KanSem Forms of K-Theory

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Introduction

Our goal in these notes is two construct two families of complex oriented cohomology theories which are determined by their associated formal group laws, following the paper [Mor89] of Morava. The first family K_α will consist of 2-periodic cohomology theories over $\mathbb{Z}_p$, as α ranges over $\mathbb{Z}_p^\times$, which are *forms* of K-theory in the sense that $K_\alpha \otimes_{\mathbb{Z}_p} W \cong K_{S\mathbb{F}_p}$ for some explicit ring W,[1] where $K_{S\mathbb{F}_p}$ is the p-completion of complex K-theory, i.e. its localization at the Moore spectrum $S\mathbb{F}_p$.

Maybe it's supposed to be localization at $H\mathbb{F}_p$ instead? I'm like 70% sure these have the same acyclics, but who knows?

The second family K_q will consist again of 2-periodic cohomology theories, but this time as q ranges over $\mathbb{C}$. Rather than being forms of K-theory, these cohomology theories come from the formal group law of an elliptic curve E_q, and the rings over which they are defined depend on the set of primes at which E_q is 'ordinary' (in the usual sense of elliptic curves).

Constructing these cohomology theories and verifying some of their basic properties is not too difficult once one knows that these belong to a large class of 'ordinary' (in the sense used in [Mor89]) cohomology theories which are determined by a formal group law. Hence, we will begin by defining and studying ordinary cohomology theories in general, and then specify to obtain the two families alluded to above.

1 Ordinary K-theories

1.1 Formal Group Laws, Complex Genera, and Cohomology Theories

It will be helpful to recall the relationship between formal group laws and complex genera.

[1] In other contexts, one might call such an object a twist.

Notation 1.1. Given a ring spectrum E, we will denote the two gradings on its coefficient ring by $E_* := E_*(pt)$ and $E^* := E^*(pt)$.

Recall 1.2. Let L denote the Lazard ring which supports the universal formal group law. Let MU denote the complex cobordism spectrum. This has a natural complex orientation $x \in MU^2(\mathbb{CP}^\infty)$ such that the map $L \to MU^* = MU^*(pt)$ induced by the formal group law induced by x is an isomorphism. Consequently, given a ring R, a choice of formal group law on R can be identified with a map $MU^* \to R$ from the complex cobordism ring.

Definition 1.3. Let R be a ring which is torsion-free. An **R-valued (complex) genus** is a map $\rho : MU^* \to R$. Since $MU^* \otimes \mathbb{Q}$ is generated by complex projective spaces, such a genus is determined by the values $\rho_n = \rho([\mathbb{CP}^n])$ and hence by either

- the **logarithm of** ρ
$$\log_\rho(X) := \sum_{n\geq 1} \rho_{n-1}\frac{X^n}{n}; \text{ or}$$
- the **zeta function of** ρ
$$\zeta_\rho(s) := \sum_{n\geq 1} \rho_{n-1} n^{-s}.$$

Remark 1.4. Some justification is required for calling this powers series the *logarithm* of ρ. Given a formal group law F over a $\mathbb{Q}$-algebra A (e.g. $R \otimes \mathbb{Q}$), there exists a unique isomorphism between F and the additive group law $(x, y) \mapsto x + y$, i.e. a unique power series $\log X \in A[\![X]\!]$ such that

$$F(X, Y) = \exp(\log X + \log Y)$$

where exp is the composition inverse of log. Our definition of logarithm above is justifed by **Miščenko's Theorem** which says that the universal formal group law over $MU^* \otimes \mathbb{Q}$ is $(x, y) \mapsto \exp_{MU}(\log_{MU} x + \log_{MU} y)$ where
$$\log_{MU} x = \sum_{n\geq 1} [\mathbb{CP}^{n-1}]\frac{x^n}{n} \in (MU^* \otimes \mathbb{Q})[\![x]\!].$$

Remark 1.5. Let R be torsion-free, and let $\rho : MU^* \to R$ be a genus. The formal group law associated to ρ is isomorphic, over $R \otimes \mathbb{Q}$, to the group law

$$F_\rho(X, Y) := \exp_\rho(\log_\rho X + \log_\rho Y)$$

where $\exp_\rho$ is the compositional inverse of $\log_\rho$.

Remark 1.6. If you start with a formal group law $F(x, y)$ over torsion-free R, then you can easily recover its logarithm. The logarithm defines an isomorphism (over $R \otimes \mathbb{Q}$) with the additive group law, so one wishes to solve the equation
$$\log_F(F(x, y)) = \log_F x + \log_F y.$$

Taking the derivative with respect to y at the point $(x, 0)$ and doing a little rearranging, one sees that

$$\log_F(x) = \int_0^x \frac{\mathrm{d}t}{\frac{\partial F}{\partial y}(t, 0)}.$$

Recovering the zeta function function from a group law is a little less straightforward. On the flipside though, starting with ζ_ρ can be more natural for obtaining group laws with nice integrality properties.

