Outline

- Canonical Models and Complex Multiplication, vaguely
- Basics of CM Abelian Varieties
- Shimura-Taniyama and Main Theorem (w/o proofs)
- Canonical Models and Complex Multiplication, an example

Notation 1.

- $\mathbb{A}_f = \prod'_p(\mathbb{Q}_p, \mathbb{Z}_p) = \widehat{\mathbb{Z}} \otimes_{\mathbb{Z}} \mathbb{Q}$ is the ring of finite adeles over $\mathbb{Q}$, and $\mathbb{A}_{f,K}$ is the ring of finite adeles over K.
- $\mathbb{A}, \mathbb{A}_K$ is the full ring of adeles over $\mathbb{Q}, K$, respectively.
- If A, B are abelian varieties, then $\mathrm{Hom}^0(A,B) := \mathrm{Hom}(A,B) \otimes_{\mathbb{Z}} \mathbb{Q}$, and its invertible elements are called `"isogenies"`.
- Given a field F, we set $G_F := \mathrm{Gal}(F^s/F)$.
- If E is a CM field (to be defined), we let F denote its maximal totally real subfield.

1 Can Models and CM, vaguely

Question 2. *What does today's discussion of CM have to do with our broader treatment of Shimura varieties?*

Recall 3. Given a Shimura datum (G, X), we have defined an inverse system

$$\mathrm{Sh}_K(G,X) := G(\mathbb{Q})\backslash X \times G(\mathbb{A}_f)/K$$

of complex algebraic varieties (K ranging over sufficiently small compact open subgroups of $G(\mathbb{A}_f)$). ⊙

We would like to show that this system is, in fact, defined over some number field $E = E(G,X)$. Given a variety $V/\mathbb{C}$, here are two strategies for descending it to a variety V_E/E

(1) V represents some functor $\mathrm{Sch}_{\mathbb{C}}^{\mathrm{op}} \to \mathrm{Set}$. If one understands this functor well enough (e.g. if V is naturally a moduli space), then they can extend this to a functor $\mathrm{Sch}_E^{\mathrm{op}} \to \mathrm{Set}$, and check representability of the resulting extension.

Example 4. Let (G, X) be the Siegel Shimura datum attached to a symplectic space (V, ψ) over $\mathbb{Q}$ which supports a unimodular lattice $V(\mathbb{Z}) \subset V$. Let

ψ restricts to a pairing $V(\mathbb{Z}) \times V(\mathbb{Z}) \to \mathbb{Z}$ w/ discriminant ± 1

$$K(N) = \left\{ g \in G(\mathbb{A}_f) : gV(\widehat{\mathbb{Z}}) = V(\widehat{\mathbb{Z}}) \text{ and } g \mapsto 1 \in \mathrm{GL}(V(\widehat{\mathbb{Z}})/NV(\widehat{\mathbb{Z}})) \right\}.$$

Then, $\mathrm{Sh}_{K(N)}(G,X)$ parameterizes principally polarized abelian varieties (over $\mathbb{C}$) equipped w/ a full level N-structure (i.e. basis for N-torsion). We saw in the beginning of the semester than this moduli problem is representable over $\mathbb{Q}$ (in fact, over $\mathbb{Z}[1/N]$) if $N \geq 3$, so $\mathrm{Sh}_{K(N)}(G,X)$ has a model over $E(G,X) = \mathbb{Q}$. △

(2) Alternatively, one can specify an action of $\mathrm{Aut}(\mathbb{C}/E)$ on V, and then use (something like) Galois descent to show this action must come from a model of V over E.

General Shimura varieties are not known to be moduli spaces, so we'll use strategy **(2)** to get models for them over number fields. The main theorem of complex multiplication will tell us what "Galois action" to require by describing a canonical action of a dense subset of special points of $\mathrm{Sh}_K(G, X)$ (analogous to CM points on Siegel modular varieties).

