Outline

- Notation + Goal of next two talks
- Application of amplification principle
- Lemmas useful for the goal

1 Goal

We start with a mountain of notation. Fix integers $g \geq 2$ and $\ell \geq 3$.

- Let $\mathbb{M}_g$ denote the (fine) moduli space of smooth curves of genus g with full level ℓ structure.

  $\mathbb{M}_g$ is a regular, quasi-projective variety of dimension $3g-3$. We implicitly fix a choice of ℓth root of unity so that $\mathbb{M}_g$ is geometrically irreducible.

  Suffices to think of this as a $\overline{\mathbb{Q}}$-scheme

- Let $\mathcal{C}_g/\mathbb{M}_g$ denote the universal curve, so $\mathcal{C}_g \to \mathbb{M}_g$ is smooth and projective and its fibers are genus g curves (equipped with full level ℓ structure).
- Let $\mathrm{Jac}(\mathcal{C}_g)/\mathbb{M}_g$ denote the Jacobian of $\mathcal{C}_g \to \mathbb{M}_g$, an abelian scheme equipped w/ a natural prinipal polarization and full level ℓ structure.
- Let $\mathbb{A}_g$ denote the (fine) moduli space of g-dimensional PPAVs w/ level ℓ structure.

  $\mathbb{A}_g$ is regular, quasi-projective, and geometrically irreducible.
- Let $\pi : \mathcal{U}_g \to \mathbb{A}_g$ denote the universal abelian scheme, so π is smooth and projective.

Since $\mathbb{A}_g$ is a fine moduli space, $\mathrm{Jac}(\mathcal{C}_g) \to \mathbb{M}_g$ must be pulled back from $\mathcal{U}_g \to \mathbb{A}_g$ along a morphism $\tau : \mathbb{M}_g \to \mathbb{A}_g$, called the Torelli map,

$$\begin{tikzcd} \mathrm{Jac}(\mathcal{C}_g) \arrow[r] \arrow[d] \arrow[dr, phantom, "\ulcorner", very near start] & \mathcal{U}_g \arrow[d, "\pi"] \\ \mathbb{M}_g \arrow[r, "\tau"] & \mathbb{A}_g. \end{tikzcd}$$

Fact. τ is quasi-finite (would be injective on $\mathbb{C}$ points w/o level structure)

We're not done yet.

- Choose an embedding $\mathbb{A}_g \hookrightarrow \mathbb{P}^m_{\overline{\mathbb{Q}}}$, let $\overline{\mathbb{A}}_g$ denote the closure of $\mathbb{A}_g$, and let $\mathcal{M} = \mathscr{O}(1)|_{\overline{\mathbb{A}}_g}$. Choose a height function

  No need to name line bundles. They don't show up anywhere

  $$h_{\overline{\mathbb{A}}_g, \mathcal{M}} : \overline{\mathbb{A}}_g(\overline{\mathbb{Q}}) \longrightarrow \mathbb{R}.$$

  (Note $\overline{\mathbb{A}}_g$ is possibly non-regular)
- Choose a symmetric relatively very ample line bundle $\mathscr{L}$ on $\mathcal{U}_g/\mathbb{A}_g$ giving rise to a closed immersion $\mathcal{U}_g \overset{\text{closed}}{\subset} \mathbb{P}^n_{\overline{\mathbb{Q}}} \times \mathbb{A}_g$ over $\mathbb{A}_g$. Let

  $$\widehat{h} : \mathcal{U}_g(\overline{\mathbb{Q}}) \longrightarrow \mathbb{R}_{\geq 0}$$

  denote the induced fiberwise Néron-Tate height.

Theorem 1.1 (Goal for Next Two Talks)**.** *Let S be an irreducible closed subvariety of $\mathbb{M}_g$ defined over $\overline{\mathbb{Q}}$. There exists positive constants c_1, c_2, c_3, c_4 depending on the choices made above and on S with the following property. Let $s \in S(\overline{\mathbb{Q}})$ with $h_{\overline{\mathbb{A}}_g, \mathcal{M}}(\tau(s)) \geq c_1$. There exists a subset $\Xi_s \subset \mathcal{C}_s(\overline{\mathbb{Q}})$ with $\#\Xi_s \leq C_2$ such that any $P \in \mathcal{C}_s(\overline{\mathbb{Q}}) \setminus \Xi_s$ satisfies*

A curve of large moduli

$$\#\left\{Q \in \mathcal{C}_s(\overline{\mathbb{Q}}) : \widehat{h}(Q - P) \leq h_{\overline{\mathbb{A}}_g, \mathcal{M}}(\tau(s))/c_3\right\} < c_4.$$

Ξ_s set of bad points. In small ball (depending on s) around good points, get bounded number of rational points. Finitely many such balls cover radius of small points. Wave hands. Happy.

