18.737 (Algebraic Groups) Notes

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These are my course notes for "Algebraic Groups" at MIT. Each lecture will get its own "chapter." These notes are live-texed or whatever, so there will likely to be some (but hopefully not too much) content missing from me typing more slowly than one lectures. They also, of course, reflect my understanding (or lack thereof) of the material, so they are far from perfect.[1] Finally, they contain many typos, but ideally not enough to distract from the mathematics. With all that taken care of, enjoy and happy mathing.

The instructor for this class is Bjorn Poonen.

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[1] In particular, if things seem confused/false at any point, this is me being confused, not the speaker

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1 Lecture 1 (2/17)

1.1 Administrative/Course stuff

Office hours (listed on webpage)

- Wed 5-6pm
- Sat 1-2pm
- Sun 2-3pm

When doing problems, list the sources you've consulted. The first problem set is on Canvas right now, and is due this Sunday on Gradescope.

1.2 Content

The goal of this course is to classify algebraic groups (at least, to try to).

Terminology:

- If X is a k-scheme and $L \supset k$, the **base change** is $X_L := X \times_{\operatorname{spec} k} \operatorname{spec} L$
- $\operatorname{spec} A = \{\text{prime ideals of } A\}$ with its sheaf of rings. Milne uses max spec instead
- A **k-variety** is a separated, finite type k-scheme. Milne requires k-varieties to also be geometrically reduced

For reduced varieties over $k = \overline{k}$

- a subvariety Y of a given (reduced) X is determined by the subset $Y(k) \subset X(k)$
- morphisms are also determined by what they do on k-points

More generally, each k-scheme X gives rise to its functor of points

$$\begin{array}{rccc} h^X : & \{k\text{-algebras}\} & \longrightarrow & \text{Set} \\ & R & \longmapsto & X(R) := \operatorname{Hom}_{\mathrm{Sch}/k}(\operatorname{spec} R, X) \end{array}$$

Warning 1.1. We're ignoring some set-theoretic issues here (e.g. there's no 'set' of k-algebras). Milne worries about these in the beginning of his book, but we won't stress them in lecture.

This functor of points determines the schemes X, Y (think, Yoneda).

Definition 1.2. A **k-group scheme** is a group object in the category of k-schemes. That is, it is a k-scheme G together with morphisms

(multiplication) $m : G \times G \to G$ (note $G \times G = G \times_k G$, product as k-schemes)

(inverse) $i : G \to G$

(identity) $e : \operatorname{spec} k \to G$

satisfying the usual group axioms. By Yoneda, an equivalent definition is that a k-group scheme is a k-scheme G together with a factorization

$$\begin{tikzcd} \{k\text{-algs}\} \arrow[r] \arrow[rr, bend left, "h^G"] & \mathrm{Grp} \arrow[r] & \mathrm{Set} \end{tikzcd}$$

Example. Consider the functor

$$\begin{array}{ccc} \{k\text{-algs}\} & \longrightarrow & \mathrm{Set} \\ R & \longmapsto & \mathrm{GL}_2(R) \end{array}$$

It is representable by the scheme

$$\mathrm{GL}_2 := \operatorname{spec} k[a,b,c,d]\left[\frac{1}{ad-bc}\right]$$

We claim this is a group scheme. One way to show this would be to write down the multiplication map $\mathrm{GL}_2 \times \mathrm{GL}_2 \to \mathrm{GL}_2$ (as well as inversion/identity), but it's easier to just note that $\mathrm{GL}_2(R)$ is always a group (functorially in R), and then conclude that GL_2 is a group scheme by Yoneda.

Example. The multiplicative group is the functor $\mathbb{G}_m(R) := R^\times$. This is representable by $\mathbb{G}_m = \operatorname{spec} k[t, 1/t]$. Note that $\mathbb{G}_m = \mathrm{GL}_1$.

Example. There's the group of nth roots of unity $\mu_n(R) := \{r \in R : r^n = 1\}$ which is representable by $\mu_n = \operatorname{spec} k[t]/(t^n - 1)$. Note this is not the trivial group even if $\operatorname{char} k \mid n$.

Example. Say $\operatorname{char} k = p > 0$. We can consider the group functor

$$\alpha_p(R) := \{r \in R : r^p = 0\}$$

(under addition). This is represented by $\alpha_p = \operatorname{spec} k[x]/(x^p)$. Notice that this example is not reduced.

Definition 1.3. A **k-algebraic group** is (separated) finite type k-group scheme, i.e. a group object in the category of k-varieties.

Remark 1.4. Above, separated is in parentheses since it follows automatically from the other conditions. This is because the diagonal $\Delta \subset G \times G$ is closed since it is cut out by the equation $gh^{-1} = e$. So group schemes are always separated.

1.3 Review of two notions of nonsingularity

Regular

Recall 1.5. Let A be a noetherian local ring with maximal ideal $\mathfrak{m}$, and let $\kappa := A/\mathfrak{m}$ be the residue field. Then,

$$\dim_\kappa \mathfrak{m}/\mathfrak{m}^2 \geq \dim A.$$

We say A is a **regular local ring** if we have equality above.

Definition 1.6. Let X be a variety with a point $x \in X$. We say X is **regular at** x if $\mathscr{O}_{X,x}$ is regular local. We say X is **regular** if it is regular at every x.

For regularity, it is enough to check at closed points only.

Smooth

Definition 1.7. Fix $r \geq 0$. An arbitrary morphism $f : X \to S$ of schemes is **smooth at** x ($x \in X$) if

(1) Suppose that f is
$$\operatorname{spec} \frac{A[t_1, \ldots, t_n]}{(g_{r+1}, \ldots, g_n)} \to \operatorname{spec} A.$$
We say f is **obviously smooth of relative dimension** r **at** x if
$$\left(\frac{\partial}{\partial g_i} t_j(x)\right) \in M_{(n-r)\times n}(\kappa(x))$$
has rank $n - r$ (i.e. maximal rank).

(2) In general, f is **smooth of relative dimension** r **at** x if it looks locally like above near x. That is, we can find open affine neighborhoods $X' \subset X$ of x and $S' \subset S$ of $f(x)$ such that f restricts to a map $X' \to S'$ which is obviously smooth of relative dimension r at x.

Notation 1.8. We let $X^{sm} := \{x \in X : X \to S \text{ is smooth at } x\}$ denote the smooth locus of X.

1.3.1 Relating the Notions over Varieties

"My general philosophy for this class is that if I feel like using some facts from algebraic geometry, I'll just use them. And I'll pretend that y'all learned them in your algebraic geometry class."

Consider a k-variety $X \to \operatorname{spec} k$.

Fact (smoothness for k-varieties)**.**

(1) smooth at $x \implies$ regular at x. Conversely, regular at $x \implies$ smooth at x *if* $\kappa(x)/k$ is separable.

Corollary 1.9. *If k is perfect, then regular = smooth.*

(2) regularity can be lost upon inseparable field extension.

(3) smoothness is unchanged by field extension.

Corollary 1.10. *geometrically regular = geometrically smooth = smooth.*

In fact, this is how Milne defines smooth.

(4) X^{sm} is open in X (holds for any morphism, defined by non-vanishing of minors of a matrix)

(5) If X is geometrically reduced, then X^{sm} is dense in X.

Example. Say $k = \mathbb{F}_p(t)$. Then, $X : y^2 = x^p - t$ is regular, but not smooth.

Proposition 1.11. *For an algebraic group G, G is smooth $\iff$ G is geometrically reduced.*

Proof. WLOG $k = \overline{k}$. Then, smooth = geometrically regular $\implies$ geometrically reduced $\implies$ some open dense subset is smooth $\implies$ smooth at one k-point $\implies$ smooth at all k-points (via translation) $\implies$ smooth (G is quasi-compact[2]). We've come full circle. ∎

[2] every (closed subset of a) quasi-compact scheme contains a closed point.

1.4 Galois Theory according to Grothendieck

Definition 1.12. An **étale k-algebra** is a k-algebra of the form $L_1 \times \ldots \times L_n$ where each L_i/k is a finite separable field extension. This are exactly the k-algebras A such that $A \otimes_k k^s \simeq (k^s)^n$ for some n, as k^s-algebras.

Let $\mathfrak{g} := \mathrm{Gal}(k^s/k)$.

Poonen did not actually use mathfrak, but this is the closest I could get to the g he did write down

Theorem 1.13. *There is an equivalence of categories*

$$\left\{\begin{matrix} \textit{étale} \\ k\textit{-algs} \end{matrix}\right\}^{op} \longleftrightarrow \left\{\begin{matrix} \textit{fin. étale} \\ k\textit{-schemes} \end{matrix}\right\} \longleftrightarrow \left\{\begin{matrix} \textit{finite sets w/} \\ \textit{continuous } \mathfrak{g}\textit{-action} \end{matrix}\right\}$$

Above, étale = smooth of rel dim 0 (note finite $\implies$ affine). The assignments above are via

$$\begin{array}{ccccc} \mathscr{O}(Z) & \longleftarrow & Z & \longmapsto & Z(k^s) \\ A & \longleftarrow & \operatorname{spec} A & \longmapsto & \mathrm{Hom}_{k\text{-}alg}(A, k^s) \end{array}$$

2 Lecture 2 (2/19)

2.1 Last time

A k-group scheme is the data of

- a k-scheme G; and
- a group structure on the set $G(R)$ for each k-algebra R s.t. for each k-algebra homomorphism $R \to R'$, the map $G(R) \to G(R')$ is a group homomorphism

A k-subscheme $N \subset G$ is a **subgroup scheme** if for each k-algebra R, the subset $N(R)$ of $G(R)$ is a subgroup. A subgroup scheme N is **normal** in G if $N(R)$ is normal in $G(R)$ for each R.

Remark 2.1. smooth $\implies$ regular. The converse holds at any x with $\kappa(x)/k$ separable.

Remark 2.2. Algebraic groups in characteristic 0 are automatically smooth (proved by Cartier).

Theorem 2.3. *There is an equivalence of categories*

$$\left\{\begin{matrix} \textit{étale} \\ k\textit{-algs} \end{matrix}\right\}^{op} \longleftrightarrow \left\{\begin{matrix} \textit{fin. étale} \\ k\textit{-schemes} \end{matrix}\right\} \longleftrightarrow \left\{\begin{matrix} \textit{finite sets w/} \\ \textit{continuous } \mathfrak{g}\textit{-action} \end{matrix}\right\}$$

Above, étale = smooth of rel dim 0 (note finite $\implies$ affine). The assignments above are via

$$\begin{array}{ccccc} \mathscr{O}(Z) & \longleftarrow & Z & \longmapsto & Z(k^s) \\ A & \longleftarrow & \operatorname{spec} A & \longmapsto & \mathrm{Hom}_{k\text{-}alg}(A, k^s) \end{array}$$

2.2 This time

Say X is a k-variety (separated scheme of finite type). We can base change it to a separable closure X_{k^s}, and consider its connected components. Note that this set of components is a finite set with (continuous) $\mathrm{Gal}(k^s/k)$-action. By Grothendieck's formulation of Galois theory, this corresponds to some finite étale

k-scheme which we denote $\pi_0(X)$. It also corresponds to some étale k-algebra, which will be the maximal étale k-algebra in $\mathscr{O}(X)$. Including this étale k-algebra into $\mathscr{O}(X)$ gives a map

$$X \to \pi_0(X)$$

whose fibers are the connected components of X (not of X_{k^s}).

Example. Take $X = \operatorname{spec} \frac{\mathbb{Q}[x,y]}{(x+3)(x^2-2)}$ over $\mathbb{Q}$. This looks like the line $x = -3$ unioned with the "pair of lines" $x = \pm\sqrt{2}$. Hence, $\pi_0(X)(\mathbb{Q})$ has three points, one for each geometric component, and there's a Galois action interchanging the lines $x = \pm\sqrt{2}$. As a $\mathbb{Q}$-scheme, we have

$$\pi_0(X) = \operatorname{spec} \mathbb{Q} \sqcup \operatorname{spec} \frac{\mathbb{Q}[x]}{(x^2-2)}.$$

The correspond étale algebra is the affine coordinate ring of $\pi_0(X)$, i.e. it is $\mathbb{Q} \times \mathbb{Q}(\sqrt{2})$.

Let G be an algebraic group over k. Let G^0 denote the connected component of G containing the identity $e \in G(k)$. This component will be stable under the Galois group (since e is), so G^0 will be defined over k. Here are some facts in this case...

- {connected comps of G_{k^s}} is a finite group with $\mathrm{Gal}(k^s/k)$-action. Hence, $\pi_0(G)$ is a finite étale group scheme over k.

  We'll later show that there's actually an exact sequence $1 \to G^0 \to G \to \pi_0(G) \to 1$, but we haven't even defined exact sequence yet.

- $(G_L)^0 = (G^0)_L$ for any field extension $L \supset k$. Hence, G^0 is geometrically connected.

- The connected components of G = the irreducible components of G. This is because every point has to look the same.

Corollary 2.4. *For an algebraic group, G is geometrically irreducible $\iff$ G is geometrically connected $\iff$ $G^0 = G$ $\iff$ G is connected $\iff$ G is irreducible.*

2.3 Kernels

Proposition 2.5. *Let $\varphi : G \to H$ be a homomorphism of group schemes. Then, there exists a unique normal subgroup scheme $K \lhd G$ such that*

$$K(R) = \ker(G(R) \to H(R))$$

functorially in R.

Proof. Take the fiber product

$$\begin{tikzcd} K \arrow[r] \arrow[d] & G \arrow[d, "\varphi"] \\ \operatorname{spec} k \arrow[r, "e"] & H \end{tikzcd}$$

∎

Definition 2.6. K above is called the **kernel** and denote $K =: \ker \varphi$.

Warning 2.7. Although G_{red} is a closed subscheme of G, it is not necessarily a subgroup scheme. However, it will be one if G_{red} is geometrically reduced.

Example. Consider $k = \mathbb{F}_2(t)$, and let $G = \ker\left(\mathbb{G}_a \xrightarrow{x \mapsto x^4 - tx^2} \mathbb{G}_a\right)$. As a scheme, G is $x^2(x^2 - t) = x^4 - tx^2 = 0$ in $\mathbb{A}^1$. Hence, G_{red} is $x(x^2 - t) = 0$ in $\mathbb{A}^1$ (point at origin no longer fat). This is not a subgroup scheme.[3]

2.4 Closures

Let G be a k-group with $k = \overline{k}$, and set $|G| = G(k)$.

Lemma 2.8. *Let U, V be dense open subsets of $|G|$. Then, $UV = |G|$ (pairwise product on LHS).*

Proof. Given $g \in |G|$, need $u \in U$ and $v \in V$ such that $uv = g$. This is the same as solving $u = gv^{-1}$. This is solvable since $U \cap gV^{-1} \neq \emptyset$ since these are both dense and open (inversion and translation are both homeomorphisms). ■

Lemma 2.9. *Let H be a subgroup of $|G|$. Let $\overline{H}$ be the Zariski closure of H in G. Then, $\overline{H}$ is an algebraic subgroup of G.*

Proof. Let $x \in |G|$. Left-translation by x is a homeomorphism $G \to G$, so it commutes with taking closures: $x\overline{H} = \overline{xH}$. For each $x \in H$, we see that $x\overline{H} = \overline{xH} = \overline{H}$. Thus, $H\overline{H} = \overline{H}$. For each $y \in \left|\overline{H}\right|$, we know $Hy \subset \overline{H}$. Taking closures gives $\overline{H}y = \overline{Hy} \subset \overline{H}$. Thus, $\overline{H}\left|\overline{H}\right| \subset \overline{H}$ as well. Since $\overline{H}$ is reduced over an algebraically closed field, it is determined by its k-points, so $\overline{H}\,\overline{H} \subset \overline{H}$. Similarly, $\overline{H}^{-1} = \overline{H}$ and $e \in \overline{H}$ are easy. ■

Remark 2.10. When we take the Zariski closure of some set of points, we'll always give it its reduced closed subscheme structure.

Warning 2.11. Say H is a subgroup of $|G|$, let Y be Zariski closure of H in G, just as a topological space, and let $\mathscr{I}_Y$ be its associated ideal sheaf, so

This came up as an audience question, so maybe I should have written this using question/answer blocks, but oh well. Too late to change it now

$$\mathscr{I}_Y(U) = \{f \in \mathscr{O}_G(U) : f(y) = 0 \in \kappa(y) \text{ for all } y \in Y \cap U\}.$$

Let $i : Y \hookrightarrow G$ be the inclusion, and let $Y_n = (Y, i^{-1}(\mathscr{O}_G/\mathscr{I}_Y^n))$ be the ring space structure on Y with structure sheaf $\mathscr{O}_{Y_n} = i^{-1}(\mathscr{O}_G/\mathscr{I}_Y^n)$. Then, Y_n is a closed subscheme of G, and $Y_1 = \overline{H}$ is the closed subscheme considered in the previous lemma. It is tempting to think that Y_n will be an algebraic subgroup of G for all n, but this is *not* the case in general, e.g. because Cartier showed that algebraic groups in characteristic 0 are always smooth (hence reduced).

Lemma 2.12. *Let G be an algebraic group over $k = \overline{k}$, and let H be a subgroup of $|G|$. If H contains a dense open subset U of $\left|\overline{H}\right|$, then H is closed in $|G|$.*

Proof. We know $H = HH \supset UU = \left|\overline{H}\right|$ (by first lemma). ■

[3] G has 'order' 4 since its coordinate ring is rank 4 over k, but G_{red} has 'order' 3, and $3 \nmid 4$, so should not exact G_{red} to be a subgroup scheme.

2.5 Images

Assumption. Again assume $k = \overline{k}$.

Recall 2.13. If $\varphi : X \to Y$ is a morphism of k-varieties, then $\varphi(|X|)$ need not be open or closed inside $\overline{Y}$. However, Chevalley's theorem tells us that $\varphi(|X|)$ will be **constructible**, a boolean combination of varieties (union of locally closed subsets).

Corollary 2.14. *$\varphi(|X|)$ contains a dense open subset of its closure.*

Proposition 2.15. *Let $\varphi : G \to G'$ be a homomorphism of k-groups. Then, $\varphi(|G|)$ is closed in $|G'|$.*

Proof. Apply previous lemma to the subgroup $\varphi(|G|) \subset |G'|$. ∎

Corollary 2.16. *Any algebraic subgroup of an algebraic group is closed (true even if $k \neq \overline{k}$ since can check closeness after base extension).*

3 Lecture 3 (2/22)

Problem set 2 due on Sunday. Next office hours Wednesday 4 – 5pm.

3.1 Last time

- For an algebraic group, connected = irreducible = geometrically connected = geometrically irreducible
- Every algebraic group G fits into an exact sequence
$$1 \longrightarrow G^0 \longrightarrow G \longrightarrow \pi_0(G) \longrightarrow 1$$
with $\pi_0(G)$ finite étale over ground field.
- The kernel K of a homomorphism $G \to H$ is characterized by $K(R) = \ker(G(R) \to H(R))$ for all R.
- Any algebraic subgroup of an algebraic group is a closed subscheme.

3.2 Exactness

Definition 3.1. Let $1 \to A \xrightarrow{i} B \xrightarrow{s} C$ be a sequence of homomorphisms of algebraic groups. This is called **exact** if i is an isomorphism of A onto $\ker s$.

Definition 3.2. We say $1 \to A \xrightarrow{i} B \xrightarrow{s} C \to 1$ is **exact** if in addition s is surjective and flat, i.e. faithfully flat.

Question: Is this equivalent to $B \to C$ being an epimorphism of sheaves (on the fppf site)?

Example. Say $\operatorname{char} k = p$. Consider $\{1\} \hookrightarrow \mu_p$. This is surjective (on underlying sets), but $1 \to \mu_p \to 1$ should not be considered exact.

Note 1. Apparently, we're supposed to be reading the text book ahead of lecture...

Answer: If

Theorem 3.3 (**Homomorphism theorem**). *Every homomorphism of algebraic groups $\varphi : G \to H$ factors uniquely into homomorphisms*

$$G \twoheadrightarrow I \hookrightarrow H$$

with $G \twoheadrightarrow I$ faithfully flat, and $I \hookrightarrow H$ a closed immersion. We call I the ***image*** *of φ.*

Proof. Later. ∎

3.3 Group actions

Let G be a k-group (algebraic group over k), and let X be a k-variety.

Definition 3.4. A **(left) action of G on X** is a morphism $a : G \times X \to X$ satisfying the usual axioms. By Yoneda, this is just saying we have functorial group actions $a(R) : G(R) \times X(R) \to X(R)$, of $G(R)$ acting on the set $X(R)$, for each k-algebra R.

Definition 3.5. Given $x \in X(k)$, get (here, $g \in G(R)$ for some k-algebra R)

$$\begin{array}{rccc} \varphi_* : & G & \longrightarrow & X \\ & g & \longmapsto & gx \end{array}$$

The **orbit of x**, sometimes denotes $G \cdot x$ or Gx, is the image $\varphi_*(G)$ of this morphism.

Proposition 3.6. *$G.x$ is a locally closed subset of X.*

Proof. WLOG can assume $k = \overline{k}$. Chevalley's theorem tells us that $G.x$ is a constructible set, so it contains a dense open subset U of $\overline{G.x}$. Then,

$$G.x = \bigcup_{g \in G(k)} gU,$$

so $G.x$ is open in $\overline{G.x}$ since it's a union of opens in $\overline{G.x}$. ∎

Theorem 3.7 (**Borel's Orbit Lemma**). *Assume $k = \overline{k}$. Every orbit of minimum dimension is closed.*

Proof. Let O be an orbit of minimal dimension. Then, O is a dense open in $\overline{O}$, so $\overline{O} \setminus O$ is a proper, closed subset of $\overline{O}$, and so has strictly lower dimension. Since G preserves O and $\overline{O}$, it also preserves $\overline{O} \setminus O$, so $\overline{O} \setminus O$ is a union of orbits of smaller dimension than O, a contradiction to O's minimality. ∎

Example. $\mathbb{G}_m \times \mathbb{G}_m \curvearrowright \mathbb{A}^2$ by scaling coordinates. The orbits are $\{0\}, X \setminus 0, Y \setminus 0, \mathbb{A}^2 \setminus (X \cup Y \cup \{0\})$ where $X = x$-axis and $Y = y$-axis. Most orbits not closed, but the smallest one $\{0\}$ is.

3.4 Stabilizer, normalizer, centralizer, center

Let G be an algebraic group acting on a k-variety X. Let Y, Z be two closed subschemes of X. The question is, which group elements map Y into Z.

Proposition 3.8. *There is a closed subscheme $T = T_G(Y, Z) \subset G$, called the* ***transporter****, whose functor of points is*

$$T(R) = \{g \in G(R) : gY_R \subset Z_R\}.$$

Proof. Skipped. "Not especially hard, just annoying." ∎

Corollary 3.9. *There exists a subgroup scheme* $S = \mathrm{Stab}_G(Y) \leq G$ *such that*

$$S(R) = \{g \in G(R) : gY_R = Y_R\}.$$

*This is the **stabilizer of** Y.*

Proof. $S = T_G(Y, Y) \cap \mathrm{inv}(T_G(Y, Y))$. ∎

Corollary 3.10. *Let G be an algebraic group with $H \leq G$ a subgroup. Then, there exists a subgroup $N = N_G(H) \leq G$ such that*

$$N(R) = \left\{g \in G(R) : gHg^{-1} = H\right\} = \left\{g \in G(R) : gH(R')g^{-1} = H(R') \text{ for all } R\text{-algs. } R'\right\}.$$

*This is the **normalizer of** H **in** G.*

Proof. $N = \mathrm{Stab}_G(H)$ where G acts by conjugation on G. ∎

Note $H \subset G$ is closed since it's a subgroup

Corollary 3.11. *Again let $H \leq G$. Then there is a subgroup $C = C_G(H) \leq G$ such that*

$$C(R) = \{g \in G(R) : g \text{ commutes with every } h \in H(R') \text{ for all } R\text{-algs. } R'\}$$

*This is the **centralizer of** H **in** G.*

Proof. Let $G \curvearrowright G \times G$ via $g(x, y) := (x, gyg^{-1})$. Then, $C = \mathrm{Stab}_G(\Delta_H)$ where $\Delta_H \subset H \times H \subset G \times G$ is the diagonal of H. This is closed since algebraic groups are varieties, and varietes are separated. ∎

Definition 3.12. The **center** $Z \leq G$ of an algebraic group G is the centralizer

$$Z = C_G(G)$$

of the whole group. Note that $Z(R) \subset$ center of $G(R)$ for all k-algebras R.

Definition 3.13. We say G is a **torus** if $G_{k^s} \simeq (\mathbb{G}_{m,k^s})^n$ for some $n \geq 0$.

3.5 Some subgroups of GL_n

Example. We let $B \subset \mathrm{GL}_n$ denote the subgroup of upper triangular matrices. This is an example of a Borel subgroup.

Example. We let $T \subset \mathrm{GL}_n$ denote the subgroup of diagonal matrices. Note that $T \simeq \mathbb{G}_m^n$ is a (maximal) torus, and also $T \subset B$.

Example. We let $U \subset \mathrm{GL}_n$ denote the subgroup of upper triangular matrices with 1's along the diagonal. This is an example of a unipotent group. Note that $U \subset B$.

Notation 3.14. Milne uses the notation $B = \mathbb{T}_n$, $T = \mathbb{D}_n$, and $U = \mathbb{U}_n$.

Remark 3.15. One can define semi-direct products (split short exact sequences), and $B = U \rtimes T$.

Note that T is composed of copies of $\mathbb{G}_m$ while U is composed of copies of $\mathbb{G}_a$. One can try to understand the interactions between T, U combinatorially, and working this out recovers the usual fact that every matrix is a product of elementary matrices (or something like this). Can get a similar sort of statement for reductive groups, more generally.

3.6 The split classical groups (up to isogeny)

Example. There's $\mathrm{SL}_{n+1} := \ker(\mathrm{GL}_{n+1} \xrightarrow{\det} \mathbb{G}_m)$.

Example (SO_{2n+1})**.** GL_{2n+1} acts on $\mathbb{A}^{\binom{2n+2}{2}}$, the space of quadratic forms via $(T \circ q)(x, y) = q(T \cdot (x, y))$. Let $q := x_0^2 + x_1y_1 + \cdots + x_ny_n$ (not the sum of squares quadratic form[4]). We define

$$O_{2n+1} := \mathrm{Stab}_{\mathrm{GL}_{2n+1}}(q) \text{ and } \mathrm{SO}_{2n+1} := \ker\left(O_{2n+1} \xrightarrow{\det} \{\pm 1\}\right)$$

(note definition of SO above only makes sense when char $\neq 2$). This gives the **odd (split) orthogonal group**.

Example (Sp_{2n})**.** There's $\mathrm{Sp}_{2n} := \mathrm{Stab}_{\mathrm{GL}_{2n}}(\varphi)$ where φ is the alternating bilinear form represented by the matrix $\begin{pmatrix} & I_n \\ -I_n & \end{pmatrix} =: S$. Hence,

$$\mathrm{Sp}_{2n} = \left\{g \in \mathrm{GL}_{2n} : g^t S g = S\right\}.$$

This is the **symplectic group**.

Example (SO_{2n})**.** We get SO_{2n} the same way we got SO_{2n+1}, but using the form $q := x_1y_1 + \cdots + x_ny_n$.

Fact. These are all connected.

4 Lecture 4 (2/24)

4.1 Last time

We introduced the split classical groups (up to isogeny)

- (A_n) SL_{n+1}
- (B_n) SO_{2n+1}. Let GL_{2n+1} act on quadratic forms, and let $q = x_0^2 + x_1y_1 + \cdots + x_ny_n$. Define

$$O_{2n+1} := \mathrm{Stab}_{\mathrm{GL}_{2n+1}}(q) \text{ and } \mathrm{SO}_{2n+1} := \ker(O_{2n+1} \xrightarrow{\det} \mu_2).$$

- (C_n) Sp_{2n}. Let $S = \begin{pmatrix} & I_n \\ -I_n & \end{pmatrix}$. Define the bilinear form $\varphi(\mathbf{x}, \mathbf{y}) = \mathbf{x}^t S \mathbf{y}$. This φ is skew-symmetric and alternating with diagonal entries 0. Define

$$\mathrm{Sp}_{2n} := \mathrm{Stab}_{\mathrm{GL}_{2n}}(\varphi) = \left\{g \in \mathrm{GL}_{2n} : g^t S g = S\right\}.$$

[4] they agree over an algebraically closed field

- (D_n) SO_{2n}. Let $q = x_1y_1 + \cdots + x_ny_n$. Let $O_{2n} = \mathrm{Stab}_{\mathrm{GL}_{2n}}(q)$. If $\operatorname{char} k \neq 2$, let

$$\mathrm{SO}_{2n} := \ker\left(O_{2n} \xrightarrow{\det} \mu_2\right).$$

(if $\operatorname{char} k = 2$, replace det with the "Dickson homomorphism" $O_{2n} \twoheadrightarrow (\mathbb{Z}/2\mathbb{Z})_k$)

Each is connected, and indexed so that the maximal torus is $\mathbb{G}_m^n$.

You can get coordinate-free versions of these

- Let V be a f.d. k-vector space. Gives rise to GL_V with

$$\mathrm{GL}_V(R) := \mathrm{Aut}_R(V \otimes_k R).$$

- Suppose $q : V \to k$ is a **nondegenerate quadratic form**, i.e. if V is identified with k^n, q is a degree 2 homogeneous polynomial that defines a smooth quadric in $\mathbb{P}V$. Then, (V, q) gives rise to $\mathrm{SO}(q) \leq O(q) \leq \mathrm{GL}_V$.

4.2 More algebraic group definitions

Definition 4.1. An algebraic group G is **finite** if $G \to \operatorname{spec} k$ is a finite morphism, so $G = \operatorname{spec} A$ where A is a f.d. k-vector space. For finite G, its **order** is defined to be $\#G := \dim_k A$.

Example. μ_n is finite with order $\#\mu_n = n$. The affine coordinate ring is $k[x]/(x^n - 1)$.

Example. α_p is order p

Example. Let H be an abstract finite group. The constant group scheme H_k over k has order $\#H$. Here, the affine coordinate ring is $\prod_{h \in H} k$.

Example. If E is an elliptic curve, then $\#E[p] = p^2$, even in characteristic p.

Definition 4.2. Say G, H are smooth, connected algebraic groups. An **isogeny** $\varphi : G \to H$ is a surjective homomorphism with finite kernel.[5] We also define the **degree** of φ to be $\deg \varphi := \# \ker \varphi$ (order as a group scheme).

Remember: for algebraic groups, smooth = geometrically reduced, and connected = geometrically irreducible

Remark 4.3. For any faithfully flat homomorphism $\varphi : G \to H$, φ is étale $\iff$ $\ker \varphi$ is étale as a k-scheme. Similarly, φ is smooth of relative dimension r $\iff$ $\ker \varphi$ is smooth of relative dimension r over k.

Definition 4.4. An **abelian variety** is a a smooth, connected algebraic group which is proper over k.

Some people use 'complete' to mean proper over a field

Example. Elliptic curves are 1-dimensional abelian varieties. The trivial group scheme is a 0-dimensional abelian variety.

Theorem 4.5. *Any abelian variety is commutative and projective.*

We won't talk too much about abelian varieties in this class. If you want to know more about them, there's Silverman's 'Arithmetic of elliptic curves' and Mumford's 'Abelian varieties'.

[5] Note: H smooth $\implies$ H reduced $\implies$ φ flat above some dense open in H (generic flatness) $\implies$ φ is flat everywhere (H a group)

4.3 Frobenius

Recall 4.6. Let $i : k \hookrightarrow L$ be a field map. Given a k-scheme X, you can form the base change

$$\begin{tikzcd} X_L \arrow[r] \arrow[d] & X \arrow[d] \\ \operatorname{spec} L \arrow[r] & \operatorname{spec} k. \end{tikzcd}$$

In conrete terms, given X, apply i to all coefficients of the equations defining X in order to get equations for X_L.

Now let k be a field of characteristic $p > 0$, and let X be a k-scheme.

Definition 4.7. The **absolute Frobenius morphism** $\sigma_X : X \to X$ is the morphism given by the identity on topological spaces, and Frobenius on the $\mathscr{O}_X$, i.e. $f \mapsto f^p$ on each $\mathscr{O}_X(U)$.

Warning 4.8. Absolute Frobenius is a morphism of schemes, but not a morphism of *k-schemes*. It belongs is a (non-Cartesian) commutative square

$$\begin{tikzcd} X \arrow[r, "\sigma_X"] \arrow[d] & X \arrow[d] \\ \operatorname{spec} k \arrow[r, "\sigma_{\operatorname{spec} k}"] & \operatorname{spec} k \end{tikzcd}$$

with $\sigma_{\operatorname{spec} k}$ induced by the pth power map $k \to k$.

Definition 4.9. The actual fiber product in the above diagram is defined $X^{(p)}$, i.e. it is the pullback

$$\begin{tikzcd} X^{(p)} \arrow[r] \arrow[d] & X \arrow[d] \\ \operatorname{spec} k \arrow[r, "\sigma_{\operatorname{spec} k}"] & \operatorname{spec} k \end{tikzcd}$$

The induced k-morphism $F_X : X \to X^{(p)}$ is called the **relative Frobenius morphism**. This *is* a morphism of k-schemes.

Remark 4.10. Say X =affine. What do these maps do on rings?

- $X^{(p)} \to X$ raising constants to the pth power.
- $X \to X^{(p)}$ raises variables to the pth power.
- $\sigma_X : X \to X$ raises everything to the pth power.

If G is an algebraic group, F_G is a homomorphism.

Proposition 4.11. *If G is smooth and connected of dimension n, then F_G is an isogeny of degree p^n.*

Proof. WLOG $k = \overline{k}$, so k is perfect. Hence, $\sigma_{\operatorname{spec} k} : \operatorname{spec} k \to \operatorname{spec} k$ is an isomorphism, so $X^{(p)} \to X$ is an isomorphism (of abstract schemes). Hence, $F_G : G \to G^{(p)}$ is a bijection. In particular, F_G is surjective (and flat), and $\ker F_G$ is $\{e\}$ as a set. Thus, F_G is an isogeny.

To compute the degree, we can work with the local rings, and even with the completed local rings. We have a k-algebra morphism $\widehat{\mathscr{O}}_{G^{(p)},e} \to \widehat{\mathscr{O}}_{G,e}$. Since G is smooth, these are power series rings. Say $\widehat{\mathscr{O}}_{G,e} \simeq k[\![t_1,\dots,t_n]\!]$ and $\widehat{\mathscr{O}}_{G^{(p)},e} \simeq k[\![u_1,\dots,u_n]\!]$. The induced map between them is simply $u_i \mapsto t_i^p$. Hence, the fiber above $e \in G^{(p)}$ is simply

$$\operatorname{spec} \frac{k[\![t_1,\dots,t_n]\!]}{(t_1^p,\dots,t_n^p)}$$

which is visibly of dimension p^n as a k-vector space. ■

Given $\varphi : X \to G$ with $X = k$-variety and $G = k$-algebraic group. What is the algebraic subgroup generated by X? We want to say there is a smallest subgroup containing the image of X. By Yoneda, φ factors through a closed subgroup $H \le G$ iff $\varphi(X(R)) \subset H(R)$ for all R. Hence, the **algebraic subgroup generated by** φ is the intersection of all such H.

4.4 Restriction of scalars

Example. Take $L = \mathbb{Q}(\sqrt{2})$ over $k = \mathbb{Q}$. Let $X : xy + (5+7\sqrt{2}) = 0$ in $\mathbb{A}^2_L$. How do we make a $\mathbb{Q}$-variety Y s.t. $Y(\mathbb{Q}) = X(L)$?

Substitute $x = x_1 + x_2\sqrt{2}$ and $y = y_1 + y_2\sqrt{2}$ to get now

$$(x_1y_1 + 2x_2y_2 + 5) + (x_1y_2 + x_2y_2 + 7)\sqrt{2} = 0.$$

Define Y to be the $\mathbb{Q}$-variety cut out by the two equations

$$Y : \begin{cases} x_1y_1 + 2x_2y_2 + 5 = 0 \\ x_1y_2 + x_2y_1 + 7 = 0 \end{cases} \quad \text{in } \mathbb{A}^4_{\mathbb{Q}}$$

Then, $Y(\mathbb{Q}) \simeq X(L)$. More generally, $Y(R) \simeq X(R \otimes_{\mathbb{Q}} L)$ for any $\mathbb{Q}$-algebra R.

Definition 4.12. Given a finite extension of fields $k \subset L$, and a quasi-projective L-variety X, then the **restriction of scalars** $Y = \operatorname{Res}_{L/k} X$ is characterized by

$$Y(R) = X(R \otimes_k L)$$

for all k-algebras R.

Remark 4.13. If X is affine, the construction proceeds in the same way as in the example. If X is not affine, apply the construction to each open affine, and glue them together. There is some difficulty in doing this, but it's always possible when X quasi-projective. It's not always possible for general schemes.

Example. If L/k is separable, then the k-variety $Y = \operatorname{Res}_{L/k} X$ satisfies

$$Y_{k^s} \simeq \prod_{\sigma \in \operatorname{Hom}_k(L,k^s)} {}^{\sigma}X$$

To show this, use Yoneda and use $L \otimes_k k^s \xrightarrow{\sim} \prod_\sigma k^s$.

In particular, note that $\dim Y = [L : k] \dim X$ in this context (i.e. when L/k separable).

If G is an algebraic group over L, then $\mathrm{Res}_{L/k} G$ is also an algebraic group (now over k), by Yoneda.

4.5 (right) Torsors

Let's start with a warm-up.

Say G is an abstract group. The trivial G-torsor is $\mathbb{G} := G$ equipped with right G-action given by translation. In general, a G-torsor (or principal homogeneous space) will be a right G-set X which is abstractly isomorphic to $\mathbb{G}$. Note that we have not specified an isomorphism, so there is no identity/base-point in X. If you choose a point $x \in X$, then this determines a specific (right) G-set iso $\mathbb{G} \xrightarrow{\sim} X$ via $g \mapsto xg$.

Now let's do the definition for real.

Definition 4.14. let G be an algebraic group over k. The **trivial torsor** is $\mathbb{G} := G$ with right (algebraic) G-action by translation. A general (right) G**-torsor** is a k-variety X with a right G-action such that $X_{\overline{k}} \simeq \mathbb{G}_{\overline{k}}$ as $\overline{k}$-varieties with right $G_{\overline{k}}$-action.

As before, if you have a rational point $x \in X(k)$, then you get an iso $\mathbb{G} \xrightarrow{\sim} X$ via $g \mapsto xg$.

Remark 4.15. A G-torsor does not need to be isomorphic to $\mathbb{G}$ over k.

Torsors can be defined more generally.

Definition 4.16. Say $G \to S$ is an fppf[6] group scheme. In this setting, a *G-torsor* is an fppf S-scheme $X \to S$ with a right G-action (over S) such that there exists some fppf base change $T \to S$ such that $X_T \simeq \mathbb{G}_T$ as T-schemes with right G_T-action. It turns out this is equivalent to $X_X \simeq \mathbb{G}_X$ as X-schemes with right G_X-action.

Another reference for torsors is Poonen's 'Rational Points on Varieties'.

5 Lecture 5 (2/26)

5.1 Last time

TODO: Fill this section out

- If $\operatorname{char} k = p$, and X is a k-variety, we get

5.2 Geometry-algebra dictionary

"I don't really like Hopf algebras very much."

Definition 5.1. A **Hopf algebra** over k is a k-algebra A with maps $\Delta : A \to A \otimes_k A$ (**comultiplication**), $S : A \to A$ (**antipode**), $\varepsilon : A \to A$ (**co-identity**) satisfying whatever axioms are needed for A to represent a group-valued functor.

[6] faithfully flat and (locally) of finite presentation

Geometry	Algebra
affine k-variety $X = \operatorname{spec} A$	f.g. k-algebra $A = \mathscr{O}(X)$
closed subscheme $\operatorname{spec}(A/I)$	ideal I of A
affine algebraic group G/k	f.g. (commutative) Hopf algebra over k
closed subgroup $H = \operatorname{spec} A/I$ (e.g. trivial subgroup $\{e\}$)	Hopf ideal I (e.g. **augmentation ideal** $\ker \varepsilon$)
group homomorphism $G \xrightarrow{\varphi} G'$	$A \xleftarrow{f} A'$ Hopf algebra homomorphism
factorization $G \xrightarrow{fp} \operatorname{im} \varphi \overset{closed}{\hookrightarrow} G'$	factorization $A \hookleftarrow A'/I \twoheadleftarrow A'$ where $I := \ker f$ is a Hopf ideal

Table 1: Geometry-algebra dictionary

5.3 Cartier's Theorem

Theorem 5.2 (Cartier's Theorem). *If* $\operatorname{char} k = 0$*, any* k*-algebraic group* G *is smooth.*

Proof. WLOG $k = \overline{k}$. Hence, G_{red} is a closed subgroup of G. We would like to show $G_{\text{red}} = G$. If G is not smooth, there exists a closed subscheme $Z = \operatorname{spec} k[t]/(t^2) \hookrightarrow G$ supported at the identity, and a "function" $f \in \mathscr{O}_{G,e}$ vanishing along G_{red} (hence nilpotent, so $f^n = 0$ for some n), but not on Z. WLOG we can assume

$$\begin{array}{ccc} \mathscr{O}_{G,e} & \longrightarrow & k[t]/(t^2) \\ f & \longmapsto & t \end{array}$$

Consider the multiplication map $G \times G \times \ldots \times G \to G$ with n factors on the LHS. Consider the embedding $G \hookrightarrow G \times G \times \ldots \times G$ of G into the ith slot. Restrict this to

$$Z \overset{i\text{th}}{\hookrightarrow} Z \times Z \times \ldots \times Z \to G.$$

What's happening at the level of rings? We have

$$\frac{k[t]}{(t^2)} \overset{\substack{t \leftarrow t_i \\ 0 \leftarrow t_j}}{\twoheadleftarrow} \frac{k[t_1, \ldots, t_n]}{(t_1^2, \ldots, t_n^2)} \leftarrow \mathscr{O}_{G,e}.$$

Where does $f \in \mathscr{O}_{G,e}$ go under these maps? Note that the composition $Z \hookrightarrow Z \times Z \times \ldots \times Z \to G$ is the natural inclusion, so f ends up at $t \in k[t]/(t^2)$. This is true for any i which means

$$f \mapsto (t_1 + \cdots + t_n) + (\text{higher order terms}) \in \frac{k[t_1, \ldots, t_n]}{(t_1^2, \ldots, t_n^2)}.$$

This is an issue since now

$$0 = f^n \mapsto n! \cdot t_1 t_2 \ldots t_n \neq 0 \in \frac{k[t_1, \ldots, t_n]}{(t_1^2, \ldots, t_n^2)}.$$

This is a contradiction. ∎

Proposition 5.3. *Say* $\operatorname{char} k = p$*, and let* G *be an affine algebraic group over* k*. Then the image of* $F^r : G \to G^{(p^r)}$ *is smooth (= geometrically reduced) for* $r \gg 0$*.*

Remark 5.4. The kernel of F^r will be a finite group scheme, so even in positive characteristic, every (affine) algebraic group is close to being smooth.

