KanSem Formal Group Laws & MU Notes

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November 2020

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Introduction

We wish to introduce the relevance of formal group laws to the study of algebraic topology. We will see that attached to certain "oriented" cohomology theories E is a power series $\mu_E(x,y) \in \pi_*(E)[\![x,y]\!]$ enjoying many properties reminiscent of group laws. Such cohomology theories will be especially amenable to calculations, and we will see that complex cobordism MU is universal among them. In fact, the complex cobordism spectrum MU will enjoy two universal properties in the present context. These are the following.

Theorem A. *Let E be a ring spectrum. Then, a complex orientation of E is the same thing as a map $MU \to E$ of ring spectra.*

Theorem B. *The natural map $L \to \pi_*(MU)$, from the Lazard ring L supporting the universal formal group law to the complex cobordism ring, is an isomorphism.*

We will sketch proofs of both of these, as well as make their statements more precise. To simplify the exposition of the latter theorem, we take as a given Milnor's computation of $\pi_*(MU)$.

1 Formal Group Laws

We will study 1-*dimensional commutative formal group laws*. The motivation for the definition is this: given a 1-dimensional (commutative) Lie group G, a chart near the identity allows one to write multiplication in G as a power series

$$\mu(x,y) = \sum_{i,j\geq 0} a_{ij}x^i y^j,$$

convergent for small x, y, and the fact that this represents multiplication will cause it to satisfy the following definition.

Definition 1.1. Let R be a ring.[1] A **formal group law**[2] over R is a formal power series

$$\mu(x,y) = \sum_{i,j\geq 0} a_{ij}x^i y^j \in R[\![x,y]\!],$$

satisfying

$$\mu(x,0) = x = \mu(0,x) \text{ and } \mu(x,\mu(x,y)) = \mu(\mu(x,y),z)$$

as well as $\mu(x,y) = \mu(y,x)$.

Example. Key examples include the **additive group law** $\mu(x,y) = x + y$ and the **multiplicative group law** $\mu(x,y) = x + y + xy$.

Remark 1.2. The facts that $\mu(x,0) = x$ and $\mu(0,y) = y$ tell us that μ must actually be of the form

$$\mu(x,y) = x + y + \sum_{i,j\geq 1} a_{ij}x^i y^j.$$

Remark 1.3. Attached to any formal group law μ is a **formal inverse** ι which is a power series

$$\iota(x) = \sum_{j\geq 1} a'_j x^j \in R[\![x]\!]$$

such that $\mu(x,\iota(x)) = 0 = \mu(\iota(x),x)$. One can determine the a'_j via induction starting with $a_1 = -1$, i.e. $\iota(x) = -x + \sum_{j\geq 2} a'_j x^j$.

Example. The inverse for the additive group law is $\iota(x) = -x$ while the inverse for the multiplicative group law is $\iota(x) = -x + x^2 - x^3 + x^4 - \ldots$.

Definition 1.4. Let μ, ν be formal group laws over R. A **homomorphism of formal group laws** over R from μ to ν is a power series $f(x) = \sum_{i\geq 1} c_i x^i \in R[\![x]\!]$ such that

$$\nu(f(x_1), f(x_2)) = f(\mu(x_1,x_2)).$$

If the coefficient c_1 is invertible, then f^{-1} exists and f is an isomorphism.

Notation 1.5. Given a formal group law μ, we may denote its power series as $x +_\mu y = \mu(x,y) \in R[\![x,y]\!]$. Using this notation, a homomorphism f must satisfy $f(x +_\mu y) = f(x) +_\nu f(y)$.

[1] All our rings are commutative with unit

[2] What we define should really be called a *1-dimensional, commutative* formal group law

2 Complex Oriented Cohomology Theories

2.1 A Brief Review of Spectra

Definition 2.1. A **spectrum** E is a sequence of based CW-complexes E_n ($n \in \mathbb{N}$) along with structure maps $\Sigma E_n \to E_{n+1}$.

I will not go into the details of various types of "mappings" between spectrum[3], but will instead rely on intuition coming from spaces. You can see [Ada95, Part III] for the details. Just trust that it makes sense to talk about the set $[E, F]$ of "homotopy classes of maps between the spectra E, F" and that this can naturally be graded so that $[E, F]_r$ corresponds to morphisms raising the degree by r (i.e.[4] E_i maps to F_{i+r}).

Warning 2.2. We will sometimes be lazy and forget that maps have degrees, so we may say e.g. "a map $E \to F$" when we really mean "a degree 0 map $E \to F$" or write "$[E, F]$" where we should really write "$[E, F]_0$". When we really want to emphasize that the set of homotopy classes of maps $E \to F$ is graded, we will write it as $[E, F]_*$.

Example. Given a based CW-complex X, one can form its **suspension spectrum** $\Sigma^\infty X$ given by $(\Sigma^\infty X)_n = \Sigma^n X$ and whose structure maps are the identity maps.

Fact. The smash product of based spaces extends to a bifunctor $\wedge$ on the category of spectra, turning it into a symmetric monoidal category with unit object $\mathbb{S} = \Sigma^\infty S^0$, the **sphere spectrum**. This has no nice description in general, but when E is a spectrum and X is a CW-complex, one has that $E \wedge X := E \wedge \Sigma^\infty X$ is given by

$$(E \wedge X)_n = E_n \wedge X$$

with the induced structure maps.

Definition 2.3. Let E be a spectrum. Then, its **(stable) homotopy groups** are

$$\pi_n(E) := [S^n, E]_0 := [\Sigma^\infty S^n, E]_0 = [\mathbb{S}, E]_{-n} = \varinjlim_k [S^{n+k}, E_k] = \varinjlim_k \pi_{n+k}(E_k)$$

where $[-,-]_r$ denotes homotopy classes of degree r maps. Given another spectrum F, its **E-(co)homology groups** are

$$E_n(F) = \pi_n(E \wedge F) = [\mathbb{S}, E \wedge F]_{-n} = [\Sigma^n \mathbb{S}, E \wedge F]_0 \text{ and } E^n(F) = [F, E]_n = [\Sigma^{-n} F, E]_0.$$

Given a *space* X, its **(un)reduced E-(co)homology groups** are

$$E_n(X) := E_n(\Sigma^\infty(X_+)),\ \widetilde{E}_n(X) := E_n(\Sigma^\infty X),\ E^n(X) := E^n(\Sigma^\infty(X_+)), \text{ and } \widetilde{E}^n(X) := E^n(\Sigma^\infty X).$$

The **coefficient groups** of E are $E^{-n}(*) = E_n(*) = \pi_n(E) = [S^n, E] =: \pi^{-n}(E)$.[5]

[3] One can define functions, maps, and morphisms between spectra as three slightly different concepts

[4] Adams using the opposite convention where "degree r" means E_i maps to F_{i-r}, so that $\pi_r(E) = [\mathbb{S}, E]_r$ for him. For us, $\pi_r(E) = [\mathbb{S}, E]_{-r}$

[5] This upper indexing on homotopy groups is non-standard and only introduced so I don't have to write $\pi_{-*}(E)$ later when looking at the Atiyah-Hirzebruch spectral sequence in cohomology. A better (and more standard) choice would have been to say $E^n = E_{-n} = \pi_{-n}(E)$, but too late to change things now.

