, $\tilde{E}_7 = T_{1,3,3}$, and $\tilde{E}_8 = T_{1,2,5}$. Order p, q, r s.t. $1 \leq p \leq q \leq r$. If $p \geq 2$, then G contains $\tilde{E}_6$, so assume $p = 1$. If $q \geq 3$, then G contains $\tilde{E}_7$, and if $q = 1$, then G is contained in some $\tilde{D}_N$; hence assume $q = 2$. Finally, since $p = 1$ and $q = 2$, it's clear that either G is contained in $\tilde{E}_8$ or G contains $\tilde{E}_8$. As such, the claim holds. ■

This provides us with a mechanism of relating the intersection form of an arbitrary graph to that of an extended Dynkin diagram. We now use this to show that if we start with an arbitrary graph G whose associated form is negative semi-definite with one-dimensional kernel, then in fact G must itself be one of the extended Dynkian diagrams. As in the proof above, the proof of this fact will handle the cases of $\widetilde{A}_0, \widetilde{A}_1$ (i.e. of G containing a loop or multi-edge) separately from the general case of simple G.

Theorem 4.2.3. *Let G be a connected graph whose associated form is negative semidefinite, with kernel of dimension one. Then, G is an extended Dynkin diagram.*

Proof. First suppose that G has a loop. We then wish to show that $G = \tilde{A}_0$, i.e. that G must only have a single vertex. Let $v \in V(G)$ be a vertex with a loop, and let $w \in V(G)$ be any vertex adjacent to v. Recall that $v^2 = -2 + 2\#\{\text{loops at v}\}$, so since $v^2 \leq 0$ by assumption, we must actually have that $v^2 = 0$ and v has only a single loop. In order to show that v is the only vertex of G, it suffices to show that $vw = 0$ from which it will follow that $v = w$ (so v is connected to nothing other than itself). Observe that for $k \in \mathbb{Q}$, we have

$$0 \geq (k \cdot v + w)^2 = 2k \cdot vw + w^2 \implies vw \leq -\frac{w^2}{2k} \leq \frac{1}{k}$$

since $w^2 \geq -2$. Taking $k = 2$, for example, we see that $vw < 1$, so $vw = 0$ which proves that $G = \tilde{A}_0$ in this case. As we shall shortly see, the remaining cases are handled similarly, ultimately resting on considering an expression of the form $(2v + w)^2$ where $v, w \in V_G$.

Now, assume G has no loops, but suppose it has multiple edges joining say v to w, i.e. $vw \geq 2$. We wish to show that $G = \tilde{A}_1$. First note that

$$0 \geq (v + w)^2 = v^2 + 2vw + w^2 = -4 + 2vw \implies vw \leq 2,$$

so there are exactly two edges joining v to w and $(v + w)^2 = 2vw - 4 = 0$. Proceeding as in the previous case where we knew $v^2 = 0$ for an edge with a loop, suppose that u is a vertex other than v, w and observe that

$$0 \geq (2(v + w) + u)^2 = 4(vu + wu) + u^2 = 4(vu + wu) - 2 \implies vu + wu \leq \frac{1}{2} \implies vu, wu = 0,$$

so u is connected to neither v nor w. Since G is connected, this is a contradiction, so v, w must be the only two vertices and $G = \tilde{A}_1$.

We may now suppose that G is simple, and so contains or is contained in a simple extended Dynkin diagram. First suppose that G contains an extended Dynkin diagram D, and let $x_0 \in V_D \subset V_G$ denote its square zero class. Because x_0 has a positive coefficient on all the vertices of D, we must have $V(G) = V(D)$. Indeed, if u is a vertex of G but not of D then, repeating the arguments made above, we see that

$$0 \geq (2x_0 + u)^2 \implies x_0 u < 1 \implies x_0 u = 0,$$

so u is not adjacent to any vertex in D. Since this applies to any u not in D and since G is connected, we conclude that $V(G) = V(D)$. Because G has no multiple edges, this then implies that $G = D$.

Finally, if G is contained in a (simple) extended diagram D, then the generator x of the kernel of V_G must be a multiple of the square zero class x_0 of D. Since x_0 has a strictly positive coefficient on all the vertices of the diagram, G must contain all those vertices, and so, since G has no multiple edges, we again see that $G = D$. ■

We have now shown that the extended Dynkin diagrams are exactly the graphs whose intersection forms are negative semi-definite with one-dimensional kernel! This means have finished most of the hard work of obtaining Kodaira's classification. In particular, it is clear now, by combining Zariski's Lemma with Theorem 4.2.3, that any fiber of an elliptic fibration gives rise to some extended Dynkan diagram! However, this mapping is not quite injective (e.g. any irreducible fiber, singular or not, corresponds to $\widetilde{A}_0$). Hence, we need to determine which geometric situations can give rise to a given extended Dynkin diagram. This is described in Table 1 and proven in Theorem 4.2.5.

Type	Description	Dual Graph
I_0	Smooth elliptic curve	$\widetilde{A}_0$
I_1	Rational curve with a node, i.e. an ordinary double point	$\widetilde{A}_0$
I_2	Two smooth rational curves meeting transversely at two point	$\widetilde{A}_1$
$\mathrm{I}_n, n \geq 3$	n smooth rational curves meeting in a cycle	$\widetilde{A}_{n-1}$
${}_m\mathrm{I}_n$	Topologically an I_n, but appearing with multiplicity m	$\widetilde{A}_{n-1}$
I_n^*	$n+5$ smooth rational curves with dual graph $\widetilde{D}_{n+4}$	$\widetilde{D}_{n+4}$
II	Rational curve with a cusp	$\widetilde{A}_0$
II^*	Nine smooth rationals meeting with dual graph $\widetilde{E}_8$	$\widetilde{E}_8$
III	Two smooth rationals meeting at one point with multiplicity 2	$\widetilde{A}_1$
III^*	Eight smooth rationals meeting with dual graph $\widetilde{E}_7$	$\widetilde{E}_7$
IV	Three smooth rationals meeting at one point	$\widetilde{A}_2$
IV^*	Seven smooth rationals meeting with dual graph $\widetilde{E}_6$	$\widetilde{E}_6$

Table 1: Kodaira's table of singular elliptic fibres.

The following technical lemma will be needed at the end of the proof of Kodaira's classification.

Lemma 4.2.4. *Let X_0 be a fiber of $\pi : X \to C$ of multiplicity m, and write $X_0 = mF$. If F is simply connected, then $m = 1$.*

Proof. By Lemma 2.5.11, this will follow from showing that $\operatorname{Pic} F$ is torsion-free. Note that the exponential exact sequences gives rise to the exact sequence

$$\mathrm{H}^1(F;\mathbb{Z}) = 0 \to \mathrm{H}^1(F;\mathscr{O}_F) \to \operatorname{Pic} F \to \mathrm{H}^2(F;\mathbb{Z}) \to 0,$$

but $\mathrm{H}^1(F,\mathscr{O}_F)$ is a vector space, hence torsion free, and $\mathrm{H}^2(F;\mathbb{Z}) \simeq \mathbb{Z}$ since F is compact, connected, so $\operatorname{Pic} F$ is the extension of something torsion free by something torsion free. This implies that $\operatorname{Pic} F$ is torsion free. ■

Theorem 4.2.5 (**Kodaira's Classification of Singular Fibers of Elliptic Fibrations**). *The only possible fibers for a smooth minimal elliptic surface are those listed Table 1.*

Proof. Write a fiber X_0 as mF with $m \in \mathbb{Z}$ the multiplicity of X_0, and write $F = \sum r_i C_i$. If F is irreducible, then, since its arithmetic genus is one, its either a smooth elliptic curve (type I_0 if $m = 1$), a nodal rational curve (type I_1 if $m = 1$), or a cuspidal rational curve (type II if $m = 1$). Hence, assume F reducible. By the adjunction formula, $K_X X_\eta = 0$ for a general fiber X_η. Since $mF = X_0$ is a fiber, we see that $m \cdot K_X F = 0 \implies K_X F = 0$ as well, so

$$0 = \sum r_i C_i K_X = \sum r_i \left(2g(C_i) - 2 - C_i^2\right).$$

We claim that all the coefficients $2g(C_i) - 2 - C_i^2$ are non-negative. Indeed, if $C_i K_X = 2g(C_i) - 2 - C_i^2 < 0$, then

$$-2 \le 2g(C_i) - 2 < C_i^2 \le -1$$

(rightmost inequality coming from Zariski's lemma), so we would have $g(C_i) = 0$ (making C_i smooth rational) and $C_i^2 = -1$, contradicting minimality. Now, since each $r_i > 0$, we must have $2g(C_i) - 2 - C_i^2 = 0$ so C_i is smooth rational with self-intersection -2. Finally, let G be the dual graph to the fiber F which must be one of the extended Dynkin diagrams (but not $\tilde{A}_0$). If G is $\tilde{A}_1$, then F must be either I_2 (if the two components meet at two points) or III (if they meet at one point with multiplicity 2). If G is $\tilde{A}_2$, then F is either I_3 (if the components meet in a cycle) or IV (if they all meet at one point). In all other cases, there is no ambiguity to how the components meet, and we obtain the types I_N, $N \ge 4$ ($\tilde{A}_{N-1}$), I_N^* ($\tilde{D}_{N+4}$), and the types $\mathrm{IV}^*, \mathrm{III}^*, \mathrm{II}^*$ ($\tilde{E}_6, \tilde{E}_7$, and $\tilde{E}_8$).

This completes the analysis when $m = 1$. If $m \neq 1$, then F must not be simply connected by the previous lemma, so only $F = \mathrm{I}_n$, $n \ge 0$ are allowed. This gives the types ${}_m\mathrm{I}_n$ and so we finish. ■

4.3 Local Monodromy

Now that we have a list of potential singular fibers, we seek to understand some properties of (smooth, relatively minimal) elliptic surfaces near each type of fibration. In particular, because our base curve is necessarily smooth, the local picture always looks like a fibration $\pi : X \to \Delta$ whose only nonsingular fiber is $X_0 = \pi^{-1}(0)$. Hence, the restriction $X \setminus X_0 \xrightarrow{\pi} \Delta \setminus \{0\}$ is a fibre bundle by Ehresman's lemma. As a general fact, such a situation gives rise to an action of the fundamental group $\pi_1(\Delta \setminus \{0\}) \simeq \mathbb{Z}$ of the base on the homology $\mathrm{H}_1(X_s)$ of a nearby nonsingular fiber. Indeed, since $X \setminus X_0 \to \Delta \setminus \{0\}$ is a locally trivial fibration, a loop $\gamma \in \pi_1(\Delta \setminus \{0\})$ induced a continuous map $L_\gamma : F \to F$ where $F = \pi^{-1}(\gamma(0))$ and $L_\gamma(x) = \widetilde{\gamma}(1)$ for the unique lift $\widetilde{\gamma} : [0,1] \to X \setminus X_0$ of $\gamma : [0,1] \to \Delta \setminus \{0\}$ based at x – i.e. $\pi \circ \widetilde{\gamma} = \gamma$ and $\widetilde{\gamma}(0) = x$. This map then naturally induces a map $\mathrm{H}_1(X_s) \to \mathrm{H}_1(X_s)$ on homology.

Technical Aside 4.3.1. One can think of this induced action more algebraically/sheaf theoretically. Letting $f = \pi|_{X \setminus X_0} \to \Delta \setminus \{0\}$ be the restriction of π, Proposition A.3 shows that the sheaf $f_{*1}\underline{\mathbb{Z}}_{X \setminus X_0}$ on $\Delta \setminus \{0\}$ is locally constant, and its stalk at a nonzero point $s \in \Delta$ is $\mathrm{H}^1(X_s; \mathbb{Z})$. By the discussion following that proposition, this sheaf then gives rise to an action of $\pi_1(\Delta \setminus \{0\})$ on $\mathrm{H}^1(X_s; \mathbb{Z})$. Because X_s is a compact curve, Poincaré duality allows us to identify $\mathrm{H}^1(X_s) \simeq \mathrm{H}_1(X_s)$ and so $f_{*1}\underline{\mathbb{Z}}_{X \setminus X_0}$ equally gives rise to an action on homology, which agrees with the one described above topologically.

