RPHDV Notes

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These are notes on talks given in "Rational Points on Higher Dimensional Varieties Seminar" which took place at ICMS. Unfortunately for the reader, these notes are live-texed and so their quality is upper bounded both by my (quite) limited ability to understand the material in real time and by my typing speed. With that in mind, they are doubtlessly missing content/insight present in the talks and certainly contain confusions not present in the talks. Despite all this, I hope that you can still find some use of them. Enjoy and happy mathing.

The website for this seminar is available here. Maybe worth mentioning that many of these were slide talks, and for all of those, my notes are quite incomplete

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1 Day 1 (4/25)

1.1 Jennifer Berg (Bucknell University): Rational Points on constant subvarieties of abelian varieties

(joint w/ Felipe Voloch)

Let $F = \mathbb{F}_q$ ($q = p^n$) be a finie field. Let D be a smooth, projective curve over $\mathbb{F}$, and let X be another nice[1] $\mathbb{F}$-variety.

Question 1.1. *When do you have a nonconstant separable morphism $D \to X$?*

Let's relate to arithmetic. Let $K = \mathbb{F}(D)$ be the field of rational functions of K.

Definition 1.2. A K-variety is called **constant** if it is isomorphic to the base change of an $\mathbb{F}$-variety

Example. $K = \mathbb{F}_p(t)$ with $p \nmid 6$. Then, $E : y^2 = x^3 + t^6$ is constant since it's iso to $E_0 : y^2 = x^3 + 1$ via $[x, y] \mapsto [x/t^2, y/t^3]$. △

Remark 1.3. If $X/\mathbb{F}$, then X_K is constant and $X_K(K) = X(K) = \mathrm{Mor}_{\mathbb{F}}(D, X)$. ○

Local information ($D/\mathbb{F}$ curve, $K = \mathbb{F}(D)$)

- Missed...

$$X(K) \hookrightarrow X(\mathbb{A}_K)^{\pi} \hookrightarrow \prod_{v \in D^1} X(K_v).$$

Today, we're interested in $X(K) \neq \emptyset$ and gaining arithmetic info from intermediate set. Points in $X(\mathbb{A}_K)^{\pi}$ are said to be unobstructed (or to survive).

Could ask

- Is $\overline{X(K)} = X(\mathbb{A}_K)^{\pi}$?
- How are various obstruction sets related?

Conjecture 1.4 (Poonen, Voloch)**.** *Let $K = \mathbb{F}(D)$ and let A be an abelian variety over K. For any closed K-subscheme X of A, one has*

$$\overline{X(K)} = X(\mathbb{A}_K) \cap \overline{A(K)} \subset A(\mathbb{A}_K),$$

and in particular, $X(K)$ is nonempty (if the middle intersection is nonempty).

There's some evidence for curves and for non-constant/isotrivial higher dimensional X.

Example. Let X be a nice curve of genus $g \geq 2$ over K. Assume $X \hookrightarrow J$ and $Ш(J)$ is finite.

Theorem 1.5. $\overline{X(K)} = X(\mathbb{A}_K) \cap \overline{J(K)}$ *if J has no isotrivial factor (implies X not constant) and a condition on the pth power torsion holds.*

Theorem 1.6. *Suppose X is defined over $\mathbb{F}$. Consider the constant variety X_K. If $g(D) < g$, then* $\overline{X(K)} = X(\mathbb{A}_K) \cap \overline{J(K)}$.

(Still open for $g(D) \geq g$)

For both, info about $\overline{X(K)}$ is obtained via $X(\mathbb{A}_K) \cap \overline{J(K)} = X(\mathbb{A})^{\mathrm{isog}}$. △

[1] smooth, proper, geometrically integral

Obstruction sets from torsors Consider the category $\mathrm{Cov}(X_K)$ of (fppf) X_K-torsors under finite group schemes over K. Given, $(X' \xrightarrow{\pi} x, G) \in \mathrm{Cov}(X_K)$, the associated obstruction set is

$$X(\mathbb{A})^{\pi} := \left\{ (P_v) \in \prod_v X(K_v) \,\middle|\, \begin{array}{c} [\pi^{-1}(P_v)] \in \prod_v \mathrm{H}^1(K_v, G) \text{ is in the image of} \\ \mathrm{H}^1(K, G) \longrightarrow \prod_v \mathrm{H}^1(K_v, G) \end{array} \right\}$$

(so (P_v) lifts to an adelic point on a twist of (X', G)).

Obstruction sets from isogenies If X is a subvariety of an abelian variety $A/\mathbb{F}$, we can consider the subset of torsors in $\mathrm{Cov}(X_K)$ which arise as pullbacks of isogenies $A' \to A$ defined over $\mathbb{F}$. Denote the corresponding obstruction set by $X(\mathbb{A})^{\mathrm{isog}}$.

If $X/\mathbb{F}$ is a curve or abelian variety, one has

$$X(\mathbb{A}_K)^{isog} = X(\mathbb{A}_K)^{\mathrm{Br}} = X(\mathbb{A}_k)^{\mathrm{FinAb}}$$

(analogous result over number fields due to Stoll, over global function fields due to Creutz, Voloch, third person)

Question 1.7. *How much information can we obtain from an obstruction set built from minimal information (e.g. a single torsor and its twists)? In this setting, if an adelic point is unobstructed, when is it already in $X(K) = \mathrm{Hom}_{\mathbb{F}}(D, X)$?*

They ask this when $X' \xrightarrow{\pi} X$ comes from Frobenius isogeny on A.

Results, informally Recall $K = \mathbb{F}(D)$ for $D/\mathbb{F}$ a nice curve.

(1) $C/\mathbb{F}$ a nice curve, embedded in $J = \mathrm{Jac}(C)$. Let $X = C^{(2)}$ be symmetric square. Study $X(K) = \mathrm{Mor}_{\mathbb{F}}(D, C^{(2)})$.

- bound the number of points of $X(K)$ which are "non-horizontal" by $(p^r - 1)/(p-1)$ where r is the Mordell Weil rank of $J(K)$
- under suitable hypotheses on C, any adelic points of $C^{(2)}$ unobstructed by Frobenius descent is global, unless it is a "reduced adelic point" or the sum of a global point and a reduced adelic point of C.

  The reduced adelic points are $X(\mathbb{A}_{K,\mathbb{F}}) = \prod_v X(\mathbb{F}_v)$ with $\mathbb{F} \subset \mathbb{F}_v \subset \mathscr{O}_v \subset K_v$ (v a place, $\mathbb{F}_v$ the residue field)

(2) $X/\mathbb{F}$ subvariety of an abelian variety A

- give geometric conditions under which unobstructed points give rise to non-constant, separable morphisms $D \to X$
- also give explicit constructions of X w/ non-global points unobstructed

Frobenius descent Let $A/\mathbb{F}$ be an abelian variety, $\dim A = g$, and X a subvariety. Consider relative Frobenius $F : A^{(1/p)} \to A$ (and let $v : A \to A^{(1/p)}$ so $Fv : A \to A$ is $[p]$). Zariski locally, defining equations of A are obtained from those defining $A^{(1/p)}$ by taking pth powers and F is given by raising coords to their pth powers.

The pullback of $X \hookrightarrow A \xleftarrow{F} A^{(1/p)}$ yields a torsor $(X', \ker F) \in \mathrm{Cov}(X_K)$ under the finite abelian group scheme $\ker F \subset A^{(1/p)}$. What can we say about $(x_v) \in \prod_v X(K_v)$ which survives $(X', \ker F)$?

For a separable extension L/K, the Kummer sequence associated to F is an exact sequence of (flat) cohomology groups, w/ connecting homomorphisms

$$0 \longrightarrow A(L)/F(A^{(1/p)}(L)) \xrightarrow{\delta_{F,L}} \mathrm{H}^1(L, \ker F)$$

which has an explicit and useful description.

Lemma 1.8. *There is an injective group homomorphism*

$$\Phi_L : \mathrm{H}^1(L, \ker F) \to \mathrm{Hom}(\Omega^1_{A/L}, \Omega_{L/\mathbb{F}}) \simeq \Omega^{\oplus g}_{L/\mathbb{F}}$$

such that the composition $\Phi_L \circ \delta_{F,L}$ sends $x \in A(L)$ to $x^*(\omega_1, \ldots, \omega_g)$ (with $\omega_1, \ldots, \omega_g$ basis of holomorphic differentials). We write $\mu_L : A(L) \to \Omega^{\oplus g}_{L/\mathbb{F}}$ for this composition...

So, $(x_v) \in \prod_v X(K_v)$ survies F-descent means there's some $\xi \in \Omega^{\oplus g}_{K/\mathbb{F}}$ s.t. for each v, $\mu_{K_v}(x_v) = \xi$. If $\neq 0$, then $[\xi] \in \mathbb{P}(\Omega^{\oplus g}_{K/\mathbb{F}}) \cong \mathbb{P}^{g-1}_K$.

From a point surviving F-descent, we obtain an associated point of $\mathbb{P}^{g-1}(K)$, i.e. a morphism $D \to \mathbb{P}^{g-1}$.

Idea. Use a Gauss map to take points of X to points of $\mathbb{P}^{g-1}$ (represeting subspace of tangent space at originin A).

