1 Sites and Sheaves

We want to define "étale cohomology" but that requires a notion of "étale sheaves" which in turn require an "étale topology."

Definition 1. Let $\mathcal{C}$ be a category with all fiber products. A **Grothendieck topology** on $\mathcal{C}$ is that data of, for each object $X \in \mathrm{ob}(\mathcal{C})$, a collection $\mathrm{Cov}(X)$ of sets $\{X_\alpha \to X\}_\alpha$ of morphsims, called **covering families**, such that

Have written on jamboard at start already

- "intersections of covers are covers"

$$\{X_\alpha \to X\} \in \mathrm{Cov}(X) \implies \{X_\alpha \times_X Y \to Y\} \in \mathrm{Cov}(Y)$$

  for any $Y \to X$

- "refinements of covers are covers"

$$\{X_\alpha \to X\} \in \mathrm{Cov}(X) \text{ and } \{X_{\alpha\beta} \to X_\alpha\} \in \mathrm{Cov}(X_\alpha) \text{ for each } \alpha \implies \{X_{\alpha\beta} \to X_\alpha \to X\}_{\alpha,\beta} \in \mathrm{Cov}(X)$$

- "isomorphisms are covers"

$$\{Y \xrightarrow[f]{\sim} X\} \in \mathrm{Cov}(X) \text{ if } f \text{ an iso}$$

A category equipped with the data of a Grothendieck topology is called a **site**, and usually denoted τ.

Example. Let X be a scheme. The **small Zariski site** on X, denote X_{zar} or $\mathrm{Open}(X)$, is the category of Zariski open embeddings $U \hookrightarrow X$ (objects are $U \hookrightarrow X$ open immersion, and morphisms are $V \hookrightarrow U$ open immersions) where $\{U_\alpha \hookrightarrow U\}$ is a cover if $\bigcup U_\alpha = U$. △

The other examples we'll care about are the flat and étale sites.

Definition 2. A morphism $f : X \to Y$ of schemes is called

- **fppf** if it is faithfully flat and locally of finite presentation.
- **fpqc** if it is faithfully flat and every qc open $V \subset Y$ is the image of some qc open $U \subset X$.

This suffices for descent since it's all that's need to reduce to the affine case

Definition 3.

- A **Zariski open cover** is a collection $\{Y_\alpha \hookrightarrow Y\}$ of open immersions so that $\bigcup Y_\alpha = Y$.
- An **étale cover** is a collection $\{f_\alpha : Y_\alpha \to Y\}$ so that each f_α is étale and $\bigcup f_\alpha(Y_\alpha) = Y$.
- An **fppf cover** is a collection $\{f_\alpha : Y_\alpha \to Y\}$ so that $\bigsqcup f_\alpha : \bigsqcup Y_\alpha \to Y$ is fppf.
- An **fpqc cover** is a collection $\{f_\alpha : Y_\alpha \to Y\}$ so that $\bigsqcup f_\alpha : \bigsqcup Y_\alpha \to Y$ is fpqc.

Remark 4. Zariski open cover $\implies$ étale cover $\implies$ fppf cover $\implies$ fpqc cover.

For the last implication, flat + locally of finite presentation $\implies$ open. Let $\{U_i\}_{i\in I}$ be an affine cover of $f^{-1}(V)$. Then, $V = \bigcup f(U_i)$ has a finite open subcover, indexed by $F \subset I$. Set $U = \bigcup_{i\in F} U_i$. ○

We now define various sites.

Definition 5. Let Sch/X denote the category of X schemes.

- The **small étale site** on X, denoted $X_{\text{ét}}$, is the full subcategory of Sch/X consisting of étale X-schemes, and whose covers are étale covers.
- The **big étale site** on X, denoted $X_{\text{Ét}}$, is the category Sch/X equipped with étale covers.
- The **big Zariski site** on X, denoted X_{Zar}, is the category Sch/Z equipped with Zariski open covers.
- The **fppf site** on X, denoted X_{Fppf}, is the category Sch/X equipped with fppf covers.
- The **fpqc site** on X, denoted X_{Fpqc}, is the category Sch/X equipped with fpqc covers.

We really only care about the fppf, small étale, and small Zariski sites.