Draw picture

2 Amplification

Unsurprisingly, the proof of Theorem 1.1 will involve the height inequality from the previous talks. Given this, the first thing we'll do is produce some non-degenerate varieties.

More notation

- For $M \geq 1$, we set

  $$\mathcal{U}_g^{[M]} := \underbrace{\mathcal{U}_g \times_{\mathbb{A}_g} \ldots \times_{\mathbb{A}_g} \mathcal{U}_g}_{M \text{ factors}},$$

  so $\mathcal{U}_g^{[M]} \to \mathbb{A}_g$ is an abelian scheme. One gets a closed immersion $\mathcal{U}_g^{[M]} \hookrightarrow (\mathbb{P}^n_{\overline{\mathbb{Q}}})^M \times \mathbb{A}_g$ whose induced Néron-Tate height is the sum of the heights of the M coordinates.

- For $M \geq 1$, one has the **Faltings-Zhang morphism**

  $$D_M : \mathcal{C}_g^{[M+1]} \longrightarrow \operatorname{Jac}(\mathcal{C}_g/M_g)^{[M]}$$

  over $\mathbb{M}_g$ which, on the fiber $C = \mathcal{C}_{g,x}$ over a point $x \in \mathbb{M}_g(\overline{\mathbb{Q}})$, is the map

  $$C(k)^{M+1} \ni (P_0, \ldots, P_M) \longmapsto (P_1 - P_0, \ldots, P_M - P_0) \in \operatorname{Jac}(C)^M.$$

- If $S \to \mathbb{M}_g$ is a morphism of schemes, we write $\mathcal{C}_S := \mathcal{C}_g \times_{\mathbb{M}_g} S$ and $\mathcal{C}_S^{[M]} := \mathcal{C}_g^{[M]} \times_{\mathbb{M}_g} S$.

  Fact. If S is irreducible, then so is $\mathcal{C}_S^{[M]}$

  Induct of M using that $\mathcal{C}_S \to S$ is smooth and proper with irreducible fibers and an irreducible base

  Note that we then get an induced Faltings-Zhang morphism

  $$\begin{tikzcd} \mathcal{C}_S^{[M+1]} \arrow[r, "\mathcal{D}_M"] \arrow[dr] & \operatorname{Jac}(\mathcal{C}_g/\mathbb{M}_g)^{[M]} \times_{\mathbb{M}_g} S \arrow[r, equal] \arrow[d] & \mathcal{U}_g^{[M]} \times_{\mathbb{A}_g} S \\ & S & \end{tikzcd}$$

  which is proper by cancellation (diagonal arrow proper + vertical arrow separated).

- If $x \in \mathbb{M}_g(\overline{\mathbb{Q}})$ and $P \in \mathcal{C}_x(\overline{\mathbb{Q}})$, we set

  Move to before degree stuff or skip?

$$\mathcal{C}_x - P := \mathcal{D}_1(\{P\} \times \mathcal{C}_x) \subset \mathrm{Jac}(\mathcal{C}_x) = \mathfrak{U}_{g,\tau(s)} \subset \mathbb{P}^n_{\overline{\mathbb{Q}}}.$$

We first want to show that $D_M(\mathcal{C}_S^{[M+1]}) \subset \mathfrak{U}_g^{[M]} \times_{\mathbb{A}_g} S$ is non-degenerate for $M \gg 0$. To get this, one appeals to the following theorem of Gao (to be covered in a later talk)

Theorem 2.1 (Gao '20, Theorem 1.3)**.** *Let S be an irreducible variety with a quasi-finite map $\iota : S \to \mathbb{M}_g$ such that $\mathcal{C}_S \to S$ admits a section. Let $\mathcal{A} = \mathfrak{U}_g \times_{\mathbb{A}_g} S$ and let $X \subset \mathcal{A}$ be an irreducible subvariety such that*

(a) $\dim X > \dim S$

(b) *For each $s \in S(\mathbb{C})$, X_s generates $\mathcal{A}_s$ (in particular, X dominates S)*

(c) *$X + \mathcal{A}' \not\subset X$ for any non-isotrivial abelian subscheme $\mathcal{A}'$ of $\mathcal{A} \to S$*

Curves generate Jacobians by universal property or because degree 0 divisors generated by things of the form $P - Q$

Then, $\mathcal{D}_M^{\mathcal{A}}(X^{[M+1]})$ is non-degenerate in $\mathcal{A}^{[M]}$ for all $M \geq \dim X$.