Example. Say $X = C$ non-hyperelliptic. For $x = (x_v) \in C(\mathbb{A}_K)$ surviving $(C', \ker F)$, the image of the corresponding $D \to \mathbb{P}^{g-1}$ coincides w/ the canincal embedding $C \hookrightarrow \mathbb{P}^{g-1}$. They conclude $(x_v) \in X(K)$

Sketch; $(x_v) \in C(\mathbb{A}_K)$ survives implies $\exists \xi \in \mathrm{H}^1(K, \ker F)$ s.t. for each v, $\mu_{K_v}(x_v) = \Phi_K(\xi) \in K^g$. So

$$[\mu_{K_v}(x_v)] = (f_1(x_v), \ldots, f_g(x_v)) \in C(K) \subset \mathbb{P}^{g-1}(K)$$

with $\omega_i = f_i(t)\mathrm{d}t$. △

Example. Symmetric squares of curves. $C, D/\mathbb{F}$ nice curves and say $g = g(C) \geq 2$. Let $C^{(2)}$ be the symmetric square, and let $J = \mathrm{Jac}(C)$. Want to consider maps (note $C^{(2)} \hookrightarrow J$)

$$D \to C^{(2)}.$$

For non-constant separable map $f : D \to J$, get a map $D \to \mathbb{P}^{g-1}$ as follows. Take $P \in D$ and consider

$$T_{f(P)} f(D) \subset T_{f(P)} J \simeq T_0(J).$$

Translating such subspaces to tangent spaces to J at origin, get $\varphi : D \to \mathbb{P}^{g-1} = \mathbb{P}(T_0 J)$. From $C \hookrightarrow J$, we recover the canonical embedding $C \subset \mathbb{P}^{g-1}$. △

Remark 1.9. Traditional Gauss maps $X \to \mathrm{Gr}(\dim X, \dim A)$ sends a smooth point $x \in X$ to the point represeting $T_x X \subset T_x A$. In our setting, we're getting a map from the projectivized tangent bundle P

of X to $\mathbb{P}^{g-1}$ w/ $\dim P = 2\dim X - 1$ and the fiber at $x \in X$ the space of 1-dim subspaces of $T_x X$ (I think) $\circ$

Theorem 1.10. *Assume that C has no g^1_2, g^1_3 or g^1_4. The number of maps $D \to C^{(2)}$ which are separable but not horizontal is bounded by $(p^r - 1)/(p-1)$, where*

$$r = \dim_{\mathbb{F}_p} \mathrm{Mor}(D, J)/F(\mathrm{Mor}(D, J^{(1/p)}))$$

w/ $J^{(1/p)} \to J$ the relative Frobenius map.

Fact (Rimeann Kempf)**.** The image of the tangent plane $T_{P+Q}C^{(2)}$ in $\mathbb{P}^{g-1}$ is the secant line $\overline{PQ}$ on the canonical curve $C \subset \mathbb{P}^{g-1}$. Thus, the image $\varphi_f(D)$ is contained in the secant variety S of the canonical embedding $C \hookrightarrow \mathbb{P}^{g-1}$.

Idea. From a point $R \in S \setminus C \subset \mathbb{P}^{g-1}$ can recover $P, Q \in C$ with $R \in \overline{PQ}$... (use no g^1_4 and no g^1_3 = no trisecant)

1.2 Ashvin Swaminathan (On the distribution of 2-Selmer groups of hyperelliptic Jacobians)

(joint w/ Manjul Bhargava and Arul Shankar)

Let $f(x, y) \in \mathbb{Z}[x, y]$ be a separable form of even degree $n \geq 4$. Consider $C_f : z^2 = f(x, y)$ with Jacobian $J(C_f)$.

Definition 1.11. A **2-cover** of C_f is a cover of C_f with automorphic group isomorphic to $J(C_f)[2]$ as $\mathrm{Gal}(\overline{\mathbb{Q}}/\mathbb{Q})$-modules. The **2-Selmer group** $\mathrm{Sel}_2(J(C_f))$ is the set of locally solvable 2-covers of $J(C_f)$.

Conjecture 1.12. *Let $n \geq 6$ with $n \equiv 2 \pmod 4$. When binary n-ic forms f are ordered by the max norm on their coefficients, we have* $\mathrm{Aut}\, \# \,\mathrm{Sel}_2(J(C_f)) = 6$.

2 steps

- Parameterize arithmetic objects of interest in terms of integral/rational orbits of a coregular[2] representation $G \curvearrowright V$; if rational, check they have integral representations

  Example. $V = \{\text{binary quadrtic forms}\}$ and $G = \mathrm{PGL}_2$ w/ ring of invariants $Z\langle I, J\rangle$. $\triangle$

- Use geometry-of-numbers to count integral points in fundamental domain

Example. Say $C_f(\mathbb{Q}) \neq \emptyset$, so then $\mathrm{Pic}^1(C_f) \simeq J(C_f)$ so locally solvablue 2 covers of $J(C_f)$ are those of $\mathrm{Pic}^1(C_f)$.

Theorem 1.13. *Locally solvable 2-covers of* $\mathrm{Pic}^1(C_f)$ *embed into the set of pairs* $(A, B) \in (\mathrm{Sym}_2 \mathbb{Z}^n) \oplus \mathrm{Sym}_2 \mathbb{Z}^n$ *s.t.* $\det(xA + yB) = f(x, y)$*, up to* $(\mathrm{SL}_n / \mu_2)(\mathbb{Z})$ *action*

(apparently the proof of this theorem does not quite right. For example, it only works if the curve is monic. Correction to show up in Ashvin's thesis)

If $C_f(\mathbb{Q}) = \emptyset$, then there may not exists (A, B) as above with $\det(xA + yB) = f(x, y)$.

[2]ring of invariants freely and finitely generated

Idea. Create a $\mathbb{Q}$-point by replacing f with $f^{\text{mon}} := f_0^{-1} \times f(x, f_0 y)$ with f_0 the leading coefficient of f. $C_{f^{\text{mon}}}(\mathbb{Q}) \neq \emptyset$ (two points at infinity), twist of C_f by $\mathbb{Q}(\sqrt{f_0})$. Finally, $J(C_f)[2] \simeq J(C_{f^{\text{mon}}})[2]$. So $\text{Sel}_2(J(C_f))$ identified with certain 2-covers of $J(C_{f^{\text{mon}}})$.

△

Remark 1.14 (Parametrization). Let $R_f := \text{H}^0\left(\text{Proj}\,\mathbb{Z}[x,y]/(f(x,y))\right)$ and $K_f = \text{Frac}(R_f)$. Let D_f be the codifferent of R_f.

Theorem 1.15. *Let $f \in \mathbb{Z}[x,y]$ be binary form of even degree $n \geq 4$, and suppose C_f is loc. sol. if $n \equiv 0 \pmod 4$. Elements of $\text{Sel}_2(J(C_f))$ correspond to certain pairs (I, α) where*

- *I = ideal class of R_f; $\alpha \in K_f^\times/(K_f^\times)^2$*
- *$I^2 \subset \alpha \times D_f$ and $N(I)^2 = N(\alpha)N(D_f)$*

Ellenberg on MathOverflow in 2011: Can one... ○

Theorem 1.16 (S., 2020)**.** *Let $f \in \mathbb{Z}[x,y]$ beinary n-ic form with leading coeffcient $f(1,0) = f_0 \neq 0$. Then, we have an injection from pairs (I, α) as before to pairs (A, B) with $\det(xA + yB) = f^{mon}(x,y)$, up to $\text{SL}_2^{\pm}(\mathbb{Z})$-action.*

Theorem 1.17 (Bhargava, Shankar, and S., 2022)**.** *Let $n \geq 4$ even. Consider binary n-ic forms f w/ fixed nonzero leading coefficient such that C_f is loc. sol. if $n \equiv 0 \pmod 4$. When such f are ordered by "height," we have $\text{Avg}\,\#\,\text{Sel}_2(J(C_f)) \leq^* 6$.*

- Shows Poonen-Rains conjecture is robust
- Previously proven by Shankar and Wang (2018) for leading coefficient 1
- Bhargava-Gross proved analaogous result for leading coefficient 0

What if leading coeff varies?

- Goal: Compute $\text{Avg}\,\#\,\text{Sel}_2(J(C_f))$ over all f (loc. sol. if $4 \mid n$)
- Naive approach: determine asymptotic count for each fixed f_0, then sum
- Given $f_0 \in \mathbb{Z} \setminus \{0\}$, let $S_{f_0}(X) := \{f : H^*(f) < X, f(1,0) = f_0\}$ where
$$H^*(f) = \max_i \left\{ \left| f_0^{i-1} f_i \right|^{1/i} \right\}.$$
Then,
$$\sum_{f \in S_{f_0}(X)} \#\,\text{Sel}_2(J(C_f)) \ll f_0^{-\frac{n(n-1)}{2}} X^{\frac{n(n+1)}{2}} + error$$
- Problem: natural height on binary forms is $H(f) = \max_i\{|f_i|\}$
- ...

Note 1. I need to stop trying to tex notes during slide talks

1.3 Tim Santens (KU Leuven): Quartic diagonal surfaces w/ a Brauer-Manin obstruction

Consider the surface $X_{\vec{a}} \subset \mathbb{P}^3$ defined by

$$a_0x_0^4 + a_1x_1^4 + a_2x_2^4 + a_3x_3^4 = 0 \text{ where } \vec{a} = (a_0, a_1, a_2, a_3).$$

This is a K3 surface (so $\omega_X \cong \mathscr{O}_X$ and $\mathrm{H}^1(X, \mathscr{O}_X) = 0$).