Definition 4.3.1. Let $\pi : X \to \Delta$ be an elliptic fibration with smooth fibers except possibly above $0 \in \Delta$, and let $\gamma \in \pi_1(\Delta \setminus \{0\}) \simeq \mathbb{Z}$ be the generator circling the origin once in the counterclockwise direction. Then, the automorphism $T = T_\gamma : \mathrm{H}_1(X_s) \to \mathrm{H}_1(X_s)$ induced by γ acting on homology is called the **local monodromy** around 0.

The aim of this section is to calculate local monodromy for many of the types of singular fibers appearing in Kodaira's classification. In order to keep the section a reasonable length, we will not give detailed calculations of the monodromy of all types of singular fibers. However, we will say a few words concerning the types not handled here at the end. Our strategy for calculating these monodromy actions comes from expanding the arguments outlined in [2, Ch. V, Sect. 8–10].

Since our fibrations are elliptic, after choosing a basis for $\mathrm{H}_1(X_s) \simeq \mathbb{Z}^2$, we may identify the local monodromy map with a matrix $T \in \mathrm{GL}_2(\mathbb{Z})$, and so we will present the results of our calculations in the form of matrices. Note that, viewed as a matrix, monodromy is only defined up to conjugation (with ambiguity coming from the choice of basis of $\mathrm{H}_1(X_s)$). The strategy for performing these calculations will depend on the type. The types I_n are special in that they are so-called "stable fibrations". For our purposes, this simply means the only singularities on the singular fibre are ordinary double points which, because they have a canonical local form, can be dealt with explicitly. For the remaining types of singular fibres, one calculates their monodromy somewhat indirectly by realising them as arising in a quotient of a stable elliptic fibration by a cyclic group.

4.3.1 Monodromy around an ordinary double point

Setup. Let $f : X \to \Delta$ be a fibration of a surface X over the unit disk $\Delta = \{z \in \mathbb{C} : |z| < 1\}$. Assume that $X_0 = f^{-1}(0)$ is the only singular fibre of f and that X_0 is reduced with no singularities other than ordinary double points.

The goal of this section is to describe, mostly without proof, how to determine the monodromy action in this case. We will see that it is controlled by the existence of a number of "vanishing cycles," homology classes in X_s which degenerate to a singular point of X_0 as $s \to 0$.

Let $p^{(i)} \in X_0$ be an ordinary double point. Since $X_0 = \{f = 0\} \subset X$, by Theorem 2.4.7, there are coordinates (x, y) near $p^{(i)}$ such that $f(x, y) = x^2 + y^2$. Let $B^{(i)}$ be the ball $B^{(i)} = \left\{|x|^2 + |y|^2 \le 1\right\}$ of radius 1, and let $B^{(i)}_t = B^{(i)} \cap f^{-1}(t)$, its fiber over $t \in \Delta$. For each nonzero $t \in \Delta$, the fiber $B^{(i)}_t$ is smooth and contains the circle

$$\begin{aligned} S^{(i)}_t &= \left\{(x, y) \in B^{(i)} : x^2 + y^2 = t, |x|^2 + |y|^2 = |t|\right\} \\ &= \left\{\left(a\sqrt{t}, b\sqrt{t}\right) \in B^{(i)} : (a, b) \in \mathbb{R}^2 \text{ and } a^2 + b^2 = 1\right\} \end{aligned}$$

of radius $\sqrt{|t|}$, where above, we've fixed a particular $\sqrt{t} \in \mathbb{C}$ (The choice of $\sqrt{t}$ does not affect the $S^{(i)}_t$ as a topological space, but does affect its orientation). As $t \to 0$, the circle $S^{(i)}_t$ contracts to the point $p^{(i)} \in X_0$.

One can imagine formally setting $t = 0$ in the above description of $S_t^{(i)}$ to obtain $S_0^{(i)} = \{(0,0)\} = \{p^{(i)}\}$ (recall (x,y) are coordinates on X centered at $p^{(i)}$). Alternatively, one can think of $S_0^{(i)}$ as the result of taking the topological closure $\overline{\bigcup\limits_{\substack{t\in\Delta\\ t\neq 0}} S_t^{(i)}} \subset X$ of the union of all the $S_t^{(i)}$ and then looking at its restriction to the fiber X_0 above 0. In this case, we once again have[7]

$$S_0^{(i)} = \overline{\bigcup_{\substack{t\in\Delta\\ t\neq 0}} S_t^{(i)}} \cap X_0 = \{(0,0)\} = \{p^{(i)}\}.$$

In either case, the circles $S_t^{(i)}$ contract to a point as $t \to 0$, and so we call $S_t^{(i)}$ – or rather, its homology class in $\mathrm{H}_1(X_t;\mathbb{Z})$ – a **vanishing cycle**. These vanishing cycles, one for each double point (defined only up to sign), control the monodromy action by way of the following result.

Theorem 4.3.1 (Picard-Lefschetz Formula). *Let $\beta^{(i)} \in \mathrm{H}_1(X_s;\mathbb{Z})$ represent the vanishing cycle associated to $p^{(i)} \in X_0$. Then, the monodromy map is given by*

$$T(\alpha) = \alpha - \sum_i (\alpha \cdot \beta^{(i)})\beta^{(i)},$$

where $\alpha \in \mathrm{H}_1(X_s;\mathbb{Z})$ and $\alpha \cdot \beta^{(i)}$ denotes their intersection number.

Proof. [14, Section 3.2] ■

The proof of this theorem is reasonably involved, so including it would detract too much from the goal of seeing examples of performing monodromy calculations. For this reason, we have omitted the proof here, and instead recommend reading it in the book by Voisin.

4.3.2 Singular Fibers of Type I_n

We now have enough tools under our belt to compute a few examples. We start with the simplest of cases: an elliptic fibration with no singular fibers.

Example (Type I_0)**.** Let $z : \Delta \to \mathfrak{H}$ be a holomorphic function from the unit disk to the upper half plane $\mathfrak{H} = \{z \in \mathbb{C} : \mathrm{Im} z > 0\}$, and let $\mathbb{Z}\times\mathbb{Z}$ act on $\mathbb{C}\times\Delta$ via

$$(m,n)\cdot(c,s) = (c + m + nz(s), s).$$

Form the quotient $X = \mathbb{C}\times\Delta/\mathbb{Z}\times\mathbb{Z}$ which is a non-singular surface fibred over Δ. By construction $X_s \simeq \mathbb{C}/(\mathbb{Z}\oplus\mathbb{Z}z(s))$ is an elliptic curve. The monodromy here is trivial because $X \to \Delta$ is a fibration with contractible base.

The next example will be a little more interesting. Before getting to it, recall that there exists an analytic isomorphism $j : \mathfrak{H}/\mathrm{SL}_2(\mathbb{Z}) \to \mathbb{C}$ from the moduli space of smooth elliptic curves to $\mathbb{C}$ such that $j(z) = 0$ if z is in the same orbit as $\mathbf{e}(\frac{1}{6}) = \exp(2\pi i/3)$ under the $\mathrm{SL}_2(\mathbb{Z})$-action, and $j(z) = 1$ if z is in the same orbit

[7] If you have a sequence of points $(x_n, y_n) \in \bigcup S_t^{(i)}$ such that $(x,y) := \lim(x_n,y_n) \in X_0$ (i.e. $f(x,y) = x^2+y^2 = 0$), then $\lim f(x_n,y_n) = 0$ so also, by definition of the $S_t^{(i)}$'s, $|x|^2+|y|^2 = \lim(|x_n|^2+|y_n|^2) = 0$, so $(x,y) = (0,0)$.

as i. Explicitly, given an elliptic curve of the form $E : y^2 = 4x^3 - g_2 x - g_3$, we have

$$j(E) = \frac{g_2^3}{g_2^3 - 27g_3^2}.$$

The fact that the denominator is nonzero turns out to be equivalent to smoothness of E. With this in mind, given an elliptic fibration $\pi : X \to S$, we define its **period map** (or **functional invariant**) to be $J(s) = j(X_s)$.

Remark 4.3.1. Consider a fibration in Weierstrass normal form

$$X = \left\{ \left([z_0 : z_1 : z_2], s \right) \in \mathbb{P}^2 \times \Delta \mid z_0 z_2^2 = 4z_1^3 - g_2(s) z_0^2 z_1 - g_3(s) z_0^3 \right\},$$

for holomorphic $g_2, g_3 : \Delta \to \mathbb{C}$, then,

$$J(s) = \frac{g_2(s)^3}{g_2(s)^3 - 27g_3(s)^2},$$

with X_s non-singular if and only if $g_2^3(s) \neq 27g_3^2(s)$.

Example (Type I_1)**.** Let X be as in the previous remark with $g_2(s) = 3 - s$ and $g_3(s) = 1 - s$, so

$$X = \left\{ \left([z_0 : z_1 : z_2], s \right) \in \mathbb{P}^2 \times \Delta \mid z_0 z_2^2 = 4z_1^3 + (s-3) z_0^2 z_1 + (s-1) z_0^3 \right\}.$$

We first show that X is non-singular. Consider the function $F(z_0, z_1, z_2, s) = 4z_1^3 + (s-3)z_0^2 z_1 + (s-1)z_0^3 - z_0 z_2^2$ whose partials are

$$\begin{array}{rclrcl} \frac{\partial F}{\partial s} & = & z_0^2 z_1 + z_0^3 & \frac{\partial F}{\partial z_2} & = & -2z_0 z_2 \\ \frac{\partial F}{\partial z_1} & = & 12z_1^2 + (s-3)z_0^2 & \frac{\partial F}{\partial z_0} & = & 2(s-3)z_0 z_1 + 3(s-1)z_0^2 - z_2^2 \end{array}$$

which only all vanish when $(z_0, z_1, z_2) = 0$, and so never all vanish at a point of $X = \{F = 0\}$. Thus, X is non-singular. We claim also that $X_0 = \{F_0 := F(\cdot, \cdot, \cdot, 0) = 0\}$ is an irreducible rational with node at $\left[1 : -\frac{1}{2} : 0\right]$, and that X_s is non-singular elliptic for $s \neq 0$. This last bit can be verified by looking at the functional invariant

$$J(s) = \frac{(3-s)^3}{s(27 - 18s - s^2)}$$

whose denominator vanishes only when

$$s(27 - 18 - s^2) = 0 \iff s = 0 \text{ or } s = -9 \pm 6\sqrt{3},$$

but the latter values do not lie in the unit disk. Returning to X_0, its partials are

$$\frac{\partial F_0}{\partial z_0} = -6z_0 z_1 - 3z_0^2 - z_2^2 \qquad \frac{\partial F_0}{\partial z_1} = 12z_1^2 - 3z_0^2 \qquad \frac{\partial F_0}{\partial z_2} = -2z_0 z_2$$

which all vanish at $[z_0 : z_1 : z_2] = [1 : -\frac{1}{2} : 0]$ and nowhere else. This is the only singular point of X_0, so we only need to show that it is a node to verify that X_0 is of type I_1. Indeed, X_0 is the (projectivization of the) curve $E : y^2 = 4x^3 - 3x - 1 = (2x+1)^2(x-1)$ which is visibly a nodal cubic.