2 CM basics

Definition 5. A CM field is a number field E which is a totally imaginary, quadratic extension of a totally real field F. A CM-type Φ for E is a subset $\Phi \subset \mathrm{Hom}(E, \mathbb{C})$ such that $\mathrm{Hom}(E, \mathbb{C}) = \Phi \sqcup \overline{\Phi}$. ◇

Notation 6. With E, F as above,

- The generator of $\mathrm{Gal}(E/F)$ will be denote $\alpha \mapsto \overline{\alpha}$ as it corresponds to complex conjugation under any embedding $E \hookrightarrow \mathbb{C}$.
- We also set $E_{\mathbb{C}} = E \otimes_{\mathbb{Q}} \mathbb{C}$, $E_{\mathbb{R}} = E \otimes_{\mathbb{Q}} \mathbb{R}$, and $E_\ell = E \otimes_{\mathbb{Q}} \mathbb{Q}_\ell$.

Definition 7. Let E be a CM field of degree $2g$ over $\mathbb{Q}$. Let $A/\mathbb{C}$ be an abelian variety of dimension g, and let i be a homomorphism $E \to \mathrm{End}^0(A)$. If

$$\mathrm{Lie}\, A := T_0(A) \simeq \mathbb{C}^{\Phi} \text{ as } E_{\mathbb{C}}\text{-modules}$$

for some CM-type Φ, then we say that (A, i) is of CM-type (E, Φ). ◇

Remark 8. Let $A/\mathbb{C}$ be a g-dimensional abelian variety equipped w/ a morphism $E \to \mathrm{End}^0(A)$ for some degree $2g$ CM field E. Then, (A, i) will be of CM-type (E, Φ) for some Φ. Indeed, $A \cong \mathrm{Lie}\, A/\mathrm{H}_1(A, \mathbb{Z})$, so

$$\mathrm{H}_1(A; \mathbb{C}) \simeq (\mathrm{H}_1(A, \mathbb{Z}) \otimes_{\mathbb{Z}} \mathbb{R}) \otimes_{\mathbb{R}} \mathbb{C} = \mathrm{Lie}(A) \otimes_{\mathbb{R}} \mathbb{C} = \mathrm{Lie}(A) \oplus \overline{\mathrm{Lie}(A)}.$$

At the same time, $\mathrm{H}_1(A, \mathbb{Q})$ is a 1-dimensional E-vector space, so $\mathrm{Lie}(A) \oplus \overline{\mathrm{Lie}(A)} \simeq \mathrm{H}_1(A; \mathbb{C})$ is a 1-dimensional $E_{\mathbb{C}}$-vector space. Hence, the set of $\varphi : E \hookrightarrow \mathbb{C}$ occurring in $\mathrm{Lie}(A)$ must form a CM-type. ○

Maybe just state first two sentences as fact w/o explanation? Better yet, just mention this aloud and write nothing

Example 9. Fix an odd prime p, and let $E = \mathbb{Q}(\zeta_p)$. Consider the hyperelliptic curve $C : y^2 = x^p + 1$ and let $A = \mathrm{Jac}(C)$. Then, A has CM by E, where $\zeta_p \in E$ acts on A as the pullback of the map $[\zeta_p] : (x, y) \mapsto (e^{2\pi i/p}x, y)$. One can compute that the associated CM-type here is

$$\Phi = \left\{ \zeta_p \mapsto e^{2\pi i k/p} : k = 1, \ldots, \frac{p-1}{2} \right\}. \quad \triangle$$

Maybe omit?

We'll show three nice properties of CM abelian varieties: **(i)** they're "all" isogenous, **(ii)** they are defined over number fields, and **(iii)** they have everywhere potentially good reduction.