Corollary 2.2 (DGH, Theorem 6.2)**.** *Let S be an irreducible variety w/ a (not necessarily dominant) quasi-finite morphism $S \to \mathbb{M}_g$. Assume $g \geq 2$ and $M \geq 3g - 2 = \dim \mathbb{M}_g + 1$. Then, $D_M(\mathcal{C}_S^{[M+1]}) \subset \mathfrak{U}_g^{[M]} \times_{\mathbb{A}_g} S$ is non-degenerate.*

Proof. If S has a section $\varepsilon : S \to \mathcal{C}_S$, apply Lemma to $X = j_\varepsilon(\mathcal{C}_S) \hookrightarrow \mathrm{Jac}(\mathcal{C}_S/S)$. Check hypotheses by hand. Assume no section. The generic fiber of $\mathcal{C}_S \to S$ has a rational point over some finite extension of $K(S)$, so get a quasi-finite, étale dominant $\rho : S' \to S$ s.t. S' is irreducible and $\mathcal{C}_{S'}$ has a section. Then, $\mathcal{D}_M(\mathcal{C}_{S'}^{[M+1]})$ is non-degenerate, so can choose some $\Delta' \subset (S')^{\mathrm{an}}$ and some $x' \in (X')^{sm,an}$ s.t.

$$\mathrm{rank}\left(\mathrm{d}b_{\Delta'}|_{(X')^{sm,an}}\right)_{x'} = 2\dim X' \text{ where } b_{\Delta'} : \mathcal{A}'_{\Delta'} \to \mathbb{R}^{2g}/\mathbb{Z}^{2g}.$$

Elements of $\mathcal{A}'(\mathbb{C})$ give automorphisms. Genus $g \geq 2$ curves have finitely many automorphisms

Let $\mathcal{A} = \mathfrak{U}_g^{[m]} \times_{\mathbb{A}_g} S$ and define $\mathcal{A}'$ similarly. Since ρ is étale, we can shrink Δ' to assume that it's mapped diffeomorphically onto its image $\Delta = \rho(\Delta')$. Then, $\rho_{\mathcal{A}} : \mathcal{A}' \to \mathcal{A}$ also maps $\mathcal{A}'_{\Delta'}$ diffeomorphically onto its image, so

$$\mathrm{rank}\left(\mathrm{d}b_{\rho(\Delta')}|_{X^{sm,an}}\right)_{\rho_{\mathcal{A}}(x')} = \mathrm{rank}\left(\mathrm{d}b_{\Delta'}|_{(X')^{sm,an}}\right)_{x'} = 2\dim X' = 2\dim X$$

where $b_\Delta = b_{\Delta'} \circ \rho_{\mathcal{A}}$ ∎

Now we want to prove some technical lemmas which will combine with this to prove the new gap principle.

3 Technical Lemmas

Recall 3.1. The height inequality proved in the previous talks will provide a dense open subset $U \subset \mathcal{D}_M(\mathcal{C}_S^{[M+1]})$ on which we get a lower bound for the heights of points.

Points of this look like $(Q_1 - P, \ldots, Q_M - P)$

In order to show that there are not too many points of $\mathcal{C}_s$ near a given $P \in \mathcal{C}_s(\overline{\mathbb{Q}})$, it'd be nice to be able to bound the size of set $\subset \mathcal{C}_s - P$ whose Mth power avoids U.

Theorem 3.2 (B'ezout, Example 8.4.6 in Fulton's Intersection Theory)**.** *Let $V_1, \dots, V_r$ be subvarieties of $\mathbb{P}^n$. Let $Z_1, \dots, Z_t$ be the irreducible components of $V_1 \cap \dots \cap V_r$. Then,*

$$\sum_{i=1}^{t} \deg(Z_i) \le \prod_{j=1}^{r} \deg(V_j).$$

Lemma 3.3 (Lemma 6.3)**.** *Let $k = \overline{k}$ be a field. Let $C \subset \mathbb{P}^n$ be an irreducible curve defined over k, and let $Z \subset (\mathbb{P}^n_k)^M$ be a Zariski closed subset such that $C^M = C \times \ldots \times C \not\subset Z$. Then, there is a number $B = B(M, \deg C, \deg Z)$ such that*

Have in mind the contrapositive

$$\Sigma \subset C(k) \text{ has cardinality } \ge B \implies \Sigma^M \not\subset Z(k).$$

Proof. We induct on M. The case $M = 1$ holds by B'ezout (take $B = \deg Z \deg C$).