Finally, we wish to calculate the monodromy which we claim is given by

$$T = \begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix}.$$

For real $s \in \Delta$ (i.e. $s \in (0,1)$), the polynomial $f_s(x) = 4x^3 + (s-3)x + (s-1)$ has three real roots $s_1 < s_2 < s_3$ since its discriminant is nonzero for such s and since it has 3 real roots when $s = 1$. As $s \to 0$, we see that $s_1, s_2 \to -\frac{1}{2}$ and $s_3 \to 1$ since these are the roots of $f_0(x) = (2x-1)^2(x+1)$. Now, consider the projection $p : ([z_0 : z_1 : z_2], s) \mapsto (z_1 : z_0)$ (which is a rational map, not a morphism). For a fixed $s \neq 0$, this exhibits X_s as a double cover of $\mathbb{P}^1 = \{[0:1]\} \cup \mathbb{C}$ ramified at s_1, s_2, s_3, and $\infty = [0:1]$. Let $a = p^{-1}([-\infty, s_1])$ and $b = p^{-1}([s_1, s_2])$, two simple closed loops in X_s. From our earlier remark on the roots of f_s, we see that b is a vanishing cycle. Furthermore, a, b generate $\mathrm{H}_1(X_s) \simeq \mathbb{Z}^2$. To see that they are linearly independent it is enough to note that their intersection number $a \cdot b$ is ± 1 since they meet in a single point. Hence, given a vanishing linear combination $na + mb = 0 \in \mathrm{H}_1(X_s)$ (with $n, m \in \mathbb{Z}$), we have

$$0 = (na + mb) \cdot b = \pm n \implies 0 = na + mb = mb \implies m = 0,$$

so $n = m = 0$. Above, we used $b^2 = 0$ since its fundamental cohomology class $[b]^* \in \mathrm{H}^1(X_s; \mathbb{Z})$ lives in odd-dimensional cohomology, so $[b]^* \smile [b]^* = -[b]^* \smile [b]^*$ which makes b^2 a torsion element of $\mathbb{Z}$. Thus, a, b give a basis for $\mathrm{H}_1(X_s)$, and Picard-Lefschetz shows that $T(b) = b - (b^2)b = b$ while $T(a) = a - (a \cdot b)b$. In order to completely determine the monodromy action, we only need to calculate $a \cdot b$.

We will do this rather explicitly. Let $E = \left\{(z_1, z_2) \in \mathbb{C}^2 : z_2^2 = 4z_1^3 + (s-3)z_1 + (s-1)\right\} =$ "$X_s \cap \mathbb{C}^2$" be an affine slice of the projective curve X_s. In these coordinates, the projection p above becomes the map $E \to \mathbb{C}$ given by $p(z_1, z_2) = z_1$. Letting $(x_1, y_1, x_2, y_2) = (\mathrm{Re} z_1, \mathrm{Im} z_1, \mathrm{Re} z_2, \mathrm{Im} z_2)$ denote the real and imaginary parts of z_1, z_2, we can express E in real coordinates as

$$E = \left\{ (x_1, y_1, x_2, y_2) \,\middle|\, \begin{array}{lcc} x_2^2 - y_2^2 & = & 4x_1^3 + (s-3)x_1 + (s-1) - 12x_1y_1^2 \\ 2x_2y_2 & = & y_1(12x_1^2 - 4y_1^2 + s - 3) \end{array} \right\} \subset \mathbb{R}^4.$$

We are interested in calculating the intersection number between the curves $a = p^{-1}((-\infty, s_1])$ and $b = p^{-1}([s_1, s_2])$. As both of these are contained in $E_{\mathbb{R}} := p^{-1}(\mathbb{R})$, $y_1 = 0$ on a, b. That is, on the part of E we care about, we have $2x_2y_2 = 0$, so $x_2 = 0$ or $y_2 = 0$. Furthermore, since x_2, y_2 vary continuously with x_1, the coordinate that must be 0 can only change over branch points, e.g. if $x_2 = 0$ over any point in a not lying over s_1, then $x_2 = 0$ on all of a. Recall that

$$f_s(x_1) = 4x_1^3 + (s-3)x_1 + (s-1).$$

Since $x_2^2 - y_2^2 = f_s(x_1)$ on a and $f_s(x_1) \to -\infty$ as $x_1 \to -\infty$, we see that $x_2 = 0$ on all of a. Similarly, that $f_s(x)$ is increasing near s_1 (since $-y_2^2 = f_s(x) < 0$ when $x < s_1$ but $f_s(s_1) = 0$), let's us see that $y_2 = 0$ on all of b. All together (i.e. using that $y_1 = 0 = x_2$ on a), this means that $a = p^{-1}((-\infty, s_1])$ is given by

$$a = \left\{ (x_1, y_1, x_2, y_2) \,\middle|\, \begin{array}{lcccc} -y_2^2 & = & 4x_1^3 + (s-3)x_1 + (s-1) & = & f_s(x_1) \\ y_1 & = & x_2 & = & 0 \end{array} \right\} \subset \mathbb{R}^4,$$

so we can think of (the "front half" of) a as the curve $\alpha(t) = \left(t, 0, 0, \sqrt{-f_s(t)}\right)$ and (the "top half" of) b as the curve $\beta(t) = \left(t, 0, \sqrt{f_s(t)}, 0\right)$. Using these parameterizations, we can calculate the tangent vectors of a, b at their point of intersection $q := (s_1, 0, 0, 0)$, and use this to determine $a \cdot b$. With the given parameterization, the (unit/normalized) tangent vector to a at q is

$$\lim_{t \to s_1^-} \frac{\alpha'(t)}{\|\alpha'(t)\|} = \lim_{t \to s_1^-} \frac{\left(1, 0, 0, -\frac{1}{2} f_s'(t)/\sqrt{-f_s(t)}\right)}{\sqrt{1 - \frac{1}{4} f_s'(t)^2/f_s(t)}}.$$

Since[8]

$$-\frac{1}{2} \lim_{t \to s_1^-} \frac{f_s'(t)}{\sqrt{-f_s(t)}\sqrt{1 - \frac{1}{4} f_s'(t)^2/f_s(t)}} = -\frac{1}{2} \lim_{t \to s_1^-} \frac{f_s'(t)}{\sqrt{\frac{1}{4} f_s'(t)^2 - f_s(t)}} = \frac{-\frac{1}{2} f_s'(s_1)}{\sqrt{\frac{1}{4} f_s'(s_1)^2}} = -1,$$

and

$$\lim_{t \to s_1^-} \left(1 - \frac{1}{4} \frac{f_s'(t)^2}{f_s(t)}\right)^{-1/2} = 0$$

we see that the normalized tangent vector to a at q, in more standard notation, is $-\frac{\partial}{\partial y_2}\Big|_q$. Similarly, the normalized tangent vector to b at p is

$$\begin{aligned}
\lim_{t \to s_1^+} \frac{\beta'(t)}{\|\beta(t)\|} &= \lim_{t \to s_1^+} \frac{\left(1, 0, \frac{1}{2} f_s'(t)/\sqrt{f_s(t)}, 0\right)}{\sqrt{1 + \frac{1}{4} f_s'(t)/f_s(t)}} \\
&= \lim_{t \to s_1^+} \left(\left(1 + \frac{1}{4} \frac{f_s'(t)}{f_s(t)}\right)^{-1/2}, 0, \frac{1}{2} \frac{f_s'(t)}{\sqrt{f_s(t) + \frac{1}{4} f_s'(t)}}, 0 \right) \\
&= (0, 0, 1, 0).
\end{aligned}$$

In more standard notation, it is $\frac{\partial}{\partial x_2}\Big|_q$. Finally, the tangent space $T_q E$ to E at $q = p^{-1}(s_1)$ has (ordered) basis $\frac{\partial}{\partial x_2}\Big|_q, \frac{\partial}{\partial y_2}\Big|_q$ while we have just shown that the intersection $a \cdot b$ gives rise to the (ordered) basis $-\frac{\partial}{\partial y_2}\Big|_q, \frac{\partial}{\partial x_2}\Big|_q$. Since this is the same orientation as that on $T_q E$ (i.e. since, in $\bigwedge^2 T_q E$, we have $-\frac{\partial}{\partial y_2}\Big|_q \wedge \frac{\partial}{\partial x_2}\Big|_q = \frac{\partial}{\partial x_2}\Big|_q \wedge \frac{\partial}{\partial y_2}\Big|_q$), we conclude that $a \cdot b = 1$.

Now that we have calculated $a \cdot b = 1$, Picard-Lefschetz shows that $T(a) = a - (a \cdot b)b = a - b$, and we previously saw that $T(b) = b$. Recalling that the monodromy matrix is only defined up to conjugation, in order to obtain the particular matrix that is standard to write down in the case of type I_1, we use the basis $\{b, -a\}$. In this basis, our monodromy matrix is

$$T = \begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix}$$

as claimed.

This covers types I_0 and I_1. In order to ensure this section has finite length, instead of continuing this pattern of bumping the index by one with each example, we will handle the cases I_n for all $n \geq 2$ at

[8] Below, we use $-f_s(t) > 0$ for $t < s_1$, $f_s(s_1) = 0$, and $f_s'(s_1) > 0$. The last fact is true because f_s is strictly increasing near s_1.

once. With many examples of these monodromy calculations, the main difficulty lies not in determining the monodromy itself, but in constructing a fibration with a singular fiber of a given type. For type I_n, we are partially luckily, because these can be constructed from a fibration of type I_1, which we constructed above, by taking (square, cube, etc.) "roots" of the I_1 fibration. However, while the construction itself is easy to give, we will see that verifying that it gives a fiber of type I_n still takes some time.

Example (Type I_n)**.** Let $f : X \to \Delta$ be an elliptic fibration with a single singular fiber $X_0 = f^{-1}(0)$ of type I_1, and let $\delta_n : \Delta \to \Delta, s \mapsto s^n$ be the nth power map. Consider the below commutative diagram

$$\begin{tikzcd}
X^{(n)} \arrow[r, "\tau''"] \arrow[d, "f^{(n)}"] \arrow[rrr, bend left, "\tau^{(n)}"] & X'' \arrow[r, "\tau'"] \arrow[d, "f''"] & X' \arrow[r, "\tau"] \arrow[d, "f'"] & X \arrow[d, "f"] \\
\Delta \arrow[r, equal] & \Delta \arrow[r, equal] & \Delta \arrow[r, "\delta_n"] & \Delta
\end{tikzcd}$$

where $X' = X \times_\Delta \Delta$ is the fibre product, X'' is the normalization of X', and $X^{(n)}$ is the minimal desingularization of X'', which we claim has a singular fibre of type I_n. We first describe the singularity in the fibre X_0'' of X'' above $0 \in \Delta$. Let $p \in X_0$ be the double point. so, by Theorem 2.4.7, near p we have coordinates (x, y) such that $f(x, y) = x^2 + y^2$. Hence, near the unique point $(p, 0) \in X' = \{(z, s) \in X \times \Delta : f(z) = \delta_n(s)\}$ above p, the fibre product is isomorphic to the affine surface

$$S := \left\{(s, x, y) : s^n = x^2 + y^2\right\} \subset \mathbb{C}^3,$$

Because the above surface is normal[9] the same is true about X'', i.e. near its unique point $q \in X''$ over $p \in X$, it is isomorphic to S. Now, S has a singularity of type A_{n-1} at $(0, 0, 0)$ but is otherwise nonsingular. To make showing this slightly easier, we perform the change of variables $u = x + iy$ and $v = x - iy$ in order to rewrite S as

$$S \simeq \{(s, u, v) : s^n = uv\} \subset \mathbb{C}^3.$$

Now, for $\zeta \in \mathbb{C}^\times$ a primitive nth root of unity, $g : \mathbb{C}^2 \to \mathbb{C}^2, (a, b) \mapsto (\zeta a, \zeta^{-1} b)$ the canonical action giving rise to an A_{n-1} singularity (recall Remark 2.3.8), and $G = \langle g \rangle$, we have an isomorphism

$$\begin{array}{rccc}
\phi : & \mathbb{C}^2/G & \longrightarrow & S \\
& [(a, b)] & \longmapsto & (ab, a^n, b^n)
\end{array}$$

whose inverse is

$$\begin{array}{rccc}
\psi : & S & \longrightarrow & \mathbb{C}^2/G \\
& (s, u, v) & \longmapsto & \left[\left(u^{1/n}, v^{1/n}\right)\right]
\end{array}$$

This shows that S has an A_{n-1} singularity at $(0, 0, 0)$ and consequently, that X'' has one at q, its unique point over the double point $p \in X_0$. The point q lies in $X_0'' = (\tau \circ \tau')^{-1}(X_0)$ and, because it is an A_{n-1} singularity, is resolved under τ'' by a string $C_2, \ldots, C_n$ of (-2)-curves. Letting $C_1 = \overline{(\tau'')^{-1}(X_0'' \setminus \{q\})}$ be the proper transform of X_0'', we have that

$$X_0^{(n)} = \left(f^{(n)}\right)^{-1}(0) = mC_1 + \sum_{i=2}^{n} C_i$$

[9]To avoid interrupting the flow of things, we will show this below the current example

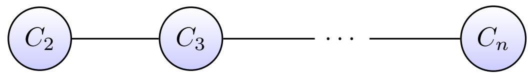

Figure 7: The dual graph of $(\tau'')^{-1}(q)$

for some $m > 0$, and we know C_iC_j when $i, j \geq 2$. We claim that $m = 1$, that $C_1C_2 = C_1C_n = 1$, and that $C_1C_i = 0$ for all $i \in \{3, \ldots, n-1\}$, showing that X_0'' is of type I_n (i.e. showing the C_i meet in a cycle of length n). Indeed, for any $j \in \{3, \ldots, n-1\}$, we have

$$0 = X_0^{(n)}C_j = mC_1C_j + \sum_{i=2}^{n} C_iC_j = mC_1C_j + C_{j-1}C_j + C_{j+1}C_j + C_j^2 = mC_1C_j + 1 + 1 - 2 = mC_1C_j.$$

Since $m > 0$, we conclude that $C_1C_j = 0$. Similarly,

$$0 = X_0^{(n)}C_2 = mC_1C_2 - 2 + 1 = mC_1C_2 - 1.$$

Since $m > 0$ and $C_1C_2 \in \mathbb{Z}$, we conclude that $m = C_1C_2 = 1$. Finally, $C_1C_n = 1$ as well since

$$0 = X_0^{(n)}C_n = C_1C_n + 1 - 2 = C_1C_n - 1.$$

Thus, the dual graph to X_0'' is a cycle of length n, making it a singularity of type I_n.