Example. If $f : X \to Y$ is a morphism of schemes, then taking preimages defines a natural functor $f^{-1} : Y_{\mathrm{zar}} \to X_{\mathrm{zar}}$ which sends covers to covers. △

Definition 6. Let τ_1, τ_2 be sites. A **continuous map** $f : \tau_1 \to \tau_2$ is a functor $f^{-1} : \tau_2 \to \tau_1$ which preserves fiber products and open coverings.

A morphism requires the inverse image functor on the category of sheaves to be exact. Bjorn claims the examples below are morphisms, and sites a non-existent stacks project tag

Example. There are natural continuous maps of sites (all the functors do nothing)

$$\begin{tikzcd} X_{\mathrm{Fpqc}} \arrow[r] & X_{\mathrm{Fppf}} \arrow[r] & X_{\text{Ét}} \arrow[r] \arrow[d] & X_{\mathrm{Zar}} \arrow[d] \\ & & X_{\text{ét}} \arrow[r] & X_{\mathrm{zar}} \end{tikzcd}$$

△

1.1 Sheaves

Definition 7. Let $\mathcal{C}, \mathcal{D}$ be categories. A **$\mathcal{D}$-valued presheaf on** $\mathcal{C}$ is a contravariant functor $F : \mathcal{C} \to \mathcal{D}$.

Definition 8. Let τ be a site. A **sheaf** $\mathscr{F}$ on τ is a presheaf $\mathscr{F}$ such that

$$\mathscr{F}(U) \longrightarrow \prod_{\alpha} \mathscr{F}(U_\alpha) \rightrightarrows \prod_{(\alpha,\beta)} \mathscr{F}(U_\alpha \times_U U_\beta)$$

is an equalizer diagram for all covers $\{U_\alpha \to U\}$ in τ. If $\{U' \to U\}$ is a single morphism, then the above becomes

$$\mathscr{F}(U) \longrightarrow \mathscr{F}(U') \rightrightarrows \mathscr{F}(U' \times_U U').$$

Exactness on the left says that sections are determined locally and exactness on the right (the middle?) says that sections glue.

Example. Let X be a scheme with qcoh sheaf $\mathscr{F}$. Consider the presheaf $\mathscr{F}_{\mathrm{Fpqc}}$ on X_{Fpqc} defined by

$$\mathscr{F}_{\mathrm{Fpqc}}(Y \xrightarrow{f} X) := \Gamma(Y, f^*\mathscr{F}).$$

This is a sheaf. Indeed, if $S' \xrightarrow{p} S$ is fpqc (with $S \xrightarrow{f} X$ an X-scheme) and $\mathscr{F}_S := f^*\mathscr{F}$, then descent tells us that

$$\mathrm{Hom}_{\mathscr{O}_S}(\mathscr{G}, \mathscr{F}_S) \longrightarrow \mathrm{Hom}_{\mathscr{O}_{S'}}(p^*\mathscr{G}, p^*\mathscr{F}_S) \rightrightarrows \mathrm{Hom}_{\mathscr{O}_{S''}}(q^*\mathscr{G}, q^*\mathscr{F}_S)$$

is an equalizer for all qcoh $\mathscr{G}$ on S. Taking $\mathscr{G} = \mathscr{O}_S$ turns this to

$$\mathscr{F}_{\mathrm{Fpqc}}(S) \longrightarrow \mathscr{F}_{\mathrm{Fpqc}}(S') \rightrightarrows \mathscr{F}_{\mathrm{Fpqc}}(S''),$$

and exactness of this is precisely the sheaf condition. (Get sheaves on the other sites via push forward) △

Example. Let X be a scheme, and let $Z \to X$ be an X-scheme. Consider the presheaf h_Z on X_{Fpqc} defined by

$$h_Z(Y \to X) := \mathrm{Hom}_{\mathrm{Sch}_X}(Y, Z).$$

This is a sheaf. Morally equivalently, if $X' \to X$ is fpqc, then the functor from X-schemes to X'-schemes with descent datum is fully faithful. We need