Proof. WLOG assume $k = \overline{k}$. Write $G = \operatorname{spec} A$. Then, $F^r : G \to G^{(p^r)}$ corresponds to $A \leftarrow A \otimes_k k$ (with $k \curvearrowright k$ via p^rth power map) taking

$$A \ni a^{p^r} c \leftarrow\!\shortmid a \otimes c \in A \otimes_k k.$$

The image of this is A^{p^r} since $k = \overline{k}$ is perfect (so $c \in k^{p^r}$). We want this image to be a reduced ring. Choose r such that $\operatorname{nil}(A)^{p^r} = 0$, and then A^{p^r} is reduced, so we win. ∎

5.4 Nonabelian group cohomology

(Reference: Serre, 'Galois cohomology')

Let G be an abstract group (e.g. a Galois group). Let A be a G-**group**, so A is a (not-necessarily-abelian) group A equipped with a left G-action; each $\sigma \in A$ acts as a group automorphism of A. We say $a \in A$ is G-**invariant** if ${}^\sigma a = a$ for all $\sigma \in G$.

Notation 5.5. We let

$$\mathrm{H}^0(G, A) = A^G := \{G\text{-invariant elts. of } A\} \leq A.$$

To define H^1, we will define cocycles.

Definition 5.6. A 1-**cocycle** of G in A is a map $G \to A, \sigma \mapsto f_\sigma$ of sets such that

$$f_{\sigma\tau} = f_\sigma \cdot {}^\sigma(f_\tau) \in A.$$

Remark 5.7. The set {1-cocycles} has a right action of A,

$$(g_\sigma), b \mapsto \left(b^{-1} g_\sigma {}^\sigma b\right),$$

(thinking of 1-cocycle as a tuple, and choosing $b \in A$).

Definition 5.8. Two 1-cocycles f and g are called **cohomologous** if f, g are in the same orbit of this right A-action, i.e. $\exists b \in A$ s.t. $f_\sigma = b^{-1} g_\sigma {}^\sigma b$ for all $\sigma \in G$. We first $f \sim g$ when they are cohomologous.

Notation 5.9. We let

$$\mathrm{H}^1(G, A) := \{\text{1-cocycles}\} / \sim$$

with is a pointed set, with distinguished/neutral element being the class of the trivial cocycle

$$\begin{array}{ccc} G & \longrightarrow & A \\ \sigma & \longmapsto & 1. \end{array}$$

Example. If $g \curvearrowright A$ trivially, then 1-cocyles = homomorphisms, and

$$\mathrm{H}^1(G, A) = \operatorname{Hom}(G, A)/\text{conjugation by elts. of } A.$$

If G is a profinite group, $G = \varprojlim G_\alpha$ (with G_α finite), then recall its topologized by saying that $U_\alpha = \ker(G \to G_\alpha)$ form a basis of nbhds of 1. We require A to have the discrete topology, and we require $G \times A \to A$ be continuous ($\iff$ each $a \in A$ fixed by some U_α). We also require 1-cocycles to be continuous. With these requirements in place, we define $\mathrm{H}^0, \mathrm{H}^1$ as before, and they satisfy

$$\mathrm{H}^i(G, A) = \varinjlim_{\alpha} \mathrm{H}^i(G_\alpha, A^{U_\alpha}) \text{ for } i = 0, 1.$$

Example. Say L/k is a (possibly infinite) Galois extension. Then, $L = \bigcup_{k_\alpha/k \text{ fin,Gal}} k_\alpha$ and

$$\underbrace{\mathrm{Gal}(L/k)}_{G} = \varprojlim_{k_\alpha} \underbrace{\mathrm{Gal}(k_\alpha/k)}_{G_\alpha}.$$

Note that G acts continuously on $L^\times$ (each $\ell \in L^\times$ lies in some k_α and hence is fixed by U_α). One can compute

$$\mathrm{H}^0(G, L^\times) = k^\times \text{ and } \underbrace{\mathrm{H}^1(G, L^\times) = \{1\}}_{\textbf{Hilbert's Theorem 90}}.$$

'Hilbert's Theorem 90' was proved by Hilbert when G cyclic and by Emmy Noether when G finite.

6 Lecture 6 (3/1)

6.1 Last time

We introduced nonabelian Galois cohomology. Let $G = \varprojlim G_\alpha$ be a profinite group. A G-group A is a discrete gorup with continuous left G-action. For such a thing, we define

$$\mathrm{H}^0(G, A) := A^G$$

and

$$\mathrm{H}^1(G, A) := \frac{\{\text{continuous } f : G \to A \mid f_{\sigma\tau} = f_\sigma \cdot {}^\sigma f_\tau \text{ for all } \sigma, \tau \in G\}}{(f_\sigma) \sim (b^{-1} f_\sigma {}^\sigma b) \text{ for } b \in A}.$$

If $1 \to A \to B \to C \to 1$ is an exact sequence of G-groups, there is a "not-so-long exact sequence" of pointed sets

$$1 \to \mathrm{H}^0(G, A) \to \mathrm{H}^0(G, B) \to \mathrm{H}^0(G, C) \to \mathrm{H}^1(G, A) \to \mathrm{H}^1(G, B) \to \mathrm{H}^1(G, C).$$

6.2 Galois descent

(Reference: Poonen, *Rational points on varieties*, Sections 1.3.4, 4.4, 4.5)

Let $L \supset k$ be a finite Galois extension with Galois group G.

Theorem 6.1 (Descent theorem for vector spaces). *There are equivalences of categories*

Don't have to assume vector spaces are finite dimensional

$$\{k\text{-vector spaces}\} \longleftrightarrow \{L\text{-vector spaces with semilinear } G\text{-action}\}$$

via $V \mapsto V \otimes_k L$ *and* $W^G \leftarrow W$.

Definition 6.2. A **semilinear** G**-action** means that

$$^{\sigma}(\ell v) = {}^{\sigma}\ell^{\sigma}v$$

(instead of $\ell \cdot {}^{\sigma}v$) for all $\ell \in L$ and $v \in W$.

Corollary 6.3. *There are equivalences of categories*

(1)

$$\{k\text{-}algebras\} \longleftrightarrow \{L\text{-}algebras\ with\ G\text{-}action\ compatible\ with\ action\ on\ L\}\,.$$

(2)

$$\{affine\ k\text{-}schemes\} \longleftrightarrow \{affine\ L\text{-}scheme\ with\ right\ G\text{-}action\ compatible\ with\ \operatorname{spec} L\}\,.$$

(3)

$$\{quasi\text{-}proj\ k\text{-}schemes\} \longleftrightarrow \{quasi\text{-}proj\ L\text{-}schemes\ with\ right\ G\text{-}action\ compatible\ \operatorname{spec} L\}\,.$$

Remark 6.4. There's some subtelty in going from **(2)** to **(3)**. Need to know that every quasi-projective L scheme with G-action is a union of affine L-schemes *with G-action* (i.e. with G-invariant affine L-subschemes). So want something like the orbit of every point is contained in an affine. For finite group acting on a quasi-projective scheme, we're in luck since every finite subset of a quasi-proj scheme is contained in any affine.

Note that giving the action (in **(3)**) on Y/L amounts to giving, for each $\sigma \in G$, a morphism $\widetilde{\sigma} : Y \to Y$ over $\sigma : \operatorname{spec} L \to \operatorname{spec} L$, such that $\widetilde{\sigma\tau} = \widetilde{\sigma}\widetilde{\tau}$ always. Note in particular that $\widetilde{\sigma} : Y \to Y$ is *not* an L-morphism (unless $\sigma = 1$). To get an L-morphism instead, consider the diagram

$$\begin{tikzcd}
Y \arrow[dr, "\widetilde{\sigma}"] & & \\
{}^{\sigma}Y \arrow[u, dashed, "f_\sigma"'] \arrow[r] & Y \arrow[dr, "\widetilde{\tau}"] & \\
{}^{\sigma\tau}Y \arrow[u, dashed, "{}^{\sigma}f_\tau"'] \arrow[uu, dashed, bend left, "f_{\sigma\tau}"] \arrow[r] \arrow[d] & {}^{\tau}Y \arrow[u, dashed, "f_\tau"] \arrow[r] \arrow[d] & Y \arrow[d] \\
\operatorname{spec} L \arrow[r, "\sigma^*"] & \operatorname{spec} L \arrow[r, "\tau^*"] & \operatorname{spec} L
\end{tikzcd}$$

The upshot is that giving the $\widetilde{\sigma} : Y \to Y$ is equivalent to instead giving $f_\sigma : {}^{\sigma}Y \xrightarrow{\sim} Y$ for all $\sigma \in G$. The group compatibility condition now becomes to **cocycle condition**

$$f_{\sigma\tau} = f_\sigma \cdot {}^{\sigma}(f_\tau)$$

always.

Now saw you have two such data, so you have Y with action given by f_σ and Z with action given by

g_σ. An isomorphism of such that is an iso $b : Y \xrightarrow{\sim} Z$ such that

$$\begin{tikzcd} Y \arrow[r, "b"] \arrow[d, "f_\sigma"'] & Z \arrow[d, "g_\sigma"] \\ {}^\sigma Y \arrow[r, "{}^\sigma b"] & {}^\sigma Z \end{tikzcd}$$

commutes for all σ, i.e. $f_\sigma = b^{-1} g_\sigma {}^\sigma b$.

Consider the special case $Y = X_L$ for some quasi-projective X/k. Then Y descends to a k-scheme, but in how many ways?

Definition 6.5. A k-scheme X' such that $(X')_L \simeq X_L$ is call an **L/k-twist of** X.

Theorem 6.6. *For quasi-projective X/k,*

$$\{L/k\text{-twists of } X\}_{/k\text{-isom}} \leftrightarrow \mathrm{H}^1(\mathrm{Gal}(L/k), \mathrm{Aut}\, X_L).$$

Proof. Descending X_L to a k-scheme amounts to giving a Galois action which we saw above amounts to giving $f_\sigma : {}^\sigma X_L \xrightarrow{\sim} X_L$ satisfying the cocycle condition $f_{\sigma\tau} = f_\sigma \cdot {}^\sigma f_\tau$. Note that ${}^\sigma X_L = X_L$ since X is defined over k, so these f_σ live in $\mathrm{Aut}\, X_L$, and the cocycle condition says exactly that $(f_\sigma)_{\sigma \in \mathrm{Gal}(L/k)}$ gives a 1-cocycle with values in $\mathrm{Aut}\, X_L$. Finally, we saw above that two such pieces of descent data are isomorphic exactly when they are cohomologous. ■

Explicitly, if X' is an L/k-twist of X, choose an iso $\varphi : X_L \xrightarrow{\sim} (X')_L$, and let $f_\sigma = \varphi^{-1} \circ {}^\sigma \varphi \in \mathrm{Aut}\, X_L$. The 1-cocycle $(f_\sigma)_\sigma$ has a class in $\mathrm{H}^1(\mathrm{Gal}(L/k), \mathrm{Aut}\, X_L)$ (changing φ gives a cohomologous cocycle, so this class is well-defined).

Some remarks

- All of descent theory works for infinite Galois extensions too (say continuous in the right places).
- Also, one can consider varieties with extra structure

  Example. G could be a (quasi-projective) algebraic groups over k. A **twist of** G will be another algebraic group H/k s.t. $H_{k^s} \cong G_{k^s}$ (as algebraic groups over k^s). Then,

  We'll later see algebraic groups are always quasi-proj

  $$\{\text{twists of } G\}_{/\text{isom}} \leftrightarrow \mathrm{H}^1(\mathrm{Gal}(k^s/k), \mathrm{Aut}\, G_{k^s}).$$

  Example. Let G be a (q-proj) algebraic group (let's assume k perfect or G smooth). Then, a right G-torsor is simply a twist of the trivial right G-torsor $\mathbb{G}$ (by definition). Since $\mathrm{Aut}\, \mathbb{G}_{k^s} = G(k^s)$ (acting by multiplication on the left[7]), we get that

  Remember: Nonempty smooth varieties will always have some point over a separable closure

  $$\{\text{right } G\text{-torsors}\}_{/k\text{-isom}} \longleftrightarrow \mathrm{H}^1(\mathrm{Gal}(k^s/k), G(k^s)) =: \mathrm{H}^1(k, G).$$

6.3 Affine algebraic groups sit in GL_n

We want to eventually (soon?) prove that all affine algebraic groups can be realized as closed subgroups of GL_n. To build up to this, we need some more results.

[7] Note that any automorphism is determined by where the identity goes, and it can go anywhere

Proposition 6.7. *Say $f : X \to Y$ is a morphism of finite type k-schemes. If $X(R) \to Y(R)$ is injective for all k-algebras, and f is faithfully flat, then f is an isomorphism.*

Proof. You can check whether something is an isomorphism after fpqc[8] base change. Consider the basechange

$$\begin{tikzcd} X \times_Y X \arrow[r] \arrow[d] & X \arrow[d] \\ X \arrow[r] & Y \end{tikzcd}$$

Note that

$$(X \times_Y X)(R) = X(R) \times_{Y(R)} X(R) = \{(x_1, x_2) \in X(R) \times X(R) : f(x_1) = f(x_2) \in Y(R)\} = \{(x,x) : x \in X(R)\} \xrightarrow{\sim} X(R)$$

(third equality coming from $X \to Y$ being monic). Thus, $X \times_R X \to X$ is an isomorphism, so $X \to Y$ is as well. ∎

Corollary 6.8. *Let $\varphi : G \to H$ be a homomorphism of affine algebraic groups over k. If $\ker \varphi = \{e\}$, then φ is a closed immersion.*

(It's not a priori obvious that φ is an immersion)

Example. Say C is a nodal cubic with unique node $p \in C$. Let $\varphi : \widetilde{C} \to C$ be the normalization, and let $q \in \widetilde{C}$ be one of the two points above C. Then, $\widetilde{C} \setminus \{q\} \xrightarrow{\varphi} C$ is not an immersion, but is injective.

Proof. By homomorphism theorem, we know φ factors as

$$G \to I \hookrightarrow H$$

with $I \hookrightarrow H$ the scheme-theoretic image (and in particular a closed immersion). We now only need to show that $G \to I$ is faithfully flat (since $\ker(G \to I) = \{e\}$ by assumption), but this is part of the homomorphism theorem? ∎

Example (Example of "generic point")**.** $\mathrm{SL}_2 = \operatorname{spec} A$, where $A = k[a,b,c,d]/(ad - bc - 1) \ni \overline{a}, \overline{b}, \overline{c}, \overline{d}$. Then,

$$\eta = \begin{pmatrix} \overline{a} & \overline{b} \\ \overline{c} & \overline{d} \end{pmatrix} \in \mathrm{SL}_2(A).$$

Also, $\mathrm{SL}_2(A \otimes_k A)$ has two independent "generic points" η, η'.

In general, if $G = \operatorname{spec} A$, then $G(A) = \mathrm{Hom}_{k\text{-schemes}}(\operatorname{spec} A, G) = \mathrm{Hom}_{k\text{-schemes}}(G, G)$ has a "generic element" η corresponding to the identity $\mathrm{id} : G = G$. Then, any $g \in G(R) = \mathrm{Hom}_{k\text{-alg}}(A, R)$ gives a map $G(A) \to G(R)$ satisfying $\eta \mapsto g$.

7 Lecture 7 (3/3)

Pset 3 due next Tuesday. Last time, we obtained

[8] fidelment plat et quasi-compact

Corollary 7.1. *Let $\varphi : G \to H$ be a homomorphism of (affine) algebraic groups over k. If $\ker \varphi = \{e\}$, then φ is a closed immersion.*

We also discussed the generic/universal element $\eta \in G(A)$ of $G = \operatorname{spec} A$.

7.1 Representations

Let V be a k-vector space (possibly infinite dimensional). Get functor

$$\begin{array}{cccc} \mathrm{GL}_V : & \{k\text{-algs}\} & \longrightarrow & \mathrm{Grp} \\ & R & \longmapsto & \mathrm{Aut}_{R\text{-modules}}(V \otimes_k R) \end{array}$$

If V is finite dimensional, then GL_V is represented by an algebraic group (though it's not when $\dim V = \infty$).

Definition 7.2. Let G be an algebraic group. A **representation of** G is a k-vector space V equipped with a homomorphism of group-valued functors

$$r : G \to \mathrm{GL}_V$$

(i.e. a compatible system of group homomorphisms $G(R) \to \mathrm{GL}_V(R)$)

Definition 7.3. A **morphisms of representations** $V \to W$ is a k-linear map $T : V \to W$ such that for all k-algebras R, the induced map

$$T_R : V \otimes R \to W \otimes R$$

is $G(R)$-equivariant.

Remark 7.4. For a fixed G, reps of G forms an *abelian* category (e.g. have kernels, cokernels, simple=irreducible objects, semisimple objects, Jordan-Hölder theorem, etc.). It's even better than just an abelian category; it also has a tensor product, so it's what's called a *tensor category.* The subcat of finite dimensional representations is also an abelian category.

Definition 7.5. A representation r is called **faithful** if $\ker r = \{e\}$ is trivial, i.e. $G(R) \hookrightarrow \mathrm{GL}_V(R)$ is always injective.

If V is finite dimensional, can associate to it an affine space $\mathbb{V}$ (satisfying $\mathbb{V}(R) = V \otimes R$), and we get an induced action $G \times \mathbb{V} \to \mathbb{V}$. If $W \subset V$ is a subspace, get closed subgroup

spec(Sym* V), I believe

$$\mathrm{Stab}_G(W) := \mathrm{Stab}_G(\mathbb{W})$$

Assumption. Now suppose $G = \operatorname{spec} A$, so A is some Hopf algebra.

In this case, r is determined by its value $r(\eta) \in \mathrm{GL}_V(A)$ on the generic element $\eta \in G(A)$. Note that $r(\eta)$ is an A-linear automorphism $V \otimes A \xrightarrow{\sim} V \otimes A$, so it's determined by a k-linear map $\rho : V \to V \otimes A$, $v \mapsto \eta v$. In order for r to be a homomorphism, ρ needs to satisfy certain axioms which are written down in the book. When ρ satisfies these axioms, we call (V, ρ) an A**-comodule** and ρ is called the **co-action**.

Remark 7.6. A subrepresentations corresponds to a subcomodule. Other rep theory notions translate to those of comodules.

Proposition 7.7. *Every representation of $G = \operatorname{spec} A$ is a filtered*[9] *union of finite dimensional representations.*

Proof Sketch. Given $v \in V$, we need to find a f. dim subrep containing it. This will suffice. Let (a_i) be a k-basis of A. Write

$$\eta v = \rho(v) = \sum_{\text{finite}} v_i \otimes a_i \in V \otimes A.$$

Since the sum is finite, ηv lives in $W \otimes A$, where $W = \operatorname{span}\{v, \text{all } v_i\text{'s appear above}\}$ is finite-dimensional. Note that specializing η to any $g \in G(R)$ gives $gv \in W \otimes R$. Let η' be an independent generic element in $G(A \otimes A)$. We compute

$$\begin{aligned}\eta'\eta v &= \underbrace{(\eta'\eta)}_{\in G(A\otimes A)} v \in W \otimes (A \otimes A)\\ &= \eta'(\eta v) = \sum_{\text{finite}} (\eta' v_i) \otimes a_i\end{aligned}$$

Since the a_i's form a basis for the last A, the only way to have $\sum(\eta' v_i) \otimes a_i \in W \otimes A \otimes A$ is to have $\eta'(v_i) \in W \otimes A$ for all i. Since a generic element maps all v_i's into W (and maps v into W), we see that W must be a subrepresentation. ■

Because of this, we'll mostly focus on finite dimensional reps.

Remark 7.8. Apparently any representation of G will factor through some affine quotient.

Question: Why?

Answer: GL_V is affine, so $G \to \mathrm{GL}_V$ factors through $\operatorname{spec} \mathscr{O}(G)$

Theorem 7.9. *Let G be an algebraic group. Then, G is affine $\iff$ G is isomorphic to a closed subgroup of GL_n for some n. When G is a closed subgroup of GL_n, people also say that G is a **linear algebraic group** (this term is more common than 'affine algebraic group')*

Proof. $(\leftarrow)$ Closed subschemes of affines are always affine.

$(\rightarrow)$ Suppose $G = \operatorname{spec} A$. We need a faithful representation of G. The right translation of G on G induces a faithful left action of G on the vector space $V := A$ (acting by k-algebra homomorphisms). Let V_0 be a finite set of generators for the k-algebra V. By previous prop, there exists a finite dimensional subrep $W \leq V$ containing V_0. We claim that G acts faithfully on W.

Remember: algebraic groups are finite type

If $g \in G(R)$ acts trivially on $W \otimes R$, then is an R-algebra homomorphism preserving V_0 (pointwise), so g acts trivially on $R[V_0] = V \otimes R$, so $g = 1$ as G acts faithfully on V.

Thus, $G \hookrightarrow \mathrm{GL}_W \simeq \mathrm{GL}_{\dim W}$ is an injection on R-valued points for all R, and so is a closed immersion. ■

7.2 Isotypic components and characters

Let G be an algebraic group, V be a representation of G, and r be an irrep of G.

Definition 7.10. The **isotypic component** $V_r :=$ sum of all subreps of V which are isomorphic to r.

[9] any f. number of them is contained in a larger one (e.g. their sum)

Proposition 7.11.

(1) *V_r is a direct sum of copies of r.*

(2) *The V_r are independent of each other as subspaces, i.e.*

$$\bigoplus_{\text{irred } r} V_r \hookrightarrow V.$$

Above, equality holds $\iff$ V is semisimple.

Proof. Abelian category stuff. ■

Definition 7.12. A **character of** G is a 1-dimensional representation $\chi : G \to \mathbb{G}_m$. If $G = \operatorname{spec} A$, this is equivalently a choice of $a \in A^\times$ such that $\Delta(a) = a \otimes a$ (this a is called a **grouplike element**).[10]

The χ-isotypic component V_χ of a G-rep V is also called the **χ-eigenspace** of V.

We say G **acts on** V **through** χ if $V_\chi = V$, i.e. $gv = \chi(g)v$ always. When $G = \operatorname{spec} A$, this amounts to saying that $\eta v = \chi(\eta)v$ for all $v \in V$ ($\iff$ $\rho(v) = v \otimes a_\chi \in V \otimes A$ for all $v \in V$).

8 Lecture 8 (3/5): Constructing G/H

No class on Monday. Instead, class on Tuesday at 10am ('Monday Schedule'). Problem set 3 due Tuesday night.

8.1 Last time

Let G be an algebraic group. A **representation of** G is a vector space V equipped with a homomorphism of functors $r : G \to \mathrm{GL}_V$.

Proposition 8.1. *Every rep of G is a filtered union of finite-dimensional reps*

Theorem 8.2. *An algebraic group G is affine iff it is linear (i.e. iso to a closed subgroup of GL_n for some n)*

Open Question 8.3. *Does the same theorem hold for finite, flat group schemes over $k[\varepsilon]/(\varepsilon^2)$?*

8.2 Chevalley's stabilizer theorem

Our goal is to construct quotients G/H. The idea is to use the orbit-stabilizer theorem, $G/\operatorname{Stab}(x) = \operatorname{Orbit}(x)$. The question the becomes: can every subgroup H be realized as a stabilizer?

Theorem 8.4 (**Chevalley's Theorem**)**.** *Let G be a linear (i.e. affine) algebraic group with (closed) subgroup $H \le G$. Then, there exists a finite dimensional representation V of G, and a 1-dimensional subspace (not a subrep) $L \le V$ such that $H = \operatorname{Stab}_G(L)$.*

Proof. Write $G = \operatorname{spec} A$ and $H = \operatorname{spec} A/I$ (I a Hopf ideal).

[10] $\chi \rightsquigarrow \chi(\eta) \in A^\times$ is grouplike

(Attempt 1) We now G acts by right translation on itself, and $H = \mathrm{Stab}_G(H)$. Can we translate this into Hopf land and get some vector spaces? G acts on the left on A, and $H = \mathrm{Stab}_G(I)$.[11] If A were finite-dimensional (and I was 1-dimensional), we could take $(V, L) = (A, I)$.

(Attempt 2) Let $V \le A$ be a finite dimensional subrep containing a (finite) set ideal generators I_0 of I. Let $W = I \cap V$, a subspace (but not subrep since G does not preserve I).

We claim that $\mathrm{Stab}_G(W) = H$. For $g \in G(k)$, we know

$$g \in \mathrm{Stab}_G(W)(k) \iff gW = W \iff gW \subset W \iff gW \subset I \implies gI_0 \subset I \implies gI = I \implies gW \subset I$$

(above, keep in mind that $\dim V < \infty$ and that g acts via algebra automorphism of A). Thus, $g \in \mathrm{Stab}_G(W)(k) \iff gI \subset I \iff g \in H(k)$. What about R-points? Do the same argument with more cumbersome notation (tensor all vector spaces with R).

If $\dim W = 1$, then we could take $(V, L) = (V, W)$, so close, but no cigar.

(Attempt 3) Let $r = \dim W$. Replace V by $\bigwedge^r V$ and W by $\bigwedge^r W$ (which is now 1-dimensional). Linear algebra says $\mathrm{Stab}_G\left(\bigwedge^r W\right) = \mathrm{Stab}_G(W) = H$. Now we win. ■

Same idea as showing Grassmanians are projective

Before constructing quotients, we will need a tool from descent theory.

Theorem 8.5 (fpqc descent for morphisms). *Given an fpqc morphism $S' \to S$ and another scheme Y, in the diagram*

$$\begin{tikzcd} S' \times_S S' \arrow[r, shift left, "\mathrm{pr}_1"] \arrow[r, shift right, "\mathrm{pr}_2"'] & S' \arrow[r, dashed, "g"] \arrow[d] & Y \\ & S, \arrow[ur, dashed, "f"'] & \end{tikzcd}$$

giving f is the same as giving g such that $g \circ \mathrm{pr}_1 = g \circ \mathrm{pr}_2$.

Example. Let $\{U_i\}$ be an open cover of S, and let $S' = \bigsqcup_i U_i$. Then, $S' \to S$ is fpqc, and giving $g : S' = \bigsqcup_i U_i \to Y$ s.t. $g \circ \mathrm{pr}_1 = g \circ \mathrm{pr}_2$ is simply giving morphisms $g_i : U_i \to Y$ which agree on overlaps $U_i \times_S U_j = U_i \cap U_j \subset S$.

Example. Say $S' = S_L$ for some finite Galois extension L/k. Giving $S \to Y$ is equivalent to giving $S_L \to Y$ that is $\mathrm{Gal}(L/k)$-invariant.

8.3 Quotients

Definition 8.6. Let $H \le G$ be a closed subgroup of an algebraic group over k.

(1) We say $q : G \to X$ (X a k-scheme) is H**-invariant** if the two compositions in

$$G \times H \underset{\mathrm{pr}_1}{\overset{m}{\rightrightarrows}} G \xrightarrow{q} X$$

agree (above, m is multiplication).

[11] If you have something outside H, it'll translate H to a different closed subscheme, so translate I to a different (regular) ideal

(2) We say $q : G \to X$ is a **quotient of** G **by** H if

  (1) q is faithfully flat

  (2) q is H-invariant

  (3) The map $G \times H \to G \times_X G, (g, h) \mapsto (g, gh)$ is an isomorphism of k-schemes.

  $G \to X$ is an H-torsor

  In this situation, we write $G/H := X$.

Remark 8.7. If $H \curvearrowright Y$, some scheme, can define the quotient in the same way.

Proposition 8.8 (Universal property of G/H). *Given*

$$\begin{tikzcd} G \arrow[r, "H\text{-invariant}"] \arrow[d, "quotient"'] & Y \\ X = G/H \arrow[ur, dashed, "\exists!"'] & \end{tikzcd}$$

Proof. We have a diagram

$$\begin{tikzcd} (g,h) \arrow[d, mapsto] & \in & G \times H \arrow[d, "\wr"] \arrow[r, shift left, "\mathrm{pr}_1"] \arrow[r, shift right, "m"'] & G \arrow[r] \arrow[d, equal] & Y \arrow[d, equal] \\ (g, gh) & \in & G \times_X G \arrow[r, shift left, "\mathrm{pr}_1"] \arrow[r, shift right, "\mathrm{pr}_2"'] & G \arrow[r] \arrow[d, "\mathrm{fpqc}"] & Y \\ & & & X = G/H \arrow[ur, dashed] & \end{tikzcd}$$

The dashed arrow above exists precisely because of fpqc descent. The compositions agree becase $G \to Y$ is H-invariant. ■

Corollary 8.9. *G/H is unique, if it exists.*

Corollary 8.10. *If H is normal in G, then G/H is an algebraic group, and $G \to G/H$ is a group homomorphism with kernel H.*

Proof. Consider

$$\begin{tikzcd} G \times G \arrow[r, "m"] \arrow[d] & G \arrow[d] \\ G/H \times G/H \arrow[r] & G/H \end{tikzcd}$$

One can check that the left vertical map is a quotient by $H \times H$, and that the composition $G \times G \xrightarrow{m} G \to G/H$ is $(H \times H)$-invariant (using H is normal). Hence, the universal property gives the bottom arrow $G/H \times G/H \to G/H$, and it inherits the group axioms from m. Similarly get an inverse map. ■

Theorem 8.11 (Existence and properties of quotients). *Let $H \leq G$ be a closed subgroup of G, over k. We assume that G is smooth and linear. Then,*

These hypotheses are not necessary, but make the proof easier

(1) A quotient G/H exists.

(2) G/H is a quasi-projective variety.

(3) $(G/H)(L) = G(L)/H(L)$ *for algebraically closed fields* $L = \overline{L}$.

(4) G/H *is smooth (even if* H *is not smooth).*

Example. If $\operatorname{char} k = p$, there is an exact sequence

$$1 \longrightarrow \mu_p \longrightarrow \mathbb{G}_m \xrightarrow{(-)^p} \mathbb{G}_m \longrightarrow 1.$$

The quotient $\mathbb{G}_m$ is smooth even though the kernel μ_p is not smooth.

We'll prove this next time (on Tuesday).

9 Lecture 9 (3/9): Existence and properties of quotients

9.1 Last time: quotients

Setup. Let

- G be an algebraic group over k
- $H \leq G$ be a (not necessarily normal) closed subgroup scheme
- X be a k-variety

Recall 9.1.

- $q : G \to X$ is *H-invariant* if the two compositions in

$$G \times H \underset{\mathrm{pr}_1}{\overset{m}{\rightrightarrows}} G \xrightarrow{q} X$$

are equal, i.e. $q(gh) = q(g)$.

- $q : G \to X$ is called a *quotient of G by H* if q is faithfully flat and H-invariant, and

$$\begin{array}{ccc} G \times H & \longrightarrow & G \times_X G \\ (g, h) & \longmapsto & (g, gh) \end{array}$$

is an isomorphism. Then we write $G/H := X$.

Recall 9.2 (Univ. Prop of G/H)**.** Every H-invariant morphism from G to a k-scheme Y factors uniquely through $G \to G/H$.

9.2 Today: Existence and properties of quotients

Theorem 9.3 (**Existence and properties of quotients**)**.** *Let H be a closed subgroup scheme of a smooth linear algebraic group G. Then,*

(1) *A quotient G/H exists (so any $G \to X$ satisfying the univ property is a quotient)*

(2) G/H *is a quasi-projective variety*

(3) $(G/H)(L) = G(L)/H(L)$ *for any algebraically closed field* $L \supset k$

(4) G/H *is smooth (even if* H *is not).*

Remark 9.4. The smooth and linear hypotheses are not necessary for above result (except need smoothness for **(4)**), but we include them to make the proof easier.

Proof. **(1)** Chevalley's stabilizer theorem provides a finite-dimensional representation V of G along with a 1-dimensional subspace $L \subset V$ such that $H = \mathrm{Stab}_G(L)$. Note that G acts on $\mathbb{P}(V)$ and $H = \mathrm{Stab}_G([L])$. Let $X :=$ orbit of $[L]$ in $\mathbb{P}V$, a locally closed[12] subset of $\mathbb{P}V$; we make it a scheme by giving it the reduced scheme structure. The morphism $G \to \mathbb{P}V, g \mapsto g[L]$ factors though a morphism $q : G \to X$ (since G is reduced).

Remember: We use the convention that $\mathbb{P}(V)$ parametrizes "lines in V"

We want to show that q is a quotient. By definition, q is surjective. Furthermore, q is flat via generic flatness + homogeneity. Next, q is H-invariant since $H = \mathrm{Stab}[L]$. This just leaves showing that $G \xrightarrow{q} X$ is an H-torsor. For $g, g' \in G(R)$, if $q(g) = q(g') \in X(R)$, then $g[L] = g'[L]$ (definition of q), so $gL = g'L$ as submodules of $V \otimes R$. Hence, $L = g^{-1}g'L$ so $g^{-1}g' \in \mathrm{Stab}_G(L)(R) = H(R)$ which exactly says $\exists! h \in H(R)$ s.t. $(g, g') = (g, gh)$. Hence, $G \times H \to G \times_X G, (g, h) \mapsto (g, gh)$ is an isomorphism.

(2) G/H being quasi-projective follows from the construction. We constructed X as a locally closed subscheme of $\mathbb{P}V$, so it is quasi-projective.

(3) If $x \in X(L)$, then $q^{-1}(x)$ is a nonempty L-variety (recall $L = \overline{L}$), so it has an L-point. By definition of quotient, two points $g, g' \in G(L)$ have the same iamge in $X(L)$ iff $g' = gh$ for some $h \in H(L)$. Thus, $G(L)/H(L) \xrightarrow{\sim} X(L)$.

(4) WLOG $k = \overline{k}$ (does not affect smoothness). Note that X is reduced, by the construction in **(1)**. Since $k = \overline{k}$, X is in fact geometrically reduced. Thus, it has a dense open subscheme U which is smooth. Finally, we use homogeneity (the G-action on X) to see that $X = \bigcup_{g \in G(k)} gU$ is smooth. ∎

Remark 9.5. Above, if had $W \subset V$ with $\mathrm{Stab}_G(W) = H$, but $\dim W > 1$, then we could run the same argument with a Grassmannian in place of $\mathbb{P}V$. However, to show the quotient is quasi-projective, you need to know Grassmannians are projective which is shown via the same top wedge power trick.

Fact. Theorem holds even for nonsmooth G.

Corollary 9.6.

- *The algebraic groups satisfy isomorphism theorems (e.g.* $G/\ker\varphi \simeq \mathrm{im}\,\varphi$*). Also get Jordan-Hölder theorem, etc. [see Wyler paper cited in Milne's errata list]*
- *The category of* $\{$*commutative algebraic groups*$\}$ *is abelian.*

Question 9.7. *If* $H \leq G$ *are both affine, is* G/H *also affine?*

There are two answers.

[12] See Proposition 3.6.

Answer (No)**.** No, in general. For example, consider $G := \mathrm{GL}_n$ acting on $\mathbb{P}^{n-1}$, and let $H := \mathrm{Stab}_G(1 : 0 : \cdots : 0)$ consisting of matrices of the form

$$\begin{pmatrix} * & * & * \\ \vdots & * & * \\ 0 & * & * \end{pmatrix}$$

(first column vector a multiple of $(1, 0, \ldots, 0)$. Everything else arbitrary). The G action is transitive, so $G/H = \mathbb{P}^{n-1}$ which is not affine if $n \geq 2$.

Answer (Yes)**.** However, if H is *normal* in G, then G/H is affine. For example, $\mathrm{PGL}_n := \mathrm{GL}_n / \mathbb{G}_m$ is affine. As varieties $\mathrm{GL}_n = \mathbb{A}^{n^2} \setminus \{\det = 0\}$ and $\mathrm{PGL}_n = \mathbb{P}^{n^2-1} \setminus \{\det = 0\}$, but the complement of a hypersurface in any $\mathbb{P}^N$ is automatically affine.

Proof of Yes(assuming G smooth). [13] WLOG $k = \overline{k}$ since affineness is unchanged by base field extension. Also, WLOG we may assume G is connected.[14]

Choose $L \leq V$ a 1-dim subspace of a f.d. G-rep V s.t. $H = \mathrm{Stab}_G L$ (via Chevalley). WLOG shrink V to $\mathrm{span}\{gL : g \in G(k)\}$ (the subrep generated by L). Now, $V = g_1 L \oplus \cdots \oplus g_n L$ for some $g_1, \ldots, g_n \in G(k)$. For each $g \in G$,

$$\mathrm{Stab}(gL) = g\,\mathrm{Stab}(L) g^{-1} = gHg^{-1} = H$$

with last equality coming from normality of H. Choosing a basis of V compatible with $V = \bigoplus_{i=1}^n g_i L$, we see that H maps into $\mathbb{G}_m^n \hookrightarrow \mathrm{GL}_V$. Therefore,

$$V = \bigoplus_{\text{chars } \chi \text{ of } H} V_\chi \text{ as } H\text{-reps}$$

(V as an H-rep is given by putting n characters along the diagonal). A calculation shows each $g \in G(k)$ maps V_χ to $V_{\chi \circ \mathrm{inn}_{g^{-1}}|_H}$ so $G(k)$ acts on the finite set $\{\text{nonzero } V_\chi\}$. We make two claims about the action.

(1) This action is transitive.

Pf: each nonzero V_χ contains some $g_i L$ and G acts transitively on the set of $g_i L$.

(2) This action is trivial.

Pf: G is a connected group, and we have map $G(k) \to \mathrm{Aut}\{\text{nonzero } V_\chi\} \simeq S_n$ to a discrete group. It must land in the identity.

Question: Is this map obviously continuous?

Taken together, this means there can only be one nonzero V_χ, so H acts by a single character. To finish we claim that

$$H = \ker\left(G \to \mathrm{GL}_V \twoheadrightarrow \mathrm{PGL}_V\right).$$

Each $h \in H$ acts as a scalar on $V_\chi = V$, so it maps to 1 in PGL_V. On the other hand, anything in the kernel acts as a scalar on V, so $gL = L$ so $g \in H$. ■

[13] Don't need G to be smooth, but simplifies proof

[14] In general, G^0 is of finite index in G. WE can take the images of the components in G/H to see that $G^0/(H \cap G^0)$ is of finite index in G/H, so it's affine iff $G^0/(H \cap G^0)$ is.

Corollary 9.8. *G/H is the image of $G \to \mathrm{PGL}_V$, a closed subgroup of the affine PGL_V.*

Remark 9.9. Should be able to prove this via Hopf algebras as well (without smoothness assumption). There was some discussion of this during lecture, but the details weren't worked out fully.

Let's end with a potential research project. We mentioned a paper of Wyler before. It axiomatizes what one needs about the category of groups to prove the isom theoerems, jordan-hölder, etc.

Open Question 9.10. *If you start with the category of all algebraic groups, you can pass to the category of algebraic groups up to isogeny[15] by inverting isogenies. Does this new category satisfy Wyler's axioms? You get one such category for each ground field, but is it invariant under separable field extension?*

10 Lecture 10 (3/10)

Pset 4 out, due next Thursday (not tomorrow).

10.1 Last time

Theorem 10.1. *Let H be a closed subgroup scheme of a smooth (only needed for **(4)**) linear (only needed for **(5)**) algebraic group G. Then,*

- ***(1)*** *A quotient G/H exists*
- ***(2)*** *G/H is a quasi-projective variety*
- ***(3)*** *$(G/H)(L) = G(L)/H(L)$ for any algebraically closed field $L \supset k$*
- ***(4)*** *G/H is smooth (even if H is not)*
- ***(5)*** *If G is affine and H is normal, then G/H is affine.*

Definition 10.2. Let $\varphi : G \to G'$ be a homomorphism between *any* algebraic groups. Then, φ is an **isogeny** if $\ker \varphi$ and $G'/\operatorname{im} \varphi$ are finite.

Definition 10.3. We say G is **isogenous** to G' if there exists a zig-zag of isogenies from G to G', i.e. something like

$$G \to G_1 \leftarrow G_2 \to G_3 \leftarrow \cdots \to G_n \leftarrow G'$$

10.2 This time: group theory

Definition 10.4. Let G be an algebraic group. A **subnormal series** or **filtration** is a sequence of normal subgroups

$$G = G_0 \triangleright G_1 \triangleright G_2 \triangleright \cdots \triangleright G_s = \{1\}.$$

This gives quotients G_0/G_1, G_1/G_2, $\ldots G_{s-1}/G_s$.

We call such a series a **normal series** if furthermore $G_i \triangleleft G$.

[15]A paper by Michel Brion does this in commutative case, but no reason can't do it in non-commutative case as well

Remark 10.5. If you have such a filtration and $H \leq G$ is any subgroup, you get a filtration $H_i := G_i \cap H$ of H. Here, the quotients are $H_{i-1}/H_i \hookrightarrow G_{i-1}/G_i$.

If $G \twoheadrightarrow Q$ is a quotient map (so faithfully flat homomorphism), get a filtration $Q_i := \operatorname{im}(G_i \to Q)$ on Q. Here, the quotients are $G_{i-1}/G_i \twoheadrightarrow Q_{i-1}/Q_i$.

In finite group theory, you like to find filtrations which are as long as possible. This won't always be possible for algebraic groups.

Example.

$$\mathbb{G}_m \triangleright \mu_{\ell^n} \triangleright \mu_{\ell^{n-1}} \triangleright \cdots \triangleright \mu_\ell \triangleright \{1\}$$

can be arbitrarily long.

The solution to this problem is to just ignore all the finite quotients.

Definition 10.6. A **composition series** is a filtration with

$$\dim G_0 > \dim G_1 > \cdots > \dim G_s,$$

and in which you cannot insert another subgroup while keeping this property.

Theorem 10.7 (**Jordan-Hölder Theorem, up to isogeny**). *If $(G_i)_{i=0}^s$ and $(H_j)_{j=0}^t$ are composition series for $G = G_0 = H_0$, then the quotients $G_0/G_1, \ldots, G_{s-1}/G_s$ are isogenous to $H_0/H_1, \ldots, H_{t-1}/H_t$ after reordering. In particular, $s = t$.*

Example. One composition series of GL_n is $\mathrm{GL}_n \triangleright \mathrm{SL}_n \triangleright \{1\}$. Another one is $\mathrm{GL}_n \triangleright \mathbb{G}_m \triangleright \{1\}$. The first one has quotients

$$\mathbb{G}_m, \mathrm{SL}_n\,.$$

The second one has quotients

$$\mathrm{PGL}_n, \mathbb{G}_m.$$

We have $\mathbb{G}_m = \mathbb{G}_m$ and an isogeny $\mathrm{SL}_n \twoheadrightarrow \mathrm{PGL}_n$ with kernel $\mathrm{SL}_n \cap \mathbb{G}_m = \mu_n$.

