Stage Spring '23 Notes

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These are notes on talks given in "STAGE" which took place at Simons Foundation in NY. Unfortunately for the reader, these notes are live-texed and so their quality is upper bounded both by my (quite) limited ability to understand the material in real time and by my typing speed. With that in mind, they are doubtlessly missing content/insight present in the talks and certainly contain confusions not present in the talks. Despite all this, I hope that you can still find some use of them. Enjoy and happy mathing.

The website for this seminar is available here.

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1 Kenta (MIT): Abelian Varieties and Complex Tori

1.1 Elliptic curves

Say we're trying to classify elliptic curves. Consider the upper half space

$$\mathfrak{H} := \{\tau \in \mathbb{C} : \operatorname{Im} \tau > 0\}.$$

For any $\tau \in \mathfrak{H}$, can associate the elliptic curve $\mathbb{C}/(\mathbb{Z} \oplus \mathbb{Z}\tau)$.

Observation 1.1. If $\begin{pmatrix} a & b \\ c & d \end{pmatrix} \in \mathrm{SL}_2(\mathbb{Z})$, then τ and $\frac{a\tau+b}{c\tau+d}$ give rise to isomorphic elliptic curves. Conversely, any two τ's giving rise to isomorphic curves differ by an element of $\mathrm{SL}_2(\mathbb{Z})$ in this way.

The upshot is that $\mathrm{SL}_2(\mathbb{Z})\backslash\mathfrak{H}$ is the (coarse) moduli space of elliptic curves over $\mathbb{C}$. We want to (eventually) generalize this story to abelian varieties.

1.2 Complex tori

Given an n-dimensional $\mathbb{C}$-vector space V along with a discrete lattice $U \subset V$ (so $\operatorname{rank}_{\mathbb{Z}} U = 2n$), one can form the quotient $X = V/U$, a `complex torus`. Note that X is diffeomorphic to $(S^1)^{2n}$, and so is a (complex) compact Lie group.

Example 1.2. All elliptic curves are complex tori $\triangle$

Example 1.3. Let C be a smooth projective curve/$\mathbb{C}$ of genus g. Then, the `Picard variety` (or `Jacobian`) of C is a complex torus of dimension g. $\triangle$

Let's recall some facts about line bundles.

Recall 1.4. When X is a complex manifold, its set of (isomorphism classes of) holomorphic line bundles is isomorphic to the group $\mathrm{H}^1(X, \mathscr{O}_X)$. However, its set of (isomorphism classes of) topological line bundles is isomorphic to $\mathrm{H}^2(X, \mathbb{Z})$. The forget map $c_1 : \mathrm{H}^1(X, \mathscr{O}_X) \to \mathrm{H}^2(X; \mathbb{Z})$ is called the `first Chern class`. An alternative construction of this map is to start with the `exponential exact sequence`

$$0 \longrightarrow \mathbb{Z} \xrightarrow{2\pi i} \mathscr{O}_X \xrightarrow{\exp} \mathscr{O}_X^\times \longrightarrow 0$$

and then look at the sequence in cohomology. Furthermore, if C is a Riemann surface, then

$$\mathrm{Pic}^0(C) = \ker\big(c_1 : \mathrm{H}^1(C, \mathscr{O}_C^\times) \to \mathrm{H}^2(C, \mathbb{Z})\big)$$

(and c_1 above is the degree map $\deg : \mathrm{Pic}(C) \to \mathbb{Z}$). This is a torus because it is isomorphic to $\mathrm{H}^0(C, \mathscr{O}_C)/\,\mathrm{H}^1(X, \mathbb{Z})$. $\odot$

1.3 Cohomology of $X = V/U$

Lemma 1.5. *There is a canonical isomorphism*

$$\mathrm{H}^r(X; \mathbb{Z}) \simeq \operatorname{Hom}\Big(\bigwedge^r U, \mathbb{Z}\Big)$$

Proof. Since $X = V/U$ (and V simply connected), we see that $\pi_1(X) = U$, so $\mathrm{H}^1(X; \mathbb{Z}) \simeq \operatorname{Hom}(U, \mathbb{Z})$ by universal coefficients. Now, cup product gives us a map

$$\bigwedge^r \mathrm{H}^1(X, \mathbb{Z}) \longrightarrow \mathrm{H}^r(X, \mathbb{Z}).$$

This is an isomorphism because $X \simeq (S^1)^{2n}$ is a product of circles. ■

Recall we have Chern class

$$c_1 : \mathrm{H}^1(X, \mathscr{O}_X^\times) \longrightarrow \mathrm{H}^2(X; \mathbb{Z}).$$

This admits the following explicit description.

Construction 1.6. Say $\mathscr{L}$ is a line bundle on X. Let $\pi : V \to X$ be the natural quotient. Because V is contractible, $\pi^*\mathscr{L}$ is trivial, so *choose* an isomorphism $s : \pi^*\mathscr{L} \xrightarrow{\sim} \mathscr{O}_V$. Given $u \in U$, also let u denote the map $V \to V, v \mapsto v + u$. For each $u \in U$, we get maps

$$\mathscr{O}_V = u^*\mathscr{O}_V \underset{u^*s}{\overset{\sim}{\longleftarrow}} u^*\pi^*\mathscr{L} = \pi^*\mathscr{L} \underset{s}{\overset{\sim}{\longrightarrow}} \mathscr{O}_V.$$

The composition is given by some $e_u \in \Gamma(V, \mathscr{O}_V^\times)$. These satisfy the cocycle condition

$$e_{u+u'}(z) = e_u(z + u')e_u(z).$$

For this, stare at the diagram

$$\begin{tikzcd}
\mathscr{L}_{\pi(z+u+u')} \arrow[r, equal] \arrow[d, "\wr"] & \mathscr{L}_{\pi(z+u')} \arrow[r, equal] \arrow[d, "\wr"] & \mathscr{L}_{\pi(z)} \arrow[d, "\wr"] \\
\mathbb{C} \arrow[r, "e_u(u'+z)"] \arrow[rr, bend right, "e_{u+u'}(z)"'] & \mathbb{C} \arrow[r, "e_{u'}(z)"] & \mathbb{C}
\end{tikzcd}$$

Write $e_u(z) = e^{2\pi i f_u(z)}$. One sees that

$$F(u_1, u_2) := f_{u_2}(z + u_1) + f_u(z) - f_{u_1+u_2}(z) \in \mathbb{Z}$$

This is some 2-cycle. Now $c_1(\mathscr{L})$ is represented by the 2-form

$$E(u_1, u_2) := F(u_1, u_2) - F(u_2, u_1)$$

(recall $\mathrm{H}^2(X, \mathbb{Z}) \simeq \mathrm{Hom}(\bigwedge^2 U, \mathbb{Z})$) ○

Definition 1.7. Let the Néron-Severi group $\mathrm{NS}(X)$ be the image of $c_1 : \mathrm{H}^1(X, \mathscr{O}_X^\times) \to \mathrm{H}^2(X, \mathbb{Z})$. ⋄

(These are the topological line bundles which can be given a holomorphic structure.

Lemma 1.8. $\mathrm{NS}(X) \subset \mathrm{Hom}(\bigwedge^2 U, \mathbb{Z})$ *consists of the* $E : \bigwedge^2 U \to \mathbb{Z}$ *such that*

$$E(iu_1, iu_2) = E(u_1, u_2),$$

where E *denotes also the induced map* $\bigwedge^2 V \to \mathbb{R}$.

Proof. From the exponential exact sequence, we have

$$\mathrm{NS}(X) \simeq \ker\left(\mathrm{H}^2(X, \mathbb{Z}) \longrightarrow \mathrm{H}^2(X, \mathscr{O}_X)\right).$$

Note that this map factors through $\mathrm{H}^2(X, \mathbb{C})$. We'll now use some Hodge theory.

Recall 1.9. Let X be a Kähler manifold. Then, there's a canonical decomposition

$$\mathrm{H}^n(X,\mathbb{C}) \simeq \bigoplus_{p+q=n} \mathrm{H}^q(X,\Omega^p).$$

$\odot$

In the present case, $\mathrm{H}^1(X,\mathbb{C}) = \mathrm{Hom}_{\mathbb{Z}}(U,\mathbb{C}) = \mathrm{Hom}_{\mathbb{R}}(V,\mathbb{C})$. Furthermore,

$$\mathrm{H}^0(X,\Omega^1) \simeq V_{\mathbb{C}}^{\vee} = \mathrm{Hom}_{\mathbb{C}}(V,\mathbb{C}) \text{ and } \mathrm{H}^1(X,\mathscr{O}_X) \simeq \overline{V}_{\mathbb{C}}^{\vee} = \mathrm{Hom}_{\mathbb{C}}(\overline{V},\mathbb{C}).$$

Thus, the Hodge decomposition (in degree 1) for X is simply

$$\mathrm{Hom}_{\mathbb{R}}(V,\mathbb{C}) \simeq \mathrm{Hom}_{\mathbb{C}}(V,\mathbb{C}) \oplus \mathrm{Hom}_{\mathbb{C}}(\overline{V},\mathbb{C}).$$

The Hodge decomposition in higher degrees comes from taking wedge powers, e.g. in degree 2 it is

$$\underbrace{\mathrm{Hom}_{\mathbb{R}}\left(\bigwedge^2 V,\mathbb{C}\right)}_{\mathrm{H}^2(X,\mathbb{C})} \simeq \underbrace{\mathrm{Hom}_{\mathbb{C}}\left(\bigwedge^2 V,\mathbb{C}\right)}_{\mathrm{H}^0(X,\Omega^2)} \oplus \underbrace{\mathrm{Hom}_{\mathbb{C}}(V\otimes\overline{V},\mathbb{C})}_{\mathrm{H}^1(X,\Omega^1)} \oplus \underbrace{\mathrm{Hom}_{\mathbb{C}}\left(\bigwedge^2 \overline{V},\mathbb{C}\right)}_{\mathrm{H}^2(X,\mathscr{O}_X)}$$

(Note: This is not literally what Kenta wrote down, but it's equivalent). The theorem is now a computation. The relevant projection map $\mathrm{H}^2(X,\mathbb{Z}) \hookrightarrow \mathrm{H}^2(X,\mathbb{C}) \twoheadrightarrow \mathrm{H}^2(X,\mathscr{O}_X)$ is explicitly given by

$$F \mapsto \left(v\wedge w \mapsto \frac{1}{4}(F(v\wedge w) + iF(iv\wedge w) + iF(v\wedge iw) - F(iv\wedge iw))\right),$$

whose kernel consists of F satisfying $F(v\wedge w) = F(iv\wedge iw)$ and $F(iv\wedge w) = -F(v\wedge iw)$. If F is $\mathbb{Z}$-valued, the first of these is sufficient. ■

Remark 1.10. Say we have a 2-form $E : \bigwedge^2 U \to \mathbb{Z}$ satisfying $E(iu_1, iu_2) = E(u_1,u_2)$. From this, one gets a `Hermitiam form` λ on V via $\lambda(x,y) := E(ix,y) + iE(x,y)$. In fact, this is Hermitian iff E satisfying $E(iu_1,iu_2) = E(u_1,u_2)$. $\circ$

Corollary 1.11.

$$\mathrm{NS}(X) \simeq \left\{ \begin{matrix} \textit{Hermitian forms } \lambda \textit{ on } V \textit{ s.t.} \\ (\mathrm{Im}\,\Lambda)(U\times U) \subset \mathbb{Z} \end{matrix} \right\}$$

Theorem 1.12 (`Appell-Humbert`)**.** $\mathrm{Pic}(X)$ *is in bijection with*[1]

$$\left\{\lambda \in \mathrm{NS}(X), \alpha : U \to S^1 \mid \alpha(u_1+u_2) = e^{i\pi(\mathrm{Im}\,\lambda)(u_1,u_2)}\alpha(u_1)\alpha(u_2)\right\}$$

(Remark: $e^{blah} = \pm 1$ above). For every (λ,α) as above, the line bundle $L(\lambda,\alpha)$ is given by the `A-H cocycle`

$$e_u(z) = \alpha(u)e^{\pi\lambda(z,u)+\frac{1}{2}\pi\lambda u,u}.$$

In other words, $L(\lambda,\alpha) = U\backslash(\mathbb{C}\times V)$ where U acts via $\varphi_u(\lambda,z) = (e_u(z)\lambda, z+u)$.