Example. Given an abelian group A, one can form the **Eilenberg-Maclane spectrum** HA with $HA_n = K(A,n)$ and structure maps adjoint to the equivalences $K(A,n) \simeq \Omega K(A,n+1)$. This spectrum represents ordinary singular (co)homology with coefficients in A.

Example. The BU-spectrum K with $K_{\text{even}} = BU$ and $K_{\text{odd}} = U$ represents complex K-theory.

Example. The complex Thom spectrum MU represents complex cobordism. This is given by

$$MU_{2n} = MU(n) = \mathrm{Th}\left(EU(n) \times_{U(n)} \mathbb{C}^n \longrightarrow BU(n)\right)$$

(and $MU_{2n+1} = \Sigma MU(n)$) with structure maps $\Sigma^2 MU(n) \to MU(n+1)$ induced by the natural maps

$$\underline{\mathbb{C}} \oplus (EU(n) \times_{U(n)} \mathbb{C}^n) \to EU(n+1) \times_{U(n+1)} \mathbb{C}^{n+1}.$$

All three of the previous examples actually have more structure than just that of a spectrum. They all represent cohomology theories that support various products. This is not the case for general spectra, but is the case for the class of spectra introduced below.

Definition 2.4. A **ring spectrum**[6] is a spectrum E equipped with a (degree 0) product map $\mu : E \wedge E \to E$ which is associative and commutative up to homotopy as well as a (degree 0) map $i : \mathbb{S} \to E$ which acts as a (two-sided) unit up to homotopy, i.e. we require the following diagrams to commute up to homotopy:

$$\begin{tikzcd}
E \wedge E \wedge E \arrow[r, "\mu\wedge 1"] \arrow[d, "1\wedge\mu"] & E \wedge E \arrow[d, "\mu"] \\
E \wedge E \arrow[r, "\mu"] & E
\end{tikzcd}
\qquad
\begin{tikzcd}
E \arrow[r, "\sim"] \arrow[d, "\wr"] \arrow[ddrr, equal] & \mathbb{S} \wedge E \arrow[r, "i\wedge 1"] & E \wedge E \arrow[dd, "\mu"] \\
E \wedge \mathbb{S} \arrow[d, "1\wedge i"] & & \\
E \wedge E \arrow[rr, "\mu"] & & E
\end{tikzcd}$$

as well as

$$\begin{tikzcd}
E \wedge E \arrow[r, "\text{switch}"] \arrow[d, "\mu"'] & E \wedge E \arrow[d, "\mu"] \\
E \arrow[r, "="] & E
\end{tikzcd}.$$

Ring spectra represent multiplicative (co)homology theories.

Remark 2.5. Let X be a spectrum, and let E be a ring spectrum. Then, there is a natural map

$$T : E^*(X) \longrightarrow \mathrm{Hom}_{\pi_*(E)}(E_*(X), \pi_*(E)).$$

For $h \in E^*(X)$ (so $h : X \to E$, not necessarily of degree 0) and $k \in E_*(X) = \pi_*(E \wedge X)$ (so $k : \mathbb{S} \to E \wedge X$, not necessarily of degree 0), $T(h)(k) \in \pi_*(F) = [\mathbb{S}, E]_*$ is the composition

$$T(h)(k) : \mathbb{S} \xrightarrow{k} E \wedge X \xrightarrow{1\wedge h} E \wedge E \xrightarrow{\mu} E.$$

[6]Technically, this is a commutative ring spectrum

We usually denote $T(h)(k)$ more succinctly by $\langle h, k\rangle \in \pi_*(E)$. Note that if $h \in E^r(X)$ and $k \in E_s(X)$, then $\langle h, k\rangle \in \pi_{r-s}(E)$.

2.2 Complex Oriented Spectra

We wish to study a particularly nice class of ring spectra.

Remark 2.6. Let E be any spectrum. Then, one has isomorphisms

$$\widetilde{E}^*(\mathbb{CP}^1) = \widetilde{E}^*(S^2) \simeq \widetilde{E}^{*-2}(S^0) = E^{*-2}(*) = \pi^{*-2}(E) = \pi_{2-*}(E).$$

If E is furthermore a ring spectrum (so $\pi_*(E)$ is a graded ring), then $\widetilde{E}^*(\mathbb{CP}^1)$ is free of rank 1 over $\pi_*(E)$ with a natural generator in degree 2 represented by the unit map $i : S^0 \to E$ in $\pi_0(E)$. Equivalently, the natural generator $\gamma \in \widetilde{E}^2(\mathbb{CP}^1)$ is represented by the degree two map

$$\Sigma^\infty \mathbb{CP}^1 = \Sigma^\infty S^2 = \Sigma^2\mathbb{S} \to \mathbb{S} \xrightarrow{i} E.$$

Definition 2.7. A ring spectrum E is called **complex orientable** if the natural reduction map $\widetilde{E}^*(\mathbb{CP}^\infty) \to \widetilde{E}^*(\mathbb{CP}^1)$ is surjective. In this case, an element $x \in \widetilde{E}^*(\mathbb{CP}^\infty)$ is called a **complex orientation** if $x|_{\mathbb{CP}^1} \in \widetilde{E}^*(\mathbb{CP}^1)$ generates it as a module over $\pi_*(E)$. In particular, x *does not* have to be in degree 2. A pair (E, x^E) is called a **complex oriented ring spectrum** if $x^E \in \widetilde{E}^*(\mathbb{CP}^\infty)$ is a complex orientation.

Notation 2.8. Let E be an oriented ring spectrum, and let $\gamma \in \widetilde{E}^2(\mathbb{CP}^1)$ be the natural generator of $\widetilde{E}^*(\mathbb{CP}^1)$ (so $\gamma \mapsto 1 \in \pi_0(E)$ under the natural iso $\widetilde{E}^2(\mathbb{CP}^1) \xrightarrow{\sim} \pi_0(E)$). Then, one has $x^E|_{\mathbb{CP}^1} = u^E\gamma$ for some unit $u = u^E \in \pi_*(E)^\times$.

Example. If $E = H := H\mathbb{Z}$, we can take $x^H \in \widetilde{H}^2(\mathbb{CP}^\infty)$ to be the usual generator and $u^H = 1$.

Example. If $E = K$, we identity $\mathbb{CP}^\infty \simeq BU(1)$ and write ξ for the universal line bundle. We take $x^k = [\xi] - 1 \in \widetilde{K}^0(\mathbb{CP}^\infty)$. The unit $u^K \in \pi_2(K)$ here is the usual Bott element generating $\pi^{-2}(K) = \pi_2(K) \simeq \mathbb{Z}$.