Theorem 1.7 (Honda, [Hon70])**.** *Suppose $\rho : MU^* \to \mathbb{Q}$ (or $\mathbb{Q}_p$) is a genus whose ζ-function has an Euler product*

$$\zeta_\rho(s) = \prod_p \left(1 + b_{1,p}p^{-s} + \cdots + b_{n,p}p^{n(1-s)-1} + \cdots\right)^{-1} = \prod_p \left(1 + b_{1,p}p^{-s} + \cdots + (b_{n,p}p^{n-1})(p^{-s})^n + \cdots\right)^{-1}$$

in which $b_{ij} \in \mathbb{Z}$ (or $b_{ij} \in \mathbb{Z}_p$). Then the formal group law

$$F_\rho(X,Y) = \exp_\rho(\log_\rho X + \log_\rho Y) \in \mathbb{Q}[\![X,Y]\!] \ (or\ \mathbb{Q}_p[\![X,Y]\!])$$

is actually defined over $\mathbb{Z}$ (or over $\mathbb{Z}_p$).

The group laws defining the cohomology theories mentioned in the intro will be given in terms of their zeta functions.

Assumption. All cohomology theories are assumed multiplicative.

Recall 1.8. Let E be a complex orientable cohomology theory. Then, a choice of complex orientation $x \in E^2(\mathbb{CP}^\infty)$ gives rise to a formal group law over $E^* = E^*(pt)$, and so to a genus $\rho_E : MU^* \to E^*$.

Definition 1.9. Let $\rho : MU^* \to R$ be a genus (so R torsion-free), and let p be a prime number. We say ρ is **ordinary at** p if either

- $p \in R^\times$; or
- $\rho(\mathbb{CP}^{p-1})$ becomes a unit in R/pR.

We call ρ an **ordinary genus** if it is ordinary at all primes, and we call an oriented cohomology theory E an **ordinary K-theory** if its associate genus ρ_E is ordinary.

Example. Let K be complex K-theory. Give it the orientation $1 - [L] \in K^0(\mathbb{CP}^\infty)$ where $L \to \mathbb{CP}^\infty$ is the universal line bundle. The associated formal group law is[2] $x +_K y = x + y - xy$, the multiplicative formal group law[3]. One quickly calculates that the associated logarithm is

$$\sum_{n\geq 1} \rho_K(\mathbb{CP}^{n-1})\frac{x^n}{n} := \log_K x = \int_0^x \frac{\mathrm{d}t}{\frac{\partial F}{\partial y}(x+y-xy)\Big|_{(x,y)=(t,0)}} = \int_0^x \frac{\mathrm{d}t}{1-t} = -\log(1-x) = \sum_{n\geq 1} \frac{x^n}{n}.$$

Thus, $\rho_K(\mathbb{CP}^{n-1}) = 1$ for all n so the associated genus to K is the Todd genus which is visibly ordinary. Hence, complex K-theory is an ordinary K-theory.

Definition 1.10. Let $\rho : MU^* \to R$ be an ordinary genus. We define the associated graded genus $\rho_* : MU^* \to R[t, t^{-1}]$, where t is an indeterminate of dimension 2, by $\rho_*(M) = \rho(M)t^{-\dim_{\mathbb{C}} M}$ on homogeneous elements. We define the associated K-theory to be $K^*_\rho(X) = MU^*(X) \otimes_{\rho_*} R[t, t^{-1}]$. These will turn out to be cohomology theories.

[2] If $m : \mathbb{CP}^\infty \times \mathbb{CP}^\infty \to \mathbb{CP}^\infty$ represents tensor products of line bundles, then $m^*[L] = [L_1][L_2]$. Since $[L] = 1 - x$ with $x := 1 - [L]$, this let's you compute the formal group law $m^*x = x_1 + x_2 - x_1x_2$.

[3] If you choose $[L] - 1$ as your orientation, you get $x + y + xy$ which is also called the multiplicative formal group law. Unsurprisingly, these are isomorphic via the isomorphism $x \mapsto -x$

Example. If you give K-theory the orientation $\beta(1-[L]) \in K^2(\mathbb{CP}^\infty)$ where $\beta \in K^* = \mathbb{Z}[\beta, \beta^{-1}]$ is the Bott element, then the associated formal group law is $x +_K y = x + y - \beta^{-1}xy$, and the associated logarithm is

$$\log_K x = -\beta \log(1 - \beta^{-1}x) = \sum_{n \geq 1} \beta^{1-n} \frac{x^n}{n} = \sum_{n \geq 1} \rho_*(\mathbb{CP}^{n-1}) \frac{x^n}{n}.$$

Thus, the genus associated to this choice of orientation is precisely the associated graded genus ρ_* defined above (where ρ is the Todd genus).

It is a theorem of Conner and Floyd [CF66, Theorem 10.1] that the K-theory associated to the Todd genus as above is in fact complex K-theory. For more general genera, one instead appeals to Landweber's exactness theorem.

Recall 1.11. Let $x +_F y \in R[\![x, y]\!]$ for a formal group law over some ring R. For $m \geq 1$, its m**-series** is the formal power series

$$[m]_F(x) := \underbrace{x +_F x +_F \cdots +_F x}_{m \text{ times}} = mx + \cdots \in R[\![x]\!]$$

representing multiplication by m.