$$\begin{tikzcd}
\mathrm{Hom}_S(S, Z_S) \arrow[r] \arrow[d, equal] & \mathrm{Hom}_{S'}(S', Z_{S'}) \arrow[r, shift left] \arrow[r, shift right] \arrow[d, equal] & \mathrm{Hom}_{S''}(S'', Z_{S''}) \arrow[d, equal] \\
\mathrm{Hom}_X(S, Z) \arrow[r] & \mathrm{Hom}_X(S', Z) \arrow[r, shift left] \arrow[r, shift right] & \mathrm{Hom}_X(S'', Z)
\end{tikzcd}$$

exact for all fpqc $S' \to S$. We can replace Z with an affine cover to assume $Z = \operatorname{Spec} R$ is affine, so Z_S is affine over S, i.e. $Z_S = \mathbf{Spec}\,\mathscr{A}$ for some sheaf of $\mathscr{O}_S$-algebras $\mathscr{A}$ (and similarly for $Z_{S'}, Z_{S''}$). Thus, the above becomes

We know h_Z is a Zariski sheaf

$$\begin{tikzcd}
\mathrm{Hom}_S(S, Z_S) \arrow[r] \arrow[d, equal] & \mathrm{Hom}_{S'}(S', Z_{S'}) \arrow[r, shift left] \arrow[r, shift right] \arrow[d, equal] & \mathrm{Hom}_{S''}(S'', Z_{S''}) \arrow[d, equal] \\
\mathrm{Hom}_{\mathscr{O}_S}(\mathscr{A}, \mathscr{O}_S) \arrow[r] & \mathrm{Hom}_{\mathscr{O}_{S'}}(\mathscr{A}', \mathscr{O}_{S'}) \arrow[r, shift left] \arrow[r, shift right] & \mathrm{Hom}_{\mathscr{O}_{S''}}(\mathscr{O}_{S''}, \mathscr{A}'')
\end{tikzcd}$$

which is exact by descent for qcoh sheaves. △

Remark 9. Descent for qcoh sheaves is always effective. Descent for schemes is usually not effective. However, as the above suggests, it will be effective when your schemes are determined by qcoh sheaves (e.g. descending affine schemes or closed subschemes). More generally, if S, S' are affine, descent datum φ on an S'-scheme X' is effective iff X' can be covered by quasi-affine open subschemes which are stable under φ. ○

i.e. φ restricts to $p_1^*U \xrightarrow{\sim} p_2^*U$

Example. Commutative algebraic groups gives abelian sheaves

- $\mathbb{G}_m : U \mapsto \mathscr{O}_U(U)^\times$
- $\mu_\ell : U \mapsto \{f \in \mathscr{O}_U(U) : f^\ell = 1\}$
- $\underline{\mathbb{Z}/\ell\mathbb{Z}} : U \to \mathrm{Hom}_{cts}(U, \mathbb{Z}/\ell\mathbb{Z})$
- $\alpha_p : U \to \{f \in \mathscr{O}_U(U) : f^p = 0\}$ for $\mathbb{F}_p$-schemes

are all sheaves. △

Proposition 10. *Let τ be a site. There is a sheafification functor* $\mathrm{Psh}(\tau) \to \mathrm{Sh}(\tau)$ *which is left adjoint to the forgetful functor* $\mathrm{Sh}(\tau) \to \mathrm{Psh}(\tau)$.

Proof Sketch. Let F be a presheaf on τ. Consider the presheaf F^+ defined by

“What I’m showing you is the real thing; it’s the X-rated version.” - Brian Conrad

$$F^+(U) := \varinjlim_{\{U_\alpha \to U\} \in \mathrm{Cov}(U)} \mathrm{Eq}\left(\prod_\alpha F(U_\alpha) \rightrightarrows \prod_{(\alpha,\beta)} F(U_\alpha \times_U U_\beta)\right)$$

(covers ordered by refinement). Then, F^{++} is the sheafification of F. ■

Corollary 11. *The category* $\mathrm{Ab}(\tau)$ *of abelian sheaves is an abelian category.*

(Use sheafification to get cokernels)

Definition 12. Let $f : \tau_1 \to \tau_2$ be a continuous map of sites. Get a **pushforward** functor $f_* : \mathrm{Sh}(\tau_1) \to \mathrm{Sh}(\tau_2)$ via

$$(f_*\mathscr{F})(U) = \mathscr{F}(f^{-1}(U)).$$

Also get an **inverse image functor** $f^* : \mathrm{Sh}(\tau_2) \to \mathrm{Sh}(\tau_1)$ which is left adjoint to pushforward.

sheafification of limit over $V \to f^{-1}(U)$

1.1.1 Stalks and Enough Injectives in the étale topology

I wanna be brief about this stuff. In the Zariski topology on X, if you zoom in as far as possible, you get a point $x \in X$. In the étale topology, you can zoom in even further on a point by taking separable field extensions.