Definition 10.8. Given subgroups $H_1, H_2 \leq G$, their **commutator** $[H_1, H_2]$ is the algebraic subgroup generated by

$$\begin{array}{ccc} H_1 \times H_2 & \longrightarrow & G \\ (a,b) & \longmapsto & [a,b] := aba^{-1}b^{-1} \end{array}$$

Definition 10.9. Note that $[G,G]$ is the smallest normal subgroup N s.t. G/N is commutative. This is sometimes denoted $\mathcal{D}G := [G,G]$ and called the **derived group** (or **commutator subgroup**) of G. The Quotient $G^{\mathrm{ab}} := G/\mathcal{D}G$ is called the **abelianization** of G.

Definition 10.10. The **derived series** of G is

$$G \triangleright \mathcal{D}G \triangleright \mathcal{D}\mathcal{D}G \triangleright \cdots \triangleright \mathcal{D}^n G \triangleright \ldots$$

Assumption. Say G is a linear algebraic group.

Definition 10.11. We say G is **solvable** if there exists a subnormal series (G_i) with commutative quotients G_{i-1}/G_i. Equivalently, $\mathcal{D}^n G = \{1\}$ for some n.

Definition 10.12. We say G is **unipotent** if every nonzero representation of G has a nonzero fixed vector. Equivalently, every finite dimensional representation $G \to \mathrm{GL}_V$ has image in[16]

$$U_n = \begin{pmatrix} 1 & & * \\ & \ddots & \\ & & 1 \end{pmatrix}$$

after choosing a suitable basis for V. Get this equivalence by quotienting by the subreg generated by the fixed vector (so it's linear span) and then repeating.

Remark 10.13. Equivalent to require nonzero fixed vectors only for finite representations.

Remark 10.14. We'll later prove this is the same as saying G is isomorphic to some closed subgroup of U_n for some n.

Definition 10.15. Let G be a smooth, connected linear algebraic group. Its **radical** $R(G)$ is the largest smooth connected solvable normal subgroup. Its **unipotent radical** $R_u(G) \le R(G)$ is the largest smooth connected unipotent normal subgroup (of G).

Warning 10.16. These notions do not respect inseparable base field extensions.

Definition 10.17. We call G **semisimple** $\iff$ $R(G_{\overline{k}}) = 1$ is trivial. We call G **reductive** $\iff$ $R_u(G_{\overline{k}}) = 1$ is trivial.

Note semisimple $\implies$ reductive.

Example. Let $G = \mathrm{GL}_n$. GL_n has lots of solvable subgroups, e.g. the upper triangle matrices (but this isn't normal so not contained in $R(G)$). It turns out that $R(G) = \mathbb{G}_m$ (see this later). $\mathbb{G}_m$ is not unipotent and indeed $R_u(G) = 1$. Hence, $G = \mathrm{GL}_n$ is reductive, but not semisimple.

Example. Let $G = \mathrm{SL}_n$. Then $R(G) = 1$ (can't be e.g. μ_n since it has to be connected and smooth). Hence, SL_n is semisimple.

Remark 10.18. Bjorn drew a picture of various algebraic groups. Maybe I'll come back and add it at some point...

10.3 Results we're skipping over

We're skiping chapters 7 and 8 of the text, but we'll need some results from there.

Let G be a smooth connected algebraic group. Can always define $G_{\textbf{affine}}$, the largest smooth connected affine normal subgroup.

Theorem 10.19 (Barsotti-Chevalley Theorem). *Let k be a perfect field. Then, G/G_{affine} is an abelian variety. Equivalently, every G fits into an exact sequence*

$$1 \longrightarrow \textit{affine} \longrightarrow G \longrightarrow \textit{ab. var} \longrightarrow 1.$$

(proof in chapter 8)

[16] 1's along the diagonal. Anything above the diagonal.

10.3.1 Structure of algebraic groups over a perfect field k

"I'll continue to assume that the field is perfect because things get a little more complicated if you don't assume that. Actually, they get a lot more complicated."

Let G be an algebraic group over k. We get a lattice of (normal, characteristic) subgroups

labels on arrows give properties of quotients

TODO: Make this look nice

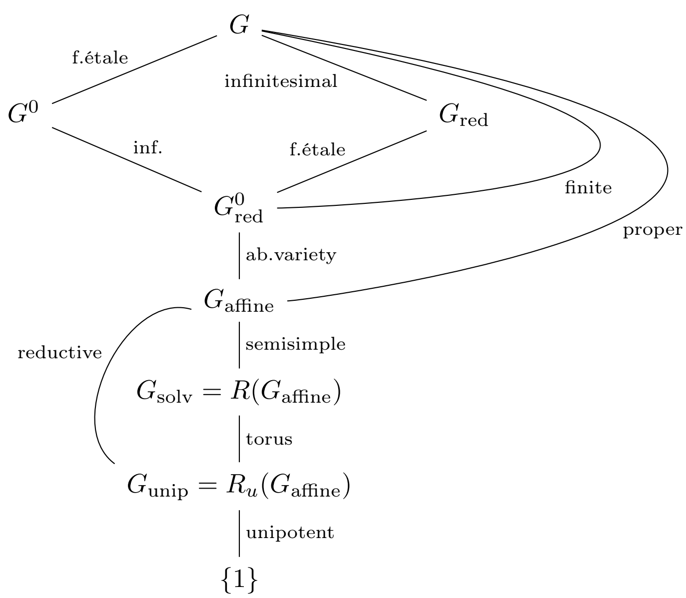

Figure 1: A lattice of various pieces of an algebraic group over a perfect field

(Above, **infitesimal** means supported on one point)

The focus for the rest of the class will be on the piece below G_{affine}. Classifying semisimple groups and torii will be doable. Classifying unipotent groups is much harder.

10.4 Tannakian formalism

Let G be a linear algebraic group over k. Let $\text{Rep}(G)$ denote its **representation category** so objects are *f. dim* reps of G and the morphisms are G-equivariant linear maps.

What kind of category is this? It's an abelian category, but even better, it's a **k-linear** category, i.e. Hom-sets are k-vector spaces and composition is k-bilinear (+ kernels, cokernels, etc. from being abelian). It's even more than this. It supports tensor products, so it's also a *tensor category* (can tensor any number of objects together). It's also a *rigid category* which means you can talk about dual objects. So it's a *rigid k-linear tensor category.*

It has a few other properties. It satisfies $\text{End}\,\mathbf{1} = k$, where $\mathbf{1}$ is the trivial representation (tensor of 0 objects). It also has a *fiber functor* $\omega : \text{Rep}(G) \to \text{Vect}_k$ given by forgetting the G-action which is an *exact, faithful, k-linear tensor functor* (such a thing is called a (k-valued) **fiber functor**).

A rigid k-linear tensor category with the extra properties of the last paragraph is what's called a **neutral Tannakian category** over k.

Question 10.20. *Can you recover G from* $\text{Rep}(G)$ *with all this extra structure?*

11 Lecture 11 (3/12)

Daylight savings on Sunday. Don't be late on Monday.

11.1 More Tannakian formalism, Rep(G)

Definition 11.1. Fix a field k.

- An **abelian category** is a category enriched with some extra structure
  - each $\mathrm{Hom}(A, B)$ is an abelian group with composition bi-additve
  - kerels and cokernels exist and satisfy some axioms;
- A **k-linear category** is an abelian category s.t.
  - each $\mathrm{Hom}(A, B)$ is a k-vector space with composition bilinear
- A **tensor category** is a category equipped with a functor $\otimes : C \times C \to C$, an object $\mathbf{1}$, and other structure allowing one to define $A_1 \otimes \cdots \otimes A_n$ for any $n \geq 0$
- A tensor category is **rigid** if every object has a dual (staisfying some axioms)

Example. Both $\mathrm{Rep}(G)$ and Vect are rigid k-linear tensor categories.

Definition 11.2. A **k-valued fiber functor** on a k-linear tensor category C is an exact faithful k-linear tensor functor $\omega : C \to \mathrm{Vect}$.

Example. The forgetful functor $\omega : \mathrm{Rep}(G) \to \mathrm{Vect}$ is a fiber functor

Definition 11.3. A **neutral Tannakian category** is a rigid k-linear tensor category such that

- $\mathrm{End}\,\mathbf{1} = k$; and
- there exists a k-valued fiber functor ω

Example. $\mathrm{Rep}(G)$ is a neutral Tannakian category.

Theorem 11.4 (**Reconstruction theorem**)**.** *A linear algebraic group G can be recovered from* $(\mathrm{Rep}(G), \otimes, \omega)$.

Proof Sketch. Given g, you get a whole system $(g|_V)_{V \in \mathrm{Rep}(G)}$ of k-linear maps on each representation. So, we have a map from $G(k)$ to systems of k-linear isos[17] $(\lambda_V : V \xrightarrow{\sim} V)_{V \in \mathrm{Rep}(G)}$ such that

(a) for all maps of reps $\varphi : V \to W$, the square

$$\begin{tikzcd} V \arrow[r, "\lambda_V"] \arrow[d, "\varphi"'] & V \arrow[d, "\varphi"] \\ W \arrow[r, "\lambda_W"] & W \end{tikzcd}$$

commutes.

(b) $\lambda_{V \otimes W} = \lambda_V \otimes \lambda_W$.

[17] It would suffice to just consider systems of maps

(c) In particular $\lambda_{\mathbf{1}} = \mathrm{id}$.

We claim this map is actually a bijection.

Idea: Given such a system, can build λ_V for every ∞-dimensional V too (since they're all unions of finite representations). In particular, if $G = \operatorname{spec} A$, then you get $\lambda_A : A \xrightarrow{\sim} A$. This is suppose to be the action of some group element, check that it defines $G = \operatorname{spec} A \leftarrow \operatorname{spec} A = G$ compatible with the left G-action (use λ respects tensor products) and so is a right multiplication by some element of $G(k)$.

Note that these compatible systems are precisely the elements of the group $\mathrm{Aut}^{\otimes}(\omega)$ of automorphisms of the k-linear tensor functor ω. We've seen these recover the k-points $G(k)$. What about the R-points for R a k-algebra? These are precisely the elements of $\underline{\mathrm{Aut}}^{\otimes}(\omega)(R)$, i.e. systems $(\lambda_V : V_R \xrightarrow{\sim} V_R)$ satisfying **(a)**,**(b)**,**(c)** from before. Here, $\underline{\mathrm{Aut}}^{\otimes}(\omega)$ is a functor $\{k\text{-algs.}\} \to \mathrm{Grp}$ and $G \xrightarrow{\sim} \underline{\mathrm{Aut}}^{\otimes}(\omega)$ as group-valued functors.

Thus, given $(\mathrm{Rep}(G), \otimes, \omega)$, the group-valued functor $\underline{\mathrm{Aut}}^{\otimes}(\omega)$ turns out to be the functor of points of G. ■

This argument generalizes to projective limits of linear algebraic groups (certain pro-algebraic groups) $G = \varprojlim G_i$ with each G_i an algebraic group. On the level of Hopf algebras, $A = \varinjlim A_i$, each A_i a f.g. commutative Hopf algebra. So $G = \operatorname{spec} A$ is some (non-finite type) group scheme. All affine k-group schemes are of this form $G = \varprojlim G_i$. Every finite-dimensional rep of G will factor through some G_i, so $\mathrm{Rep} G = \varinjlim \mathrm{Rep}(G_i)$.

Theorem 11.5.

$$\{\textit{affine } k\textit{-group schemes}\} \longrightarrow \left\{ \begin{array}{c} \textit{neutral Tannakian category} \\ \textit{equipped with a } k\textit{-valued fiber functor } \omega \end{array} \right\}$$

via $G \mapsto (\mathrm{Rep}(G), \otimes, \textit{forget})$ *is an equivalence of categories. The functor in the other direction is* $\underline{\mathrm{Aut}}^{\otimes}(\omega) \leftarrow\!\shortmid (C, \otimes, \omega)$

Example.

$$\mathbb{G}_m \longleftrightarrow \left\{ \begin{array}{c} \text{f.dim } \mathbb{Z}\text{-graded } k\text{-vector spaces} \\ V = \bigoplus_{n\in\mathbb{Z}} V_n \end{array} \right\}$$

$\lambda \in \mathbb{G}_m$ acts as λ^n on V_n, so the decomp of V is just the isotypic decomposition of V into its characters.

Example.

$$\text{The } \textbf{Deligne torus } \mathbb{S} := \mathrm{Res}_{\mathbb{C}/\mathbb{R}}(\mathbb{G}_m) \longleftrightarrow \{\mathbb{R}\text{-Hodge structure}\}$$

These are f.dim $\mathbb{C}$-vector spaces which are $\mathbb{Z}\times\mathbb{Z}$-graded, i.e. $V = \bigoplus_{(p,q)\in\mathbb{Z}^2} V^{p,q}$, s.t. $V^{pq} = \overline{V^{qp}}$.

Example.

$$? \longleftrightarrow \{\mathbb{Q}\text{-Hodge structure}\}$$

"If you try to think about the group that corresponds to it, it'll make your head hurt."

Definition 11.6. Say V is a f.dim k-vector space. Let $a \in \operatorname{End} V$ be some endomorphism of V.

- a is **diagonalizable** $\iff$ V has a basis consisting of eigenvectors for a.
- a is **semisimple** $\iff$ a becomes diagonalizable after field extension.

- a is **nilpotent** $\iff$ $a^s = 0$ for some $s \geq 1$ $\iff$ all its eigenvalues are 0.
- a is **unipotent** $\iff$ $a - 1$ is nilpotent $\iff$ all eigenvalues are 1.

Example. The matrix

$$\begin{pmatrix} 2 & 1 & & & & \\ & 2 & & & & \\ & & 5 & 1 & & \\ & & & 5 & 1 & \\ & & & & 5 & \\ & & & & & 5 \end{pmatrix} = \begin{pmatrix} 2 & & & & & \\ & 2 & & & & \\ & & 5 & & & \\ & & & 5 & & \\ & & & & 5 & \\ & & & & & 5 \end{pmatrix} + \begin{pmatrix} 0 & 1 & & & & \\ & 0 & & & & \\ & & 0 & 1 & & \\ & & & 0 & 1 & \\ & & & & 0 & \\ & & & & & 0 \end{pmatrix}$$

can be decomposed as a semisimple matrix + a nilpotent matrix (and the two commute).

12 Lecture 12 (3/15)

12.1 Last time: Tannakian formalism, $\mathrm{Rep}(G)$

Theorem 12.1 (Reconstruction Theorem). *Given an affine k-group scheme G, consider $(\mathrm{Rep}(G), \otimes, \textit{forgetful})$. Then, $G \simeq \underline{\mathrm{Aut}}^{\otimes}(\textit{forgetful})$.*

Characterization of reprsentation categories

TODO: Go over this slide and fill things in

12.2 Today: Jordan decomposition

Assumption. From now on k is a *perfect* field.

Recall 12.2. Say V is a f.dim k-vector space. Let $a \in \mathrm{End}\, V$ be some endomorphism of V.

- a is **diagonalizable** $\iff$ V has a basis consisting of eigenvectors for a.
- a is **semisimple** $\iff$ a becomes diagonalizable after field extension.
- a is **nilpotent** $\iff$ $a^s = 0$ for some $s \geq 1$ $\iff$ all its eigenvalues are 0.
- a is **unipotent** $\iff$ $a - 1$ is nilpotent $\iff$ all eigenvalues are 1.

Theorem 12.3 (additive Jordan decomposition, in $\mathrm{End}\, V$**).** *Given $a \in \mathrm{End}\, V$, there exists a unique decomposition $a = a_s + a_n$ with a_s semisimple, a_n nilpotent, and $a_s a_n = a_n a_s$. Moreover, $a_s, a_n \in k[a] \subset$* $\mathrm{End}\, V$.

Proof. Case 1 ($k = \overline{k}$). For existence, we use structure theorem of modules over a PID. Note that

$$k[a] \xleftarrow{\sim} \frac{k[T]}{(\text{min. poly of } a)} \xrightarrow{\sim} \frac{k[T]}{(T - \lambda_1)^{e_1}} \times \ldots \times \frac{k[T]}{(T - \lambda_r)^{e_r}}$$

with RHS coming from factoring the minimal polynomial of a. On the RHS, we have

$$T \leftrightarrow (T, \ldots, T) = \underbrace{(\lambda_1, \ldots, \lambda_r)}_{a_s} + \underbrace{(T - \lambda_1, \ldots, T - \lambda_r)}_{a_n}.$$

This finishes existence. a_s above acts diagonally while a_n is nilpotent, and they commute since they do use factorwise.

Now let's do uniqueness. If $a = b_s + b_n$ is another decomposition, then b_s commutes with b_s and with b_n, hence with a. Since $a_s, a_n \in k[a]$ are polynomials in a, b_s will also commute with them. Now, since a_s, b_s are commuting and semisimple, they have a simultaneous basis of eigenvectors, so their difference $b_s - a_s$ is semisimple. At the same time, $b_s - a_s = a_n - b_n$ which is nilpotent (a_n, b_n commute + binomial expansion). Thus, $b_s - a_s = a_n - b_n = 0$ since this is the only nilpotent, semisimple endomorphism. The finishes the algebraically closed case. This leaves the general case.

Case 2 (k any perfect field). The a_s, a_n over $\overline{k}$ are $\mathrm{Gal}(\overline{k}/k)$-invariant by uniqueness, so they're defined over k. ■

Here are some nice properties of the Jordan decomp.

- If $W \le V$ is an a-stable subspace (i.e. $a(W) \subset W$), then the Jordan decomp of a on V induces the Jordan decomp of $a|_W$ on W and of a on V/W.
- Given linear maps

$$\begin{tikzcd} V \arrow[r, "\varphi"] \arrow[d, "a"'] & W \arrow[d, "b"] \\ V \arrow[r, "\varphi"] & W, \end{tikzcd}$$

  the J.D. of a is compatible with that of b, i.e. there are two more squares with a, b replaced by a_s, b_s or a_n, b_n.

Recall our running assumption that k is perfect.

Theorem 12.4 (**Multiplicative Jordan decomposition**, in alg. gps)**.** *There is a unique way to define, for every linear algebraic group G/k and $g \in G(k)$, a factorization into commuting elements $g = g_s g_u = g_u g_s \in G(k)$ such that*

(1) *If $G = \mathrm{GL}_V$, then g_s is semisimple and g_u is unipotent.*

(2) *For any homomorphism $\varphi : G \to G'$ and $g \in G(k)$, one has*

$$\varphi(g_s) = \varphi(g)_s \text{ and } \varphi(g)_u = \varphi(g_u).$$

Proof. (Existence) We first give the definition for $g \in \mathrm{GL}_V(k) = \mathrm{GL}(V)$. Write $g = g_s + g_n$ using the additive Jordan decomposition (note g_s invertible since it has same eigenvalues as g), so $g = g_s(1 + g_s^{-1} g_n)$. Now, $g_s^{-1} g_n$ is nilpotent (since they commute), so $1 + g_s^{-1} g_n$ is unipotent, so we just set $g_u = 1 + g_s^{-1} g_n$ to get our decomposition.

Now let's show existence for $g \in G(k)$ where G is any linear algebraic group. Consider the system $(r(g)_s)_{r \in \mathrm{Rep}(G)}$ of semisimple parts of all representations of g. By second nice property of Jordan decompositions, this is a compatible system, so the reconstruction theorem tells us there must exist a unique element $g_s \in G(k)$ so that $r(g_s) = r(g)_s$ for all reps $r \in \mathrm{Rep}(G)$. Can define g_u in the same way. Then, $g = g_s g_u = g_u g_s$ (can check using any faithful[18] representation $r : G \hookrightarrow \mathrm{GL}_V$). Note that when $G = \mathrm{GL}_V$ these two definitions agree since $r(g_s) = r(g)_s$ and $r(g_u) = r(g)_u$ e.g. for $r = \mathrm{id} : \mathrm{GL}_V = \mathrm{GL}_V$.

[18] In fact, one faithful representation already tells you what the decomposition has to be

(Compatibility) Let $\varphi : G \to G'$ be a homomorphism of linear algebraic groups, and fix some $g \in G(k)$. For each $r : G' \to \mathrm{GL}_V$, get $G \xrightarrow{\varphi} G' \xrightarrow{r} \mathrm{GL}_V$, a rep of G. Observe that $r(\varphi(g_s)) = r(\varphi(g))_s$. Thus, $\varphi(g_s)$ satisfies the defining property of $\varphi(g)_s$, so $\varphi(g_s) = \varphi(g)_s$ as desired. Similar argument shows $\varphi(g_u) = \varphi(g)_u$.

(Uniqueness) Follows from stuff above. ■

Definition 12.5. We call $g \in G(K)$ a **semisimple element** if $g = g_s$, and we call it a **unipotent element** if $g = g_u$.

12.3 Review of Lie algebras and Lie groups

Definition 12.6. A **Lie algebra** if a vector space $\mathfrak{g}$ equipped with a bilinear map $[-,-] : \mathfrak{g} \times \mathfrak{g} \to \mathfrak{g}$, the **Lie bracket**, satisfying

- $[x, x] = 0$.
- $[x, [y, z]] + [y, [z, x]] + [z, [x, y]] = 0$ (**Jacobi identity**).

Example. Let V be any vector space. Then, $\operatorname{End} V$ with Lie bracket given by the commutator $[A, B] = AB - BA$ is a Lie algebra.

Definition 12.7. Let A be a k-algebra. A **derivation** of A is a k-linear map $D : A \to A$ such that

$$D(fg) = D(f)g + fD(g)$$

(**Liebniz rule**).

Example. Let M be a C^∞ manifold. Then, there's a bijection

$$\{\text{vector fields on } M\} \xrightarrow{\sim} \{\text{derivations of } C^\infty(M)\} \subset \operatorname{End}(C^\infty(M)).$$

Recall the commutator bracket on $\operatorname{End}(C^\infty(M))$. It is a fact that this preserves the space of derivations, i.e. $[D, E] = DE - ED$ is a derivation if D, E are, so the set of derivations of $C^\infty(M)$ is also a Lie algebra (hence, vector fields on M form a Lie algebra too).

Example. Now say G is a Lie group (C^∞-manifold with group law). Then the tangent space at the identity, via (right) translations, is identified

$$T_e G \xrightarrow{\sim} \{\text{left-invariant vector fields on } F\} \xrightarrow{\sim} \{\text{left-invariant derivations on } C^\infty(G)\}$$

with the space of left-invariant vector fields. This space is preserved by the Lie bracket, so $T_e G$ inherits the structure of a Lie algebra. This is the **Lie algebra of** G.

Next time we'll talk about an algebraic version of this.

Let X be a k-variety. For $x \in X(k)$ the **Zariski cotangent space** at x is $\mathfrak{m}_x/\mathfrak{m}_x^2$ and the **Zariski tangent space** is the dual $T_x X := \operatorname{Hom}_k(\mathfrak{m}_x/\mathfrak{m}_x^2, k)$. Equivalently, the tangent space is the preimage of x under $X(k[\varepsilon]/\varepsilon^2) \to X(k)$.

13 Lecture 13 (3/17)

Note 2. *a minute or two late*

13.1 Lie algebras of an algebraic group

Fix a k-variety X and some $x \in X(k)$.

- The **Zariski cotangent space** at x is $T_x^*X := \mathfrak{m}_x/\mathfrak{m}_x^2$
- The **Zariski tangent space** at x is $T_xX := \operatorname{Hom}_k(\mathfrak{m}_x/\mathfrak{m}_x^2, k)$
- Let $\mathcal{I}$ be the ideal sheaf of the diagonal $X \overset{\Delta}{\hookrightarrow} X \times X$. Then view the $\mathscr{O}_{X\times X}/\mathcal{I}$ module $\mathcal{I}/\mathcal{I}^2$ as an $\mathscr{O}_X$-module Ω_X, the **cotangent bundle**.

  Slogan. The diagonal is sort of like the family over X that parameterizes points on X.

- A **derivation on** $\mathscr{O}_X$ is a homomorphism of sheaves $D : \mathscr{O}_X \to \mathscr{O}_X$ (not $\mathscr{O}_X$-linear) such that for every open $U \subset X$, the map $D(U) : \mathscr{O}_X(U) \to \mathscr{O}_X(U)$ is a derivation.

Proposition 13.1. $\operatorname{Der}(\mathscr{O}_X) \simeq \operatorname{Hom}_{\mathscr{O}_X}(\Omega_X, \mathscr{O}_X) \simeq \Gamma(X, \mathcal{T}_X)$ *(with last iso if X is smooth*[19] *(so Ω_X locally free))*

Proof. There's the natural, universal differential $\mathrm{d} : \mathscr{O}_X \to \Omega_X$. Any derivation $D : \mathscr{O}_X \to \mathscr{O}_X$ will factor through a unique $\mathscr{O}_X$-linear map $\Omega_X \to \mathscr{O}_X$. ∎

Proposition 13.2. *Let G be an algebraic group. Then, $\Omega_G \simeq \mathfrak{m}_e/\mathfrak{m}_e^2 \otimes_k \mathscr{O}_G$ is a free $\mathscr{O}_G$-module.*

Proof. There's an automorphism $G \times G \xrightarrow{\sim} G \times G, (x, y) \mapsto (xy^{-1}, y)$ sending the diagonal to the vertical axis $\{e\} \times G$. Hence, the ideal sheaf $\mathscr{I}$ of the diagonal will get mapped to $\mathfrak{m}_e \otimes_k \mathscr{O}_G$, the ideal sheaf of the vertical axis. Hence,

$$\Omega_G = \mathscr{I}/\mathscr{I}^2 \leftrightarrow \frac{\mathfrak{m}_e}{\mathfrak{m}_e^2} \otimes_k \mathscr{O}_G.$$

∎

Corollary 13.3. $\operatorname{Der}(\mathscr{O}_G) = T_eG \otimes \mathscr{O}(G)$.

Corollary 13.4. $\operatorname{Der}_{\textit{left-inv}}(\mathscr{O}_G) = T_eG = \ker\left(G(k[\varepsilon]) \to G(k)\right)$.

The left-invariant functions in $\mathscr{O}(G)$ are the constant functions.

As before, there is a Lie bracket on the space of derivations, so we get a corresponding bracket on T_eG, giving it the structure of a Lie algebra. We call this Lie algebra $\operatorname{Lie} G$.

Example. $\mathfrak{gl}_n := \operatorname{Lie}\mathrm{GL}_n = \ker\left(\mathrm{GL}_n(k[\varepsilon]) \to \mathrm{GL}_n(k)\right)$. These will be the matrices[20]

$$\{1 + X\varepsilon : X \in M_n(k)\} \simeq M_n(k)$$

(multiplication on the LHS and addition and the RHS). The bracket on this Lie algebra is $[X, Y] = XY - YX$.

[19] Usually don't define tangent bundle when X is not smooth

[20] $1 + X\varepsilon$ invertible iff it's determinant is a unit after reducing mod ε

Example. $\mathfrak{gl}_V = \operatorname{End} V$ if you wanna be coordinate-free

Fact (on homework)**.** Any homomorphism $\varphi : G \to H$ of algebraic groups induces a homomorphism $\mathrm{d}\varphi : \operatorname{Lie} G \to \operatorname{Lie} H$ of Lie algebras.

Definition 13.5. A **representation of a Lie algebra** $\mathfrak{g}$ is a Lie algebra homomorphism

$$\mathfrak{g} \longrightarrow \mathfrak{gl}_V$$

for some vector space V.

13.2 Adjoint representation

Let G be an algebraic group with Lie algebra $\mathfrak{g} := \operatorname{Lie} G$.

There is a conjugation action of G on G which induces an action of G on $\mathfrak{g}$ and hence a representation $\operatorname{Ad} : G \to \mathrm{GL}_{\mathfrak{g}}$ called the **Adjoint representation**. You can take is derivative to get another representation $\operatorname{ad} : \mathfrak{g} \to \mathfrak{gl}_{\mathfrak{g}} = \operatorname{End} \mathfrak{g}$ (now of Lie algebras) called the **adjoint representation**.

Fact. Tracing through the definitions shows that

$$\operatorname{ad} x(y) = [x, y]$$

for all $x, y \in \mathfrak{g}$.

Motivation. Given a finite abelian group G, can form the dual/character group $G^\vee := \operatorname{Hom}(G, \mathbb{C}^\times)$. Is there a group scheme analogue of this?

Theorem 13.6 (**Cartier duality**)**.** *Consider the (abelian) category $\mathcal{C} = \{$commutative finite group schemes$/k\}$. Then there exists an exact equivalence of categories $\mathcal{C} \to \mathcal{C}^{op}, G \mapsto G^\vee$ which is its own essential inverse (i.e. there's a functorial iso $G \xrightarrow{\sim} G^{\vee\vee}$).*

Here are two constructions of $G^\vee$

- G is finite, so $G = \operatorname{spec} A$ for some A with A a Hopf algebra which is f.d. as a k-vector space. We have k-linear maps
$$A \otimes A \xrightarrow{\mu} A, \;\; k \xrightarrow{i} A, \;\; A \xrightarrow{\Delta} A \otimes A, \;\text{ and }\; A \xrightarrow{\varepsilon} k.$$
If you take k-linear duals everywhere, then we get
$$A^\vee \to A^\vee \otimes A^\vee, \;\; A^\vee \to k, \;\; A^\vee \otimes A^\vee \to A^\vee, \;\text{ and }\; k \to A^\vee,$$
so $A^\vee$ is also a f.dim k-Hopf algebra! Let $G^\vee = \operatorname{spec} A^\vee$.
- There is a group scheme $G^\vee := \mathscr{H}om(G, \mathbb{G}_m)$ whose functor of points is
$$G^\vee(R) = \operatorname{Hom}_{R\text{-gp schemes}}(G_R, (\mathbb{G}_m)_R).$$
Note there's a natural pairing $G \times G^\vee \to \mathbb{G}_m$.

Example. $\mathbb{Z}/n\mathbb{Z} \leftrightarrow \mu_n$ (always, even if $n = p = \operatorname{char} k$ for example)

Example. $\alpha_p \leftrightarrow \alpha_p$

Example. For an elliptic curve E, $E[n] \leftrightarrow E[n]$. For an abelian variety, $A[n] \leftrightarrow A^\vee[n]$ with $A^\vee$ the dual abelian variety (different notion of duality). The pairing in these examples is the Weil pairing.

13.3 Diagonalizable groups

Recall 13.7. Let $G = \operatorname{spec} A$ be an affine algebraic group. Then, an element $a \in A^\times$ is called **group-like** iff $\Delta(a) = a \otimes a \iff G \xrightarrow{a} \mathbb{G}_m$ is a character (i.e. homomorphism).

Definition 13.8. G is **diagonalizable** if A is the k-span of its group-like elements.

Fact. The group-like elements are always k-linearly independent.

Example. For $G = \mathbb{G}_m$, $A = k[t, t^{-1}]$. The group-likes are t^n ($n \in \mathbb{Z}$), and they do span the coordinate ring.

Proposition 13.9. *There exists an exact equivalence of categories*

$$\{\textit{f.g. abelian groups}\}^{op} \longleftrightarrow \{\textit{diagonalizable algebraic groups}/k\}$$

Secretly, the functor in both directions is $\operatorname{Hom}(-, \mathbb{G}_m)$. *Other notation is*

$$M \mapsto D(M) \text{ and } X(G) \leftarrow\!\shortmid G.$$

Above, $X(G)$ is the ***character group*** *of G, i.e. $X(G) = \operatorname{Hom}(G, \mathbb{G}_m)$, and the functor of points of $D(M)$ is $D(M)(R) = \operatorname{Hom}(M, R^\times)$.*

Proof Sketch. Let's first show $D(M)$ is representable. If $M = \mathbb{Z}$, then $D(M) = \mathbb{G}_m$. If $M = \mathbb{Z}/n\mathbb{Z}$, then $D(M) = \mu_n$. In the general case, you get products of these. Alternatively, $D(M) = \operatorname{spec} k[M]$ where $k[M]$ is the group algebra of M.[21] By construction, $D(M)$ is diagonalizable since the group-likes e_m span $k[M]$. ■

14 Lecture 14 (3/19)

No class Monday/Tuesday next week for 'Spring break'

Next homework due on Sunday (not in two days, I hope)

14.1 Last time: Diagonalizable groups

For a linear algebraic group $G = \operatorname{spec} A$

$$\{\text{characters } \chi : G \to \mathbb{G}_m\} \longleftrightarrow \left\{\text{group-like elements } a \in A^\times\right\}$$

Recall 14.1. G is *diagonalizable* if A is the k-span of its group-like elements.

[21] The comulitplication sends a basis element e_m ($m \in M$) to $e_m \mapsto e_m \otimes e_m$. The coidentity sends $e_m \mapsto 1$ and the coinverse is $e_m \mapsto e_{m^{-1}}$

Theorem 14.2. *Fix a field k. There exists an exact equivalence of categories*

$$\{\textit{f.g. abelian groups}\}^{op} \longleftrightarrow \{\textit{diagonalizable algebraic groups}/k\}$$

sending $M \mapsto D(M)$ and $X(G) \leftarrow\!\shortmid G$.

Three equivalent desriptions of $D(M)$:

- Explicitly construction: $D(\mathbb{Z} \oplus \mathbb{Z} \oplus \mathbb{Z}/3\mathbb{Z}) = \mathbb{G}_m \times \mathbb{G}_m \times \mu_3 \hookrightarrow \mathrm{GL}_3$, etc.
- Its functor points is $R \mapsto \mathrm{Hom}(M, R^\times)$
- It is $\operatorname{spec} k[M]$, where $k[M]$ is the group algebra.

14.2 Some things we didn't get to last time

Let's mention a few more things about the proof.

- If $G = \operatorname{spec} A$ is diagonalizable, then $G \simeq D(M)$ for some M:

  Let M be the set of group-like elements in A. They are automatically independent. Hence, if they span, then $k[M] \xrightarrow{\sim} A$. Check that this is an iso of Hopf algebras.

- $\mathrm{Hom}_{k\text{-group scheme}}(D(M), D(M')) \simeq \mathrm{Hom}_{\text{groups}}(M', M)$

  Suffices to check the cases when each of M, M' is either $\mathbb{Z}$ or $\mathbb{Z}/n\mathbb{Z}$. Let's do the case where $M = M' = \mathbb{Z}$, so $D(M) = D(M') = \mathbb{G}_m$.

  Remark 14.3. A better version of this theorem computes Hom after any base change, so you get a $\underline{\mathrm{Hom}}$-functor
  $$\underline{\mathrm{Hom}}\,(D(M), D(M')) \simeq \underline{\mathrm{Hom}(M', M)}$$
  (LHS is functor whose R points is R-group scheme homomorphisms $D(M)_R \to D(M')_R$ while the RHS is the constant group scheme over k).

  We'll prove that version here, i.e. we'll show
  $$\mathrm{Hom}(\mathbb{G}_{m,R}, \mathbb{G}_{m,R}) = \underline{\mathbb{Z}}(R).$$

  For simplicity, suppose the following equivalent conditions hold:

  - $\operatorname{spec} R$ is connected
  - the solutions to $a^2 = a$ in R are 0 and 1
  - $\underline{\mathbb{Z}}(R) = \mathbb{Z}$

  Remember: A scheme is connected iff its global sections has no nontrivial idempotents

  Any homomorphism $\mathbb{G}_{m,R} \to \mathbb{G}_{m,R}$ is of the form $t \mapsto f(t) = \sum_{n \in \mathbb{Z}} a_n t^n$ for some $f \in R[t, 1/t]^\times$ such that $f(x)f(y) = f(xy)$ in $R[x, 1/x, y, 1/y]$. Equate coefficients of $x^n y^m$:
  $$a_n a_m = \begin{cases} a_n & \text{if } n = m \\ 0 & \text{otherwise.} \end{cases}$$

Thus each a_n is an idempotent, so each $a_n \in \{0, 1\}$. Furthermore, the product of any two is 0, so at most one of them is 1. Finally, $f = 0$ is not a unit, so they can't all be 0. Hence, $f = t^n$ for some $n \in \mathbb{Z}$. Thus, $\mathrm{Hom}(\mathbb{G}_{m,R}, \mathbb{G}_{m,R}) = \mathbb{Z}$ as desired.

Proposition 14.4. *Subgroups and quotient groups of diagonalizable groups are diagonalizable.*

Proof. (subgroups) Say $H \leq G$ with G diagonalizable. Then, H is a closed subscheme, so have some surjection $\mathscr{O}(G) \twoheadrightarrow \mathscr{O}(H)$. The group-likes in G span $\mathscr{O}(G)$, so their images span $\mathscr{O}(H)$ and are still group-like, so $\mathscr{O}(H)$ is diagonalizable.

(quotients) Say $1 \to H \to G \to Q \to 1$ with G diagonalizable. By the previous case, we can write $G = D(M)$ and $H = D(M'')$. The map $H \hookrightarrow G$ corresponds to a map $M \twoheadrightarrow M''$, so let $M' := \ker(M \to M'')$. Then exactness of the functor gives an exact sequence

$$1 \to D(M'') \to D(M) \to D(M') \to 1,$$

so we conclude $Q = D(M')$. ∎

Corollary 14.5. *G is diagonalizable $\iff$ $G \leq \mathbb{G}_m^n$ (think of as diagonal $n \times n$ matrices) for some $n \geq 0$.*

14.3 Representations, groups of multiplicative type

Theorem 14.6. *Let G be diagonalizable, and let V be a representation of G. Then,*

(1) *V is a direct sum of 1-dimensional representations*

(like reps of a finite abelian group G with $\mathrm{char}\, k \nmid \#G$)

(2) *$V = \bigoplus_{\chi \in X(G)} V_\chi$ is a sum of its isotypic components.*

Definition 14.7. The χ for which $V_\chi \neq 0$ are called the **weights of** V. V_χ is also called the **χ-eigenspace** or **χ-weight space**.

(3) *Under Tannakian formalism*

$$D(M) = G \longleftrightarrow (\mathrm{Rep}(G), \otimes, \textit{forgetful})$$

with $\mathrm{Rep}(G)$ the category of M-graded vector spaces. (Note $M = X(G)$)

Recall 14.8. G is a **torus** $\iff$ $G_{\overline{k}} \simeq \mathbb{G}_{m,\overline{k}}^n$ for some $n \geq 0$.

Definition 14.9. More generally, we say G is **of multiplicative type** $\iff$ $G_{\overline{k}}$ is diagonalizable. G is **split** $\iff$ G is diagonalizable over k.

Proposition 14.10. *If $G_{\overline{k}} \simeq D(M)_{\overline{k}}$, then $G_{k^s} \simeq D(M)_{k^s}$.*

Proof. Inseparable descent.[22] Can reduce to showing that for $L = k[x]/(x^p - a)$, if $G_L \xrightarrow{\sim} D(M)_L$, then $G \xrightarrow{\sim} D(M)$.

[22]The intuition comes from showing that a morphism over L is defined over k by showing that it is $\mathrm{Gal}(L/k)$-invariant

Let $S' = \operatorname{spec} L$ and $S = \operatorname{spec} k$. Let $S'' = S' \times_S S'$. Consider the diagram

$$\begin{tikzcd}
& S' \arrow[d, "\Delta"] \arrow[ddl, bend right, equal] \arrow[ddr, bend left, equal] & \\
& S'' \arrow[dl, "p_1"'] \arrow[dr, "p_2"] & \\
S' \arrow[dr] & & S' \arrow[dl] \\
& S &
\end{tikzcd}$$

Note that $L \otimes_k L \simeq \frac{k[x,y]}{(x^p - a, y^p - a)} \simeq \frac{L[\varepsilon]}{(\varepsilon^p)}$ where $x - y \leftarrow\!\shortmid\ \varepsilon$. How do we do the descent now? We need "agreement on the overlap," i.e. we need the two isomorphisms

$$G_{L\otimes L} \underset{p_2^*\varphi}{\overset{p_1^*\varphi}{\rightrightarrows}} D(M)_{L\otimes L}$$

to coincide.

Consider their difference $(p_1^*\varphi)(p_2^*\varphi)^{-1} \in \operatorname{Aut} D(M)_{L\otimes L} \simeq \operatorname{Aut} M$ (since $L \otimes L$ is connected). If we pull back along the diagonal Δ, then we get $\varphi\varphi^{-1} \in \operatorname{Aut} D(M)_L = \operatorname{Aut} M$ (this is "reduce mod ε"). Thus, we have a square

$$\begin{tikzcd}
\operatorname{Aut} D(M)_{L\otimes L} \arrow[r, "="] \arrow[d, "\Delta^*"'] & \operatorname{Aut} M \arrow[d, "="] \\
\operatorname{Aut} D(M)_L \arrow[r, "="] & \operatorname{Aut} M
\end{tikzcd}$$

where $(p_1^*\varphi)(p_2^*\varphi)^{-1} \in \operatorname{Aut} D(M)_{L\otimes L}$ gets mapped to $\mathrm{id} = \varphi\varphi^{-1} \in \operatorname{Aut} D(M)_L$. Thus, it must have been the identity all along, so we win. ■

Remark 14.11. $\operatorname{spec} L \to \operatorname{spec} k$ in the previous proof is not étale but it is fpqc.

Corollary 14.12. *For each finite Galois extension k'/k, there exists an exact equivalence of categories*

$$\begin{array}{ccccc}
\left\{\begin{array}{c}\textit{algebraic groups } G/k \\ \textit{s.t. } G_{k'} \textit{ is diagonalizable}\end{array}\right\} & \longleftrightarrow & \left\{\begin{array}{c}\textit{diagonal groups } /k' \\ \textit{with semilinear } \mathrm{Gal}(k'/k)\textit{-action}\end{array}\right\} & \longleftrightarrow & \left\{\begin{array}{c}\textit{f.g. ab groups equipped} \\ \textit{with a } \mathrm{Gal}(k'/k)\textit{-action}\end{array}\right\} \\
G & \longmapsto & G_{k'} & \longmapsto & X(G_{k'})
\end{array}$$

Take a direct limit over all $k' \subset k^s$.

Corollary 14.13. *There exists an exact equivalence of categories*

$$\{\textit{groups of mult. type}/k\} \xrightarrow{\sim} \left\{\begin{array}{c}\textit{f.g. ab groups equipped w/} \\ \textit{a continuous } \mathrm{Gal}(k^s/k)\textit{-action}\end{array}\right\}$$

*via $G \mapsto X(G_{k^s}) =: X^*(G)$ (also called the **character group of** G).*

15 Lecture 15 (3/24)

15.1 Last time

Recall 15.1. G is a *torus* $\iff G_{\overline{k}} \simeq \mathbb{G}^n_{m,\overline{k}}$ for some $n \geq 0$.

G is *of multiplicative type* $\iff G_{\overline{k}} \simeq D(M)_{\overline{k}}$ for some f.g. abelian group M.

Inseparable descent allows one to replace $\overline{k}$ with k^s above.