Theorem 1.12 (**Landweber Exactness Theorem**). *For a given prime p, let $v_{p,n} \in MU^*$ be the coefficient of x^{p^n} in the p-series $[p]_{MU}x$ of the universal formal group law (so $v_{p,0} = p$). Then, given a genus $\rho : MU^* \to R$, the functor*[4]

$$X \mapsto MU^*(X) \otimes_{MU^*} R^*$$

defines a (complex orientable) cohomology theory iff the sequences $(p, v_{p,1}, v_{p,2}, \dots)$ are regular in R for all primes p.

Remark 1.13. Recall that the complex cobordism logarithm is

$$\log_{MU}(x) = \sum_{n \geq 1} [\mathbb{CP}^{n-1}] \frac{x^n}{n},$$

so its p-series (at least over $MU^* \otimes \mathbb{Q}$) is

$$[p]_{MU}(x) = \exp_{MU}(p \log_{MU} x) = \sum_{n \geq 0} b_n \left(px + \frac{p}{2}[\mathbb{CP}^1]x + \frac{p}{3}[\mathbb{CP}^2]x + \cdots + [\mathbb{CP}^{p-1}]x^p + \dots \right)^{n+1}$$

where $b_0 = 1$, and the higher b_n are irrelevant for current purposes. Let $u = v_{p,1}$ be the coefficient of x^p in the above expansion. We claim that $u \equiv [\mathbb{CP}^{p-1}] \pmod{p}$. The "mod p" map factors as

$$MU^* \to MU^* \otimes \mathbb{Z}_{(p)} \to MU^* \otimes \mathbb{F}_p.$$

[4]Technically speaking, since we are working with cohomology instead of homology, this will only define a cohomology theory on finite complexes/spectra, but extending it so the infinite case is not difficult (it just won't literally be a tensor product)

For any $k \in \{0, \ldots, p-1\}$, we have

$$\frac{p[\mathbb{CP}^{k-1}]}{k} = [\mathbb{CP}^{k-1}] \otimes \frac{p}{k} \in MU^* \otimes p\mathbb{Z}_{(p)},$$

which gets killed by the mod p map. Thus, mod p, we have

$$[p]_{MU}(x) \equiv \sum_{n \geq 0} b_n \left([\mathbb{CP}^{p-1}]x^p + \ldots\right)^{n+1} \equiv b_0[\mathbb{CP}^{p-1}]x^p + \cdots \equiv [\mathbb{CP}^{p-1}]x^p + \ldots \pmod{p}.$$

The upshot is that $v_{p,1} = [\mathbb{CP}^{p-1}] \in MU^*/pMU^*$, so for the purposes of applying Landweber, we may as well pretend $v_{p-1} = [\mathbb{CP}^{p-1}]$.

Corollary 1.14. *Let $\rho : MU^* \to R$ be an ordinary genus (so R torsion-free). Then,*

$$X \mapsto MU^*(X) \otimes_\rho R$$

(or even $X \mapsto MU^*(X) \otimes_{\rho_*} R[t, t^{-1}]$) is a cohomology theory.

1.2 The Category of Ordinary K-theories

Definition 1.15. Let E be an ordinary K-theory. A **normalization** of E is a graded isomorphism

$$E^* \xrightarrow{\sim} R[t, t^{-1}] \text{ with } |t| = 2 \text{ where } R = E^0.$$

Of course, E may not be normalizable (it doesn't even have to be periodic), but one can always let $B^* = E^*[t, t^{-1}]$ (with the induced grading) and then form the functor $E^*(-) \otimes_{E^*} B^*$ which is now a normalizable ordinary K-theory.

The benefit of working with normalize ordinary K theories is that they are determined by their group law in a strong sense.

Definition 1.16. Let $\mathscr{K}$ be the category whose objects are normalized K-theories over torsion-free rings. For $K_0, K_1 \in \mathscr{K}$, a morphism $\varphi : K_0 \to K_1$ is multiplicative transformation of graded cohomology theories (i.e. a morphism of ring spectra).

Remark 1.17. Every ordinary K-theory E is naturally oriented since it comes with a preferred choice of (ordinary) genus $\rho_E : MU^* \to E^*$.

Definition 1.18. Let $K_0 \in \mathscr{K}$ with $K_0^* = R[t, t^{-1}]$. Since K_0 is oriented, there's a canonical $T_0 \in K_0^*(\mathbb{CP}^\infty)$ such that $K_0^*(\mathbb{CP}^\infty) = R[t, t^{-1}] [\![T_0]\!]$ with diagonal

$$\Delta(T_0) = F_0(T_0 \otimes 1, 1 \otimes T_0) \in R[t, t^{-1}] [\![T_0 \otimes 1, 1 \otimes T_0]\!] ,$$

where F_0 is a formal group law over $R[t, t^{-1}]$. Let $\overline{F}_0$ be the formal group law on R induced by $t \mapsto 1$, so it corresponds to the genus

$$MU^* \longrightarrow R[t, t^{-1}] \xrightarrow{t \mapsto 1} R.$$

Example. Let K_0 be complex K-theory with orientation $\beta(1 - [L])$. We saw earlier that then $F_0(X, Y) = X + Y - \beta^{-1}XY$ and $\overline{F}_0(X, Y) = X + Y - XY$.