Definition 13. Let $\overline{x} \to X$ be a geometric point of a scheme X, and let $\mathscr{F}$ be a sheaf on $X_{\text{ét}}$. The **stalk** $\mathscr{F}_{\overline{x}}$ of $\mathscr{F}$ at $\overline{x}$ is the colimit

$$\mathscr{F}_{\overline{x}} := \varinjlim_{(U,\overline{u})} \mathscr{F}(U)$$

over diagrams

$$\begin{tikzcd} \overline{u} \arrow[r] \arrow[d] & U \arrow[d] \\ \overline{x} \arrow[r] & X \end{tikzcd}$$

(with $U \to X$ étale). This is equivalently, the pullback of $\mathscr{F}$ along $\overline{x} \to X$.

Fact. A sequence $\mathscr{F}' \to \mathscr{F} \to \mathscr{F}''$ of abelian sheaves on $X_{\text{ét}}$ is exact iff it's exact on all stalks (iff it's exact on one stalk above each scheme point of X).

Proposition 14. *Let τ be a site. The category* $\mathrm{Ab}(\tau)$ *of abelian sheaves on τ has enough injectives.*

Proof Sketch when $\tau = X_{ét}$. Let $\mathscr{F}$ be a sheaf on $X_{\text{ét}}$. Choose a geometric point $\iota_{\overline{x}} : \overline{x} \to X$ over each scheme point $x \in X$. Choose an embedding $\mathscr{F}_{\overline{x}} \hookrightarrow I(\overline{x})$ into an injective abelian group. Let $\mathscr{I} := \prod_{\overline{x}} \iota_{\overline{x},*} I(\overline{x})$. Note that, for any sheaf $\mathscr{G}$, one has

$$\mathrm{Hom}(\mathscr{G}, \mathscr{I}) = \prod_{\overline{x}} \mathrm{Hom}(\mathscr{G}, \iota_{\overline{x},*} I(\overline{x})) = \prod_{\overline{x}} \mathrm{Hom}(\mathscr{G}_{\overline{x}}, I(\overline{x}))$$

from which we see that $\mathrm{Hom}(-, \mathscr{I})$ is exact and that there's a natural injective $\mathscr{F} \hookrightarrow \mathscr{I}$ (check injectivity on stalks). ■

Corollary 15. *For $\mathscr{F}$ a sheaf on $X_{\acute{e}t}$, we get cohomology groups,* $\mathrm{H}^i(X_{\acute{e}t}, \mathscr{F})$.

2 Cohomology Computations

2.1 Cohomology of a point

Let k be a field, and let $G = \mathrm{Gal}(k^s/k)$.

Proposition 16. *There is an equivalence of (abelian) categories*

$$\begin{array}{rccc} \iota: & \mathrm{Ab}((\mathrm{Spec}\, k)_{\acute{e}t}) & \longrightarrow & \textit{Discrete } G\textit{-modules} \\ & \mathscr{F} & \longmapsto & \varinjlim_{k^s/L/k} \mathscr{F}(\mathrm{Spec}\, L) =: \mathscr{F}(k^s) \end{array}$$

In the opposite direction, say M is a discrete G-module. If $V \to \mathrm{Spec}\, k$ is étale, then $V = \bigsqcup \mathrm{Spec}(k_i)$ for some finite separable k_i/k. We send M to the sheaf $\mathscr{F}_M$ with

$$\mathscr{F}_M(V \to \mathrm{Spec}\, k) := \prod_i M^{\mathrm{Gal}(k^s/k_i)}.$$

Corollary 17.

$$\mathrm{H}^i(\mathrm{Spec}\, k_{\acute{e}t}, \mathscr{F}) = \mathrm{H}^i_{Grp}(G, \mathscr{F}(k^s))$$