Proof. We have an exact sequence $0 \to \mathrm{Pic}^0(X) \to \mathrm{Pic}(X) \to \mathrm{NS}(X) \to 0$ by definition. The first of these objects is

$$\mathrm{Pic}^0(X) = \mathrm{H}^1(X,\mathscr{O}_X)/\,\mathrm{H}^1(X,\mathbb{Z})$$

[1] The group operation is the naive one

$$
\begin{aligned}
&= \overline{V}_{\mathbb{C}}^{\vee}/U_{\mathbb{Z}}^{\vee} \\
&= V_{\mathbb{R}}^{\vee}/U_{\mathbb{Z}}^{\vee} \\
&= \mathrm{Hom}(U, S^1)
\end{aligned}
$$

(one calls $\mathrm{Pic}^0(X)$ the `dual complex torus`). The isomorphism now follows from comparison with the exact sequence

$$0 \longrightarrow \mathrm{Hom}(U, S^1) \longrightarrow \{\text{A-H data}\} \longrightarrow \mathrm{NS}(X) \longrightarrow 0$$

and using Corollary 1.11 (I guess you need this already to know what the quotient above is). ■

Why have we been talking about line bundles? We want to know when a complex torus is algebraic. This is when it supports an ample line bundle.

Theorem 1.13. *The line bundle $L(\lambda, \alpha)$ is ample if and only if λ is positive definite. In that case,*

$$\dim \mathrm{H}^0(X, L(\lambda, \alpha)) = \sqrt{\det E}$$

(recall that $E = \mathrm{Im}\,\lambda$)

Proof. From our explicit description, one sees that sections of $L(\lambda, \alpha)$ are equivalent to holomorphic functions $\theta : V \to \mathbb{C}$ such that

$$\theta(z+u) = e_u(z)\theta(z) = \alpha(u)e^{\pi\lambda(z,u)+\frac{1}{2}\pi\lambda(u,u)}\theta(z).$$

- First suppose λ is degenerate.

  Let $W = \{x \in V : \lambda(x, y) = 0 \text{ for all } y \in V\} = \{x \in V : E(x, y) = 0 \text{ for all } y \in V\}$. E is integral on $U \times U$, so $W \cap U \subset W$ is a lattice. At the same time

  $$\theta(z+u) = \alpha(u)\theta(z) \text{ for all } u \in W \cap U.$$

  Restrict θ to a W-cost in V (where $|\theta(z+u)| = |\theta(z)|$). Since $W/(W \cap U)$ is compact, the maximum modulus principle means that θ must be constant on W-cosets. Hence, the map to projective space collapses W-cosets, so is not an embedding. The same holds for all $n\lambda$, so such a λ can't be ample.

- Now suppose λ is not positive definite (and non-degenerate)

  Let $W \subset V$ be a negative definite subspace, i.e. $\lambda(w, w) < 0$ for all nonzero $w \in W$. Let $K \subset V$ be a compact subset w/ $V = U + K$. Given $z_0 \in V$ and $w \in W$, we can write $w = d + u$ with $d \in K$ and $u \in U$. We have

  $$|\theta(z_0+w)| = |\theta(z_0+d+u)| = \left|\theta(z_0+d)e^{\pi\,\mathrm{Re}\,\lambda(z_0+d,u)+\frac{1}{2}\pi\lambda(u,u)}\right| = |\theta(z_0+d)|\, e^{\pi\left[\frac{1}{2}\lambda(w,w)+\mathrm{Re}\,\lambda(z_0,w)+c(d,z_0\right]},$$

  where c is some bounded function. The exponent on the last expression looks like

  $$(\text{neg. def. quad. form}) + (\text{linear function}) + (\text{bounded function}),$$

  so it goes to $-\infty$ as $w \hookrightarrow \infty$. Again, by maximum modulus principle, we must have $\theta(z_0+w) = 0$, so $L(\lambda, \alpha)$ won't be ample.

  Remark 1.14. Seemingly this in fact proves that $\mathrm{H}^0(X, L(\lambda, \alpha)) = 0$ in this case. ○

- Now suppose λ is positive definite.

The idea is to choose a sublattice $U' \subset U$ and then modify the function θ so that it becomes U'-periodic. Then write down a Fourier expansion of this modified θ-function and use this to see that there are many such θ's.

One chooses U' a rank n sublattice of U such that $E(U', U') = 0$ and $(\mathbb{R}U') \cap U = U'$. Now, $W \cap iW \subset V$ is a $\mathbb{C}$-subspace where E acts as 0. Since E is non-degenerate, this implies that $V = W \oplus iW$. At the same time $E(w, w) = 0$ for all $w \in W$, so there is a unique bilinear form B on V with $B|_{W\times W} = \lambda|_{W\times W}$. Now, $\alpha|_{U'}$ is an honest homomorphism to S^1, so $\alpha(u) = e^{2\pi i \ell(u)}$ for some linear functional ℓ. Consider

λ Hermitian, so not a priori linear in second slot

$$\widetilde{\theta}(z) = e^{-\frac{1}{2}\pi B(z,z) - 2\pi i \ell(z)}\theta(z).$$

This is U'-periodic, so has a Fourier expansion

$$\widetilde{\theta}(z) = \sum_{\chi \in \widehat{U}'} c_\chi e^{2\pi i \chi(z)}.$$

The functional equation gives a relationship $c_{\chi+\widehat{u}} = (blah)c_\chi$ from which one obtains

$$\dim \mathrm{H}^0(X, \mathscr{L}(\lambda, \alpha)) = \left[\widehat{U}' : \mathrm{Im}(U \to \widehat{U}')\right] = \sqrt{\det(E)}.$$

From here, one sees that power of L have tons of global sections. ■

2 Meng Yan (Brandeis University): Quotients of the Siegal upper half space

Let $X = V/U$ be a complex torus of dimension n.

Definition 2.1. Say X is an `abelian variety` if there exists an embedding $X \hookrightarrow \mathbb{P}^N$ for some N. ◇

Definition 2.2. A map $\lambda : \bigwedge^2 U \to \mathbb{Z}$ is a `polarization` if there exists nonzero integers $d_1, \dots, d_n$ s.t. $d_1 \mid \cdots \mid d_n$, we can choose a $\mathbb{Z}$-basis $e_1, \dots, e_n, f_1, \dots, f_n \in U$ s.t.

$$\lambda(e_i, f_j) = \delta_{ij} d_i, \quad \lambda(e_i, e_j) = 0, \quad \text{and} \quad \lambda(f_i, f_j) = 0.$$

W.r.t. to this basis, we can represent λ as a matrix

$$\lambda = \begin{pmatrix} & D \\ -D & \end{pmatrix} \text{ where } D = \begin{pmatrix} d_1 & & \\ & \ddots & \\ & & d_n \end{pmatrix}.$$

D is called the `type of the polarization`. If D is the identity matrix, λ is a `principal polarization`. A symplectic basis is called a `canonical basis`. ◇

Theorem 2.3 (Riemann). *X is an abelian variety iff there exists a positive definite hermitian form λ s.t. $\mathrm{Im}\,\lambda|_U$ is an integral symplectic 2-form (the condition here is just that it is integer valued).*

Proof. See Theorem 1.13. ■

Remark 2.4. Meng almost gave an alternate proof, but then we (= audience + speaker) decided to skip this. However, the start of the alternate proof contained the following criterion for ampleness of a line bundle $\mathscr{L}$ (assuming I understood correctly): $\forall$ subtori $Y \subset X$, $\exists N$, $\sigma \in \mathrm{H}^0(X, L^N)$, $x_1, x_2 \in X$ satisfying $x_1 - x_2 \in Y$, $\sigma(x_1) = 0$, $\sigma(x_2) \neq 0$. ○

Choose a $\mathbb{C}$-basis $e_1, \dots, e_n$ of V and $\mathbb{Z}$-basis $u_1, \dots, u_{2n}$ of U. Write $u_i = \sum \lambda_{ij} e_j$. Then,

$$\Pi = (\lambda_{ij})$$

is called a `period matrix`. Riemann's condition holds iff

$$\Pi E^{-1} \Pi^t = 0 \text{ and } i\Pi E^{-1} \overline{\Pi}^t > 0,$$

where $E = \operatorname{Im} \lambda$. The first condition should make λ Hermitian while the second should make it positive definite.

Note 1. Meng wrote down a proof of this, but I did not follow. Some takeaways

- $\lambda(x, y) = E(ix, y) + iE(x, y)$
- λ Hermitian $\iff$ $E(ix, iy) = E(x, y)$

Maybe worth mentioning that this (and the previous) talk had the early sections (1.1 and 1.2) of Genestier and Ngo as the main reference.

Corollary 2.5. *Fix a polarization type $D = \operatorname{diag}(d_1, \dots, d_n)$. There is a bijection*

$$\{(X, E, (u, v)\} \longleftrightarrow \left\{\Pi_{\mathbb{R}} : U \otimes \mathbb{R} \xrightarrow{\sim} V : E\big(\Pi_{\mathbb{R}}^{-1} iu, \Pi_{\mathbb{R}}^{-1} iv\big) = E\big(\Pi_{\mathbb{R}}^{-1} u, \Pi_{\mathbb{R}}^{-1} v\big) \text{ and } E\big(\Pi_{\mathbb{R}}^{-1} iu, \Pi_{\mathbb{R}}^{-1} v\big) > 0)\right\} / \operatorname{GL}_{\mathbb{C}}(V)$$

(LHS: polarized complex tori of fixed type D equipped w/ a symplectic basis)

Let's say what this bijection is. First, choose a $\mathbb{C}$-basis $e_1, \dots, e_n$ of V (this is why we get $\operatorname{GL}_{\mathbb{C}}(V)$ orbits on the RHS). Using this, we can define a period matrix $\Pi \in M_{n,2n}(\mathbb{C})$. Write this as $\Pi = \begin{pmatrix} \Pi_1 & \Pi_2 \end{pmatrix}$ with $\Pi_i \in M_n(\mathbb{C})$. Take real an imaginary parts, and so write

$$\Pi_1 = \Pi_{11} + i\Pi_{21} \text{ and } \Pi_2 = \Pi_{12} + i\Pi_{22}$$

with $\Pi_{ij} \in M_n(\mathbb{R})$. From these, we produce

$$\Pi_{\mathbb{R}} := \begin{pmatrix} \Pi_{11} & \Pi_{12} \\ \Pi_{21} & \Pi_{22} \end{pmatrix} \in M_{2n}(\mathbb{R}).$$

3 Hao Peng (MIT): Moduli Space of abelian varieties I – Didn't Go

4 Zifan Wang (MIT): Moduli Space of abelian varieties II

Last time we talked about abelian schemes. An `abelian scheme` X/S is a smooth, proper group scheme w/ geometrically connected fibers.