Recall 10. Let $M = \mathbb{C}^n/\Lambda$ be a complex torus, and let J denote the induced complex structure on $V := R \otimes_{\mathbb{Z}} \Lambda \simeq \mathbb{C}^n$. Then, M is an abelian variety if and only if it supports a Riemann form, i.e. an alternating form $E : \Lambda \times \Lambda \to \mathbb{Z}$ whose extension $E_{\mathbb{R}}$ to V satisfies

$$E(Ju, Jv) = E(u, v) \text{ and } E_J(u, v) := E(u, Jv) \text{ is positive-definite.} \quad \odot$$

This is a polarization on the integral Hodge structure $\mathrm{H}_1(M, \mathbb{Z})$

Proposition 11 ((i))**.** *Let Φ be a CM-type on E. Abuse notation by letting Φ denote also the natural map $E \to \mathbb{C}^{\Phi}, a \mapsto (\varphi(a))_{\varphi \in \Phi}$. Then,*

(a) *The image $\Phi(\mathscr{O}_E) \subset \mathbb{C}^{\Phi}$ is a lattice.*

(b) *The quotient*

$$A_\Phi : \mathbb{C}^\Phi/\Phi(\mathscr{O}_E)$$

is an abelian variety of CM-type (E, Φ) *for the natural homomorphism* $i_\Phi : E \to \mathrm{End}^0(A_\Phi)$.

(c) *Any other pair* (A, i) *of CM-type* (E, Φ) *is* E*-isogenous to* (A_Φ, i_Φ).

Proof. **(a)** This holds simply because

$$\mathscr{O}_E \otimes_{\mathbb{Z}} \mathbb{R} \simeq E \otimes_{\mathbb{Q}} \mathbb{R} = E_{\mathbb{R}} \xrightarrow[\Phi]{\sim} \mathbb{C}^\Phi.$$

(b) $A_\Phi = \mathbb{C}^\Phi/\Phi(\mathscr{O}_E)$ is a complex torus by definition. To show that it is an abelian variety, one needs to show it supports a Riemann form. Choose some $\alpha \in E$ so that $\operatorname{Im}\varphi(\alpha) > 0$ for all $\varphi \in \Phi$.[1] We may scale α to assume that $\alpha \in \mathscr{O}_E$. Let $F \subset E$ be the maximal totally real subfield. Let $v \mapsto \overline{v}$ denote the nontrivial element of $\mathrm{Gal}(E/F)$. Then,

$$\psi(u, v) := \mathrm{Tr}_{E/\mathbb{Q}}(\alpha u \overline{v}) = \sum_{\varphi \in \Phi} \mathrm{Tr}_{\mathbb{C}/\mathbb{R}}\Big(\varphi(\alpha)\varphi(u)\overline{\varphi(v)}\Big) \text{ for } u, v \in \mathscr{O}_E$$

is a Riemann form, so A_Φ is an abelian variety. Furthermore, $i_\Phi : \mathscr{O}_E \to \mathrm{End}(A_\Phi)$ sending $\alpha \in \mathscr{O}_E$ to multiplication by $\Phi(\alpha)$ ends to the desired $i_\Phi : E \to \mathrm{End}^0(A_\Phi)$. Finally, $T_0(A_\Phi) = \mathbb{C}^\Phi$ as $E_{\mathbb{C}}$-modules, by construction, so (A_Φ, i_Φ) is of CM-type (E, Φ).

(c) Say (A, i) is of CM-type (E, Φ). Then, $T_0(A) \simeq \mathbb{C}^\Phi$ as $E_{\mathbb{C}}$-modules, so $A \simeq \mathbb{C}^\Phi/\Lambda$ w/ $\mathbb{Q}\Lambda \subset \mathbb{C}^\Phi$ stable under the E-action. We must therefore have $\mathbb{Q}\Lambda = \Phi(E) \cdot \lambda$ for some $\lambda \in E_{\mathbb{R}}^\times$, so $\Lambda = \Phi(\mathscr{O}) \cdot \lambda$ for some lattice $\mathscr{O} \subset E$. Choose N such that $N\mathscr{O} \subset \mathscr{O}_E$. Then, we have isogenies

$$A = \mathbb{C}^\Phi/\Lambda = \mathbb{C}^\Phi/(\Phi(\mathscr{O}) \cdot \lambda) \xrightarrow{N} \mathbb{C}^\Phi/(\Phi(N\mathscr{O}) \cdot \lambda) \xleftarrow{\cdot\lambda} \mathbb{C}^\Phi/\Phi(\mathscr{O}_E).$$ ■