Proof. These are both right derived functors of $\mathrm{H}^0(\mathrm{Spec}\, k_{\text{ét}}, \mathscr{F}) = \mathscr{F}(k) = \mathscr{F}(k^s)^G$. ∎

2.2 Čech cohomology

Say we have a cover $U = \bigsqcup_i U_i \to X$ in X_* where $* \in \{\text{ét, Fppf, zar, ...}\}$, and let $\mathscr{F} \in \mathrm{Sh}(X_*)$ be an abelian sheaf. From this, we get the diagram (a simplicial X-sheaf?)

$$X \longleftarrow U \leftleftarrows U \times_X U \Lleftarrow U \times_X U \times_X U \;\cdots$$

(think of as double, triple, etc. intersections). Applying our sheaf, we get a diagram (a cosimplicial sheaf of abelian groups)

$$\mathscr{F}(U) \underset{\mathrm{d}^1}{\overset{\mathrm{d}^0}{\rightrightarrows}} \mathscr{F}(U \times_X U) \Rrightarrow \mathscr{F}(U \times_X U \times_X U) \;\cdots$$

The usual alternating sum construction flattens this into a chain complex, called the **Čech complex**,

$$\check{C}^\bullet(U/X, \mathscr{F}) : 0 \to \mathscr{F}(U) \to \mathscr{F}(U \times_X U) \to \cdots$$

whose nth differential

$$\mathrm{d}_n = \sum_{i=0}^{n} (-1)^i d^i$$

is given by the alternating sum of the differentials in the diagram from before. The **total Čech complex** is

$$\check{C}^\bullet(X_{\text{ét}}, \mathscr{F}) := \varinjlim_{\{U_i \to X\}_i} \check{C}^\bullet(U/X, \mathscr{F}),$$

with direct limit taken over all covering families.

Definition 18. We define **Čech cohomology** as

$$\check{\mathrm{H}}^i(U/X, \mathscr{F}) = \mathrm{H}^i(\check{C}^\bullet(U/X, \mathscr{F}))$$

and

$$\check{\mathrm{H}}^i(X_*, \mathscr{F}) = \mathrm{H}^i(\check{C}^\bullet(X_*, \mathscr{F})).$$

Proposition 19. $\check{\mathrm{H}}^1(X_*, \mathscr{F}) = \mathrm{H}^1(X_*, \mathscr{F})$ *always.*

Proposition 20. $\check{\mathrm{H}}^i(U/X, \mathscr{I}) = \check{\mathrm{H}}^i(X_*, \mathscr{I}) = 0$ *if* $i > 0$ *and* $\mathscr{I}$ *is injective.*

Theorem 21 (Milne, III)**.** *If X is quasi-compact, and if any finite subset of X is contained in an affine (e.g. if X is quasi-projective), then Čech cohomology computes étale cohomology for all i.*

(These conditions guarantee that Čech cohomology is a δ-functor or whatever)

2.3 H^1, torsors, and twists

Let X be a scheme, and let $*$ be a topology on X (e.g. $* =$ét).

Definition 22. Let G be a sheaf of groups on X (e.g. G could be representable by a group scheme). A G**-torsor** is a sheaf $T \in \mathrm{Sh}(X_*)$ equipped with a G-action $a : G \times_X T \to T$ such that

$$\begin{array}{rccc} (a, \pi_2): & G \times_X T & \longrightarrow & T \times_X T \\ & (g, t) & \longmapsto & (gt, t) \end{array}$$

is an isomorphism. Equivalently, the induced action $G(S) \curvearrowright T(S)$ is free and transitive for all $S \to X$.

Example. The trivial G-torsor is G acting on itself by multiplication. Note that any G-torsor automorphism of G is given by multiplication by some $g \in G(X)$. △

Remark 23. $G \times_X T \xrightarrow{\sim} T \times_X T$, so any G-torsor T becomes trivial after pulling back to itself. More generally, T is trivial over $S \to X$ if and only if $T(S)$ is nonempty. ◦

Definition 24. We say a G-torsor T is **split by a cover** $U \to X$ if T_U is trivial, i.e. if $T(U) \neq \emptyset$. We say T is **locally split** if it's split by some cover.

