Outline

- Selmer Sets + Weak MW
- Descent obstruction
- Comparison with Brauer-Manin

(All cohomology is fppf cohomology and all torsors are fppf-locally trivial right torsors)

Notation 1. For any field k

- G/k will denote a smooth (affine) algebraic group

We will use the following notation when k is a global field

- v will denote a place of k. k_v the completion and $\mathscr{O}_v \subset k_v$ its ring of integers.
- S will denote a nonempty finite set of places of k which includes all the archimedean places
- $\mathbb{A}_S = \mathbb{A}_{k,S} := \prod_{v \in S} k_v \times \prod_{v \notin S} \mathscr{O}_v$ the S-adeles
- $\mathbb{A} = \mathbb{A}_k := \varinjlim_S \mathbb{A}_{k,S}$ the adeles

All the above on the board already at the start

1 Selmer

Recall 2 (Evaluation map)**.** Let k be a field, let X be a k-variety, and let G be a smooth algebraic group over k. Let $Z \xrightarrow{f} X$ be a G-torsor with class $\zeta = [Z] \in \mathrm{H}^1(X, G)$. Then, there is an evaluation map

variety = separated + f.type

$$\zeta : X(k) \longrightarrow \mathrm{H}^1(k, G)$$

defined via pullbacks, i.e. $\zeta(x)$ is represented by the fiber $Z_x \to \operatorname{Spec} k$ over x.

Now, given $\tau \in \mathrm{H}^1(k, G)$, represented by some $T \to \operatorname{Spec} k$, let $f^\tau : Z^\tau \to X$ be the corresponding twisted G^τ-torsor[1], i.e $Z^\tau := Z \overset{G_X}{\times}_X T_X^{-1} = Z \overset{G}{\times}_k T^{-1}$.[2]

Should assume G is affine so Z^τ is a scheme and not just an algebraic space

Theorem 3. *With notation as above,* $\{x \in X(k) : \zeta(x) = \tau\} = f^\tau(Z^\tau(k))$, *so*

$$X(k) = \bigsqcup_{\tau \in \mathrm{H}^1(k,G)} f^\tau(Z^\tau(k)).$$ ⊙

This reduces the problem of finding rational points on X to that of finding rational points on twists of some torsor over X. In general, $\mathrm{H}^1(k, G)$ can be rather big.[3] Hence, we'd like to know when we can actually reduce to only looking at finitely many twists.

Give example in footnote?

In example last week, only 2 different twists, corresponding to $\tau = 1, 2 \in \mathbb{Q}^\times/(\mathbb{Q}^\times)^2$

[1] $G^\tau = T \overset{G}{\times} G$ with $G \curvearrowright G$ via conjugation

[2] $G \curvearrowright T^{-1}$ on the left via acting by g^{-1} on the right. Note, T^{-1} is a $G - G^\tau$-bitorsor

[3] If $G = A[m]$ with $(m, \operatorname{char} k) = 1$ and A an abelian variety, and $L = k(A[m])$, then inflation-restriction gives $\mathrm{H}^1(k, A[m]) \to \mathrm{Hom}_{\mathrm{cts}}(G_L, \mathbb{Z}/m\mathbb{Z})^{2g}$ with finite kernel

Definition 4. Let k be a global field, and let G be a smooth algebraic group. Let $[Z \to X] = \zeta \in \mathrm{H}^1(X, G)$ be G-torsor over a k-variety X. We define the **Selmer set**

$$\mathrm{Sel}_Z(k, G) := \left\{ \tau \in \mathrm{H}^1(k, G) : \tau_v \in \mathrm{im}\left(X(k_v) \xrightarrow{\zeta} \mathrm{H}^1(k_v, G) \right) \text{ for all } v \right\}. \qquad \diamond$$

Example. Say $\varphi : A \to B$ is an isogeny of abelian varieties with kernel $G := \ker \varphi$. Then, φ is a G-torsor, and the exact sequence $0 \to G \to A \xrightarrow{\varphi} B \to 0$ gives rise to