Lemma 4.1 (`Rigidity`)**.** *Let $\alpha : X \to Y$ be a map of abelian schemes such that α maps the identity to the identity. Then, α is a homomorphism.*

Some consequences

- If you fix the identity section $e \in A(S)$, there's only one group structure of A (apply rigidity to $\mathrm{id} : A \to A$)
- Abelian schemes are commutative (apply rigidity to $- : A \to A$)

Let $\mathrm{Pic}_{X/S} : \mathrm{Sch}_S \to \mathrm{Ab}$ be the functor sending an S-scheme T to the isomorphism classes of `rigidified line bundles` $(\mathscr{L}, u)$ over T, i.e. $\mathscr{L} \in \mathrm{Pic}(X \times_S T)$ and $u : e_T^* \mathscr{L} \to \mathscr{O}_T$ is an isomorphism of line bundles on T.

Example 4.2. Equivalently, $\mathrm{Pic}_{X/S}(T) = \mathrm{Pic}(X \times_S T)/\mathrm{Pic}(T)$. $\triangle$

The `dual of an abelian scheme` X/S is $\widehat{X} = \mathrm{Pic}^0_{X/S}$, the connected component containing the trivial bundle $\mathscr{O}_X \in \mathrm{Pic}_{X/S}(S)$.

Lemma 4.3. *Given $\alpha : X \to Y$, one gets a dual map $\widehat{\alpha} : \widehat{Y} \to \widehat{X}$ and a perfect pairing*

$$\ker \alpha \times \ker \widehat{\alpha} \longrightarrow \mathbb{G}_m.$$

Definition 4.4. Given $\mathscr{L}$ on X w/ rigidification u, get `Mumford bundle`

$$m^*\mathscr{L} \otimes \mathrm{pr}_1^* \mathscr{L}^{-1} \otimes \mathrm{pr}_2^* \mathscr{L}^{-1},$$

a rigidified line bundle on $X \times X$. This gives a map $X \to \mathrm{Pic}_{X/S}$ which must factor through the identity component, resulting in

$$\lambda_{\mathscr{L}} : X \longrightarrow \widehat{X}.$$

Let $K(\mathscr{L}) := \ker \lambda_{\mathscr{L}}$. $\diamond$

Definition 4.5. A `polarization` λ of X is a symmetric isogeny $\lambda : X \to \widehat{X}$ such that, étale locally (on S), $\lambda = \lambda_{\mathscr{L}}$ for some ample $\mathscr{L}$. $\diamond$

Remark 4.6. Above, can replace 'symmetric isogeny' with 'homomorphism'. $\circ$

Because $\lambda_{\mathscr{L}}$ is symmetric, it gives rise to perfect pairing

$$K(\mathscr{L}) \times K(\mathscr{L}) \longrightarrow \mathbb{G}_m,$$

which is furthermore symplectic (i.e. anything paired w/ itself is trivial).

Remark 4.7. Suppose S is connected, and that $d = \deg \lambda$ is invertible on S. For any geometric point $\overline{s} \to S$, we can write

$$K(\mathscr{L})_{\overline{s}} \simeq (\mathbb{Z}/d_1\mathbb{Z} \times \ldots \times \mathbb{Z}/d_n\mathbb{Z})^2 \quad \text{where} \quad d_1 \mid d_2 \mid \cdots \mid d_n$$

with these d_i's independent of $\overline{s}$. $\circ$

4.1 Constructing the moduli space

Let's define our moduli problem.

Setup 4.8. Fix positive integers n, N. Fix a type $D = (d_1, \ldots, d_n)$ (such that $d_i \mid d_j$ if $i \leq j$) such that $\gcd(d_n, N) = 1$. Let $\mathcal{A} : \mathrm{Sch}_{\mathbb{Z}[1/(Nd_n)]} \to \mathrm{Set}$ be the functor sending a scheme S to isomorphism classes of triples

$$(X, \lambda, \eta)$$

where

- X is an abelian scheme
- $\lambda : X \to \widehat{X}$ is a polarization of type D
- $\eta : (\mathbb{Z}/N\mathbb{Z})^{2n} \xrightarrow{\sim} X[N]$ is a symplectic isomorphism. The symplectic structure on the LHS comes from $\begin{pmatrix} & I \\ -I & \end{pmatrix}$ while on the RHS it is given by the Weil pairing

$$X[N] \times X[N] \longrightarrow X[N] \times \widehat{X}[N] \longrightarrow \mathbb{G}_m.$$

Note 2. Missed a few things the speaker said

Theorem 4.9. *$\mathcal{A}$ is representable by a smooth, quasi-projective scheme over $\mathbb{Z}[1/(Nd_n)]$, for $N \gg 1$.*

Proof. This will be long with many parts.

- Given an abelian scheme X/S with polarization λ, can pullback the Poincaré bundle along

$$X \xrightarrow{\Delta} X \times X \xrightarrow{(1,\lambda)} X \times \widehat{X}$$

  to get some line bundle $\mathscr{L}^{\Delta}(\lambda)$ on X. We claim that $\lambda_{\mathscr{L}^{\Delta}(\Lambda)} = 2\lambda$, and $\mathscr{L}^{\Delta}(\lambda)$ is relatively ample. Can check these both étale locally, so may assume $\lambda = \lambda_{\mathscr{L}}$ for some ample $\mathscr{L}$.

  Exercise. Check things by hand in this case.

  In this case,
$$\mathscr{L}^{\Delta}(\lambda) = \Delta^*\big(m^*\mathscr{L} \otimes \mathrm{pr}_1^*\mathscr{L}^{-1} \otimes \mathrm{pr}_2^*\mathscr{L}^{-1}\big) = [2]^*\mathscr{L} \otimes \mathscr{L}^{\otimes -2}.$$

  Use that $\lambda_{[2]^*\mathscr{L}} = 2^2\lambda_{\mathscr{L}}$, so $\lambda_{\mathscr{L}^{\Delta}(\lambda)} = 2\lambda$. To see that this is relatively ample, decompose that $\mathscr{L}$ into symmetric and antisymmetric pieces (really, decompose $\mathscr{L}^2$, but whatever). Above is obviously ample for $\mathscr{L}$ symmetric, and is trivial for $\mathscr{L}$ antisymmetric.

- Say $\mathscr{L}$ is a line bundle on an abelian variety X (over a field). Then, $\mathscr{L}$ ample $\implies$ $\mathscr{L}^3$ very ample. We'll accept this without proof.

  As a consequence, $\mathscr{L}^{\Delta}(\lambda)^{\otimes 3}$ from the previous bullet point is relatively very ample.

- Use `cohomology and base change`: Say $\pi : X \to Y$ is proper, finitely presented and $\mathscr{F}$ is a coherent sheaf on X, flat over Y, and $q \in Y$ is a point, you get a map

$$\varphi_q^p : (R^p\pi_*\mathscr{F}) \otimes \kappa(q) \longrightarrow \mathrm{H}^p(X_q, \mathscr{F}_q).$$

  If φ_q^p is surjective, then,

  (1) φ_q^p is an isomorphism; and

  (2) $(R^p\pi_*\mathscr{F})_q$ is a free $\mathscr{O}_{Y,q}$-module $\iff$ φ_q^{p-1} is surjective.

  Taking $p = 0$, we see that if φ_q^0 is surjective for every point, then $\pi_*\mathscr{F}$ is a vector bundle.

  We'll also need `Mumford's Vanishing Theorem`: Let $\mathscr{L}$ be a line bundle on X such that $K(\mathscr{L})$ is finite. Then, there is a unique index $i = i(\mathscr{L})$ satisfying $0 \le i \le n = \dim X$ such that $\mathrm{H}^j(X, \mathscr{L}) \neq 0 \iff j = i$. Furthermore, $\mathscr{L}$ is ample $\iff$ $i(\mathscr{L}) = 0$.

  The upshot of these two is that $\pi_*\mathscr{L}^{\Delta}(\lambda)$ is a locally free sheaf. Similarly, $\pi_*\big(\mathscr{L}^{\Delta}(\lambda)^{\otimes 3}\big)$ is locally free.

- Let $\mathscr{M} := \pi_*\big(\mathscr{L}^\Delta(\lambda)^{\otimes 3}\big)$. We claim that $\mathscr{M}$ has rank $6^n d$, where $d := d_1 \ldots d_n$.

  Use Riemann-Roch for abelian varieties, $\chi(\mathscr{O}_A(D)) = (D^g)/g!$ on a g-dimensional abelian variety.

  Exercise. Fill in details (hint: also relate Euler characteristic of line bundle to the degree of the induced map)

- We have $X \hookrightarrow \mathbb{P}_S(\mathscr{M})$. We would like an embedding in an actual projective space $\mathbb{P}^m_S$.

  Definition 4.10. A `linear rigidification` of (X, λ) is an isomorphism

  $$\mathbb{P}_S(M) \xrightarrow{\sim} \mathbb{P}^m_S. \qquad \diamond$$

  Consider functor $\mathcal{H} : \mathrm{Sch}_{\mathbb{Z}[1/(Nd_n)]} \to \mathrm{Set}$ with $\mathcal{H}(S) = \{(X, \lambda, \eta, \alpha)\}/\sim$, with (X, λ, η) as above, and α a linear rigidification. The forgetful map

  $$\mathcal{H} \longrightarrow \mathcal{A}$$

  is a $\mathrm{PGL}(m+1)$-torsor. Hence, we want to show that $\mathcal{H}$ is representable, and then take GIT quotients to show that $\mathcal{A}$ is representable.

  The Hilbert polynomial is $Q(t) = 6^n dt^n$, so we get a map $f : \mathcal{H} \to \mathrm{Hilb}^{Q(t),1}_m$ of 1-pointed flat, closed subschemes of $\mathbb{P}^m$ w/ Hilbert polynomial $Q(t)$. One uses this to show that $\mathcal{H}$ is representable and quasi-projective.

  Proposition 4.11. *f identifies $\mathcal{H}$ w/ an open subfunctor of $\mathrm{Hilb}^{Q(t),1}_m$, so $\mathcal{H}$ is an open subscheme of this Hilbert scheme.*

  In the proof of this, one appeals to

  Theorem 4.12. *Let S be connected, locally Noetherian. Let $\pi : X \to S$ be smooth, projective w/ section $e : S \to X$. If, for some geometric point $\overline{s} \to S$, the fiber $X_{\overline{s}}$ is an abelian variety w/ identity $e(\overline{s})$, then X is an abelian scheme w/ identity e.*

  Proof Sketch. Show theorem holds when $S = \operatorname{Spec} A$ and $S_0 = \operatorname{Spec} A/I$, with A Artinian local and $I\mathfrak{m} = 0$. Show can extend abelian scheme structures from X_0/S_0 to X/S.

  Second step is to consider functor $F : \mathrm{LocNoethSch}_S \to \mathrm{Set}$ sending T/S to the set of abelian scheme structures on X_T/T w/ identity pullback back from e. Then, show that F is represented by some open subscheme $U \subset S$.

  Final step is to show that U is closed. Apply the valuative criterion for universal closedness, or something like this? ■

  Exercise. Prove Proposition 4.11 (Hint: essentially identify $\mathcal{H}$ w/ the subfunctor representing smooth, closed subschemes of $\mathbb{P}^m$)

- Accept GIT stuff w/o worrying about it

- How do we show $\mathcal{A}$ is smooth?

  Say R is a ring with ideal I satisfying $I^2 = 0$. Let $S = \operatorname{Spec} R$ and $\overline{S} = \operatorname{Spec} R/I$. One needs to show that an abelian scheme (+ extra data) $\overline{X}/\overline{S}$ always lifts to X/S. This is the infinitesimal lifting criterion.

  To verify this, use Grothendieck-Messing theory. Time's up... ■

5 Kush Singhal (Harvard): Review of reductive algebraic groups

Real title: Algebraic Groups, Reductive Groups & Lie Algebras (& maybe some other stuff too).

Outline

(1) Definitions, basic examples, and constructions

(2) Reductive Groups

(3) Lie algebras

Let k be a commutative ring (for now).