Remark 25. If G is an affine group scheme, then any locally split G-torsor is representable (by a scheme). ◦

Proposition 26. *There is a natural bijection*

$$\left\{ \begin{array}{c} G\text{-torsors } T \\ \text{split by } \mathcal{U} \to X \end{array} \right\} \longleftrightarrow \check{\mathrm{H}}^1(\mathcal{U}/X, G).$$

Proof Sketch. Say $\varphi : T|_{\mathcal{U}_{\text{ét}}} \xrightarrow{\sim} G|_{\mathcal{U}_{\text{ét}}}$ as torsors. Let $\pi_1, \pi_2 : \mathcal{U} \times_X \mathcal{U} \rightrightarrows \mathcal{U}$ be the two projections. Since T is a sheaf on $X_{\text{ét}}$, we get a commutative diagram

$$\begin{tikzcd} \pi_1^* T|_{\mathcal{U}_{\text{ét}}} \arrow[r, "\sim"] \arrow[d, "\pi_1^*\varphi"'] & \pi_2^* T|_{\mathcal{U}_{\text{ét}}} \arrow[d, "\pi_2^*\varphi"] \\ \pi_1^* G \arrow[r, "\sim"] & \pi_2^* G. \end{tikzcd}$$

The bottom arrow in the above diagram is a G-torsor automorphism over $U \times_X U$, so is multiplication by some element of $\Gamma(\mathcal{U} \times_X \mathcal{U}, G)$. This will be a Čech 1-cocycle, and so give a cohomology class. ■

Corollary 27. *There is an isomorphism*

$$\{\textit{locally trivial } G\textit{-torsors}\} \xrightarrow{\sim} \check{\mathrm{H}}^1(X_*, G).$$

Corollary 28. *Let G be a finite abstract group. Let $\underline{G}$ be the associated constant sheaf on $X_{ét}$. Then,*

$$\mathrm{H}^1(X_{ét}, \underline{G}) = \check{\mathrm{H}}^1(X_{ét}, \underline{G}) = \mathrm{Hom}_{cts}(\pi_1^{ét}(C), G)$$

Remark 29. More generally, if you have Y and $G = \underline{\mathrm{Aut}}(Y)$ is its automorphism sheaf, $\check{\mathrm{H}}^1(U/X, G)$ is the set of descent data for sheaves which become isomorphic to Y over U, so $\check{\mathrm{H}}^1(X_*, G)$ is the set of twists of Y (assuming the relevant descent problem is always effective). ○

Example (Hilbert 90). An (fppf or étale) GL_n-torsor is the same thing as a rank n vector bundle (i.e. a twist of $\mathscr{O}^{\oplus n}$), so $\check{\mathrm{H}}^1(X_*, \mathrm{GL}_n)$ is the set of rank n vector bundles. In particular,

uses faithful flat descent for vector bundles

$$\mathrm{H}^1(X_{\text{ét}}, \mathbb{G}_m) = \mathrm{H}^1(X_{\text{Fppf}}, \mathbb{G}_m) = \mathrm{H}^1(X_{\text{zar}}, \mathbb{G}_m) = \mathrm{Pic}(X)$$

Explicitly, if $\mathscr{V}$ is a rank n vector bundle, the sheaf

$$(U \xrightarrow{f} X) \mapsto \mathrm{Isom}(f^*\mathscr{V}, \mathscr{O}_U^{\oplus n})$$

is a $\mathrm{GL}_n = \underline{\mathrm{Aut}}(\mathscr{O}_U^{\oplus n})$-torsor. If T is a GL_n-torsor, then the quotient

$$T \overset{\mathrm{GL}_n}{\times} \mathscr{O}_X^{\oplus n} := (T \times \mathscr{O}_X^{\oplus n}) / \mathrm{GL}_n$$

is a vector bundle (GL_n acts diagonally). △

(The above is false for other algebraic groups)

Corollary 30. *Let k be a field with absolute Galois group G_k. Then,*

$$\mathrm{H}^1_{Grp}(G_k, (k^s)^\times) = \mathrm{H}^1(\mathrm{Spec}\, k_{ét}, \mathbb{G}_m) = \mathrm{Pic}(k) = 1$$

2.3.1 Curve computations

Let $k = \overline{k}$ be a field of characteristic $p > 0$. Let $\ell \neq p$ be a prime. Let C/k be a smooth, proper, geometrically integral k-curve. Note that $\mu_\ell \cong \underline{\mathbb{Z}/\ell\mathbb{Z}}$ as k-group schemes since k contains an ℓth root of unity.