Omit and leave as (easy) exercise? Yes

$$\begin{tikzcd}
0 \arrow[r] & B(k)/\varphi(A(k)) \arrow[r] \arrow[d] & \mathrm{H}^1(k, G) \arrow[r] \arrow[d] & \mathrm{H}^1(k, A)[\varphi] \arrow[r] \arrow[d] & 0 \\
0 \arrow[r] & \prod_v B(k_v)/\varphi(A(k_v)) \arrow[r] & \prod_v \mathrm{H}^1(k_v, G) \arrow[r] & \prod_v \mathrm{H}^1(k_v, A)[\varphi], \arrow[r] & 0
\end{tikzcd}$$

from which we see that

$$\mathrm{Sel}_A(k, G) = \left\{ \tau \in \mathrm{H}^1(k, G) : \tau_v \in \ker\left(\mathrm{H}^1(k_v, G) \to \mathrm{H}^1(k_v, A) \right) \text{ for all } v \right\} =: \mathrm{Sel}_\varphi(A)$$

agrees with the Selmer groups usually defined in the context of abelian varieties. $\triangle$

Note that (by the recall applied to τ_v and k_v)

$$\mathrm{Sel}_Z(k, G) = \left\{ \tau \in \mathrm{H}^1(k, G) : Z^\tau(k_v) \neq \emptyset \text{ for all } v \right\} \supset \left\{ \tau \in \mathrm{H}^1(k, G) : Z^\tau(k) \neq \emptyset \right\},$$

so

$$X(k) = \coprod_{\tau \in \mathrm{Sel}_Z(k, G)} f^\tau(Z^\tau(k)).$$

That is, we do not need to look at the whole (possibly infinite) set $\mathrm{H}^1(k, G)$ to obtain the rational points of X, but only the (smaller) set $\mathrm{Sel}_Z(k, G)$. To really be impressed by this, we better hope that $\mathrm{Sel}_Z(k, G)$ is actually small. Before proving that Selmer sets are finite, we'll need the following lemma.

Lemma 5 (Krasner's Lemma). *Let k be a local field. Let $f : Y \to X$ be a finite étale morphism of k-varieties. Then, the isomorphism type of the étale k-scheme $f^{-1}(x)$ is locally constant as x varies over $X(k)$ in the analytic topology.*

Proposition 6. *Let k be a local field. Let X be a proper k-variety, and let F be a finite étale algebraic group over k. Let $f : Z \to X$ be an F-torsor over X. Then, the image of $X(k) \to \mathrm{H}^1(k, F)$ is finite.*

Proof. Note that $Z \xrightarrow{f} X$ is a finite étale morphism since F is a finite étale k-scheme (descent: $Z \to X$ is fppf-locally isomorphic to $F \times_k X \to X$ by definition). Krasner's lemma then tells us that the isomorphism type of its fibers varies locally constantly, so evaluation $X(k) \to \mathrm{H}^1(k, F)$ is continuous when the target is given the discrete topology and the domain is given the analytic topology. Finally, $X(k)$ is compact, so the image is both compact and discrete. ∎

Only give proof sketch, and don't write down statement of Krasner.

Theorem 7. *Say X is a proper variety over the global field k. Then, $\mathrm{Sel}_Z(k, G)$ is finite.*

Proof. The key to finiteness will be that torsors coming from rational points will be unramified, owing to a comparison with the evaluation on integral points.

Let F be the (finite étale) component group of G. For a suitable finite nonempty set S of places containing all archimedean ones, we can spread out X to a proper $\mathscr{O}_{k,S}$-scheme $\mathcal{X}$, spread G out to a smooth f.type separated group scheme $\mathcal{G}$ over $\mathscr{O}_{k,S}$, and spread out Z to a $\mathcal{G}$-torsor over $\mathcal{X}$. Staring at the square

$$\begin{tikzcd} \mathcal{X}(\mathscr{O}_v) \arrow[r, "="] \arrow[d] & X(k_v) \arrow[d] \\ \mathrm{H}^1(\mathscr{O}_v, \mathcal{G}) \arrow[r] & \mathrm{H}^1(k_v, G) \end{tikzcd}$$