Example. Let $E = MU$. There is a natural homotopy equivalence $\omega : \mathbb{CP}^\infty \to MU(1)$. Indeed, $MU(1)$ is quotient of the universal disc-bundle over $BU(1)$ by the universal circle-bundle over it, but this universal circle bundle is precisely $EU(1)$ which is contractible. Hence, $MU(1)$ is homotopy equivalent to the universal disc bundle over $BU(1)$ which is in turn homotopy equivalent to $BU(1)$; that is ω is the composition of the natural homotopy equivalences

$$\mathbb{CP}^\infty = BU(1) \xleftarrow{\sim} D(EU(1) \times_{U(1)} \mathbb{C}) \xrightarrow{\sim} MU(1).$$

In any case, we take $x^{MU} \in \widetilde{MU}^2(\mathbb{CP}^\infty) = [\Sigma^\infty \mathbb{CP}^\infty, MU]_2$ to be the class of $\omega : \mathbb{CP}^\infty \xrightarrow{\sim} MU(1) = MU_2$. We have $u^{MU} = 1$.

Fix an oriented spectrum E. We now justify our earlier claim that E-(co)homology is calculable by performing various calculations of E-(co)homology. First, a bit on passing from cohomology of finite complexes to those of infinite complexes.

Theorem 2.9 (**Milnor Exact Sequence**)**.** *Let X_n be pointed CW complexes and let $X = \varinjlim X_n$. Then one has short exact sequences (one for each value of $*$)*

$$0 \longrightarrow \varprojlim_n{}^1 \widetilde{E}^{*-1}(X_n) \longrightarrow \widetilde{E}^*(X) \longrightarrow \varprojlim_n \widetilde{E}^*(X_n) \longrightarrow 0.$$

Furthermore, for any fixed value $* = k$, if the maps $\widetilde{E}^k(X_n) \to \widetilde{E}^k(X_{n-1})$ are surjective for all n, then the kernel above vanishes, so $\widetilde{E}^k(X) \xrightarrow{\sim} \varprojlim_n \widetilde{E}^k(X_n)$.

Proof. See [nLa20]. ■

Remark 2.10. E is a ring spectrum, so there are two ways of viewing $E^*(X)$ as a ring (instead of a multiplicative sequence of groups). We can set

$$E^*(X) = \bigoplus_{n\in\mathbb{Z}} E^n(X) \text{ or } E^*(X) = \prod_{n\in\mathbb{Z}} E^n(X).$$

We choose the latter (i.e. $E^*(X) = \prod E^n(X)$). Note that this means e.g. that $\mathrm{H}^*(\mathbb{CP}^\infty;\mathbb{Z}) = \mathbb{Z}[\![x]\!]$ is a power series ring, not a polynomial ring. One reason is that products commute with (inverse) limits while coproducts do not, so this convention allows us to interpret the Milnor sequence as actually being a single sequence for $\widetilde{E}^*$ (instead of a series of sequences, one for each $\widetilde{E}^k$).

Let $x = x^E \in \widetilde{E}^*(\mathbb{CP}^\infty)$ and $u = u^E \in \pi_*(E)^\times$ (so $u^{-1}x|_{\mathbb{CP}^1}$ is the usual generator), and note that the projections $\mathbb{CP}^\infty \times \mathbb{CP}^\infty \rightrightarrows \mathbb{CP}^\infty$ give two elements $x_1, x_2 \in \widetilde{E}^*(\mathbb{CP}^\infty \times \mathbb{CP}^\infty)$, each pulled back from x.

Lemma 2.11.

$$\begin{aligned} E^*(\mathbb{CP}^n) &= \pi^*(E)[x]/(x^{n+1}) \\ E^*(\mathbb{CP}^\infty) &= \pi^*(E)[\![x]\!] \\ E^*(\mathbb{CP}^n \times \mathbb{CP}^m) &= \pi^*(E)[x_1,x_2]/(x_1^{n+1}, x_2^{m+1}) \\ E^*(\mathbb{CP}^\infty \times \mathbb{CP}^\infty) &= \pi^*(E)[\![x_1,x_2]\!] \end{aligned}$$

Proof Sketch. Consider the Atiyah-Hirzebruch spectral sequence (AHSS) for $\mathbb{CP}^n$; this is

$$E_2^{p,q} = \mathrm{H}^p(\mathbb{CP}^n; \pi^q(E)) \implies E^{p+q}(\mathbb{CP}^n).$$

The algebra on the E_2-page is

$$\pi^*(E)[y]/(y^{n+1})$$

with $y = 1 \in E_2^{2,0} = \mathrm{H}^2(\mathbb{CP}^n, \pi^0(E)) = \pi^0(E)$. By construction $u^{-1}x|_{\mathbb{CP}^1}$ is in filtration 2 and reduces mod filtration 3 to $y \in E_2^{2,0}$ (a generator of $\mathrm{H}^2(\mathbb{CP}^n;\pi^*(E))$). Thus, y survives to the end. Since all differentials are $\pi^*(E)$-linear and derivations, we see that the spectral sequence degenerates on E_2. From this, it is easy to see that the natural map $\pi^*(E)[x]/(x^{n+1}) \to E^*(\mathbb{CP}^n)$ is an isomorphism.

The case of $\mathbb{CP}^n \times \mathbb{CP}^m$ is handled similarly. For $\mathbb{CP}^\infty$ and $\mathbb{CP}^\infty \times \mathbb{CP}^\infty$, one applies the Milnor sequence, noting that the the maps $E^*(\mathbb{CP}^n) \to E^*(\mathbb{CP}^{n-1})$ are surjective. In fact, I think that strictly speaking, one does not need the Milnor sequence as the AHSS degenerate on E_2 in any of these cases. ■

Remark 2.12. We now can see how to attach a formal group law μ_E to our oriented ring spectrum E. Consider the map $m : \mathbb{CP}^\infty \times \mathbb{CP}^\infty \to \mathbb{CP}^\infty$ which represents the tensor product of line bundles. This induces a map

$$m^* : \pi^*(E)\,[\![x]\!] \simeq E^*(\mathbb{CP}^\infty) \longrightarrow E^*(\mathbb{CP}^\infty \times \mathbb{CP}^\infty) \simeq \pi^*(E)\,[\![x_1, x_2]\!]$$

in cohomology. In particular, it gives rise to a power series $\mu_E(x_1, x_2) := m^*x \in \pi^*(E)\,[\![x_1, x_2]\!]$. Since tensor product gives a group operation on line bundles, it is easy to see that $\mu_E(x_1, x_2)$ is in fact a formal group law. For notational convenience, we may also denote this as $x_1 +_E x_2 := \mu_E(x_1, x_2)$.

Example. When $E = H$, it is classical that the associate formal power series is the additive power series $x +_H y = x + y$.

Example. When $E = K$, recall that $x = x^K = [\xi] - 1$ for $\xi \to \mathbb{CP}^\infty$ the universal line bundle. Note that $m^*[\xi] = [\xi_1][\xi_2] \in K^0(\mathbb{CP}^\infty \times \mathbb{CP}^\infty)$ so $m^*(1 + x) = (1 + x_1)(1 + x_2)$ which means that

$$m^*x = x_1 + x_2 + x_1x_2$$

is the multiplicative formal group law.

Lemma 2.13.