Question 1.18. *When is $X_{\vec{a}}(\mathbb{Q})$ nonempty, and when does the Hasse principle fail?*

Conjecture 1.19 (Sko.)**.** *The Brauer-Manin obstruction is the only obstruction to the Hasse principle for K3 surfaces.*

Question 1.20. *Is often is there a B-M obstruction?*

(This question is made with the thought in mind that the previous conjecture is true)

Theorem 1.21 (Bright, spelling?)**.** *There is a Brauer-Manin obstruction for 0% of $\vec{a}$?*

Theorem 1.22 (S.)**.** *There exists $A, B > 0$ s.t.*

$$\#\left\{X_{\vec{a}} \text{ with a B-M obstruction with } |a_i| \le T\right\} \sim AT^2 (\log T)^{\frac{33}{8}}$$

and

$$\#\left\{\begin{array}{c} X_{\vec{a}} \text{ with a B-M obstruction with } |a_i| \le T \\ \text{and } a_0a_1a_2a_3 = -\square \end{array}\right\} \sim AT^2(\log T)^{\frac{33}{8}}$$

and

$$\#\left\{\begin{array}{c} X_{\vec{a}} \text{ with a B-M obstruction with } |a_i| \le T \\ \text{and } a_0a_1a_2a_3 \neq -\square \end{array}\right\} \sim BT^2(\log T)^{\frac{45}{16}}$$

Remark 1.23. The set of $\vec{a}$ s.t. $a_0a_1a_2a_3 = -\square$ form a thin set of $\mathbb{A}^4$. For Manin-type conjectures, one usually wants to throw away a thing set to get the "real answer," so the last statement of the theorem can be seen as what's really happening. ◦

Consider Brauer-Manin pairing

$$\begin{array}{ccc} X(\mathbb{A}_k) \times \mathrm{Br}(X) & \longrightarrow & \mathbb{Q}/\mathbb{Z} \\ ((x_v)_v, A) & \longmapsto & \sum_v \mathrm{inv}_v(A(x_v)) \end{array}$$

The left kernel of this pairing is, by definition, the **Brauer-Manin set** $X(\mathbb{A}_k)^{\mathrm{Br}}$ of X.

Definition 1.24. X **has a Brauer-Manin obstruction** if $X(\mathbb{A}_k) \neq \emptyset$, but $X(\mathbb{A}_k)^{\mathrm{Br}} = \emptyset$.

Note that the structure map $X \to \mathbb{Q}$ induces a map $\mathrm{Br}\,\mathbb{Q} \to \mathrm{Br}\,X$, and the image of this map doesn't really play a role in the Brauer-Manin obstruction. All that matters is $\mathrm{Br}(X)/\mathrm{Br}(\mathbb{Q})$, which is often finite.

Question: Why is $\mathrm{Br}(\mathbb{Q})$ in the right-kernel of the pairing?

Definition 1.25. An element $A \in (\mathrm{Br}\,X)[n]$ is **prolific at** v if the map

$$X(\mathbb{Q}_v) \to \frac{\mathbb{Z}}{n\mathbb{Z}} \subset \frac{\mathbb{Q}}{\mathbb{Z}}$$

Answer: These are "constant"

(sending $x_v \mapsto \mathrm{inv}_v(A(x_v))$) is surjective.

Theorem 1.26. *If A is prolific at v, then it does not contribute to the Brauer-Manin obstruction.*

I'm not 100% sure exactly what's meant by this

(there's also an analogue for multiple elements)

If I'm reading correctly, A prolific at v $\Longleftarrow$ certain varieties over $\mathbb{F}_v$ have $\mathbb{F}_v$-points $\iff$ These varieties are geometrically irreducible.

Fact. $\mathrm{Br}(X_{\vec{a}})/\operatorname{Br}\mathbb{Q} \cong \mathbb{Z}/2\mathbb{Z}$ most of the time with generator $A \in \operatorname{Br} X_{\vec{a}}$ having an explicit descriptions, but not a uniform one.

Theorem 1.27 (S.)**.** *If A is nowhere prolific, then $X_{\vec{a}}$ has a 50% chance of having a Brauer-Manin obstruction.*

Proof idea. Use subfamilies on which A above has a uniform description. ∎

Say A is nowhere prolific. This turns out to be equivalent to $\vec{a}$ being a weak Compana point + some other conditions.

Theorem 1.28. *If $\pi : X \to \mathbb{P}^n$ a smooth family. Then,*

$$\#\left\{P \in \mathbb{P}^n : \pi^{-1}(P)(\mathbb{A}_k) \neq \emptyset, |X_i| \leq T\right\} \ll T^{n+1}(\log T)^{-\Delta}$$

for $\Delta = \sum_{D \subset \mathbb{P}^{n+1}} \delta_\pi(D)$

(Assuming I copied down the statement correctly)

Corollary 1.29. *$\#\vec{a}$ weak compana point, ... $\approx T^2(\log T)^{6-\Delta(\pi)}$ where $6 - \Delta(\pi) = 45/16$*

2 Day 2 (4/26)

2.1 Marta Pieropan (Utrecht University): Campana points on Fano Varieties

Let's first talk about numbers. Define

$$D_m := \{m\text{th powers}\} \text{ and } C_m = \{m\text{-full numbers}\}$$

An m**-full numbers** are numbers n s.t. $p \mid n \implies p^m \mid n$. Note that an mth power is one such that $p \mid m \implies p^{md} \mid\mid n$ for some $d = d(p)$. Note

$$\begin{tikzcd}
\mathbb{N} \arrow[r, equal] & D_1 \arrow[d, equal] & \supseteq & D_2 \arrow[d, phantom, sloped, "\subseteq"] & & D_3 \arrow[d, phantom, sloped, "\subseteq"] & & & \supseteq & \{1\} \arrow[d, equal] \\
& C_1 & \supseteq & C_2 & \supseteq & C_3 & \supseteq & \dots & \supseteq & \{1\}
\end{tikzcd}$$

Warning 2.1. $D_3 \not\subseteq D_2$.

Keeping with numbers, recall

$$\mathbb{Q} = \left\{\frac{a}{b} : b > 0, \gcd(a,b) = 1\right\} \supset \left\{\frac{a}{b} : b = 1\right\} = \mathbb{Z}.$$

We can then consider the chain (in every slot below, always assume $b \neq 0$)

$$\begin{array}{ccccccccc}
\mathbb{Q} = & \{b \in \mathbb{N}_{\neq 0}\} & & & & & \{b = 1\} & = \mathbb{Z} \\
& \text{IU} & & & & & \| & \\
(*) & \{b \in C_2\} & \supseteq & \{b \in C_3\} & \supseteq & \dots \supseteq & \bigcap_{m>0} \{b \in C_m\} & \\
& \text{IU} & & \text{IU} & & & \| & \\
(**) & \{b \in D_2\} & & \{b \in D_3\} & & & \bigcap_{m>0} \{b \in D_m\} &
\end{array}$$

Remark 2.2. The row $(*)$ gives Campana points for $\left(\mathbb{P}^1_{(a:b)}, \left(1 - \frac{1}{m}\right)(1:0)\right)$ over $\mathbb{Q}$.

The sets (**) give Darmon points on the same $\mathbb{Q}$-orbifold $\left(\mathbb{P}^1_{(a:b)}, \left(1 - \frac{1}{m}\right)(1:0)\right)$ ◦

(We'll define Campana points later)

Slogan. Campana points interpolate between rational points and integral points.

Setup. Let k be a number field, let X be a smooth projective k-variety. Let

$$D = \sum_{i=1}^{m} D_i$$

be a strict normal crossing divisor. Also choose some paramaters

$$\underline{n} = (n_1, \dots, n_m) \in \mathbb{N}^m_{\neq 0}.$$

To talk about Campana points, need an integral model, so say $\mathcal{X}/\mathscr{O}_{k,S}$ is a proper, regular model of X, and let $\mathcal{D}_i = \overline{D}_i \subset \mathcal{X}$.

(I think we'll mostly ignore the extra baggage coming from looking at S-integral points in this talk)

Definition 2.3. Let $\alpha \in X(k) = \mathcal{X}(\mathscr{O}_k)$ and write $\alpha : \operatorname{spec} \mathscr{O}_k \to \mathcal{X}$.

Assumption. Below, we implicitly assume $\alpha(\operatorname{spec} \mathscr{O}_k) \not\subset \mathcal{D}_i$ in order for the condition we wrote to be the right one.

- α is a **Campana point** if
$$\alpha^* \mathcal{D}_i = \sum_{\mathfrak{p}} n_{\mathfrak{p},i} \mathfrak{p}$$
satisfies $n_{\mathfrak{p},i} \neq 0 \implies n_{\mathfrak{p},i} \geq n_i$ always.

  α is a **Darmon point** if above $n_i \mid m_{\mathfrak{p},i}$ for all $\mathfrak{p}$ and all i.

- The above was a geometric definition. Alternatively, if f_i is a local equation for $\mathcal{D}_i$ around α (and say f_i defines a subscheme with good reduction), then $m_{\mathfrak{p},i} = v_{\mathfrak{p}}(f_i(\alpha))$. Thus, α is a *Campana point* if $f_i(\alpha) \in C_{m_i}$ for every i, and it is a *Darmon point* if $f_i(\alpha) \in D_{m_i}$ for all $i = 1, \dots, m$.

Example. Take $\left(\mathbb{P}^2, \left(1 - \frac{1}{m}\right)\{x_0 = 0\}\right)$. Is $\alpha = (4:5:3)$ a Campana point?