Theorem 15.2. *Fix a field k. Then there is an exact equivalence of categories*

$$\{\text{groups of mult. type}/k\} \longleftrightarrow \{\text{f.g. abelian groups with cts. } \mathrm{Gal}(k^s/k)\text{-actions}\}$$

via $G \mapsto X^*(G)$. Here, $X^*(G) := \mathrm{Hom}(G_{k^s}, \mathbb{G}_{m,k^s})$ is the ***character group****, and $X_*(G) := \mathrm{Hom}(\mathbb{G}_{m,k^s}, G_{k^s}) = \mathrm{Hom}_{\mathbb{Z}}(X^*(G), \mathbb{Z})$ is the* ***cocharacter group****.*

Remark 15.3. Above, $\mathrm{Hom}_{\mathbb{Z}}(X^*(G), \mathbb{Z}) = \mathrm{Hom}_{\mathbb{Z}}(X^*(G), X^*(\mathbb{G}_{m,k^s}))$.

15.2 Representations of these

Theorem 15.4. *Let G be a group of multiplicative type over k. Then, there's an equivalence of categories*

$$\{k\text{-reps of } G\} \quad \longleftrightarrow \quad \left\{\begin{matrix} k\text{-vector spaces } V \text{ equipped with decomp} \\ V_{k^s} = \bigoplus_{\chi \in X^*(G)} V_\chi \text{ that is Gal-compatible} \end{matrix}\right\}$$

(each V_χ some k^s-vector space). Above being ***Galois-compatible*** *means that*

$$^{\sigma}(V_\chi) = V_{^\sigma\chi} \text{ for all } \sigma \in \mathrm{Gal}(k^s/k).$$

(think of the M-graded vector spaces from before).

Corollary 15.5.

$$\{\text{f.dim } k\text{-reps of } G\}_{/\simeq} \leftrightarrow \left\{\begin{matrix} \text{formal finite sums } \sum_{\chi \in X^*(G)} n_\chi \chi \\ \text{with } n_\chi \in \mathbb{Z}_+ \text{ and } n_{^\sigma\chi} = n_\chi \text{ for all } \chi \text{ and } \sigma \in \mathrm{Gal} \end{matrix}\right\}.$$

Corollary 15.6.

$$\{\text{irred. reps}\}_{/\simeq} \leftrightarrow \{\mathrm{Gal}\text{-orbits in } X^*(G)\}.$$

Definition 15.7. A linear algebraic group G is **linearly reductive** iff every f.dim rep is **semisimple** (i.e. a direct sum of irreps).

Warning 15.8. This is not quite the same as the definition of reductive we gave earlier.

Example. Any group of multiplicative type is linearly reductive (by theorem(s) at beginning of section)

Fact. If $\mathrm{char}\, k = 0$, G is linearly reductive $\iff G^0$ is reductive (i.e. unipotent radical trivial)

Fact. If $\mathrm{char}\, k = p$ and G smooth, then G linearly reductive $\iff G^0$ is a torus and $p \nmid (G : G^0)$.

Remark 15.9. If G is linearly reductive, any representation (f.dim or not) is a direct sum of irreps. Also, any short exact sequence of representations automatically splits.

Theorem 15.10 (Theorem 7.1 in the text)**.** *Let X be a variety, and let G be an algebraic group acting on X. Then,*

- *There exists a largest closed subscheme $X^G \subset X$ on which G acts trivially.*
- $X^G(R) = \{x \in X(R) : gx = x \text{ for all } g \in X(R') \text{ for all } R \to R'\}$
- *If $x \in X^G(k)$, then $T_x(X^G) = (T_xX)^G$.*
- *If G is smooth and $k = k^s$, then*
$$X^G = \bigcap_{g \in G(k^s)} X^g,$$
descended to k, where
$$X^g(R) = \{x \in X(R) : gx = x\}.$$

A priori this intersection is a k^s-scheme, but it's Gal-invariant so it descends

Theorem 15.11. *Let X be a smooth variety, and let G be a linearly reductive group acting on X. Then, X^G is smooth.*

Proof Sketch. (Affine case) First suppose X is the affine space associated to a representation V of G. Then, X^G is the affine space associated to V^G $(= V^{G(k)})$.

(General case) WLOG assume $k = \overline{k}$ since this doesn't affect smoothness or linear reductiveness. Choose some $x \in X^G(k)$. Get local ring $\mathscr{O}_{X,x} \supset \mathfrak{m}_x =: \mathfrak{m}$. Consider the short exact sequence

$$0 \to \mathfrak{m}^2 \to \mathfrak{m} \to \mathfrak{m}/\mathfrak{m}^2 \to 0.$$

Note that, since G fixes x, it acts on everything here. Every exact sequence of G-reps splits, so we get a G-equivariant splitting $\mathfrak{m}/\mathfrak{m}^2 \to \mathfrak{m}$. In particular, a basis $t_1, \dots, t_n \in \mathfrak{m}/\mathfrak{m}^2$ will lift to local coordinates $f_1, \dots, f_n \in \mathfrak{m}$.[23] These define a rational map

$$(f_1, \dots, f_n) : X \dashrightarrow \mathbb{A}^n$$

which is (defined and) étale at x. We also have G-actions on both sides for which this map is equivariant. Note that $(\mathbb{A}^n)^G$ is some affine subspace. Since $X \to \mathbb{A}^n$ is étale at x, we conclude that X^G is smooth at x.[24] ■

Question: Why is G acting on $\mathbb{A}^n$?

Answer: This $\mathbb{A}^n$ is given by the representation that is the span of those f_i's

Corollary 15.12. *Say $T \leq G$ with T of multiplicative type and G a smooth algebraic group. Then,* **(a)** *$C_G(T)$ and* **(b)** *$N_G(T)$, the centralizer and normalizer, are smooth.*

Proof. **(a)** $C_G(T) = G^T$ where T acts by conjugation, so win by thm (T linearly reductive).

(b) Define $N_G(T) \xrightarrow{\varphi} \underline{\mathrm{Aut}}(T)$ via $g \mapsto \mathrm{inn}_g|_T$. Since T is of multiplicative type, $\underline{\mathrm{Aut}}(T) = \underline{\mathrm{Aut}}(X^*(T))$ (think of as an étale group scheme), so $\{1\}$ is an open subgroup. Then, $\ker\varphi$ is open in $N_G(T)$. By definition $\ker\varphi = C_G(T)$. Since $C_G(T)$ is smooth, $N_G(T)$ is as well (by translation). ■

This also shows $C_G(T) \leq N_G(T)$ of finite index

[23] By Nakyama, they give a minimal set of generators for the maximal ideal

[24] Sounds like we're saying $X^G \to (\mathbb{A}^n)^G$ is étale at x, so $X^G \to (\mathbb{A}^n)^G \to \operatorname{spec} k$ is smooth at x

15.3 Limits

Suppose we have $\varphi : \mathbb{G}_m \to X$ with X a k-variety (in particular, X is separated). Since X is separated, there is at most one extension of φ to $\widehat{\varphi} : \mathbb{A}^1 \to X$. If such an extension exists, then we define the **limit**

$$\lim_{t \to 0} \varphi(t) := \widetilde{\varphi}(0) \in X(k).$$

Can make the same definition for R-schemes $\mathbb{G}_{m,R} \to X$ (with X a separated R-scheme).

Suppose that $\mathbb{G}_m$ acts on an affine variety $X = \operatorname{spec} B$, so $\mathbb{G}_m$ acts on B as well. Then we get a decomposition

$$B = \bigoplus_{n \in \mathbb{Z}} B_n$$

as $\mathbb{G}_m$-reps. Given, $f = \sum_n f_n \in B$ and $t \in \mathbb{G}_m$, then

$$t.f = \sum_n t^n f_n.$$

For $x \in X(R)$, TFAE

- $\lim\limits_{t \to 0} t \cdot x$ exists
- $\lim\limits_{t \to 0} f(t.x)$ exists for all $f \in B$. $f(t.x) = (t.f)(x)$
- $\lim\limits_{t \to 0} \sum_n t^n f_n(x)$ exists for all $f = \sum f_n \in B$.
- In the last bullet point, we now have some polynomial, so the condition is equivalent to: for all $n < 0$ and $f_n \in B_n$, $f_n(x) = 0$ (no negative exponents appearing in previous expression).

Note that this is a closed condition (given by common zero set of the $f \in B_{<0}$).

Corollary 15.13.

(1) *The functor*

$$F \mapsto \left\{ x \in X(R) : \lim_{t \to 0} t.x \ \textit{exists} \right\}$$

is represented by a closed subscheme P of X.

(2) *There's a morphism $P \to X^{\mathbb{G}_m} \subset X$ given by $x \mapsto \lim\limits_{t \to 0} t.x$.*

The usual set up for us using this will be the following. Let G be a linear algebrac group, and let $\lambda : \mathbb{G}_m \to G$ (homomorphism) be a **co-character/1 parameter subgroup**. Then we get a $\mathbb{G}_m$-action on G: $t.g := \lambda(t) g \lambda(t)^{-1}$. We can now define 3 groups:

- $P(\lambda) := \left\{ g \in G : \lim\limits_{t \to 0} t.g \ \text{ exists } \right\}$
- $U(\lambda) := \left\{ g \in G : \lim\limits_{t \to 0} t.g = e \right\}$
- $Z(\lambda) := \{ g \in G : t.g = g \ \text{ for all } \ t \in \mathbb{G}_m \} = C_G(\lambda(\mathbb{G}_m))$.

16 Lecture 16 (3/26)

Warning 16.1 (Correction to something from long ago)**.** Even over an algebraically closed field, G_{red} is not necessarily normal in G (Example: if $G = \alpha_{p^2} \rtimes \mathbb{G}_m$, then $G_{\mathrm{red}} = \mathbb{G}_m$). It is always preserved by conjugation by elements of $G(k)$, but not always by conjugation by elements of $G(R)$.

Remember: Apparently G_{red} doesn't even have to be a group if k is not perfect

16.1 Last time: limits

Let G be a linear algebraic group. Let $\lambda : \mathbb{G}_m \to G$ be a homomorphism. Get conjugation $\mathbb{G}_m$-action on G via $t.g := \lambda(t)g\lambda(t)^{-1}$.

Recall 16.2.

- $P(\lambda) := \left\{ g \in G : \lim_{t\to 0} t.g \text{ exists } \right\}$
- $U(\lambda) := \left\{ g \in G : \lim_{t\to 0} t.g = e \right\}$
- $Z(\lambda) := \{ g \in G : t.g = g \text{ for all } t \in \mathbb{G}_m \} = C_G(\lambda(\mathbb{G}_m))$.

16.2 Properties of $P(\lambda), U(\lambda), Z(\lambda)$

Example. Let $G = \mathrm{GL}_3$, and define $\lambda : \mathbb{G}_m \to G$ by $\lambda(t) := \begin{pmatrix} t^7 & & \\ & t^7 & \\ & & t^2 \end{pmatrix}$. Each $t \in \mathbb{G}_m$ acts on a matrix G by multiplying entries as follows

$$\begin{pmatrix} \cdot 1 & \cdot 1 & \cdot t^5 \\ \cdot 1 & \cdot 1 & \cdot t^5 \\ \cdot t^{-5} & \cdot t^{-5} & \cdot 1 \end{pmatrix}.$$

One sees that

$$P(\lambda) = \begin{pmatrix} * & * & * \\ * & * & * \\ & & * \end{pmatrix}, \quad Z(\lambda) = \begin{pmatrix} * & * & \\ * & * & \\ & & * \end{pmatrix}, \quad U(\lambda) = \begin{pmatrix} 1 & & * \\ & 1 & * \\ & & 1 \end{pmatrix}, \quad \text{and } U(-\lambda) = \begin{pmatrix} 1 & & \\ & 1 & \\ * & * & 1 \end{pmatrix}$$

(Above $-\lambda$ really means λ^{-1}, but we tend to think of characters additively).

Proposition 16.3. *Say G, λ as above. Then,*

(1) *$P(\lambda), U(\lambda), Z(\lambda)$ are closed subgroups of G. If G is smooth (resp. connected), then so are they.*

(2) *$P(\lambda) \cap P(-\lambda) = Z(\lambda)$*

(3) *$P(\lambda) = Z(\lambda) \ltimes U(\lambda)$*

(4) *$U(-\lambda) \times P(\lambda) \to G$ is an open immersion.*

(5) $U(\lambda)$ *is isomorphic to a subgroup of*

$$U_n := \begin{pmatrix} 1 & & * \\ & \ddots & \\ & & 1 \end{pmatrix}$$

(6) *The* $\mathbb{G}_m$*-action on* G *induces a* $\mathbb{G}_m$*-action on* $\operatorname{Lie} G = \bigoplus_{n\in\mathbb{Z}} \mathfrak{g}_n$*. Using this decomposition, one has*

$$\begin{aligned}
\operatorname{Lie} Z(\lambda) &= \mathfrak{g}_0 \\
\operatorname{Lie} U(\lambda) &= \bigoplus_{n\geq 1} \mathfrak{g}_n \\
\operatorname{Lie} P(\lambda) &= \bigoplus_{n\geq 0} \mathfrak{g}_n \\
\operatorname{Lie} U(-\lambda) &= \bigoplus_{n\leq -1} \mathfrak{g}_n
\end{aligned}$$

Proof. **(1)** One can check from the definition that these are group subfunctors of G. We showed last time (Corollary 15.13) that $P(\lambda)$ is a closed subscheme. We'll finish **(1)** later...

(2) If $x \in (P(\lambda)\cap P(-\lambda))(R)$, then $\mathbb{G}_{m,R} \to G_R, t \mapsto t.x$ extends to a morphism $\mathbb{P}^1_R \to G_R$. Note that $\mathbb{P}^1_R$ is projective (so proper) while G_R is affine. This factors through $\operatorname{spec} R$ (so is constant), so the original map on $\mathbb{G}_{m,R}$ also factors through $\operatorname{spec} R$. Hence, $x \in Z(\lambda)(R)$. This proves **(2)**.

(3) Consider the homomorphism

$$\begin{aligned}
P(\lambda) &\longrightarrow Z(\lambda) \\
x &\longmapsto \lim_{t\to 0} t.x
\end{aligned}$$

The inclusion $Z(\lambda) \hookrightarrow P(\lambda)$ is a section of this homomorphism, so we get a split exact sequence

The map below is faithfully flat since it's split. It's surjective on R-valued points for every R so a epimorphism of fppf sheaves

$$1 \to U(\lambda) \to P(\lambda) \to Z(\lambda) \to 1.$$

This tells us that $U(\lambda)$ is a closed subgroup, and that we have a semi-direct product. Hence we finish **(3)** and the first part of **(1)**.

Remark 16.4. If $G \hookrightarrow G'$, then $P_G(\lambda) = P_{G'}(\lambda) \cap G$.

Pick a faithful representation $G \hookrightarrow \mathrm{GL}_V$, and diagonal the action of $\mathbb{G}_m$ on V given by $\mathbb{G}_m \xrightarrow{\lambda} G \hookrightarrow \mathrm{GL}_V$, with weights in decreasing order. By explicit calculation (as in the example), **(4)**,**(5)** hold for GL_V, so they hold also for any subgroup of GL_V (e.g. V).

If G is smooth, then $U(-\lambda)\times P(\lambda)$ is smooth **(4)**, so $U(-\lambda), P(\lambda)$ are smooth. Also, $P(\lambda) = Z(\lambda)\times U(\lambda)$ as varieties. Since $P(\lambda)$ is smooth, $Z(\lambda)$ is too. If G is connected, go through the same argument. This finishes **(1)**.

(6) For $g \in \mathfrak{g} \subset G(k[\varepsilon]/(\varepsilon^2))$, how do we get $g \in \operatorname{Lie} P(\lambda)$? By definition,

$$g \in \operatorname{Lie} P(\lambda) \iff \lim_{t\to 0} t.c \text{ exists } \text{ in } G(k[\varepsilon]) \iff g \in \bigoplus_{n\geq 0} \mathfrak{g}.$$

The rest of the cases are done similarly. ∎

16.3 Unipotent Groups and Representations

Recall 16.5 (Theorem 4.14 in the book)**.** Let G be a linear algebraic group, and let V be a faithful f.dim rep. Then, every f.dim rep of G can be obtained from V by taking $\otimes, \oplus, (-)^\vee$, subreps, quotients.

Sounds like every rep generator V like this is automatically faithful, so the existence of such a thing gives a check for a neutral Tannakian category to be associated to an algebraic group

Definition 16.6. Let V be a f.dim G-rep (G = linear algebraic group). We say V is a **unipotent representation** if $\exists$ subreps

$$0 = V_0 \subset V_1 \subset \cdots \subset V_n = V \text{ with } V_i/V_{i-1} \simeq \mathbf{1} \text{ for all } i,$$

i.e. the successive quotients are the trivial reps. Equivalently, all Jordan-Hölder factors for this rep are trivial.

If you choose a basis compatible with the filtration above, then every element of g acts by an upper triangular matrix with 1's along the diagonal.

Remark 16.7. Unipotent representations are preserved by all of the operations in Recall 16.5.

Recall 16.8. We say a linear algebraic group G is *unipotent* if every nonzero representation has a nonzero fixed vector.

Theorem 16.9. *For a linear algebraic group G, TFAE*

(1) *G is unipotent.*

(2) *Every f.dim representation of G is unipotent.*

(3) *G has some faithful unipotent representation, i.e. G is isomorphic to a closed subgroup of U_n.*

(4) *(When G smooth) Every $g \in G(\overline{k})$ is unipotent.*

Proof of some parts. (**(3)** $\implies$ **(2)**) Uses Recall 16.5. Every rep of G is built from a unipotent one, so they're all unipotent.

(**(4)** $\implies$ **(1)**) WLOG $k = \overline{k}$ (since $\dim V^G$ unchanged by field extension). Let V be a nonzero representation. WLOG V is irreducible. We may replace G by its image in GL_V. We now make use of the following:

Theorem 16.10 (**Burnside's theorem**)**.** *If V is an irrep over $k = \overline{k}$, then the elements $h \in G(k)$ span* $\operatorname{End} V$ *as a k-vector space.*

If $g \in G(k)$, then $\mathrm{Tr}(g) = 1 + 1 + \cdots + 1 = n := \dim V$ since it is unipotent. Hence, $\mathrm{Tr}((1-g)h) = \mathrm{Tr}(h) - \mathrm{Tr}(gh) = n - n = 0$ for all $h \in G(k)$. Thus, Burnside tells us that $\mathrm{Tr}((1-g)a) = 0$ for all $a \in \operatorname{End} V$. Thus forces $1 - g = 0$, so $g = 1$. Thus, after all our reductions, we arrived at $G = \{1\}$ (since G smooth and $k = \overline{k}$, $G(k) = 1 \implies G = 1$) which almost certainly has a nonzero fixed vector. ∎

Corollary 16.11. *The property "G is unipotent" is unchanged by field extension (e.g. by **(3)** + **(1)**).*

Corollary 16.12. *Subgroups and quotient groups of unipotent groups are unipotent (e.g. by **(1)** + **(3)**)*

Example. $\mathbb{G}_a = U_2 = \begin{pmatrix} 1 & * \\ & 1 \end{pmatrix}$ is unipotent.

Example. If $\operatorname{char} k = p$, then $\underline{\mathbb{Z}/p\mathbb{Z}}, \alpha_p, \mathbb{W}_2$ (the **Witt ring scheme** with $\mathbb{W}_2(R)$ being the length 2 Witt vectors in R) are all unipotent.

17 Lecture 17 (3/29)

Last time, we talked about unipotent groups.

Recall 17.1. For a linear algebraic group G, TFAE

(1) G is *unipotent*, i.e. every nonzero rep has a nonzero fixed vector

(2) Every f.d. rep is *unipotent*, i.e. all Jordan-Hölder factors are $\mathbf{1}$ (trivial rep)

(3) G has a *faithful unipotent* representation, i.e. $G \hookrightarrow U_n$

(4) (when G smooth) every $g \in G(\overline{k})$ is unipotent (in the sense of Jordan decomposition)

Corollary 17.2. *The property "G is unipotent" is unchanged by field extension. Also subgroups, quotients, and extensions of unipotent groups are unipotent.*

Example. U_n, powers of $\mathbb{G}_a$, and (in char p) $\mathbb{Z}/p\mathbb{Z}$, α_{p^n},[25] $\mathbb{W}_n$ are all unipotent groups.

17.1 More on unipotent groups

Proposition 17.3. *If G is unipotent (in particular, linear algebraic) and of multiplicative type, then $G = \{1\}$.*

Proof. WLOG $k = \overline{k}$. Since it's diagonalizable, every representation is a direct sum of 1-dim reps. Since it is unipotent, each 1-dim rep is trivial, so every rep of G is trivial. Hence, $G = \{1\}$. ∎

Corollary 17.4. *Any homomorphism $U \to M$ or $M \to U$ (with U unipotent and M of multiplicative type) is trivial.*

Proof. The image is a quotient/subgroup of a unipotent/multiplicative type group so it is both unipotent and of multiplicative type. ∎

Proposition 17.5. *Let G be a smooth, connected unipotent group over a perfect field k. Then, G has a filtration (G_i) with quotients $G_{i-1}/G_i \simeq \mathbb{G}_a$. Moreover, it can be chosen to be a* ***central series****, i.e. $G_{i=1}/G_i \subset Z(G/G_i)$ for all i.*

Proof. Embed $G \hookrightarrow U_n$ (upper triangular matrices with 1's along the diagonal). We know (some homework) that U_n has a central series with quotients $\mathbb{G}_a$. Intersecting with G gives an induced filtration

$$G = G_0 \supset G_1 \supset \cdots \supset G_n = \{1\}$$

with $G_{i-1}/G_i \leq \mathbb{G}_a$. Since k is perfect, we can form $H_i = (G_i^0)_{\mathrm{red}}$ which is now smooth and connected.[26] This gives a new filtration

If A is normal in B, then A_{red} is normal in B_{red} (consider the conjugation action morphism)

$$G = H_0 \supset H_1 \supset \cdots \supset H_n = \{1\}.$$

We still have $H_i \triangleleft G$, and one can verify that $H_{i-1}/H_i \subset Z(G/H_i)$, and H_{i-1}/H_i is smooth, connected (since H_{i-1} is) and unipotent of dimension ≤ 1. You can classify all smooth, connected, unipotent groups of dimension ≤ 1 (homework); the only ones are $\mathbb{G}_a$ or $\{1\}$. ∎

[25] $\mathbb{W}_n$ is length n Witt vectors, e.g. $\mathbb{W}_n(\mathbb{F}_p) = \mathbb{Z}/p^n\mathbb{Z}$

[26] Want to avoid $G_{i-1}/G_i = \alpha_p$ or $\mathbb{Z}/p\mathbb{Z}$ or something like that

Definition 17.6. We say G is **split unipotent** if it has a filtration whose successive quotients are all $\mathbb{G}_a$.

Remark 17.7. If $\operatorname{char} k = 0$, things are even better.

- $\{$unipotent algebraic groups$/k\} \longleftrightarrow \{$f.dim nilpotent Lie algebra$/k\}$ is an equiv of cats.

  Remark 17.8. Cartier's theorem says algebraic groups in characteristic 0 are automatically smooth. Unipotent algebraic groups are also automatically connected (since $\pi_0(G)$ is finite and unipotent in characteristic 0)

  This equivalence is given by $G \mapsto \operatorname{Lie} G$ in one direction. In the other direction it sends (affine space assoc. to $\mathfrak{g}$)$\leftarrow\!\shortmid\ \mathfrak{g}$ with group law given by the Baker-Campbell-Hausdorff formula.[27]

- $\{$commutative unipotent alg. gps.$/k\} \longleftrightarrow \{$f.dim vector spaces$/k\}$ is an equiv of cats. The equivalence is given by $\mathbb{G}_a^n \leftrightarrow k^n$.

  Non-example. In char p, the Witt group scheme $\mathbb{W}_n$ is commutative but not $\mathbb{G}_a^n$ (it's not even p-torsion).

17.2 Commutative (linear) algebraic groups

Lemma 17.9. *Let $k = \overline{k}$, and let V be a f.dim k-vector space. Let $S \subset \operatorname{End} V$ be a set of pairwise commuting endomorphisms. There, there exists a basis of V w.r.t which every $s \in S$ is upper triangular, and also every semisimple $s \in S$ is diagonal.*

Proof. If all $s \in S$ act as scalars, we're done.

If there is some semisimple non-scalar $s \in S$, then any other $t \in S$ must preserve the eigenspaces of s (which span all of V since s semisimple). Then apply induction to these eigenspaces.

If there is a non-scalar s, choose one eigenspace W and apply induction to both W and V/W. This gives something block upper-triangular with upper triangular blocks on the diagonal (so it is upper triangular). ■

Theorem 17.10. *Let k be perfect. Let G be a commutative, smooth linear algebraic group over k. Then,*

Can relax this assumption

- ***(1)*** *The semisimple elements in $G(\overline{k})$ are the $\overline{k}$-points of a unique, smooth, closed subgroup $G_s \leq G$ of multiplicative type.*
- ***(2)*** *The unipotent elements in $G(\overline{k})$ are the $\overline{k}$-points of a unique, smooth, unipotent, closed subgroup $G_u \leq G$.*
- ***(3)*** *Multiplication gives an isomorphism $G_s \times G_u \xrightarrow{\sim} G$.*
- ***(4)*** *If G is connected, then G_s, G_u are connected too.*

Proof. WLOG $k = \overline{k}$ (uniqueness let's us apply Galois descent). Choose a faithful representation $G \hookrightarrow \mathrm{GL}_V$. Choose a basis of V as in Lemma 17.9 (so semisimple elements are all diagonal). We define $G_s := (G \cap T_n)_{\mathrm{red}}$ (where T_n is the diagonal torus). It is a group of multiplicative type (contained in T_n)

[27] If $\mathfrak{g} \leq \mathfrak{gl}_n$, then $G(k) = \{\exp(v) : v \in \mathfrak{g}\}$ and the only question is how to multiply two exponentials

which is smooth and closed with the right $\overline{k}$-points (by definition[28]), so this gives **(1)**. For **(2)**, similarly define $G_u := (G \cap U_n)_{\text{red}}$.

(3) For injectivity, the kernel is $G_s \cap G_u \subset T_n \cap U_n = \{1\}$. Let's do surjectivity first on k-points. This is true by Jordan decomposition (every matrix is a product of something semisimple and something unipotent). Since G is smooth (and $k = \overline{k}$), surjectivity on k-points is enough to conclude that $G_s \times G_u \to G$ is faithfully flat.

(4) The maps $G \simeq G_s \times G_u \twoheadrightarrow G_s, G_u$ show that G connected $\implies G_s, G_u$ connected. ■

Remark 17.11. Every smooth connected *nilpotent* algebraic group G over a perfect field k also has the form $U \times T$ with U unipotent and T a torus.

(Compare: A finite group is nilpotent $\iff$ it is a product of its Sylow p-groups)

17.3 Trigonalizable

We're skipping chapter 15 and going to chapter 16.

"I never heard [the word 'trigonalizable'] before I saw it in this book" (paraphrase)

Principal Example.

$$B_n := \left\{ \begin{pmatrix} * & & * \\ & \ddots & \\ & & * \end{pmatrix} \right\} \subset \mathrm{GL}_n$$

is the subgroup of (invertible) upper triangular matrices.

Definition 17.12. A f.dim rep V is **trigonalizable** $\iff$ $G \to \mathrm{GL}_V$ has image in B_n, for a suitable choice of basis. Equivalently, there exists a full flag

$$0 = V_0 \subset \cdots \subset V_n = V$$

of subreps with $\dim V_i = i$. Equivalently, the Jordan-Hölder factors of the representation V are all 1-dimensional.[29]

Theorem 17.13. *For a linear algebraic group G, TFAE*

(1) *Every irrep of G is 1-dimensional.*

(2) *Every f.dim rep of G is trigonalizable.*

(3) *G has a faithful trigonalizable representation, i.e. G is isomorphic to a subgroup of some B_n.*

(4) *There exists an exact sequence*

$$1 \longrightarrow U \longrightarrow G \longrightarrow D \longrightarrow 1$$

with U unipotent and D diagonalizable.

*If any (all) of these equivalent conditions hold, we call G **trigonalizable**.*

[28] Taking the reduction doesn't affect $\overline{k}$-points since $\overline{k}$ is itself reduced

[29] The entries along the diagonal give characters of G. These are the Jordan-Hölder factors which will appear

Proof. **(1)** $\iff$ **(2)** $\iff$ **(3)** Same proof as for unipotent representations (using Recall 16.5).

(3) $\implies$ **(4)** Take the standard filtration $B_n \rhd U_n \rhd 1$ (with quotients T_n and U_n), and intersect it with G.

(4) $\implies$ **(1)** Let V be an irrep of G. We want to show $\dim V = 1$. Since $U \leq G$ is unipotent, $V^U \neq 0$. At the same time, V^U is a nonzero rep of $G/U = D$. Since D is diagonalizable, V^U is a direct sum of characters, each of which can be viewed as a 1-dim rep W of G. Since V is irreducible, we conclude $W = V$, so $\dim V = 1$. ■

18 Lecture 18 (3/31)

Always check Bjorn's website for updates in office hours.

18.1 Last time: trigonalizable groups

Keep in mind the exact sequence

$$1 \longrightarrow U_n \longrightarrow B_n \longrightarrow T_n \longrightarrow 1.$$

Recall 18.1. A representation V is *trigonalizable* if any of the following equivalent conditions hold:

(1) $G \to \mathrm{GL}_V$ has image in B_n for a suitable basis of V

(2) there is a full flag of G-equivariant subspace $0 = V_0 < \cdots < V_n = V$

(3) the Jordan-Hölder factors of the rep V are all 1-dimensional

Recall 18.2. A linear algebraic group G is *trigonalizable* if any of the following equiv conditions hold

(1) Every irrep of G is 1-dimensional

(2) every f.dim rep of G is trigonalizable

(3) G has a faithful trigonalizable rep, i.e. $G \hookrightarrow B_n$ for some n

(4) there is an exact sequence $1 \to U \to G \to D \to 1$ with U unipotent and D diagonalizable

Remark 18.3. In **(4)** above, *every* unipotent subgroup U' of G will be contained in U. This is simply because the composition $U' \hookrightarrow G \twoheadrightarrow D$ is trivial (mapping from unipotent to multiplicative type, Corollary 17.4).

Notation 18.4. $G_U := U$ as in **(4)**

18.2 Splittings

Theorem 18.5. *Say* $k = \overline{k}$.

(1) *Any exact sequence*

$$1 \longrightarrow U \longrightarrow G \longrightarrow D \longrightarrow 1$$

with U *unipotent and* D *diagonalizable splits, so* $G \simeq U \rtimes D$.

(2) *Any two splittings are conjugate an element $u \in U(k)$.*

(3) *The maximal diagonalizable subgroups of G are the groups $s(D)$ as s varies over the splittings.*

These first two are equivalent to the vanishing of some non-abelian cohomology group, I think. This would be true if these were abstract groups; probably still true here?

Compare the above to the following theorem in group theory.

Theorem 18.6 (**Schur-Zassenhaus**).

(1) *Any exact sequence*

$$1 \longrightarrow N \longrightarrow G \longrightarrow Q \longrightarrow 1$$

of finite groups with $\gcd(|N|, |Q|) = 1$ *(and N or Q solvable*[30]*) splits.*

(2) *Any two splittings are conjugate by an element of N.*

(3) *The maximal subgroups of G or order dividing $|Q|$ are the groups $s(Q)$.*

In some sense, unipotent and diagonalizable groups are relatively prime; "one only divisible by $\mathbb{G}_a$'s and the other only divisible by $\mathbb{G}_m$'s."

Proof Sketch of Theorem 18.6, when N solvable. **(a)** We first do the case that N is abelian. The obstruction for a set-theoretic splitting to be a homomorphism is a 2-cocycle, so the class of the obstruction lives in $\mathrm{H}^2(Q, N)$. This group is killed by $|Q|$ and by $|N|$, so it must be trivial. Similarly, splittings up to conjugacy are in bijection the cohomology group $\mathrm{H}^1(Q, N)$ which vanishes for the same reason.

Remember: Extensions of Q by N are classified by $\mathrm{H}^2(Q, N)$

(b) If N is solvable, the last nontrivial term in its derived series is an abelian group $A \triangleleft G$. Apply induction to G/A to reduce to case **(a)**. ■

The proof of Theorem 18.5 is completely analogous to this. To find the right analogue for group cohomology, see chapter 15 of the book (we skipped this).

Remark 18.7. If $|Q| = p^n$, then S-Z is much easier: choosing a splitting in this case is the same as choosing a Sylow p-subgroup of G.

Question 18.8. *Do any of the proofs of the Sylow theorems adapt to alg. groups, to give a "Sylow $\mathbb{G}_m$-subgroup"?*

That is, can you adapt of proof of Sylow to construct the splitting in Theorem 18.5?

Remark 18.9. Sounds like Bjorn was able to make one of them work at least if D is a torus. The tricky part is that many proofs of Sylow use counting mod p, so how does one make sense of "counting mod $\mathbb{G}_m$"? Apparently, can reduce to finite field case and then count points mod q^n (or $q-1$ or $q^n - 1$; I don't remember what he said) to get things to work out. However, less clear about what to do in "general case."

[30]This extra condition was in the original version of the theorem. However, this is automatic by the **Feit-Thompson Theorem** which says any group of odd order is solvable

18.3 Solvable groups

Keep in mind that "solvable" implicitly implies "linear" (e.g. elliptic curves are not "solvable"). This is just a convention.

Proposition 18.10. *Let G be a smooth, connected solvable group over $k = \overline{k}$. Then, G has a filtration with quotients isomorphic to $\mathbb{G}_a$ or $\mathbb{G}_m$.*

Proof. The derived series has smooth, connected commutative quotients. Every smooth connected commutative quotient in this case is of the form $T \times U$ with T a torus and U unipotent. Since $T = \mathbb{G}_m^n$ and U has a filtration with quotients $\mathbb{G}_a$, we can refine to win. ■

Definition 18.11. Let k be a potentially non-algebraically closed field. A **split solvable group** is an algebraic group G with a filtration whose quotients are isomorphic to $\mathbb{G}_a$ or $\mathbb{G}_m$ (over the ground field k).

Theorem 18.12 (**Borel fixed point theorem**)**.** *Let G be a split solvable group. Let X be a proper k-variety with a k-point $x \in X(k)$. Say G acts on X. Then, there exists a G-fixed point $z \in X(k)$.*

Proof. As usual, we induct on the number of steps in the filtration (or equivalently on the dimension of G).

(1) If $G = \mathbb{G}_a$ or $G = \mathbb{G}_m$, let $z = \lim_{t \to \infty} t.x$. This limit exists since X is proper; the valuative criterion tells us that $\mathbb{G}_m \to X$ or $\mathbb{G}_a \to X$ extends to $\mathbb{P}^1 \to X$. This limit will be fixed.

(2) If $N \triangleleft G$ is such that N and G/N are split solvable groups for which the result holds, then it also holds for G. Indeed, N acts on X, so there is an N-fixed point y. Now G/N acts on $\overline{G.y}$ which is proper, so there is a G/N-fixed point $z \in \overline{G.y} \subset X$. Hence, z is fixed by G as well. ■

Corollary 18.13 (**Lie-Kolchin theorem**)**.** *If G is a split solvable group, then G is trigonalizable.*

Proof. Let V be a f.dim rep of G. We will show that V is trigonalizable. Let X be the **flag variety** of V, so $X(k) = \{\text{full flags in } V\}$. This variety is projective (so proper) since it embeds $X \hookrightarrow \prod_{m=1}^{\dim V} \mathrm{Gr}(m, V)$. Thus, Borel says G fixes a full flag in V, so V is trigonalizable, so we win (since V arbitrary). ■

Corollary 18.14. *For a smooth, connected algebraic group over $k = \overline{k}$, solvable = trigonalizable.*

18.4 Borel subgroups

We've sort of done the easy part so far. In the next couple of weeks, we'll move onto the hard part. We've been talking about solvable groups, but let's switch to more general smooth connected linear algebraic groups.

Assumption. Always assume G is a smooth connected linear algebraic group over an arbitrary field k. We won't keep repeating this from now on.

Definition 18.15.

(1) If $k = \overline{k}$, a **Borel subgroup** of G is a maximal [smooth connected *solvable* subgroup of G].

(2) For general k, $B \leq G$ is **Borel** if $B_{\overline{k}} \leq G_{\overline{k}}$ is Borel.

Warning 18.16. Borel subgroups do not always exist over k.

Example. The upper triangular matrices $B_n \leq \mathrm{GL}_n$ is a Borel subgroup. (Every smooth connected solvable group over $k = \overline{k}$ is trigonalizable).

Note that conjugating B_n will give another Borel subgroup (and all of them arise in this way).

Next time we'll define parabolic subgroups.

19 Lecture 19 (4/2)

Last time talked about trigonalizable and solvable groups.

Recall 19.1 (Lie-Kolchin theorem)**.** If G is a split solvable group, then G is trigonalizable ($G \hookrightarrow B_n$).

Corollary 19.2. *For a smooth, connected group G over $k = \overline{k}$, solvable = trigonalizable*

Recall 19.3 (Borel fixed point theorem)**.** If a split solvable group G acts on a proper k-variety X with a k-point x, then there exists a G-fixed point in $X(k)$.

Note 3. We're halfway through the class

19.1 Borel subgroups

Assumption. Always assume G is a smooth connected linear algebraic group over an arbitrary field k. We won't keep repeating this from now on.

Recall 19.4.

(1) If $k = \overline{k}$, a *Borel subgroup* of G is a maximal [smooth connected *solvable* subgroup of G].

(2) For general k, $B \leq G$ is *Borel* if $B_{\overline{k}} \leq G_{\overline{k}}$ is Borel.

Remark 19.5. The *radical* $R(G)$ is the maximal *normal* smooth connected solvable subgroup, so the radical is contained in any Borel subgroup.

Definition 19.6. A smooth subgroup $P \leq G$ is **parabolic** if the k-variety G/P is proper (we'll late see such P are automatically connected).

Remark 19.7. Borel and parabolic subgroups are usually *not* normal.

Example. Let GL_n acts on $\mathbb{P}^{n-1}$, and set $P = \mathrm{Stab}(1:0:0:\cdots:0)$, so elements of P are block upper triangular

$$\begin{pmatrix} a & b \\ & D \end{pmatrix}$$

with a a 1×1 matrix, b a $1 \times (n-1)$ matrix, and D a $(n-1) \times (n-1)$ matrix. Note that $G/P \simeq$ orbit $= \mathbb{P}^{n-1}$, so P is parabolic.

Can do something similar by setting $P = \mathrm{Stab}_{\mathrm{GL}_V}(W)$ for some $W \leq V$. Even more generally, could look at the stabilizer of some flag.

Remark 19.8. Let G be a smooth connected subgroup over $k = \overline{k}$...

TODO: Add the picture from Bjorn's notes

Remark 19.9. Not every subgroup of G is solvable or parabolic. For example a product of a solvable group with a parabolic groups is probably neither.

Theorem 19.10. ***(a)*** *If $B \leq G$ is a Borel subgroup, then G/B is proper.*

(Borel $\implies$ parabolic)

(b) *If $k = \overline{k}$, then all Borel subgroups are $G(k)$-conjugate.*

(c) *If $k = \overline{k}$, a smooth subgroup $P \leq G$ is parabolic $\iff$ P contains some Borel subgroup.*

Proof. WLOG $k = \overline{k}$ since all properties here unchanged by field extension.

(Step 1) For any solvable subgroup $B \leq G$, there exists a representation V of G and a full flag F in V, i.e. a k-point of the flag variety X (parametrizing full flags), such that $B = \mathrm{Stab}_G(F)$.

Proof. By Chevalley's theorem, there's a rep V of G and a 1-dim subspace $L \leq V$ s.t. $B = \mathrm{Stab}\, L$. View V/L as a rep of B. Since B is solvable, Lie-Kolchin implies that it stabilizers some full flag in V/L. Taking inverse images in V gives a full flag – $F : 0 \leq V_1 \leq V_2 \leq \cdots \leq V_n = V$ with $V_1 = L$ – with $B \subset \mathrm{Stab}(F) \subset \mathrm{Stab}(L) = B$, so $B = \mathrm{Stab}(F)$. ■

(Step 2) If B is a Borel of maximum dimension, G/B is proper.

Proof. Choose V, F as in step 1, so F a full flag with $\mathrm{Stab}_G(F) = B$. For any other flat $F' \in X(k)$, $\mathrm{Stab}(F')$ is trigonalizable $=$ solvable, so $\dim \mathrm{Stab}(F') \leq \dim \mathrm{Stab}(F) = \dim B$. Orbit stabilizer then tells us that $\dim \mathrm{orbit}(F') \geq \dim \mathrm{orbit}(F)$, so the orbit of F is the orbit of smallest dimension! Our old friend Borel's Orbit Lemma (Theorem 3.7) now shows that $G/B \simeq \mathrm{orbit}(F) \hookrightarrow X$ is a closed subvariety, so proper (projective even!) since the flag variety X is. ■

(Step 3) If B' is any other Borel, then $B' = gBg^{-1}$ for some $g \in G(k)$.

Proof. Let B' act by left multiplication on G/B. By Borel fixed point theorem, there is a B'-fixed point $x = gB$ for some $g \in G(k)$. Then, $B' \leq \mathrm{Stab}_G(x) = gBg^{-1}$. Since gBg^{-1} is another Borel (automorphisms preserve Borels), this inclusion must be an equality. ■

(b) Follows from step 3. **(a)** follows from steps 2 + 3. This just leaves...

(c) ($\rightarrow$) If P is parabolic, argue as in step 3 to get $B' \subset gPg^{-1}$ for some g, so $P \supset g^{-1}B'g$ which is a Borel. ($\leftarrow$) If P contains a Borel subgroup B', then $G/B' \twoheadrightarrow G/P$ with G/B' proper, so G/P is proper as well. ■

Remark 19.11. Above, we could replace proper with projective.

Remember: (Quotients of) algebraic groups are always quasi-projective (Theorem 9.3)

Theorem 19.12. *Say $k = \overline{k}$. All maximal tori in G are $G(k)$-conjugate.*

Proof. Every maximal torus T of G is solvable, so also a maximal torus of some Borel B. Since all Borels are conjugate, it suffices to show that all maximal tori within a given B are $B(k)$-conjugate. This follows from the algebraic group analogue of Schur-Zassenhaus (maximal tori in B are the same as splittings of the relevant exact sequence). ■

This proof shows a little more: it shows the pairs (B, T) of a Borel with a maximal torus inside of it are all conjugate.

Definition 19.13. For now, if $k \neq \overline{k}$, a k-torus in G is called **maximal** if $T_{\overline{k}}$ is maximal among $\overline{k}$-tori in G.

(Later: above $\iff$ maximal among k-tori). From now on,

Assumption. $G \geq B \geq T$ with

- G a smooth connected linear algebraic group
- B a (fixed) Borel in G
- T a (fixed) maximal torus

We'll fix this notation for the next few lectures. Upcoming is an "army of lemmas."