Definition 5.1 (`Algebraic Group`)**.**

(1) It is a group object G in Sch_k which is furthermore of finite type over k. In particular, G comes equipped with morphisms

$$m : G \times G \to G, \;\; \varepsilon : \operatorname{Spec} k \to G, \;\; \text{and} \;\; \mathrm{inv} : G \to G$$

satisfying the expected commutative diagrams.

(2) (Functor of points perspective) For G a f.type k-scheme, have associated Yoneda functor

$$G(-) = \mathrm{Hom}(\mathbf{Spec}(-), G) : \mathrm{CAlg}_k \longrightarrow \mathrm{Set}.$$

An `algebraic group structure` on G is a factorization of this functor through the forgetful functor $\mathrm{Grp} \to \mathrm{Set}$. $\diamond$

Definition 5.2. An `algebraic subgroup` $H \leq G$ is a closed subscheme H of G such that $H(R) \leq G(R)$ for all $R \in \mathrm{CAlg}_k$. $\diamond$

Example 5.3. $\mathrm{GL}_{n,k}$ is an affine algebraic group. In fact,

$$\mathrm{GL}_{n,k} = \operatorname{Spec} k[x_{ij}][1/\det].$$

If V is a vector space over k, then

$$\mathrm{GL}_V : R \rightsquigarrow \mathrm{GL}(V \otimes_k R)$$

is representable by a group scheme, which is furthermore algebraic iff V is finite dimensional. $\triangle$

Example 5.4. The `symplectic group`

$$\mathrm{Sp}_{2n,k} : R \rightsquigarrow \left\{ X \in \mathrm{GL}_{2n}(R) : X^t J X = J \right\},$$

where $J \in \mathrm{GL}_{2n}(k)$ is some fixed skew symmetric matrix (usually, $J = \begin{pmatrix} 0 & I_n \\ -I_n & 0 \end{pmatrix}$). $\triangle$

Example 5.5. The `neutral component` G^0 is the connected component of the identity section. $\triangle$

(Probably want k a field, or at least want $\operatorname{Spec} k$ connected above)

Example 5.6. $\mathbb{G}_a = \operatorname{Spec} k[t]$ w/ additive group law, $\mathbb{G}_m = \operatorname{Spec} k[t, t^{-1}]$ w/ multiplicative group law. $\triangle$

Assumption. Go ahead and assume k is a field from now.

Definition 5.7. A torus T is an algebraic group such that $T_{k^s} \cong \mathbb{G}_m^r$ for some r, called the rank of the torus. $\diamond$

Example 5.8. Assume $\operatorname{char} k \neq 2$.[2] Then,

$$\mathrm{SO}_2 : R \mapsto \left\{ \begin{pmatrix} a & b \\ -b & a \end{pmatrix} \in \mathrm{GL}_2(R) : a^2 + b^2 = 1 \right\}$$

is a torus. It is split iff $\sqrt{-1} \in k$, and it has rank 1. $\triangle$

Example 5.9. The Deligne torus, often denoted $\mathbb{S}$, is the $\mathbb{R}$-torus

$$\mathbb{S}(R) = (\mathbb{C} \otimes_{\mathbb{R}} R)^{\times}.$$

For example, $\mathbb{S}(\mathbb{R}) = \mathbb{C}^{\times}$ and $\mathbb{S}_{\mathbb{C}} \simeq \mathbb{G}_{m,\mathbb{C}}^2$. $\triangle$

Definition 5.10. If $\varphi : G_1 \to G_2$ is a homomorphism of algebraic groups, the its kernel if $\ker \varphi$ defined by

$$(\ker \varphi)(R) := \ker(\varphi(R) : G_1(R) \longrightarrow G_2(R)).$$

For example, $\ker(\det : \mathrm{GL}_n \to \mathbb{G}_m) = \mathrm{SL}_n$. $\diamond$

Definition 5.11 (Normalizer)**.** If $H < G$, can define

$$N_G(H)(R) = \left\{ g \in G(R) : gH(R')g^{-1} = H(R') \text{ for all } R' \in \mathrm{CAlg}_k \right\}. \quad \diamond$$

Proposition 5.12. *$N_G(H)$ is an algebraic group.*

Remark 5.13. $N_G(H)(R) = \left\{ g \in G(R) : gH_R g^{-1} = H_R \right\}$ $\circ$

Definition 5.14. A subgroup $H < G$ is normal if $N_G(H) = G$. Equivalently, $H(R)$ is normal in $G(R)$ for all R. $\diamond$

Theorem 5.15 (Milne Theorem 5.21)**.** *If $N \triangleleft G$ and G affine, then there exists an affine algebraic group G/N and a morphism $\pi : G \to G/N$ such that*

(1) $\ker \pi = N$

(2) *$(G/N, \pi)$ is universal (= initial) for this property*

One idea towards proving this is to (fppf-)sheafify the group functor $R \mapsto G(R)/N(R)$.

Warning 5.16. Sheafifying is necessary. Say $G = \mathrm{GL}_{n,\mathbb{Q}}$ and $N = \mathbb{G}_{m,\mathbb{Q}}$ with $N \hookrightarrow G$ the scalar matrices. Then, $G/N \cong \mathrm{PGL}_n \cong \mathrm{PSL}_n$. However, $\mathrm{SL}_n(\mathbb{Q})/\mu_n(\mathbb{Q}) \subsetneq \mathrm{GL}_n(\mathbb{Q})/\mathbb{Q}^{\times}$.

As another example, $\mathbb{G}_m/\mu_2 \cong \mathbb{G}_m$ via the squaring map $\mathbb{G}_m \xrightarrow{2} \mathbb{G}_m$. $\bullet$

Remark 5.17. Given $N \triangleleft G$, have exact sequence $1 \to N \to G \to G/N \to 1$ of fppf sheaves, so get exact sequence

$$1 \longrightarrow \frac{G(R)}{N(R)} \longrightarrow (G/N)(R) \longrightarrow \mathrm{H}^1_{\mathrm{fppf}}(R, N). \quad \circ$$

[2] When $\operatorname{char} k = 2$, this is not reduced, so not a torus.

Definition 5.18. The `center` of an algebraic group G is

$$Z_G(R) := \{g \in G(R) : gh = hg \text{ for all } h \in G(R') \text{ for all } R' \in \mathrm{CAlg}_R\}. \qquad \diamond$$

Definition 5.19 (`Adjoint Group`)**.** $G^{\mathrm{ad}} := G/Z_G$. $\diamond$

5.1 Reductive Groups

Definition 5.20. A `representation` is a morphism of group schemes $\rho : G \to \mathrm{GL}_V$. It is said to be `faithful` if ρ is a monomorphism. $\diamond$

Definition 5.21. An algebraic group G is `linear` if it admits a faithful f.dim representation, i.e. is a subgroup of some GL_n. $\diamond$

Theorem 5.22. *An algebraic group is linear iff it is affine.*

Open Question 5.23 (Apparently)**.** *Are all affine algebraic groups over $k[\varepsilon]/(\varepsilon^2)$ linear?*

Definition 5.24. A `linearly reductive` group G is an algebraic group all of whose (f.dim) representations are semisimple. $\diamond$

On the other end of the spectrum, we have unipotent groups.

Definition 5.25. A group G is `unipotent` if every representation of G has a fixed vector. $\diamond$

Remark 5.26. If $\rho : G \to \mathrm{GL}_V$ is both semisimple and unipotent, then ρ is a direct sum of trivial reps (i.e. is a big trivial rep). $\circ$

Example 5.27. $U_n = \left\{\begin{pmatrix} 1 & & * \\ & \ddots & \\ & & 1 \end{pmatrix}\right\} \hookrightarrow \mathrm{GL}_n$ is unipotent. In fact, every unipotent group is a subgroup of U_n for some n. For example, $\mathbb{G}_a \cong U_2$ is unipotent. $\triangle$

Proposition 5.28. *If $U \triangleleft G$ is a normal unipotent subgroup, then $U < \ker \rho$ for all semisimple reps ρ of G.*

Definition 5.29. The `unipotent radical` $R_u(G)$ of a smooth, affine algebraic group G is the maximal Zariski-connected normal unipotent subgroup. We say such a G is `reductive` if $R_u(G_{\overline{k}}) = 1$. $\diamond$

Warning 5.30. In general, unipotent radicals do not commute with field extensions. However, they do commute w/ separable field extensions. $\bullet$

Theorem 5.31 (Milne 22.138)**.** *Assume $\operatorname{char} k = 0$. For G connected, linearly reductive is equivalent to reductive.*

Example 5.32. $\mathrm{GL}_n, \mathrm{SL}_n, \mathrm{SO}_n, \mathrm{Sp}_{2n}$, all tori – these are all reductive $\triangle$

Non-example. U_n is not reductive. $P_n \subset \mathrm{GL}_n$ (all upper triangle matrices) is not reductive ($R_u(P_n) = U_n$) $\triangledown$

Lemma 5.33. *For any smooth, affine algebraic group G, $G/R_u(G)$ is reductive.*

Example 5.34. $P_n/U_n \cong \mathbb{G}_m^n$ $\triangle$

5.2 Lie Algebras

Definition 5.35. The Lie algebra of an algebraic group G is

$$\mathrm{Lie}(G) := \ker\left(G\left(\frac{k[\varepsilon]}{(\varepsilon^2)}\right) \xrightarrow{\varepsilon\mapsto 0} G(k)\right).$$

This is the tangent space at the identity. ◇

Example 5.36. $\mathrm{Lie}\,\mathrm{GL}_n = \{I + B\varepsilon : B \in M_n(k)\} \cong M_n(k)$. △

Example 5.37. $\mathrm{Lie}\,\mathbb{G}_m = k$ ($n = 1$ in prev example) △

Where is the Lie algebra structure (= Lie bracket) on $\mathrm{Lie}\,G$?

Suppose G is affine, $G = \mathrm{Spec}\,\mathscr{O}(G)$. Let $\varepsilon : \mathscr{O}(G) \to k$ be the counit. Note that $x \in \mathrm{Lie}\,G$ if and only if

$$\begin{tikzcd} \mathrm{Spec}\,k \arrow[r, "\delta\mapsto 0"'] \arrow[rr, bend left, "e"] & \mathrm{Spec}\,k[\delta]/\delta^2 \arrow[r, "x"'] & G \end{tikzcd}$$

commutes, i.e. iff

$$\begin{tikzcd} \mathscr{O}(G) \arrow[r, "x^*"'] \arrow[rr, bend left, "\varepsilon"] & k[\delta]/\delta^2 \arrow[r, "\delta\mapsto 0"'] & k \end{tikzcd}$$

commutes. This is the case iff x^* is a k-derivation of $\mathscr{O}(G)$, i.e. it is a k-linear map $D : \mathscr{O}(G) \to k$ such that $D(fg) = \varepsilon(f)D(g) + \varepsilon(g)D(f)$.

Proposition 5.38. *For G affine, there is a natural bijection* $\mathrm{Lie}(G) \simeq \mathrm{Der}_{k,\varepsilon}(\mathscr{O}(G), k)$.

Let $\Delta : \mathscr{O}(G) \to \mathscr{O}(G) \otimes_k \mathscr{O}(G)$ be the comultiplication map.

Definition 5.39. A derivation $D : \mathscr{O}(G) \to \mathscr{O}(G)$ is left-invariant if

$$\Delta \circ D = (\mathrm{id} \otimes D) \circ \Delta.$$ ◇

Proposition 5.40. *There is an isomorphism*

$$\mathrm{Der}_k^{\textit{left-inv}}(\mathscr{O}(G), \mathscr{O}(G)) \xrightarrow{\sim} \mathrm{Der}_{k,\varepsilon}(\mathscr{O}(G), k)$$

via $D \mapsto \varepsilon \circ D$.