(1) *$E^*(\mathbb{CP}^n)$ and $E_*(\mathbb{CP}^n)$ are dual finitely-generated free modules over $\pi_*(E)$*

(2) *There is a unique element $\beta_n \in E_*(\mathbb{CP}^n)$ such that*

$$\langle x^i, \beta_n \rangle = \begin{cases} 1 & \text{if } i = n \\ 0 & \text{otherwise} \end{cases}.$$

We can then consider the image of β_n in $E_*(\mathbb{CP}^m)$ for $m \geq n$ and also in $E_*(\mathbb{CP}^\infty)$; these images are also dented β_n.

(3) *$E_*(\mathbb{CP}^n)$ is free over $\pi_*(E)$ on generators $\beta_0, \dots, \beta_n$. $E_*(\mathbb{CP}^\infty)$ is free over $\pi_*(E)$ on generators $\{\beta_i\}_{i \in \mathbb{N}}$. $E_*(\mathbb{CP}^n \times \mathbb{CP}^m)$ is free over $\pi_*(E)$ with a base consisting of the external products $\beta_i\beta_j$ for $0 \leq i \leq n, 0 \leq j \leq m$. $E_*(\mathbb{CP}^\infty \times \mathbb{CP}^\infty)$ is free over $\pi_*(E)$ with base consisting of the external products $\beta_i\beta_j$ for $i, j \in \mathbb{N}$.*

(4) *The external product*

$$E_*(\mathbb{CP}^\infty) \otimes_{\pi_*(E)} E_*(\mathbb{CP}^\infty) \to E_*(\mathbb{CP}^\infty \times \mathbb{CP}^\infty)$$

is an isomorphism.

Proof Sketch. All of this comes from considering the pairing on the two (co)homology Atiyah-Hirzebruch spectral sequences. ■

The previous two results say, in a strong sense, that E-(co)homology of complex projective spaces "looks like" their singular (co)homology. The same is true for some other spaces closely related to $\mathbb{CP}^\infty$. It is probably fine to skip the rest of this section. The main point is that the E-homology of $\mathbb{CP}^\infty$, $BU(n)$, and of MU all look like their singular homology.

Recall that BU is an H-space with product induced by the maps $BU(n) \times BU(m) \to BU(n+m)$ representing the direct sum of vector bundles, so $E_*(BU)$ has a ring structure. The inclusion $\mathbb{CP}^\infty = BU(1) \to BU$ allows us to via the classes $\beta_i \in E_*(\mathbb{CP}^\infty)$ as E-homology classes on BU, which we still denote by β_i. The element β_0 is the unit for the products.

Lemma 2.14. *$E_*(BU)$ is free over $\pi_*(E)$, with a basis consisting of the monomials*

$$\beta_{i_1}\beta_{i_2}\dots\beta_{i_r}$$

with $i_j > 0$ for all j and $r \leq n$ ($r = 0$ allowed). In the limit, $E_*(BU)$ is the polynomial algebra

$$E_*(BU) \simeq \pi_*(E)[\beta_1, \beta_2, \dots, \beta_i, \dots].$$

Proof Sketch. The AHSS will degenerate on the E^2-page with $\pi_*(E)$ basis given by the relevant collection of monomials. ∎

We would like to dualize this calculation in order to get a theory of Chern classes in E-cohomology, analogous to the situation for singular cohomology. In order to do so, it will be nice to have the following technical lemma.

Lemma 2.15. *Let X be a spectrum with $\pi_r(X) = 0$ if $r \ll 0$. Suppose that $\mathrm{H}_*(X; \pi_*(E))$ is free over $\pi_*(E)$, and that the spectral sequence $\mathrm{H}_*(X; \pi_*(E)) \implies E_*(X)$ is trivial. Then, the spectral sequence*

$$\mathrm{H}^*(X; \pi^*(E)) \implies E^*(X)$$

is trivial, and the map

$$E^*(X) \longrightarrow \mathrm{Hom}_{\pi_*(E)}(E_*(X), \pi_*(E))$$

is an isomorphism.

Proof Sketch. Considering the pairing

$$E_r^{p,-s} \otimes E^r_{p,t} \to \pi_{s+t}(E)$$

between the cohomological and homological AHSS's allows one to conclude that the cohomological spectral sequence is trivial. From here, one does some more work to conclude what they want. See [Ada69, pg. 20, Prop 17] for the details. ∎

Lemma 2.16.

(1) *$E^*(BU)$ contains a unique element c_i such that*

$$\langle c_i, (\beta_1)^i \rangle = 1$$

and

$$\langle c_i, m \rangle = 0$$

*when m is any monomial $\beta_1^{i_1}\beta_2^{i_2}\dots\beta_r^{i_r}$ distinct from β_1^i. We have $c_0 = 1$. These are called **generalized Chern classes**.*

(ii) *The restriction of c_1 to $BU(1)$ is x^E, our chosen orientation.*

(iii) *The restriction of c_i to $BU(n)$ is zero for $i > n$ (Otherwise, the image of c_i in $E^*(BU(n))$ is also denoted c_i).*

(iv) *$E^*(BU(n))$ is the ring of formal power series*

$$\pi^*(E)\,[\![c_1, c_2, \ldots, c_n]\!]$$

and $E^*(BU)$ is the ring of formal power series

$$\pi^*(E)\,[\![c_1, c_2, \ldots, c_i, \ldots]\!]\,.$$

Proof Sketch. The definition of c_i in **(i)** is legitimate by Lemma 2.15 applied to $X = BU$. It also easy to see that the unit $1 \in E^*(BU)$ plays the role laid down for c_0. Part **(ii)** is essentially by definition of β_1. Part **(iii)** holds since E-homology of $BU(n)$ contains no monomial (in the β_j's) of degree $> n$ (so 0 satisfies the role played by c_i and we can apply lemma 2.15 to $BU(n)$). Part **(iv)** comes from Lemma 2.15 + showing that the pairing

$$\pi^*(E)\,[\![c_1, c_2, \ldots, c_n]\!] \otimes_{\pi_*(E)} E_*(BU) \longrightarrow \pi_*(E)$$

is perfect. ■

Finally, we determine the E-homology of MU. Note that $E_*(MU)$ is a ring, and that the "inclusion" $MU(1) = MU_2 \to MU$ gives rise to a degree 2 map $\Sigma^\infty MU(1) \to MU$, and so induces a homomorphism

$$\widetilde{E}_k(MU(1)) \longrightarrow E_{k-2}(MU).$$

Recalling that $MU(1) \simeq \mathbb{CP}^\infty$, we let $b_i^E \in E_*(MU)$ be the image of $u^E\beta_{i+1}^E \in \widetilde{E}_*(MU(1))$ under this map (for $i \geq 0$). The factor of u^E ensures that $b_0^E = 1 \in E_0(MU)$.