Morava [Mor89, Section 3] proves that the assignment $K_0 \mapsto \overline{F}_0$ gives a fully faithful functor $\mathscr{K} \to \mathscr{FG}$ to the category of formal group laws, and he characterizes when a morphism of formal group laws arises form a morphism of normalized K-theories. This is the sense in which normalized K-theories are determined by their formal group laws.

Before stating this as a theorem, we should probably say what this so-called functor does to morphisms, and we should probably say a little more about what formal groups are. We will tackle these in reverse order.

Remark 1.19. Let $F(x,y) \in R[\![x,y]\!]$ be a formal group law over a ring R. Then, $(R[\![T]\!], \Delta_F)$ is a Hopf R-algebra with diagonal/comultiplication map $\Delta_F : R[\![T]\!] \to R[\![T \otimes 1, 1 \otimes T]\!]$ determined by

$$\Delta_F(T) = F(T \otimes 1, 1 \otimes T).$$

That Δ_F is a map of algebras is by construction; however, that it is co-commutative and co-associative and has a co-unit $R[\![T]\!] \to R$ (sending $T \mapsto 0$) exactly encodes the fact that F is a (commutative, 1-dimensional) formal group law. Thus, one can say that a **formal group** over R is a Hopf R-algebra G whose underlying algebra is isomorphic to $R[\![T]\!]$, and then a *choice* of $T \in G$ (i.e. choice of iso $G \xrightarrow{\sim} R[\![T]\!]$) determines a **formal group law**.

Neither of our categories, $\mathscr{K}$ and $\mathscr{FG}$, has a single ground ring, but the simplest morphisms $A \to B$ ($A, B \in \mathscr{K}$ or $A, B \in \mathscr{FG}$) are those where A, B are both defined over the same ring. To reduce to the case of dealing with these, we prove the following factorization lemmas.

Lemma 1.20.

(i) *Any morphism φ in $\mathscr{K}$ can be canonically factored in the form*

$$K_0^* \xrightarrow{\varphi''} K_{0,1}^* \xrightarrow{\varphi'} K_1^*,$$

where φ' induces the identity map $K_{0,1}^0(pt) = K_1^0(pt)$, and φ'' is an extension of scalars.

(ii) *A morphism $\psi : (A[\![T_0]\!], \Delta_0) \to (B[\![T_1]\!], \Delta_1)$ of Hopf algebras (i.e. of formal group laws) can be factored similarly as*

$$(A[\![T_0]\!], \Delta_0) \xrightarrow{\psi''} (B[\![T_0]\!], \Delta_0) \xrightarrow{\psi'} (B[\![T_1]\!], \Delta_1)$$

where ψ' is a morphism of formal group laws over B. In particular, since $\psi'(T_0) \in B[\![T_1]\!]$, there is a unique power series $\widetilde{\psi}$ s.t. $\widetilde{\psi}(T_1) = \psi'(T_0)$.

Proof. Use $K_{0,1}^*(X) = K_0^*(X) \otimes_{K_0^0(pt)} K_1^0(pt)$ in the first case. In the second case, ψ'' is induced by $\psi|_A : A \to B$. ∎

Definition 1.21. A morphism $\varphi : K_0^* \to K_1^*$ is called **strict** if $\varphi = \varphi'$, i.e. if $K_0^0(pt) \xrightarrow{\varphi} K_1^0(pt)$ is the identity.

Lemma 1.22. *Any morphism $\varphi : K_0^* \to K_1^*$ in $\mathscr{K}$ induces a morphism $\widetilde{\varphi} : \overline{F}_1 \to \overline{F}_0$ in $\mathscr{FG}$.*

Proof. By comparing canonical factorizations in $\mathscr{K}$ and $\mathscr{FG}$, we see that we may assume φ is strict. The proof is now completely formal. We have a commutative diagram[5]

$$\begin{array}{ccc} K_0^*(\mathbb{CP}(\infty)) & \xrightarrow{\Delta_0} & K_0^*(\mathbb{CP}(\infty))\widehat{\otimes}K_0^*(\mathbb{CP}(\infty)) \\ \varphi\big\downarrow & & \big\downarrow\varphi\otimes\varphi \\ K_1^*(\mathbb{CP}(\infty)) & \xrightarrow{\Delta_1} & K_1^*(\mathbb{CP}(\infty))\widehat{\otimes}K_1^*(\mathbb{CP}(\infty)) \end{array}.$$

Let $\widetilde{\varphi}'$ be the formal power series with coefficients in $K_1(pt)$ so that

$$\varphi(T_0) = \widetilde{\varphi}'(T_1) \in K_1^*(pt)\,[\![T_1]\!] \simeq K_1^*(\mathbb{CP}(\infty)).$$

The above diagram then says that

$$F_0(\widetilde{\varphi}'(T_1)\otimes 1, 1\otimes\widetilde{\varphi}'(T_1)) = F_0(\varphi(T_0)\otimes 1, 1\otimes\varphi(T_0)) = (\varphi\otimes\varphi)\Delta_0(T_0) = \Delta_1\varphi(T_0) = \Delta_1\widetilde{\varphi}'(T_1) = \widetilde{\varphi}'F_1(T_1\otimes 1, 1\otimes T_1),$$

i.e. that $\widetilde{\varphi}' : F_1 \to F_0$ is a morphism of formal group laws over $A[t,t^{-1}] = K_1(pt)$. Setting $t = 1$ then gives the desired morphism $\widetilde{\varphi} : \overline{F}_1 \to \overline{F}_0$ of formal group laws over A. ■