Hence, $\mathrm{Lie}\,G \simeq \mathrm{Der}_k^{\text{left-inv}}(\mathscr{O}(G), \mathscr{O}(G))$. The RHS has an obvious Lie algebra structure, so we get one on the LHS via transfer of structure.

Let's end w/ the Adjoint representation. Let $\mathfrak{g} := \mathrm{Lie}\,G$. There is an exact sequence

$$1 \longrightarrow \mathfrak{g} \otimes_k R \longrightarrow G\left(\frac{R[\delta]}{\delta^2}\right) \longrightarrow G(R) \longrightarrow 1$$

for all $R \in \mathrm{CAlg}_k$, which is moreover split (on the right). In particular, $G(R) \curvearrowright (\mathfrak{g} \otimes_k R)$ via conjugation, so we get a natural map $\mathrm{Ad} : G \to \mathrm{GL}_{\mathfrak{g}}$, called the Adjoint representation. Applying the Lie functor $\mathrm{Lie} : \mathrm{AlgGrp} \to \mathrm{LieAlg}$, we get

$$\mathrm{ad} = \mathrm{Lie}(\mathrm{Ad}) : \mathfrak{g} = \mathrm{Lie}\,G \longrightarrow \mathrm{Lie}(\mathrm{GL}_{\mathfrak{g}}) = \mathrm{End}(\mathfrak{g}).$$

Proposition 5.41. *For all $x, y \in \mathfrak{g}$,* $\operatorname{ad}(x)(y) = [x, y]$.

Proposition 5.42 (Milne Exercise 22.2)**.** *If G is reductive, then* $\ker \operatorname{Ad}_G \cong Z_G$. *In particular, G^{ad} is the image of* Ad_G.

6 Gefei Dang (MIT): Hodge Structures

The endgoal of today is to state a theorem telling us that Hermitian symmetric domains can be realized as parameter spaces of variations of Hodge structures.

Let V be an $\mathbb{R}$-vector space. Define complex conjugation on $V(\mathbb{C}) = \mathbb{C} \otimes_{\mathbb{R}} V$ by

$$\overline{c \otimes v} = \overline{c} \otimes v.$$

Definition 6.1. A Hodge decomposition of an $\mathbb{R}$-vector space V is a decomposition

$$V(\mathbb{C}) = \bigoplus_{p,q \in \mathbb{Z}} V^{p,q}$$

such that $\overline{V^{p,q}} = V^{q,p}$. A Hodge structure is an $\mathbb{R}$-vector space equipped w/ a Hodge decomposition. Given such a thing, the pairs $(p, q) \in \mathbb{Z}^2$ such that $V^{p,q} \neq 0$ are called the types of the Hodge structure. ⋄

Remark 6.2. In any Hodge structure, $\bigoplus_{p+q=n} V^{p,q}$ is invariant under complex conjugation. Thus, there is always a real subspace $V_n \subset V$ such that

$$V_n(\mathbb{C}) = \bigoplus_{p+q=n} V^{p,q}.$$

Furthermore,

$$V = \bigoplus_n V_n,$$

and this is called the weight decomposition. ◦

Definition 6.3. An integral (resp. rational) Hodge structure is a free $\mathbb{Z}$-module (resp. $\mathbb{Q}$-vector space) V equipped with a Hodge decomposition of $V(\mathbb{R})$ such that the weight decomposition is defined over $\mathbb{Z}$ (resp. $\mathbb{Q}$). ⋄

Example 6.4. Let $J : V \to V$ be a complex structure on a real vector space V, i.e. $J^2 = -1$. Let $V^{-1,0}$ and $V^{0,-1}$ denote the $\pm i$ eigenspaces ($V^{-1,0}$ the $+i$ eigenspace). Then,

$$V(\mathbb{C}) = V^{-1,0} \oplus V^{0,-1}$$

is a Hodge structure of type $(-1, 0), (0, -1)$. Conversely, any Hodge structure of this type comes from a complex structure, with $J : V \to V$ defined as $(v^{-1,0}, v^{0,-1}) \mapsto (iv^{-1,0}, -iv^{0,-1})$.[3] △

Example 6.5. Let X be a nonsingular projective $\mathbb{C}$-variety. Then, there is a Hodge decomposition of its de Rham cohomology:

$$\mathrm{H}^n_{\mathrm{dR}}(X, \mathbb{C}) = \bigoplus_{p+q=n} \mathrm{H}^{p,q} \quad \text{where} \quad \mathrm{H}^{p,q} = \mathrm{H}^q(X, \Omega^p).$$ △

[3]Note that $v^{0,-1} = \overline{v^{-1,0}}$

Example 6.6 (Tate Hodge structure)**.** Let $R = \mathbb{Z}, \mathbb{Q}$, or $\mathbb{R}$. Let $R(m)$ be the unique 1-dimensional R-Hodge structure of weight $-2m$. Note that $R(m)$ has type $(-m, -m)$. △

Definition 6.7. The Hodge filtration associated to a Hodge structure of weight n is the (decreasing) filtration

$$F^p = \bigoplus_{r \geq p} V^{r,n-r} \subset V(\mathbb{C}). \qquad \diamond$$

Remark 6.8. The Hodge filtration determines the Hodge structure. If $p + q = n$, then

$$\overline{F_q} = \bigoplus_{r \geq q} \overline{V^{r,n-r}} = \bigoplus_{r \geq q} V^{n-r,r} = \bigoplus_{s \leq p} V^{s,n-s}.$$

Thus, $V^{p,q} = F^p \cap \overline{F^q}$. ◦

6.1 Hodge structures as reps of $\mathbb{S}$

Let T be a torus over k. If T is split, every representation V of T is of the form

$$V \simeq \bigoplus_{\chi \in X^*(T)} V_\chi \text{ where } X^*(T) = \mathrm{Hom}(T_{K^{\mathrm{sep}}}, \mathbb{G}_m).$$

Above, V_χ is the χ-eigenspace of V.

If T is non-split, then it splits over some Galois extension K/k. Let V be a k-vector space with T-action $\rho : T_K \to \mathrm{GL}_{K \otimes_k V}$ defined over K. When does this descend to an action over k? Because T_K is split, we can decompose

$$K \otimes_k V = \bigoplus_{\chi \in X^*(T)} V_\chi.$$

This ρ is defined over k if and only if

$$\sigma V_\chi = V_{\sigma\chi} \text{ for all } \sigma \in \mathrm{Gal}(K/k), \chi \in X^*(T).$$

Above, σ acts on characters via $\sigma\chi = \sigma \circ \chi \circ \sigma^{-1}$. Thus, T-reps over k are gradations $K \otimes V = \bigoplus V_\chi$ that satisfy the above condition.

Definition 6.9. The Deligne torus is $\mathbb{S} := \mathrm{Res}_{\mathbb{C}/\mathbb{R}} \mathbb{G}_m$. ⋄

In particular, $\mathbb{S}(\mathbb{R}) = \mathbb{C}^\times$ and $\mathbb{S}_\mathbb{C} \simeq \mathbb{G}_m^2$. Let's describe its character lattice. Fix an identification

$$\mathbb{S}_\mathbb{C} \xrightarrow{\sim} \mathbb{G}_m \times \mathbb{G}_m$$

so that the induced map

$$\mathbb{C}^\times = \mathbb{S}(\mathbb{R}) \to \mathbb{S}(\mathbb{C}) \xrightarrow{\sim} \mathbb{C}^\times \times \mathbb{C}^\times$$

is

$$z \mapsto (z, \overline{z}),$$

and so that complex conjugation on $\mathbb{S}(\mathbb{C}) \simeq \mathbb{C}^\times \times \mathbb{C}^\times$ is given by

$$\overline{(z_1, z_2)} = (\overline{z}_2, \overline{z}_1).$$

Define the weight homomorphism

$$w : \mathbb{G}_m \to \mathbb{S}$$

whose map on $\mathbb{R}$ points is

$$\begin{array}{rccc} w(\mathbb{R}): & \mathbb{R}^\times & \longrightarrow & \mathbb{C}^\times \\ & r & \longmapsto & r^{-1}. \end{array}$$

With this identification,

$$X^*(\mathbb{S}) = X^*(\mathbb{S}_\mathbb{C}) = \{\chi_{r,s} : (z_1, z_2) \mapsto z_1^r z_2^s : r, s \in \mathbb{Z}\} \cong \mathbb{Z} \times \mathbb{Z},$$

and complex conjugation acts by swapping the coordinates. Indeed,

$$\overline{\chi_{p,q}}(z_1, z_2) = \overline{\chi_{p,q}\Big(\overline{(z_1, z_2)}\Big)} = \overline{\overline{z}_2^p \overline{z}_1^q} = z_1^q z_2^p = \chi_{q,p}(z_1, z_2).$$

Suppose (h, V) is an $\mathbb{R}$-rep of $\mathbb{S}$.[4] Then, we get a decomposition

$$V(\mathbb{C}) = \bigoplus_{p,q\in\mathbb{Z}} V^{p,q} \text{ where } V^{p,q} := V_{\chi_{-p,-q}} = \left\{v \in V(\mathbb{C}) : h(z)v = z^{-p}\overline{z}^{-q}v\right\}$$

such that $\overline{V^{p,q}} = V^{q,p}$. That is, an $\mathbb{R}$-rep of $\mathbb{S}$ is a Hodge structure. Let $w_h = h \circ w : \mathbb{G}_m \to \mathrm{GL}_V$. The weight decomposition is given by

$$V_n = \{v \in V : w_n(r) = r^n v\}.$$

Let $C = h(i) \in \mathrm{GL}(V(\mathbb{C}))$. We call this the `Weil operator`. Note that C acts on $V^{p,q}$ via i^{q-p}.

Example 6.10. The Tate Hodge structure $\mathbb{Q}(m)$ corresponds to the representation

$$\begin{array}{rccc} h: & \mathbb{S} & \longrightarrow & \mathbb{G}_{m,\mathbb{R}} \\ & z & \longmapsto & (z\overline{z})^m. \end{array}$$

Keep in mind that this is of weight $-2m$. △

6.2 Hodge tensors and polarizations

Let V be a weight n Hodge structure, and let W be a weight m Hodge structure. We define the Hodge structure $V \otimes W$ with decomposition

$$(V \otimes W)^{p,q} := \bigoplus_{\substack{r+r'=p \\ s+s'=q}} V^{r,s} \otimes W^{r',s'}.$$

In terms of representations (V, h_V) and $(W, h_W$, this is simply $(V \otimes W, h_V \otimes h_W)$.

Definition 6.11. A `morphism of Hodge structures` $f: V \to W$ is a linear map s.t. $f(V^{p,q}) \subset W^{p,q}$ for all $p, q \in \mathbb{Z}$. ⋄

Definition 6.12. Let $R = \mathbb{Z}, \mathbb{Q}, \mathbb{R}$, and let (V, h) be an R-Hodge structure of weight n. A `Hodge tensor` is a morphism

$$t : V^{\otimes r} \to R\Big(-\frac{nr}{2}\Big)$$

of Hodge structures. Equivalently, t is a Hodge tensor if

$$t(h(z)v_1, h(z)v_2, \ldots, h(z)v_r) = (z\overline{z})^{-nr/2} t(v_1, v_2, \ldots, v_r).$$

[4] $h : \mathbb{S} \to \mathrm{GL}_V$

In particular ($z = i$), we must have $t(Cv_1, \ldots, Cv_r) = t(v_1, \ldots, v_r)$. $\diamond$

Note that both structures above have the same weight, so you can have a nonzero Hodge tensor only if $V^{\otimes r}$ is non-trivial in middle degree.