Lemma 2.17. *$E_*(MU)$ is the polynomial algebra*

$$\pi_*(E)[b_1, b_2, \ldots, b_i, \ldots].$$

Proof Sketch. As one may expect, the point is that the monomials in the b_i form a $\pi_*(E)$-base for the E_2-term of the AHSS, so the sequence degenerates on the second page and one obtains this result. ■

3 Universality of MU

3.1 A Proof of Theorem A

We wish to show that complex orientations on a ring spectrum E correspond to maps $MU \to E$ from the complex Thom spectrum. We will see that this essentially follows from our computation of the E-homology ring $E_*(MU)$. Before stating and proving the result formally, we make the following remark.

Remark 3.1. Let E be a complex oriented ring spectrum, and let $g : MU \to E$ be a map of ring spectra. Note that $g_* x^{MU} \in \widetilde{E}^*(\mathbb{CP}^\infty)$ is also a complex orientation, so we necessarily have

$$g_* x^{MU} = \sum_{i \geq 0} d_i \left(x^E\right)^{i+1} =: f(x^E) \in \pi^*(E) \llbracket x^E \rrbracket = E^*(\mathbb{CP}^\infty)$$

with $d_0 \in \pi^*(E)$ a unit. We claim that $u^E d_k = \left\langle g, b_k^E \right\rangle$ where we view $g : MU \to E$ as an element of $E^*(MU)$. Intuitively, this is because "$b_k^E = u^E \beta_{i+1}^K$" and $d_i = \left\langle g_* x^{MU}, \beta_{i+1}^E \right\rangle$. The latter equality holds literally by definition of β_{i+1}^E, but the first one requires some care.

Let $\omega : \mathbb{CP}^\infty \to MU$ be the map representing $x^{MU} \in MU^*(\mathbb{CP}^\infty)$. Note that this map satisfies $\omega_*(u^E \beta_{k+1}^E) = b_k^E$. This means that $\left\langle g, \omega_*(u^E \beta_{k+1}^E) \right\rangle = u_E \left\langle g_* x^{MU}, \beta_{k+1}^E \right\rangle$ which gives the claimed equality. The most straightforward way to see this is to observe that $\left\langle g, \omega_*(u^E \beta_{k+1}^E) \right\rangle \in \pi^*(E)$ is represented by the composition

$$\mathbb{S} \xrightarrow{u^E \beta_{k+1}^E} E \wedge \mathbb{CP}^\infty \xrightarrow{1 \wedge \omega} E \wedge MU \xrightarrow{1 \wedge g} E \wedge E \xrightarrow{\mu} E$$

while $u^E \left\langle g_* x^{MU}, \beta_{k+1}^E \right\rangle = \left\langle g_* x^{MU}, u^E \beta_{k+1}^E \right\rangle$ is represented by

$$\mathbb{S} \xrightarrow{u^E \beta_{k+1}^E} E \wedge \mathbb{CP}^\infty \xrightarrow{1 \wedge (\omega \circ g)} E \wedge E \xrightarrow{\mu} E.$$

These two compositions are the same, so we have the claimed equality. In particular, note that $u^E d_0 = \left\langle g, b_0^E \right\rangle = \langle g, 1 \rangle = 1$.

Theorem 3.2. *Let E be a complex oriented cohomology theory, and suppose we are given an invertible formal power series*[7]

$$f(x^E) = \sum_{i \geq 0} d_i \left(x^E\right)^{i+1} \in \widetilde{E}^2(\mathbb{CP}^\infty)$$

with $u^E d_0 = 1$. Then there is one and (up to homotopy) only one (degree 0) map of ring spectra $g : MU \to E$ such that $g_* x^{MU} = f(x^E)$.

Proof. Lemma 2.15 applies to $X = MU$ to give an isomorphism

$$E^*(MU) \xrightarrow{\sim} \mathrm{Hom}_{\pi_*(E)}(E_*(MU), \pi_*(E)).$$

In particular, there is a bijection between homotopy classes of degree 0 maps $g : MU \to E$ and maps $\theta : E_*(MU) \to \pi_*(E)$ which are linear over $\pi_*(E)$ and of degree 0. Similarly for maps $MU \wedge MU \to E$. Hence a map $g : MU \to E$ make the following diagram homotopy-commutative (i.e. is a map of ring spectra)

$$\begin{array}{ccc} MU \wedge MU & \xrightarrow{g \wedge g} & E \wedge E \\ \downarrow & & \downarrow \\ MU & \xrightarrow{\quad g \quad} & E \end{array}$$

[7] i.e. a choice of complex orientation in degree 2

iff the following corresponding diagram commutes.

$$\begin{tikzcd} E_*(MU) \otimes_{\pi_*(E)} E_*(MU) \arrow[r, "\theta\otimes\theta"] \arrow[d] & \pi_*(E) \otimes \pi_*(E) \arrow[d] \\ E_*(MU) \arrow[r, "\theta"] & \pi_*(E) \end{tikzcd}.$$

This is the case iff θ is map of algebras over $\pi_*(E)$. Now, by remark 3.1, the condition

$$g_* x^{MU} = \sum_{i\geq 0} d_i \left(x^E\right)^{i+1} = f(x^E)$$

is equivalent to

$$\theta(b_i^E) := \langle g, b_i^E \rangle = u^E d_i \text{ for } i \geq 0$$

(note that $b_0^E = 1$, so we must have $u^E d_0 = \theta(b_0^E) = 1$). There is one and only one map of $\pi_*(E)$-algebras satisfying this condition, since $E_*(MU)$ is the polynomial algebra $\pi_*(E)[b_1, b_2, \ldots, b_i, \ldots]$, so we win. ■

Remark 3.3. I think one should be able to make sense of the above theorem when the power series $f(x^E)$ appears in other degrees, but I'm not 100% sure it makes sense to talk about a "map of ring spectra" $g : MU \to E$ if it is not of degree 0. In particular, a "map of ring spectra" should satisfy $g \circ i_{MU} = i_E$ (where $i_{MU} : \mathbb{S} \to MU$ and $i_E : \mathbb{S} \to E$ are the units), so additivity of degrees really wants to say g is of degree 0.

I guess the point is just that if you have (X, x^E) (with $u^E \in \pi_*(E)^\times$) defined as before, then you get some $g : MU \to E$ such that $g_* x^{MU} = \left(u^E\right)^{-1} x^E$, so as we've defined it, a complex orientation is not just a map $MU \to E$, but actually a map $MU \to E$ *and* a choice of unit $u^E \in \pi_*(E)^\times$. Some people require their complex orientations to satisfy $x^E|_{\mathbb{CP}^1} = \gamma^E$ (i.e. $u^E = 1$) so for them they only need the map from MU.

This shows that the pair (MU, x^{MU}) is universal (specifically, initial) among pairs (X, x^E) which satisfy $u^E = 1$.

3.2 A Proof Sketch of Theorem B

We first define the Larzard ring L appearing in the statement of Theorem B. This is the home of the universal formal group law.

Definition 3.4. Let $R = \mathbb{Z}[a_{ij} \mid i, j \geq 0]$. The **Lazard ring** L is the largest quotient of R such that the formal power series $\mu(x, y) = \sum_{i,j\geq 0} a_{ij} x^i y^j \in R[\![x, y]\!]$ becomes a formal group law. That is, $L = R/I$ where I is generated by a_{i0} $(i \geq 1)$, a_{0j} $(j \geq 1)$, $a_{ij} - a_{ji}$ $(i, j \geq 0)$, and whatever nasty polynomials are needed to guarantee associativity.