(for $v \notin S$) shows that $\mathrm{Sel}_Z(k,G) \subset \mathrm{H}^1_S(k,\mathcal{G})$. Consider now the map

classes unramified away from S. Unramified means in image of $\mathrm{H}^1(\mathscr{O}_v, \mathcal{G}) \to \mathrm{H}^1(k_v, G)$

$$\mathrm{H}^1_S(k,\mathcal{G}) \longrightarrow \prod_{v\in S} \mathrm{H}^1(k_v, F).$$

By a previous talk, this map has finite fibers. By Proposition 6, $\mathrm{Sel}_Z(k,G)$ has finite image under this map (since $\#S < \infty$). Taken together, we conclude that the Selmer set is finite. ■

Composition $\mathrm{H}^1_S(k,\mathcal{G}) \to \mathrm{H}^1_S(k,\mathcal{F}) \to \prod_v \mathrm{H}^1(k_v, F)$

Remark 8. In fact, it is possible to show that $\mathrm{Sel}_Z(k,G)$ is effectively computable. ○

Corollary 9. *There exists a finite separable extension k'/k such that $X(k) \subset f(Z(k'))$.*

Proof. Since $\mathrm{Sel}_Z(k,G)$ is finite, there exists some finite separable k' such that $f^\tau : Z^\tau \to X$ becomes isomorphic to $f : Z \to X$ over k' for all $\tau \in \mathrm{Sel}_Z(k,G)$. ■

Corollary 10 (**Weak Mordell-Weil**)**.** *Let A be an abelian variety over a global field k, and let m be a positive integer coprime to* $\mathrm{char}\, k$ *(so $A[m]$ is smooth!). Then, $A(k)/mA(k)$ is finite.*

Proof. Multiplication $[m] : A \to A$ gives A the structure of a $G = A[m]$-torsor over itself, so we get an evaluation map[4]

$$A(k) \longrightarrow \mathrm{H}^1(k, A[m])$$

with image contained in the (finite!) Selmer set $\mathrm{Sel}_A(k, A[m])$. The kernel of this map consists of those points $x \in A(k)$ for which $[m]^{-1}(x)$ is the trivial $A[m]$-torsor, i.e. for which $[m]^{-1}(x)$ has a k-point, i.e. for which $x \in mA(k)$. ■

Note 1. Time for break?

2 Descent Obstruction

Recall 11 (Obstructions from functors)**.** Let k be a global field with adele ring $\mathbb{A} = \mathbb{A}_k$. Let $F : \mathrm{Sch}_k^{\mathrm{op}} \to$ Set be a functor. Let X be a k-variety. Given $A \in F(A)$, we get an obstruction set

$$X(\mathbb{A})^A := \{x \in X(\mathbb{A}) : A(x) \in \mathrm{im}\,(F(k) \to F(\mathbb{A}))\} \supset X(k).$$

We then set $X(\mathbb{A})^F := \bigcap_{A \in F(X)} X(\mathbb{A})^A$, so $X(k) \subset X(\mathbb{A})^F \subset X(\mathbb{A})$. We say there is an F-obstruction to local-global if $X(\mathbb{A}) \neq \emptyset = X(\mathbb{A})^F$, and an F-obstruction to rational points if $X(\mathbb{A})^F = \emptyset$. ⊙

[4]This is a homomorphism, because addition induces an isomorphism $+ : A_x \overset{A[m]}{\times} A_y \xrightarrow{\sim} A_{x+y}$ of $A[m]$-torsors for $x, y \in A(k)$

The Brauer-Manin obstruction was obtained by applying this construction to $F = \mathrm{Br}$. We now apply it to $F = \mathrm{H}^1(-, G)$ for G a smooth (affine) algebraic group. With notation as in the recall, we define

$$\begin{aligned} X(\mathbb{A})^{\mathrm{H}^1(-,G)} &:= \bigcap_{\text{all } G\text{-torsors } Z \xrightarrow{f} X} X(\mathbb{A})^f \\ X(\mathbb{A})^{\mathrm{descent}} &:= \bigcap_{\text{all smooth affine } G} X(\mathbb{A})^{\mathrm{H}^1(-,G)}. \end{aligned}$$

Let's connect this to the first half of this talk.