One local equation is $f = x_0/\ell$ for some linear form ℓ s.t. $\ell(\alpha) = 1$ (working in open set $U = D_+(\ell)$).[3] Here, we take $f = x_0/(x_0 - x_2)$. Looking at p-adic valuation is the same as looking at reduction modulo

[3] Not always possible to find ℓ with $\ell(\alpha) = 1$, but easier when you can

prime powers. For example, we have $(4:5:3) \equiv (0:5:3) \mod 4$, but not mod 8, so $(4:5:3)$ is Campana for $m=2$ but not for $m \geq 3$. △

2.1.1 Motivation

We won't say much about Darmon points, but note that they correspond to points on orbifold stacks.

Example. Say $\mathcal{X} = \mathbb{P}^1$ with a $\frac{1}{2}$ point at ∞ and $\frac{1}{3}$ points at 0. Points on this stack correspond to Darmon points associated to the pair $(\mathbb{P}^1, \frac{1}{2}(1:0) + \left(1-\frac{1}{3}\right)(0:1))$.

Darmon points here are rational numbers a/b with b a square and a cube? (Audience remark, I think) △

Campana points are supposed to be related to images of rational points under fibrations.

Say $X \xrightarrow{f} Y$ is a fibration of nice varieties. Some fibers won't be nice (may have multiple components with varying multiplicities). Note we care about fibers over divisors here. Given a divisor $D \subset Y$, can write

$$f^*D = \sum_{E \subset X} n_E(f^*D) \cdot E$$

(E ranging over prime divisors). Say $\mathcal{X} \to \mathcal{Y}$ an integral model of f, and say $\alpha \in \mathcal{X}(\mathbb{Z})$. Consider

$$\alpha^* f^* \mathcal{D} = \sum_{\mathcal{E}} m_{\mathcal{E}}(f^*\mathcal{D}) \alpha^* \mathcal{E} = \sum_p \left(\underbrace{\sum_{\mathcal{E}} m_E(f^*\mathcal{D}) m_p(\mathcal{E})}_{m_p(\alpha^* f^* D)} \right) p$$

with coefficient of p

$$\geq \underbrace{\inf_E m_E(f^*D)}_{m_D} \sum_E m_p(E)$$

(above is a divisor in $\operatorname{spec} \mathbb{Z}$, not an integer, i.e. the sum over p is formal). In this setup, Campana associated to this the data

$$\left(Y, \sum_D \left(1 - \frac{1}{m_D}\right) D \right)$$

and calls this the **orbifold base** of f.

Remark 2.4. With this choice of orbifold base, all rational points on X have image in Y a Campana point. ○

Remark 2.5. The sum in the definition of orbifold base should always be finite (since X has integral generic fiber?). Also, I kinda missed this remark, but the number m_D above is really just the infinimum of the multiplicities of the components of the fiber above D. ○

Example. Consider

$$X = \left\{ax^2 - byz = 0\right\} \subset \mathbb{P}^2_{(x:y:z)} \times \mathbb{P}^1_{(a:b)} \to \mathbb{P}^1,$$

and let $f : X \to \mathbb{P}^1$ be the projection. The orbifold base in this case is $(\mathbb{P}^1, (1-\frac{1}{2})(1:0))$. Note that X is not smooth, but we'll ignore this by restricted to points satisfying $\gcd(x, yz) = 1$. A point $(1:b) \in f(X(\mathbb{Q}))$ (with preimage satisfying $\gcd(x, yz) = 1$) only if b is a square and so 2-full. △

Remark 2.6. The definition of Campana/Darmon points imposing conditions at each prime, but does so independently. Hence, we can make analogous definitions for (non-archimedean) local fields. One can also give a notion of adelic Campana/Darmon points. ○

(Maybe a bit of this in the next talk)

2.1.2 Some Open Problems

(Only about Campana points)

Open Question 2.7 (**Orbifold-Mordell Conjecture**). *Say X a smooth projective curve. If the pair $(X, \sum\left(1-\frac{1}{m_i}\right)D_i)$ is a Campana pair with log canonical divisor*

$$K_X + \sum_{i=1}^{m}(1-\frac{1}{m_i})D_i$$

ample, then the set of Campana points is not Zariski dense.

This is known in the function field case by Campana in characteristic 0, and by Kebehus-Pereisa-Smeets (spelling?) in positive characteristic case. However, it is open for number fields. However, it follows from the abc conjecture.

Open Question 2.8. *In the same setting as above, if*

$$-\left(K_X + \sum_{i=1}^{m}\left(1-\frac{1}{m_i}\right)D_i\right)$$

is ample, then we expect potential density (get density after base field extension).

Open Question 2.9. *With ample log-canonical divisor, under which conditions is the set of Campana points (potentially) not thin?*

Example. Consider $\mathbb{N} \supset C_m \supset D_m \supset \{1\}$. The integers are not thin in $\mathbb{A}^1_{\mathbb{Q}}$, and C_m is also not thin in $\mathbb{A}^1$. However, D_m is thin in $\mathbb{A}^1$. △

Maybe makes one expect that Campana condition should give non-thin sets, while Darmon condition should give thin sets.

When the set of Campana points is Zariski dense, can count them using some height function, and there's a Manin-type conjecture for their asymptotics.

2.2 Sam Streeter (University of Bristol): Semi-integral points and quadrics

(joint w/ Vlad Mitankin & Masahiro Nakahara)

Start with a recap of Campana points.

Recall 2.10. For us, an **orbifold** is a pair $(X, D)/K$ (D a $\mathbb{Q}$-divisor) with K a number field. Here, X/K is a variety and D looks like $D = \sum_{i=1}^{n}\left(1-\frac{1}{m_i}\right)D_i$. Note $m_i \in \mathbb{Z} \cup \{\infty\}$ (where $\frac{1}{\infty} = 0$).

Given such an orbifold, want a model $(\mathcal{X}, \mathcal{D})/\mathcal{O}_S$ (S a finite set of places continuing all archimedean places) over the ring of S-integers. We take $\mathcal{D} = \overline{D} \subset \mathcal{X}$.

For a point $P \in X(K_v)$ ($v \notin S$), we have multiplicities $n_v(\mathcal{D}_i, P)$. We define two types of **semi-integral points**:

- **Campana**: $n_v(D_i, P) > 0 \implies v_n(D_i, P) \geq m_i$ (think "powerful solutions")

  The set of such we'll denote by $(\mathcal{X}, \mathcal{D})^C(\mathscr{O}_v)$.

- **Darmon**: $m_i \mid n_v(D_i, P)$ (think "solutions in powers")

  The set of such we'll denote $(\mathcal{X}, \mathcal{D})^D(\mathscr{O}_v)$

For the global version, the set will be denoted $(\mathcal{X}, \mathcal{D})^*(\mathscr{O}_S)$ with $* \in \{C, D\}$

Remark 2.11. If $\mathcal{D}_i = Z(f_i)$ and $P \in \mathcal{X}(\mathscr{O}_v)$, then $n_v(\mathcal{D}_i, P) = v(f_i(P))$. ○

Goal. Study local-global principles for these semi-integral points.

Two relevant products

- $$P_1 = \prod_{v \notin S} (\mathcal{X}, \mathcal{D})^* \times \prod_{v \in S} X(K_v)$$

- Let $U := X \setminus \mathrm{supp}(D)$ and $\mathcal{U} := \mathcal{X} \setminus \mathrm{supp}(\mathcal{D})$.

$$P_2 = \left[{\prod_{v \notin S}}' ((\mathcal{X}, \mathcal{D})^*(\mathscr{O}_v) \cap \mathcal{U}(K_v), \mathcal{U}(\mathscr{O}_v)) \times \prod_{v \in S} U(K_v) \right] \sqcup \prod_v D_{\mathrm{red}}(K_v).$$

  Note that P_2 pairs w/ $\mathrm{Br}(U) \times \mathrm{Br}(D_{\mathrm{red}})$, and so one can make sense of a Brauer-Manin obstruction.

Definition 2.12. Say we have **semi-integral weak approximation** (resp. **semi-integral strong approximation**) when the global points $(\mathcal{X}, \mathcal{D})^*(\mathscr{O}_S)$ are dense in P_1 (resp. P_2). We say the **semi-integral Hasse principle** holds if either $P_i \neq \emptyset \implies (\mathcal{X}, \mathcal{D})^*(\mathscr{O}_S) \neq \emptyset$.

Now let's talk about quadrics.

Take $Q \subset \mathbb{P}_K^{n+1}$ a quadric hypersurface, and $H \subset \mathbb{P}_K^{n+1}$ a hyperplane. Get two families

- $F_1 : \left(\mathbb{P}^{n+1}, \left(1 - \frac{1}{m}\right) Q\right)$

- $F_2 : (Q, (1 - 1/m)(Q \cap H))$

(in either case, take the obvious models)

Theorem 2.13. *Orbifolds in F_1 satisfy Campana weak approximation (CWA) and always have a Campana point.*

Proof Idea. Reduce to 1-dimensional case of projective line w/ quadratic point by taking a line section in a clever way. ■

Theorem 2.14. *(Here, need $n \geq 2$). Orbifolds in F_2 satisfy CWA.*

Proof Idea. Reduce to theorem 1 via birational map $Q \dashrightarrow \mathbb{P}^n$ (projection away from a point on Q?). Get a sort of triangle

$$\begin{tikzcd} & (X, \widetilde{C}) \arrow[dl] \arrow[dr] & \\ (Q, (1 - \frac{1}{m})(Q \cap H)) \arrow[rr, dashed] & & (\mathbb{P}^n, C) \end{tikzcd}$$

■

Theorem 2.15. *If m is odd, $n \geq 2$, and $Q \cap H$ is smooth, then orbifolds in F_2 satisfy the Darmon Hasse Principle (HP).*

Question: Can $n = 1$ be deduced from this paper (audience remark)

Proof Idea. Find place where generator of $\operatorname{Br} U / \operatorname{Br} K$ ($U = Q \setminus D_{\text{red}}$) is prolofic. Use that the BM obstruction is the only one (coming from work of Colliot-Thélène-Xu on integral points) ■

2.3 Rosa Winter (King's College London): Concurrent exceptional curves and torsion points on del Pezzo surfaces of degree one

Definition 2.16. A **del Pezzo surface** X is a nice surface w/ ample anticaonical divisor $-K_X$. It's **degree** is the self-intersection K_X^2 with always between 1 and 9.