By construction, $x +_L y = \sum_{i,j\geq 0} a_{ij} x^i y^j \in L[\![x, y]\!]$ is the universal formal group law in the sense that every formal group law $x +_\mu y$ over any ring A is of the form $x +_\mu y = f(x +_L y)$ for some unique ring map $f : L \to A$. That is, a formal group law over A is the same thing as a ring map $L \to A$.

Remark 3.5. We can make into L a graded ring. To motivate this, note that if E is complex orientable, then it has an orientation in $\widetilde{E}^2(\mathbb{CP}^\infty)$ which, thinking homologically, is in degree -2.

We grade L by assigning x, y and $\mu(x, y)$ to degree -2. Hence, a_{ij} has degree $2i+2j-2 = 2(i+j-1)$. It is easy to show that the ideal I is generated by homogeneous elements, so $L = R/I$ is a graded ring which is nonzero only in even degrees.

We have the following surprising result.

Theorem 3.6. *With the above grading, L is a polynomial algebra over $\mathbb{Z}$ with one generator in each positive even degree.*

Compare the above theorem to the calculation of $\pi_*(MU)$.

Let's sketch a proof of the above theorem. Inspired by the Hurewicz map $\pi_*(MU) \hookrightarrow \mathrm{H}_*(MU) = \mathbb{Z}[b_i \mid i \geq 1]$, we consider the graded commutative ring

$$R = \mathbb{Z}[b_1, b_2, \ldots, b_i, \ldots]$$

with b_n of degree $2n$. Consider the formal power series $\exp(y) = \sum_{i\geq 0} b_i y^{i+1} \in R[\![y]\!]$, and let $\log(x) \in R[\![x]\!]$ be its inverse. We define the formal product

$$x +_R y := \exp(\log x + \log y)$$

which gives rise to a map $\theta : L \to R$. By analyzing the image of this map "one degree at a time," i.e. by looking at the induced maps $Q_{2n}(L) := (I_L/I_L^2)_{2n} \to (I_R/I_R^2)_{2n} =: Q_{2n}(R)$ (where $I_L = \sum_{n>0} L_n$ and similarly for I_R), one is able to show that θ is an injection and moreover determine the degrees of the generators of its image. Studying things "one degree at a time" helps because it reduces to understanding formal group laws on graded rings of the form $\mathbb{Z} \oplus A$ with $\mathbb{Z}$ in degree 0 and A an abelian group in degree $2n$, and understanding these is manageable. See [Ada95, Part II, Sect. 7] for details.

Accepting Theorem 3.6, how do we then show that the map $L \to MU$ characterizing the formal group law arising from the complex orientation $\omega : \mathbb{CP}^\infty \xrightarrow{\sim} MU(1)$ considered earlier is an isomorphism? We use the fact that Milnor showed us that $\pi_*(MU)$ is also a polynomial algebra over $\mathbb{Z}$ on even generators which embeds into $\mathrm{H}_*(MU) = R$. That is, we have a diagram

$$\begin{tikzcd} L \arrow[r] \arrow[d] & \pi_*(MU) \arrow[d, "h"] \\ \mathbb{Z}[b_1, b_2, \ldots, b_n, \ldots] \arrow[r, equal] & \mathrm{H}_*(MU) \end{tikzcd} \tag{3.1}$$

which we claim commutes. Using the universal property of L, saying this commutes is equivalent to saying that the pushforward of the formal group law $x +_{MU} y$ over $\pi_*(MU)$ under the Hurewicz map is $x +_R y = \exp(\log x + \log y)$. We show this below.

To understand the action of the Hurewicz map, we will want to consider the spectrum $H \wedge MU$. This naturally comes with maps $H, MU \rightrightarrows H \wedge MU$, and so we have two natural complex orientations $x^E, x^{MU} \in H \wedge MU$. Relating these two will allow us to relate the formal group laws they induce. Note the Hurewicz map is precisely the map on homotopy groups induced from the composition

$$MU \simeq S^0 \wedge MU \xrightarrow{i\wedge 1} H \wedge MU.$$

Lemma 3.7. *Let X, Y be spectra, and let E be a ring spectrum. Let $B : [X, Y]_* \to [X, E \wedge Y]_*$ send f*

to the composition $B(f) : X \simeq S^0 \wedge X \xrightarrow{i\wedge f} E \wedge Y$. Then, the following diagram commutes:

$$\begin{tikzcd}
{[X,Y]_*} \arrow[rr, "B"] \arrow[dr, "\alpha"'] & & {[X, E\wedge Y]_*} \arrow[dl, "p"] \\
& \mathrm{Hom}_{\pi_*(E)}(E_*(X), E_*(Y)) &
\end{tikzcd}$$

Here, α is defined by

$$\alpha(f) = f_* : E_*(X) \to E_*(Y),$$

while $p(h)(k) = \langle h, k\rangle \in \pi_*(E \wedge Y) = E_*(Y)$ is the composition ($h : X \to E \wedge Y$ and $k \in E_*(X)$)

$$\mathbb{S} \xrightarrow{k} E \wedge X \xrightarrow{1\wedge h} E \wedge E \wedge Y \xrightarrow{\mu\wedge 1} E \wedge Y.$$

Proof. Given $f : X \to Y$ and $k : \mathbb{S} \to E \wedge X$ (so $k \in E_*(X)$), we have that $(p \circ B)(f)(k)$ is given by the composition

$$\mathbb{S} \xrightarrow{k} E \wedge X \xrightarrow{1\wedge f} E \wedge Y \xrightarrow{\sim} E \wedge S^0 \wedge Y \xrightarrow{1\wedge i\wedge 1} E \wedge E \wedge Y \xrightarrow{\mu\wedge 1} E \wedge Y.$$

However, this involves multiplying by the unit, so it is homotopic to the composition

$$\mathbb{S} \xrightarrow{k} E \wedge X \xrightarrow{1\wedge f} E \wedge Y$$

which is precisely $\alpha(f)(k)$. ■

Corollary 3.8. *In $(H \wedge MU)^*(\mathbb{CP}^\infty) = [\mathbb{CP}^\infty, H \wedge MU]_*$ we have*

$$x^{MU} = \sum_{i\geq 0} b_i^H (x^H)^{i+1} =: \exp(x^H)$$

with coefficients $b_i^E \in \pi_*(E \wedge MU)$.

Proof. We can apply the previous lemma to the case $X = \mathbb{CP}^\infty$, $Y = MU$, and $E = H$ to see that the below diagram commutes.

$$\begin{tikzcd}
MU^*(\mathbb{CP}^\infty) \arrow[rr, "B"] \arrow[dr, "\alpha"'] & & (H\wedge MU)^*(\mathbb{CP}^\infty) \arrow[dl, "p"] \\
& \mathrm{Hom}_{\pi_*(H)}(H_*(\mathbb{CP}^\infty), H_*(MU)) &
\end{tikzcd}$$

Since x^{MU} is a reduced class, so is Bx^{MU}. By definition (recall $u^H = 1$), we have

$$\left(\alpha x^{MU}\right)\left(\beta_{i+1}^H\right) = b_i^H.$$

We also have[8]

$$\left(p(x^H)^j\right)(\beta_i^H) = \begin{cases} 1 & \text{if } i = j \\ 0 & \text{otherwise} \end{cases}.$$

[8]Adams writes b_i^E when proving (a slight generalization of) this result, but it should be β_i^E.