Finally, we calculate its monodromy. For this, note that we have the following commutative diagram

$$\begin{tikzcd}
X^{(n)} \setminus X_0^{(n)} \arrow[r, "\sim"] \arrow[d] & X' \setminus X_0' \arrow[r] \arrow[d] & X \setminus X_0 \arrow[d] \\
\Delta \setminus \{0\} \arrow[r, equal] & \Delta \setminus \{0\} \arrow[r, "\delta_n : s \mapsto s^n"] & \Delta \setminus \{0\}
\end{tikzcd}$$

so, as far as monodromy is concerned, we may think of $X^{(n)}$ as simply being an n-fold cover of $X \setminus X_0$. As such, the action of a generator $\gamma \in \pi_1(\Delta \setminus \{0\})$ on $X^{(n)}$ is just n times its action on X. More formally, the local monodromy $T : \mathrm{H}_1(X_s) \to \mathrm{H}_1(X_s)$ of $X^{(n)}$ is given by the matrix

$$T = \begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix}^n = \begin{pmatrix} 1 & n \\ 0 & 1 \end{pmatrix}.$$

We now fulfill a promised made in a footnote by showing that the surface S considered in the previous example is a normal complex space. The proof of this fact will be more "algebraic" than most of the arguments in this section, but it is good to include for completeness.

Lemma 4.3.2. *Let $S = \{(s, x, y) : s^n = x^2 + y^2\} \subset \mathbb{C}^3$. This surface is normal.*

Proof. Let $\mathscr{O}_{\mathbb{C}^3}$ denote the sheaf of holomorphic functions on $\mathbb{C}^3$, and let $\mathscr{I} \subset \mathscr{O}_{\mathbb{C}^3}$ be the ideal sheaf generated by $f(s, x, y) = x^2 + y^2 - s^n$, so $\mathscr{O}_S = \mathscr{O}_{\mathbb{C}^3}/\mathscr{I}$. Hence, for any point $p \in S$, we have

$$\mathscr{O}_{S,p} = \mathscr{O}_{\mathbb{C}^3,p}/\mathscr{I}_p \simeq \mathbb{C}\{s, x, y\}/(x^2 + y^2 - s^n),$$

so we only need show this latter ring is integrally closed. where $\mathbb{C}\{s, x, y\}$ is the ring of convergent power series in 3 variables.

Let $F = \operatorname{Frac} \mathbb{C}\{s, x\}$, and let $E = F[\![y]\!]/(y^2 - (s^n - x^2)) = F[y]/\big(y^2 - \big(s^n - x^2\big)\big)$, a quadratic extension of F. Note that any element $\alpha \in E$ can be written in the form $\alpha = g + hy$ with $g, h \in F$, and so satisfies the polynomial

$$(X - (g + hy))(X - (g - hy)) = X^2 - 2gX + (g^2 - h^2(s^n - x^2)) \in F[X].$$

Hence, $\alpha = g + hy \in A$ is integral over $R := \mathbb{C}[\![s, x]\!]$ if and only if both $2g \in R$ and $g^2 - h^2(s^n - x^2) \in R$. Since $\frac{1}{2} \in R$, we see that we have $g \in R$ and so $h^2(s^n - x^2) \in R$ as well, when α is integral over R. If $h \in \operatorname{Frac} R$ has a nontrivial denominator, then $h^2(s^n - x^2)$ will too. This is because $(s^n - x^2)$ is not a square in the UFD R, so some irreducible $\pi \in R$ divides it with odd multiplicity, but every irreducible in R divides (the denominator of) h^2 with even multiplicity. Thus, $\alpha \in E$ is integral over R iff $g \in R$ and $h \in R$ iff $\alpha \in R[y]/(y^2 - (s^n - x^2)) \simeq \mathscr{O}_{S,p}$. Therefore, $\mathscr{O}_{S,p}$ is the integral closure of R in E, and so it itself integrally closed. This proves that S is normal. ∎

The above lemma officially finishes off our monodromy calculations for elliptic fibres of type I_n, so we may now move on to other fibre types.

4.3.3 Singular Fibers of Type III and I_n^*

The Picard-Lefschetz formula, which was the main workhorse behind computing monodromy for fibers of type I_n, no longer applies for any of the other types; it is specific to calculating the monodromy around an ordinary double point. For the remaining fiber types, the strategy for calculating monodromy is in a sense "dual" to the strategy used to calculate monodromy around a fiber of type I_n, $n > 1$, in the previous section. In that case, we were able to realize a fibre of type I_n as, essentially, a pullback of a fiber of type I_1 along the nth power map $\delta_n : \Delta \to \Delta, s \mapsto s^n$. This allowed us realize the monodromy around a fiber of type I_n as the nth power of the monodromy around a fiber of type I_n.

For the remaining fiber types, we do the opposite. It is a fact ([2, Ch. III, Sect. 10] combined with Lemma 4.1.1) that every fiber type can be realized as a quotient of a fiber of type I_n, for some n, acted on by a cyclic group G. Morally, for each fiber type, if you pull it back along the nth power map for an appropriately chosen n, you end up with a fiber of type I_k for some k. Miraculously, for most of the remaining fiber types, you can arrange it so that this pullback has no singular fibers (is type I_0); in other words, most remaining fiber types can be realized as the quotient of an *elliptic surface with no singular fibers* acted on by some cyclic group. We will see how this can play out in the following example.

The only remaining (reduced) fiber type which cannot be realized as the quotient of one of type I_0 is the type I_n^* when $n \geq 1$. Instead, this is most easily constructed as the quotient of a fiber of type I_{2n}. We will perform this construction, and subsequent monodromy calculation, after doing so for type III below.

Example (Type III)**.** Let $z : \Delta \to \mathbb{C}$ be the holomorphic map $z(s) = \frac{i+is^2}{1-s^2}$, and note that

$$z(is) = \frac{i - is^2}{1 + s^2}\frac{i}{i} = \frac{s^2 - 1}{i + is^2} = -\frac{1}{z(s)}.$$

Let Y be the surface $Y = \mathbb{C} \times \Delta/(\mathbb{Z} \oplus \mathbb{Z})$ where $\mathbb{Z} \oplus \mathbb{Z}$ acts by

$$(m, n) \cdot (c, s) = (c + m + nz(s), s),$$

so in particular, $Y_0 = \mathbb{C}/(\mathbb{Z} \oplus \mathbb{Z}i)$. Note that Y has an automorphism $\mu : Y \to Y$ induced by the map

$(c, s) \mapsto \left(-\frac{c}{z(s)}, is\right) = (cz(is), is)$ on $\mathbb{C} \times \Delta$. This is well-defined on Y because

$$\mu((m, n) \cdot (c, s)) = (-n, m) \cdot \mu(c, s)$$

for any $m, n \in \mathbb{Z}$ and $(c, s) \in \mathbb{C} \times \Delta$. Let $G = \langle \mu \rangle \simeq \mathbb{Z}/4\mathbb{Z}$ by the cyclic group of automorphisms generated by μ, let $Y' = Y/G$, and let $p : Y \to Y'$ be the natural quotient map. Y' is smooth away from the images of points of $P \in Y$ with non-trivial isotropy subgroups $G_P = \{\sigma \in G : \sigma(P) = P\}$, so we begin our investigation of it by finding these fixed points.

Every non-trivial subgroup of G is generated by μ or μ^2, so we really only care about the fixed points of these two elements. We begin with μ. Any fixed point (c, s) of μ must satisfy $s = is$, and so have $s = 0$. Since $z(0) = i$ and $\mu(c, 0) = (ic, 0)$, any fixed point must also satisfy

$$c - ci = m + in \implies c = \frac{(m - n) + i(m + n)}{2}$$

for some $m, n \in \mathbb{Z}$. Thinking of $(c, 0)$ as an element of $Y_0 = \mathbb{C}/(\mathbb{Z} \oplus \mathbb{Z}i)$, we see that c gives a well-defined class in $(\frac{1}{2}\mathbb{Z} \oplus \frac{1}{2}\mathbb{Z}i)/(\mathbb{Z} \oplus \mathbb{Z}i)$, and so there are at most 4 unique points of Y fixed by μ. These are

$$P_1 = (0, 0) \qquad P_2 = \left(\frac{1 + i}{2}, 0\right) \qquad P_3 = \left(\frac{i - 1}{2}, 0\right) \qquad P_4 = (i, 0)$$

We see immediately that $P_1 = P_4$, $P_2 = P_3$, and $P_1 \neq P_2$ (as elements of Y, not $\mathbb{C} \times \Delta$), so Y has two points with isotropy subgroup $G \simeq \mathbb{Z}/4\mathbb{Z}$. One can similarly find the fixed points of μ^2 which leads them to

$$Q_1 = (0, 0) \qquad Q_2 = \left(\frac{1}{2}, 0\right) \qquad Q_3 = \left(\frac{i}{2}, 0\right) \qquad Q_4 = \left(\frac{1 + i}{2}, 0\right)$$

Here, we have that $Q_1 = P_1$ and $Q_4 = P_2$ are the μ-fixed points we already found, so $Q_2 \neq Q_3$ are the only points on Y with isotropy subgroup $\langle \mu^2 \rangle \simeq \mathbb{Z}/2\mathbb{Z}$.

Referring back to Theorem 2.3.10 (in either case, the weights are $(1, 1)$ since μ looks like multiplication by i in both slots), we can see that the points P_1, P_4 give rise to two $A_{4,1}$ singularities, $p(P_1) \neq p(P_4)$. However, the points Q_2, Q_3 have the same image $p(Q_2) = p(Q_3) \in Y'$, and so give rise to a double $A_1 = A_{2,1}$ singularity. This is to say that we can resolve Y''s singularities by a smooth surface $\overline{Y} \to Y'$ which has (distinct) smooth rationals with self-intersection -4 above $p(P_1), p(P_4)$ and a double smooth rational with self-intersection -2 above $p(Q_2) = p(Q_3)$. The spaces defined so far fit into a diagram like so

$$\begin{tikzcd}
\mathbb{C} \times \Delta \arrow[d] & & & \\
Y \arrow[r, "p"] \arrow[d, "\pi"] & Y' \arrow[d, "\pi'"] & X \arrow[l] \arrow[d] & \overline{Y} \arrow[l] \arrow[ll, bend right, "q"'] \arrow[d, "\overline{\pi}"] \\
\Delta \arrow[r, "\delta_4 : s \mapsto s^4"] & \Delta \arrow[r, equal] & \Delta \arrow[r, equal] & \Delta
\end{tikzcd}$$

where X is a relatively minimal model of $\overline{Y}$, obtained by blowing down all (-1)-curves in its fibers.