Ask audience if they want to see the proof. It was given last semester

Assume the lemma for $1, \dots, M-1$ and let $q : (\mathbb{P}^n_k)^M \to \mathbb{P}^n_k$ be projection onto the first factor. Bézout bounds the number of irreducible components of $Z \cap C^M$ along with their degrees. Let Z' be the union of the irreducible components Y of $Z \cap C^M$ with $\dim q(Y) \ge 1$, and let Z'' be the union of all other irreducible components.

Note that $q(Z') \subset C$ (components of $Z \cap C^M$). For all $P \in C(k)$, the fiber $F := q|_{Z'}^{-1}(P) = Z' \cap (\{P\} \times (\mathbb{P}^n)^{M-1})$ has dimension at most $\dim Z' - 1 \le M - 2$ ($Y \subset Z \cap C^M \subsetneq C^M$). The projection of F to the final factors $(\mathbb{P}^n)^{M-1}$ can't contain C^{M-1}, and Bézout bounds the degree of this projection in terms of $M, \deg C, \deg Z$, so induction gives a number $B' = B'(M, \deg C, \deg Z)$ such that

B' independent of $P \in C(k)$

$$\Sigma \subset C(k) \text{ has cardinality } \ge B' \implies \{P\} \times \Sigma^{M-1} \not\subset Z'(k) \text{ for all } P \in C(k).$$

Since $\dim q(Z'') = 0$, it is a finite with cardinality at most the number B'' of irreducible components of $Z \cap C^M$. We win by taking $B = \max\{B', B'' + 1\}$. ∎

To prove Theorem 1.1, we'll want to apply this lemma to curves of the form $\mathcal{C}_s - P \subset \mathcal{U}_{g,\tau(s)} \subset \mathbb{P}^n_{\overline{\mathbb{Q}}}$ with Z the complement of the open U given by the height inequality. Since the bound B in the Lemma depends on $\deg(\mathcal{C}_s - P)$, we would like a uniform bound for this quantity.

If $\widehat{h}(Q - P)$ is small for many $Q \in \mathcal{C}_s(\overline{\mathbb{Q}})$, then a tuple of such Q's would give a point of U with small height

Lemma 3.4 (Lemma 6.1)**.** *There exists a constant c such that $\deg(\mathcal{C}_s - P) \le c$ for all $s \in \mathbb{M}_g(\overline{\mathbb{Q}})$ and all $P \in \mathcal{C}_s(\overline{\mathbb{Q}})$.*

Proof. We will appeal to the following fact

Fact (EGA IV$_4$ Corollaire 17.16.3(ii))**.** Let $f : X \to S$ be a surjective smooth morphisms. Then, there is an étale morphism $g : S' \twoheadrightarrow S$ so that there exists an S-morphism $S' \to X$ (i.e. a section of $X_{S'} \to S'$). If S is quasi-compact (resp. quasi-compact and quasi-separated), then S' can be taken to be affine (resp. S' affine and g quasi-finite).

In the present case, this tells us there exists some affine S with a quasi-finite étale surjection $S \overset{f}{\twoheadrightarrow} \mathbb{M}_g$

and a morphism $\varepsilon : S \to \mathcal{C}_g$ of $\mathbb{M}_g$-schemes. Given this, consider the diagram

$$\begin{tikzcd}
\mathcal{C}_g \times_{\mathbb{M}_g} S \arrow[r, "(\mathrm{id},\varepsilon)"] \arrow[d] \arrow[rrr, bend left, "\varphi"] & \mathcal{C}_g^{[2]} \arrow[r, "\mathcal{D}_1"] & \mathcal{U}_g \times_{\mathbb{A}_g} \mathbb{M}_g \arrow[r, "\mathrm{pr}_1"] & \mathcal{U}_g \arrow[d, "\pi"] \\
S \arrow[r, "f"] & \mathbb{M}_g \arrow[rr, "\tau"] & & \mathbb{A}_g
\end{tikzcd}$$

The top composition above is $(P, s) \mapsto P - \varepsilon(s) \in \mathrm{Jac}(\mathcal{C}_{f(s)})$. Let $Z = \mathrm{im}(\varphi) \subset \mathcal{U}_g$ be the (scheme-theoretic) image. Given $x \in \mathbb{M}_g(\overline{\mathbb{Q}})$, note that we have

Finite union (of irreducible curves) since f, τ both quasi-finite

$$Z_{\tau(x)} = \bigcup_{s \in f^{-1}(\tau^{-1}(\tau(x)))} \left(\mathcal{C}_{f(s)} - \sigma(s)\right).$$

Choosing some $s \in f^{-1}(x)$, we see that there is some $P_x = \varepsilon(s)$ so that $\mathcal{C}_x - P_x$ is an irreducible component of $Z_{\tau(x)}$. Now, we want to describe $Z_{\tau(x)}$ has an intersection of projective varieties, so we can bound the degrees of its irreducible components using Bézout.