Writing $Bx^{MU} = \sum_{j\geq 0} c_j (x^E)^{j+1}$, we compute

$$b_i^H = (\alpha x^{MU})(\beta_{i+1}^H) = p(Bx^{MU})(\beta_{i+1}^H) = c_i.$$

Since p is an isomorphism (by Lemma 2.15), this proves the claim. ■

Corollary 3.9. *After applying the Hurewicz homomorphism $h : \pi_*(MU) \to \pi_*(H \wedge MU) = H_*(MU)$, we have*

$$x_1 +_{MU} x_2 = \exp^H(\log^H x_1 + \log^H x_2),$$

where

$$\exp^H(x) = \sum_{i\geq 0} b_i x^{i+1}$$

with $b_i \in \mathrm{H}_{2i}(MU)$ the usual generators coming from $\mathrm{H}_{2i+2}(MU(1))$, and $\log^H$ is the formal power-series inverse to $\exp^H$.

Proof. Let $m : \mathbb{CP}^\infty \times \mathbb{CP}^\infty \to \mathbb{CP}^\infty$ be the map representing addition of (singular) cohomology classes. Then, m^* commutes with addition and multiplication, so $m^* \exp^H(x^H) = \exp^H(m^* x^H)$. Spelled out,

$$x_1 +_{MU} x_2 = m^* x^{MU} = m^* \exp^H(x^H) = \exp^H(m^* x^H) = \exp^H(m^* \log^H(x^{MU})) = \exp^H(\log^H x_1 + \log^H x_2)$$

where the last equality holds since H's group law is the additive group law. Also, above we should probably really write x_1^{MU}, x_2^{MU} instead of just x_1, x_2, but meh. ■

The above corollary finally says that the diagram (3.1) commutes. This means that the map $L \to \pi_*(MU)$ is an injection. To show that it is an isomorphism, Adams [Ada95, Part II. Sect. 8] shows that $\mathrm{im}(\pi_*(MU) \to \mathrm{H}_*(MU)) = \mathrm{im}(L \to \mathbb{Z}[b_1, b_2, \ldots, b_n])$ by analyzing things "one degree at a time" again by looking at the induced maps on $Q_{2n}(-)$; we will not go into the details of this. This concludes the proof sketch of theorem B. The coefficient ring $\pi_*(BU)$ of the complex Thom spectrum is the Lazard ring, and the formal group law associated to complex cobordism is the universal formal group law.

4 A Glimpse of Some Consequences

In the final section, we quickly mention some consequences of our main theorems, with little regard for detail or rigour.

- The first consequence is that we can now construct a ton of cobordism invariants. Indeed, since $\pi_*(MU)$ is the Lazard ring, an R-valued complex genera $\pi_*(MU) \to R$ is the same thing as a formal group law defined over R. In particular, one can show that the L-genus showing up in Hirzebruch's signature theorem comes from the tanh formal group law

  $$\mu(x, y) = \frac{x + y}{1 + xy}$$

  (this satisfies $\mu(\tanh(x), \tanh(y)) = \tanh(x + y)$).

- For the second consequence, let E be a complex oriented spectrum. Recall that we showed this means that E comes equipped with a preferred map $MU \to E$ of ring spectra (the map associated to $f(x^E) = \left(u^E\right)^{-1} x^E$ in the notation of theorem 3.2). Thus, any complex vector bundle $V \to B$ has an associated Thom class $U_V \in E^*(B)$ and these are natural under pullback and turn direct sums of vector bundles into products of Thom classes. Using these, one can define fundamental classes for embeddings between smooth manifolds.

- For the third consequence, recall the exponential series $\exp^H(x) = \sum_{i\geq 0} b_i x^{i+1}$ and its inverse formal power series $\log^H(x) = \sum_{i\geq 0} m_i x^{i+1}$. One can write down an explicit formula for m_j in terms of the b_i's and use to this to conclude that the logarithm series for the formal product $x +_{MU} y$ on $\pi_*(MU)$ is
$$\log^H x^{MU} = \sum_{n\geq 0} \frac{[\mathbb{CP}^n]}{n+1} \left(x^{MU}\right)^{n+1}.$$
In particular, since $\mathrm{H}_*(MU)$ is generated by the b_n's (so also by $m_n = [\mathbb{CP}^n]/(n+1)$) and $\pi_*(MU) \hookrightarrow \pi_*(MU) \otimes \mathbb{Q} \simeq \mathrm{H}_*(MU) \otimes \mathbb{Q}$, we see that the complex cobordism ring is rationally generated by complex projective spaces, and that every complex genus is determined by its value on complex projective spaces; although not any set of values work since these do not give algebra generators integrally.

- Note that $MU_*(MU) = \pi_*(MU)[b_1, b_2, \dots, b_i, \dots]$ and so is flat as a (right) module over $\pi_*(MU)$. Thus, for any spectrum X, the natural map
$$MU_*(MU) \otimes_{\pi_*(MU)} MU_*(X) \to (MU \wedge MU)_*(X)$$
is an isomorphism (e.g. because both sides give homology theories agreeing when $X = S^0$). Thus, $MU_*(X)$ is naturally a comodule over $MU_*(MU)$ with coaction given by
$$MU_*(X) \xrightarrow{\sim} (MU \wedge S^0)(X) \to (MU \wedge MU)(X) \xleftarrow{\sim} MU_*(MU) \otimes_{\pi_*(MU)} MU_*(X).$$
In particular, taking $X = MU$, $MU_*(MU)$ has a diagonal map
$$MU_*(MU) \to MU_*(MU) \otimes_{\pi_*(MU)} MU_*(MU).$$
This gives additional structure to $MU_*(MU)$, and one can determine explicitly what this diagonal map does.[9]

- We have saved the best (and longest to explain) consequence for last. Let R be a ring with a given formal group law $\mu(x, y) = x +_\mu y$, so it is naturally a $\pi_*(MU)$ module. One can ask whether the functor
$$X \longmapsto MU_*(X) \otimes_{\pi_*(MU)} R$$
gives a homology theory. The main thing to worry about is maintaining long exact sequences, since tensoring is usually only right exact. Thus, at the very least, it is clear that this gives homology

[9] In fact, $MU_*(MU)$ has two diagonal maps, the other one making use of the isomorphism $MU \xrightarrow{\sim} S^0 \wedge MU$ with S^0 on the left. These two diagonal maps can be used to turn the pair $(\pi_*(MU), MU_*(MU))$ into a Hopf algebroid, a groupoid valued functor on rings.

theory when R is flat over $\pi_*(MU)$, but in fact even more is true. The point is that $MU_*(X)$ is not just a module over $\pi_*(MU)$, but also a comodule over $MU_*(MU)$, so we do not need tensoring with R to preserve exactness for all (exact sequences of) $\pi_*(MU)$-modules, but only for those which are furthermore comodules over $MU_*(MU)$. The right condition was found by Landweber. To state it, first let $[0]_\mu x := 0$ and inductively let $[n]_\mu x := x +_\mu [n-1]_\mu x$ be the n**-series** for our formal group law.