Because $\overline{Y} \setminus \overline{Y}_0 \simeq Y' \setminus Y'_0$, the only (-1)-curves of $\overline{Y}$ are contained in the fiber $\overline{Y}_0$ above 0. By the remarks on how $\overline{Y}$ resolves Y''s singularities, we know that $\overline{Y}_0 = q^* Y'_0 + 2B + D_1 + D_2$ with B, D_1, D_2 smooth rational, $B^2 = -2$ and $D_1^2 = D_2^2 = -4$. Hence, any (-1)-curve in $\overline{Y}_0$ must originate from Y'_0. We claim that Y_0' is a (singular) rational curve of multiplicity 4, so $q^*Y_0' = 4C$ with $C \subset \overline{Y}_0$ smooth rational. To see this, consider the map $f = \pi' \circ p : Y \to \Delta, P \mapsto \pi(P)^4$, so $Y_0' = p(f^{-1}(0))$. Now, $f^{-1}(0) \subset Y$ is the analytic subspace whose ideal sheaf is

$$\mathscr{I}_{f^{-1}(0)} := f^*(\mathscr{I}_0) = \pi^*(\delta_4^*(\mathscr{I}_0)) = \pi^*(\mathscr{I}_0^4) = (\pi^*(\mathscr{I}_0))^4 = \mathscr{I}_{Y_0}^4,$$

where $\mathscr{I}_0 \subset \mathscr{O}_\Delta$ is the ideal sheaf of $0 \in \Delta$ and $\mathscr{I}_{Y_0} = \pi^*\mathscr{I}_0$ is the ideal sheaf of the fiber Y_0 of π. This shows that $Y_0' = p(f^{-1}(0))$ is the curve $p(Y_0)$ with multiplicity 4. Since Y_0 has genus 1, the arithmetic genus of $p(Y_0)$ is at most 1, but since $p(Y_0)$ is singular, its geometric genus is strictly less than its arithmetic genus, so $p(Y_0)$ has geometric genus 0, i.e it is rational. Hence, Y_0' really is rational of multiplicity 4 as claimed. As such, $\overline{Y}_0 = 4C+2B+D_1+D_2$ with B, D_1, D_2 as before and $C \simeq \mathbb{P}^1$. Note that, in addition to the self-intersections $B^2 = -2$ and $D_1^2 = D_2^2 = -4$, we have that $CB = CD_1 = CD_2 = 1$ while $BD_1 = BD_2 = D_1D_2 = 0$. Combining these with the fact that $(\overline{Y}_0)^2 = 0$, one sees that $C^2 = -1$. Returning to our formation of the minimal desingularization X of Y', we have just shown that C, the proper transform of Y_0' under the map $q : \overline{Y} \to Y'$, is the unique (-1)-curve on $\overline{Y}$. The result of blowing it down, followed by blowing down all new (-1)-curves that arise, is shown in the sequence of incidence graphs in Figure 8. Hence, we do indeed

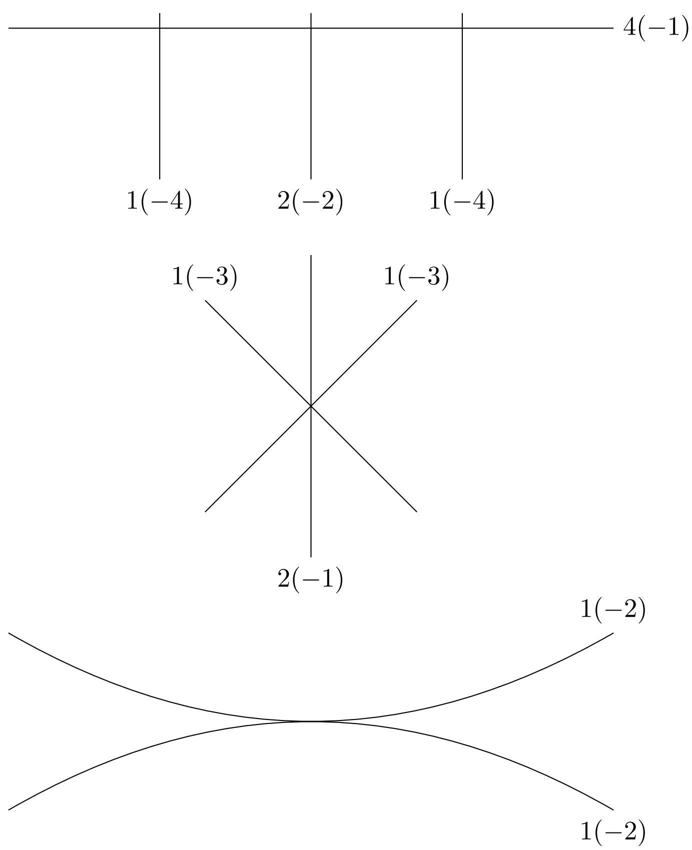

Figure 8: The incidence graphs obtained by successively blowing down the (-1)-curves of $\overline{Y}_0$. Each line represents in rational curve in the fiber above 0. A line labelled "$m(s)$" indicates that the corresponding copy of $\mathbb{P}^1$ appears with multiplicity m and has self-intersection s. To go from one graph to the next, we blowdown the (-1)-curve and then adjust self-intersections in accordance with Corollary 2.3.4. The end result is two smooth rationals meeting at a double point.

end up with a fibration $X \to \Delta$ with a single singular fibre of type III (two smooth rationals meeting at a double point).

With all that out of the way, calculating monodromy should now be simple. The purpose of the majority of the work done above was to construct a smooth elliptic fibration so as to identify which type in Kodaira's table we are going to calculate the monodromy for. However, the monodromy action is calculated from the data of the spaces away from the singular fibers. That is, we throw away the fiber above 0 in order to get a well-defined monodromy. Hence, the monodromy of $X \to \Delta$ agrees with that of $Y' \to \Delta$. Put more rigorously, because we have a commutative diagram

$$\begin{tikzcd} Y' \setminus Y_0' \arrow[dr] & & X \setminus X_0 \arrow[ll, "\sim"'] \arrow[dl] \\ & \Delta \setminus \{0\} & \end{tikzcd}$$

whose horizontal arrow is an isomorphism, the monodromy of X agrees with that of $Y' = Y/G$. To calculate the monodromy of Y', first let $\Delta^* = \Delta \setminus \{0\}$, $Z = Y \setminus Y_0$ and $Z' = Y' \setminus Y_0'$ for notational convenience, and then note that we have a commutative diagram

$$\begin{tikzcd} & Z_s \arrow[r] \arrow[d] & Z'_{s^4} \arrow[d] \\ G \arrow[r] \arrow[d, equal] & Z \arrow[r, "p"] \arrow[d, "\pi"] & Z' \arrow[d, "\pi'"] \\ G \arrow[r] & \Delta^* \arrow[r, "\delta_4"] & \Delta^* \end{tikzcd}$$

whose rows and columns are fiber sequences. Above we've fixed a basepoint $z = (c, s) \in Z$ (e.g. $s \in \Delta^*$ and $c \in \mathbb{C}/\mathbb{Z} \oplus \mathbb{Z}z(s)$) and the map $G \to Z$ sends $\mu^k \mapsto \mu^k(z) \in Z$ where $\mu : Z \to Z$ is the automorphism generating G. Keep in mind that Δ^* is homotopy equivalent to S^1, and let $\gamma \in \pi_1(\Delta^*, s^4)$ be the generator circling the origin once counterclockwise. Then, γ lifts to a map $\widetilde{\gamma} : [0, 1] \to \Delta^*$ from $\widetilde{\gamma}(0) = s$ to $\widetilde{\gamma}(1) = is$. Recall that Z is fibered trivially over Δ^*, i.e. topologically $\pi : Z \to \Delta^*$ is equivalent to the natural projection $T \times \Delta^* \to \Delta^*$ where $T \cong S^1 \times S^1$ is a torus.[10] Thus, the map $L_{\widetilde{\gamma}} : Z_s \to Z_{is}$ induced by $\widetilde{\gamma}$ acts trivially on homology. More specifically, we have a commutative diagram

$$\begin{tikzcd} T \arrow[r, equal] & T \\ Z_s \arrow[u, "\wr"] \arrow[r, "L_{\widetilde{\gamma}}"] \arrow[d] & Z_{is} \arrow[u, "\wr"'] \arrow[d] \\ Z'_{s^4} \arrow[r, "L_\gamma"] & Z'_{s^4} \end{tikzcd}$$

which will allow us to compute γ's monodromy action of Z'_{s^4}. For $a \in \mathbb{C}$ and $s \in \Delta$, let $\gamma_{a;s} : [0, 1] \to \mathbb{C} \times \Delta$

---

[10]If you take $T = \mathbb{C}/(\mathbb{Z} \oplus \mathbb{Z}i)$, then an explicit equivalence $f : Z \to T \times \Delta^*$ is given by

$$f(c, s) = \left(c + \frac{i - z(s)}{z(s) - \overline{z(s)}}(c - \overline{c}), s \right)$$

since this sends $\mathbb{Z} \oplus \mathbb{Z}z(s)$ to $\mathbb{Z} \oplus \mathbb{Z}i$, is bijective, and commutes with the projections $Z, T \times \Delta^* \rightrightarrows \Delta^*$.

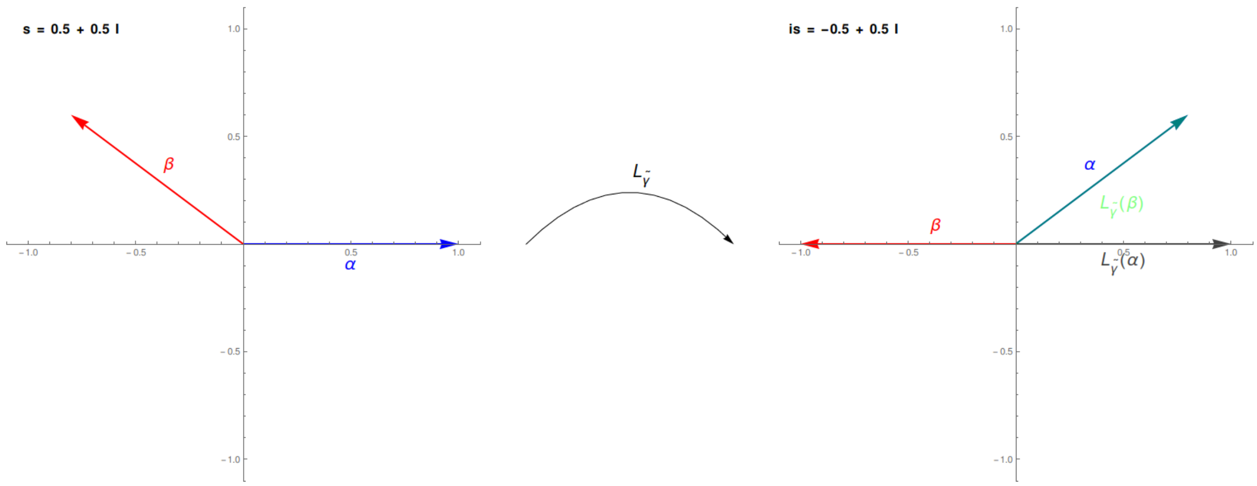

Figure 9: A visualization of how the basis α, β of $\mathrm{H}_1(Z'_{s^4}; \mathbb{Z})$ transforms under monodromy when $s = \frac{1+i}{2}$. On the left we show α, β's lifts to $\mathbb{C} \times \{s\}$. On the right, we show their lifts to $\mathbb{C} \times \{is\}$ as well as the images $L_{\widetilde{\gamma}}(\alpha), L_{\widetilde{\gamma}}(\beta)$ of their lifts (to s) under the monodromy map induced by $\widetilde{\gamma}$.

denote the path $\gamma_{a;s}(t) = (ta, s)$, i.e.

$$\gamma_{a;s} \text{ is the "straight line path from 0 to } a \text{ in } \mathbb{C} \times \{s\}\text{".}$$

Then[11], $\alpha = [\gamma_{1;s}] = [\gamma_{z(is);is}]$ and $\beta = [\gamma_{z(s);s}] = [\gamma_{-1;is}]$ generate $\mathrm{H}_1(Z'_{s^4}; \mathbb{Z})$. Using the commutative diagram above, one sees that $L_{\widetilde{\gamma}}$ sends

$$\gamma_{1;s} \longmapsto \gamma_{1;is} \text{ and } \gamma_{z(s);s} \longmapsto \gamma_{z(is);is},$$

where we've abused notation by identifying these paths with their images in Z. This action is visualized in Figure 9. Thus, on homology, L_γ sends $\alpha \mapsto -\beta$ and $\beta \mapsto \alpha$, so the monodromy matrix in the basis $\{\alpha, \beta\}$ is

$$T = \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix},$$

finally completing the calculation.