Lemma 19.14. *Let $\mathcal{C} := C_G(B)$ be the centralizer of B. Then, $Z(B) \subset \mathcal{C} = Z(G)$.*

Proof. The only nontrivial piece above is $\mathcal{C} \subset Z(G)$. Consider the commutator map $\mathcal{C} \times G \xrightarrow{[-,-]} G$. First note that this in fact factors through a morphism $\mathcal{C} \times G/B \to G$ since $\mathcal{C}$ centralizes B. Now G/B is proper (and smooth, connected) while G is affine, so any map $G/B \to G$ is constant. From this, one can show that $\mathcal{C} \times G/B \to G$ factors through a map $\mathcal{C} \to G$. At the same time $(c, 1) \mapsto 1$, so this map $\mathcal{C} \to G$ must be trivial, so we win. ■

Rigidity lemma? I guess even simpler than that since the target is affine. All the restrictions $\{c\} \times (G/B) \to G$ (for c a scheme-theoretic point) must be constant since mapping from proper to affine

Lemma 19.15. *TFAE*

(1) *$G_{\overline{k}}$ has only one maximal torus*

(2) *B is nilpotent*

(3) *G is nilpotent*

(4) *$T \leq Z(G)$.*

Moreover, when these hold, $G = B$.

*Proof of **(1)** $\implies$ **(2)**.* WLOG $k = \overline{k}$. If $g \in G(k)$, then $gTg^{-1} = T$. Thus, $N_G(T)$ contains all k-points of G, so $N_G(T) = G$, so $T \triangleleft G$. We have $B = U \rtimes T$; since T is normal, we conclude $B = U \times T$ which is nilpotent. ■

20 Lecture 20 (4/5)

20.1 Last time: Borel subgroups

Recall 20.1. A *Borel subgroup* over $k = \overline{k}$ is a maximal smooth, connected solvable subgroup. We've fixed a choice

$$\underbrace{G}_{\text{smooth connected linear}} \geq \underbrace{B}_{\text{Borel}} \geq \underbrace{T}_{\text{maximal torus}}$$

Recall 20.2. We've shown the following.

- $B = U \rtimes T$ for some smooth connected unipotent $U \triangleleft B$ (if $k = \overline{k}$)

  (consequence of the structure of solvable/trigonalizable groups)

- G/B is proper ('B is parabolic')

- All pairs (B, T) are $G(k)$-conjugate (if $k = \overline{k}$)

- $Z(B) \leq C_G(B) = Z(G)$ (we'll later show $Z(B) = Z(G)$)

- For any torus $S \leq G$, then centralizer $C_G(S)$ is smooth, and $N_G(S)/C_G(S)$ is finite étale.

Question: When did we prove this last bullet point?

Answer: Fixed point of torus action always smooth (Corollary 15.12), so centralizer smooth. The normalizer acts on S and so on the discrete, f.g. character group. Hence its action has finite image/this quotient is finite. Argument is something like this

20.2 More about G, B, T

We were in the middle of proving the following criteria for nilpotency last time.

Lemma 20.3. *TFAE*

(1) $G_{\overline{k}}$ *has only one maximal torus*

(2) B *is nilpotent*

(3) G *is nilpotent*

(4) $T \leq Z(G)$.

Moreover, when these hold, $G = B$.

Proof. We showed **(1)** $\implies$ **(2)** at the end of last lecture. Recall we may assume $k = \overline{k}$.

((**2**) $\implies$ (**3**)) We'll prove $G = B$ by induction on $\dim B$. If $\dim B = 0$, then $G = G/B$ is affine and proper, so $G = \{1\}$. Now suppose $\dim B > 0$. Then, $\dim Z(B) > 0$ (last step in central series). We know $Z(B) \leq Z(G)$ (Lemma 19.14) and that $B/Z(B)$ is Borel in $G/Z(B)$. Now, $B/Z(B)$ is nilpotent of smaller dimension, so $B/Z(B) = G/Z(B)$ by inductive hypothesis. Hence, $B = G$.

((**3**) $\implies$ (**4**)) Now assume G nilpotent, so $G = U \times T$ with U unipotent and T a maximal torus. T commutes with U since it is a direct product, and also T commutes with T since it's commutative, so $T \leq Z(G)$.

Prove this using induction on the central series.

((**4**) $\implies$ (**1**)) The maximal tori are all conjugate to each other, but conjugation does nothing since $T \leq Z(G)$, so there's only one maximal torus. ∎

Corollary 20.4. *If the only smooth connected unipotent subgroup is* $\{1\}$*, then* G *is a torus.*

Proof. In general, $B = U \rtimes T$ is a semidirect product, but U trivial by assumption, so $B = T$. Then, B is nilpotent, so $T = B = G$ by above. ∎

Corollary 20.5. *If* $\dim G \leq 2$*, then* G *is solvable.*

Proof. If $\dim B \leq 1$, then B is commutative so nilpotent ($B = \mathbb{G}_a$ or $B = \mathbb{G}_m$ or $B = \{1\}$). So G is nilpotent, hence solvable. If $\dim B = 2$, then $G = B$ and B is solvable. ∎

Non-example. This is false in higher dimensions. e.g. SL_2 is non-solvable of dimension 3.

Theorem 20.6. *Let* $S \leq G$ *be a torus. Then,* $C_G(S)$ *is smooth and connected.*

Proof. We've seen before that it's smooth. WLOG $k = \overline{k}$.

- If $S = \mathbb{G}_m$, let $\lambda : \mathbb{G}_m \hookrightarrow G$ be the inclusion. Then, $C_G(S) = Z_G(\lambda)$ which we showed before was connected.
- If $S = S_1 \times S_2 \times \ldots \times S_n$ with $S_i \simeq \mathbb{G}_m$. Consider $G \geq G_1 \geq G_2 \geq \cdots \geq G_n = C_G(S)$ with $G_1 = C_G(S_1)$ and $G_{i+1} = C_{G_i}(S_{i+1})$. There are smooth connected all the way down by induction.

■

Definition 20.7. Let $C := C_G(T)$. We call this a **Cartan subgroup**. Also let $N := N_G(T)$.

Proposition 20.8. *C is nilpotent with unique maximal torus T.*

Proof. $T \leq Z(C)$ by definition of C. Apply **(4)** $\implies$ **(3)** of theorem at beginning of section to C. ■

Proposition 20.9. *We have the following general picture (i.e. below inclusions and inequalities):*

$$\begin{tikzcd}
G \arrow[d, dash] & \\
B = N(B) \arrow[dd, dash] & \\
& N = N(C) \\
C = N^0 \arrow[ur, dash] \arrow[d, dash] \arrow[dr, dash] & \\
T & Z(G) = Z(B)
\end{tikzcd}$$

The (finite étale) quotient N/C is called the ***Weyl group*** *$W(G,T)$.*

Warning 20.10. The center $Z(G) = Z(B)$ is not necessarily smooth.

Non-example. If $G = \mathrm{SL}_n$, then $Z(G) = \mu_n$ which is not smooth if $\operatorname{char} k \mid n$.

Proof of Picture. Might as well assume $k = \overline{k}$. We prove the nontrivial parts.

($C = N^0$) This is because C is smooth, connected, and N/C is finite étale.

($C \leq B$) We've shown C is nilpotent, so solvable. Hence C is a smooth, connected solvable subgroup so it's contained in some Borel B'. Conjugate (B', T) to (B, T). Since C is the centralizer of T, and T is getting conjugated to itself, C must also get conjugated to itself, so $C \leq B$.

($Z(G) = Z(B)$) We've already shown $Z(B) \leq Z(G)$. On the other hand, $Z(G) \leq C \leq B$, so $Z(G) \leq Z(B)$.

($N = N(C)$) Any (inner) automorphism of G_R preserving T_R also preserves C_R (C_R is the normalizer of T_R). Similarly, any (inner) automorphism of G_R preserving $C_R = T_R \times U_R$ also preserves T_R (since $\mathrm{Hom}(T_R, U_R) = \{1\}$).

($N = N(B)$) This is to be proved later. Apparently everything else should be easy after this. ■

Corollary 20.11. *The set $\{$Borels B' containing $T\}$ is finite.*

Proof. WLOG $k = \overline{k}$. Any (B', T) is conjugate to (B, T), so $N(k)$ acts transitively on this set. At the same time, $C(k)$ stabilizes the element B since $C \leq B$. Thus, $\#\{B' \supset T\} \leq \# N(k)/C(k) < \infty$. ■

20.3 Union of conjugates of a subgroup

Setup. $k = \overline{k}$, and $H \leq G$ are both smooth connected linear algebraic groups. Let $\mathcal{N} := N(H)$ be the normalizer of H, and let

$$X := \bigcup_{g \in G(k)} gH(k)g^{-1} \subset G(k).$$

Recall 20.12. By some counting argument, for a finite group G with proper subgroup $H < G$, the union of conjugates of H never covers G.

Lemma 20.13.

(a) *X is a constructible subset of $G(k)$.*

(b) *If G/H is proper, then X is closed.*

(c) *If $\mathcal{N}/H$ is finite and $\exists h \in H(k)$ belong to only finitely many conjugates of H, then X contains a dense open subset of $G(k)$.*

Proof. Let $V := G/\mathcal{N}$. This variety parametrizes the conjugates of H.[31] Above this variety is the universal family $\mathcal{H} \to V$ whose fibers are the conjugates themselves. Here, $\mathcal{H} := \{(H', g) : H' \in V, g \in H'\} \subset V \times G$. Now consider the diagram

$$\begin{tikzcd} & \mathcal{H} \arrow[dl, "\mathrm{pr}_1"'] \arrow[dr, "\mathrm{pr}_2"] & \\ V & & G \end{tikzcd}$$

(a) We have $X = \mathrm{pr}_2(\mathcal{H})$ so is constructible.

(b) Assume in addition that G/H is proper. Then so its $G/\mathcal{N}$, so V is proper. Hence, $V \times G \twoheadrightarrow G$ is a proper morphism. Since $\mathcal{H} \overset{\text{closed}}{\subset} V \times G$, it has closed image in G. This exactly says that X is closed.

(c) Let's compute dimensions. First $\dim V = \dim G - \dim \mathcal{N} = \dim G - \dim H$ (second equality using $\mathcal{N}/H$ finite). Since $\mathcal{H} \to V$ with fibers conjugates of H, we also know $\dim \mathcal{H} = \dim V + \dim H = \dim G$. The second condition says that $\mathrm{pr}_2 : \mathcal{H} \to G$ is generically finite to its image, so $\dim X = \dim \mathcal{H} = \dim G$. Now, X is a constructible subset of G of the same dimension, so it better contain a dense open. ■

I realized I'm not sure if we ever defined nilpotent algebraic groups in lecture (the definition was contained in an assigned reading at some point I imagine).

Definition 20.14. An algebraic group G is **nilpotent** if it admits a **central subnormal series**, i.e. a normal series

$$G = G_0 \supset G_1 \supset \cdots \supset G_t = \{1\}$$

[31] This is basically orbit-stabilizer

such that each quotient G_i/G_{i+1} is contained in the center of G/G_{i+1} (these are also called **nilpotent series**).

21 Lecture 21 (4/7)

21.1 Last time

Notation 21.1.

$$\underbrace{G}_{\text{smooth connected linear}} \geq \underbrace{B}_{\text{Borel}} \geq \underbrace{C}_{\text{Cartan}} \geq \underbrace{T}_{\text{maximal torus}}$$

$C := C_G(T)$, $N := N_G(T)$, and $W = W(G/T) := N/C$ is the Weyl group, a finite étale group.

If $k = \overline{k}$, then W acts transitively on {Borels containing T} by conjugation (we'll later see W acts *simply* transitively). We showed this when showing this set is finite.

Example. If $G = \mathrm{GL}_n$ and $T = \mathbb{G}_m^n$, then $C = T$, $N = T \rtimes S_n$, and the Weyl group is $N/C = S_n$.

21.2 Union of conjugates of a subgroup, continued

Recall 21.2 (Lemma 20.13)**.** Suppose $k = \overline{k}$, that $H \leq G$ is smooth connected, and $\mathcal{N} := N_G(H)$. Then,

(a) X is a constructible subset of $G(k)$.

(b) If G/H is proper, then X is closed.

(c) If $\mathcal{N}/H$ is finite and $\exists h \in H(k)$ belong to only finitely many conjugates of H, then X contains a dense open subset of $G(k)$.

Lemma 21.3.

(1) *There exists some $t \in T(k)$ with $C_G(t) = C$ $(:= C_G(T))$.*

(2) *The only conjugate C' of C containing t is C itself.*[32]

Proof. Embed $G \hookrightarrow \mathrm{GL}_n$. WLOG $T \hookrightarrow G \hookrightarrow \mathrm{GL}_n$ lands in $\mathbb{G}_m^n$, i.e. this composition is

$$x \mapsto \operatorname{diag}\left(\chi_1(x), \ldots, \chi_1(x), \chi_2(x), \ldots, \chi_2(x), \ldots, \chi_r(x)\right).$$

So we have blocks on which x acts by the scalar $\chi_i(x)$. A matrix calculation shows that $C_{\mathrm{GL}_n}(T)$ consists of block-diagonal matrices. Similarly, $C_{\mathrm{GL}_n}(t)$ is the same provided that $\chi_i(t) \neq \chi_j(t)$ for all $i \neq j$. Each condition $\chi_i(t) \neq \chi_j(t)$ is telling you to avoid f.many subvarieties, so you can always find some $k = \overline{k}$ point lying outside all of them. Now note that $C_G(T) = C_{\mathrm{GL}_n}(T) \cap G = C_{\mathrm{GL}_n}(t) \cap G = C_G(t)$ so we have **(1)**. For **(2)**, suppose $t \in C'$, a conjugate of C. Keep in mind that the maximal torii are in bijection with the Cartan subgroups $\{T'\} \leftrightarrow \{C'\}$ (take torus to centralizer and centralizer to unique maximal torus). If $t \in C'$, then $t \in T'$ and so is semisimple. Thus, $T' \leq C_G(t) = C$ so $C' = C$ by the bijection between maximal tori and Cartan subgroups. ∎

[32] verifies hypothesis of Lemma 20.13 part **(c)**

Theorem 21.4. *Assume $k = \overline{k}$.*

(1) The union of the conjugates of C contains a dense open subset of G.

Example. When $G = \mathrm{GL}_n$, $C = \mathbb{G}_m^n$ and the union of conjugates of C is the set of diagonalizable matrices.

(2) Every element of $G(k)$ is contained in some Borel subgroup.

Example. When $G = \mathrm{GL}_n$, every matrix can be conjugated into something upper triangular (e.g. using Jordan canonical form)

(3) Every semisimple element of $G(k)$ is contained in some maximal torus.

Example. When $G = \mathrm{GL}_n$, since $k = \overline{k}$, semisimple $=$ diagonalizable.

Proof. **(1)** $N_G(C) = N$, Lemma 21.3 part **(2)** + Lemma 20.13 **(c)**.

(2) $\bigcup B'$ contains $\bigcup C'$ which is dense. Lemma 20.13 part **(b)** says $\bigcup B'$ is closed (since G/B proper), so $\bigcup B' = G(k)$.

(3) Let $g \in G(k)$ be a semisimple element. Consider the Zariski closure of the group is generates $\overline{\langle g \rangle}$. This group is diagonalizable (compare with GL_n case) subgroup of some Borel B. We have $B = T \rtimes U$ for some unipotent U, so all its diagonalizable subgroups are contained in T. ■

We're working towards the 'normalizer theorem,' that the normalizer of a Borel is equal to itself.

Theorem 21.5. *Assume $k = \overline{k}$. Let S be a torus in G, and let $\mathcal{C} := C_G(S)$ be its centralizer. Then,*

$$\begin{array}{ccc} \left\{ \begin{array}{c} \textit{Borel subgroups of } G \\ \textit{containing } S \end{array} \right\} & \longrightarrow & \{\textit{Borel subgroups of } \mathcal{C}\} \\ B & \longmapsto & \mathcal{C} \cap B = C_B(S) \end{array}$$

defines a surjection.

Proof. Let's first show this is well-defined, i.e. that $\mathcal{C} \cap B$ is a Borel of $\mathcal{C}$. Note that $\mathcal{C} = C_G(S)$ is automatically smooth connected, and that $\mathcal{C} \cap B = C_B(S)$ is smooth, connected, and solvable (contained in B).

Remark 21.6. Note we have some maximal torus T so that $S \leq T \leq \mathcal{C}$. Similarly, we know the Borel B fits into a short exact sequence $1 \to U \to B \to T \to 1$. We can pull this back along $S \hookrightarrow T$ to get $1 \to U \to SU \to S \to 1$.

We want this intersection $\mathcal{C} \cap B = C_B(S)$ to be a Borel. Let's restate this; TFAE

- $\mathcal{C} \cap B$ is a Borel of $\mathcal{C}$
- $\mathcal{C} \cap B$ is parabolic in $\mathcal{C}$
- $\mathcal{C}/(\mathcal{C} \cap B) \simeq \mathcal{C}B/B$ is proper
- $\mathcal{C}B/B$ is closed in G/B
- $\mathcal{C}B$ is closed in G

- $\mathcal{C}\backslash\mathcal{C}B$ is closed in $\mathcal{C}\backslash G$

  Remark 21.7. $\mathcal{C}\backslash G \hookrightarrow \mathrm{Hom}(S, G)$ via $g \mapsto [s \mapsto g^{-1}sg]$. Also,

$$\mathcal{C}\backslash\mathcal{C}B = \mathcal{C}\backslash\mathcal{C}U = (\text{image of } U \text{ in } \mathcal{C}\backslash G) \xrightarrow{\sim} \{\text{sections } \sigma \text{ of } SU \twoheadrightarrow S\}.$$

  (last iso by "Schur-Zassenhaus," all splittings are conjugate)

What's the point of all of this? The set of $g \in G$ (thought of as an element of $\mathrm{Hom}(S, G)$) such that $\mathrm{inn}_{g^{-1}}$ maps S into SU and defines a section of $SU \twoheadrightarrow S$ is closed in G (think 'transporters'). By the remark above, this is saying that $\mathcal{C}\backslash\mathcal{C}B$ is closed in $\mathcal{C}\backslash G$ and so $\mathcal{C} \cap B$ is Borel.

Now let's show the map we defined is surjective. $\mathcal{C}$ acts by conjugation on both sides, and the RHS $\{\text{Borel subgroups of } \mathcal{C}\}$ is a single orbit, so it must be surjective (the image is a union of orbits). ■

We need one for lemma before getting to the normalizer theorem. Part of its proof will be checking that $N(B) = B$ on k-points, but this won't be enough since $N(B)$ might not be smooth. For that, we'll also need to understand the Lie algebra of $N(B)$.

Lemma 21.8. *Fix any smooth H with $C \leq H \leq G$ (so H contains a Cartan subgroup). Let $\mathcal{C} := N_G(H)$. Then, $\mathcal{N}^0 = H^0$, so $\mathcal{N}$ is smooth.*

Proof. C is the centralizer of the maximal torus, so $C = G^T$ where $T \curvearrowright G$ by conjugation. Similarly, $\mathcal{N}/H = (G/H)^H$ with H also acting by conjugation. Checking what this says on $k[\varepsilon]/(\varepsilon^2)$-points tells us that

$$\mathfrak{c} := \operatorname{Lie} C = \mathfrak{g}^T.$$

Note that $\mathfrak{c} \leq \mathfrak{h} \leq \mathfrak{g}$; taking T-invariants shows $\mathfrak{g}^T = (\mathfrak{g}^T)^T \leq \mathfrak{h}^T \leq \mathfrak{g}^T$, so $\mathfrak{g}^T = \mathfrak{h}^T$. Now note that

$$\mathfrak{n}/\mathfrak{h} = (\mathfrak{g}/\mathfrak{h})^H \leq (\mathfrak{g}/\mathfrak{h})^T = \mathfrak{g}^T/\mathfrak{h}^T = 0$$

(above, used that T is of multiplicative type so linearly reductive, so all T-reps split and taking T-fixed points behaves with quotients and whatnot). Thus, $\mathfrak{n} = \mathfrak{h}$. Now we do a little dimension counting:

$$\dim \mathfrak{h} = \dim H \leq \dim \mathcal{N} \leq \dim \mathfrak{n} = \dim \mathfrak{h}.$$

This forces $\dim \mathcal{N} = \dim \mathfrak{n} = \dim \mathfrak{h}$, so $\mathcal{N}$ is smooth and we conclude that $\mathcal{N}^0 = H^0$. ■

Next time we prove the normalizer theorem: $N_G(B) = B$.

22 Lecture 22 (4/9)

22.1 Last time

Notation 22.1.

$$\underbrace{G}_{\text{smooth connected linear}} \geq \underbrace{B}_{\text{Borel}} \geq \underbrace{C}_{\text{Cartan}} \geq \underbrace{T}_{\text{maximal torus}}$$

$C := C_G(T)$, $N := N_G(T)$, and $W = W(G/T) := N/C$ is the Weyl group, a finite étale group.

Recall 22.2. If $k = \overline{k}$, all Borels are conjugate, all maximal tori are conjugate, and $B = U \rtimes T$ with U unipotent and T a maximal torus.

Recall 22.3. Suppose that $k = \overline{k}$, and $S \leq G$ is a torus. Set $\mathcal{C} = C_G(S)$. Then there is a surjection

$$\begin{array}{ccc} \{\text{Borels of } G \text{ containing } S\} & \longrightarrow & \{\text{Borels of } \mathcal{C}\} \\ B & \longmapsto & \mathcal{C} \cap B \end{array}$$

Lemma 22.4. *If $C \leq H \leq G$ with H smooth, and $\mathcal{N} = N_G(H)$, then $\mathcal{N}^0 = H^0$, so $\mathcal{N}$ is smooth too.*

22.2 Normalizer theorem

Theorem 22.5 (Normalizer theorem). $N_G(B) = B$.

Proof. WLOG $k = \overline{k}$. Since $B \supset C$, the Lemma 21.8 tells us that $N_G(B)$ is smooth, so ti suffices to check k-points. It's obvious that $B \subset N_G(B)$, so only need other direction. Pick some k-point $n \in N_G(B)(k)$. We'll show $n \in B$ by induction on $\dim G$.

Note that T and nTn^{-1} are both maximal tori in B. All maximal tori are conjugate, so $nTn^{-1} = bTb^{-1}$ for some $b \in B$ (really, $b \in B(k)$ but we'll drop the (k)'s). We may replace n by $b^{-1}n$ to assume that $nTn^{-1} = T$. The morphism $\varphi : T \to T, t \mapsto ntn^{-1}t^{-1}$ is a homomorphism (since T commutative). Note that $\ker \varphi = C_T(n)$. This is not necessarily a torus, but $S = (\ker \varphi)^0_{\text{red}}$ is a subtorus of T.

(Case 1) First suppose $1 \neq S \leq Z(G)$. Apply induction to B/S, a Borel inside G/S. This shows $\overline{n} \in B/S$ from which we see that $n \in B$.

(Case 2) Now suppose $1 \neq S \not\leq Z(G)$. Apply induction to the Borel $C_G(S) \cap B$ in $C_G(S) \neq G$. We know $n \in C_G(S)$ by definition of φ and S so induction says $n \in C_G(S) \cap B \subset B$.

(Case 3) Finally suppose $S = 1$. Then, $\dim \ker \varphi = 0$, so $\varphi(T) = T$ (since $\dim \varphi(T) = \dim T$ and T smooth, connected). By Chevalley (Theorem 8.4), $N_G(B) = \operatorname{Stab}(L)$ for some 1-dim $L = \langle v \rangle$ in a G-rep V. Since $n \in N_G(B)$ and $T \subset N_G(B)$, they both act by scalars on L. Hence, by the definition of φ, we see that $T = \varphi(T)$ fixes v. Recalling $B = U \rtimes T$; since L is 1-dim rep of $U \leq B$, it must be trivial (unipotents always have some fixed vector). Thus, we see that $B = U \rtimes T$ fixes v. Now we look at the orbit map

$$\begin{array}{ccc} G/B & \longrightarrow & V \\ g & \longmapsto & gv \end{array}$$

Note that G/B is proper, smooth, and connected; and that V ($= \operatorname{spec} \operatorname{Sym}(V)$) is affine, so this orbit map must be constant. Hence, G fixes v, so we have

$$B \overset{\text{open}}{\subset} N_G(B) = \operatorname{Stab}(L) = G,$$

which means $B \overset{\text{open}}{\subset} G$. Thus, B is clopen in the connected space G, so $B = G$. In particular, $B = N_G(B)$.

∎

Corollary 22.6. *If P is parabolic in G, then P is connected and $N(P) = P$.*

(use that parabolics contain Borels and all Borels are conjugates and stuff like this)

Over $k = \overline{k}$, all Borels are conjugate (one orbit under conjugation action), so orbit-stabilizer says that they are parameterized by $G/\operatorname{Stab}_G(B) = G/N_G(B) = G/B =: \mathcal{B}$. We call this a **flag variety**.

Example. If $G = \mathrm{GL}_n$, then B is the stabilizer of the standard full flag, so G/B parameterizes all full flags.

If $k \neq \overline{k}$, but k is perfect, can use Galois descent to say that this variety descends to a k-variety $\mathcal{B}$, but $\mathcal{B}(k)$ might be empty (even when $k = k^s$)!

Fact (Miracle, due to Grothendieck)**.** On the other hand, every (smooth, connected linear algebraic group) G contains a (torus) T over k such that $T_{\overline{k}}$ is a maximal torus.

Can't always find Borels, but can always find maximal tori.

Recall the picture (Proposition 20.9)

$$\begin{tikzcd}[arrows=dash]
G \arrow[d] \\
B \arrow[dd] \\
& N = N(C) \arrow[dl] \\
C = N^0 \arrow[d] \\
T
\end{tikzcd}$$

Proposition 22.7. $N \cap B = C$

Proof. In any semidirect product $B = U \rtimes T$ with T commutative, $N_B(T) = C_B(T)$. In our situation, we have $N_B(T) = N \cap B$ and $C_B(T) = C \cap B = C$ (since $C \subset B$). ∎

Corollary 22.8. *If $k = \overline{k}$, then W acts simply transitively on $\{$Borels containing $T\}$.*

Proof. N acts transitively by conjugation with stabilizer $\operatorname{Stab}_N(B) = N \cap N(B) = N \cap B = C$, where we used to Normalizer theorem $N(B) = B$. Thus, $W = N/C$ acts simply transitively. ∎

(Compare with Corollary 20.11)

Recall 22.9. For any torus $S \leq G$, we showed that $C_G(S)$ is smooth, connected.

This must have been in a reading at some point, because I have no recollection of seeing this in lecture.

Update: See Theorem 20.6

Update 2:

We can generalize this to other actions.

Proposition 22.10. *If S is a torus acting on G, then G^S is smooth connected.*

(if $S \curvearrowright G$ by conjugation, this is just the previous statement).

Proof. Form the semidirect product $G \rtimes S$ using the given action of S on G. Then, one can directly compute that $C_{G \rtimes S}(S) = G^S \times S$ is smooth connected, so G^S must be. ∎

Notation 22.11 (Not Standard)**.** Recall that $R_u(G)$ denoted the unipotent radical. To make things less cluttered, we instead use $G_u := R_u(G)$, the largest smooth, connected unipotent normal subgroup of G.

Theorem 22.12 (**Chevalley's formula for the unipotent radical**)**.** *Assume $k = \overline{k}$ and fix a maximal torus T, but vary $B = B_u \rtimes T$. Then,*

$$G_u \cdot T = \left(\bigcap_{B \supset T} B \right)^0_{\text{red}}$$

with intersection taken over Borels containing T. Similarly,

$$G_u = \left(\bigcap_{B \supset T} B_u \right)^0_{\text{red}}.$$

Proof. Skipped. ∎

Remark 22.13. The second part of Chevalley's formula follows from the first.

Corollary 22.14. *Still assuming $k = \overline{k}$. If $\mathcal{C}$ is the centralizer of some torus $S \le G$, then $\mathcal{C}_u = G_u \cap \mathcal{C}$.*

Proof. ($\ge$) The torus S acts on G_u by conjugation (since $G_u \triangleleft G$ normal by definition), so $G_u \cap \mathcal{C} = (G_u)^S$ is smooth connected. It's also unipotent since it's contained in a unipotent group. Since G_u is normal in G, this will also be normal in $\mathcal{C}$. All together, this exactly says $G_u \cap \mathcal{C} \le \mathcal{C}_u$.

($\le$) Choose a maximal torus T with $S \le T \le B$. We apply Chevalley's formula:

$$\mathcal{C}_u \cdot T = \left(\bigcap_{\substack{B' \supset T \\ \text{in } \mathcal{C}}} B' \right)^0_{\text{red}} \le \left(\bigcap_{B'' \supset T} B'' \right)^0_{\text{red}} = G_u \cdot T$$

since the B''s are precisely the $B'' \cap \mathcal{C}$'s. Thus, $\mathcal{C}_u \le G_u \cdot T$. Since $\mathcal{C}_u$ is unipotent, we must actually have $\mathcal{C}_u \le G_u$. ∎

Corollary 22.15. *If G is reductive (i.e. $R_u(G_{\overline{k}}) = G_{\overline{k},u} = 1$), then so is $\mathcal{C}$.*

Corollary 22.16. *In a reductive group G, we have*

$$
\begin{tikzcd}
G \arrow[d, dash] \\
B \arrow[d, dash] \\
C = T \arrow[d, dash] \\
Z(G) \arrow[r, equal] \arrow[d, dash, "\textit{finite}"] & \textit{mult. type} \\
R(G) \arrow[r, equal] \arrow[d, dash] & Z(G)^0_{\mathrm{red}} \\
R_u(G) \arrow[r, equal] & 1
\end{tikzcd}
$$

We'll show this next time.

Plan for next few lectures is to cover much of chapter 17, and then quickly go through chapter 18. Then we want to get to root data.

23 Lecture 23 (4/12)

23.1 Last time

Notation 23.1.

$$\underbrace{G}_{\text{smooth connected linear}} \geq \underbrace{B}_{\text{Borel}} \geq \underbrace{C}_{\text{Cartan}} \geq \underbrace{T}_{\text{maximal torus}}$$

$C := C_G(T)$, $N := N_G(T)$, and $W = W(G/T) := N/C$ is the Weyl group, a finite étale group.

Recall 23.2. If $k = \overline{k}$, all Borels are conjugate, all maximal tori are conjugate, and $B = U \rtimes T$ (since solvable over $\overline{k}$ = trigonalizable) with U unipotent and T a maximal torus.

Recall 23.3. If $k = \overline{k}$, then $\bigcup_{g \in G(k)} gCg^{-1}$ contains a dense open subset of $G(k)$.

Recall 23.4. The *radical* is $R(G) :=$ largest smooth connected, solvable normal subgroup

The *unipotent radical* is $R_u(G) :=$ largest smooth connected unipotent normal subgroup.

Recall 23.5. If $k = \overline{k}$, then

- G is **semisimple** if $R(G) = 1$ (e.g. $G = \mathrm{SL}_n$)
- G is **reductive** if $R_u(G) = 1$ (e.g. $G = \mathrm{GL}_n$)

If $k \neq \overline{k}$, check after base extension to $\overline{k}$ (e.g. *semisimple* $\iff R(G_{\overline{k}}) = 1$)

Proposition 23.6. *If G is reductive, then the centralizer of any torus in G is reductive.*

23.2 Reductive Groups, I think

We ended last time with the following statement.

Corollary 23.7 (of Theorem 22.12)**.** *In a reductive group G, we have*

$$\begin{tikzcd}
G \arrow[d, dash] \\
B \arrow[d, dash] \\
C = T \arrow[d, dash] \\
Z(G) \arrow[d, dash, "\textit{finite}"] \arrow[r, equal] & \textit{mult. type} \\
R(G) \arrow[d, dash] \arrow[r, equal] & Z(G)^0_{\mathrm{red}} \arrow[r, equal] & \textit{largest torus in } Z(G) \\
R_u(G) \arrow[r, equal] & 1
\end{tikzcd}$$

Furthermore, the quotient $G^{\mathrm{ad}} := G/Z(G)$ is ***adjoint*** *(i.e. trivial center) and semisimple.*

Example. $G = \mathrm{GL}_n$ is reductive. We can take $B = B_n$, $C = T = T_n$ (the diagonal torus). We have $Z(G) = \mathbb{G}_m$ which is already connected and reduced, so $R(G) = \mathbb{G}_m$ as well. Finally, since G reductive, $R_u(G) = 1$. We see in this case that $G^{\mathrm{ad}} = \mathrm{PGL}_n = \mathrm{GL}_n / \mathbb{G}_m$.

Remark 23.8. Everything in Corollary 23.7 respects (arbitrary) field extension.

Warning 23.9. In general, (unipotent) radicals do *not* respect field extensions. However, in the case of a reductive group, $R_u(G) = 1$ so we're good there, and $R(G) =$ largest torus in $Z(G)$ has a nice alternative description (the center does respect field extensions).

Proof of Corollary 23.7. WLOG assume $k = \overline{k}$.

($C = T$) C is nilpotent over $k = \overline{k}$, so we can write $C = T \times U$. Since C is reductive (by Corollary 22.15), the unipotent part must be trivial, so $C = T$.

Fact (Bonus fact)**.** $\bigcap_{\text{max tori } T'} T'$ has the same k-points as $Z(G)$

Let's prove this real quick. Since T' contains the center, one inclusion is obvious. For the other, pick some k-point $g \in \bigcap T'$. Then, g commutes with every Cartan subgroup C'. These cover a dense subset of $G(k)$, so $C_G(g)$ must be closed and dense, so $g \in Z(G)$.

($R(G)$ is a torus) $R(G)$ is smooth, connected solvable, so $R(G) = \mathcal{U} \rtimes \mathcal{T}$ for some unipotent $\mathcal{U}$ and torus $\mathcal{T}$. Furthermore, $\mathcal{U}$ will be normal in G, so $\mathcal{U} \subset R_u(G) = 1$, so $R(G) = \mathcal{T}$ is a torus.

($R(G) \leq Z(G)$) Consider conjugation action of G on $R(G)$. Since $R(G)$ is a torus, $\operatorname{Aut} R(G)$ is discrete (automorphisms same as automorphisms of character group); since G is connected, this forces the conjugation action to be trivial, so $R(G) \leq Z(G)$.

($Z(G/Z(G)) = 1$) The inverse image of $Z(G/Z(G))$ in G is some normal subgroup $Z' \triangleleft G$ containing $Z(G)$. The conjugation actions of G on $Z(G), Z'/Z(G)$ are both trivial, so G acts on Z' through $\operatorname{Hom}(Z'/Z(G), Z(G))$. These are both groups of multiplicative type, so this Hom is discrete. Since G is connected, the image must be trivial, so the conjugation action of G on Z' is trivial, so $Z' = Z$.

■

Go back to k being an arbitrary field.

Definition 23.10. We say G is **quasi-split** if $\exists$ Borel $B \leq G$ defined over k.

Warning 23.11. If $k \neq \overline{k}$, there exists non-quasi-split groups over k.

Definition 23.12. We say (a smooth, connected, linear algebraic group) G is **split** if there exists a Borel $B \leq G$ which is a split solvable group over k.

Recall 23.13. A solvable group is called *split* if it has a subnormal series whose factors are all $\mathbb{G}_a$ or $\mathbb{G}_m$.

Fact.

- G split $\implies$ G has a split maximal torus.

  The converse holds if G is reductive (later).

  Corollary 23.14. *A reductive group becomes split after a separable extension.*

  Remember: In general, it takes an inseparable extension to split a unipotent group

- Quotients of split groups are split

  Warning 23.15. Subgroups of split groups may not be split.

  Example. Take $G = \operatorname{GL}_2/\mathbb{Q}$. This is split since $B_2 = \mathbb{G}_a \rtimes \mathbb{G}_m$ is split. Let $[L : \mathbb{Q}] = 2$ be a quadratic extension. Then you get an embedding $L^\times \hookrightarrow \operatorname{Aut}_{\mathbb{Q}}(L) = \operatorname{GL}_2(\mathbb{Q})$. Can upgrade this to $\operatorname{Res}_{L/\mathbb{Q}} \mathbb{G}_m \hookrightarrow \operatorname{GL}_2$. This is a 2-dim torus, so maximal. However, it is not split (e.g. since $\operatorname{Gal}(\overline{\mathbb{Q}}/\mathbb{Q})$ acts nontrivially on its character group).

- All split Borel subgroups are $G(k)$-conjugate.

- Every split solvable subgroup is contained in a split Borel.

- If G is split, all split maximal tori are $G(k)$-conjugate.

As long as you restrict to split Borel subgroups, things behave a lot like they do over $\overline{k}$.

23.3 Simply connected groups

Motivation. Let X be a connected manifold with base point $x \in X$. Then one says that X is *simply connected* if its only connected covering space is itself. In general, there always exist some simply connected cover $\widetilde{X} \to X$, and the *fundamental group* $\pi_1(X, x)$ is the group of deck transformations (automorphisms of the covering). $\widetilde{X}$ is called the *universal cover*. If X is a Lie group, then $\widetilde{X}$ will be as well.

We want an analogue of this for algebraic groups. Before getting to that, let's talk about how one usually generalizes this to algebraic varieties.

Motivation. Say $k = k^s$, and let X be a connected k-variety with basepoint $\operatorname{spec} k \xrightarrow{x} X$. Then, one says X is *simply connected* if the only connected, *f.étale* cover of X is X itself. In this situation, universal covers do not always exist (since we only talk about finite covers), but there is a projective system of finite covers and so can use this to build a profinite *étale fundamental group*.

Example. If $\operatorname{char} k = 0$ and $X = \mathbb{G}_m$, the nth power map $\mathbb{G}_m \xrightarrow{x^n} \mathbb{G}_m$ is finite étale. Each one has covering group $\mathbb{Z}/n\mathbb{Z}$, and one sees that $\pi_1(\mathbb{G}_m, 1) = \varprojlim \mathbb{Z}/n\mathbb{Z} = \widehat{\mathbb{Z}}$.

Fun fact: there exist (infinite) groups whose profinite completions are trivial

For algebraic groups, we won't use either of the these two settings. There's a third setup that works well for (semisimple) algebraic groups (wanna avoid things like $\mathbb{G}_m$ with their 'infinite covers').

Setup. Say G is a semisimple algebraic group over $k = k^s$.

Let's start with examples.

Example. There's a natural homomorphism $\mathbb{G}_m \times \mathrm{SL}_n \to \mathrm{GL}_n$. This is surjective[33] (and automatically flat since GL_n smooth). It is not injective, e.g. since $\mu_n \hookrightarrow \mathbb{G}_m \cap \mathrm{SL}_n \subset \mathrm{GL}_n$ (in fact this μ_n is the whole intersection). So this is some nontrivial isogeny $\mathbb{G}_m \times \mathrm{SL}_n \twoheadrightarrow \mathrm{GL}_n$. We can mod out by $\mathbb{G}_m$ to get an isogeny

$$\mathrm{SL}_n \twoheadrightarrow \mathrm{PGL}_n$$

with kernel μ_n. Is this a cover? Well, μ_n is not always étale (e.g. imagine $n = p = \operatorname{char} k$) so it's not an étale cover. However, we do want it to be a 'cover' in the current setting.

The upshot is we want to allow non-étale covers. However, we should not allow any isogney to count as a 'cover'. This would prevent anything from being simply connected, e.g. we always have Frobenius $\mathrm{SL}_n \to \operatorname{Fr} \mathrm{SL}_n$. Hence, Frobenius should not count as a cover.

Definition 23.16. Let G, G' be smooth connected groups. An isogeny $\varphi : G' \to G$ is **central** if $\ker \varphi \leq Z(G')$. It is **multiplicative** if $\ker \varphi$ is of multiplicative type (recall this means it's diagonalizable over $\overline{k}$, i.e. a product of copies of[34] μ_n over $\overline{k}$).

Remark 23.17. Most people in the literature talk about central isogenies, but Milne uses multiplicative ones.

Remark 23.18. multiplicative $\implies$ central. The conjugation action of G' on $\ker \varphi$ is trivial by the same argument as before (the character group is discrete).

When G' is reductive, the converse holds as well (since $Z(G')$ is of multiplicative type).

Remark 23.19. In characteristic 0, all isogenies are étale (since all finite groups are étale), multiplicative, and central.

Remember: algebraic groups in char 0 are automagically smooth (by Theorem 5.2)

[33] fppf locally, can form $(\det A)^{-1/n} A \in \mathrm{SL}_n$

[34] Could be some $\mathbb{G}_m$'s too except the kernel of an isogeny is finite

24 Lecture 24 (4/14)

24.1 Last time

Due date for pset moved to Sunday April 25.

Notation 24.1.

$$\underbrace{G}_{\text{smooth connected linear}} \geq \underbrace{B}_{\text{Borel}} \geq \underbrace{C}_{\text{Cartan}} \geq \underbrace{T}_{\text{maximal torus}}$$

$C := C_G(T)$, $N := N_G(T)$, and $W = W(G/T) := N/C$ is the Weyl group, a finite étale group.

Recall 24.2. The *radical* is $R(G) :=$ largest smooth connected, solvable normal subgroup

The *unipotent radical* is $R_u(G) :=$ largest smooth connected unipotent normal subgroup.

Recall 24.3. If $k = \overline{k}$, then

- G is **semisimple** if $R(G) = 1$ (e.g. $G = \mathrm{SL}_n$)
- G is **reductive** if $R_u(G) = 1$ (e.g. $G = \mathrm{GL}_n$)

If $k \neq \overline{k}$, check after base extension to $\overline{k}$ (e.g. *semisimple* $\iff R(G_{\overline{k}}) = 1$)

Recall 24.4. Let $\varphi : G' \to G$ be an isogeny between smooth connected groups. We call φ *central* if $\ker\varphi \leq Z(G')$ and *multiplicative* if $\ker\varphi$ is of multiplicative type.

In general, multiplicative $\implies$ central. The converse holds if G' is reductive. In char 0, all isogenies are étale, multiplicative, and central.

We only use multiplicative isogenies from now on.

24.2 Fundamental Groups and Stuff about Semisimple/Reductive Groups

Definition 24.5. A smooth, connected, *semisimple*[35] G is **simply connected** if every multiplicative isogeny $G' \to G$ (so G' smooth, connected) is an isomorphism.

Recall 24.6. G (as above) is **adjoint** if $Z(G) = 1$.

Theorem 24.7. *For G semisimple $/\ k = k^s$,*

Remember: semisimple includes 'smooth connected' in its definition

(1) *There exists a simply connected $\widetilde{G}$ fitting into a diagram*

$$\begin{tikzcd} \widetilde{G} \arrow[d, no head] \\ G \arrow[d, no head] \\ G^{\mathrm{ad}} \arrow[r, equal] & G/Z(G) \end{tikzcd}$$

(2) *Every multiplicative isogeny $G' \to G$ is **dominated by** $\widetilde{G}$ in the sense that $\widetilde{G} \to G$ uniquely factors as $\widetilde{G} \xrightarrow{\exists!} G' \to G$.*

[35] Wanna avoid something like $G = \mathbb{G}_m$. This won't be simply connected, and won't have a universal cover.