Example 6.13. Let (V, h) be a Hodge structure of type $(-1, 0), (0, -1)$. Then,

$$t : V \otimes V \longrightarrow \mathbb{R}(1)$$

iff $t(Ju, Jv) = t(u, v)$ with J the complex structure defined earlier. $\triangle$

Definition 6.14. Let (V, h) be a Hodge structure of weigh n. A polarization is a Hodge tensor

$$\psi : V \times V \longrightarrow \mathbb{R}(-n)$$

such that

Does ψ_C spit out real numbers? If so, is that clear?

$$\psi_C(u, v) := (2\pi i)^n \psi(u, Cv).$$

is symmetric and positive definite. $\diamond$

As a consequence of the definition, $\psi(u, v) = (-1)^n \psi(v, u)$.

6.3 Grassmannians and Flags

Let V be an n-dimensional vector space over k. For any $d < n$, the Grassmannian $G_d(V)$ is the set of all d-dimensional subspaces of V. There is a map

$$\begin{array}{ccc} G_d(V) & \longrightarrow & \mathbb{P}\left(\bigwedge^d V\right) \\ W & \longmapsto & \bigwedge^d W \end{array}$$

identifying $G_d(V)$ as a closed subvariety of $\mathbb{P}\left(\bigwedge^d V\right)$. Hence, $G_d(V)$ is also projective.

For a point $W_0 \in G_d(V)$, let S be a complementary subspace to $W_0 \subset V$ (i.e. $V = W_0 \oplus S$), and define

$$G_d(V)_S = \{W \in G_d(V) : W \cap S = 0\}.$$

This is an open subvariety of $G_d(V)$ and admits a bijection

$$G_d(V)_S \simeq \mathrm{Hom}(W_0, S)$$

via $\Gamma_\varphi \leftarrow\!\shortmid \varphi$ (Γ_φ is the graph of φ). The upshot is that $G_d(V)_S$ is an affine space and that $T_{W_0} G_d(V) = \mathrm{Hom}(W_0, S) = \mathrm{Hom}(W_0, V/W_0)$.

Let $\vec{d} := (n > d_1 > \cdots > d_r > 0)$ be a decreasing sequence of numbers. Define

$$G_{\vec{d}}(V) = \left\{V \supset V^1 \supset \cdots \supset V^r \supset 0 : \dim V^i = d_i.\right\}$$

Note that this naturally injects

$$G_{\vec{d}}(V) \hookrightarrow \prod_i G_{d_i}(V)$$

into a product of Grassmannians, so $G_{\vec{d}}(V)$ is also projective. One can show that the tangent space to

$G_{\vec{d}}(V)$ at a flat F can be identified with

$$\left\{ \left(\varphi^i : V^i \to V/V^i : 1 \le i \le r \right) : \varphi^i|_{V^{i+1}} = \varphi^{i+1} \mod V^{i+1} \right\}.$$

6.4 Continuation: Variations of Hodge Structures

Note 3. Stealing a bit of time from the following week's talk.

See Milne's notes for more on today's stuff. These were the main reference.

Let S be a connected complex manifold, and let V be an $\mathbb{R}$-vector space. Suppose that for all $s \in S$, we have a Hodge structure h_s on V of weight n. Let $V_s^{p,q} = V_{h_s}^{p,q}$ and $F_s^p = F_{h_s}^p V$.

Definition 6.15. A family of Hodge structures $(h_s)_{s\in S}$ is continuous if $\dim V_s^{p,q} = d(p,q)$ is constant (i.e. independent of s), and the maps

$$\begin{array}{ccc} S & \longrightarrow & G_{d(p,q)}(V(\mathbb{C})) \\ s & \longmapsto & V_s^{p,q} \end{array}$$

are continuous (for all p, q). ◇

Definition 6.16. $(h_s)_{s\in S}$ is holomorphic if it is continuous, and the map

$$\begin{array}{cccc} \varphi : & S & \longrightarrow & G_{\vec{d}}(V(\mathbb{C})) \\ & s & \longmapsto & F_s^\bullet \end{array}$$

is holomorphic, where $\vec{d} = (\ldots, d(p), \ldots)$, $d(p) = \dim F_s^p$. ◇

Let $(h_s)_{s\in S}$ be a holomorphic family of Hodge structures. Since φ above is holomorphic, it has a well-defined differential

$$\mathrm{d}\varphi : T_s S \longrightarrow T_{F_s^\bullet} G_{\vec{d}}(V(\mathbb{C})) \subset \bigoplus_p \mathrm{Hom}(F_s^p, V(\mathbb{C})/F_s^p).$$

We want to impose a condition on this differential. We say $(h_s)_s$ satisfies Griffiths transversality if $\mathrm{im}(\mathrm{d}\varphi_s)$ lies in the subspace

$$\bigoplus_p \mathrm{Hom}\left(F_s^p, F_s^{p-1}/F_s^p\right) \subset \bigoplus_p \mathrm{Hom}(F_s^p, V(\mathbb{C})/F_s^p).$$

Definition 6.17. A variation of Hodge structures is a holomorphic family satisfying Griffiths transversality. ◇

Example 6.18. Let $f : X \to S$ be a smooth, projective morphism over a complex analytic variety S. For each $s \in S$, there is a Hodge decomposition

$$\mathrm{H}^n_{\mathrm{dR}}(X_s) \otimes \mathbb{C} = \bigoplus_{p+q=n} \mathrm{H}_s^{p,q} \quad \text{where} \quad \mathrm{H}_s^{p,q} = \mathrm{H}^q(X_s, \Omega^p)$$

of the de Rham cohomology. The family $(\mathrm{H}_s^{p,q})_{s\in S}$ is a variation of Hodge structures. △

Question: Variation of Hodge structure technically needs a fixed vector space. What to do if S is not simply connected?

Definition 6.19. Let V be a real vector space. Let T be a family of $\mathbb{R}$-valued multilinear maps on V, containing a nondegenerate bilinear form t_0. Choose $d : \mathbb{Z} \times \mathbb{Z} \to \mathbb{N}$ satisfying

- $d(p,q) = 0$ for almost all p, q

- $d(p,q) = d(q,p)$ always
- $\exists n : d(p,q) = 0$ if $p+q \neq n$.

Define $S(d,T)$ to be the set of all Hodge structures h on V such that

- $\dim V_h^{p,q} = d(p,q)$
- each $t \in T$ is a Hodge tensor for h (i.e. a morphism of Hodge structures)
- t_0 is a polarization for h

We topologize $S(d,T)$ via

$$S(d,T) \subset \prod_{d(p,q)\neq 0} G_{d(p,q)}(V(\mathbb{C})). \qquad \diamond$$

Theorem 6.20. *Let S^+ be a connected component of $S(d,T)$.*

(a) *S^+ has a unique complex structure for which (h_s) is a holomorphic family of Hodge structures*

(b) *S^+ is a hermitian symmetric domain if (h_s) is a variation of Hodge structures*

(c) *Every irreducible hermitian symmetric domain if of the form S^+ for some $V; d, T$*

Proof Sketch of ***(a)***. Let G be the smallest algebraic subgroup of $\mathrm{GL}(V)$ that contains $h(\mathbb{S})$ for all $h \in S^+$. Then, the identity component $G(\mathbb{R})^+$ acts transitively on S^+ w/ some stabilizer K_0. Hence, $S^+ \cong G(\mathbb{R})^+/K_0$ is a smooth manifold. Somehow the complex structure comes from the map φ and its derivative. ■

7 Anne Larsen (MIT): Hermitian symmetric domains and locally symmetric varieties

Note 4. Things were kinda rushed, so double check what's below w/ Milne's notes

Let (M,g) be a complex manifold M equipped with a Hermitian metric g.

Definition 7.1. We say (M,g) is `homogeneous` if $\mathrm{Aut}(M,g)$ acts transitively on M. We say it is `symmetric` if it is homogeneous and for each (equivalently, some) point $p \in M$, there is an involution $s_p \in \mathrm{Aut}(M)$ s.t. p is an isolated fixed point. $\diamond$

Can forget complex structure by considering $(M^\infty, \mathrm{Re}\, g)$. One can use differential geometry to show that s_p above is unique and corresponds to "flipping geodesics" $(\gamma(t) \rightsquigarrow \gamma(-t))$.

Example 7.2.

(1) Upper halfplane $\mathfrak{H}_1$ w/ action by $\mathrm{SL}_2(\mathbb{R})/\pm 1$. Near i, have $z \mapsto -1/z$

"negatively curved" "noncompact type"

(2) $\mathbb{C}/\Lambda$ "flat" with $z \mapsto -z$ near 0'

"Euclidean"

(3) $\mathbb{P}^1_{\mathbb{C}}$ "positive" w/ rotation by $180°$ around north pole

"compact type" △

These are (basically) all types in dimension 1.

Fact. $\mathrm{Aut}(M, g)$ is a real Lie group. compact/noncompact above refers to its identity component.

Fact. Every Hermitian symmetric space decomposes as a product

$$M = M^{+} \times M^{0} \times M^{-}$$

into a compact, Euclidean, and non-compact type factors.

We mainly care about non-compact types.

Definition 7.3. A `hermitian symmetric domain` is a hermitian symmetric space of noncompact type. ◇

Example 7.4. `Siegel upper half space`

$$\mathfrak{H}_g = \left\{ Z \in M_g(\mathbb{C}) : Z = Z^t \ \text{ and } \ \operatorname{Im} Z > 0 \right\}$$

$\mathrm{Sp}_{2g}(\mathbb{R})$ acts transitively on this space. Furthermore, have involution $\begin{pmatrix} 0 & -I \\ I & 0 \end{pmatrix}$ fixing iI. △

Fact. Any hermitian symmetric *domain* will be biholomorphic to some connected, convex open subset of some $\mathbb{C}^n$.

7.1 Hermitian symmetric domains & locally symmetric varieties

Proposition 7.5. *Let (M, g) be a hermitian symmetric domain. The following 3 groups have equal identity components*

$$\mathrm{Aut}(M, g), \quad \mathrm{Aut}(M^{\infty}, \operatorname{Re} g), \quad \textit{and} \quad \mathrm{Hol}(M).$$

Furthermore, there is a connected, adjoint (= trivial center), semisimple real algebraic group $G/\mathbb{R}$ s.t. their identity components are $G(\mathbb{R})^{+}$.

Fact. $\mathrm{Aut}(M, g)^{+}$ acts transitively on M. Given $p \in M$, $K_p := \operatorname{Stab} p$ is compact.

We want to algebraize the situation.

Notation 7.6. We'll write D for the Hermitian symmetric domain (M, g).

Theorem 7.7. *If D is a HSM and $p \in D$, then there is a unique homomorphism $u_p : U_1 \to G$ such that $v_p(z)$ fixes p and acts by multiplication by z on the tangent space $T_p D$.*

U_1 is the circle group (1x1 unitary matrices)

Proof Sketch. Look at 'geodesic normal coordinates' around p. ■

Definition 7.8. If G is a real algebraic group and θ is an involution, then θ is `Cartan` if

$$G^{(\theta)}(\mathbb{R}) = \{ g \in G(\mathbb{C}) : g = \theta(\overline{g}) \}$$

is compact. ◇

Fact. Connected $G/\mathbb{R}$ has a Cartan involution $\iff$ it is reductive. In this case, all Cartan involutions are conjugate.

Example 7.9. If $G = \mathrm{GL}_n$, then $M \rightsquigarrow (M^t)^{-1}$ is a Cartan involution as $G^{(\theta)}(\mathbb{R}) = U_n$. △

Example 7.10. The identity map is a Cartan involution iff $G(\mathbb{R})$ is compact △

Theorem 7.11. *Let D be a HSD w/ associated real algebraic G. Then, the homomorphism $u_p : U_1 \to G$ we defined (for some fixed $p \in D$) satisfies*

(1) Note U_1 acts on $\mathrm{Lie}(G)_{\mathbb{C}}$ via $\mathrm{Ad} \circ u_p$. The only characters of this action are $1, z, z^{-1}$.