Now we have an actual functor. Morava proves

Theorem 1.23 ([Mor89], Sect. 3 Main Theorem)**.** *The functor $M : \mathscr{K} \to \mathscr{FG}$ sending a normalized ordinary K-theory to its underlying formal group law is faithful. Furthermore, given $K_0, K_1 \in \mathscr{K}$ with respective formal group laws $\overline{F}_0, \overline{F}_1$, a morphism $\overline{F}_0 \to \overline{F}_1$ is in the image of M iff its strict part is an isomorphism.*

Remark 1.24. This is (roughly) the statement Morava gives for his main theorem. It is a bit of a mouthful. If one let's $\mathscr{OG}$ denote the category of ordinary genera whose morphisms are strict isomorphisms between their corresponding formal group law, then Morava's main theorem can be more simply stated as: the functor $M : \mathscr{K} \to \mathscr{OG}$ defines an equivalence of categories.

Most of the work that goes into proving this result is completely formal. If you want to see the details, I recommend just taking a look at Morava's paper. We will not go over the proof here. However, we will offer some additional perspective on one piece of Morava's proof (his "main lemma") which appears more involved than the others. This is the following claim.

Lemma 1.25 ([Mor89], Main Lemma)**.** *For any $f(T) = T+$ higher order terms $\in A\,[\![T]\!]$, there is a natural (ungraded) ring homomorphism $[f] : MU^*(-)\otimes A \to MU^*(-)\otimes A$. In fact there is a natural action of the group $\Gamma_0(A)$ of all such formal power series on $MU^*(-)\otimes A$.*

Morava gives a hands on proof of this fact, explicitly constructing the morphism $[f]$. We offer the following two bullet points as potentially more conceptual routes for obtaining $[f]$.

- The ring $MU_*(MU)$ hosts the universal strict isomorphism[6] between formal group laws. Thus, $f(T)$ is pushed from a map $MU_*(MU) \to A$, and so $[f]$ is simply the composition

$$MU_*(-) \to MU_*(-) \otimes_{\pi_*(MU)} MU_*(MU) \to MU_*(-) \otimes_{\pi_*(MU)} A,$$

[5]Here, $\widehat{\otimes}$ is a completed tensor product. If R is complete with respect to the ideal I and S is complete with respect to J, then $R\widehat{\otimes}S$ is the completion of $R\otimes S$ with respect to the ideal $I\otimes 1 + 1\otimes J$. Technically Morava does not define this notation, but I think this definition works well enough for our purposes.

[6]An isomorphism between formal group laws is called 'strict' if its leading coefficient is 1 (as opposed to some other unit)

where the first map is the natural coaction of $MU_*(MU)$ on $MU_*(X)$, linearly extended to A. This gives $[f]$ in MU-homology, and obtaining it in MU-cohomology is now not difficult.

- Alternatively, one can informally think of $f(T) \in A[\![T]\!]$ as a choice of complex orientation, and so as a map from MU. This induces $MU \to MUA$ which then induces $[f]$. More formally...

  Let MUA be "MU with coefficients," that is $MUA = MU \wedge SA$ where SA is A's Moore spectrum. Now, given such a thing, we have (not necessarily split) short exact sequences [Ada95, Prop III.6.6]

$$0 \longrightarrow \pi_n(MU) \otimes A \longrightarrow \pi_n(MUA) \longrightarrow \mathrm{Tor}_1^{\mathbb{Z}}(\pi_{n-1}(MU), A) \longrightarrow 0$$

$$0 \longrightarrow MU_n(X) \otimes A \longrightarrow (MUA)_n(X) \longrightarrow \mathrm{Tor}_1^{\mathbb{Z}}(MU_{n-1}(X), A) \longrightarrow 0$$

  Since $\pi_*(MU)$ is nonzero only in even degrees, we see that $\pi_*(MUA) = \pi_*(MU) \otimes A$ is a polynomial algebra over A, so it supports a complex orientation x^{MU} (coming from the natural map $MU \to MU \wedge SA$) and $f(x^{MU})$ is another choice of said orientation. This then corresponds to a different map $[f] : MU \to MU \wedge SA$ (really [Ada95, Lemma II.4.6], a map $[f] : MUA \to MUA$ with $[f]_*(x^{MU}) = f(x^{MU})$). Further, since A is torsion-free, the Tor term vanishes so $MU_n(X) \otimes A \xrightarrow{\sim} MUA_n(X)$ always, and we get a map

$$MU_*(X) \otimes A \xrightarrow{\sim} MUA_*(X) \xrightarrow{[f]} MUA_*(X) \xleftarrow{\sim} MU_*(X) \otimes A.$$

2 Constructing The New Cohomology Theories

2.1 Results on the Classification of Formal Group Laws

Thanks to the previous section, we are essentially reduced to the problem of classifying and understanding certain formal group laws (those corresponding to ordinary genera). Luckily for us, other people have done the hard parts already.