Proposition 12 ((**ii**))**.** *Let* (A, i) *be an abelian variety of CM-type* (E, Φ) *over* $\mathbb{C}$*. Then,* (A, i) *has a model over* $\overline{\mathbb{Q}}$*, which is unique up to isomorphism.*

Proof. Uniqueness is easy: the functor $A \to_{\mathbb{C}}$: $\mathrm{AV}(\overline{\mathbb{Q}}) \to \mathrm{AV}(\mathbb{C})$ is fully faithful. Indeed, a morphism $A \to B$ between $\mathbb{C}$-abelian varieties is determined by its action on the (Zariski dense) subset of torsion points, but $A(\overline{\mathbb{Q}})_{\mathrm{tors}} = A(\mathbb{C})_{\mathrm{tors}}$, so any morphism is fixed by the action of $\mathrm{Aut}(\mathbb{C}/\overline{\mathbb{Q}})$, and so defined over $\overline{\mathbb{Q}}$.

For existence, suppose (A, i) is of CM-type (E, Φ). It clearly has a model over some subring $R \subset \mathbb{C}$ which is finitely generated over $\overline{\mathbb{Q}}$. Let $\mathfrak{m} \subset R$ be a maximal ideal where A has good reduction. Then, $R/\mathfrak{m} \simeq \overline{\mathbb{Q}}$, so (A, i) specializes to some (A', i') over $\overline{\mathbb{Q}}$, which is still of CM-type (E, Φ).[2] Proposition 11**(c)** shows that there is an isogeny $(A', i')_{\mathbb{C}} \to (A, i)$. Its kernel will be defined over $\overline{\mathbb{Q}}$ since $A'(\overline{\mathbb{Q}})_{\mathrm{tors}} = A'(\mathbb{C})_{\mathrm{tors}}$, so $(A'/H, i')$ is a model of (A, i) over $\overline{\mathbb{Q}}$. ■

Lemma 13 (Corollary 1, [ST68])**.** *Let* F *be a local field with residue characteristic* p*, and let* $G_F :=$ $\mathrm{Gal}(F^s/F)$*. Let* A/F *be an abelian variety. If, for some* $\ell \neq p$*, the image of* G_F *in* $\mathrm{Aut}(T_\ell(A))$ *is abelian, then* A *has potential good reduction at* v*.*

Proposition 14 ((**iii**))**.** *Let* (A, i) *be an abelian variety of CM-type* (E, Φ) *over some number field* $K \subset \mathbb{C}$*. Then,* A *has potential good reduction over all* $\mathfrak{p} \in \operatorname{Spec}\mathscr{O}_K$*.*

[1] E/F quadratic, so can choose α s.t. $E = F(\alpha)$ and $\alpha^2 \in F^\times$. This forces α^2 to be totally negative (otherwise, get some real embedding of E)

[2] Write $E = \mathbb{Q}(\alpha)$. Then, the eigenvalues of α acting on $T_0(A)$ determine the CM type.

Proof. Fix some $\mathfrak{p} \in \operatorname{Spec} \mathscr{O}_K$, as well as a rational prime $\ell \notin \mathfrak{p}$. Let $G_K = \operatorname{Gal}(K^s/K)$, and consider the ℓ-adic Tate representation

$$\rho_{A,\ell} : G_K \longrightarrow \operatorname{Aut}(V_\ell(A)) \qquad V_\ell(A) := T_\ell(A) \otimes_{\mathbb{Z}_\ell} \mathbb{Q}_\ell.$$

Let $E_\ell := E \otimes_{\mathbb{Q}} \mathbb{Q}_\ell$, and note that $V_\ell(A)$ is a free E_ℓ-module of rank 1.[3] Therefore, $\rho_{A,\ell}$ really lands in $\operatorname{Aut}_{E_\ell}(V_\ell(A)) \simeq E_\ell^\times$. In particular, it has abelian image, so we win by Lemma 13. ■

3 S-T + Main Theorem of CM

We know by Proposition 12 that all CM abelian varieties A have a model defined over some number field. We further know, by Proposition 14, that we can always choose this number field so that A has everywhere good reduction. Given such a setup, one may ask, "How does Frobenius act on the reductions of A at various primes?"