The example of type III exhibits all the techniques one needs to calculate the monodromy around all of the remaining fiber types with the single possible section of type I_n^* (for $n \geq 1$). As was noted earlier, in this case, one has to take the quotient of a fiber of type I_{2n} instead of one of type I_0, so there is a bit of extra difficulty in carrying out the initial construction. However, as we will soon see, once this construction has been made, verifying that you obtain a fiber of the expected type and calculating its monodromy are both done analogously to the work above.

Example (Type I_n^*, $n \geq 1$)**.** Let $Y \xrightarrow{p} \Delta$ be an elliptic fibration with a single non-smooth fiber, whose type is I_1, let $\delta : s \mapsto s^{2n}$ be the $(2n)$th power map on Δ, and consider the spaces $Y', Y^{(2n)}$ where $Y' = Y \times_\Delta \Delta$ is the pullback of $Y \to \Delta$ along δ and $Y^{(2n)}$ is the minimal desingularization of Y'. Hence, we have a

[11] $[\gamma_{1;s}] = [\gamma_{z(is);is}]$ since $\mu(1, s) = (z(is), is)$ and $[\gamma_{z(s);s}] = [\gamma_{-1;is}]$ since $\mu(z(s), s) = (-1, is)$.

commutative diagram

$$\begin{tikzcd} Y^{(2n)} \arrow[r, "\tau'"] \arrow[d, "p^{(2n)}"] & Y' \arrow[r, "\tau"] \arrow[d, "p'"] & Y \arrow[d, "p"] \\ \Delta \arrow[r, equal] & \Delta \arrow[r, "\delta"] & \Delta \end{tikzcd}$$

and $Y^{(2n)}$ is a fibration with a singular fiber of type I_{2n}. We seek to construct an involution $\iota^{(2n)} : Y^{(2n)} \to Y^{(2n)}$, so that the quotient $Y^{(2n)}/(y \sim \iota(y))$ has fiber of type I_n^*. First note that there is an involution $\iota' : Y' \to Y'$ given by negating both factors: Y and Δ. More formally, there is a (fiber-preserving) map $\iota : Y \to Y$ which negates each fiber[12] so we get a map $\iota' : Y' \to Y'$ given by $\iota' : (y, s) \mapsto (\iota(y), -s)$ where $Y' = \{(y, s) \in Y \times \Delta : p(y) = \delta(s)\}$ which is well-defined since $\delta(s) = s^{2n} = \delta(-s)$. To see that this map is holomorphic we use the universal property of fiber products (recall Proposition 2.1.4) to realize that ι' is simply the map $Y' \to Y'$ induced by the compositions

$$Y' \xrightarrow{\tau} Y \xrightarrow{\iota} Y \text{ and } Y' \xrightarrow{p'} \Delta \xrightarrow{s \mapsto -s} \Delta,$$

and so is holomorphic. Clearly $\iota' \circ \iota' = \mathrm{Id}_Y$. Thus, we are interested in lifting ι' to an involution $\iota^{(2n)} : Y^{(2n)} \to Y^{(2n)}$. In particular, since $\tau' : Y^{(2n)} \to Y'$ is an isomorphism away from the singular point y_0 of $Y'_0 = (p')^{-1}(0)$, we are interested in lifting ι' to an involution around $Y_0^{(2n)} = \left(\tau^{(2n)}\right)^{-1}(0)$. For this, recall from our original construction of a fiber of type I_{2n} that near y_0 the surface Y' looks like the affine surface

$$S = \left\{(s, u, v) \in \mathbb{C}^2 : s^{2n} = uv\right\} \simeq \mathbb{C}^2/((x, y) \sim (\zeta x, \zeta^{-1} y)),$$

where $\zeta \in \mathbb{C}^\times$ is a primitive $(2n)$th root of unity, and y_0 corresponding to the singular point of type A_{2n-1} in S. In this local picture, the involution $Y' \xrightarrow{\iota'} Y'$ constructed above is given on S by $(s, u, v) \mapsto (-s, -u, -v)$ or $(x, y) \mapsto (-x, -y)$ depending on which description you like better. Recall from Remark 2.3.3 that this A_{2n-1} singularity can be resolved by a sequence of blow ups. Since the map ι' is locally given by negating each coordinate, it is possible to successively lift it to each blow up encountered in the process of resolving this singularity, and so obtain our desired lifted involution $\iota^{(2n)} : Y^{(2n)} \to Y^{(2n)}$. Doing this, one sees that, writing $Y_0^{(2n)} = \sum_{i=1}^{2n} C_i$ (with $C_i C_j = 1$ if $i - j \equiv \pm 1 \pmod{2n}$ but $C_i C_j = 0$ otherwise for $i \neq j$), the involution $\iota^{(2n)}$ interchanges C_i and C_{2n+2-i} (when $1 < i < n+1$) and has two fixed points on each of C_1 and C_{n+1}. Let $\overline{Y} = Y^{(2n)}/\{\mathrm{Id}, \iota^{(2n)}\}$, so we have a commutative diagram

$$\begin{tikzcd} Y^{(2n)} \arrow[r, "f"] \arrow[d, "p^{(2n)}"'] & \overline{Y} \arrow[d, "\overline{p}"] & X \arrow[l] \arrow[d, "\pi"] \\ \Delta \arrow[r, "\delta_2 : s \mapsto s^2"] & \Delta \arrow[r, equal] & \Delta \end{tikzcd}$$

where X is a relatively minimal model of $\overline{Y}$. Since $\iota^{(2n)}$ identified the $(2n-2)$ "central" (-2)-curves $C_2, C_2, \ldots, C_n, C_{n+2}, C_{n+3}, \ldots, C_{2n+2} \subset Y_0^{(2n)}$ in pairs, these give rise to $(n-1)$ copies of $\mathbb{P}^1$, $C'_2, \ldots, C'_n \subset \overline{Y}_0$. In addition to these, we have the images $C'_1, C'_{n+1} \subset \overline{Y}_0$ of $C_1, C_{n+1} \subset Y_0^{(2n)}$, respectively, which are also copies of $\mathbb{P}^1$. Furthermore, these curves $C'_1, \ldots, C'_{n+1}$ meet each other in a path, not a cycle, since $C'_1 C'_{n+1} = 0$ unless $n = 1$ (when $n = 1$, the two points where C_1 and C_2 meet are identified in the quotient,

[12] Thinking of Y as the explicit example given before

$$Y = \left\{([z_0 : z_1 : z_2], s) \in \mathbb{P}^2 \times \Delta \mid z_0 z_2^2 = 4z_1^3 + (s-3) z_0^2 z_1 + (s-1) z_0^3\right\}$$

this is just the (holomorphic) map $([z_0 : z_1 : z_2], s) \mapsto ([z_0 : z_1 : -z_2], s)$.

so $C_1'C_2' = 1$). We claim that each of these $(n+1)$ copies of $\mathbb{P}^1$ occur with multiplicity 2 within $\overline{Y}_0$. Indeed, $\overline{Y}_0 = \overline{p}^{-1}(0) = f\left(\left(\delta_2 \circ p^{(2n)}\right)^{-1}(0)\right)$ and $\left(\delta_2 \circ p^{(2n)}\right)^{-1}(0) \subset Y_0^{(2n)}$ has ideal sheaf

$$\mathscr{I}_{\left(\delta_2 \circ p^{(2n)}\right)^{-1}(0)} = \left(p^{(2n)}\right)^*(\delta_2^* \mathscr{I}_0) = \left(\left(p^{(2n)}\right)^*(\mathscr{I}_0)\right)^2 = \mathscr{I}^2_{Y_0^{(2n)}},$$

so $\overline{Y}_0$ is the curve $f(Y^{(2n)})$ with multiplicity 2. That is,

$$\overline{Y}_0 = 2\left(C_1' + C_2' + C_3' + \cdots + C_n' + C_{n+1}'\right).$$

On each of C_1' and C_{n+1}' there are two singular points arising from the fixed points of $\iota^{(2n)}$. Since this involution stems from a map which negated each coordinate (in terms of Theorem 2.3.10, there is only one possible nonzero weight of a $\mathbb{Z}/2\mathbb{Z}$ action), we see that the singularities it gives rise to are each of type A_1, i.e. are each resolved by a single (-2)-curve. Hence, X arises from $\overline{Y}$ by attaching 4 distinct (-2)-curves B_1, B_2 (meeting C_1') and D_1, D_2 (meeting C_{n+1}'). Letting $X_0 = \pi^{-1}(0)$ be the central fiber, this means that[13]

$$X_0 = B_1 + B_2 + 2\left(C_1' + C_2' + C_3' + \cdots + C_n' + C_{n+1}'\right) + D_1 + D_2$$

with $B_1^2 = B_2^2 = D_1^2 = D_2^2 = -2$. We claim that also $(C_i')^2 = -2$ for all i, so X is relatively minimal. Indeed, X_0 is smooth, so

$$\begin{aligned}
0 &= X_0 C_i' = 2(C_{i-1}'C_i' + C_{i+1}'C_i') + 2(C_i')^2 && = 4 + 2(C_i')^2 \qquad (2 \le i \le n)\\
0 &= X_0 C_1' = B_1C_1' + B_2C_1' + 2(C_1')^2 + 2C_2'C_1' && = 4 + 2(C_1')^2\\
0 &= X_0 C_{n+1}' = D_1C_{n+1}' + D_2C_{n+1}' + 2C_n'C_{n+1}' + 2(C_{n+1}')^2 && = 4 + 2(C_{n+1}')^2.
\end{aligned}$$

Hence $(C_i')^2 = -2$ for all i as claimed. This makes X relatively minimal. Since the C_i's met in a path, and we adjoined two (-2)-curves to each end of that path, we see that X_0 is indeed of type I_n^*, i.e. its dual graph is $\widetilde{D}_{n+4}$, pictured in Figure 10. Now all that remains is to perform the monodromy calculation. To ease

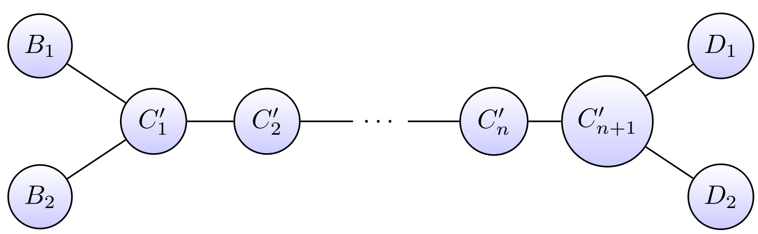

Figure 10: The dual graph $\widetilde{D}_{n+4}$ of a singular fiber of type I_n^*.

notation, let $\Delta^* = \Delta \setminus \{0\}$, let $Z = Y^{(2n)} \setminus Y_0^{(2n)}$, and let $Z' = X \setminus X_0 \simeq \overline{Y} \setminus \overline{Y}_0$. Also, let $q : Z \to \Delta^*$ be the restriction of the map $p^{(2n)} : Y^{(2n)} \to \Delta$, and let $q' : Z' \to \Delta^*$ be the restriction of the map $\pi : X \to \Delta$.