Theorem 12. *Let k be a global field. Let X be a k-variety. Let G be a smooth affine algebraic group over k, and let $f : Z \to X$ be a G-torsor. Then,*

$$X(\mathbb{A})^f = \bigcup_{\tau \in \mathrm{H}^1(k,G)} f^\tau(Z^\tau(\mathbb{A})) = \bigcup_{\tau \in \mathrm{Sel}_Z(k,G)} f^\tau(Z^\tau(\mathbb{A})).$$

If X is furthermore proper, then $X(\mathbb{A})^f$ is closed in $X(\mathbb{A})$.

Proof. By the general formalism, we have

$$X(\mathbb{A})^f = \left\{ x \in X(\mathbb{A}) : Z_x \in \mathrm{im}\left(\mathrm{H}^1(k, G) \to \mathrm{H}^1(\mathbb{A}, G)\right)\right\}.$$

By Recall 2, for any $\tau \in \mathrm{H}^1(k, G)$, we have

$$\left\{ x \in X(\mathbb{A}) : [Z_x] = \tau_{\mathbb{A}} \in \mathrm{H}^1(\mathbb{A}, G)\right\} = f^\tau(Z^\tau(\mathbb{A})),$$

from which the first part of the theorem follows. For the second part, it suffices to show that $f^\tau(Z^\tau(\mathbb{A})) \subset X(\mathbb{A})$ is closed, and then remark that $\mathrm{Sel}_Z(k, G)$ is finite when X is proper. This is because $f^\tau : Z^\tau \to X$ is smooth and so open (even in the analytic topology), so

$$f^\tau(Z^\tau(\mathbb{A})) = X(\mathbb{A}) \setminus \bigcup_{\zeta \in \mathrm{H}^1(\mathbb{A},G)\setminus\{\tau\}} f^\zeta(Z^\zeta(\mathbb{A}))$$

is closed. ∎

Corollary 13. *If X is proper, then $X(\mathbb{A})^{descent}$ is closed in $X(\mathbb{A})$.*

This gives a description of the descent obstruction. How does it compare to our other favorite obstruction, the Brauer-Manin one? First note that in addition to the full descent obstruction, we can get weaker obstructions by restricting the set of algebraic groups under consideration, e.g.

$$X(\mathbb{A})^{\mathrm{PGL}} := \bigcap_{n \geq 1} X(\mathbb{A})^{\mathrm{PGL}_n}.$$

We claim that the PGL-obstruction equals the Brauer-Manin obstruction (so descent is stronger than Brauer-Manin).

Lemma 14. *Let G be a smooth algebraic group over a global field k. Then, the natural map*

Make exercise if low on time

$$\mathrm{H}^1(\mathbb{A}, G) \longrightarrow \prod_v \mathrm{H}^1(k_v, G)$$

is injective.

Proof. Let $Z \to \operatorname{Spec} \mathbb{A}$ be a G-torsor.[5] Spread this out to a G-torsor $Z_S \to \operatorname{Spec} \mathbb{A}_S$ for some finite set S of places including all archimedean ones. Suppose that $Z_S(k_v) \neq \emptyset$ for all v. We'll show that $Z_S(\mathbb{A}_S)$ is nonempty (so also is $Z_S(\mathbb{A}) = Z(\mathbb{A})$). Choose some $(z_v)_v \in \prod_v Z_S(k_v)$. For each $v \notin S$, since $G(k_v) \curvearrowright Z_S(k_v)$ transitively, we can choose some $g_v \in G(k_v)$ so that $z_v \cdot g_v \in \operatorname{im}(Z_S(\mathscr{O}_v) \to Z_S(k_v))$. For $v \in S$, we set $g_v = 1 \in G(k_v)$. Then, $(z_v \cdot g_v)_v \in Z_S(\mathbb{A}_S)$. ∎

This let's us replace $\mathrm{H}^1(\mathbb{A}, G)$ with $\prod_v \mathrm{H}^1(k_v, G)$ in the definition of our obstruction sets.

Fact. $\operatorname{Br}(\mathbb{A}) \simeq \bigoplus_v \operatorname{Br}(k_v) \hookrightarrow \prod_v \operatorname{Br}(k_v)$.