(2) $\mathrm{Ad}(u_p(-1))$ is a Cartan involution.

(3) $u_p(-1)$ does not project to 1 in any factor of G (recall G is semisimple)

Conversely, given a real `adjoint` *(= semisimple w/ trivial center) algebraic group G and a homomorphism $u : U_1 \to G$ satisfying these properties, the set D of conjugates of u by elements of $G(\mathbb{R})^+$ is a HSD.*

Proof Sketch. **(1)** $G(\mathbb{R})^+/K_p \cong D$ as smooth manifolds. We claim that conjugation by $u_p(z)$ fixes K_p pointwise. We also claim that $v_p(z)$ acts by $\cdot z$ on T_pD. When you complexify, the characters you get are $1, z, z^{-1}$.

(2) General fact about spaces of negative curvature

(3) G is noncompact (so no compact factors)

In the other direction, K_u is compact, essentially by **(2)**. By staring at $gv(z)g^{-1} = v(z)$, can show that K_u is contained in the compact $G^{(\theta)}$. This is enough to give $D := G(\mathbb{R})^+/K_u$ a smooth structure. **(1)** is used to give a complex structure on T_uD. To get Hermtiian form, average over K_u and move around by $G(\mathbb{R})^+$, or something like this. ∎

As a consequence, can classify Hermitian symmetric domains in terms of algebraic groups data/Dynkin diagrams.

7.2 Locally symmetric varieties

Proposition 7.12. *If D is a HSD, and $\Gamma \subset \mathrm{Hol}(D)^+$ is discrete + torsion-free, then the quotient $\Gamma\backslash D$ has a complex structure.*

Definition 7.13. Subgroups H_1, H_2 of G are `commensurable` if $H_1 \cap H_2$ has finite index in both H_1, H_2. ⋄

Definition 7.14. If $G/\mathbb{Q}$ is an algebraic group, then $\Gamma \subset G(\mathbb{R})$ is an `arithmetic subgroup` if its commensurable w/ $G(\mathbb{Q}) \cap \mathrm{GL}_n(\mathbb{Z})$ for some (any) embedding $G \hookrightarrow \mathrm{GL}_n$. ⋄

Definition 7.15. If H is a real algebraic group, a subgroup $\Gamma \leq H(\mathbb{R})$ is `arithmetic` if there is an algebraic group $G/\mathbb{Q}$ and a surjection $G(\mathbb{R})^+ \twoheadrightarrow H(\mathbb{R})^+$ w/ compact kernel along with an arithmetic subgroup Γ_G s.t. $\Gamma_G \cap G(\mathbb{R})^+$ maps onto $\Gamma \cap H(\mathbb{R})^+$. ⋄

Proposition 7.16. *If H is a semisimple real Lie group w/ a faithful, f.dim representation, then every arithmetic subgroup of H is discrete, of finite covolume, and contains a torsion-free subgroup of finite index.*

Theorem 7.17 (Bailey-Borel)**.** *Let $D(\Gamma) := \Gamma\backslash D$ be the quotient of a HSD by a torsion-free arithmetic subgroup of $\mathrm{Hol}(D)^+$. Then, $D(\Gamma)$ has a canonical structure of a quasi-projective algebraic variety. It is open in a canonical projective variety $D(\Gamma)^*$.*

(Potentially, the boundary is of codimension ≥ 2)

Theorem 7.18 (Borel)**.** *If $D(\Gamma)$ is as above and V is a nonsingular quasi-projective variety over $\mathbb{C}$, then any holomorphic map $f : V^{an} \to D(\Gamma)^{an}$ is algebraic.*

Theorem 7.19. *Let $D(\Gamma)$ be as before. Then, $D(\Gamma)$ only has finitely many automorphisms as a complex manifold.*

8 Eunsu Hur (MIT): Shimura data and Shimura varieties – Didn't Go

9 Dylan Pentland (Harvard): Examples of Shimura Varieties

Starts with the classics: modular curves

Say $K \leq \mathrm{GL}_2(\mathbb{A}_f)$ is a compact open. Consider

$$Y_K(\mathbb{C}) := \mathrm{GL}_2(\mathbb{Q})\backslash\mathbb{H}^{\pm} \times \mathrm{GL}_2(\mathbb{A}_f)/K.$$

Which K do we pick to get something that's more familiar as a modular curve?

Example 9.1. Take $K = \widehat{\Gamma}(N) := \ker\Big(\mathrm{GL}_2(\widehat{\mathbb{Z}}) \longrightarrow \mathrm{GL}_2(\mathbb{Z}/N\mathbb{Z})\Big)$. You might suspect that this K literally recovers $Y(N) = \Gamma(N)\backslash\mathbb{H}$. That's not quite true. Instead, $\pi_0(Y_{\widehat{\Gamma}(N)}) = \widehat{\mathbb{Z}}^{\times}/\det\widehat{\Gamma}(N) = (\mathbb{Z}/N\mathbb{Z})^{\times}$, and each component is isomorphic to the usual $Y(N)$. △

Warning 9.2. There's no Shimura datum for GL_n when $n \geq 3$. •

To get to the next example, we'll look at (general) symplectic groups $\mathrm{GSp}(\psi)$.

Definition 9.3. A `Shimura datum` (G, X) consists of a connected reductive group $G/\mathbb{Q}$ along with a ($G(\mathbb{R})$-)conjugacy class X of homomorphisms $\mathbb{S} \to G_{\mathbb{R}}$ satisfying

(SV1) For any $h \in X$, the action of $\mathbb{C}^{\times}$, via $\mathrm{Ad}\circ h$, on $\mathrm{Lie}(G_{\mathbb{C}})$ only has the characters $z \mapsto z/\overline{z}$, $\overline{z}/z$, or 1.

(SV2) For any $h \in X$, the element $\mathrm{ad} \circ h(i)$ is a Cartan involution.

(SV3) G^{ad} has no $\mathbb{Q}$-factor where h is trivial ◇

Example 9.4. Modular curves come from the Shimura datum $(\mathrm{GL}_2, \mathbb{H}^{\pm})$. Here, $i \in \mathbb{H}^{\pm}$ corresponds to the homomorphism $\mathbb{S} \to \mathrm{GL}_2$ given by the action of $\mathbb{C}^{\times}$ on $\mathbb{C} = \mathbb{R}^{\oplus 2}$. △

9.1 Siegal Shimura Datum

Definition 9.5. Let V be a $\mathbb{Q}$-vector space, give it a symplectic form ψ (non-degenerate, alternating). The `general symplectic group` $\mathrm{GSp}(\psi)/\mathbb{Q}$ is defined by

$$\mathrm{GSp}(\psi)(R) := \left\{g \in \mathrm{GL}_V(R) : \psi(gv, gv) = c(g)\psi(v, v) \text{ for some } c(g) \in R^{\times}, \text{ for all } v \in V\right\}.$$

Note that $\mathrm{Sp}(\psi) = \mathrm{GSp}(\psi)^{\mathrm{der}}$. ◇

Lemma 9.6. *There are bijections*

$$\left\{\begin{matrix}\textit{complex structures}\\ \textit{on } V\end{matrix}\right\} \longleftrightarrow \left\{\begin{matrix}\textit{Hodge structures of type}\\ (-1,0),(0,-1)\end{matrix}\right\} \longleftrightarrow \left\{\begin{matrix}\textit{reps } h : \mathbb{S} \to \mathrm{GL}_V \textit{ where}\\ h_{\mathbb{C}} \curvearrowright V_{\mathbb{C}} \textit{ only via } z, \overline{z}\end{matrix}\right\}$$

- If J is a complex structure. The Hodge structure comes from the $(\pm i)$-eigenspaces. The representation $h_J : \mathbb{S} \to \mathrm{GL}_V$ is given by $h_J(a + bi) = a + bJ$.
- Given a rep h, the complex structure is given by $J := h(i)$.

Note 5. Let $S^{\pm}$ denote the set of complex structures J on V such that $\mathrm{im}\, h_J \subset \mathrm{GSp}(\psi)(\mathbb{R})$.

It's easier to understand things if you define $S^{\pm}$ as the union of the J's s.t. ψ_J is

This will be our conjugacy class of homomorphisms.

Proposition 9.7. *$(\mathrm{GSp}(\psi), S^{\pm})$ is a Shimura datum.*

Proof.

(SV1) Take a single $h_J \in S^{\pm}$, and write $V = \mathbb{Q}^{2n}$. We can decompose, according to J,

$$V_{\mathbb{C}} \simeq V^{-1,0} \oplus V^{0,-1},$$

where $z \in \mathbb{C}^{\times} = \mathbb{S}(\mathbb{R})$ acts by z on the first factor and by $\overline{z}$ on the second. Now, we want to understand $\mathrm{Ad} \circ h$. Observe $\mathrm{Ad}_g(X) = gXg^{-1}$. We'll compute how $g = h_J(z)$ acts on $\mathrm{End}(V_{\mathbb{C}}) \supset \mathrm{Lie}(\mathrm{GSp}(\psi)_{\mathbb{C}})$. Note that

$$\mathrm{End}(V_{\mathbb{C}}) = \mathrm{Hom}(V^{-1,0}, V^{-1,0}) \oplus \mathrm{Hom}(V^{-1,0}, V^{0,-1}) \oplus \mathrm{Hom}(V^{0,-1}, V^{-1,0}) \oplus \mathrm{Hom}(V^{0,-1}, V^{0,-1}).$$

Can directly compute how z acts on each piece, and so win.[5]

(SV2) Need to show that $\mathrm{ad} \circ h(i)$ is a Cartan involution of $G^{\mathrm{ad}}_{\mathbb{R}}$.

Let J be a complex structure on V coming from $S^{\pm}$. The condition $\mathrm{im}\, h_J \subset \mathrm{GSp}(\psi)(\mathbb{R})$ actually forces $\psi(Ju, Jv) = \psi(u, v)$ (exercise[6]), so $\mathrm{ad}J \in \mathrm{Sp}(\psi)$.

Recall 9.8. An involution θ of $G/\mathbb{R}$ is `Cartan` if

$$G^{(\theta)}(\mathbb{R}) := \{g \in G(\mathbb{C}) : g = \theta(\overline{g})\}$$

is compact. ⊙

Lemma 9.9. *If a faithful f.dim $\mathbb{R}$-rep of $G_{\mathbb{R}}$ carries a positive definite symmetric G-invariant form, then G is compact.*

One can easily check that

$$\psi(Ju, Jv) = \psi(u, v) \iff \psi_J(u, v) := \psi(u, Jv) \text{ is symmetric.}$$

Somehow, one can show that it is positive-definite as well.