Theorem 15 (Shimura-Taniyama Formula)**.** *Let (A, i) be an abelian variety of CM-type (E, Φ) over a number field $K \subset \mathbb{C}$. Let $\mathfrak{p} \in \operatorname{Spec} \mathscr{O}_K$ be a prime of good reduction for A, say to $(\overline{A}, \bar{i})$ over $\mathbb{F}_q = \mathscr{O}_K/\mathfrak{P}$. Let $\operatorname{Fr} \in \operatorname{End}(\overline{A})$ denote the qth power Frobenius map.*[4] *Then,*

- ***(a)*** *There exists a unique $\pi \in E$ such that $\bar{i}(\pi) = \operatorname{Fr}$.*
- ***(b)*** *Assume that $K/\mathbb{Q}$ is Galois and contains all conjugates of E. Then, for all primes v of E lying over $p = \operatorname{char} \mathbb{F}_q$,*

$$\frac{\operatorname{ord}_v(\pi)}{\operatorname{ord}_v(q)} = \frac{\#(\Phi \cap H_v)}{\# H_v} \quad \textit{where } H_v = \left\{\rho : E \to K : \rho^{-1}(\mathfrak{p}) = \mathfrak{p}_v\right\}.$$

Proof. Omitted.[5] ■

The main utility of this theorem, for us, is that it can be used in the proof of the main theorem of complex multiplication. However, we won't prove this main theorem in this talk. Instead, we'll state it, and then, via an example, relate it to canonical models of Shimura varieties.

Definition 16. Let (E, Φ) be a CM-type. Its reflex field E^* is the smallest subfield of $\overline{\mathbb{Q}}$ such that there exists an $E \otimes_{\mathbb{Q}} E^*$-module V satisfying

$$V \otimes_{E^*} \overline{\mathbb{Q}} \simeq \bigoplus_{\varphi \in \Phi} \overline{\mathbb{Q}}_\varphi \text{ as } E \otimes_{\mathbb{Q}} \overline{\mathbb{Q}}\text{-modules.}$$ ◇

Remark 17.

(1) V above is uniquely determined up to isomorphism. Furthermore, letting $T^E := \operatorname{Res}_{E/\mathbb{Q}} \mathbb{G}_m$ and $T^{E^*} := \operatorname{Res}_{E^*/\mathbb{Q}} \mathbb{G}_m$, there is a reflex norm $N_\Phi : T^{E^*} \to T^E$ whose action on $\mathbb{Q}$-points is

$$\begin{array}{ccccccc} (E^*)^\times & = & T^*(\mathbb{Q}) & \longrightarrow & T(\mathbb{Q}) & = & E^\times \\ a & & & \longmapsto & & & \det_E(a \mid V) \end{array}$$

[3] e.g. because $V_\ell A \simeq \mathrm{H}_1(A(\mathbb{C}), \mathbb{Q}) \otimes \mathbb{Q}_\ell$

[4] This one is a k-morphism

[5] The key to **(a)** is to realize it suffices to show that Frobenius is in the image of $E \otimes_{\mathbb{Q}} \mathbb{Q}_\ell \to \operatorname{End}^0(\overline{\mathbb{Q}}) \otimes \mathbb{Q}_\ell \hookrightarrow \operatorname{End}(V_\ell(\overline{A}))$. Since $V_\ell(\overline{A})$ is free of rank one over E_ℓ, any endomorphism commuting with the E-action is in this image.