*(3) We define the **fundamental group** of G to be $\pi_1(G) := \ker\left(\widetilde{G} \to G\right)$, a finite group of multiplicative type (so diagonal[36] since $k = k^s$). Furthermore, the Picard group of G is $\operatorname{Pic} G \simeq X^*(\pi_1(G))$, the character group of the fundamental group (in particular, $\operatorname{Pic} G$ finite abelian).*

I think the intuition here is that $X^*(\pi_1(G))$ is like $\mathrm{H}^1(G, \mathbb{G}_m)$

Example. Consider $G = \mathrm{SO}_n$. In this case, one gets $\pi_1(G) \simeq \mu_2$ so $\operatorname{Pic} G = \mathbb{Z}/2\mathbb{Z}$. The universal cover is a μ_2 algebra called the **Spin group** $\widetilde{G} = \mathrm{Spin}_n$. Furthermore, the center is

$$Z(G) = \begin{cases} 1 & \text{if } n \text{ odd} \\ \mu_2 & \text{if } n \text{ even} \end{cases}$$

Hence, $G^{\mathrm{ad}} = \mathrm{PSO}_n$ if n even. Furthermore,

$$Z(\widetilde{G}) = \pi_1(G^{\mathrm{ad}}) = \begin{cases} \mu_2 & \text{if } n \text{ odd} \\ \mu_4 & \text{if } n \equiv 2 \pmod 4 \\ \mu_2 \times \mu_2 & \text{if } n \equiv 0 \pmod 4 \end{cases}$$

We'll be able to calculate this stuff once we have root data.

Remark 24.8. If $G^{\mathrm{ad}} \to H$ is a multiplicative isogeny, then $\ker(G^{\mathrm{ad}} \to H) \subset Z(G^{\mathrm{ad}}) = 1$, so this map is an isomorphism.

Remark 24.9. If you want to understand things over non-separably closed fields, draw the picture over the separable closure, and then keep track of all the Galois actions.

Proposition 24.10. *Let $k \le L$ be fields. Let G be a smooth connected linear algebraic group over k. Then, G is semisimple $\iff$ G_L is semisimple.*

Proof. WLOG $k = \overline{k}$ and $L = \overline{L}$ (because semisimple means $R(G_{\overline{k}}) = 1$). That is, the interesting case is in moving between algebraically closed fields (e.g. $k = \overline{\mathbb{Q}}$ and $L = \mathbb{C}$).

($\to$) Suppose that $H \le G_L$ is a nontrivial smooth connected normal solvable subgroup. Consider the k-algebra

$$A := k\begin{bmatrix}\text{all the f.many coeffs involved in defining } H \\ \text{and in checking it has the right properties}\end{bmatrix} \subset L.$$

This is a f.g. k-algebra over which H is defined, i.e. $H = \mathcal{H}_L$ for some subgroup scheme $\mathcal{H} \le G_A$. We now want a subgroup over k, so let's reduce mod some maximal ideal. Let $\mathfrak{m}$ be a maximal ideal of A. Then, $A/\mathfrak{m}$ is a finite extension of k (by weak Nullstellensatz), so $A/\mathfrak{m} \simeq k$. Hence, $\mathcal{H}_{A/\mathfrak{m}}$ is a nontrivial smooth connected normal solvable subgroup of G. ■

Fact.

- $R(G)$ respects *separable* field extensions (by Galois descent).
- $R_u(G)$ also respects *separable* field extensions.
- If G is known to be reductive (i.e. $R_u(G_{\overline{k}}) = 1$), then $R(G)$ respects *arbitrary* field extensions (since $R(G) =$ max torus in $Z(G)$.[37]), and $G/R(G)$ is semisimple.

[36] i.e. (since it's finite) it'll be a copy of μ_m's (possibly with $p \mid m$)

[37] In a group of multiplicative type, can canonically recover maximal torus from torsion-free quotient of character group

- Quotients of semisimple/reductive groups are semisimple/reductive
- Being semisimple/reductive preserved under multiplicative isogenies
- If G is semisimple, then $Z(G)$ is finite and G is **perfect** (i.e. $G^{\mathrm{der}} := [G,G] = G$).[38]

  Pf. $Z(G)^0_{\mathrm{red}} = R(G) = 1$ (so $Z(G)$ finite). The proof of perfect is something we'll do later; the main point is that G is generated by images of SL_2, which are all perfect.
- More properties in chapter 19 of the book

Example. $\mathbb{G}_m \cdot \mathrm{SL}_n = \mathrm{GL}_n$ and $\mathbb{G}_m \cap \mathrm{SL}_n = Z(\mathrm{SL}_n) = \mu_n$.

We want to generalize this to arbitrary reductive groups.

Theorem 24.11. *Let G be reductive.*

(a) $Z(G) \cdot G^{\mathrm{der}} = G$

(b) $Z(G) \cap G^{\mathrm{der}} = Z(G^{\mathrm{der}})$.

(c) *Above, $Z(G)$ is of multiplicative type, G^{der} is semisimple, and $Z(G^{\mathrm{der}})$ is a finite group of mult. type.*

Proof, assuming semisimple groups are perfect. **(a)** We'll show this generate G by quotient by both of them (note they're both normal).

$$\begin{tikzcd} & G^{\mathrm{ab}} := G/G^{\mathrm{der}} \arrow[dr, two heads] & \\ G \arrow[ur, two heads] \arrow[dr, two heads] & & \dfrac{G}{Z(G)\cdot G^{\mathrm{der}}} \\ & G^{\mathrm{ad}} := G/Z(G) \arrow[ur, two heads] & \end{tikzcd}$$

Note that G^{ab} is commutative while G^{ad} is semisimple (by Corollary 23.7) so perfect. Thus, $G/Z(G)\cdot G^{\mathrm{der}}$ is commutative and perfect, but this forces it to be 0.

(b) $Z(G^{\mathrm{der}}) \le Z(G)$ follows from **(a)**.

(c) We already know $Z(G)$ is of multiplicative type.

Let's show that $Z(G^{\mathrm{der}}) = Z(G) \cap G^{\mathrm{der}}$ is finite (we know it's of multiplicative type). For this, we may assume $k = \overline{k}$. Choose a faithful representation V of G. Decompose into characters of the center: $V = \bigoplus_{\chi \in X^*(Z(G))} V_\chi$. Each V_χ is a G-rep (by homework problem), so $\mathrm{im}(G \hookrightarrow \mathrm{GL}_V)$ consists of block-diagonal matrices (lands in $\bigoplus \mathrm{GL}_{V_\chi} \subset \mathrm{GL}_V$). On each V_χ, $Z(G)$ acts by a character, so the image of $Z(G)$ in GL_V looks like blocks of diagonal matrices $\bigoplus_\chi \mathbb{G}_m$. For G^{der}, can take commutator block-by-block so lands in $\bigoplus_\chi \mathrm{SL}_{V_\chi}$. Thus, the intersection lands in $\bigoplus_\chi (\mathbb{G}_m \cap \mathrm{SL}_{V\chi}) = \bigoplus_\chi \mu_{\dim V_\chi}$ which is finite.

Now, we know $G^{\mathrm{der}} \twoheadrightarrow G/Z(G) = G^{\mathrm{ad}}$ is a multiplicative isogeny (kernel is $Z(G) \cap G^{\mathrm{der}}$) to a semisimple group. Thus, G^{der} must itself be semisimple (one of the facts stated before). ■

[38] Poonen uses G_{der} instead of G^{der}, but I already have a command for the latter, so I'm sticking with a supersciprt

Remark 24.12 (Audience)*.* Here's another proof that G^{der} is semisimple. We know it is reductive since it is normal in G. Then, $R(G^{\mathrm{der}}) = Z(G^{\mathrm{der}})^0_{\mathrm{red}} = 1$ since $Z(G^{\mathrm{der}})$ is finite multiplicative type.

Remark 24.13*.* There's a variant of the above where you use the radical $R(G) = Z(G)^0_{\mathrm{red}}$ in place of the center $Z(G)$.

Corollary 24.14. *Reductive groups are those of the form*

$$\frac{\textit{torus} \times \textit{semisimple}}{\textit{finite}}.$$

In particular, they are always isogeneous to a product of a torus and a semisimple group. Furthermore, WLOG, the finite piece in the quotient can be taken to be to graph of an isomorphism between central finite subgroups of the torus and the semisimple factors.

Up next, let's try and classify semisimple groups.

25 Lecture 25 (4/16)

No lecture on Monday. Problem set due Sunday (not in two days).

25.1 Last time

Notation 25.1.

$$\underbrace{G}_{\text{smooth connected linear}} \geq \underbrace{B}_{\text{Borel}} \geq \underbrace{C}_{\text{Cartan}} \geq \underbrace{T}_{\text{maximal torus}}$$

$C := C_G(T)$, $N := N_G(T)$, and $W = W(G/T) := N/C$ is the Weyl group, a finite étale group.

Recall 25.2. The *radical* is $R(G) :=$ largest smooth connected, solvable normal subgroup

The *unipotent radical* is $R_u(G) :=$ largest smooth connected unipotent normal subgroup.

Recall 25.3.

- G is **semisimple** if $R(G_{\overline{k}}) = 1$ (e.g. $G = \mathrm{SL}_n$)
- G is **reductive** if $R_u(G_{\overline{k}}) = 1$ (e.g. $G = \mathrm{GL}_n$)
- If G semisimple, then $Z(G)$ finite and G is perfect (haven't proved the perfect part ye)
- If G is reductive, then $R(G)$ is a torus, $Z(G)$ is of multiplicative type, $G^{\mathrm{der}} = [G, G]$, and $G^{\mathrm{ad}} = G/Z(G)$ and $G/R(G)$ are semisimple
- $$\text{reductive} = \frac{\text{torus} \times \text{semisimple}}{\text{finite}}$$
  where the torus $R(G)$ and semisimple group G^{der} are subgroups generating G, and their intersection is a finite group contained in the center of both.

25.2 Classifying split semisimple groups of rank ≤ 1

Definition 25.4. Let G be some smooth, connected linear algebraic group, and fix a maximal torus $T \leq G$.

- If $k = \overline{k}$, the **rank** of G is $\operatorname{rank} G := \dim T$.
- If $k \neq \overline{k}$, the (geometric) **rank** of G is $\operatorname{rank} G := \operatorname{rank}(G_{\overline{k}})$. The **$k$-rank** of G is the dimension of the largest *split* torus of G.

Remark 25.5. We claimed earlier that $G_{\overline{k}}$ always has a maximal torus defined over k, so $\operatorname{rank} G = \dim T$ always.

Definition 25.6. If G is reductive over $\overline{k}$, we define its **semisimple rank** to be $\operatorname{rank}(G/R(G))$. For general k, its **semisimple rank** is $\operatorname{s.s.rank}(G_{\overline{k}})$, while its **semisimple k-rank** is the k-rank of $G/R(G)$.

Remark 25.7. These are all invariant under isogeny.

"Let's play a game. I give you a group, and you tell me what the rank is. It's not a very fun game, but what do you expect" (paraphrased)

Example. $\operatorname{rank}(\mathrm{GL}_n) = n$, $\operatorname{rank}(\mathrm{SL}_n) = n-1$, $\operatorname{s.s.rank}(\mathrm{GL}_n) = \operatorname{rank}(\mathrm{PGL}_n) = n-1$ (PGL_n isogeneous to SL_n)

Theorem 25.8.

- *rank* $0 \iff$ *unipotent (contains no nontrivial torus).*
- *s.s.rank* $0 \iff$ *solvable ($R(G)$ and $G/R(G)$ both solvable).*
- *reductive and s.s. rank* $0 \iff$ *torus ($G = R(G)$ torus).*

Let's do rank one next.

Lemma 25.9. *Say $k = \overline{k}$. Suppose $\mathbb{G}_m$ acts linearly on a f.dim k-vector space V. Let $Y \subset \mathbb{P}V$ be a $\mathbb{G}_m$-stable subvariety of dimension d. Then,*

$$\# Y^{\mathbb{G}_m}(k) \geq d+1.$$

This is an improvement on Borel's fixed point theorem (which gives at least 1 fixed point) in this case.

Proof. We induct on $d = \dim Y$. WLOG can assume $\mathbb{P}V = \mathbb{P}^n$ with each $t \in \mathbb{G}_m$ acting as $(x_0 : \cdots : x_n) \mapsto (t^{m_0}x_0 : \cdots : t^{m_n}x_n)$ with $m_0 \leq \cdots \leq m_n$ in $\mathbb{Z}$. Also, we may as well scale to assume that $m_0 = 0$. Finally, we can assume that Y is irreducible ($\mathbb{G}_m$ must preserve the irred components since $\mathbb{G}_m$ connected[39]).

Write $\mathbb{P}^n = \mathbb{P}^{n-1} \sqcup \mathbb{A}^n$ (with the $\mathbb{P}^{n-1}$ where $x_0 = 0$). If $Y \subset \mathbb{P}^{n-1}$, can start over with smaller V. Otherwise, Y has $\geq d$ fixed points in $\mathbb{P}^{n-1}$ (by inductive hypothesis applied to $Y \cap \mathbb{P}^{n-1}$). Hence, we only need to show it has ≥ 1 fixed point in $\mathbb{A}^n$. For this, choose some $y = (1 : a_1 : \cdots : a_n) \in Y \cap \mathbb{A}^n$, and take the limit $\lim_{t \to 0} t.y$ (exists since all exponents are positive); this lies in $\mathbb{A}^n$ and is fixed by $\mathbb{G}_m$. ∎

Use projective dimension theorem

[39]have homomorphism from $\mathbb{G}_m$ into permutation group of components. Alternatively, if you have a point of y, its orbit is connected/irreducible.

Theorem 25.10. *Let G be semisimple with split maximal torus $T \le G$. If* $\operatorname{rank} G = 1$*, then* $G \simeq \mathrm{SL}_2$ *or* PGL_2.

Proof.

(Step 1) If $k = \overline{k}$,[40] then $\dim G/B = 1$.

Proof. G acts on G/B by left translation. Note that our split maximal torus is isomorphic to $T \simeq \mathbb{G}_m$ (since rank 1), and recall that G/B is in bijection with the set $\{\text{Borels in } G\}$ ($gB \leftrightarrow gBg^{-1}$), as a consequence of the normalizer theorem. Note that $\dim G/B > 0$ (otherwise, $G = B$ is solvable and semisimple so $G = R(G) = 1$, a contradiction). Hence,

$$2 \le \dim G/B + 1 \le \#\{T\text{-fixed points in } G/B\} = \#\{\text{Borels containing } T\} = \#W \le 2.$$

For second inequality, embed G/B in some $\mathbb{P}V$ using Chevalley[41] and use Lemma 25.9. For the first equality, T fixes $gB \iff T \le gBg^{-1}$ (B is stabilizer of B). For the last inequality, $W = N/C \le \operatorname{Aut} T = \operatorname{Aut} \mathbb{G}_m = \mathrm{GL}_1(\mathbb{Z}) = \{\pm 1\}$. Since the same number is at both ends, equality holds everywhere; in particular, $\dim G/B = 1$. ■

(Step 2) If $k = \overline{k}$, then $G/B \simeq \mathbb{P}^1$ (WLOG the T-fixed points are 0 and ∞).

Proof. We have a torus acting on it with only 2 fixed points. Hence, we have a nonconstant morphism $\mathbb{G}_m \simeq T \to G/B$ (orbit of non-fixed points). Hence, G/B is dominated by a rational curve, so it must be rational. Since it's smooth, projective over $\overline{k}$, it must be $\mathbb{P}^1$. ■

(Step 3) If $k = \overline{k}$, then the resulting homomorphism

For last iso below, check that $\operatorname{Aut}(\mathbb{P}^1_R) = \mathrm{PGL}_2(R)$ for all k-algebras R. There's a section of the book that does this.

$$\varphi : G \to \underline{\operatorname{Aut}}(G/B) = \underline{\operatorname{Aut}}(\mathbb{P}^1) \simeq \mathrm{PGL}_2$$

has finite kernel.

Proof. The kernel $\ker \varphi \le \bigcap_{g \in G(k)} gBg^{-1}$ is contained in all the stabilizers of points of G/B. Furthermore,

$$\left(\bigcap_{g \in G(k)} gBg^{-1} \right)^0_{\mathrm{red}} = R(G) = 1,$$

so $\ker \varphi$ must be finite. ■

(Step 4) Define the co-character $\lambda : \mathbb{G}_m \simeq T \hookrightarrow G$. Then, $P_G(\lambda)$ (points whose limit exists[42]) is a Borel in G. Note that we are over k, not $\overline{k}$ in this step.

Proof. WLOG $k = \overline{k}$. In general, for an isogeny $\varphi : G \to G'$ and $H \le G$, all smooth connected, H is solvable (resp. parabolic, resp. Borel) $\iff$ $\varphi(H)$ is solvable (resp. parabolic, resp. Borel). For solvable, look at the derived series.

[40] We don't know yet that G has a Borel over k

[41] Theorem 8.4

[42] See Proposition 16.3 for a reminder of this stuff

The upshot is that it suffices to check that $\varphi(P_G(\lambda))$ is a Borel in PGL_2. We know $\varphi(P_G(\lambda)) = P_{\mathrm{PGL}_2}(\lambda')$ where $\lambda' = \varphi \circ \lambda$ is nontrivial. We know what the cocharacters of PGL look like. Since $T = \mathbb{G}_m$ fixes $0, \infty \in \mathbb{P}^1$, we get

$$\begin{array}{rccc} \lambda' : & \mathbb{G}_m & \longrightarrow & \mathrm{PGL}_2 \\ & t & \longmapsto & \begin{pmatrix} t^r & \\ & 1 \end{pmatrix} \end{array}$$

for some $r \neq 0 \in \mathbb{Z}$. At this point, one can just calculate that

$$P_{\mathrm{PGL}_2}(\lambda') = \begin{pmatrix} * & * \\ & * \end{pmatrix} \text{ or } \begin{pmatrix} * & \\ * & * \end{pmatrix} \text{ in } \mathrm{PGL}_2,$$

and these are both Borel. ■

(Step 4') Same for $P_G(-\lambda)$.

(Step 5) Let $B = P_G(\lambda)$. Then, $G/B \simeq \mathbb{P}^1$ over k.

Proof. This quotient is a genus 0 curve with a k-point (since B defined over k), so it must be $\mathbb{P}^1$. ■

(Step 6) Get $\varphi : G \to \mathrm{PGL}_2$ over k. This is a multiplicative isogeny.

Proof. WLOG $k = \overline{k}$. Note

$$\ker \varphi \leq \bigcap gBg^{-1} \leq P_G(\lambda) \cap P_G(-\lambda) = Z_G(\lambda) = C_G(T) = T$$

(last equality using G reductive). ■

(Step 7) $G \simeq \mathrm{SL}_2$ or PGL_2.

Proof. SL_2 is the universal cover of PGL_2 with $\pi_1(\mathrm{PGL}_2) = \mu_2$.[43] ■

That finishes the proof. ■

Question 25.11 (Audience)**.** *How to show that* SL_2 *is the universal cover.*

Answer. Can use fact that the character group of the fundamental group is always the Picard group, and then try to compute the Picard group instead. That is, show that $\operatorname{Pic} \mathrm{SL}_2 = 0$. One way to do this is to show that $\mathrm{SL}_2 \times \mathbb{G}_m = \mathrm{GL}_2$ as varieties, and $\mathrm{GL}_2 \overset{\text{open}}{\subset} \mathbb{A}^{n^2}$, so these all have trivial Picard group. Once you know this, we know $\pi_1(\mathrm{PGL}_2) = \ker(\mathrm{SL}_2 \to \mathrm{PGL}_2) = Z(\mathrm{SL}_2) = \mu_2$.

[43] Only talked above universal covers over separably closed fields. Can first prove this over k^s and then use Galois descent

26 Lecture 26 (4/21)

26.1 Last time

Notation 26.1.

$$\underbrace{G}_{\text{smooth connected linear}} \geq \underbrace{B}_{\text{Borel}} \geq \underbrace{C}_{\text{Cartan}} \geq \underbrace{T}_{\text{maximal torus}}$$

$C := C_G(T)$, $N := N_G(T)$, and $W = W(G/T) := N/C$ is the Weyl group, a finite étale group.

Recall 26.2. The *radical* is $R(G) :=$ largest smooth connected, solvable normal subgroup

The *unipotent radical* is $R_u(G) :=$ largest smooth connected unipotent normal subgroup.

Recall 26.3.

- G is **semisimple** if $R(G_{\overline{k}}) = 1$ (e.g. $G = \mathrm{SL}_n$)
- G is **reductive** if $R_u(G_{\overline{k}}) = 1$ (e.g. $G = \mathrm{GL}_n$)
- If G semisimple, then $Z(G)$ finite and G is perfect, i.e. $[G, G] = G$
- If G is reductive, then $R(G)$ is a torus, $Z(G)$ is of multiplicative type, and the groups $G^{\mathrm{der}} = [G, G]$, $G^{\mathrm{ad}} = G/Z(G)$, and $G/R(G)$ are semisimple
- $$\text{reductive} = \frac{\text{torus} \times \text{semisimple}}{\text{finite}}$$

  where the torus $R(G)$ and semisimple group G^{der} are subgroups generating G, and their intersection is a finite group contained in the center of both.

Recall 26.4. If G is a semisimple group of rank 1 with a split maximal torus T, then $G \simeq \mathrm{SL}_2$ or PGL_2.

Definition 26.5. A **split reductive group** is a pair (G, T) with G reductive and T a split maximal torus

Fact (Next week's homework)**.** Every split reductive group of semisimple rank 1 is one of

$$\mathbb{G}_m^{r-1} \times \mathrm{SL}_2\,, \;\; \mathbb{G}_m^{r-1} \times \mathrm{PGL}_2, \;\; \text{and} \;\; \mathbb{G}_m^{r-2} \times \mathrm{GL}_2\,.$$

26.2 Forms of $\mathrm{GL}_2, \mathrm{SL}_2$ and PGL_2

Warning 26.6. In the book, there's a theorem saying what all the (non-split) reductive groups of semisimple rank 1 are, but it's incorrect as currently stated.

Recall 26.7. Let k be a field with separable closure $K := k^s$ and Galois group $\mathrm{Gal} := \mathrm{Gal}(k^s/k)$. Then,

$$\begin{array}{ccc} \{k\text{-forms of } G\} & \longleftrightarrow & \mathrm{H}^1(\mathrm{Gal}, \mathrm{Aut}\, G_K) \\ \{k\text{-forms of } M_2(k)\} & \longleftrightarrow & \mathrm{H}^1(\mathrm{Gal}, \mathrm{Aut}_{K\text{-alg}}(M_2(K))) \end{array}$$

Above, a k-form of $M_2(k)$ is a k-algebra A s.t. $A \otimes_k K \simeq M_2(K)$ (these are also called **quaternion algebras** over k).

Theorem 26.8 (Skolem-Noether Theorem). *Every automorphism of a central simple algebra (over a field?) is inner.*

Corollary 26.9.
$$\mathrm{Aut}_{K\text{-}alg}\, M_2(K) = \frac{M_2(K)^\times}{Z(M_2(K))} = \frac{\mathrm{GL}_2(K)}{K^\times} \simeq \mathrm{PGL}_2(K).$$

Proposition 26.10. *Every automorphism of* $\mathrm{GL}_2, \mathrm{SL}_2, \mathrm{PGL}_2$ *over* K *is inner. Hence,*

$$\begin{aligned} \mathrm{Aut}\,\mathrm{GL}_2 &= \mathrm{GL}_2(K)/\mathit{center} &&= \mathrm{PGL}_2(K) \\ \mathrm{Aut}\,\mathrm{SL}_2 &= \mathrm{SL}_2(K)/\mathit{center} &&= \mathrm{PGL}_2(K) \\ \mathrm{Aut}\,\mathrm{PGL}_2 &= \mathrm{PGL}_2(K)/\mathit{center} &&= \mathrm{PGL}_2(K) \end{aligned}$$

Corollary 26.11. *There are natural bijections*

$$\begin{tikzcd}
 & \{k\text{-}\mathit{forms\ of}\ \mathrm{GL}_2\} \\
\{\mathit{quaternion\ algebras}/k\} \arrow[ur, leftrightarrow] \arrow[r, leftrightarrow] \arrow[dr, leftrightarrow] & \{k\text{-}\mathit{forms\ of}\ \mathrm{SL}_2\} \\
 & \{k\text{-}\mathit{forms\ of}\ \mathrm{PGL}_2\}
\end{tikzcd}$$

You can make this explicit. Given a quaternion algebra A/k, can form the algebraic groups

$$G^A(R) = (A \otimes R)^\times\,, \quad S^A(R) = \ker\left((A \otimes R)^\times \xrightarrow{\text{reduced norm}} R^\times\right), \quad \textit{and}\ P^A := G^A/\mathbb{G}_m.$$

26.3 Root data and (split) reductive groups

Let (G, T) be a split reductive group of rank r, and let $X = X^*(T) = X(T) = \mathbb{Z}^r$ be the character group. There is also the cocharacter group $X^\vee = \mathrm{Hom}(\mathbb{G}_m, T) = \mathrm{Hom}_{\mathbb{Z}}(X, \mathbb{Z}) = \mathbb{Z}^r$. We let

$$\langle -, - \rangle : X \times X^\vee \to \mathbb{Z}$$

denote the natural perfect pairing, i.e. $\langle \alpha, \lambda \rangle = \alpha \circ \lambda \in \mathrm{End}\,\mathbb{G}_m = \mathbb{Z}$.

Recall the adjoint action $G \curvearrowright \mathfrak{g} := \mathrm{Lie}\, G$. We can restrict this action to T to get a decomposition

$$\mathfrak{g} = \bigoplus_{\alpha \in X(T)} \mathfrak{g}_\alpha \ \text{ where } \ \mathfrak{g}_\alpha = \alpha\text{-eigenspace}$$

as T-reps. We can easily describe the 0-eigenspace:

$$\mathfrak{g}_0 = (\mathrm{Lie}\, G)^T = \mathrm{Lie}(G^T) = \mathrm{Lie}\, C = \mathrm{Lie}\, T =: \mathfrak{t}.$$

Definition 26.12. The nonzero $\alpha \in X$ such that $\mathfrak{g}_\alpha \neq 0$ are called the **roots**. We use Φ to denote the set of all roots, a finite subset of X.

We write

$$\mathfrak{g} = \mathfrak{t} \oplus \bigoplus_{\alpha \in \Phi} \mathfrak{g}_\alpha.$$

Recall 26.13. There's the **Weyl group** $W = N/C$ which acts by conjugation on (G, T). Hence, it acts on $X, X^\vee, \Phi, \ldots$ as well.

Example. Consider $G = \mathrm{GL}_2$ and $T = \mathbb{G}_m^2$ with character group $X = \mathbb{Z}^2$ spanned by

$$\chi_i \begin{pmatrix} t_1 & \\ & t_2 \end{pmatrix} := t_i.$$

Bjorn drew a picture of the weight lattice for the adjoint rep, but I'm too lazy to reproduce it in real time... There are two roots in this example: $\pm\alpha$ where $\alpha = \chi_1 - \chi_2$.

What about the Weyl group? Need to find a matrix which normalizes T but does not centralize it. Turns out W is generated by $s_\alpha = \left[n_\alpha := \begin{pmatrix} & -1 \\ 1 & \end{pmatrix} \right]$. This acts on X as $\chi_1 \mapsto \chi_2$ and $\chi_2 \mapsto \chi_1$ (reflection across line $x = y$ in the character lattice).

Example. Consider $G = \mathrm{SL}_2$ with $T = \left\{ \begin{pmatrix} t & \\ & t^{-1} \end{pmatrix} \right\}$ and character group $X = \mathbb{Z}$ spanned by

$$\chi \begin{pmatrix} t & \\ & t^{-1} \end{pmatrix} := t.$$

The roots in this case are -2χ and 2χ.

The Weyl group here acts as -1. The matrix n_α from before was chosen to work for all 3 of these cases.

Example. Now say $G = \mathrm{PGL}_2$ with $T = \left\{ \begin{pmatrix} t & \\ & 1 \end{pmatrix} \right\}$ whose character group $X = \mathbb{Z}$ is spanned by

$$\chi \begin{pmatrix} t & \\ & 1 \end{pmatrix} := t.$$

The roots in this case are $\pm\chi$.

The Weyl group here acts as -1.

Let's take a look at a more involved situation.

Example. Take $G = \mathrm{GL}_3$ with $T = \mathbb{G}_m^3$ the diagonal torus. Let α be a root; in fact, let's consider $\alpha = \chi_1 - \chi_2 : T \to \mathbb{G}_m$, i.e.

$$\alpha \begin{pmatrix} t_1 & & \\ & t_2 & \\ & & t_3 \end{pmatrix} = \frac{t_1}{t_2}.$$

Let

$$S_\alpha = \left\{ \begin{pmatrix} t & & \\ & t & \\ & & t_3 \end{pmatrix} \right\}$$

whose centralizer is

$$G_\alpha = \left\{ \begin{pmatrix} * & * & \\ * & * & \\ & & * \end{pmatrix} \right\} = \mathrm{GL}_2 \times \mathbb{G}_m.$$

What do things look like in general? Consider any root α, and define the subtorus

$$S_\alpha := (\ker \alpha)_t := (\ker \alpha)^0_{\mathrm{red}} \leq \underbrace{T}_{\dim r}$$

Remember: If G is a of multiplicative type, then G^0_{red} is an algebraic group (and a torus)

(the t subscript is taking the '**toric part**'). In addition, we define $G_\alpha := C_G(S_\alpha)$, the centralizer of S_α.

Proposition 26.14. *(G_α, T) is a split reductive group of semisimple rank 1.*

Proof. We already know that the centralizer of a torus (in a reductive group) is itself reductive. We also know that $T \leq C(S_\alpha) = G_\alpha$ is a maximal torus, and $S_\alpha \leq Z(G_\alpha) \leq T$ (note $Z(G_\alpha) \leq C_{G_\alpha}(T) = T$).[44] We can look at Lie algebras:

$$\operatorname{Lie} G_\alpha = \mathfrak{g}^{S_\alpha} = \bigoplus_{\beta \in \mathbb{Q}\alpha \cap X} \mathfrak{g}_\beta = \mathfrak{t} \oplus \mathfrak{g}_\alpha \oplus ?$$

Since there's a $\mathfrak{g}_\alpha$ above, T acts nontrivially on $\operatorname{Lie} G_\alpha$, so $T \not\leq Z(G_\alpha)$. Hence,

$$R(G_\alpha) = Z(G_\alpha)_t = S_\alpha.$$

Thus, $G_\alpha / R(G_\alpha)$ has maximal torus $T/R(G_\alpha) = T/S_\alpha$ of dimension 1, so G has semisimple rank 1. ■

Corollary 26.15. $\operatorname{Lie} G_\alpha = \mathfrak{t} \oplus \mathfrak{g}_\alpha \oplus \mathfrak{g}_{-\alpha}$ *with* $\dim \mathfrak{g}_{\pm\alpha} = 1$.

This is because we know all the split semisimple groups of rank 1. There are basically 3 examples, and we computed things for them earlier.

Example. If you have $G = \mathbb{G}_m^n \times \mathrm{GL}_2$, then $X = \mathbb{Z}^n \oplus X_{\mathrm{GL}_2}$ and the roots are the same as those for GL_2. Adding this $\mathbb{G}_m^n$ just makes the torus bigger, but does not really add much to the conjugation action.

Choose a cocharacter $\lambda : \mathbb{G}_m \to T$ such that $\langle \alpha, \lambda \rangle > 0$. Define

$$U_\alpha = U_{G_\alpha}(\lambda)$$

(subgroup where limit equals identity).

Example ($G = \mathrm{GL}_3$)**.** Take $\lambda : \mathbb{G}_m \to T$ given by $t \mapsto \begin{pmatrix} t & & \\ & 1 & \\ & & 1 \end{pmatrix}$. Then,

$$U_\alpha = \left\{ \begin{pmatrix} 1 & * & \\ & 1 & \\ & & 1 \end{pmatrix} \right\}.$$

[44] Note $Z(G_\alpha)$ is of multiplicative type, but not necessarily a torus

Proposition 26.16. *In general, there is an isomorphism* $u_\alpha : \mathbb{G}_a \xrightarrow{\sim} U_\alpha$ *and* $\operatorname{Lie} U_\alpha = \mathfrak{g}_\alpha$.

(Proof this by checking cases).

Can also define $U_{-\alpha} = U_{G_\alpha}(-\lambda)$. Get same conclusions: $U_{-\alpha} \simeq \mathbb{G}_a$ and $\operatorname{Lie} U_{-\alpha} = \mathfrak{g}_{-\alpha}$.

In fact, can construct a multiplicative isogeny

$$v_\alpha : \mathrm{SL}_2 \to G^\alpha := G_\alpha^{\mathrm{der}}$$

mapping $\begin{pmatrix} 1 & * \\ & 1 \end{pmatrix}$ to U_α and $\begin{pmatrix} 1 & \\ * & 1 \end{pmatrix}$ to $U_{-\alpha}$.

Remark 26.17. This v_α is uniquely determined by the choice of a nonidentity element of U_α (or equivalently of $\mathfrak{g}_\alpha$).

27 Lecture 27 (4/23)

27.1 Last time

Say (G, T) is a split reductive group of rank r.

Recall 27.1. $X := \operatorname{Hom}(T, \mathbb{G}_m) \simeq \mathbb{Z}^r$ is the character group of T.

$X^\vee := \operatorname{Hom}(\mathbb{G}_m, T) \simeq \mathbb{Z}^r$ is the cocharacter group of T.

$\langle -, - \rangle : X \times X^\vee \to \mathbb{Z}$ is the perfect paring sending (α, λ) to $\alpha \circ \lambda \in \operatorname{End} \mathbb{G}_m = \mathbb{Z}$.

Recall 27.2. G acts on G via conjugation, and on $\mathfrak{g}$ via the adjoint action. As a T-rep, we can decompose

$$\mathfrak{g} = \mathfrak{t} \oplus \bigoplus_{\alpha \in \Phi} \mathfrak{g}_\alpha,$$

where $\Psi = \{\text{roots}\} := \{\text{nonzero } \alpha \in X \text{ such that } \mathfrak{g}_\alpha \neq 0\}$. Each $\mathfrak{g}_\alpha$ is 1-dimensional.

Recall 27.3. The Weyl group $W = N/C$ acts by conjugation on (G, T), hence on $X, X^\vee, \Phi$, etc.

For each root α, we define

- $S_\alpha := (\ker \alpha)_t := (\ker \alpha)^0_{\mathrm{red}}$, subtorus of rank $r - 1$ in T
- $G_\alpha := C_G(S_\alpha)$, a split reductive group of semisimple rank 1 with
$$\operatorname{Lie} G_\alpha = \mathfrak{t} \oplus \mathfrak{g}_\alpha \oplus \mathfrak{g}_{-\alpha}.$$
- $G^\alpha := G_\alpha^{\mathrm{der}}$, a split semisimple group of rank 1, isomorphic to SL_2 or PGL_2.
- $v_\alpha : \mathrm{SL}_2 \twoheadrightarrow G^\alpha \hookrightarrow G_\alpha \hookrightarrow G$ is the universal cover of G^α, chosen so that
  - v_α maps $\begin{pmatrix} 1 & * \\ & 1 \end{pmatrix} \simeq \mathbb{G}_a$ isomorphically to a group U_α with Lie algebra $\mathfrak{g}_\alpha$; and
  - v_α maps $\begin{pmatrix} 1 & \\ * & 1 \end{pmatrix} \simeq \mathbb{G}_a$ isomorphically to a group $U_{-\alpha}$ with Lie algebra $\mathfrak{g}_{-\alpha}$.

27.2 Weyl Groups and Root {Data, Systems}

Consider the Weyl group of $G_\alpha = C_G(S_\alpha) \subset G$. We have

$$N_{\mathrm{SL}_2}(\text{torus}) \xrightarrow{v_\alpha} N_{G_\alpha}(T) \twoheadrightarrow W(G_\alpha, T) \leq W(G, T) =: W.$$

Recall 27.4. $\# W(G_\alpha, T) = 2$

What's the generator? An element of $N_{\mathrm{SL}_2}(\mathbb{G}_m)$ not contained in the diagonal torus is $\begin{pmatrix} & -1 \\ 1 & \end{pmatrix}$. This maps to some $n_\alpha \in N_{G_\alpha}(T)$ which then maps to a generator $s_\alpha \in W(G_\alpha, T)$.

Construction 27.5. Consider the composition

$$\begin{array}{rcccccc} \alpha^\vee : & \mathbb{G}_m & \longrightarrow & \mathrm{SL}_2 & \xrightarrow{v_\alpha} & G \\ & t & \longmapsto & \begin{pmatrix} t & \\ & t^{-1} \end{pmatrix} & \longmapsto & ???, \end{array}$$

called the **coroot** of α. This $\alpha^\vee \in X^\vee$ is characterized by the formula

$$\begin{array}{rccc} s_\alpha : & X & \longrightarrow & X \\ & x & \longmapsto & x - \langle x, \alpha^\vee \rangle \alpha. \end{array}$$

Since $s_\alpha(\alpha) = -\alpha$, we see that $\langle \alpha, \alpha^\vee \rangle = 2$. Note that s_α fixes (and in fact is reflection across) the hyperplane in $X_{\mathbb{R}} = X \otimes_{\mathbb{Z}} \mathbb{R}$ orthogonal to $\alpha^\vee$.

Remark 27.6. We haven't defined an inner product on this vector space $X_{\mathbb{R}}$. Sometimes people do this (and then the hyperplane is orthogonal to α), but we have kept to space and its dual separate.

Proof that s_α is given by the claimed formula. Reduces to doing a calculation in SL_2 and PGL_2, where you can do everything explicitly. ■

Question 27.7. *What's the group $\langle U_\alpha, U_{-\alpha} \rangle$ generated by $U_{\pm\alpha}$?*

The upper/lower triangular matrices generate SL_2, so these will generate the image $G^\alpha = \langle U_\alpha, U_{-\alpha} \rangle$ of SL_2. If you throw in the torus as well, then you get $G_\alpha = \langle T, U_\alpha, U_{-\alpha} \rangle$. This was for one α, but if you take all of them, then you get all of $G = \langle T, U_\alpha : \alpha \in \Phi \rangle$ since you get the whole Lie algebra (and G smooth, connected).

Proposition 27.8. $\dim G = \dim T + \#\Phi$.

(since $\dim \mathfrak{g}_\alpha = 1$ for all roots).

Let's give a name to all this data we've defined.

Definition 27.9. A **root datum** (in the sense of SGA3) is $\mathcal{R} = (X, R, X^\vee, R^\vee, \langle -, - \rangle, \alpha \mapsto \alpha^\vee)$. Here,

(1) X is a f.free $\mathbb{Z}$-module

(2) $X^\vee$ is a f.free $\mathbb{Z}$-module

(3) $\langle \, , \, \rangle : X \times X^\vee \to \mathbb{Z}$ is a perfect pairing

(4) $R \subset X$ is a finite subset (the **roots**)

(5) $R^\vee \subset X^\vee$ is a finite subset (the **coroots**)

(6) $\alpha \mapsto \alpha^\vee$ is a bijection $R \to R^\vee$.

We require these satisfy

(RD1) If $\alpha \in R$, then $\langle \alpha, \alpha^\vee \rangle = 2$.

(RD2) For each $\alpha \in R$, the reflection

$$\begin{array}{cccc} s_\alpha : & X & \longrightarrow & X \\ & x & \longmapsto & x - \langle x, \alpha^\vee \rangle \alpha \end{array}$$

preserves the set of roots, i.e. $s_\alpha(R) = R$. Similarly,

$$\begin{array}{cccc} s_{\alpha^\vee} : & X^\vee & \longrightarrow & X^\vee \\ & y & \longmapsto & y - \langle \alpha, y \rangle \alpha^\vee \end{array}$$

preserves the set of coroots.

Remark 27.10. $s_{\alpha^\vee}$ is the transpose of s_α:

$$\langle s_\alpha(x), y \rangle = \langle x, y \rangle - \langle x, \alpha^\vee \rangle \alpha, y = \langle x, s_{\alpha^\vee}(y) \rangle$$

Remark 27.11. There's an equivalent definition (what Milne calls a root datum not in the sense of SGA3) where you just give $(X, R, R \to X^\vee := \mathrm{Hom}(X, \mathbb{Z}))$ with the map $R \to X^\vee$ denoted $\alpha \mapsto \alpha^\vee$ and you require

(rd1) **(RD1)** ($\alpha^\vee(\alpha) = 2$).

(rd2) First half of **(RD2)**.

(rd3) The group generated by the s_α's is finite.

What happens if you forget the lattice and only remember the vector space $V := \mathbb{R}$-span of R? Then you get

Definition 27.12. A **root system** is (V, R) with V a f.dim $\mathbb{R}$-vector space, and $R \subset V$ a finite subset (the 'roots') satisfying

(RS1) R spans V

(RS2) For each $\alpha \in R$, there exists a $\alpha^\vee \in V^\vee$ such that $\langle \alpha, \alpha^\vee \rangle = 2$, $s_\alpha(R) \subset R$, and $\alpha^\vee(R) \subset \mathbb{Z}$.

We can use this new terminology to summarize what we know about split reductive groups.

Theorem 27.13. *Each split reductive group (G, T) gives rise to a **reduced**[45] root datum.*

[45] For each $\alpha \in R$, the only real multiples of α in R are $\pm\alpha$

Definition 27.14. The **Weyl group of a root system** (V, R) is

$$W(R) := \langle s_\alpha : \alpha \in R \rangle \leq \operatorname{Aut} X.$$

This is a finite group, and we'll later see it agrees with the other definition of Weyl group when the root system comes from a split reductive group.

Proposition 27.15.

$$Z(G) = \bigcap_{\alpha \in \Phi} \ker \alpha.$$

Proof. Recall $G = \langle T, \text{ all the } U_\alpha\text{'s} \rangle$. What commutes with all of these? First, $C(T) = T$, so $Z(G) \leq T$, and in fact

$$Z(G) = \bigcap C_T(U_\alpha) = \bigcap \ker \alpha$$

since conjugation by $t \in T$ acts on U_α as $\alpha(t)$ on U_α (since $\operatorname{Lie} U_\alpha = \mathfrak{g}_\alpha$). ∎

Corollary 27.16. *The character groups of*

$$R(G) = Z(G)^0_{\text{red}} \leq Z(G) \leq T$$

are

$$X/(\mathbb{Z}\Phi)_{sat} \twoheadleftarrow X/\mathbb{Z}\Phi \twoheadleftarrow X.$$

Above, $\mathbb{Z}\Phi$ is the $\mathbb{Z}$-span of Φ, the subgroup generated by the roots. On the other hand, its **saturation** is

$$(\mathbb{Z}\Phi)_{\text{sat}} := \{\beta : n\beta \in \mathbb{Z}\Phi \text{ for some } n \geq 1\}.$$

Example. When $G = \mathrm{SL}_2 \times \mathbb{G}_m$, one sees $\mathbb{Z}\Phi = 2\mathbb{Z} \times \{0\}$ so $(\mathbb{Z}\Phi)_{\text{sat}} = \mathbb{Z} \times \{0\}$. Since the center is diagonalizable, one sees from its character group that $Z(G) \simeq \mu_2 \times \mathbb{G}_m$.