Example.

- Smooth cubic surfaces (degree 3)
- Complete intersection of two quadrics in $\mathbb{P}^4$ (degree 4)
- For $3 \leq d \leq 9$, ...

△

Slide talks too fast, just gonna listen and not take notes...
Some things

- The number of lines on a del Pezzo surface of degree d depends only on d
- A del Pezzo surface of degree $d \geq 3$ with a rational pont is unirational over k (k any field?)
-

  Open Question 2.17. *Is there an example of a minimal del Pezzo surface of degree 1 which can be shown to be unirational or not unirational?*
- X/k with k an infinite field. Then X unirational over $k \implies X(k)$ dense in X.

2.4 Margherita Pagano (Leiden University): Brauer-Manin obstruction at primes w/ good reduction

Notation 2.18. Let k be a number field. Let Ω_k be its set of places. For every $v \in \Omega_k$, k_v will denote its completion at v.

Let X/k be a smooth, proper, geometrically integral k-variety.

We want to study $X(k)$.

Remark 2.19. For all $v \in \Omega_k$, have inclusion $X(k) \hookrightarrow X(k_v)$ which together give diagonal embedding

$$X(k) \hookrightarrow \prod_v X(k_v).$$

$\circ$

Definition 2.20. We say that X satisfies **weak approximation** if $X(k)$ is dense in $\prod_{v \in \Omega_k} X(K_v)$.

Idea (Manin). Use the Brauer group to build an intermediate set

$$X(k) \subset \left(\prod_v X(k_v)\right)^{\mathrm{Br}} \overset{\text{closed}}{\subset} \prod_v X(k_v).$$

Definition 2.21. We say that there is a **Brauer-Manin obstruction to weak approximation** on X if

$$\left(\prod_v X(k_v)\right)^{\mathrm{Br}} \subsetneq \prod_v X(k_v).$$

Let $A \in \mathrm{Br}(X) = \mathrm{H}^2_{\text{ét}}(X, \mathbb{G}_m)$. For every place v, have an evaluation map

$$\mathrm{ev}_A : X(k_v) \to \mathbb{Q}/\mathbb{Z}\,.$$

Using this, we define

$$\left(\prod_v X(k_v)\right)^{\mathrm{Br}} = \left\{(x_v) \in \prod_v X(k_v) : \sum_v \mathrm{ev}_A(x_v) = 0 \ \text{ for all } \ A \in \mathrm{Br}(X)\right\}.$$

Definition 2.22. We say that a place v **plays a role in the Brauer-Manin obstruction** if there exists some $A \in \mathrm{Br}(X)$ s.t. $\mathrm{ev}_A : X(k_v) \to \mathbb{Q}/\mathbb{Z}$ is non-constant.

Remark 2.23. If v_0 plays a role in the B-M obstruction and $A \in \mathrm{Br}(X)$ s.t. $\mathrm{ev}_A : X(K_v) \to \mathbb{Q}/\mathbb{Z}$ is non-constant, then there exists $x_0, y_0 \in X(k_v)$ with different evaluations. Let $(x_v), (y_v) \in \prod_v X(k_v)$ with $x_v = y_v$ for all $v \neq v_0$. Then,

$$\sum_v \mathrm{ev}_A(x_v) \neq \sum_v \mathrm{ev}_A(y_v),$$

so at least one among them does not belong to the Brauer-Manin set, so there's an actual obstruction in this case. $\circ$

Assume $\mathrm{Pic}(\overline{X})$ ($\overline{X} := X \times_k \overline{k}$) is torsion-free and finitely generated.

Question 2.24 (Swinnerton-Dyer)**.** *Is it true that the only places involved in the B-M obstruction to weak approximations are the archimedeans ones together with the places of bad reduction for X?*

Some results

(1) First define $\mathrm{Br}_1(X) := \ker(\mathrm{Br}(X) \to \mathrm{Br}(\overline{X}))$. Then, if v is a non-archimedean place of good reduction which does not divide $\#(\mathrm{Br}(X)/\mathrm{Br}_1(X))$, then for all $A \in \mathrm{Br}(X)$, the evaluation map $\mathrm{ev}_A : X(k_v) \to \mathbb{Q}/\mathbb{Z}$ is constant.

Corollary 2.25. *If the Brauer group is algebraic (i.e.* $\mathrm{Br}(X) = \mathrm{Br}_1(X)$*) or if all primes dividing* $\#\mathrm{Br}(X)/\mathrm{Br}_1(X)$ *are primes of bad reduction, then we get an affirmative answer to Swinnerton-Dyer's question.*

(2) (Bright-Newton) Without any assumption on $\mathrm{Pic}(\overline{X})$, but assuming $\mathrm{H}^0(X, \Omega^2_X) \neq 0$. Let $\mathfrak{p}$ be a prime of good ordinary[4] reduction w/ residue characteristic p. Then, there exists a finite field extension L/K and a prime $\mathfrak{p}'$ over $\mathfrak{p}$ along with an element $A \in \mathrm{Br}(X_L)[p]$ such that

$$\mathrm{ev}_A : X(L_{\mathfrak{p}'}) \to \mathbb{Q}/\mathbb{Z}$$

is non-constant.

We now want to give an example where **(2)** above applies.

K3 surfaces First some facts about these

- $\mathrm{Pic}(\overline{X})$ is always torsion-free and finitely generated.
- After a finite base extension, K3 surfaces have infinitely many primes of good ordinary reduction.
- $\mathrm{H}^0(X, \Omega^2_X)$ is a one-dimesnional k-vector space
- (Bright-Newton) If $\mathfrak{p}$ is a prime s.t. $e_{\mathfrak{p}} < p - 1$, then $\mathrm{ev}_A : X(\kappa_{\mathfrak{p}}) \to \mathbb{Q}/\mathbb{Z}$ is constant for all $A \in \mathrm{Br}(X)$.

By the last bullet point, in order to get an example over $\mathbb{Q}$, we'll need 2 to be a prime of good ordinary reduction (so the evaluation map has a hope of being non-constant).

Example. Take $X \subset \mathbb{P}^3_{\mathbb{Q}}$ defined by the equation

$$x^3y + y^3z + z^3w + w^3x + xyzw = 0.$$

Theorem 2.26. *The class of the quaternion algebra*

$$A = (f, g) \in \mathrm{Br}(\mathbb{Q}(X)) \ \textit{with} \ f = \frac{z^3 + w^2x + xyz}{z^3}, y = -\frac{z}{x}$$

lies in $\mathrm{Br}(X)$*, and the evaluation map* $\mathrm{ev}_A : X(\mathbb{Q}) \to \mathbb{Q}/\mathbb{Z}$ *is non-constant.*

Let $Y/\mathbb{F}_2$ be the K3 surface defined by the reduction mod 2 of the equation cutting out X. Then, $\mathrm{H}^0(Y, \Omega^2_Y)$ is a 1-dimensional $\mathbb{F}_2$-vector space. Let w be its nonzero element. The image of w along $\mathrm{H}^0(Y, \Omega^2_Y) \to \Omega^2_F$ ($F = \mathbb{F}_2(y)$) is

$$\frac{\mathrm{d}\overline{f}}{\overline{f}} \wedge \frac{\mathrm{d}\overline{g}}{\overline{g}}.$$

Theorem 2.27 (Bloch-Kato)**.** *Let* $K = \mathbb{Q}_2(X_2)$*. Then, Bloch and Kato define a decreasing filtration on* $\mathrm{Br}(K)[2]$ *as well as an isomorphism*

Question: Why is the RHS not 0?

[4]Let $Y/\kappa(\mathfrak{p})$ be the reduction mod $\mathfrak{p}$. Let $B^q_y = \mathrm{Im}\left(\mathrm{d} : \Omega^{q-1}_Y \to \Omega^q_Y\right)$. Then, Y is **ordinary** if for all $n, q \geq 0$, one has $\mathrm{H}^n(Y, B^q_Y) = 0$

$$\rho_0 : \Omega^2_{F,\log} \oplus \Omega^1_{F,\log} \xrightarrow{\sim} \frac{\mathrm{Br}(K)[2]}{\bigcup^m \mathrm{Br}(K)[2]}$$

so that $(\omega, 0) \mapsto [A]$.

Theorem 2.28. $\{A \in \mathrm{Br}(X_2)[2] \text{ s.t. } \forall L/\mathbb{Q}_2 \forall P \in X(L) : \mathrm{ev}_A : X(L) \to \mathbb{Q}/\mathbb{Z} \text{ is constant on } B(P,1)\} = \{A \in \mathrm{Br}(X_2)[2] : A \in U^2 \,\mathrm{Br}(k)[2]\}$.

Question: What?