The formal group laws we will want to study (i.e. quote results about) are those "of height one." This is a characteristic p phenomenon, so we start with a lemma on formal group laws in positive characteristic.

Lemma 2.1. *Let F be a formal group law over a ring R of characteristic p. Then there exists a formal power series φ such that $[p]_F(X) = \varphi(X^p)$, i.e. F's p-series only has terms whose exponents are p-powers.*

Proof. Let $F(T_1, T_2, \ldots, T_n)$ be defined inductively by $F(T_1, \ldots, T_{n-1}, T_n) = F(F(T_1, \ldots, T_{n-1}), T_n)$; e.g. $F(T, \ldots, T) = [n]_F(T)$ when T is repeated n times. By commutativity, $F(T_1, \ldots, T_n)$ is symmetric in $T_1, \ldots, T_n$ so can be written as $F_s(\sigma_1, \ldots, \sigma_n)$ with $\sigma_1, \ldots, \sigma_n$ the elementary symmetric functions of $T_1, \ldots, T_n$. If $T_1 = \cdots = T_n$, then

$$\sigma_k = \binom{n}{k} T^k.$$

In particular, if $n = p$, then $\sigma_1 \equiv 0, \ldots, \sigma_{p-1} \equiv 0, \sigma_p \equiv T^p \pmod{p}$. Thus over a ring of char. p, $[p]_F(T) = F_s(0, \ldots, 0, T^p)$. ∎

Definition 2.2. With F, R as above, we say that F is **of height one** if its p-series is of the form

$$[p]_F(T) = uT^p + \ldots \quad \text{with} \quad u \in R^\times.$$

If R is not of characteristic p, then we say F is **of height one at** p if its reduction to a group law over R/pR is of height one.

Warning 2.3. It is my understanding that usually one defines the height of F to be the least $n > 0$ s.t. T^{p^n} has *any nonzero* coefficient in $[p]_F(T)$, so there is no unit requirement. However, Morava makes the above definition, so we do too.

We will be concerned with formal group laws of height one.

Lemma 2.4. *Let F be a formal group law over R with corresponding genus $\rho : MU^* \to R$. Then, F is of height one at p iff ρ is ordinary at p.*

Proof. This follows from remark 1.13 which showed that the coefficient $v_{p,1}$ of T^p in the universal p-series over MU^* is $v_{p,1} \equiv \mathbb{CP}^{p-1} \pmod p$. ■

Now, we have the following useful facts (recall from the intro that our forms of K-theory will be defined over $\mathbb{Z}_p$).

Theorem 2.5 (Various people. TODO: add citations...)**.**

(1) *Two formal group laws of height one over a complete dvr R are isomorphic iff the induced laws over the residue field k are isomorphic.*

(2) *Two formal group laws of the same height over an algebraically closed field are isomorphic.*

(3) *If F is of height one over a field of char p, then the ring* $\mathrm{End}(F)$ *of endomorphisms of F is canonically isomorphic to $\mathbb{Z}_p$.*

One can use **(3)** to classify formal group laws over $\mathbb{F}_p$ (and its extensions), so also over $\mathbb{Z}_p$ (and its extensions).

Definition 2.6. Let F be a formal group law over $\mathbb{F}_p$. Then, $F(X,Y)^p = F(X^p, Y^p)$, so $\mathscr{F}(X) := X^p$ is an endomorphism of F, the **Frobenius endomorphism**. Since $\mathrm{End}(F) \simeq \mathbb{Z}_p$ (where the topology on the RHS comes from the usual valuation on formal power series) and the valuation of Frobenius is $v(\mathscr{F}) = 1$, there must exist some unique unit $u \in \mathbb{Z}_p^\times$ such that $\mathscr{F} = up$. It is straightforward to show that u is an isomorphism invariant of F, so let's call it the **invariant** of F. Note that u is also defined for F over $\mathbb{Z}_p$ in the expected way.

Given a formal group law F over $\mathbb{Z}_p$, this u is the unique p-adic unit so that $\exp_F(up \log_F x) \equiv x^p$ mod p.

Example. Say $F(x,y) = x + y - xy$ is the multiplicative group law, so $\log_F x = -\log(1-x)$ and $\exp_F x = 1 - e^{-x}$. Hence,

$$\exp_F(up \log_F x) = 1 - e^{up\log(1-x)} = 1 - (1-x)^{up} \equiv 1 - (1-x^p)^u \pmod p$$

from which we visibly see that the invariant of the multiplicative group is $u = 1$.

We end this section with one final result.

Theorem 2.7. *Two group laws of height one over $\mathbb{Z}_p$ are isomorphic iff they have the same invariant. Further, any invariant can arise: if $F_\alpha(X,Y)$, with $\alpha \in \mathbb{Z}_p^\times$, is the formal group law over $\mathbb{Z}_p$ defined by the ζ-function*

$$\zeta_\alpha(s) = \left(1 - \alpha p^{-s} + p^{1-2s}\right)^{-1},$$

then $\alpha = u^{-1} + pu$, where $u \in \mathbb{Z}_p^\times$ is the invariant defined above.