Proof of Corollary 27.16. We have an exact sequence

$$0 \longrightarrow Z(G) \longrightarrow T \xrightarrow{\Pi}_{\alpha \in \Phi} \mathbb{G}_m$$

by the proposition. Taking character groups gives

$$0 \leftarrow X(Z(G)) \leftarrow X \leftarrow \bigoplus_{\alpha \in \Phi} \mathbb{Z}.$$

Finally, $R(G) = Z(G)_t$ means that $X(R(G)) = X(Z(G))/\text{torsion}$ and this finishes the proof. ∎

Corollary 27.17. *TFAE*

(1) *G is semisimple*

(2) *$Z(G)$ is finite*

(3) *$\mathbb{Z}\Phi$ is of finite index in X*

28 Lecture 28 (4/26)

28.1 Last time

Say (G,T) split reductive of rank r.

We can decompose the adjoint rep of T on $\mathfrak{g}$:

$$\mathfrak{g} = \mathfrak{t} \oplus \bigoplus_{\alpha\in\Phi} \mathfrak{g}_\alpha.$$

These are Lie algebras of $G = \langle T, U_\alpha \text{ for } \alpha \in \Phi\rangle$. Each U_α is a copy of $\mathbb{G}_a$ and $\dim \mathfrak{g}_\alpha = 1$.

We can form the *root datum*

$$\mathcal{R}(G,T) := \left(\underbrace{X}_{\text{character group}}, \underbrace{\Phi}_{\text{roots}}, \underbrace{X^\vee}_{\text{cocharacter group}}, \underbrace{\Phi^\vee}_{\text{coroots}}, \underbrace{\langle -,-\rangle}_{X\times X^\vee\to\mathbb{Z}}, \underbrace{\alpha\mapsto\alpha^\vee}_{\text{bijection}} \right)$$

If we forget X and $X^\vee$, but remember $V := \mathbb{R}\Phi \le X_\mathbb{R}$, then we get a *root system* (V,Φ).

The *Weyl group* $W := N/C$ acts by conjugation on (G,T) and so on $X, X^\vee, \Phi$, etc. $\alpha \in \Phi$ gives rise to $s_\alpha \in W$ acting on X. The action of s_α on $V = \mathbb{R}\Phi$ is characterized by

- $s_\alpha(\alpha) = -\alpha$
- s_α acts as the identity on some hyperplane in V, and
- s_α preserves Φ

These first two conditions tell you it's a reflection

The element $\alpha^\vee \in X^\vee$ can be characterized by

$$s_\alpha(x) = x - \langle x, \alpha^\vee\rangle\, \alpha$$

so $\langle \alpha, \alpha^\vee\rangle = 2$, and s_α is the identity on the hyperplane $(\alpha^\vee)^\perp \le X_\mathbb{R}$.

28.2 Root systems (and stuff about relating groups to their algebras)

Say we are given the root system (V,Φ) of some (secret) split *semisimple*[46] group.

Question 28.1. *What are the possibilities for X?*

There are natural choices for "lower/upper bounds" on X. On one hand, X contains all the roots, so must contain the **root lattice** $Q := \mathbb{Z}\Phi$. On the other hand, $\langle \alpha, \beta^\vee\rangle \in \mathbb{Z}$ for any root α and coroot $\beta^\vee$, so X must be contained in the **weight lattice** $P := \{x \in V : \langle x, \Phi^\vee\rangle \subset \mathbb{Z}\}$, the lattice dual to $\mathbb{Z}\Phi^\vee$. Note that Q is already a full rank lattice, so $[P:Q] < \infty$. That is, we have X sandwiched

$$Q \subset X \subset P$$

between a finite index lattice inclusion, so there will only be finitely many possible choices for X. We'll see later all these choices are realizable (assuming I heard correctly). In particular, we'll have a picture

[46] So X is a lattice in V

like

$$\begin{tikzcd}
\widetilde{G} \arrow[d, two heads, "\pi_1(G)"'] & P \\
G \arrow[d, two heads, "Z(G)"'] & X \arrow[u, hook, "P/X = \operatorname{Pic} G"'] \\
G^{\mathrm{ad}} & Q \arrow[u, hook, "X/Q"']
\end{tikzcd}$$

Above,

$$\begin{array}{ccc}
\text{groups} & \xrightarrow{\text{char. gp. of max. torus}} & \text{lattices} \\
\text{kernels} & \xrightarrow{\text{char. group}} & \text{quotients}
\end{array}$$

e.g. $\widetilde{G}$ has max torus with character group P and $X(\ker(\widetilde{G} \twoheadrightarrow G)) = P/X$.

Note 4. Distracted making up diagram and missed what he said we're doing next.

Suppose $H \leq G$ is normalized by T. Get T-reps $\mathfrak{h} \leq \mathfrak{g} = \mathfrak{t} \oplus \bigoplus_{\alpha \in \Phi} \mathfrak{g}_\alpha$. Since $\mathfrak{h}$ is a subrep, it must be a sum of some of these eigenspaces. That is,

$$\mathfrak{h} = \mathfrak{t}_{\mathfrak{h}} \oplus \bigoplus_{\alpha \in \Psi} \mathfrak{g}_\alpha \text{ for some } \mathfrak{t}_{\mathfrak{h}} \leq \mathfrak{t} \text{ and } \Psi \subset \Phi.$$

Can we reconstruct $\mathfrak{h}$ from this Lie algebra?

Proposition 28.2 (Yes)**.** *Assume $H \leq G$ is a smooth subgroup such that T normalizes H (e.g. $T \leq H$) Then, for each root $\alpha \in \Phi$, H contains $U_\alpha \iff \mathfrak{h} := \operatorname{Lie} H$ contains $\operatorname{Lie} U_\alpha = \mathfrak{g}_\alpha$.*

Proof. ($\rightarrow$) Easy.

($\leftarrow$) Suppose that $\mathfrak{h}$ contains $\mathfrak{g}_\alpha$. WLOG replace G by $G_\alpha := C_G(S_\alpha)$ and H by $H_\alpha := C_H(S_\alpha) = G_\alpha \cap H$.

Why is this ok? Centralizers of tori are smooth, so we have not lost smoothness doing this. The Lie algebras of G_α, H_α contain $\mathfrak{g}_\alpha$, so the hypotheses are preserved. If $U_\alpha \leq H_\alpha$ then $U_\alpha \leq H$ since $H_\alpha \leq H$, so everything is fine.

The point of this reduction is that we now have a group of semisimple rank 1. The Lie algebra of $G = G_\alpha$ now looks like $\mathfrak{t} \oplus \mathfrak{g}_\alpha \oplus \mathfrak{g}_{-\alpha}$. Choose a cocharacter λ s.t. $\langle \alpha, \lambda \rangle > 0$ so $U_G(\lambda) = U_\alpha$ (this is where[47] we use that we reduced to $G = G_\alpha$). We intersect with H to see that $U_H(\lambda) = H \cap U_\alpha \leq U_\alpha$ is a smooth connected subgroup of $U_\alpha \cong \mathbb{G}_a$; hence, it's either trivial or everything. It's Lie algebra is $\mathfrak{h} \cap \mathfrak{g}_\alpha = \mathfrak{g}_\alpha$ is nontrivial, so $U_H(\lambda) = H \cap U_\alpha = U_\alpha$ and we win. ■

Remember: This limit stuff is apparently good for picking out pieces of the Lie algebra and finding corresponding groups

I will never remember what this limit stuff is off the top of my head

Corollary 28.3. *If H is smooth connected and $H \geq T$, then*

$$H = \langle T, \textit{the } U_\alpha \textit{ with } \mathfrak{g}_\alpha \leq \mathfrak{h} \rangle$$

Proof. Both are smooth connected with the RHS contained in the LHS and their Lie algebras agreeing. ■

28.2.1 Weyl chambers

Fix a root system (V, Φ).

[47] Looking at the Lie algebra $U_G(\lambda)$, this is picking out the piece where λ positive, but that's simply $\mathfrak{g}_\alpha$ since there's nothing else

Note 5. It would be nice if I were to draw pictures and add them here...

We have roots in V, and coroots in $V^\vee$. Given a root $\alpha \in V$, we can form a hyperplane $\alpha^\perp \subset V^\vee$. These separate $V^\vee$ into different pieces, called Weyl chambers.

Definition 28.4. A **Weyl chamber** is a connected component of

$$V^\vee - \bigcup_{\alpha\in\Phi} \alpha^\perp.$$

Definition 28.5. A $\lambda \in V^\vee$ is called **regular** if it lies in some Weyl chamber, i.e. $\langle \alpha, \lambda\rangle \neq 0$ for all $\alpha \in \Phi$. In this case, we set

$$\Phi^+ := \{\alpha \in \Phi : \langle \alpha, \lambda\rangle > 0\}.$$

We call this a **system of positive roots**:

- For all α, exactly one of $\pm\alpha$ is positive (i.e. is in Φ^+)
- For all positive α, β, their sum $\alpha + \beta$ is positive *if* it is a root.

Given such a system, we can pick out the "smallest" positive roots.

Definition 28.6. A **base for** (V, Φ) is a basis $\Delta \subset \Phi$ of V such that each $\alpha \in \Phi$ is of the form $\sum_{s\in\Delta} m_s s$ with either all $m_s \in \mathbb{Z}_{\geq 0}$ or all $m_s \in \mathbb{Z}\leq 0$. Elements of Δ are called the **simple roots**.

Theorem 28.7. *Say we have a root system (V, Φ) of some split reductive group (G, T). Then,*

(1) *There exists bijections*

$$\begin{tikzcd}
\{\textit{Borels in } G \textit{ containing } T\} \arrow[dr, "\{\alpha:\mathfrak{g}_\alpha\leq \mathrm{Lie}\, B\}"'] & & \{\textit{Weyl chambers in } V^\vee\} \arrow[ll, "P(\lambda)\ \textit{for}\ \lambda\in \textit{chamber}"'] \arrow[dl, "\{\alpha:\langle\alpha,\lambda\rangle>0\}"] \\
& \{\textit{systems of positive roots}\} \arrow[d, bend left, "\{\textit{positive roots not sum of 2 positive roots}\}"] & \\
& \{\textit{bases}\} \arrow[u, bend left, "\Phi\cap\mathbb{Z}_{\geq 0}\Delta"] &
\end{tikzcd}$$

(2) *The bijections are W-equivariant.*

(3) *W acts simply transitively on each set.*

Next time we'll explain some of the details of the proof of this.

29 Lecture 29 (4/28)

Note 6. *10 (5?) minutes late*

29.1 More Root Stuff

"The point of view of this class is that combinatorics is an easy subject that anyone can do, and the whole of algebraic groups is all the hard stuff" (paraphrased)

Recall we had ended last time by stating the following theorem.

Theorem 29.1. *Say we have a root system (V, Φ) of some split reductive group (G, T). Then,*

(1) *There exists bijections*

$$\begin{tikzcd}
\{\textit{Borels in } G \textit{ containing } T\} \arrow[dr, "\{\alpha:\mathfrak{g}_\alpha \leq \mathrm{Lie}\, B\}"'] & & \{\textit{Weyl chambers in } V^\vee\} \arrow[ll, "P(\lambda)\ \textit{for}\ \lambda\in \textit{chamber}"'] \arrow[dl, "\{\alpha:\langle\alpha,\lambda\rangle>0\}"] \\
& \{\textit{systems of positive roots}\} \arrow[d, bend left, "\{\textit{positive roots not sum of 2 positive roots}\}"] & \\
& \{\textit{bases}\} \arrow[u, bend left, "\Phi\cap\mathbb{Z}_{\geq 0}\Delta"] &
\end{tikzcd}$$

(2) *The bijections are W-equivariant.*

(3) *W acts simply transitively on each set.*

Proof Sketch of not-purely-combinatorial parts. ($\leftarrow$ arrow on top well-defined) Why is $P(\lambda)$ a Borel? We know it is smooth connected, and that it is a semidirect product $P(\lambda) = Z(\lambda) \ltimes U(\lambda)$. We computed the Lie algebras of these pieces earlier, so we know $Z(\lambda)$ has Lie algebra $\mathfrak{t}$ ($\implies Z(\lambda) = T$ is a maximal torus). Furthermore, $U(\lambda)$ is unipotent, so $P(\lambda)$ is solvable. We claim it is maximal solvable. Suppose $P(\lambda) < H \leq G$ with H smooth connected; we'll show H is not solvable. Then, $\mathfrak{h} := \mathrm{Lie}\, H$ contains $\mathfrak{t}$ (since $T \leq H$), $\mathfrak{g}_\alpha$ for all α with $\lambda(\alpha) > 0$ (since $U(\lambda) \leq H$), and $\mathfrak{g}_{-\alpha}$ for some α with $\lambda(\alpha) > 0$ (since $P(\lambda) \subsetneq H$). Thus, H contains U_α and $U_{-\alpha}$, so H contains a copy of SL_2 or PGL_2. Neither of these are solvable, so H is not solvable either. Thus, $P(\lambda)$ is a maximal solvable smooth connected subgroup, i.e. a Borel.

(the upper triangle commutes) This is by definition slash by Proposition 16.3 **(6)**.

(**(2)** + **(3)**) They are all W-equivariance basically since they're all canonical. This is not hard to check. We've shown before that W acts transitively on the Borels, so it does so too on each of the other sets. ∎

Corollary 29.2. *For each Borel $B \supset T$, $\exists!$ Borel B^-, the **opposite Borel**, such that $B \cap B^- = T$. Furthermore, the variety morphism*

$$B_u^- \times T \times B_U \longrightarrow G$$

given by multiplication is an open immersion.

Remark 29.3. The image of this morphism contains both $B = T \cdot B_u$ and $B^- = T \cdot B_u^-$.

Proof. If $B = P(\lambda)$, then define $B^- := P(-\lambda)$. Its unipotent radical will be $B_u^- = U(-\lambda)$. Recall (Proposition 16.3 **(4)**), we showed before that $U(-\lambda) \times P(\lambda) \to G$ is an open immersion. ∎

Definition 29.4. This open subset $B_u^- \times T \times B_u \hookrightarrow G$ is called the **big cell**.

Example. Take $G = \mathrm{GL}_n$ and $B = B_n$ the upper triangular matrices. Then, $B^- =$ lower triangular matrices. This statement is saying almost every matrix is the product of a lower triangular matrix and an upper triangular matrix (LU decomposition).

Corollary 29.5. *For any cocharacter λ, regular or not (i.e. λ potentially on boundary of a Weyl chamber), $P(\lambda)$ is parabolic.*

Proof. Let λ' be a regular cocharacter whose direction is sufficiently close to that of λ. Then, $P(\lambda) \supset P(\lambda')$ contains a Borel, so is parabolic. ■

Fact. Fix a Borel B (so also a Weyl chamber and also a system Δ of simple roots). Then,

$$\begin{array}{ccc} \left\{\begin{array}{c}\text{parabolics}\\ \text{containing } B\end{array}\right\} & \longleftrightarrow & \{\text{subsets of } \Delta\} \\ P & \longmapsto & \{\alpha \in \Delta : \mathfrak{g}_{-\alpha} \leq \mathfrak{p}\} \\ \begin{array}{c} P(\lambda) \text{ with } \lambda \in \overline{\text{chamber}} \\ \text{s.t. } \{\alpha : \langle \alpha, \lambda \rangle = 0\} = I \end{array} & \longleftarrow & I \end{array}$$

Moreover, $P(\lambda) = Z(\lambda) \ltimes U(\lambda)$ with $U(\lambda)$ the unipotent radical of $P(\lambda)$, and $Z(\lambda)$ called the **Levi subgroup**; it will be another reductive group (since centralizer of some subtorus).

Remark 29.6. Something something the Weyl chamber looks like $\mathbb{R}_+^r$ where r is the rank of your root system. A point on the boundary lies on some subset of the r hyperplanes bounding your chamber. Each hyperplane it lands on causing it to pick up an extra root.

Example. Take $G = \mathrm{GL}_5$ and $\lambda(t) = \begin{pmatrix} tI_2 & \\ & I_3 \end{pmatrix}$. Then,

$$P(\lambda) = \begin{pmatrix} *_{2\times 2} & *_{2\times 3} \\ 0_{3\times 2} & *_{3\times 3} \end{pmatrix}, \; Z(\lambda) = \mathrm{GL}_2 \times \mathrm{GL}_3, \text{ and } U(\lambda) = \begin{pmatrix} I_2 & *_{2\times 3} \\ & I_3 \end{pmatrix}.$$

Corollary 29.7. *In a reductive group with $G \geq B \geq T$, one has*

$$\dim B = \frac{\dim G + \dim T}{2}.$$

Proof. Just look at the Lie algebras. We have $\dim G = \dim T + \#\Phi$ while $\dim B = \dim T + \frac{1}{2}\#\Phi$. ■

Corollary 29.8. *The Weyl group W is generated by the reflections $\{s_\alpha\}$ coming from the roots.*

Proof. $\langle s_\alpha \rangle$ corresponds to $\langle s_{\alpha^\vee} \rangle$ acting on $V^\vee$ which acts transitively on {Weyl chambers}. since the Weyl group acts simply transitively on the Weyl chambers, to show the s_α's generate the whole group, it suffices to show they can be used to get from any Weyl chamber to another one.

We do this in steps. Just take a path from one Weyl chamber to another. Each time the path crosses a wall, apply s_α for α the root corresponding to that wall. This swaps the chambers adjacent to that wall, so you get from one chamber to any other by a series of reflections along the path. ■

Example. For GL_n, the s_α's are the transpositions, while $W = S_n$. Hence, we recover the classical fact that S_n is generated by transpositions.

Remark 29.9. The Weyl group $W = N(T)/C(T)$ is a priori a finite étale algebraic group, but even more is true. It is always discrete. This is because N acts on the torus, giving $N \to \underline{\mathrm{Aut}}T = \mathrm{Aut}\, X = \mathrm{GL}_n(\mathbb{Z})$ with kernel C, so $W = N/C \hookrightarrow \mathrm{GL}_n(\mathbb{Z})$ and hence W is discrete.

Hence, (G, T) is split reductive, W is furthermore a constant group (there's no Galois action).

Recall 29.10. For a split reductive (G, T), $G = \langle T, \text{all the } U_\alpha\text{'s}\rangle$.

Proposition 29.11. *Let (G,T) be a split semisimple group. Then,*

(1) $\mathbb{Z}\Phi$ *is of f. index in* X

(2) $\mathbb{Z}\Phi^\vee$ *is of f. index in* $X^\vee$

(3) $G = \langle G^\alpha : \alpha \in \Phi\rangle$ *where* $G^\alpha = G_\alpha^{\mathrm{der}}$ *is a copy of* SL_2 *or* PGL_2

(4) $G = \langle U_\alpha : \alpha \in \Phi\rangle$

(5) G *is perfect, i.e.* $G = [G,G]$.

Prove this next time.

30 Lecture 30 (4/30)

30.1 Last time

Say (G,T) split reductive group of rank r.

Notation 30.1. $X := \mathrm{Hom}(T, \mathbb{G}_m) \simeq \mathbb{Z}^r$ is character group of T
$X^\vee := \mathrm{Hom}(\mathbb{G}_m, T) \simeq \mathbb{Z}^r$ is cocharacter group of T.
$\langle\, ,\, \rangle : X \times X^\vee \to \mathbb{Z}$ the perfect pairing.

We decompose

$$\mathfrak{g} = \mathfrak{t} \oplus \bigoplus_{\alpha\in\Phi} \mathfrak{g}_\alpha \text{ so } G = \langle T, \text{all the } U_\alpha\rangle\,.$$

For each root α, have $\mathrm{SL}_2 \twoheadrightarrow G^\alpha = G_\alpha^{\mathrm{der}}$ restricting to

$$\begin{pmatrix} * & \\ & *^{-1}\end{pmatrix} \simeq \mathbb{G}_m \xrightarrow{\alpha^\vee} T$$
$$\begin{pmatrix} 1 & * \\ & 1\end{pmatrix} \simeq \mathbb{G}_a \xrightarrow{\sim} U_\alpha$$
$$\begin{pmatrix} 1 & \\ * & 1\end{pmatrix} \simeq \mathbb{G}_a \xrightarrow{\sim} U_{-\alpha}.$$

Set $\Phi^\vee = \{\alpha^\vee : \alpha \in \Phi\} \subset X^\vee$ as well as $V = \mathbb{R}\Phi$ and $V^\vee = \mathbb{R}\Phi^\vee$.

30.2 Decomposition of semisimple groups

We ended last time with the statement of the following proposition.

Proposition 30.2. *Let (G,T) be a split semisimple group. Then,*

(a) $\mathbb{Z}\Phi$ *is of f. index in* X

(b) $\mathbb{Z}\Phi^\vee$ *is of f. index in* $X^\vee$

(c) $G = \langle G^\alpha : \alpha \in \Phi\rangle$ *where* $G^\alpha = G_\alpha^{\mathrm{der}}$ *is a copy of* SL_2 *or* PGL_2

(d) *$G = \langle U_\alpha : \alpha \in \Phi \rangle$*

(e) *G is perfect, i.e. $G = [G, G]$.*

Proof. **(a)** We've proved this before.

(b) Choose a positive definite inner product on $V = \mathbb{R}\Phi$. Average it over the finite group W to make it W-invariant. This gives an identification $V \xrightarrow{\sim} V^\vee$ with W acting as orthogonal transformations. Now, $s_\alpha \in W$ is literally a(n) (orthogonal) reflection, sending $\alpha \mapsto -\alpha$. Its fixed hyperplane is $(\alpha^\vee)^\perp$, but also now $\alpha^\perp$. Hence, $\alpha^\vee$, viewed as an element of V, must be a nonzero real multiple of α. Thus, the coroots span something of full rank since the roots do.

(c) By **(b)**, there are $r = \dim T$ independent elements $\alpha^\vee$, so the groups $\alpha^\vee(\mathbb{G}_m)$ generate T. Hence, $\langle \text{all } G^\alpha \rangle \supset T$. It also contains all the U_α's, so it must be everything.

(d) First note that $\begin{pmatrix} 1 & * \\ & 1 \end{pmatrix}$ and $\begin{pmatrix} 1 & \\ * & 1 \end{pmatrix}$ generate SL_2, so U_α and $U_{-\alpha}$ generate G^α. Thus, if you take all the U_α's they generate the same group as all the G^α's, so we win by **(c)**.

(e) By **(c)**, G is generated by the perfect groups G^α (images of SL_2 which is perfect). ■

For later, we state a useful proposition.

Proposition 30.3. *Let $N \lhd G$ be a smooth connected normal subgroup of a semsimple group G. Then, N is semisimple too.*

(same for reductive)

Definition 30.4. We say a semisimple group G is **almost simple** if its only nontrivial smooth connected normal subgroup is G itself.

Example. SL_n is (going to turn out to be) almost simple, but $\mu_n \lhd \mathrm{SL}_n$ is a (non-connected) nontrivial normal subgroup.

Example. In characteristic p, $\ker \mathrm{Frob}$ is always some nontrivial normal subgroup scheme.

Theorem 30.5. *Let G be a semisimple algebraic group.*

(1) *G has only f.many minimal nontrivial smooth connected normal subgroups, say $G_1, \ldots, G_n$.*

(2) *The G_i's are almost simple.*

(3) *The G_i pairwise commute, i.e. $g_i g_j = g_j g_i$ if $g_i \in G_i$ and $g_j \in G_j$ and $i \neq j$.*

(4) *The multiplication map*

$$G_1 \times \ldots \times G_n \longrightarrow G$$

is an isogeny.

(5) *The smooth connected normal subgroup of G are the groups*

$$\langle G_i : i \in I \rangle \ \text{ for some } \ I \subset \{1, \ldots, n\}.$$

Proof sketch. WLOG $k = k^s$ (use descent, grouping factors into Galois orbits). Next, group theory + G_i's being perfect (since it's semisimple) + the following lemma

Lemma 30.6. *If $N \lhd G$ is a smooth connected normal subgroup, there exists another smooth connected normal $N' \lhd G$ such that multiplication $N \times N' \to G$ is an isogeny.*

Proof of Lemma. Choose a maximal torus T_N of N, and then enlarge it to a maximal torus T of G. Then, $N \cap C_G(N) = Z(N)$ is finite since N semisimple.

Claim: each U_α of G belongs to exactly one of N, $C(N)$.

(Then $N = \langle U_\alpha : U_\alpha \le N \rangle$ so can define $N' := \langle U_\alpha : U_\alpha \le C(N) \rangle = \langle U_\alpha : U_\alpha \not\le N \rangle$. These have finite intersection and $G = NN' = \langle U_\alpha : \text{all } \alpha \rangle$.)

Let's prove this claim now. First consider the commutator group $[T_N, U_\alpha] \subset [T, U_\alpha] \subset U_\alpha$ (since U_α normalized by T). Since $[T_N, U_\alpha]$ smooth connected and $U_\alpha \simeq \mathbb{G}_a$, we have two cases.

($[T_N, U_\alpha] = U_\alpha$) Then, $U_\alpha = [T_N, U_\alpha] \le [N, G] \le N$.

($[T_N, U_\alpha] = 1$) We know T_N and U_α commute. We claim U_α normalizes U_β for any root β of N. Enough to check this for k-points,[48] i.e. that each $u \in U_\alpha(k)$ normalizes U_β. Conjugation by u is the identity on T_N (since $[T_N, U_\alpha] = 1$), so it maps $T_N \ltimes U_\beta$ to some $T_N \ltimes \mathbb{G}_a$ defined by the same T_N-action on $\mathbb{G}_a$ (since u acts by identity on T_N). There is only one $\mathbb{G}_a$ on which T_N acts by this particular character (since each root space 1-dimensional), so this $\mathbb{G}_a$ must be U_β. Hence, U_α normalizes U_β.

This gives conjugation action $U_\alpha \to \underline{\mathrm{Aut}} U_\beta = \mathbb{G}_m$ (last equality since $U_\beta \simeq \mathbb{G}_a$). There are no nontrivial homomorphisms $\mathbb{G}_a \to \mathbb{G}_m$, so this must be trivial, i.e. U_α centralizes U_β. Since N is generated by the U_β's, we see U_α centralizes N, i.e. $U_\alpha \le C_G(N)$.

This finishes the proof of the claim (and so of the lemma). ■

With this lemma in hand, the rest of the theorem is basically just group theory. ■

Fact. Decompositions of semisimple group up to isogeny $\longleftrightarrow$ decompositions of root system.

Example. Say $G = \frac{\mathrm{SL}_2 \times \mathrm{SL}_2}{\mu_2}$ with $\mu_2 \overset{\Delta}{\hookrightarrow} \mu_2 \times \mu_2 \hookrightarrow \mathrm{SL}_2 \times \mathrm{SL}_2$ embedded diagonally. The root system of G is pictured below in Figure 2.

Theorem 30.7. *Let H be a smooth connected algebraic group with an action of a torus T. Choose $U_1, \ldots, U_n \le H$, each isom to $\mathbb{G}_a$, preserved by T such that*

$$\mathfrak{h} = \mathfrak{u}_1 \oplus \cdots \oplus \mathfrak{u}_n$$

and T acts by a different character on each $\mathfrak{u}_i$. Then,

$$U_1 \times \ldots \times U_n \to H$$

is an isomorphism of varieties (but not of algebraic groups since the U_i may not commute with each other), and moreover H is split unipotent.

Application. Start with split reductive $G \ge B \ge T$. Take $H = B_u$, the unipotent radical of B, and choose $U_1, \ldots, U_n$ to be the U_α's contained in H (i.e. $\alpha \in \Phi^+$).

[48]since they're Zariski dense (everything smooth and $k = k^s$) and everything reduced

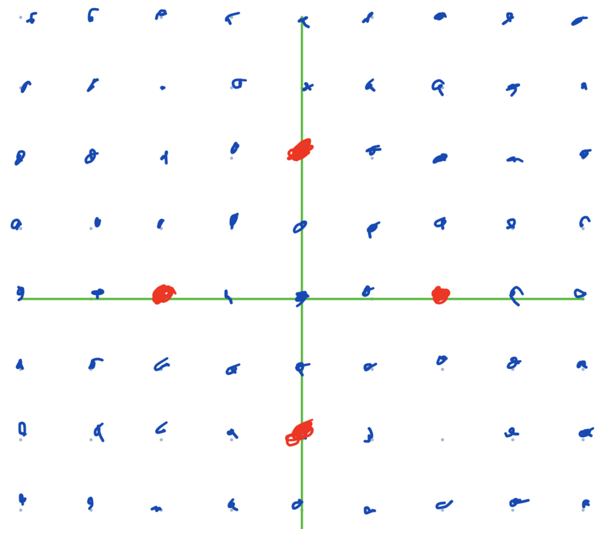

Figure 2: The root system of $G = (\mathrm{SL}_2 \times \mathrm{SL}_2)/\mu_2$.

Example. When $G = \mathrm{GL}_n$ and $H = U_n$, this is saying we can write any upper triangular unipotent matrix as a product of elementary matrices with 1's along the diagonal and at most one nonzero entry elsewhere.

We'll prove the theorem next time.

31 Lecture 31 (5/3)

31.1 Last time

We ended last time with the following statement.

Theorem 31.1 (Direct spanning theorem). *Let H be a nilpotent*[49] *smooth connected algebraic group with an action by a torus T. Let $U_1, \dots, U_n \le H$ be subgroups isomorphic to $\mathbb{G}_a$, preserved by T, such that*

$$\mathfrak{h} = \mathfrak{u}_1 \oplus \cdots \oplus \mathfrak{u}_n.$$

Suppose that T acts by a different character on each $\mathfrak{u}_i$. Then, the multiplication map

$$U_1 \times \ldots \times U_n \to H$$

is an isomorphism of varieties, and H is split unipotent.

Application. For split reductive $G \ge B \ge T$, let $H := B_u$ and let $U_1, \dots, U_n$ be the U_α's with $\alpha \in \Phi^+$. We conclude that B_u, B are *split solvable* groups, i.e. $\exists$ filtration with quotients $\mathbb{G}_a$ and $\mathbb{G}_m$

[49]It's possible this follows from the other hypotheses. Unclear

31.2 Direct Spanning

Proof of Theorem 31.1. We induct on $\dim H$ and split into cases.

(Case 1: H commutative) The multiplication map is a homomorphism

$$U_1 \times U_2 \times \ldots \times U_n \longrightarrow H.$$

It is injective since the kernel of the map on Lie algebras is trivial. It is surjective since the map of Lie algebras is surjective + the fact that H is smooth and connected.

(Case 2: H non-comm) Form the descending central series

$$H > \cdots > Z > 1$$

so $Z \leq Z(H)$. Each group is smooth connected, normalized by T. Form the quotient

$$1 \longrightarrow Z \longrightarrow H \longrightarrow H/Z \longrightarrow 1.$$

Since Z is normalized by T, $\operatorname{Lie} Z$ will be a sum of characters of T, so we must have $\operatorname{Lie} Z = \bigoplus_{i \in I} \mathfrak{u}_i$ for some $I \subsetneq [n] = \{1, \ldots, n\}$. Let $J = [n] \setminus I$, so also $\operatorname{Lie}(H/Z) = \bigoplus_{j \in J} \mathfrak{u}_j$. For $j \in J$, the quotient map $H \to H/Z$ sends $U_j \xrightarrow{\sim} \overline{U}_j$ with $\operatorname{Lie} \overline{U}_j = \mathfrak{u}_j$. By case 1, $\prod_{i \in I} U_i \xrightarrow{\sim} Z$ (since Z commutative). By the inductive hypothesis, $\prod_{j \in J} \overline{U}_j \xrightarrow{\sim} H/Z$ as varieties. Now, the map

$$H \to H/Z \xleftarrow{\sim} \prod_{j \in J} \overline{U}_j$$

of varieties has a splitting $\prod_{j \in J} \overline{U}_j \xrightarrow{\sim} \prod_{j \in J} U_j \hookrightarrow H$ as varieties, so we conclude that

$$\prod_{i \in I} U_i \times \prod_{j \in J} U_j \xrightarrow{\sim} H$$

as varieties, with product taken in this order. This may not be the order $1, 2, \ldots, n$ we wanted. However, we have the U_i's in $Z \subset Z(H)$, so we can move them around to put them in the right order (the U_j's are ordered among themselves by inductive hypothesis). Finally H split reductive since Z and H/Z are.

■

Proposition 31.2. *Let (G, T) be split reductive. Fix $u_\alpha : \mathbb{G}_\alpha \xrightarrow{\sim} U_\alpha$ for each root α. Order the roots. Then if $\alpha \neq \pm\beta$, one has*

$$[u_\alpha(x), u_\beta(y)] = \prod_{\substack{i,j \in \mathbb{Z}_{>0} \\ i\alpha + j\beta \in \Phi}} u_{i\alpha + j\beta}\left(c_{\alpha,\beta,i,j} x^i y^i\right)$$

with product taken in the given ordering of the roots. The $c_{\alpha,\beta,i,j}$'s above are constants in k.

Proof. WLOG assume $\beta < \alpha < \ldots$ (in the fixed ordering). Choose Φ^+ containing both α, β and let B be the associated Borel. First note that $U_\alpha U_\beta \leq B_u = \prod_{\gamma \in \Phi^+} U_\gamma \simeq \mathbb{A}^n$ (with last two equalities as a variety). Hence, the multiplication map is just some map $\mathbb{A}^n \times \mathbb{A}^n \to \mathbb{A}^n$, so we can write

$$u_\alpha(x)u_\beta(y) = \prod_\gamma u_\gamma(f_\gamma(x,y)) \text{ for some } f_\gamma(x,y) \in k[x,y].$$

If we conjugation by $t \in T$, we get

$$u_\alpha(\alpha(t)x)u_\beta(\beta(t)y) = \prod_\gamma u_\gamma(\gamma(t)f_\gamma(x,y)).$$

Alternatively, we can substitute $(x,y) \mapsto (\alpha(t)x, \beta(t)y)$ to see

$$u_\alpha(\alpha(t)x)u_\beta(\beta(t)y) = \prod_\gamma u_\gamma(f_\gamma(\alpha(t)x, \beta(t)y)).$$

We conclude that $\gamma(t)f_\gamma(x,y) = f_\gamma(\alpha(t)x, \beta(t)y)$. If you expand these into monomials and compare coefficients and use that α, β are linearly independent, you see that one must have

$$f_\gamma(x,y) = \begin{cases} c_{\alpha,\beta,i,j}x^iy^j & \text{if } \gamma = i\alpha + j\beta \\ 0 & \text{otherwise.} \end{cases}$$

Note above that $i, j \geq 0$ (i.e. not necessarily positive yet). If we set $y = 0$, then we see $f_\alpha(x, 0) = x$ and we similarly see $f_\beta(0, y) = y$. The conclusion is (recall $\beta < \alpha$)

$$u_\alpha(x)u_\beta(y) = u_\beta(y)u_\alpha(x) \prod_{\substack{\gamma = i\alpha + j\beta \in \Phi^+ \\ i,j>0}} (blah).$$

■

Chevalley proved that you can arrange $c_{\alpha,\beta,i,j} \in \{\pm 1, \pm 2, \pm 3\}$ so you can use this to define G over $\mathbb{Z}$.

31.3 Bruhat decomposition

Given $A \in \mathrm{GL}_n(k)$, one can scale rows and add them to *earlier* rows, and scale columns and add columns to *later* columns, to get a uniquely determined permutation matrix: $U_1AU_2 = P$ with U_1, U_2 upper triangular matrices and P a permutation matrix.

Theorem 31.3.

$$\mathrm{GL}_n = \bigsqcup_{P \in S_n} B_nPB_n$$

is a disjoint union of these double cosets (which are locally closed subschemes).

Remark 31.4. These double cosets have different dimensions. For example, $B1B = B$ is closed in GL_n. On the other hand,

$$B \begin{pmatrix} & & 1 \\ & \iddots & \\ 1 & & \end{pmatrix} B$$

is open in GL_n ("generic" case of Gaussian elimination).

Theorem 31.5 (**Bruhat Decomposition**). *Say $G \geq B \geq T$ split reductive. Then,*

$$G = \coprod_{w \in W} BwB.$$

Remark 31.6. Note wB is well-defined, independent of the choice of lift of w in $N(T)$.

Proof of Bruhat Decomposition. Skipped. "I can skip what I want to." ■

31.4 Representations of SL_2

Let

$$T = \left\{ \begin{pmatrix} t & \\ & t^{-1} \end{pmatrix} \right\} \leq \mathrm{SL}_2$$

and define

$$\begin{array}{rccc} \chi : & T & \longrightarrow & \mathbb{G}_m \\ & \begin{pmatrix} t & \\ & t^{-1} \end{pmatrix} & \longmapsto & t, \end{array}$$

so $X \simeq \mathbb{Z}$ generated by χ.

Let SL_2 act on the left on $\mathbb{A}^2$ in the usual way. Hence, it acts on the right on $k[x, y]$ as well as on $k[x, y]_n =: V_n$, the homogeneous polynomials of degree n.

Remark 31.7. Note that $V_n = \mathrm{Sym}^n(V_1)$ and $k[x, y] = \mathrm{Sym}^*(V_1)$.

Thus, V_n is an SL_2-rep (with SL_2 acting on the right).

Remark 31.8. If you want, you can turn this into a left action by letting g act on the left via g^{-1} on the right.

How does the torus act? $\begin{pmatrix} t & \\ & t^{-1} \end{pmatrix} \in T$ acts as $x \mapsto tx$ and $y \mapsto t^{-1}y$. Thus it acts on $x^a y^b$ as $x^a y^b \mapsto (a - b)x^a y^b$. Note that $a - b \equiv n \pmod 2$ (where $n := a + b$) so the weights of $T \curvearrowright V_n$ are the integers

$$n, n-2, n-4, \ldots, 0, -2, -4, \ldots, 4-n, 2-n, -n$$

spaced by 2. Furthermore, each weight space is 1-dimensional.

Note 7. 6 lectures left. In the remaining time, we'll try to go over chapter 22 of Milne, and at least state the main theorems of chapter 23.

32 Lecture 32 (5/5)

Note 8. No class on Friday. Problem set due next Tuesday.

32.1 Last time

Let $T = \left\{ \begin{pmatrix} t & \\ & t^{-1} \end{pmatrix} \right\} \leq \mathrm{SL}_2$ with character group $X \simeq \mathbb{Z}$ generated by $\chi \begin{pmatrix} t & \\ & t^{-1} \end{pmatrix} := t$.

SL_2 acts on $\mathbb{A}^2$ so acts on $k[x,y]$, so acts on $V := k[x,y]_n = \operatorname{span}\{x^n, x^{n-1}y, \dots, y^n\}$. We can restrict this action to the torus to see

$$\begin{array}{ccccc} x^n & \mapsto & t^n x^n & & \text{weight } n \\ x^{n-1}y & \mapsto & t^{n-2}x^{n-1}y & & \text{weight } n-2 \\ \vdots & & & & \vdots \\ y^n & \mapsto & t^{-n}y^n & & \text{weight } -n \end{array}$$

So, as a T-rep, V has weights $n, n-2, \dots, -n$.

32.2 Reps of SL_2

Keep notation from the 'Last time' section.

Lemma 32.1. *Any nonzero SL_2-subrep $W \leq V$ contains x^n.*

Proof. W contains the monomials of any $f \in W$ (because W is a T-rep, so a direct sum of weight spaces). Therefore, W contains $x^i y^j$ for some $i + j = n$. Let's act by $\begin{pmatrix} 1 & \\ 1 & 1 \end{pmatrix} \in U_{-2}$. This sends $x^i y^j \mapsto x^i (x+y)^j = x^n + \dots$. W must contain any monomial of any of its elements, so W contains x^n. ■

Lemma 32.2. *If $\operatorname{char} k = 0$, then V is irreducible.*

Proof. Suppose $W \leq V$ is a nonzero subrep. Then, $x^n \in W$. Now act by $\begin{pmatrix} 1 & 1 \\ & 1 \end{pmatrix}$. This sends $x^n \mapsto (x+y)^n \in W$. Hence, all monomials in $(x+y)^n$ must be in W. In char 0, all binomial coefficients are nonzero, so $x^n, x^{n-1}y, \dots, y^n \in W$ which shows $W = V$. ■

Example. If $\operatorname{char} k = p$ and $n = p$, then $\operatorname{span}\{x^p, y^p\}$ is a subrep, so V is not irreducible.

In any characteristic, there exists an irreducible representation with highest weight n.

Fact. $\{\text{irreps of } \mathrm{SL}_2\} \leftrightarrow \mathbb{Z}_{\geq 0}$

32.3 Reps of other groups

Fix a split reductive $G \geq B \geq T$ (with choice of Borel, i.e. choice of positive roots). We get $X, X^\vee$, and $\Phi = \Phi^+ \sqcup \Phi^-$. We also get a set $\Delta \subset \Phi^+$ of simple roots. Both B and the opposite Borel B^- are split solvable, so we get the usual exact sequences

$$1 \to U \to B \to T \to 1 \text{ and } 1 \to U^- \to B^- \to T \to 1.$$

The subvariety UTU^- is a dense open in G, i.e. $U \times T \times U^- \hookrightarrow G$ is an open immersion.

Now let's do something new, we'll define a partial order on X.

Definition 32.3. We say $\lambda \geq \mu$ to mean $\lambda - \mu = \sum_{\alpha \in \Delta} m_\alpha \alpha$ for some $m_\alpha \in \mathbb{Z}_{\geq 0}$.

Definition 32.4. We say $\lambda \in X$ is **dominant** $\iff$ $\langle \lambda, \alpha^\vee \rangle \geq 0$ for all $\alpha \in \Phi^+$ $\iff$ $\langle \lambda, \alpha^\vee \rangle \geq 0$ for all $\alpha \in \Delta$ $\iff$ $\lambda \geq \mu$ for all $\mu \in W\lambda$.

Given a G-rep V, restriction action to T, and decompose into eigenspaces for T-action ('**weight spaces**'). The **weights of V** are the $\mu \in X$ such that $V_\mu \neq 0$.

Definition 32.5. We say V has **highest weight** λ if $\lambda \geq \mu$ for all weights μ of V.

(then λ must be dominant since set of weights closed under Weyl action).