Since $[A]$ is nonzero, we have $A \notin U^1 \,\mathrm{Br}(k)[2] \implies A \notin \mathrm{Br}(k)[2] \implies \mathrm{ev} -a4$ is not constant mod 2 (after field extension). △

2.5 Alec Shute (Institute of Science and Technology Austria): Polynomials Represented by norm forms via the beta sieve

Let K be a number field of degree n over $\mathbb{Q}$. Let $\alpha \in K$ and consider $m_\alpha : K \to K$, $x \mapsto \alpha x$. The **field norm** is the map

$$\begin{aligned} \mathrm{N}_{K/\mathbb{Q}} : \quad K &\longrightarrow \quad \mathbb{Q} \\ \alpha &\longmapsto \det m_\alpha. \end{aligned}$$

Question 2.29. *Given a polynomial $f \in \mathbb{Z}[t]$, does the equation $f(t) = \mathrm{N}_{K/\mathbb{Q}}(\alpha)$ have a solution with $t \in \mathbb{Q}$ and $\alpha \in K$?*

By choosing a basis for $K/\mathbb{Q}$, we can turn this into a diophantine problem over the rationals.

Definition 2.30. The **norm form** is

$$N(x_1, \dots, x_n) = \mathrm{N}_{K/\mathbb{Q}}(\omega_1 x_1 + \cdots + \omega_n x_n)$$

where $\omega_1, \dots, \omega_n$ is a $\mathbb{Q}$-basis for K.

So we want rational solutions to

$$f(t) = N(x_1, \dots, x_n) \tag{2.1}$$

In order to avoid trivial solutions, we want $N(x_1, \dots, x_n) \neq 0$. In order to have rational solutions, it first needs solutions everywhere locally.

Definition 2.31. The **Hasse principle** holds for (2.1) if (2.1) being every locally solvable implies that it is soluble over $\mathbb{Q}$.

Example (Iskovskikh (spelling?), 1971)**.** Consider $(t^2-2)(3-t^2) = x_1^2 + x_2^2$. Here, $f(t) = (t^2-2)(3-t^2)$ and $K = \mathbb{Q}(i)$. In this example, the HP fails, as can be checked using the Brauer-Manin obstruction. △

Conjecture 2.32 (Colliot-Thélène, 2003)**.** *The BMO is the only obs to HP for any smooth projective model of (2.1)*

Known results The above conjecture holds if

- $f(t)$ is a product of linear polynomials and K is arbitrary. This is due to Browning, Mathissen (spelling?) (2017) using Green-Tao-Ziegler

- $f(t)$ is an irreducible quadratic and K is arbitrary. This is due to Derenthal, Smeets, and Wei (2015)
- $f(t)$ is an irreducible cubic and $[K:\mathbb{Q}] \leq 3$. This is due to Colliot-Thélène, Sansuc, Swinnerton-Dyer (K degree 2) and Salberger (+ Colliot-Thelene?) (K degree 3), 1987 + '89
- (Irving 2017) HP holds for (2.1) provided that
  (1) $f(t) \in \mathbb{Z}[t]$ is an irreducible cubic
  (2) K satisfies the Hasse norm principle HNP, i.e. $\mathbb{Q}^\times \cap \mathrm{N}_{\mathrm{K}/\mathbb{Q}}(\mathbb{A}_K^\times) = \mathrm{N}_{\mathrm{K}/\mathbb{Q}}(K^\times)$
  (3) There exists a prime $q \geq 7$ s.t. for all but finitely many primes $p \not\equiv 1 \pmod{q}$, the inertia degrees of p in $K/\mathbb{Q}$ are coprime ($\gcd(a_1, \ldots, a_r) = 1$)
  We can exclude finitely many primes, so can assume p is unramified, so write $(p) = \prod_{i=1}^r \mathfrak{p}_i$. The inertia degrees are $a_i := [K_{\mathfrak{p}_i} : \mathbb{Q}_p]$
  (4) $\mathbb{Q}[t]/(f(t)) \not\subset \mathbb{Q}(\zeta_q)$

2.5.1 New results

Let $\widehat{K}$ be the Galois closure of K, and let $G = \mathrm{Gal}(\widehat{K}/\mathbb{Q})$. Let

$$T(G) := \frac{1}{\#G} \# \{\sigma \in G : \text{the cycle lengths of } \sigma \text{ are not coprime}\}$$

Theorem 2.33 (S. 2022). *Say $f \in \mathbb{Z}[t]$, all of whose irreducible factors have degree ≤ 2. Suppose that HNP holds for K and that*

$$T(G) < \frac{0.39006\ldots}{\deg f + 1}.$$

Then, HP holds for (2.1)

Note there's no assumption on the number of irreducible factors of f.

Corollary 2.34. *Say $G = S_n$ above and $f(t) = Q_1(t)Q_2(t)$ is a product of two (non-proportional) irreducible quadratics. Suppose further that $\widehat{K} \cap \mathbb{Q}[t]/(f(t)) = \mathbb{Q}$. Then, HP holds for (2.1) provided that*

$$n \notin \{2, \ldots, 12, 14, \ldots, 42, 48\}.$$

Remark 2.35. If $n \geq 3$, then $\mathrm{Br}\, X = \mathrm{Br}\, \mathbb{Q}$, so there's no Brauer-Manin obstruction. ∘

Proof Sketch. First HNP holds when $G = S_n$ (Kunyavsky, Voskyeski (spelling?), 1984). Then one computes $T(S_n)$ and there is an explicit formula for this. Then plug into a compute and work out for which values of n, the quatntity is too large for the theorem to apply. ■

2.5.2 HP⇝Sieves

Let $f(a, b)$ be the homogenization of $f(t)$ ($t = a/b$).

Assumption. HNP holds for number field K, and (2.1) is everywhere locally soluble.

We want to show that (2.1) has a rational solution. It suffices to show $\exists a, b \in \mathbb{Z}$ s.t. $b, f(a,b) \in \mathrm{N}_{\mathrm{K}/\mathbb{Q}}(K^\times)$ since $f(t) = b^{-\deg f} f(a,b)$ and norms are multiplicative.

Notation 2.36. Let $c = f(a,b) \neq 0$.

We want to show that $c \in \mathrm{N}_{\mathrm{K}/\mathbb{Q}}(K^\times) \iff c \in \mathrm{N}_{\mathrm{K}/\mathbb{Q}}\left(\mathbb{A}_K^\times\right)$, i.e. for all places p of $\mathbb{Q}$, there exists $(x_v)_{v|p}$ with $x_v \in K_v^\times$ so that

$$\prod_{p|v} \mathrm{N}_{K_v/\mathbb{Q}_p}(x_v) = c \tag{2.2}$$

and $(x_v)_{v \in \Omega_K} \in \mathbb{A}_K^\times$ (i.e. $x_v \in \mathscr{O}_v^\times$ for almost all v).

It turns out we'll be able to ignore some finite set S of places. Below, assume $p \notin S$.

Lemma 2.37. *If $p \nmid c$, then $\exists (x_v)_{v|p}$ s.t. $x_v \in \mathscr{O}_v^\times$ and (2.2) holds.*

Lemma 2.38. *If $p \mid c$, and $\gcd(a_1, \ldots, a_r) = 1$ (inertia degrees coprime), then (2.2) holds for $x_v \in K_v^\times$.*

Proposition 2.39. *Suppose HNP holds and (2.1) is everywhere locally soluble. Then, there's a finite set S of primes, integers $a_0, b_0, \Delta \in \mathbb{N}$, and real numbers $r, \zeta > 0$ so that if $\exists a, b \in \mathbb{Z}$ with*

(1) $a \equiv a_0, b \equiv b_0 \pmod{\Delta}$

(2) $|a/b - r| < \zeta$

(3) *$bf(a,b)$ has no prime factors in*

$$\mathcal{P} := \{p \notin S : \gcd(a_1, \ldots, a_r) > 1\}$$

Then, (2.1) has a solution/$\mathbb{Q}$.

There was more after this, but I stopped taking notes...

3 Day 3

3.1 Leonhard Hochfilzer (Goettingen University): Diagonal cubic forms over $\mathbb{F}_q(t)$ and a question of Davenport

(joint w/ Jakob Glas)

Let F be a nonsingular cubic form in n variables over $k \in \{\mathbb{Q}, \mathbb{F}_q(t)\}$. Let $X = \{F = 0\} \subset \mathbb{P}^{n-1}$.

Question 3.1.

- *Does the Hasse principle hold for X?*
- *Does weak approximation hold for X?*

For $P \in k$, can consider the counting function

$$N(P) = \#\left\{\underline{x} \in \mathscr{O}_k^n \mid F(\underline{x}) = 0, |x_i| \leq |P|\right\} \text{ where } |f/g| = q^{(\deg f)(\deg g)}.$$

Remark 3.2. If $n \geq 5$, expect $N(P) \sim c\,|P|^{n-3}$ with the constant c encoding info about HP? ◦

Previous work

- Hooley (1986): $n \geq 9$, get HP and WA
- Hooley (2014): $n \geq 8$, get HP and WA (conditional on certain hypotheses on certain Hasse-Weil L-functions)
- Heath-brown (1998): Say $F = \sum_{i=1}^n F_i x_i^3$ a diagonal cubic form.
  - $n = 6 \implies N(P) \ll |P|^{3+\varepsilon}$
  - $n = 4 \implies N_0(P) \ll |P|^{3/2+\varepsilon}$ (counting points away from rational lines)

  (again dependent on some HW L-func hypotheses)

In the function field setting, one should have these HW L-funtion hypotheses via Deligne's work.