[13] Below we're abusing the notation C_i' by identifying a curve on $\overline{Y}_0$ with its strict transform in X

These spaces fit into the commutative diagram

$$\begin{tikzcd}
 & Z_s \arrow[r] \arrow[d] & Z'_{s^2} \arrow[d] \\
G \arrow[r] \arrow[d, equal] & Z \arrow[r, "f"] \arrow[d, "q"] & Z' \arrow[d, "q'"] \\
G \arrow[r] & \Delta^* \arrow[r, "\delta_2 : s \mapsto s^2"] & \Delta^*
\end{tikzcd}$$

whose rows and columns are fiber sequences. Here $G = \left\langle \iota^{(2n)} \right\rangle \simeq \mathbb{Z}/2\mathbb{Z}$, we have fixed a basepoint $z \in Z$, and the map $G \to Z$ sends $\iota_k^{(2n)} \mapsto \iota_k^{(2n)}(z) \in Z$ where $\iota_0^{(2n)} = \mathrm{Id}$ and $\iota_1^{(2n)} = \iota^{(2n)}$. Let $s = q(z)$, let $\gamma \in \pi_1(\Delta^*, s^2)$ be the loop circling the origin once counter clockwise. Then, γ lifts along δ_2 to a path $\widetilde{\gamma} : [0, 1] \to \Delta^*$ from $\widetilde{\gamma}(0) = s$ to $\widetilde{\gamma}(1) = -s$. The maps $L_{\widetilde{\gamma}} : Z_s \to Z_{-s}$ and $L_\gamma : Z'_{s^2} \to Z'_{s^2}$ can be compared via the following commutative square.

$$\begin{tikzcd}
Z_s \arrow[r, "L_{\widetilde{\gamma}}"] \arrow[d, "g_s"'] & Z_{-s} \arrow[d, "g_{-s}"] \\
Z'_{s^2} \arrow[r, "L_\gamma"] & Z'_{s^2}
\end{tikzcd}$$

Let $\alpha, \beta \in \mathrm{H}_1(Z_s; \mathbb{Z})$ be a basis so the monodromy T of Z_s sends

$$T(\alpha) = \alpha \ \text{ and } \ T(\beta) = 2n\alpha + \beta.$$

Note that $Z = Y^{(2n)} \setminus Y^{(2n)} \simeq Y \times_\Delta \Delta^*$ is literally a fiber product, so

$$Z = \left\{ (y, t) \in Y \times \Delta^* : p(y) = t^{2n} \right\},$$

and we see that $L_{\widetilde{\gamma}}(y, t) = (y, -t)$. Thus, letting $\alpha' = g_{s,*}(\alpha), \beta' = g_{s,*}(\beta) \in \mathrm{H}_1(Z'_{s^2}; \mathbb{Z})$ be a basis, we get that $\alpha' = -g_{-s,*}(L_{\widetilde{\gamma}}(\alpha))$, and so $L_{\gamma,*}(\alpha') = -\alpha'$. Furthermore, the path $2\gamma \in \pi_1(\Delta^*, s^2)$ lifts to a loop $\widetilde{2\gamma}$ on Δ^* based at s which circles the origin once. Combining these two remarks, in terms of monodromy, we get that the monodromy action $T' \in \mathrm{GL}_2(\mathrm{H}_1(Z'_{s^2}; \mathbb{Z}))$, when written as a matrix in the basis $\{\alpha', \beta'\}$, must satisfy

$$T' \begin{pmatrix} 1 \\ 0 \end{pmatrix} = \begin{pmatrix} -1 \\ 0 \end{pmatrix} \ \text{ and } \ (T')^2 = \begin{pmatrix} 1 & 2n \\ 0 & 1 \end{pmatrix}.$$

Writing $T' = \begin{pmatrix} a & b \\ c & d \end{pmatrix}$, the first condition shows $(a, c) = (-1, 0)$, while the second condition says

$$\begin{pmatrix} 1 & 2n \\ 0 & 1 \end{pmatrix} = (T')^2 = \begin{pmatrix} -1 & b \\ 0 & d \end{pmatrix}^2 = \begin{pmatrix} 1 & bd - b \\ 0 & d^2 \end{pmatrix}.$$

Since $d^2 = 1$, we know $d = \pm 1$, but also $b(d - 1) = 2n \neq 0$, so $d = -1$. Hence, $-2b = 2n$, so $b = -n$. Thus, the monodromy matrix for a singular fiber of type I_n^* is

$$T' = \begin{pmatrix} -1 & -n \\ 0 & -1 \end{pmatrix} = -\begin{pmatrix} 1 & n \\ 0 & 1 \end{pmatrix}.$$

4.3.4 Remarks on the Remaining Fiber Types

We would like to end by giving some indication of how one would go about calculating the monodromy action for the remaining fiber types. For some more details, beyond what it given here, we recommend [2, Ch. V, Sect. 9–10].

For type ${}_m\mathrm{I}_n$, the fiber is topologically equivalent to one of type I_n, so the monodromy matrix is again

$$T = \begin{pmatrix} 1 & n \\ 0 & 1 \end{pmatrix},$$

up to conjugation. However, knowing this alone does not guarantee that fibers of this type exist. We will give a way to construct fibers of type ${}_m\mathrm{I}_0$ in Table 2 below, but for general ${}_m\mathrm{I}_n$ (i.e. when both $n \neq 0$ and $m \neq 1$), we recommend the book by Barth et al. for a construction.

For the remaining fiber types, include I_0^*, the calculation can proceed completely analogously to how we calculated the monodromy around a fiber of type III. One starts with a fibration $Y = \mathbb{C} \times \Delta / \mathbb{Z} \oplus \mathbb{Z}z(s) \to \Delta$ with only smooth fibers for some appropriately chosen holomorphic map $z(s)$. This $z(s)$ is chosen such that a cyclic group $G = \langle \mu \rangle$ acts on Y, and after resolving Y/G's singularities and blowing down its vertical (-1)-curves, one can determine the fiber type of the resulting relatively minimal elliptic surface X. Then, since X was constructed by quotienting a trivial fibration, its monodromy is not too difficult to calculate. A possible choice of $z(s)$ along with a map $\mu : \mathbb{C} \times \Delta \to \mathbb{C} \times \Delta$ inducing a generator of G is given in Table 2. In this table, recall that $\mathbf{e}(x) = \exp(2\pi i x)$.

Type	$z : \Delta \to \mathbb{C}$	$\mu : \mathbb{C} \times \Delta \to \mathbb{C} \times \Delta$
${}_m\mathrm{I}_0$	$s \mapsto s^m$	$(c, s) \mapsto \left(c + \frac{1}{m}, \mathbf{e}\left(\frac{1}{m}\right) s\right)$
I_0^*	$s \mapsto s^2$	$(c, s) \mapsto (-c, -s)$
II	$s \mapsto \frac{\mathbf{e}(1/3) - \mathbf{e}(2/3)s^2}{1-s^2}$	$(c, s) \mapsto \left(-\frac{c}{z(s)}, \mathbf{e}\left(\frac{1}{6}\right) s\right)$
II^*	$s \mapsto \frac{\mathbf{e}(1/3) - \mathbf{e}(2/3)s^4}{1-s^4}$	$(c, s) \mapsto \left(\frac{c}{z(s)+1}, \mathbf{e}\left(\frac{1}{6}\right) s\right)$
III^*	$s \mapsto \frac{i+is^2}{1-s^2}$	$(c, s) \mapsto \left(\frac{c}{z(s)}, is\right)$
IV	$s \mapsto \frac{\mathbf{e}(1/3) - \mathbf{e}(2/3)s^2}{1-s^2}$	$(c, s) \mapsto \left(-\frac{c}{z(s)+1}, \mathbf{e}\left(\frac{1}{3}\right) s\right)$
IV^*	$s \mapsto \frac{\mathbf{e}(1/3) - \mathbf{e}(2/3)s}{1-s}$	$(c, s) \mapsto \left(\frac{c}{z(s)}, \mathbf{e}\left(\frac{1}{3}\right) s\right)$

Table 2: Functions used to construct elliptic fibrations with singular fibers of various types.

The complete list of monodromy matrices (which we recall are only defined up to conjugacy) is shown in Table 3. As a final mathematical remark, one sees from Table 3 that, assuming one is concerned with a reduced fiber, the monodromy action completely determines the singular fiber type.

Type	Monodromy Matrix
I_0	$\begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}$
I_n	$\begin{pmatrix} 1 & n \\ 0 & 1 \end{pmatrix}$
${}_m\mathrm{I}_n$	$\begin{pmatrix} 1 & n \\ 0 & 1 \end{pmatrix}$
I_n^*	$\begin{pmatrix} -1 & -n \\ 0 & -1 \end{pmatrix}$
II	$\begin{pmatrix} 1 & 1 \\ -1 & 0 \end{pmatrix}$
II^*	$\begin{pmatrix} 0 & -1 \\ 1 & 1 \end{pmatrix}$
III	$\begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}$
III^*	$\begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix}$
IV	$\begin{pmatrix} 0 & 1 \\ -1 & -1 \end{pmatrix}$
IV^*	$\begin{pmatrix} -1 & -1 \\ 1 & 0 \end{pmatrix}$

Table 3: The monodromy matrices for each type of singular elliptic fiber.

Acknowledgments

This paper was written to fulfill the thesis portion of Stanford's honors math degree. It would not have been possible without an immense amount of help and support from my advisor, Dr. Rafe Mazzeo. I thank him for guiding and advising me as I attempted to learn the material that went into this thesis. I especially appreciate the patience he has shown whenever I have been struggling to grasp this material, and the helpful comments he has given me on this document.

Appendices

A Higher Direct Image Sheaves and Locally Constant Sheaves

In this appendix, we cover the basics of a particular topic from sheaf theory: higher direct image sheaves. These give a sort of relative cohomology attached to a morphism $X \to Y$, where we think of normal sheaf cohomology as being attached to the unique map $X \to \mathrm{pt}$ to the 1-pt space (with trivial structure sheaf). These higher direct image sheaves show up from time to time in this document, so this appendix serves as a place to get acquainted with some of their facets. Before defining and studying higher direct images, we recall some basic definitions.

Recall A.1. A **ringed space** is a pair $X = (|X|, \mathscr{O}_X)$ consisting of a topological space $|X|$ along with a sheaf $\mathscr{O}_X$ of (commutative) rings on it. The space $|X|$ is called the **underlying (topological) space** of X, and $\mathscr{O}_X$ is called its **structure sheaf**

Recall A.2. For a continuous map $f : X \to Y$ between ringed spaces and a sheaf $\mathscr{F}$ on X, one obtains a sheaf $f_*\mathscr{F}$ on Y, called the **pushforward/direct image sheaf**, given on an open $U \subset Y$ by

$$f_*\mathscr{F}(U) = \mathscr{F}(f^{-1}(U)).$$

Recall A.3. Given two ringed spaces $X = (|X|, \mathscr{O}_X)$ and $Y = (|Y|, \mathscr{O}_Y)$ a **morphism of ringed spaces** $f = (|f|, \widetilde{f})$ is a pair consisting of a continuous map $|f| : |X| \to |Y|$ and a morphism $\widetilde{f} : \mathscr{O}_Y \to |f|_* \mathscr{O}_X$ of sheaves on $|Y|$.

Example. If $|X|$ is a complex manifold, then we can take $\mathscr{O}_X$ to be its sheaf of holomorphic, C^∞, or continuous functions to $\mathbb{C}$, depending on how much structure we want $\mathscr{O}_X$ to see. That is, for $U \subset |X|$ open, we can set

$$\mathscr{O}_X(U) = \{f : U \to \mathbb{C} : f \text{ is “nice”}\}$$

where we choose “nice” to mean one of holomorphic, C^∞, or continuous.

Example. If $f : X \to \{x\}$ is the map to a one-point space and $\mathscr{F}$ is any sheaf on X, its direct image $f_*\mathscr{F}$ is simply its global sections $f_*\mathscr{F}(x) = \mathscr{F}(f^{-1}(x)) = \mathscr{F}(X)$.