Remark 2.8. Assume $p \neq 2$. Note that the above theorem uniquely determines u, given α. One has $pu^2 - u\alpha + 1 = 0$, so u must be one of

$$\frac{\alpha \pm \sqrt{\alpha^2 - 4p}}{2p}.$$

Since $\alpha^2 - 4p \equiv \alpha^2 \pmod{p}$ is a square mod p (and $\alpha \neq 0$), Hensel's lemma gives some nonzero $\beta \in \mathbb{Z}_p^\times$ such that $\beta^2 = \alpha^2 - 4p$ and $\beta \equiv \alpha \pmod{p}$. Since u is integral, it cannot have any factors of p in its denominator, so we must have $u = (\alpha - \beta)/(2p)$ as $(\alpha + \beta)/(2p)$ does not lie in $\mathbb{Z}_p$ (since $\alpha + \beta \equiv 2\alpha \not\equiv 0 \pmod{p}$).

2.2 Forms of K-theory

We've spent all this time looking at formal group laws. The payoff is that now we can define new cohomology theories completely by magic.

Let W be the ring of Witt vectors over $\overline{\mathbb{F}}_p$., so W is the completion of the ring $W' = \mathbb{Z}_p[\zeta_n : p \nmid n]$.[7] The important thing to note is that this W is a complete dvr containing $\mathbb{Z}_p$ such that $W/pW = \overline{\mathbb{F}}_p$.

Warning 2.9. It is tempting to believe that $\mathbb{Z}_p[\zeta_n : p \nmid n]$ is already complete. However, see this answer to a question on Math Overflow.

Proposition 2.10. *Any two formal group laws of height* 1 *over W are isomorphic.*

Proof. This follows from Theorem 2.5 **(1)** and **(2)**. ■

We will now actually construct our first family. Theorem 1.7 tells us that the ζ-function

$$\zeta_\alpha(s) = \left(1 - \alpha p^{-s} + p^{1-2s}\right)^{-1}$$

defines a formal group law over $\mathbb{Z}_p$ when α is a p-adic integer. One can visibly see that the associated genus ρ_α satisfies $\rho_\alpha(\mathbb{CP}^{p-1}) = \alpha$ (note the sign change) and so is ordinary iff α is a p-adic unit. Thus, when $\alpha \in \mathbb{Z}_p^\times$, Landweber applies to tell us that

$$K_\alpha^*(X) := MU^*(X) \otimes_{\rho_{\alpha,*}} \mathbb{Z}_p[t, t^{-1}]$$

is an ordinary, normalized K-theory. Appealing to various other results, we obtain...

Theorem 2.11. *There is a family K_α ($\alpha \in \mathbb{Z}_p^\times$) of* 2*-periodic multiplicative cohomology theories, taking values in the category of $\mathbb{Z}_p$-modules, with the following properties*

[7] This W' is the ring of integers (i.e. the integral closure of $\mathbb{Z}_p$) in the maximal unramified (i.e. p remains prime in the ring of integers) extension of $\mathbb{Q}_p$.

(1)

$$K^q_\alpha(S^n) = \begin{cases} \mathbb{Z}_p & \textit{if } q \equiv n \pmod 2 \\ 0 & \textit{otherwise} \end{cases}.$$

(2) When $\alpha = 1+p$, K_{1+p} is canonically equivalent to complex K-theory, completed at p.

(3) If $\alpha, \beta \in \mathbb{Z}_p^\times$ are distinct, then K_α, K_β are not isomorphic (as multiplicative theories), but

(4) $K_\alpha(-)\otimes_{\mathbb{Z}_p} W$ and $K_\beta(-)\otimes_{\mathbb{Z}_p} W$ are isomorphic as multiplicative theories (though not canonically).

Indeed, **(1)** is by construction. For **(2)**, Theorem 2.7 tells us that K_{1+p}'s invariant u satisfies $u^{-1} + up = 1+p$, so $u = 1$ (the other solution is $u = 1/p$, but this is not in $\mathbb{Z}_p^\times$); the multiplicative group law also has $u = 1$, so K_{1+p} is (naturally isomorphic to) p-completed K-theory. **(3)** follows from Theorem 2.7 as well. Finally, **(4)** follows from the fact that W is (faithfully) flat over $\mathbb{Z}_p$ (so these are indeed cohomology theories) along with the fact that group laws over W are determined by their height at p.

In the end, we just had to write down a zeta function, and then some cohomology fell into our laps.

Remark 2.12. If one did not have access to Landweber's theorem, they could still show that the K_α defined above are cohomology theories using the Conner-Floyd result that $K(X) = MU^* \otimes_{\rho_*} \mathbb{Z}[\beta, \beta^{-1}]$ where K is complex K-theory. Indeed, given any ordinary genus $\rho : MU^* \to \mathbb{Z}_p$, the ring W is faithfully flat over $\mathbb{Z}_p$, so the functor $K_\rho : X \mapsto MU^* \otimes_{\rho_*} \mathbb{Z}_p[t, t^{-1}]$ is a cohomology theory (i.e. supports long exact sequences) iff the functor $K'_\rho : X \mapsto MU^* \otimes_{\rho_*} W[t, t^{-1}]$ is. But, we know that all ordinary genera over W are associated to the multiplicative group law, so K'_ρ is naturally isomorphic to p-completed complex K-theory, a cohomology theory, and so K_ρ itself must also be a cohomology theory.