Over $\mathbb{F}_q(t)$

- Browning-Vishe (spelling) 2015: Say $\operatorname{char} \mathbb{F}_q \neq 2, 3$. Get HP + WA if $n = 8$

Theorem 3.3 (Glas-H.). *Assume* $\operatorname{char} \mathbb{F}_q \neq 2, 3$. *Let* $F = \sum F_i x_i^3$ *be diagonal. Then,*

- *If* $n = 6$, *then* $N(P) \ll |P|^{3+\varepsilon}$
- *If* $n = 4$, *then* $N_0(P) \ll |P|^{3/2+\varepsilon}$

In particular,

$$\#\left\{\underline{x} \in \mathbb{F}_q[t]^6 : x_1^3 + x_2^3 + x_3^3 = x_4^3 + x_5^3 + x_6^3, |x_i| \leq |P|\right\} \ll |P|^{3+\varepsilon}$$

(which was a question Davenport asked a PhD student of his in the 60's)

Sounds like one can handle characteristic 3 as well (and that it's easier?).

TODO: Transpose table

Let's review the function field/number field dictionary

$\mathbb{Q}$	$\mathbb{Z}$	$\lvert\cdot\rvert_\infty$	$\mathbb{R}$	$S^1 = \mathbb{R}/\mathbb{Z}$	$e : \mathbb{R} \to \mathbb{C}^\times$
$\mathbb{F}_q(t)$	$\mathbb{F}_q[t] = \mathscr{O}$	$\lvert f/g\rvert_\infty = p^{\deg f - \deg g}$	$\mathbb{F}_q\left(\left(t^{-1}\right)\right) = k_\infty$	$K_\infty/\mathscr{O} = \left\{\sum_{i \leq -1} a_i t^i\right\}$	$\psi : K_\infty \to \mathbb{C}^\times$, $\sum_{i \leq N} \alpha_i t^i \mapsto e\Big($

Consider

$$S(\alpha) = \sum_{\substack{\underline{x} \in \mathscr{O}^n \\ |\underline{x}| \leq |P|}} \psi(\alpha F(\underline{x}))$$

so that

$$\int_{\mathbb{T}} S(\alpha) \mathrm{d}\alpha = N(P).$$

Note that

$$S(\alpha) = \sum_{\underline{x} \in \mathscr{O}^n} \psi(\alpha F(\underline{x})) \varpi(\underline{x}/P) \text{ where } \varpi = \mathbf{1}_{\mathbb{T}}$$

(more useful description for Poisson summation).

Fact (Dirichelt approximation, using ultrametric inequality)**.** For $Q \geq 1$ ($\widehat{Q} = q^Q$), one has

$$\mathbb{T} = \bigsqcup_{|r| \leq \widehat{Q} : r \text{ monic}} \bigsqcup_{|a| < |r| : (a,r) = 1} \left\{\theta : \left|\frac{a}{r} - \theta\right| < \frac{1}{|r|\,\widehat{Q}}\right\}$$

It's getting hard to follow all these expressions, so I'm gonna stop taking notes...

3.2 Julian Demeio (Max Planck Institute, Bonn): On the distribution of rational points on ramified covers of abelian varieties

(joint w/ Corvajo, Javonpeylear, Lorbardo, Zannier (spelling?))

Let A/k be a g-dimensional abelian variety of a finitely generated field k of characteristic 0.

Theorem 3.4. *Let $\pi : Y \to A$ be a ramified cover, and let $\Omega \subset A(k)$ be a finitely generated Zariski dense subgroup. Then, there exists a coset $C \subset \Omega$ of finite index such that $C \cap \operatorname{Im} Y(k) = \emptyset$.*

(Y above should be normal, and π should not be unramified?)

Remark 3.5.

- This proves the weak Hilbert property for ab. vars/k
- *Corollary* 3.6. *If π is Galois w/ group G, then G is a Galois group over k.*

$\circ$

A bit of history

- Say $g = 1$. Then, Y is a cover of genus $g \geq 2$ (π ramified), so $\operatorname{Im} Y(k)$ is finite by Faltings' (proof of Mordell-Lang?)
- Say $A = E^m$ with E a non-CM EC. Then this was proven by Zannier in 2010 (used Serre's open image theorem, so non-CM assumption)
- $A = E_1 \times E_2$. Proven by Jov. 2021

3.2.1 The Proof

Start with some reductions.

First reduce for k f.gen field to K a number field. Since k is f.gen over $\mathbb{Q}$, we must have $k = K(S)$ for some variety S/K w/ K a number field. Spread A/k to an abelian scheme $\mathcal{A} \to S$ and assume $\Omega \subset \mathcal{A}(S)$. Want the specialization of Ω to also be Zariski dense. This can be achieved using results of Serre, Nook from '95 (specialization of End rings) along with work of Masser in '89 (specialization of f.gen subgroups). The upshot is that Ω_s is Zariski dense in $\mathcal{A}_s$ for almost all $s \in S(\overline{K})$.

Then reduced from f.gen Ω to cyclic (Zariski dense) $\Omega = \langle P \rangle$ (P non-torsion). This reduction is technical enough that we don't want to go over it here.

Now the main idea of the proof. Want to use a sort of local-global principle.

Simplifying Assumption. All reductions above work and $K = \mathbb{Q}$.

Consider

$$\begin{array}{ccc} Y(\mathbb{Q}) & \longrightarrow & A(\mathbb{Q}) \\ \downarrow & & \downarrow \\ Y(\mathbb{F}_p) & \longrightarrow & A(\mathbb{F}_p) \end{array}$$

and write $\widetilde{\Omega}_p := \operatorname{Im}\Omega$. Note that $\ker(\Omega \to \widetilde{\Omega}_p)$ is a finite index subgroup of Ω (since $A(\mathbb{F}_p)$ finite). From commutativity of the diagram, is suffices to find one $a \in \widetilde{\Omega}_p$ s.t. $a \notin \operatorname{Im} Y(\mathbb{F}_p)$.

Construction of p and a

- 1st step

  Find $\zeta \in A(\overline{\mathbb{Q}})_{\text{tor}}$ s.t. $\zeta \notin \operatorname{Im} Y(\mathbb{Q}(\zeta))$. In essence, find a (torsion) point over $\mathbb{F}_q$ for some q not lifting to an $\mathbb{F}_q$-point of Y, and then lift this a torsion point over $\overline{\mathbb{Q}}$.

- We look for ∞ many primes p s.t.

  - p splits in $\mathbb{Q}(\zeta)$
  - $\exists r \in \mathbb{N}$ s.t. $rP \equiv \overline{\zeta} \mod p$

    $(\implies \overline{\zeta} \in \widetilde{\Omega}_p)$
  - $\overline{\zeta} \notin \operatorname{Im} Y(\mathbb{F}_p)$

  Want to enforce these using Kummer theory. Let $T_\ell(A)$ be its ℓ-adic Tate module and let $V_\ell A = T_\ell A \otimes_{\mathbb{Z}_\ell} \mathbb{Q}_\ell$. Note that $V_\ell(A)/T_\ell(A) = A[\ell^\infty]$. Question: Why?

  Simplifying Assumption. Assume $\operatorname{ord}(\zeta) = \ell^k$ is a prime power

  Consider field extensions
  $$\mathbb{Q}(\ell^{-\infty}P, A[\ell^\infty])/\mathbb{Q}(A[\ell^\infty])/\mathbb{Q}$$

  Let H_ℓ be Galois group of top extension and G_ℓ be the Galois group of the bottom extension. Then,
  $$G_\ell \hookrightarrow \operatorname{Aut}(T_\ell A) \cong \mathrm{GL}_{2g}(\mathbb{Z}_\ell) \text{ and } H_\ell \hookrightarrow T_\ell A \cong \mathbb{Z}_\ell^{2g}.$$

  One can check that the total extension of the tower above is Galois with group $H_\ell \rtimes G_\ell$. Ribet '79 showed that $H_\ell \xhookrightarrow{\sim} T_\ell A$ for $\ell \gg 0$.

  Fact (Pink '03)**.** Let $p \neq \ell$ be a prime unramified for A, and consider $\mathrm{Frob}_p \in H_\ell \rtimes G_\ell$. Write $\mathrm{Frob}_p = (t_p, \varphi_p)$. Then,
  $$[P \mod p]_\ell \equiv (\varphi_p - 1)^{-1} t_p \mod T_\ell A$$

  (the subscript $_\ell$ denotes the ℓ-torsion part in $A(\mathbb{F}_p)$)

  There was more after this, but I stopped taking notes...

3.3 Tim Browning (Institute of Science and Technology Austria): Manin-Peyre for some quadric bundle threefolds

(joint w/ Z. Huang and D. Bonolis)

Let X be a smooth projective Fano 3-fold over $\mathbb{Q}$.

Assumption. Assume $X(\mathbb{Q})$ is Zariski dense. Let $H : X(\mathbb{Q}) \to \mathbb{R}$ be an anticanonical height function.

There are certain thin sets of rational points which can be problematic from the point of view of counting points. Sounds like these are $W(\mathbb{Q})$ on $W \subsetneq X$ (W rational?) and $\pi(Y(\mathbb{Q}))$ for $Y \to X$ of degree > 1 (something special about Y?)

Conjecture 3.7 (Manin-Peyre)**.** *There exists a thing set $Z \subset X(\mathbb{Q})$ s.t.*

$$N_Z(B) = \#\{x \in X(\mathbb{Q}) \setminus Z : H(x) \le B\} \sim cB(\log B)^{\rho-1}$$

where $\rho = \operatorname{rank} \operatorname{Pic} X$.