Example.

$$\mathrm{H}^i(C_{\text{ét}}, \underline{\mathbb{Z}/\ell\mathbb{Z}}) = \mathrm{H}^i(C, \mu_\ell) = \begin{cases} \mathbb{Z}/\ell\mathbb{Z} & \text{if } i = 0 \\ (\mathbb{Z}/\ell\mathbb{Z})^{2g} & \text{if } i = 1 \\ ? & \text{otherwise.} \end{cases}$$

The Kummer sequence (of sheaves)

$$1 \longrightarrow \mu_\ell \longrightarrow \mathbb{G}_m \xrightarrow{t \mapsto t^\ell} \mathbb{G}_m \longrightarrow 1$$

is exact in the étale topology since $\ell \in k^\times$. Thus, we get an exact sequence

$x^\ell - 1$ is a separable polynomial

$$0 \to \mathbb{G}_m(C)/\mathbb{G}_m(C)^\ell \to \mathrm{H}^1(C_{\text{ét}}, \mu_\ell) \to \mathrm{H}^1(C_{\text{ét}}, \mathbb{G}_m)[\ell] \to 0.$$

Since $\mathbb{G}_m(C) = k^\times$ and $k = \overline{k}$, this turns into an isomorphism

$$\mathrm{H}^1(C_{\text{ét}}, \mu_\ell) = \mathrm{Pic}(C)[\ell] = (\mathbb{Z}/\ell\mathbb{Z})^{2g}.$$

The same argument computes $\mathrm{H}^1(C_{\text{Fpqc}}, \mu_\ell) = \mathrm{Pic}(C)[\ell]$ even if $\ell = p$. △

Example. Say E/k is an elliptic curve.

$$\mathrm{H}^1(E_{\text{Fppf}}, \alpha_p) = \begin{cases} 0 & \text{if } E \text{ ordinary} \\ \mathbb{F}_p & \text{otherwise.} \end{cases}$$

Consider the exact sequence

$$0 \longrightarrow \alpha_p \longrightarrow \mathbb{G}_a \xrightarrow{t \mapsto t^p} \mathbb{G}_a \longrightarrow 0.$$

This gives

$$0 \longrightarrow k/k^p \longrightarrow \mathrm{H}^1(E_{\text{ét}}, \alpha_p) \longrightarrow \ker\left(\mathrm{Frob}^* \curvearrowright \mathrm{H}^1(E, \mathbb{G}_a)\right) \longrightarrow 0.$$

$k = k^p$ since $k = \overline{k}$. Furthermore, $\mathbb{G}_a = (\mathscr{O}_E)_{\text{ét}}$ as sheaves, by definition. Accepting that $\mathrm{H}^1(E_{\text{Fppf}}, \mathbb{G}_a) = \mathrm{H}^1(E, \mathscr{O}_E)$, we win by definition of ordinary.

Similarly,

$$\mathrm{H}^1(E_{\text{ét}}, \underline{\mathbb{F}}_p) = \begin{cases} \mathbb{F}_p & \text{if } E \text{ ordinary} \\ 0 & \text{otherwise.} \end{cases}$$

The collection of maps $\cdot n : E \to E$ gives a universal cover (use RH + isogeny theory) with automorphism groups $E[n]$. Thus,

$$\pi_1^{\text{ét}}(E) = \varprojlim_n E[n] = \begin{cases} \mathbb{Z}_p \times \prod_{\ell \neq p} \mathbb{Z}_\ell & \text{if } E \text{ ordinary} \\ \prod_{\ell \neq p} \mathbb{Z}_\ell & \text{otherwise.} \end{cases}$$

Hence,

$$\mathrm{H}^1(E_{\text{ét}}, \underline{\mathbb{F}}_p) = \mathrm{Hom}(\pi_1^{\text{ét}}(E), \mathbb{F}_p) = \begin{cases} \mathbb{F}_p & \text{if } E \text{ ordinary} \\ 0 & \text{otherwise.} \end{cases}$$

$\triangle$

2.4 qcoh sheaves

Use Čech-to-derived spectral sequence