(SV3) Want to check that G^{ad} has no $\mathbb{Q}$-factor on which h is trivial.[7] The group $\mathrm{Sp}(\psi)$ is simple, but not compact over $\mathbb{R}$. ■

Remark 9.10. There are additional Shimura axioms (see Milne), and the Siegal Shimura datum satisfies them as well. ○

9.2 Other Examples

Slogan. Siegel ⊂ PEL ⊂ Hodge ⊂ Abelian

Theorem 9.11. *Let $K \subset \mathrm{GSp}(\psi)(\mathbb{A}_f)$ compact open. Then,*

$$\mathrm{Sh}_K(\mathbb{C}) = \mathrm{GSp}(\psi)(\mathbb{Q}) \backslash S^{\pm} \times \mathrm{GSp}(\psi)(\mathbb{A}_f)/K$$

[5] It acts by $1, z/\overline{z}, \overline{z}/z, 1$

[6] Hint: $\mathbb{C}^{\times} = \mathbb{S}(\mathbb{R}) \to \mathrm{GSp}(\psi)(\mathbb{R}) \to \mathbb{R}^{\times}$ lands in the identity component

[7] $\mathbb{S} \xrightarrow{h} G_{\mathbb{R}} \to G^{\mathrm{ad}}_{\mathbb{R}} \to$ component

parameterizes (equivalence classes of) triples $(A, s, \eta K)$, where

- *$A/\mathbb{C}$ is an abelian variety*
- *$\pm s$ is a polarization of* $\mathrm{H}_1(A, \mathbb{Q})$
- *ηK is a K-orbit of $\mathbb{A}_f$-linear isos*

$$V(\mathbb{A}_f) \xrightarrow{\sim} V_f(A) := \mathrm{H}_1(A, \mathbb{Q}) \otimes_{\mathbb{Q}} \mathbb{A}_f$$

such that ψ maps to some $\mathbb{A}_f^\times$-multiple of s.

Remark 9.12. This forces $\dim A = \dim V/2$. ○

Let's end by describing the map

$$(A, s, \eta K)/\sim \longmapsto [ah, a \circ \eta]$$

where

- $a : \mathrm{H}_1(A, \mathbb{Q}) \longrightarrow V$ where $\psi \rightsquigarrow \mathbb{Q}^\times$ multiple of s
- $ah : z \mapsto a \circ h(z) \circ a^{-1}$ for h a Hodge structure on $H_1(A, \mathbb{Q})$
- $a \circ \eta$; just pick some isomorphism η

10 Aaron Landesman (MIT): Canonical Models of Shimura Varieties

"The most important thing in a talk is to make sure everyone is awake."

So far, Shimura varieties for us have been defined over the complex numbers. We would like to define a canonical model for them over some number field.

Example 10.1. The Siegal modular varieties $\mathbb{A}_g$ (attached to GSp_{2g}) can be defined over the rationals, where it is still a moduli space for PPAVs. △

Definition 10.2. Let $G/\mathbb{Q}$ be a reductive group, and let X be a $G(\mathbb{R})$-conjugacy class of maps $\mathrm{Res}_{\mathbb{C}/\mathbb{R}} \mathbb{G}_m = \mathbb{S} \to G_{\mathbb{R}}$. The pair (G, X) is called a *Shimura datum* when they satisfy three axioms. ⋄

Remark 10.3 (Audience Remark). Axiom 1 says the family of Hodge structures satisfy Griffith transversality. Axiom 2 is related to being polarized. ○

Definition 10.4. Given a Shimura datum (G, X) and a compact open $K \subset G(\mathbb{A}_f)$, get a `Shimura variety associated to` K:

$$\mathrm{Sh}_K(G, X) := G(\mathbb{A}) \backslash X \times G(\mathbb{A}_f)/K.$$

The `Shimura variety` (G, X) is the inverse system $\mathrm{Sh}_K(G, X)$. ⋄

Example 10.5. Take $G = \mathrm{GL}_2$ and X the conjugacy class of the map

$$\begin{array}{cccc} h : & \mathbb{S} & \longrightarrow & \mathrm{GL}_{2,\mathbb{R}} \\ & a + bi & \longmapsto & \begin{pmatrix} a & b \\ -b & a \end{pmatrix}. \end{array}$$

One can identify $X \cong \mathbb{H}^{\pm} = \mathbb{C} \setminus \mathbb{R}$ (via $ghg^{-1} \mapsto g \cdot i$). Say $K = \mathrm{GL}_2(\widehat{\mathbb{Z}})$. Then,

$$\mathrm{Sh}_K(G,X) = G(\mathbb{Q})\backslash X \times G(\mathbb{A}_f)/K = G(\mathbb{Z})\backslash X = \mathrm{GL}_2(\mathbb{Z})\backslash \mathbb{H}^{\pm} = Y(1) = M_{1,1} = \mathbb{A}^1_{\mathbb{C}} \qquad \triangle.$$

Goal. Define some canonical model for $\mathrm{Sh}_K(G,X)$ over a number field, called the `reflex field`.

Question 10.6. *Which number field?*

Remark 10.7. We have to worry about where h is defined. Is there some $h \in X$ defined over a number field? ◦

Consider some $h : \mathbb{S} \to G_{\mathbb{R}}$ and its class in $G_{\mathbb{R}}\backslash \mathrm{Hom}(\mathbb{S}, G_{\mathbb{R}})$. That is, only consider its conjugacy class.

Lemma 10.8. *$[h]$ is defined over $\overline{\mathbb{Q}}$, and hence over some number field.*

Proof. Choose a split maximal torus $T \subset G_{\overline{\mathbb{Q}}}$, and consider the `Weyl group` $W = T\backslash N_G(T)$ along with the map

$$W\backslash \mathrm{Hom}(\mathbb{G}_m, T) \longrightarrow G\backslash \mathrm{Hom}(\mathbb{G}_m, G).$$

This is a bijection, because any two maximal tori are conjugate. Since $\mathrm{Hom}(\mathbb{G}_m, T)$ is étale over $\overline{\mathbb{Q}}$, the quotient $W\backslash \mathrm{Hom}(\mathbb{G}_m, T)$ is as well. Thus, all of the points of this quotient are defined over $\overline{\mathbb{Q}}$. ■

Definition 10.9. The `reflex field` $E(G,X)$ of (G,X) is the minimal number field that a member of X is defined over. ⋄

Example 10.10. If $T/\mathbb{Q}$ is a torus and you have $h : \mathbb{S} \to T_{\mathbb{R}}$, then $E(T,h)$ is the fixed field of $\{\sigma \in \mathrm{Gal}(\overline{\mathbb{Q}}/\mathbb{Q}) : \sigma \cdot h = \sigma\}$. △

Example 10.11. Consider Siegel Shimura datum (GSp_{2g}, X). Write $V = L \oplus L'$ with L, L' maximal isotropic subspaces w.r.t. the symplectic form. Here, X is the conjugacy class of $h : \mathbb{G}_m \to \mathrm{GL}(V)$ where z acts by z on L and 1 on L'. Observe

- **(a)** There is a unique $\mathrm{GSp}_{2g}(\mathbb{R})$-orbit of (L, L'), i.e. any two such pairs are translates by $\mathrm{GSp}_{2g}(\mathbb{R})$.
- **(b)** There exists such (L, L') defined over $\mathbb{Q}$

These will tell you that $E(\mathrm{GSp}_{2g}, X) = \mathbb{Q}$. △

Definition 10.12. $x \in X$ is `special` if there exists a torus $T \subset G$ (over $\mathbb{Q}$) so that $h_x(\mathbb{C}^\times) \subset T(\mathbb{R})$. ⋄

(This is probably equivalent to $\mathrm{im}\, h_x \subset T_{\mathbb{R}}$)

Example 10.13. What are the special points for $(G,X) = (\mathrm{GL}_2, \mathbb{H}^{\pm})$. The special points here will be CM elliptic curves (i.e. the $x \in \mathbb{H}^{\pm}$ s.t. $\mathbb{Q}(x)$ is a quadratic extension of $\mathbb{Q}$).

Say $E = \mathbb{Q}(x)$ is quadratic, then x will be special, and $\mathrm{Res}_{E/\mathbb{Q}}\, \mathbb{G}_m$ will be the associated torus. Conversely, if x is special, then the torus will be rank 2 which will force $\mathbb{Q}(x)$ to be a quadratic extension. △

Remark 10.14 (Audience Remark). If $x \in \mathbb{H}^{\pm}$ generates a quadratic extension, can alternatively describe the torus T as the connected component of the stabilizer of the pair $\{x, \overline{x}\}$. ◦

10.1 Defining Canonical models

Say $(T,x) \subset (G,X)$ is special. Let $E(x)$ be the field of definition of the map $\mu_x : \mathbb{G}_m \to G_{\mathbb{C}}$ associated to h_x. Define

$$r_x : \mathbb{A}_{E(x)}^{\times} \longrightarrow T(\mathbb{A}_{\mathbb{Q},f})$$

via

$$(a_\infty, a_f) \longmapsto \sum_{\rho : E \hookrightarrow \overline{\mathbb{Q}}} \rho(\mu_x(a_f)).$$

There's also the Artin map

$$\mathrm{Art} : \mathbb{A}_{E(x)}^{\times} \longrightarrow \mathrm{Gal}(E(x)^{\mathrm{ab}}/E(x))$$

(normalized by sending a uniformizer at v to Frob_v^{-1}).

Definition 10.15. We'll say $M_K(G,X)$, a model of $\mathrm{Sh}_K(G,X)$ over $E(G,X)$, is `canonical` if for all special points $(T,x) \subset (G,X)$ and every $a \in G(\mathbb{A}_f)$, $\sigma \in \mathrm{Gal}(E(x)^{\mathrm{ab}}/E(x))$ acts via

$$\sigma[x,a]_K = [x, r_x(s)a]_K$$

for any $s \in \mathbb{A}_{E(x)}^{\times}$ mapping to σ under the Artin map. ◇

Example 10.16. For (GSp_{2g}, X), we claim $\mathcal{A}_g$ defines a canonical model over $\mathbb{Q}$. To show this, one needs to check that for each special point, the Galois group acts as dictated. If you specify a torus in GSp_{2g}, there's a CM Shimura variety associated to that torus. Thus, it will suffice to compute canonical models for tori.

Recall $\mathrm{Sh}_K(T, h_\Phi)$ parameterizes triples $(A, i, \eta K)$, where A is an abelian variety, $i : E \to \mathrm{End}^0(A)$, and $\eta : V(\mathbb{A}_f) \xrightarrow{\sim} V_f(A)$. Let M_K be the model of $\mathrm{Sh}_K(T,x)$ where $\sigma \in \mathrm{Gal}$ acts via

$$\sigma(A, i, \eta K) \mapsto (\sigma A, {}^\sigma i, {}^\sigma \eta K),$$

where

$$\begin{tikzcd} V(\mathbb{A}_f) \arrow[r, "\eta"'] \arrow[rr, bend left, "{}^\sigma\eta"] & V_f(\mathbb{A}) \arrow[r, "V_f(\sigma)"'] & V_f(\sigma A). \end{tikzcd}$$

This M_K is a canonical model of $\mathrm{Sh}_K(T,x)$.

Proof. Just need to compute the corresponding action on $[g]_K \in \mathrm{Sh}_K(T, h_\Phi) = T(\mathbb{Q})\backslash T(\mathbb{A}_f)/K$. The identification $M_K \to \mathrm{Sh}_K(T, h_\Phi)$ is via

$$(A, i, \eta) \mapsto [a \circ \eta],$$

where

$$V(\mathbb{A}_f) \xrightarrow{\eta} V_f(A) \xrightarrow[E\text{-linear}]{a} V(\mathbb{A}_f).$$

Note that $\sigma(A, i, \eta) = (\sigma A, {}^\sigma i, {}^\sigma \eta)$ maps to $[{}^\sigma \eta \circ b]$ for some E-linear b. By main theorem of CM, there exists a map $\alpha : A \to \sigma A$ such that $\alpha(N_\Phi(s) \cdot x) = \sigma(x)$, where $\mathrm{Art}_{E(x)} : s \mapsto \sigma$. Thus, we can take $b := a \circ V_f(\alpha)^{-1}$. Expanding definitions,

$$[{}^\sigma \eta \circ b] = [a \circ V_f(\alpha)^{-1} \circ \sigma \circ \eta] = [a \circ N_\Phi(s) \circ \eta] = [N_\Phi(s) a \circ \eta].$$

■

$\triangle$

11 List of Marginal Comments

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