The constant c was predicted by Peyre, and breaks up as $c = \alpha(X)\beta(X)\tau_H(X(\mathbb{A}_\mathbb{Q})^{\mathrm{Br}})$.

Example (B.-Laughran)**.** Consider

$$X = \left\{ \begin{matrix} x_1^2 + x_2^2 = x_3^2 + x_4^2 + x_5^2 + x_6^2 + 8x_1x_6 \\ 2x_1x_2 = 2x_3x_4 + x_6^2 \end{matrix} \right\} \subset \mathbb{P}^5,$$

a smooth intersection of two quadrics. Here, $\rho = 1$ and you can take

$$H(x) = \max |x_i|^2.$$

Let $C : y^2 = x^5 - x^4 - 2x^3 + 18x^2 + x + 17$, a genus 2 curve (the degeneracy locus of this pencil of quadrics). Then, the variety of lines $F_1(X)$ of this quadric is a principal homogeneous space for $J = \operatorname{Jac}(C)$, and in this case, one even has $F_1(X) \cong J$ over $\mathbb{Q}$ (i.e. there is a rational line). One can compute that J is simple and $J(\mathbb{Q})$ has rank 1, so $J(\mathbb{Q})$ is Zariski dense. Note that

$$L \in F_1(X)(\mathbb{Q}) \implies \#\{x \in L(\mathbb{Q}) : H(x) \le B\} \gg B.$$

Hence, one expects $N_Z(B) \sim cB$ if $Z = \bigcup_{L \in F_1(X)(\mathbb{Q})} L(\mathbb{Q})$ (union of rational points on rational lines). $\triangle$

Theorem 3.8 (Manin '90)**.** *For any smooth Fano 3-fold/$\mathbb{Q}$, there exists a number field $K/\mathbb{Q}$ s.t.*

$$\#\{x \in X(K) : H(x) \le B\} \gg B.$$

Question 3.9. *Can you beat $N_Z(B) \gg \sqrt{B}$ (e.g. for a smooth intersection of 2 quadrics)?*

For the example above, can get this lower bound by intersection with $x_6 = 0$. This gives a degree 4 del Pezzo surface and work of others let's you conclude this lower bound.

Today we'll talk about asymptotic formulae when $\rho = 2$. These were classified in 1980's by Mori-Mukai. In dimension 2, sounds like the Fano varieties are del Pezzos and conic bundles. In dimension 3, there are 36 iso types of Fano varieties. These types are numbered in a table

$\# = (-K_X)^3$	Manin-Peyre?
1	
$\vdots$	
24	$\sum_{i=1}^3 x_i y_i^2 = 0$ (?), again seems doable
25	Today $\mathrm{Bl}_{\text{genus }1}(\mathbb{P}^3)$
26	Seems doable, blowup of $\mathbb{P}^3$ along a twisted cubic
27	
29	blowup of quadric 3-fold at a conic (Blomer et al. 2022)
28, 30, 31	equivariant compactification of $\mathbb{G}_a^3$ (ACL-Tschinkel)
32	bidegree $(1,1)$ in $\mathbb{P}^2 \times \mathbb{P}^2$ (Spencer '08) or a flag variety
$33-36$	Toric (Batyred-Tschinkel)

(I think the entires on the RHS of the table generally refer to specific examples, not all Fano 3-folds of given degree)

Let E be the genus 1 curve cut out by two quadrics $Q_1(y_1, \ldots, y_4) = 0$ and $Q_2(y_1, \ldots, y_4) = 0$ in $\mathbb{P}^3$. The variety we'll be interested in is

$$X = \{x_1 Q_1(y) = x_2 Q_2(y)\} \subset \mathbb{P}^1 \times \mathbb{P}^3$$

which has bidegree $(1, 2)$ and $\rho = 2$. Note this has maps

$$\begin{tikzcd} & X \arrow[dl, "\pi_1"'] \arrow[dr, "\pi_2"] & \\ \mathbb{P}^1 & & \mathbb{P}^3. \end{tikzcd}$$

History. In a conference in Bristol in 2009, Peyre suggested proving the Manin-Peyre conjecture (for the above variety X?). Sounds like Elsenhaus was also at this conference, studied this numerically, and suggested that one should take

$$Z = \underbrace{\pi_2^{-1}(E)}_{\simeq \mathbb{P}^1 \times E}(\mathbb{Q}) \cup \bigcup_{\substack{x \in \mathbb{P}^1(\mathbb{Q}) \\ \pi_1^{-1}(x) \text{ split}}} \pi_1^{-1}(x)(\mathbb{Q}).$$

Theorem 3.10. *Let $L_1, \ldots, L_4 \in \mathbb{Z}[x_1, x_2]$ be linear forms. Let*

$$X = \left\{ \sum_{i=1}^4 L_i(x_1, x_2) y_i^2 = 0 \right\}$$

(A way of rewriting the equation from slightly before). Assume

- $\mathrm{Res}(L_i, L_j) \neq 0$ *(so X smooth)*
- *L_i's have coprime coefficients*
- $E(\mathbb{R}) = \emptyset$.

Then,

$$N_Z(B) \sim cB \log B$$

(with c as predicted by Peyre).

Example.

$$x_1y_1^2 + x_2y_2^2 + (x_1 + 2x_2)y_3^2 + (x_1 + 6x_2)y_4^2 = 0.$$

Here, the split quadrics are points on

$$x_1x_2(x_1 + 2x_2)(x_1 + 6x_2) = z^2$$

which has rank 1 (via LMFDB). Need to throw away all x_1, x_2's lying on this curve. $\triangle$

The proof is modelled on some earlier work of B. and Heath-Brown in '20 proving Manin-Peyre for $\sum_{i=1}^4 x_iy_i^2 = 0 \subset \mathbb{P}^3 \times \mathbb{P}^3$.

In the present context, they studied

$$N(B) = \frac{1}{4}\#\left\{(\underline{x}, \underline{y}) \in \mathbb{Z}^2_{\text{prim}} \times \mathbb{Z}^4_{\text{prim}} : \sum L_i(\underline{x})y_i^2 = 0 \text{ and } |\underline{x}|\,|\underline{y}|^2 \leq B \text{ where } \prod L_i(\underline{x}) \neq \square\right\}$$

Some comments on how to count

- When $|\underline{x}| \leq B^{1/4}$, fix $\underline{x}$ and apply circle method machinery from BHB (to fibers of π_1?)
- When $|\underline{x}| > B^{1/4+\delta}$, via as $x_1Q_1(\underline{y}) = x_2Q_2(\underline{y})$. Assume

  $$\frac{Y}{2} < |\underline{y}| < Y \text{ for } Y \leq B^{3/8+\delta/2}.$$

  Put $d = \gcd(Q_1(\underline{y}), Q_2(\underline{y}))$. Then,

  $$(x_1, x_2) = \pm\left(\frac{Q_2(y)}{d}, \frac{Q_1(y)}{d}\right),$$

  so want to bound

  $$\sum_{d=1}^{\infty} \#\left\{\underline{y} \in \mathbb{Z}^4_{\text{prime}} : |\underline{y}| \sim Y, d = \gcd(Q_1(y), Q_2(y)), \text{ and } B^{1/4+\delta} \leq \frac{\max|Q_i(\underline{y})|}{d} \leq \frac{B}{|\underline{y}|^2}\right\}$$

  $E(\mathbb{R})$ being empty allows one to show that $Y^2 \ll \max|Q_i(\underline{y})| \ll Y^2$ (upper bound easy. Lower bound from $E(\mathbb{R}) = \emptyset$). This tells you that $Y^4/B \ll d \ll Y^2/B^{1/4+\delta}$.

  There was more, but I stopped taking notes...

4 Delete Later

Say you have $\mathbb{P}^1$ with points $a_1, \dots, a_6 \in \mathbb{P}^1$ replaced by 1/2 points (form relevant root stack). Call resulting orbifold $\mathcal{X}$. The hyperelliptic curve $C : y^2 = (x - a_1)\dots(x - a_6)$ has an (unramified) map $C \to \mathcal{X}$. In fact $\mathcal{X} = [C/(\mathbb{Z}/2\mathbb{Z})]$.

Question 4.1. *If you have a $\mathbb{Q}$-point on C, what kind of point do you get on $\mathbb{P}^1$ (or on $\mathcal{X}$?)?*

Remark 4.2. Say p a prime of good reduction (the a_i's still distinct mod p), and suppose $P \in C(\mathbb{Q})$ reduces to $(a_1, 0)$ mod p, so $P = (x, y)$ with $v_p(x - a_1) > 0$. Furthermore, $v_p(x - a_i) = 0$ for all $i \neq 1$. Hence, $2v_p(y) = v_p(y^2) = v_p(x - a_1)$, so $v_p(x - a_1) \in 2\mathbb{Z}$, i.e. the power of p in the numerator of $(x - a_1)$ is even.

This is coming from the fact that the preimage of a_1 in $\mathbb{P}^1(\mathbb{Q})$ is $2(a_1, 0)$. Think of taking integral models, and then pulling back the (zero) divisor of $(x - a_1)$ along

$$\operatorname{spec} \mathbb{Z} \xrightarrow{P} \mathcal{C} \to \mathbb{P}^1_{\mathbb{Z}}$$

$\circ$

Remark 4.3. $\mathcal{X}$ has Euler characteristic

$$\chi(\mathbb{P}^1) - 6(1) + 6(1/2) = -1 < 0$$

finite étale covers multiply Euler characteristics $\circ$

5 List of Marginal Comments

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