Morava (at the very end of his section 4) has a different argument showing that ordinary genera over $\mathbb{Z}_p$ give rise to cohomology without using Landweber or Conner-Floyd. Sadly, though, I do not quite understand his argument (in particular, I do not see why an equivariant sheaf need to be locally constant; he makes it sound like this should be obvious...). If you read it and understand it, can you explain it to me?

2.3 Elliptic Cohomology Theories

There is at least one other source of formal group laws from which we might hope to eke out some cohomology theories: elliptic curves.

Remark 2.13. One normally sees a complex elliptic curve being presented as $\mathbb{C}/\Lambda$ for some lattice $\Lambda \subset \mathbb{C}$. It is not hard to show that, up to analyrc isomorphism, one may always assume $\Lambda = \Lambda_\tau := \mathbb{Z} \oplus \mathbb{Z}\tau$ for some $\tau \in \mathbb{H} := \{z \in \mathbb{C} : \operatorname{Im} z > 0\}$. In this case, note that one has an analytic isomorphism $\mathbb{C}/\Lambda_\tau \xrightarrow{\sim} \mathbb{C}^\times/q^{\mathbb{Z}}$, where $q = \exp(2\pi i\tau)$, descended from

$$\mathbb{C} \ni z \longmapsto \exp(2\pi i z) \in \mathbb{C}^\times.$$

Thus, every complex elliptic curve is of the form $E_q := \mathbb{C}^\times/q^{\mathbb{Z}}$ for some (non-unique) $q \in \mathbb{C}^\times$ with $|q| < 1$.

E_q is a (commutative, 1-dimensional) Lie group over $\mathbb{C}$. Hence, after choosing a suitable coordinate chart near the identity, one can express the addition law on E_q as a formal power series $F_q(x, y) \in \mathbb{C}[\![x, y]\!]$

over $\mathbb{C}$. This gives a family of formal group laws over $\mathbb{C}$, depending continuously on the parameter q. However, for any fixed value of q, we can actually the group law over a smaller ring.

Let

$$E_k(q) = 1 + (-1)^{k/2}\frac{2k}{b_k}\sum_{n\geq 0}\sigma_{k-1}(n)q^n$$

denote the normalized Eisenstein series of weight k, where b_k is a Bernoulli number, $k \geq 4$ is even and $\sigma_{k-1}(n) = \sum_{d|n,d\geq 1} d^{k-1}$. Let A_q denote the ring $\mathbb{Z}[E_4(q), E_6(q)] \subset \mathbb{Q}(E_4(q), E_6(q))$.

Theorem 2.14. *The group E_q determines a formal group law F_q with coefficients $a_{ij}(q) \in A_q$. If we adjoin q to the ring A_q, then we can construct an isomorphism of F_q with the multiplicative group.*

Morava attributes this theorem to Tate and Jacobi, but includes a sketch of a proof in his paper.

Given the above theory, we have a formal group law F_q defined over A_q, and so we potentially have a cohomology theory. Perhaps unsurprisingly at this point, F_q will give a cohomology theory defined over the ring $A_q[S_q^{-1}]$ where S_q is a set of "bad" primes, the ones at which F_q is not ordinary, and

$$A_q[S_q^{-1}] := \left\{\frac{a}{\prod_{p\in S_q} p^{e_p}} : a \in A_q, e_p \in \mathbb{Z}_{\geq 0} \text{ and } e_p = 1 \text{ for all but finitely many } p\right\}.$$

Thankfully, determining when F_q is ordinary at a prime p fits neatly into the theory of elliptic curves, and is actually the inspriation for the terminology "Ordinary K-theory."

Lemma 2.15. *The formal group law F_q is ordinary at a prime p (i.e. has height 1 at p) iff the elliptic curve is 'ordinary' at p in the usual sense from the theory of elliptic curves.*

Proof. This is [Sil09, Corollary IV.7.5] + [Sil09, Theorem V.3.1(a)]. ∎

Thus, to end, we have shown the following.[8]

Theorem 2.16. *Fix $q \in \mathbb{C}$ with $0 < |q| < 1$. Then, there is a multiplicative genus $\rho_q : \Omega^* \to \mathbb{Z}[E_4(q), E_6(q)]$ of almost-complex manifolds, and a corresponding cohomology theory K_q over a certain localization $A_q[S_q^{-1}]$ of $A_q = \mathbb{Z}[E_4(q), E_6(q)]$; this is the theory*

$$K_q^*(X) := MU^*(X) \otimes_{\rho_{q,*}} A_q[S_q^{-1}][t, t^{-1}].$$

Two values q, q' of the parameter give inequivalent functors as long as the modular invariants $j(q), j(q')$ are different; where $j(q) = q^{-1} + 744 + 196,844q + \dots$.

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[8]Here, the modular invariant $j(q) = j(E_q) \in \mathbb{C}$ is some complete invariant of complex elliptic curves

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