Consider the various embeddings

$$\begin{tikzcd}
Z \arrow[r, hook] & \mathcal{U}_g \arrow[r, hook] & \mathbb{P}^n_{\overline{\mathbb{Q}}} \times \mathbb{A}_g \arrow[r, hook] & \mathbb{P}^n_{\overline{\mathbb{Q}}} \times \mathbb{P}^m_{\overline{\mathbb{Q}}} \arrow[r, hook] & \mathbb{P}^{(n+1)(m+1)-1}_{\overline{\mathbb{Q}}} \\
Z_{\tau(x)} \arrow[u, hook] \arrow[rr, hook] & & \mathbb{P}^n \times \{\tau(x)\} \arrow[u, hook] & &
\end{tikzcd}$$

We see that $\mathcal{C}_x - P_x \subset \mathbb{P}^n_{\overline{\mathbb{Q}}} \subset \mathbb{P}^N_{\overline{\mathbb{Q}}}$ is an irreducible component of

$$\left[\overline{Z} \cap \left(\mathbb{P}^n_{\overline{\mathbb{Q}}} \times \{\tau(s)\}\right)\right] \subset \mathbb{P}^N_{\overline{\mathbb{Q}}}$$

so $\deg(\mathcal{C}_x - P_x) \le \deg(\overline{Z})$ for all x.

For general $P \in \mathcal{C}_x(\overline{\mathbb{Q}})$, we note that translation by $P_x - P$ does not change degrees, so

$\mathcal{U}_{g,\tau(s)} \subset \mathbb{P}^n_{\overline{\mathbb{Q}}}$ non-degenerate so translation gives an $\mathscr{O}(1)$-preserving automorphism

$$\deg \overline{Z} \ge \deg(\mathcal{C}_x - P) = \deg(\mathcal{C}_x - P_x + (P_x - P)) = \deg(\mathcal{C}_x - P).$$

$\blacksquare$

Remark 3.5. There's a notion of the height $h(\mathcal{C}_s - P_s)$ of a projective variety defined using Arakelov intersection theory, and a similar argument shows that

$$h(\mathcal{C}_s - P_s) \le c \max\left\{1, h_{\overline{\mathbb{A}}_g, \mathcal{M}}(\tau(s))\right\}$$

for all $s \in \mathbb{M}_g(\overline{\mathbb{Q}})$. This won't be used in the next talk, but may come up in the talk after that. $\circ$

Finally, there's one last lemma that will be useful in the proof of Theorem 1.1 (for bounding the number of bad points)

Lemma 3.6 (Lemma 6.4)**.** *Let $A/\overline{\mathbb{Q}}$ be an abelian variety, and suppose $C \subset A$ is a smooth curve of*

Probably won't get to

genus $g \geq 2$. If Z is an irreducible Zariski closed proper subset of $\mathcal{D}_M(C^{M+1})$, then

$$\#\left\{P \in C(\overline{\mathbb{Q}}) : (C-P)^M \subset Z\right\} \subset 84(g-1).$$

Proof. Let

$$\Xi = \left\{P \in C(\overline{\mathbb{Q}}) : (C-P)^M \subset \mathbb{Z}\right\},$$

and fix some $P_0 \in \Xi$. Say $P_i \in \Xi$, and consider $i \in \{0,1\}$. Note that $Z \subsetneq \mathcal{D}_M(C^{M+1})$ so $\dim Z < \dim \mathcal{D}_M(C^{M+1}) = M+1$ since $\mathcal{D}_M(C^{M+1})$ is irreducible. At the same time $(C-P_i)^M \subset Z$, so $(C-P_i)^M = Z$ as Z is irreducible. Applying the first projection $A^M \to A$, we see that $C - P_1 - P_0$, so translation by $P_1 - P_0$ induces an automorphism of C. By Hurwitz's theorem, there are $\leq 84(g-1)$ choices for $P_1 - P_0$. ■