(2) E^* is equivalently the fixed field of $\{\sigma \in G_{\mathbb{Q}} : \sigma\Phi = \Phi\}$

(3) If $E/\mathbb{Q}$ is quadratic imaginary, then $E^* = E$. ○

Theorem 18 (Main Theorem of Complex Multiplication)**.** *Let (A, i) be an abelian variety with CM-type (E, Φ) over $\mathbb{C}$. Let $\varphi_{E^*} : \mathbb{A}_E^\times \to G_{E^*}^{\mathrm{ab}}$ denote the Artin reciprocity map, normalized so as to send uniformizers to the inverse of Frobenius. Then, for any $\sigma \in \mathrm{Aut}(\mathbb{C}/E^*)$ and any $s \in \mathbb{A}_{f,E}^\times$ with $\varphi_{E^*}(s) = \sigma|_{(E^*)^{\mathrm{ab}}}$, there is a unique E-"isogeny" $\alpha : A \to A^\sigma$ such that*

$$\alpha(N_\Phi(s) \cdot x) = \sigma x \ \text{ for all } \ x \in V_f(A) = \mathrm{H}^1(A; \mathbb{Q}) \otimes_{\mathbb{Q}} \mathbb{A}_f.$$

Why true? A^σ has CM by $(E, \sigma\Phi) = (E, \Phi)$, so there's some isogeny $\alpha : A \to A^\sigma$. This gives two maps $V_f(A) \rightrightarrows V_f(A^\sigma) : x \mapsto \alpha(x), \sigma(x)$. Because $V_f(A)$ is a free $\mathbb{A}_{f,E}$-module of rank 1, these must differ via multiplication by some $\eta(\sigma) \in \mathbb{A}_{f,E}^\times$. The choice of isogeny α means η only gives a well-defined map $\eta : \mathrm{Aut}(\mathbb{C}/E^*) \to \mathbb{A}_{f,E}^\times/E^\times$. This only leaves computing η, which apparently one can do using Shimura-Taniyama.

4 Can Models and CM, example

Let (E, Φ) be a CM-type, and let $F \subset E$ be E's maximal totally real subfield. Define $T^E := \mathrm{Res}_{E/\mathbb{Q}} \mathbb{G}_m$ and $T^F := \mathrm{Res}_{F/\mathbb{Q}} \mathbb{G}_m$. Let $\mathrm{Nm} : T^E \to T^F$ be the norm map $a \mapsto a \cdot \overline{a}$, and consider the fiber product

$$\begin{array}{ccc} T & \longrightarrow & T^E \\ \big\downarrow & & \big\downarrow \mathrm{Nm} \\ \mathbb{G}_m & \longrightarrow & T^F, \end{array}$$

i.e. $T(R) = \left\{a \in (E \otimes_{\mathbb{Q}} R)^\times : \varphi_R(a)\overline{\varphi_R(a)} = \varphi'_R(a)\overline{\varphi'_R(a)} \in \mathbb{C} \otimes_{\mathbb{Q}} R \text{ for any } \varphi, \varphi' \in \Phi\right\}$. Finally, consider the morphism $h_\Phi : \S \to T^E_{\mathbb{R}}$ defined on $\mathbb{R}$-points by

$$\begin{array}{rccc} h_\Phi(\mathbb{R}) : & \mathbb{C}^\times & \longrightarrow & E_{\mathbb{R}}^\times \simeq \prod_{\varphi \in \Phi} \mathbb{C}_\varphi \\ & z & \longmapsto & (z, z, \ldots, z). \end{array}$$

Lemma 19. *The image of h_Φ lands in $T_{\mathbb{R}} \subset T^E_{\mathbb{R}}$, and the pair $(T, \{h_\Phi\})$ is a Shimura datum.*

Proof. Easy exercise. ■

Remark 20. Fix a purely imaginary $\alpha \in \mathscr{O}_E$ such that $\mathrm{Im}\,\varphi(\alpha) > 0$ for all $\varphi \in \Phi$, as in the proof of Proposition 11. Let $\psi : E \times E \to \mathbb{Q}$ be the bilinear form

$$\psi(x, y) := \mathrm{Tr}_{E/\mathbb{Q}}(\alpha x \overline{y}) = \sum_{\varphi \in \Phi} \mathrm{Tr}_{\mathbb{C}/\mathbb{R}}\left(\varphi(\alpha)\varphi(x)\overline{\varphi(y)}\right).$$

With this defined, $T \subset T^E$ can be described as the torus

$$T(R) := \left\{(a, b) \in (E \otimes_{\mathbb{Q}} R)^\times \times R^\times : \psi(ax, ay) = b\psi(x, y) \text{ for all } x, y \in E \otimes_{\mathbb{Q}} R.\right\}. \tag{1}$$

Note above that $b = \varphi_R(a)\overline{\varphi_R(a)}$ for any $\varphi \in \Phi$.