Lemma 32.6. *Let V be a G-rep. Then U_α maps V_λ into $\bigoplus_{i \geq 0} V_{\lambda + i\alpha}$. More precisely, $u_\alpha(x)$ maps v to*

$u_\alpha : \mathbb{G}_a \xrightarrow{\sim} U_\alpha$

$$v + \sum_{i \geq 1} x^i v_i \quad \textit{where } v_i \in V_{\lambda + i\alpha} \textit{ is independent of } x \in \mathbb{G}_a(k).$$

Proof. Every morphism $\mathbb{G}_a \to \mathbb{V}$ has the form $x \mapsto \sum x_i v_i$ for some $v_i \in V$ (map of affine spaces). Let's apply this to the map $x \mapsto u_\alpha(x).v$. For $t \in T$, apply $tu_\alpha(x) = u_\alpha(\alpha(t)x)t$ (equality in G) to $v \in V_\lambda$:

$$t \sum x^i v_i = \lambda(t) \sum (\alpha(t)x)^i v_i.$$

Equate coefficients to see that $t.v_i = \lambda(t)\alpha(t)^i v_i$ for all t, so $v_i \in V_{\lambda + i\alpha}$. This just leaves the constant term. Set $x = 0$ to get $v = 1.v = u_\alpha(0).v = v_0$. ■

Corollary 32.7. *If λ is a maximal weight of V, then $V_\lambda \subset V^U$.*

Proof. Each $v \in V_\lambda$ must be fixed by U_α for all $\alpha \in \Phi^+$ by the lemma (+ maximality of λ). These U_α's generate the unipotent radical U.[50] ■

Theorem 32.8 (Fundamental Theorem on Representations of Reductive Groups).

(a) *Every irreducible representation has a highest weight, which is dominant.*

(b) *For each dominant $\lambda \in X$, there exists a unique irrep $V = V(\lambda)$ of G with highest weight λ.*

(c) $\dim V(\lambda)_\lambda = 1$.

(following Jantzen (2003) for proof)

Proof sketch of construction. Via λ as a character of B^- via $B^- \to T \xrightarrow{\lambda} \mathbb{G}_m$.[51] Now we take the **induced representation**

$$I := \operatorname{Ind}_{B^-}^G \lambda = \left\{ f \in \mathscr{O}(G) : f(xb) = \lambda(b)^{-1} f(x) \text{ for all } x \in G, b \in B^- \right\} = \mathrm{H}^0(G/B^-, \mathscr{L}(\lambda))$$

(with $\mathscr{L}(\lambda)$ some line bundle on G/B^-), so we see I is finite dimensional. Furthermore, it has a G-action via

$$({}^g f)(x) := f(g^{-1}x).$$

We take V to be the unique irreducible G-subrep of I (also called the G-**socle**[52] of I). ■

Example. Say $\lambda = 0$. Then $I = \mathscr{O}(G)^{B^-} = \mathscr{O}(G/B^-) = k$ (since G/B^- projective and geometrically integral) with trivial G-action (constant functions invariant under translations). Hence, $V(0)$ is the trivial rep.

[50] They even directly span it, I think

[51] All characters of B^- have this form since all homomorphisms $B^- \to \mathbb{G}_m$ must kill the unipotent radical

[52] In general, this is the sum of all irreducible subreps

Weights preserved by Weyl group.

TODO: Format this

Inside G you have these elements n_α representing s_α. When you conjugate by these, it acts on everything and acts on Weyl group by s_α.

33 Lecture 33 (5/10)

What's left?

- May 10/12: Prove fundamental theorem on representations
- May 14: Grothendieck group of representations, and statement of Weyl character formula
- May 17/19: Isogeny and existence theorems, mostly without proofs (chapter 23)

33.1 Last time: representations of a split reductive group

Let $G \geq B \geq T$ be split reductive with a choice of Borel subgroup. Write $B = U \rtimes T$ and $B^- = U^- \rtimes T$. Then, $UB^- = UTU^- \hookrightarrow G$ is dense and open. Also,

$$U = \langle U_\alpha : \alpha \in \Phi^+ \rangle \text{ and } U^- = \langle U_\alpha : \alpha \in \Phi^- \rangle .$$

$\Delta = \{\text{simple roots}\}$ is the subset of Φ^+ forming a basis for $R\Phi$ so that every root is $\sum_{\alpha\in\Delta} m_\alpha \alpha$ with all $m_\alpha \in \mathbb{Z}_{\geq 0}$ or all $m_\alpha \in \mathbb{Z}_{\leq 0}$.

$\lambda \in X$ is *dominant* if $\langle \lambda, \alpha^\vee \rangle \geq 0$ for all $\alpha \in \Phi^+$ ($\iff \forall \alpha \in \Delta$)

Recall 33.1 (*Fundamental Theorem on Representations of Reductive Groups*, Theorem 32.8)**.**

(a) Every irreducible representation has a highest weight, which is dominant.

(b) For each dominant $\lambda \in X$, there exists a unique irrep $V = V(\lambda)$ of G with highest weight λ.

(c) $\dim V(\lambda)_\lambda = 1$.

33.2 This time: Rep theory

Let's continue our discussion of Theorem 32.8.

Remark 33.2 (Recipe for going from λ to V). View λ as a character of B^- via $B^- \twoheadrightarrow T \xrightarrow{\lambda} \mathbb{G}_m$. Let

$$I := \operatorname{Ind}_{B^-}^G \lambda := \left\{ f \in \mathscr{O}(G) : f(xg) = \lambda(b)^{-1} f(x) \text{ for all } x \in G, b \in B^- \right\} .$$

Then, I is a finite-dimensional G-representation. We define ${}^g f$ by $({}^g f)(x) := f(g^{-1}x)$. To finish, we let V be the G-subrepresentation of I generated by the vector space I^U.[53]

This may be surprising since we're inducing from something of infinite index, but G/B^- is proper (projective even) and this is almost as

Here are some properties of I, V above, assuming $I \neq 0$:

(a) $\dim I^U = 1$.

(b) $I^U = I_\lambda$, and all weights of I are $\leq \lambda$.

[53] This is I itself in characteristic 0

(c) Every nonzero subrep of I contains I^U.

(d) V is the unique irreducible G-subrep of I.

(e) V has highest weight λ with $V^U = V_\lambda = I^U = I_\lambda$ (1-dimensional)

Proof. **(a)** What is I^U? Suppose $f \in I^U$. If $f(1) = 1$, then

$$f(ub) = f(b) = \lambda(b)^{-1} f(1) = \lambda(b)^{-1},$$

where $u \in U$ and $b \in B^-$. In general (without assuming $f(1) = 1$), f is a multiple of $f_\lambda(ub) := \lambda(b)^{-1}$ on $UB^- \subset G$. A priori, this f_λ is a rational function on G (i.e. regular on the dense open UB^-. It may not be regular on all of G). Thus,

$$I^U = \begin{cases} \{cf_\lambda : c \in k\} & \text{if } f_\lambda \text{ regular on } G \text{ (no poles)} \\ 0 & \text{otherwise.} \end{cases}$$

Hence, $I \neq 0 \iff I^U \neq 0$ (since U unipotent[54]) and this is the case iff f_λ is regular in G. This proves **(a)** (recall assuming $I \neq 0$).

(b) Suppose $I \neq 0$. For $t \in T$, observe that

$$\left({}^t f_\lambda\right)(1) = f_\lambda(t^{-1}) = \lambda(t) \implies {}^t f_\lambda = \lambda(t) f_\lambda.$$

Thus, $f_\lambda \in I_\lambda$ and so $I^U \subset I_\lambda$. We want to show this is the whole λ-eigenspace. Let μ be a maximal weight of I. By Corollary 32.7, we must have $I_\mu \leq I^U \leq I_\lambda$, so $\mu = \lambda$ and $I^U = I_\lambda$. This finishes **(b)**.

(c) Let $J \leq I$ be a nonzero subrep. Then, J^U is a nonzero vector space contained in I^U. Since $\dim I^U = 1$, we have $J^U = I^U$, so we win.

(d) This follows from **(c)**. Any subrep contains the subrep V generated by I^U.

(e) Follows from **(b)**. ■

Lemma 33.3. *Each irreducible G-rep W is isomorphic to $V(\lambda)$ for a unique λ such that $I \neq 0$.*

Proof. Let λ be a maximal weight weight of W. Consider the maps (of T-reps)

$$W \overset{\text{projection}}{\twoheadrightarrow} W_\lambda \twoheadrightarrow \lambda$$

(λ here a 1-dim quotient with torus acting via λ). And let's go ahead and equiv W_λ, λ with the trivial U^--action. The formula for the action of U_α for $\alpha \in \Phi^-$ shows that these maps respect the U^--actions as well as the T-actions (i.e. $u_\alpha(x) \cdot w$ acts as identity on W_λ part plus some stuff in lower weight spaces, Lemma 32.6). Therefore, there is a nonzero B^--homomorphism $W \to \lambda$. By Frobenius reciprocity, there exists a nonzero G-homomorphism $W \to \operatorname{Ind}_{B^-}^G \lambda = I$. Since W is nonzero, the kernel is trivial. The image of this map is an irreducible subrep of I; there's only one of those, so $W \xrightarrow{\sim} V \hookrightarrow I$.

This just leaves uniqueness of λ. This is easy since $V(\lambda)$ and $V(\mu)$ have different highest weights so are pairwise non-isomorphic. ■

[54]Theorem 16.9

To finish the proof of the fundamental theorem (Theorem 32.8), we need to identify the λ that actually arise, i.e. for which $I \neq 0$.

Proposition 33.4. *For $\lambda \in X$, TFAE*

(1) $I \neq 0$

(2) $I^U \neq 0$

(3) *The function $f_\lambda(ub) := \lambda(b)$ on $UB^- \subset G$ extends to a regular function on G.*

(4) *$\exists$ irred. representation with highest weight λ.*

(5) *λ is dominant.*

Beginning of Proof. We've already shown **(1)** $\iff$ **(2)** $\iff$ **(3)** $\iff$ **(4)**. Furthermore, if there is an irrep with highest weight λ, it must be higher than everything in its orbit, so λ must be dominant, i.e. **(4)** $\implies$ **(5)**. In other words, if

$$\Lambda := \{\lambda : \mathbf{(1)}, \mathbf{(2)}, \mathbf{(3)}, \mathbf{(4)} \text{ hold}\},$$

then $\Lambda \subset \{\text{dominant weights}\}$, and we claim this is an equality.

Let's prove this. By **(3)**, Λ is a submonoid of X. It's even a saturated monoid (still by **(3)**). If $n\lambda \in \Lambda$ for some $n \geq 1$, then $\lambda \in \Lambda$ (if f_λ^n has no poles, then f_λ has no poles[55]). Thus it suffices to construct irreps whose highest weights generate a monoid containing a multiple of each dominant weight α. So,

- If $\varphi : G \to G'$ is an isogeny, solving the problem for G' also solves it for G (reps for G' give reps of G and they have basically the same cones of dominant weights)
- If $G = G_1 \times G_2$, solving the problem for G_1, G_2 also solves it for G.

Therefore, WLOG G is semisimple or a torus. ■

34 Lecture 34 (5/12)

34.1 Last time: reps of a split reductive group

$G \geq B \geq T$ split reductive with choice of Borel. We can write $B = U \rtimes T$ and $B^- = U^- \rtimes T$. Then, $UB^- = UTU^- \hookrightarrow G$ is a dense open.

The choice of Borel gives positive roots Φ^+ so that $U = \langle U_\alpha : \alpha \in \Phi^+\rangle$ and $U^- = \langle U_\alpha : \alpha \in \Phi^-\rangle$. We let $\Delta = \{\text{simple roots}\}$ be the subset of Φ^+ forming a basis for $\mathbb{R}\Phi$ so that every root is $\sum_{\alpha\in\Delta} m_\alpha \alpha$ with all $m_\alpha \in \mathbb{Z}_{\geq 0}$ or all $m_\alpha \in \mathbb{Z}_{\leq 0}$.

We say $\lambda \in X$ is *dominant* if $\langle \lambda, \alpha^\vee\rangle \geq 0$ for all $\alpha \in \Phi^+$ (or all $\alpha \in \Delta$).

Recall 34.1 (*Fundamental theorem on representations of a reductive group:*)**.** Given $G \geq B \geq T$, there is a bijection

$$\begin{array}{ccc} \{\text{irreducible } G\text{-reps}\} & \xrightarrow{\text{highest weight}} & \{\text{dominant weights}\} \\ V = V(\lambda) & \longleftrightarrow & \lambda \end{array}$$

[55] ring of regular functions integrally closed

Also $\dim V_\lambda = 1$.

So far we've proven that every irreducible representation is isomorphic to $V(\lambda)$ for a unique dominant weight λ. Here λ is the highest weight of V.

We still need to show that every dominant weight arises as the highest weight of some irrep. We've reduced this to

Given a split semisimple group or torus G, construct irreducible representations, enough so that their highest weights generate a monoid containing a multiple of every dominant weight.

34.2 Finishing proof of fundamental theorem

Lemma 34.2. *Let G be split semisimple with simple root α. Let ℓ be the line orthogonal to $\beta^\vee$ for all simple roots $\beta\neg\alpha$. Then there exists a representations of G with highest weight $\lambda \neq 0$ on ℓ.*

Proving this lemma will give us enough irreps to deduce the fundamental theorem. The convex hull of the (positive parts) of these lines contains all dominant weights, so having a weight on each of these lines will give a monoid whose saturation contains all dominant weights.

Proof. Choose $\mu \in X^\vee$ on the ray defined by $\langle \beta, \mu \rangle = 0$ for all $\beta \neq \alpha$ and $\langle \alpha, \mu \rangle > 0$. Let $P = P(\mu)$, a maximal parabolic subgroup. We see that P contains T, U_α and all $U_{\pm\beta}$'s (look at Lie algebra[56]). Hence, P contains B and all G_β's. Inside the G_β's are the elements $n_\beta \in G_\beta$ which normalize the torus and induce the reflection s_β.

As always, recall Proposition 16.3

We need to do something to make a representation. We use Chevalley's theorem (Theorem 8.4). It provides a G-representation V and a 1-dimensional subspace L such that $\mathrm{Stab}_G(L) = P$. WLOG we may assume L generates the whole representation V. Since $U \leq B \leq P$, U preserves L and so acts trivially on it. Similarly, T preserves L, so it acts via some character λ.

Claim 34.3. *V has highest weight λ. In particular, λ is dominant. Also* $\dim V_\lambda = 1$.

Proof. Recall L generates the representation G. Where does U^-TU map L? U maps L to L, and T maps L to L, but U^- does not preserve L. It will move λ in a "negative direction," i.e. it maps L to $L \oplus \bigoplus_{\sigma<\lambda} V_\sigma$. Since U^-TU is dense in G, we conclude that G maps L into $L \oplus \bigoplus_{\sigma<\lambda} V_\sigma$ which, since L generates V, means that λ is the highest weight V, and that $V_\lambda = L$ is 1-dimensional. ■

Claim 34.4. *$\lambda \in \ell$, i.e. $\langle \lambda, \beta^\vee \rangle = 0$ for all β $(\neq \alpha)$*

Proof. Fix simple $\beta \neq \alpha$. Note that $n_\beta \in G_\beta \leq P$, so n_β preserves $L = V_\lambda$. Looking at conjugation by n_β, we see that s_β fixes λ, i.e. $\lambda = s_\beta(\lambda) = \lambda - \langle \lambda, \beta^\vee \rangle \beta$ which means $\langle \lambda, \beta^\vee \rangle = 0$. ■

Claim 34.5. $\lambda \neq 0$

Proof. Suppose $\lambda = 0$. Pick a nonzero $v \in L$. Then, T fixes v (acting via trivial rep). We already know that U fixes v, so the whole Borel B fixes v. Hence, the orbit map $G \to \mathbb{V}, g \mapsto gv$ acts through the *proper* (smooth, connected) quotient G/B. Since $\mathbb{V}$ is affine, this forces the orbit map to be constant! Hence, G fixes v, so $P = \mathrm{Stab}_G(L) = G$, a contradiction. ■

[56] contains all roots pairing nonnegatively with μ

We're almost done. All that's left is irreducibility.

Claim 34.6. *There exists an irreducible G-rep W with the same highest weight λ.*

Proof. Choose a Jordan-Hölder series for V (i.e. chain of subreps with each quotient irreducible). Since $\dim V_\lambda = 1$, exactly one of the irreducible quotients W in this series will have $\dim W_\lambda = 1$. This W is not an irrep with highest weight W (every weight of W is a weight of V). ■

■

34.3 Grothendieck Group of Representations

Definition 34.7. We say $\mathrm{Rep}(G)$ is **semisimple** when

$$\{\text{f.dim } G\text{-reps}\}_{/\text{isom}} = \mathbb{N}\left[\{\text{irred. reps}\}_{/\text{isom}}\right],$$

i.e. the monoid of f.dim G-reps is free abelian on the irreps, i.e. every f.dim representation is completely reducible. Equivalently, one can require

$$\{\text{virtual reps}\}_{/\text{isom}} = \mathbb{Z}\left[\{\text{irred reps}\}_{/\text{isom}}\right],$$

where $\mathbb{Z}[-]$ above the the free abelian group, not the group ring.

Example. If G is a finite group, then

$$\{\text{virtual reps}/\mathbb{C}\} \xrightarrow{\sim} \{\textbf{class functions}\} := \left\{f : G \to \mathbb{C} : f(ghg^{-1}) = f(h)\right\}$$

with isomorphism given by taking the character $V \mapsto \chi_V(g) = \operatorname{Tr} \rho_V(g)$.

Definition 34.8. For any abelian category $\mathcal{A}$ (semisimple or not), one can forms its **Grothendieck group** $K(\mathcal{A})$ defined by

$$K(\mathcal{A}) := \frac{\mathbb{Z}[\{\text{objects}\}_{/\text{isom}}]}{\langle [B] = [A] + [C] \ \text{ when } \ \exists 0 \to A \to B \to C \to 0\rangle}.$$

Proposition 34.9. *If every object in $\mathcal{A}$ has finite length, then $K(\mathcal{A}) \simeq \mathbb{Z}[\{$irred. objects$\}]$.*

Proof. The Jordan-Hölder theorem. ■

This applies to $\mathrm{Rep}(G)$ (f.dim reps). Note that in this case, we really have a Grothendieck *ring* since tensor products exist.

Example. Let T be a split torus with $X = \mathrm{Hom}(T, \mathbb{G}_m)$. Then, $K(\mathrm{Rep}(T)) = \mathbb{Z}[X]$ is the free abelian group e^χ for $\chi \in X$. This notation is used to distinguish the formal sum $e^\chi + e^{\chi'}$ from their sum in the character group $e^{\chi+\chi'} = e^\chi e^{\chi'}$. The map $K(\mathrm{Rep}(T)) \xrightarrow{\sim} \mathbb{Z}[X]$ is given by $[V] \mapsto \sum_{\chi \in X} (\dim V_\chi) e^\chi =: \mathrm{ch}(V)$.

35 Lecture 35 (5/14)

Last problem set due on Wednesday.

35.1 Last time: Grothendieck group $K(\mathrm{Rep}(G))$

Recall 35.1.

$$K(\mathrm{Rep}(G)) := \frac{\mathbb{Z}[\text{f.dim } G\text{-reps}]}{\langle [V]-[V_1]-[V_2] : 0 \to V_1 \to V \to V_2 \to 0 \text{ exact}\rangle} \simeq \mathbb{Z}[\text{irreps}].$$

Example. T split torus and $X = \mathrm{Hom}(T, \mathbb{G}_m)$. Then there is a ring isomorphism

$$K(\mathrm{Rep}(T)) \simeq \mathbb{Z}[X] := \bigoplus_{\chi \in X} \mathbb{Z}e^{\chi}$$

sending $[V] \mapsto \mathrm{ch}(V) := \sum_{\chi\in X}(\dim V_\chi)e^{\chi}$.

35.2 $K(\mathrm{Rep}(G))$ for split reductive (G,T)

Can consider the composition $\mathrm{ch}_G : \mathrm{Rep}G \to \mathrm{Rep}T \xrightarrow{\mathrm{ch}} \mathbb{Z}[x]$. This is like a formal character; think of it as an 'unevaluated trace.'

Remark 35.2. This composition may seem like it's forgetting a lot of information, but as far as characters are concerned, it isn't. For reductive group, the torus is also the Cartan subgroup, and conjugates of the Cartan subgroup contain an open dense subset of G. Characters (think: traces) are supposed to be conjugation invariant, so we're not really losing much information in considering this composition.

Theorem 35.3. *$K(\mathrm{Rep}_G) \to K(\mathrm{Rep}(T))$ is $\mathbb{Z}[X]^W \hookrightarrow \mathbb{Z}[X]$. In particular, $K(\mathrm{Rep}(G)) \xrightarrow{\sim} \mathbb{Z}[X]^W$.*

Proof. Totally order X by values of some linear functions $\langle -, \mu\rangle$ taking distinct values on elements of X (e.g. choose $\mu \in X^\vee_{\mathbb{R}}$ with $\mathbb{Q}$-linearly independent coefficients), and by refining the partial ordering $\le$ we already had (e.g. choose μ so that $\langle \alpha, \mu\rangle > 0$ for all $\alpha \in \Delta$). Note that, for any $C > 0$, the set $\{$dominant weights $\lambda : |\langle \lambda, \mu\rangle| < C\}$ is finite.

Now, $K(\mathrm{Rep}(G))$ is the $\mathbb{Z}$-span of $[V_\lambda]$ as λ runs over dominant weights. Their characters $\mathrm{ch}(V_\lambda)$ will be a basis for $\mathbb{Z}[X]^W$ (note dominant weights are in bijection with W-orbits so $\mathbb{Z}[X]^W \simeq \mathbb{Z}[$dominant weights$]$). This is because λ appears in $\mathrm{ch}(V_\lambda)$ with multiplicity 1; if you start with some $p \in \mathbb{Z}[X]^W$ can kill the highest weight λ_1 using a multiple of $\mathrm{ch}(V_{\lambda_1})$, and then kill the next highest weight λ_2 using a multiple of $\mathrm{ch}(V_{\lambda_2})$, and so on... ■

For $\lambda \in X$, let $\mathcal{L}(\lambda)$ be the associated line bundle on G/B^-. Note that $\mathcal{L}(\lambda)$ is effective (i.e. has a nonzero global section) $\iff$ $\langle \lambda, \alpha^\vee\rangle \ge 0$ for all $\alpha \in \Delta$ (i.e. λ dominant). We saw this when classifying irreps.

This is described somewhere in Milne, I think

It turns out that one can also say that

- $\mathcal{L}(\lambda)$ is ample $\iff$ $\langle \lambda, \alpha^\vee\rangle > 0$ for all $\alpha \in \Delta$ (λ in interior of dominant cone).
- $T_e(G/B^-) = \frac{\mathrm{Lie}\, G}{\mathrm{Lie}\, B^-} = \bigoplus_{\alpha\in\Phi^+} \mathfrak{g}_\alpha$ as T-reps.

- The tangent bundle of G/B^- is

$$“\mathscr{L}\left(\frac{\operatorname{Lie} G}{\operatorname{Lie} B^-}\right)” := \bigoplus_{\alpha\in\Phi^+} \mathscr{L}(\alpha)$$

- The cotangent bundle of G/B^- is

$$\bigoplus_{\alpha\in\Phi^+} \mathscr{L}(-\alpha).$$

- The canonical bundle is

$$\bigotimes_{\alpha\in\Phi^+} \mathscr{L}(-\alpha) = \mathscr{L}\left(-\sum_{\alpha\in\Phi^+}\alpha\right).$$

Example. For $G = \mathrm{SL}_2$, $G/B^- = \mathbb{P}^1$. The canonical bundle is $\omega = \mathscr{O}(-2)$ which is good since the root for SL_2 is 2.

Notation 35.4. Define

$$\rho := \frac{1}{2}\sum_{\alpha\in\Phi^+}\alpha \in \frac{1}{2}X$$

(actually this is contained in X, if G is simply connected or more generally if $\operatorname{Pic} G = 0$).

I think probably $\operatorname{Pic}(G)[2] = 0$ is enough

Definition 35.5. We define the **antisymmetry operator**

$$J = \sum_{w\in W} (\det w) w \in \mathbb{Z}[W]$$

with $\mathbb{Z}[W]$ the integral group ring.

Example. If $\lambda \in X$, then

$$J(e^\lambda) = \sum_{w\in W} (\det w) e^{w(\lambda)} \in \mathbb{Z}[X].$$

Recall 35.6. These representations I from the past lecture or two are the global sections of these line bundles $\mathscr{L}(\lambda)$.

Proposition 35.7 (Euler characteristic formula). *For any $\lambda \in X$,*

$$\sum_{i\geq 0} (-1)^i \operatorname{ch}_G \mathrm{H}^i\left(G/B^-, \mathscr{L}(\lambda)\right) = \frac{J(e^{\lambda+\rho})}{J(e^\rho)} \in \mathbb{Z}[X]$$

(that this fraction lands in $\mathbb{Z}[X]$ is part of the claim)

(think of this as an equivariant Riemann-Roch, keeping track of the torus action instead of just dimensions)

We won't be proving this. We will use it to understand the character of H^0 though. The other ingredient we will need is

Theorem 35.8 (Kempf's vanishing theorem). *If λ is dominant, then $\mathscr{L}(\lambda)$ is **acyclic**, i.e. $\mathrm{H}^i(G/B^-, \mathscr{L}(\lambda)) = 0$ for all $i > 0$.*

Proof in char 0. Define a character $A \in X$ by

$$\lambda = -\sum_{\alpha \in \Phi^+} \alpha + A \in X$$

(A measure difference between $\mathscr{L}(\lambda)$ and the canonical bundle ω). For any $\beta \in \Delta$, note that

$\sum_{\alpha\in\Phi^+}\alpha$ is strictly greater than any element of its Weyl orbit. This gives that the '2' below will be a number > 0. To know that it's actually 2, use that $\sum_{\alpha\in\Phi^+}\alpha = 2\sum_{\beta\in\Delta}\omega_\beta$ where $\{\omega_\beta\}$ is the dual basis to $\{\beta^\vee\}$. To see this, use that a simple reflect s_β permutes $\Phi^+ \setminus \{\beta\}$ and sends $\beta \mapsto -\beta$, and then compute $s_\beta(\sum\alpha)$

$$\langle A, \beta^\vee \rangle = \langle \lambda, \beta^\vee \rangle + \sum_{\alpha \in \Phi^+} \langle \alpha, \beta^\vee \rangle \geq 0 + 2 > 0.$$

Hence, $\mathscr{L}(A)$ is ample. We now bring in the big guns...

Theorem 35.9 (**Kodaira vanishing** (char 0 only))**.** *If $\mathscr{L} = \omega \otimes \mathscr{L}'$ with $\mathscr{L}'$ ample (and ω the canonical bundle), then $\mathscr{L}$ is acyclic.*

We just apply this to $\mathscr{L}(\lambda) = \omega \otimes \mathscr{L}(A)$ and then win. ■

Corollary 35.10 (**Weyl character formula**)**.** *If λ is dominant character, and we let $I = \operatorname{ind}_{B^-}^G \lambda = \mathrm{H}^0(G/B^-, \mathscr{L}(\lambda))$, then*

$$\operatorname{ch}_G(I) = \frac{J(e^{\lambda+\rho})}{J(e^\rho)} \in \mathbb{Z}[X].$$

Last two lectures next week. We'll talk about isogeny and existence theorems for split reductive groups.

36 Lecture 36 (5/17)

36.1 Review of notation: (G, T) a split reductive group

We decompose the adjoint rep of T on $\mathfrak{g}$:

$$\mathfrak{g} = \mathfrak{t} \oplus \bigoplus_{\alpha \in \Phi} \mathfrak{g}_\alpha.$$

These are the Lie algebras of

$$G = \langle T, \text{ the } U_\alpha \text{ for all } \alpha \in \Phi \rangle .$$

This gives rise to a root datum

$$\underbrace{\mathcal{R}(G,T)}_{\text{root datum}} := \left(\underbrace{X}_{\text{character group}}, \underbrace{\Phi}_{\text{roots}}, \underbrace{X^\vee}_{\text{cocharacter group}}, \underbrace{\Phi^\vee}_{\text{coroots}}, \underbrace{\langle -, - \rangle}_{X \times X^\vee \to \mathbb{Z}}, \underbrace{\alpha \mapsto \alpha^\vee}_{\text{bijection}} \right).$$

For each root $\alpha \in \Phi$, we define

- $S_\alpha := (\ker \alpha)_t$, a subtorus of rank $r - 1$ in T
- $G_\alpha := C_G(S_\alpha)$, a split reductive group of semisimple rank 1, with

$$\operatorname{Lie} G_\alpha = \mathfrak{t} \oplus \mathfrak{g}_\alpha \oplus \mathfrak{g}_{-\alpha}.$$

- $G^\alpha := G_\alpha^{\mathrm{der}}$, a split semisimple group of rank 1, isomorphic to SL_2 or PGL_2

Recall 36.1 (Weyl character formula)**.** If $\lambda \in X$ is dominant and $I := \mathrm{ind}_{B^-}^G \lambda = \mathrm{H}^0(G/B^-, \mathscr{L}(\lambda))$, then

$$\mathrm{ch}_G(I) = \frac{J(e^{\lambda+\rho})}{J(e^\rho)} \in \mathbb{Z}[X] \text{ where } J(e^\lambda) = \sum_{w \in W} (\det w) e^{w(\lambda)}.$$

Note that this we can get the dimension from the character by applying the homomorphism $\mathbb{Z}[X] \to \mathbb{Z}$ sending $e^\chi \mapsto 1$ (i.e. by remembering that each character is 1-dimensional).

Open Question 36.2. *Find a formula for* $\dim V(\lambda)$ *in characteristic* p.

36.2 Isogenies

Let $\varphi : G \to G'$ be an isogeny between smooth connected algebraic groups.

Recall 36.3. For smooth connected groups, 'isogeny' $\iff$ surjective homomorphism with finite kernel.

Example. If G is a split torus T, then G' will be another split torus T'. Can understanding everything in terms of the character groups. φ here corresponds to an injective homomorphism $f : X' \to X$ with finite cokernel.

Example. If $G = \mathbb{G}_a$, then also $G' \simeq \mathbb{G}_a$ and φ is of the form $x \mapsto c_n x^{p^n} + \cdots + c_1 x^p + c_0 x$ where $p = \mathrm{char}\, k > 0$ (otherwise just have $c_0 x$). For this to be surjective, just require that some c_i is nonzero.

Notation 36.4. We let $p = \mathrm{char}\, k$. We define the notation

$$p^{\mathbb{N}} := \begin{cases} \{p^n : n \geq 0\} & \text{if } \mathrm{char}\, k = p \\ \{1\} & \text{if } \mathrm{char}\, k = 0. \end{cases}$$

(set of 'possible degrees of isogenies of $\mathbb{G}_a$)

Example. Consider $G = T \ltimes_\alpha \mathbb{G}_a$ with T a split torus with $T \curvearrowright \mathbb{G}_a$ via the nontrivial character $\alpha : T \to \mathbb{G}_m = \underline{\mathrm{Aut}}(\mathbb{G}_a)$. Say $\varphi : G \to G'$ is an isogeny. Then, $G' = T' \ltimes_{\alpha'} \mathbb{G}_a$ with T' another split torus and $\alpha' : T' \to \mathbb{G}_m$ a nontrivial character. Furthermore, φ will of the form

$$\varphi(t, x) = (\varphi(t), cx^q) \text{ for some } c \in k^\times \text{ and } q \in p^{\mathbb{N}}.$$

To show this, would first want to classify finite subgroups of G. This gives that G' must be a similar semi-direct product. Then, to get the form of φ, use that $\ker(\varphi|_{\mathbb{G}_a})$ is normalized by T; hence it is preserved by multiplication by any $g \in \mathbb{G}_m$ (since α nontrivial) which forces it to be supported at the origin.[57] Also, $\varphi|_T : T \to T'$ is given by some map $f : X' \to X$ such that

$$f(\alpha') = q\alpha \in X$$

(this is the condition that $(t, x) \mapsto (\varphi(t), cx^q)$ be a homomorphism).

[57] It has had support at any nonzero number, could move it around via $\mathbb{G}_m$ to get support everywhere, so it wouldn't be finite

Definition 36.5. Fix some $p \in \{\text{primes}\} \cup \{0\}$. An **isogeny of root data** $(X, \Phi, \dots) \to (X', \Phi', \dots)$ is an injective homomorphism $f : X' \to X$ with finite cokernel such that $\exists$ bijection $\Phi \to \Phi'$, denoted $\alpha \leftrightarrow \alpha'$, and a function $q : \Phi \to p^{\mathbb{N}}$ satisfying

$$f(\alpha') = q_\alpha \alpha \in X \text{ for all } \alpha \in \Phi$$

as well as (this really is an additional condition)

$$f^\vee(\alpha^\vee) = q_\alpha \left(\alpha'\right)^\vee \in \left(X'\right)^\vee \text{ for all } \alpha \in \Phi,$$

where $f^\vee : X^\vee \to \left(X'\right)^\vee$ is the dual homomorphism to f.

Here's the big result (not to be proved here)

Theorem 36.6 (Isogeny theorem). *Fix split reductive groups (G, T) and (G', T') over k of characteristic p. Let $(X, \Phi, \dots)$ and $(X', \Phi', \dots)$ be their corresponding root data. Then, there is a bijection*

$$_{(T'/Z')(k)\backslash}\left\{\begin{matrix} \textit{isogenies} \\ (G,T) \to (G',T') \end{matrix}\right\} \longleftrightarrow \left\{\begin{matrix} \textit{isogenies of root data} \\ (X,\Phi,\dots) \to (X',\Phi',\dots) \end{matrix}\right\}.$$

On the LHS, $Z' = Z(G')$ and $t \in (T'/Z')(k)$ acts on $\{$isogenies$\}$ as $\varphi \mapsto \mathrm{inn}_t \circ \varphi$ (this restricts to the same morphism $T \to T'$).

Remark 36.7. There's a variant of the theorem that removes the quotient on the left by choosing *pinnings*: on LHS, give a base Δ and isos $\mathbb{G}_\alpha \xrightarrow{\sim} U_\alpha$ for each $\alpha \in \Delta$; on RHS, give Δ. For isogenies of pinned $\{$reductive groups, root data$\}$, one gets a bijection no the nose.

Example. In char $p > 0$, consider isogenies from SL_2. There's, e.g. $\mathrm{SL}_2 \to \mathrm{PGL}_2$ and also Frobenius $F : \mathrm{SL}_2 \to \mathrm{SL}_2$. Can fit these into

$$\begin{tikzcd} \dots \arrow[r] & \mathrm{SL}_2 \arrow[r, "F"] \arrow[d] & \mathrm{SL}_2 \arrow[r] \arrow[d] & \dots \\ \dots \arrow[r] & \mathrm{PGL}_2 \arrow[r, "F"] & \mathrm{PGL}_2 \arrow[r] & \dots \end{tikzcd}$$

What do these look like on the root datum side? For $\mathrm{SL}_2 \to \mathrm{PGL}_2$, both character groups are rank 1, and one has $X' \hookrightarrow X$ included as an index 2 sublattice. For Frobenius $F : \mathrm{SL}_2 \to \mathrm{SL}_2$, the action on the character group is multiplication by p.

Let's end with some general comments

- φ is multiplicative/central $\iff q_\alpha = 1$ for all α.
- φ is an isomorphism $\iff q_\alpha = 1$ for all α, and f is an isomorphism.
- For (G, T) over $\mathbb{F}_p$, the Frobenius isogeny $(G, T) \xrightarrow{F} (G, T)$ corresponds to $f = p\,\mathrm{Id}$ with $\Phi \xrightarrow{\sim} \Phi$ the identity and $q_\alpha = p$ for all α.

37 Lecture 37 (5/19): Last class

37.1 Last time: the isogeny theorem

Recall 37.1. For $p \in \{\text{primes}\} \cup \{0\}$. As *isogeny of root data*

$$(X, \Phi, \dots) \to (X', \Phi', \dots)$$

is an injective homomorphism $f : X' \to X$ with finite cokernel such that there exists

- a bijection $\Phi \to \Phi'$, written $\alpha \mapsto \alpha'$, and
- a function $q : \Phi \to p^{\mathbb{N}}$ (if $p = 0$, interpret $p^{\mathbb{N}}$ as $\{1\}$)

satisfying $f(\alpha') = q_\alpha \alpha$ and $f^\vee(\alpha^\vee) = q_\alpha \left(\alpha'\right)^\vee$ for all $\alpha \in \Phi$.

37.2 Some ideas from the proof

The hard part is going from an isogeny of root data to an isogeny of reductive groups.

- First construct $G^\alpha \to (G')^\alpha$ for each $\alpha \in \Delta$.

  Recall G^α is SL_2 or PGL_2. This isogeny is one that can, more-or-less, be constructed by hand.

- Next construct $G_\alpha \to (G')_\alpha$

  Recall $G_\alpha = (G^\alpha \times T)/Z$ with T a torus and Z a finite central subgroup of the product. This is still explicit enough that you can construct the corresponding isogeny, more-or-less, by hand.

- These G_α's (along with the split maximal torus T) generate G. We need to make sure the maps on the G_α's are all compatible.

  Intuition. Think of G as the 'free product' of $\{G_\alpha : \alpha \in \Delta\}$ modulo various relations, e.g. you need to identify the copies of T in each G_α and you need $[U_\alpha, U_{-\beta}] = 1$ for all $\alpha \neq \beta$ in Δ.

  These relations turn out to be all of them. That is, to give $\varphi : G \to G'$ is to give $\varphi_\alpha : G_\alpha \to G'$ for all $\alpha \in \Delta$ such that $\varphi_\alpha|_T = \varphi_\beta|_T$ always and also the commutator relation holds.[58]

These steps give the proof of the isogeny thoerem.

37.3 Existence theorem

Theorem 37.2 (**Existence theorem**). *Given k and a (reduced) root datum $\mathcal{R}$, there exists a split reductive group (G, T) over k with root datum $\mathcal{R}$.*

Remark 37.3. (G, T) above is unique (up to isomorphism) by the isogeny theorem.

Remark 37.4. In fact, one can construct (G, T) over $\mathbb{Z}$ (so over any base scheme)

Remark 37.5. There is a complete classification of root data.

[58] $[\varphi_\alpha(x), \varphi_{-\beta}(y)] = 1$ when $\alpha \neq \pm\beta$

Theorem 37.6 (**Isogeny + Existence theorem**). *Fix a field k. Then there exists an equivalence of categories*

$$\left\{\begin{matrix}\textit{split reductive groups}/k\ \textit{ with}\\ {}_{(T'/Z')(k)\backslash}\mathrm{Isog}((G,T),(G',T'))\end{matrix}\right\} \longleftrightarrow \left\{\begin{matrix}\textit{reduced root data with}\\ \textit{isogenies of root data}\end{matrix}\right\}$$

Let's give some ideas of the proof (following Lusztig 2017, Geck 2017)

- WLOG R is semisimple (essentially because tori are easy to build). In fact, since semisimple groups exist in a tower with the universal cover at the top and the adjoint one at the bottom, it suffices to construct any one of the groups in this tower (since then you get its universal cover and so get all of them). Hence, we'll only construct the adjoint form. See Milne chpt. 18

- All affine groups are linear, so we'll try to construct G inside some GL_n, i.e. find a faithful representation. Since we want to construct the adjoint form, it'll suffice to construct the adjoint representation $\mathrm{Ad} : G \to \mathrm{GL}_{\mathfrak{g}}$ (and we'll do it over $\mathbb{Z}$).

- To do this, first construct $\mathrm{ad} : \mathfrak{g} \to \mathfrak{gl}_{\mathfrak{g}}$ and then "exponentiate".

  Note 9. Got distracted and missed some stuff.

- For $i, j \in \Phi$ with $i \neq \pm j$, define m_{ij} to be the smallest $m \in \mathbb{Z}_{\geq 0}$ such that $j - mi \notin \Phi$. Fix a base Δ. Let $M = \bigoplus_{i\in\Delta} \mathbb{C}u_i \oplus \bigoplus_{j\in\Phi} \mathbb{C}v_j$. This will become our Lie algebra $\mathfrak{g}$. Define elements $e_i, h_i \in \operatorname{End} M$. These are given by ($i \in \pm\Delta$)

$$e_i(u_j) = |\langle i, j^\vee\rangle|\, v_i \ \text{ and }\ e_i(v_j) := \begin{cases} m_{ij}v_{j+i} & \text{if } j+i \in \Phi \\ u_i & \text{if } j+i = 0 \\ 0 & \text{otherwise.}\end{cases}$$

  as well as ($i \in \Delta$)

$$h_i(u_j) = 0 \ \text{ and }\ h_i(v_j) = \langle j, i^\vee\rangle\, v_j.$$

  Let $\mathfrak{g}$ be the Lie subalgebra of $\mathfrak{gl}_N$ generated by all the e_i's and h_i's. One now checks

  - span $\{e_i, e_{-i}, h_i\}$ is a copy of $\mathfrak{sl}_2$
  - $\mathfrak{g}$ is semisimple
  - $\mathfrak{g} \to \mathfrak{gl}_M$ is the adjoint representation

- Now we construct the group (over $\mathbb{Z}$). View $e_i, h_i \in M_n(\mathbb{Z})$. Define

$$U_i = \mathrm{im}\begin{pmatrix} \mathbb{G}_a & \longrightarrow & \mathrm{GL}_n \\ t & \longmapsto & \exp(te_i)\end{pmatrix}$$

  Above, e_i is nilpotent (so the matrix exponential is a finite sum), and even better than this, one can check (from looking at the explicit matrices) that the entries of $\exp(te_i)$ live in $\mathbb{Z}[t]$ (i.e are integral, not rational). Similarly get

$$\begin{matrix} \mathbb{G}_m & \longrightarrow & \mathrm{GL}_n \\ \tau = \exp(t) & \longmapsto & \exp(th_i)\end{matrix}$$

Above, one sees that $\exp(th_i)$ is block diagonal of dimensions $(i+j)\times(i+j)$ of the form

$$\exp(th_i) = \begin{pmatrix} I_i & \\ & T \end{pmatrix} \text{ where } T = \operatorname{diag}\left(\tau^{\langle j, i^\vee\rangle}\right).$$

Let $T := \operatorname{im}\left(\mathbb{G}_m^r \to \mathrm{GL}_n\right)$ be the image of all these maps. Now check

- $\langle T, U_i, U_{-i}\rangle$ is reductive of semsimple rank 1, and $G := \langle T, \text{all } U_i\rangle$ is semisimple with the right root datum.

Since all the matrices involved had integral entries, we see this really gives a group over $\mathbb{Z}$.

37.4 Some more words

This is not the end of the theory of algebraic groups. The construction in the existence theorem gives split reductive groups. Might want to understand non-split reductive groups (forms/twists of split reductive groups). Also, the classification of root data gives some exceptional root data not attached to classical groups, so can try to understand the corresponding exceptional algebraic groups.

One can also study Kac-Moody algebras/groups, infinite-dimensional analogues of Lie algebras/-groups. There are also things called quantum groups that some people study.[59]

We talked about representations of the algebraic group G. One can also study representations of groups of points, e.g. representations of $G(\mathbb{F}_q)$ or $G(\mathbb{Q}_p)$ or $G(\mathbb{A}_K)$ or what have you.

[59]Something like starting with the (commutative) Hopf algebra of a linear algebraic group, and then deform it into a non-commutative Hopf algebra

38 List of Marginal Comments

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