Theorem 4.1 (**Landweber Exact Functor Theorem**). *For a given prime p, let $v_n \in R$ be the coefficient of x^{p^n} in the p-series $[p]_\mu x$. Then, the above construction gives a homology theory iff for every prime p, the sequence $(v_0, v_1, v_2, v_3, \dots)$ is regular in R, i.e. v_n is not a zero divisor in $R/(v_0, v_1, \dots, v_{n-1})$ for all $n \geq 0$.*

This theorem gives us a way to construct many new (complex oriented) homology theories (equivalently, many new (complex oriented) spectra). For a slightly more detailed exposition on this theorem, see these notes on a talk by Dylan Wilson [Wil18].

Example. Let $R = \mathbb{Z}[\beta^{\pm 1}]$ with the multiplicative formal group law $x +_\mu y = x + y + xy = (1+x)(1+y) - 1$. Then, its p-series is $[p]_\mu x = (1+x)^p - 1 = px + (\text{middle terms}) + x^p$ so $v_0 = p$, $v_1 = 1$, and $v_n = 0$ if $n \geq 2$. Now, p is not a zero divisor in R and 1 is not a zero divisor in $\mathbb{F}_p[\beta^{\pm 1}] = R/(p)$, and 0 is not a zero divisor in $0 = R/(p, 1)$, so $(p, 1, 0, 0, \dots)$ is indeed a regular sequence. Thus, $X \mapsto MU_*(X) \otimes_{\pi_*(MU)} \mathbb{Z}[\beta^{\pm 1}]$ is a homology theory. In fact, this is the homology theory associated to the K-spectrum.

A Some computations

This appendix is completely separate from the main part of these notes. It is mainly here just for me to practice doing some detailed computations.

Fix some complex oriented spectrum E with orientation $x = x^E$.

A.1 $E^*(\mathbb{RP}^\infty)$

We first wish to show that

$$E^*(\mathbb{RP}^\infty) \simeq \frac{\pi^*(E)\,[\![c]\!]}{(c +_E c)} \text{ with } |c| = |x|\,.$$

Say $x \in E^k(\mathbb{CP}^\infty)$ and consider $X = \mathbb{RP}^{2n}$. Note that there is a unique nontrivial map $f \in [\mathbb{RP}^{2n}, \mathbb{CP}^\infty] = \mathrm{H}^2(\mathbb{RP}^{2n}; \mathbb{Z})$, and let $c = f^*x \in E^k(\mathbb{RP}^{2n})$.

Assumption. To simplify the calculation, assume that $\pi_*(E)$ has no 2-torsion. Hence,

$$\mathrm{H}^*\left(\mathbb{RP}^{2n}; \pi^*(E)\right) \simeq \frac{\pi^*(E)[y]}{(2y, y^{n+1})} \text{ with } |y| = 2$$

by universal coefficients.

Consider the Atiyah-Hirzebruch SS

$$E_2^{p,q} = \mathrm{H}^p(\mathbb{RP}^{2n}; \pi^q(E)) \implies E^{p+q}(\mathbb{RP}^{2n}).$$

Because $x|_{(\mathbb{CP}^\infty)^1} = 0$ and $x|_{S^2}$ gives a generator of $E^*(S^2)$ over $\pi^*(E)$, we see that c is in filtration 2, represents a nonzero element of the second graded piece, and in fact reduces to an element of $E_2^{2,k-2}$ generating column 2 over $\pi^*(E)$. The powers c^m then generate the $2m$th columns while the odd columns all vanish, so the E_2-page is generated as an algebra over $\pi^*(E)$ by c. Since c survives to the end and the differential is $\pi_*(E)$-linear + multiplicative, this shows that the spectral sequence degenerates at E_2. Hence, the associated graded pieces of $E^*(\mathbb{RP}^{2n})$ are $G^0E^*(\mathbb{RP}^{2n}) = \pi^*(E)$, $G^{2i}E^*(\mathbb{RP}^{2n}) = \pi^*(E)/2$, and $G^{2i+1}E^*(\mathbb{RP}^{2n}) = 0$ (for $0 < i \leq n$).

To solve the extension problem, note that the composition

$$\mathbb{RP}^{2n} \xrightarrow{f} \mathbb{CP}^\infty \xrightarrow{\Delta} \mathbb{CP}^\infty \times \mathbb{CP}^\infty \xrightarrow{m} \mathbb{CP}^\infty$$

is trivial – where m is the map representing addition of cohomology classes and Δ is the diagonal – since $\mathrm{H}^2(\mathbb{RP}^{2n}; \mathbb{Z}) = [\mathbb{RP}^{2n}, \mathbb{CP}^\infty]$ is 2-torsion. Thus,

$$E^*(\mathbb{RP}^{2n}) \ni 0 = (m\Delta f)^*x = f^*\Delta^*m^*x = f^*\Delta^*(x_1 +_E x_2) = f^*(x +_E x) = c +_E c.$$

Hence, using that also $c^{n+1} = 0$ (since the spectral sequence is finite width), we have a map

$$A := \frac{\pi^*(E)[c]}{(c +_E c, c^{n+1})} \longrightarrow E^*(\mathbb{RP}^{2n})$$

which we claim is an isomorphism. To see this, filter the LHS via

$$F^{2i}A := \sum_{j\geq i} \pi^*(E)\cdot c^i \subset \frac{\pi^*(E)[c]}{(c +_E c, c^{n+1})}$$

and note that our map $A \to E^*(\mathbb{RP}^{2n})$ respects filtrations. It is clear that $G^0 A \simeq \pi^*(E) \xrightarrow{\sim} G^0 E^*(\mathbb{RP}^{2n})$. For $0 < i \leq 2n$, note that

$$A \ni 0 = c +_E c = 2c + \sum_{n\geq 2} \sum_{r+s=n} a_{rs} c^n \implies 2c^i = (2c)c^{i-1} = -\sum_{n\geq i+1} \sum_{r+s=i-1} a_{rs} c^{i+1} \in F^{2(i+1)}A,$$

so $G^{2i}A \simeq \pi^*(E)/2 \xrightarrow{\sim} G^{2i}E^*(\mathbb{RP}^{2n})$. Thus, the natural map $A \to E^*(\mathbb{RP}^{2n})$ induces an isomorphism on associated gradings, so is itself an isomorphism.

We have $E^*(\mathbb{RP}^{2n+2}) \twoheadrightarrow E^*(\mathbb{RP}^{2n})$ surjectively, so Milnor gives $E^*(\mathbb{RP}^\infty) \simeq \pi^*(E) [\![c]\!] /(c +_E c)$.

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