Fix a continuous map $f : X \to Y$ between ringed spaces. The operation of pushing forward gives a functor $f_* : \mathrm{Ab}(X) \to \mathrm{Ab}(Y)$ between the categories of abelian sheaves on X, Y. This functor is left exact which follows from the easily checkable fact that it preserves kernels, i.e. for any $\mathscr{F}, \mathscr{G} \in \mathrm{Ab}(X)$ and $\mathscr{F} \to \mathscr{G}$, we have

$$f_* \ker(\mathscr{F} \to \mathscr{G}) = \ker\left(f_*\mathscr{F} \to g_*\mathscr{G}\right).$$

Because the category of abelian sheaves has enough injectives [8, Ch. II, Corollary 2.3], by abstract nonsense, f_* has right derived functors $R^i f_* : \mathrm{Ab}(X) \to \mathrm{Ab}(Y)$ so that, among other things,

(1) for any exact sequence $0 \to \mathscr{A} \to \mathscr{B} \to \mathscr{C} \to 0$ of abelian sheaves on X, there is a corresponding long exact sequence

$$0 \to f_*\mathscr{A} \to f_*\mathscr{B} \to f_*\mathscr{C} \to R^1 f_*\mathscr{A} \to R^1 f_*\mathscr{B} \to R^1 f_*\mathscr{C} \to R^2 f_*\mathscr{A} \to \cdots$$

of abelian sheaves on Y.

(2) if $\mathscr{F} \in \mathrm{Ab}(X)$ is an injective sheaf, then $R^i f_* \mathscr{F} = 0$ for all $i \geq 1$.

Definition A.1. The sheaves $R^i f_*$ constructed above are called the **higher direct image sheaves** of f and, for notational convenience, are also denoted $f_{*i} = R^i f_*$.

Technical Aside A.1. As one does when studying sheaf cohomology, one can show that $f_{*i}\mathscr{G} = 0$ for all $i \geq 1$ if $\mathscr{G}$ is flasque (and not necessarily injective), and then use this to show that the derived functors of f_* are the same regardless of whether you view it as a functor $\mathrm{Ab}(X) \to \mathrm{Ab}(Y)$ between categories of abelian sheaves or as as a functor $\mathrm{Mod}(X) \to \mathrm{Mod}(Y)$ between categories of modules over the structure sheaf. The point is that flasque sheaves are acyclic for f_* (i.e. $f_{*i}\mathscr{G} = 0$ for $i \geq 1$ if $\mathscr{G}$ flasque) when viewed as either abelian sheaves or as $\mathscr{O}_X$-modules. This means that given a sheaf $\mathscr{F} \in \mathrm{Mod}(X)$ (so also $\mathscr{F} \in \mathrm{Ab}(X)$), a flasque resolution of $\mathscr{F}$ is an acylic resolution for f_* in either category, and so can be used to compute the derived functors of f_* for either category.

We claimed earlier that these higher direct image sheaves could be thought of as a relative cohomology theory so we now make this precise by showing that $f_{*i}\mathscr{F}$ is essentially a sheaf of cohomology groups. As a warmup, note that if $Y = \{y\}$ is a point then $\mathrm{Ab}(Y) \cong \mathrm{Ab}$, the category of abelian groups, and the functor $f_* : \mathrm{Ab}(X) \to \mathrm{Ab}(Y) \cong \mathrm{Ab}$ is simply the global sections functor $\mathscr{F} \mapsto \Gamma(X, \mathscr{F})$. Hence, in this case, f_* and Γ have the same right derived functors which means that $f_{*i}\mathscr{F} = \mathrm{H}^i(X, \mathscr{F})$ for all $\mathscr{F} \in \mathrm{Ab}(X)$. In general, we have the below result.

Proposition A.1. *Fix an integer $i \geq 0$ and a sheaf $\mathscr{F} \in \mathrm{Ab}(X)$. Let $\mathscr{H}^i(\mathscr{F}) \in \mathrm{Ab}(Y)$ be the sheafification of the presheaf*

$$U \longmapsto \mathrm{H}^i(f^{-1}(U), \mathscr{F}|_{f^{-1}(U)})$$

with the natural restriction maps. Then, $f_{*i}\mathscr{F} \simeq \mathscr{H}^i$.

Proof. When $i = 0$, the result holds by definition of $f_{*0} = f_*$. For larger i, we will dimension shift. Let $\mathscr{F}$ be any abelian sheaf on X and embed it $\mathscr{F} \hookrightarrow \mathscr{G}$ into a flasque sheaf. Let $\mathscr{Q} = \mathscr{G}/\mathscr{F}$ be the cokernel of this embedding, so we have an exact sequence

$$0 \longrightarrow \mathscr{F} \longrightarrow \mathscr{G} \longrightarrow \mathscr{Q} \longrightarrow 0.$$

Using that both higher direct images and cohomology vanish for flasque sheaves, and that $\mathscr{G}|_U$ is flasque for any open $U \subset X$, this exact sequence gives rise to the commutative diagram

$$\begin{tikzcd}
0 \arrow[r] & f_*\mathscr{F} \arrow[r] \arrow[d] & f_*\mathscr{G} \arrow[r] \arrow[d] & f_*\mathscr{Q} \arrow[r] \arrow[d] & f_{*1}\mathscr{F} \arrow[r] \arrow[d, dashed] & 0 \\
0 \arrow[r] & \mathscr{H}^0(\mathscr{F}) \arrow[r] & \mathscr{H}^0(\mathscr{G}) \arrow[r] & \mathscr{H}^0(\mathscr{Q}) \arrow[r] & \mathscr{H}^1(\mathscr{F}) \arrow[r] & 0
\end{tikzcd}$$

whose rows are exact. Above, the dashed map exists by appealing to the universal property of cokernels (note that $f_{*1}\mathscr{F} = \mathrm{coker}(f_*\mathscr{G} \to f_*\mathscr{Q})$) and is an isomorphism by the 5 lemma since all solid maps are isomorphisms. As $\mathscr{F}$ was arbitrary, this shows that $f_{*1}\mathscr{F} \simeq \mathscr{H}^1(\mathscr{F})$ for all abelian sheaves on X. Now, pick $i \geq 2$ and assume we know $f_{*(i-1)}\mathscr{F} \simeq \mathscr{H}^{i-1}(\mathscr{F})$ for all $\mathscr{F} \in \mathrm{Ab}(X)$. Fix any specific $\mathscr{F}$, and note that the short exact sequence we looked at in the beginning of this proof now gives rise to the commutative diagram

$$\begin{tikzcd}
0 \arrow[r] & f_{*(i-1)}\mathscr{Q} \arrow[r] \arrow[d] & f_{*i}\mathscr{F} \arrow[r] \arrow[d, dashed] & 0 \\
0 \arrow[r] & \mathscr{H}^{i-1}(\mathscr{Q}) \arrow[r] & \mathscr{H}^{i}(\mathscr{F}) \arrow[r] & 0
\end{tikzcd}$$

whose rows are again exact. The solid vertical map exists and is an isomorphism by the inductive hypothesis. Because all maps between nonzero objects above are isomorphisms, the dashed map exists and is an isomorphism too. This completes the proof. ■

The above proposition suggests a potential connection between these higher direct image sheaves $f_{*i}\mathscr{F}$ and the cohomology $\mathrm{H}^i(f^{-1}(y), \mathscr{F})$ of the fiber above a point $y \in Y$. Indeed, one may be tempted to believe that $(f_{*i}\mathscr{F})_y = \mathrm{H}^i(f^{-1}(y), \mathscr{F})$ in general. While this is not always the case, it is true when f is sufficiently nice. In fact, more than this will be true in the cases we care about. To prove the next result, it will be helpful to know when sheaf cohomology coincides with singular cohomology so we may appeal to homotopy invariance. In this vain, we recall the following.

Notation A.2. Let X be a topological space, and let A be an abelian group. Then, we let $\underline{A}_X$ denote the corresponding constant sheaf on X.

Theorem A.2. *Let X be a paracompact, Hausdorff, locally contractible topological space. Then, for any abelian group A, $\mathrm{H}^i(X, \underline{A}_X) \simeq \mathrm{H}^i_{\mathrm{sing}}(X; A)$.*

Proof. [13, Ch. 6] shows that Čech cohomology agrees with singular cohomology in this case. Combine this with the fact [1, Theorem 7.4.9] that Čech cohomology agrees with sheaf cohomology on paracompact, Hausdorff spaces. ■

Proposition A.3. *Assume that X, Y are paracompact, Hausdorff, and locally contractible. Let $f : X \to Y$ be a fibre bundle, and let $A \in \mathrm{Ab}$ be an abelian group. Then, the sheaves $f_{*i}\underline{A}_X$ are locally constant and $(f_{*i}\underline{A}_X)_y \simeq \mathrm{H}^i(f^{-1}(y), A)$ for any $y \in Y$.*

Proof. Fix any $y \in Y$ and let $U \ni y$ be a contractible, trivializing neighborhood around y, so $f^{-1}(U) \simeq U \times F$ where $F = f^{-1}(y)$ is the fiber and there exists a homotopy $h_t : U \to U$ from the identity $h_0 = \mathrm{Id}_U$ to the constant map $h_1 = y$. We can lift h_t to a deformation retraction $\widetilde{h}_t : U \times F \to U \times F, \widetilde{h}_t(a, b) = (h_t(a), b)$ from $U \times F$ onto $\{y\} \times F \cong F$. Hence,

$$\mathrm{H}^i(f^{-1}(U), \underline{A}_X) \simeq \mathrm{H}^i_{\mathrm{sing}}(U; A) \simeq \mathrm{H}^i_{\mathrm{sing}}(F; A) \simeq \mathrm{H}^i(F, \underline{A}_X).$$

Because this holds for any contractible, trivializing open in Y, because these form a basis of opens in U, and because sheaves are determined by what they do on a basis, we see that proposition A.1 shows that $f_{*i}\underline{A}_X|_U \simeq \underline{\mathrm{H}^i(F, A)}_U$ is constant with stalks $\mathrm{H}^i(f^{-1}(y), A)$. Hence, $f_{*i}\underline{A}_X$ is locally constant with the claimed stalks. ■

The main utility of showing that the pushforward of a constant sheaf along a fibre bundle is locally constant is that this endows us with the ability to transport cohomology classes between fibers. Namely, in general, if you have a locally constant sheaf $\mathscr{F}$ on a space Y, then this automatically gives rise to an action $\pi_1(Y, y) \curvearrowright \mathscr{F}_y$ of the fundamental group of Y on the stalks of $\mathscr{F}$. In the situation of the proposition above, this means that the fundamental group of the base of a fibre bundle acts on the cohomology groups of the fibers. This is the monodromy action studied towards the end of this document. The construction of this action proceeds as follows.

Let $\mathscr{F}$ be a locally constant sheaf on a locally simply connected topological space Y, and let $\gamma : [0,1] \to Y$ be a loop. Then, because $\mathscr{F}$ is locally free and because $[0,1]$ is compact, we can write $\gamma = \gamma_1\gamma_2 \dots \gamma_n$ as the composition of n paths $\gamma_i : [0,1] \to Y$ such that $\operatorname{im}\gamma_i$ lands in an open $U_i \subset Y$ on which $\mathscr{F}|_U$ is constant, say $\phi_i : \mathscr{F}|_{U_i} \xrightarrow{\sim} \underline{A_i}_U$ $(i = 1, \dots, n)$. Let $y_0 = \gamma(0)$ and let $y_i = \gamma_i(1) \in U_i \cap U_{i+1}$ for $i = 1, \dots, n$ (here, $U_{n+1} = U_1$). Then, we get the composition

$$\mathscr{F}_{y_0} \xrightarrow{\phi_1} A_1 \xrightarrow{\phi_1^{-1}} \mathscr{F}_{y_1} \xrightarrow{\phi_2} A_2 \to \cdots \to A_n \xrightarrow{\phi_n^{-1}} \mathscr{F}_{y_n}.$$

Since γ is a loop, $y_0 = y_1 = y$, so the above actually defines an action of γ on $\mathscr{F}_y$.

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