In particular, there's a morphism of Shimura datum $(T, \{h_\Phi\}) \hookrightarrow (\mathrm{GSp}(\psi), X(\psi))$. ◦

Consequently, $(T, \{h_\Phi\})$ must parameterize some polarized abelian varieties

Definition 21. For any compact open subgroup $K \subset T(\mathbb{A}_f)$, let $\mathcal{M}_K$ denote the set of isomorphism classes of quadruples $(A, j, \lambda, \eta K)$ where

- A is a complex abelian variety
- λ is a polarization, equivalently a Riemann form $\psi_\lambda : \mathrm{H}_1(A, \mathbb{Q}) \times \mathrm{H}_1(A, \mathbb{Q}) \to \mathbb{Q}$
- j is a homomorphism $E \to \mathrm{End}^0(A)$
- ηK is a K-orbit of $\mathbb{A}_{E,f}$-linear (note $\mathbb{A}_{E,f} = E \otimes_{\mathbb{Q}} \mathbb{A}_f$) isomorphisms $\eta : \mathbb{A}_{E,f} \to V_f(A)$,

and which satisfy

$$\text{There exists an } E\text{-linear isomorphism } a : \mathrm{H}_1(A, \mathbb{Q}) \to V \text{ sending } \psi_\lambda \text{ to a } \mathbb{Q}^\times\text{-multiple of } \psi. \qquad (\star)$$

An isomorphism from one tuple $(A, i, \eta K)$ to another $(A', i', \eta' K)$ is an E-"isogeny" $A \to A'$ sending η to η' modulo K. ⋄

Proposition 22. *For any compact open subgroup $K \subset T(\mathbb{A}_f)$, there is a natural bijection*

$$\mathcal{M}_K \xrightarrow{\sim} \mathrm{Sh}_K(T, \{h_\Phi\}) = T(\mathbb{Q}) \backslash T(\mathbb{A}_f)/K.$$

Theorem 23. *Let $\mathrm{Aut}(\mathbb{C}/E^*)$ act on $\mathrm{Sh}_K(T, \{h_\Phi\}) = T(\mathbb{Q}) \backslash T(\mathbb{A}_f)/K$ as follows: given $\sigma \in \mathrm{Aut}(\mathbb{C}/E^*)$, choose $s \in \mathbb{A}_{f,E^*}^\times$ such that $\mathrm{Art}_{E^*}(s) = \sigma|_{(E^*)^{\mathrm{ab}}}$, and then set*

$$\sigma[a] := [N_\Phi(s) \cdot a] \quad \text{for } a \in T(\mathbb{A}_f).$$

The bijection

$$\mathcal{M}_K \xrightarrow{\sim} \mathrm{Sh}_K(T, \{h_\Phi\})$$

is equivariant for this action.

The isomorphism between the corresponding tuples on the $\mathcal{M}_K$ side is given by the isogeny $\alpha : A \to A^\sigma$ of Theorem 18.

Remark 24. This theorem gives a canonical (because it comes from moduli) action of $\mathrm{Aut}(\mathbb{C}/E^*)$ on $\mathrm{Sh}(T, \{h_\Phi\}) \hookrightarrow \mathrm{Sh}(\mathrm{GSp}(\psi), X(\psi))$. As E ranges over CM fields, these various (zero-dimensional) Shimura subvarieties cover a dense subset of $\mathrm{Sh}(\mathrm{GSp}(\psi), X(\psi))$, and so the canonical $\mathrm{Aut}(\mathbb{C}/E^*)$-actions on all of them can be used to define a canonical action on $\mathrm{Sh}(\mathrm{GSp}(\psi), X(\psi))$. A similar strategy will work for general Shimura varieties. ◦

References

[ST68] Jean-Pierre Serre and John Tate. Good reduction of abelian varieties. *Ann. of Math. (2)*, 88:492–517, 1968. 3