Lemma 15. *Let k be a global field, and let X be a k-variety. Let $Z \xrightarrow{f} X$ be an PGL_n-torsor, and let $A \in \operatorname{Br}(X)$ be its associated Brauer class (image under[6] $\mathrm{H}^1(X, \mathrm{PGL}_n) \to \operatorname{Br}(X)[n]$). Then, $X(\mathbb{A})^f = X(\mathbb{A})^A$.*

Proof. Fix $x = (x_v) \in X(\mathbb{A})$, and consider the commutative diagram

$$\begin{array}{ccccccc}
[f] & \in & \mathrm{H}^1(X, \mathrm{PGL}_n) & \longrightarrow & \operatorname{Br}(X)[n] & \ni & A \\
 & & \big\downarrow x & & \big\downarrow x & & \\
f(x) & \in & \prod_v \mathrm{H}^1(k_v, \mathrm{PGL}_n) & \xrightarrow{(1)} & \prod_v \operatorname{Br}(k_v)[n] & \ni & A(x) \\
 & & \big\uparrow & & \big\uparrow & & \\
 & & \mathrm{H}^1(k, \mathrm{PGL}_n) & \xrightarrow{(2)} & \operatorname{Br}(k)[n] & &
\end{array}$$

Commutativity directly shows that $X(\mathbb{A})^f \subset X(\mathbb{A})^A$. To get the other inclusion, it would suffice to show that (1) is injective and (2) is surjective (in fact, both are isomorphisms). Injectivity of (1) follows from the exact sequences

$$1 = \mathrm{H}^1(k_v, \mathrm{GL}_n) \longrightarrow \mathrm{H}^1(k_v, \mathrm{PGL}_n) \longrightarrow \mathrm{H}^2(k_v, \mathbb{G}_m) = \operatorname{Br}(k_v).$$

Surjectivity of (2) is more involved and is omitted.[7] ∎

Lemma 16. $\mathrm{H}^1(k_v, \mathrm{PGL}_n) \to \operatorname{Br}(k_v)[n]$ *is an isomorphism*

Corollary 17. $X(\mathbb{A})^{descent} \subset X(A)^{\mathrm{PGL}} = X(\mathbb{A})^{\mathrm{Br}}$ *when X is a regular, quasi-proj variety over a global field k.*

Proof. The hypotheses on X ensure that every Brauer element comes from a PGL_n-torsor for some n. ∎

Can we get an obstruction even stronger than descent? Recall

$$X(\mathbb{A})^{\text{descent}} = \bigcap_{\text{all smooth affine } G} \ \bigcap_{\text{all } G\text{-torsors } Z \xrightarrow{f} X} \ \bigcup_{\tau \in \mathrm{H}^1(k,G)} f^\tau(Z^\tau(\mathbb{A})).$$

[5] $Z \to \operatorname{Spec} A$ is finitely presented by descent with G is locally of finite presentation, quasi-compact, and (quasi)separated

[6] Compare $1 \to \mathbb{G}_m \to \mathrm{GL}_n \to \mathrm{PGL}_n \to 1$ and $1 \to \mu_n \to \mathrm{SL}_n \to \mathrm{PSL}_n \to 1$ (note $\mathrm{PGL}_n = \mathrm{PSL}_n$ as algebraic groups)

[7] Prove (1) is surjective by hand using that $\operatorname{Br}(k_v) \hookrightarrow \mathbb{Q}/\mathbb{Z}$. Then use class field theory

Define

$$X(\mathbb{A})^{\text{descent, descent}} = \bigcap_{\text{all smooth affine } G} \bigcap_{\text{all } G\text{-torsors } Z \xrightarrow{f} X} \bigcup_{\tau \in \mathrm{H}^1(k,G)} f^\tau(Z^\tau(\mathbb{A})^{\text{descent}}).$$

Theorem 18 (Yang Cao '17)**.** *For any smooth quasi-projective geometrically integral variety X over a number field,*

$$X(\mathbb{A})^{descent} = X(\mathbb{A})^{descent,descent}$$