ON BRAUER GROUPS OF TAME STACKS (DRAFT)

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Abstract. We develop some general tools for computing the Brauer group of a tame algebraic stack $\mathcal{X}$ by studying the difference between it and the Brauer group of the coarse space X of $\mathcal{X}$. It is our hope that these tools will be used to simplify future computations of Brauer groups of stacks. Informally, we show that $\operatorname{Br}\mathcal{X}$ if often built from $\operatorname{Br}X$ and "information about the Picard groups of the fibers of $\mathcal{X} \to X$". Along these lines, we compute, for example, the Brauer group of the moduli stack $\mathcal{Y}(1)_S$ of elliptic curves, over any regular noetherian $\mathbb{Z}[1/2]$-scheme S as well as the Brauer groups of (many) stacky curves (allowing generic stabilizers) over algebraically closed fields.

Contents

1. Introduction

Brauer groups of fields were classically studied objects whose definition was generalized to rings in work of Azumaya, Auslander, and Goldman [Azu51, AG60], and then later to schemes in work of Grothendieck

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[Gro66a, Gro66b, Gro68]. They have been cemented as important cohomological invariants for their applications to class field theory, to understanding ℓ-adic cohomology (especially of curves), and to obstructions to rational points on varieties. In recent times, there has been growing interest in extending our understanding of Brauer groups from schemes to stacks. There is, for example, the breakthrough work of Antieau and Meier [AM20] on the Brauer group of the moduli stack $\mathcal{Y}(1)$ of elliptic curves (which inspired papers such as [Shi19, Mei18, ABJ+24]), work of Lieblich [Lie11] who used the Brauer group of $B\mu_n$ to study the period-index problem, work of Shin computing the Brauer groups of a variety of stacks [Shi21, Shi23, Shi24], work of Di Lorenzo and Pirisi computing Brauer groups via their theory of cohomological invariants [LP22, DLP21], and work of Tim Santens [San23] who has begun the study of Brauer-Manin obstructions on (generically schemey) stacky curves.

That said, many of the previously quoted papers are essentially devoted to computing the Brauer groups of very specific stacks (often even just a single stack viewed over various bases). For example, between the papers [AM20, Shi19, Mei18, LP22, ABJ+24] which compute Brauer groups of modular curves, the only two such curves considered have been $\mathcal{Y}(1)$ and the moduli stack $\mathcal{Y}_0(2)$ of elliptic curves equipped w/ an étale subgroup of order 2. It is the goal of the present paper to develop general enough tools for computing Brauer groups of (tame) stacks (especially of stacky curves; see Section 1.1 for a precise definition) so that, informally speaking, carrying out such computations in the future would take only a handful of pages instead of a whole paper. We work out some examples in Section 9 to help the reader evaluate how well we achieve this goal.

Let $\mathcal{X}$ be a tame algebraic stack (in the sense of [AOV08]; see Definition/Proposition 2.10), with coarse space $c\colon \mathcal{X} \to X$. We aim to study $\mathrm{Br}'(\mathcal{X}) \coloneqq \mathrm{H}^2(\mathcal{X}, \mathbb{G}_m)_{\mathrm{tors}}$ (or, rather, the cokernel of $c^*\colon \mathrm{Br}' X \to \mathrm{Br}' \mathcal{X}$) via the Leray spectral sequence. Our main technical result along these lines is as follows.

Theorem A. *If $\mathcal{X}$ is 'locally Brauerless' in the sense of Definition 4.13, then $\mathrm{R}^2 c_* \mathbb{G}_m = 0$. Consequently, there is an exact sequence*

$$\dots \longrightarrow \mathrm{H}^2(X, \mathbb{G}_m) \xrightarrow{c^*} \mathrm{H}^2(\mathcal{X}, \mathbb{G}_m) \longrightarrow \mathrm{H}^1(X, \mathrm{R}^1 c_* \mathbb{G}_m) \longrightarrow \dots \tag{1.1}$$

Proof. This is Theorem 4.14, and a longer version of (1.1) appears in Corollary 4.16. ■

Remark 1.1. In words, Theorem A says that understanding the Brauer group $\mathrm{H}^2(\mathcal{X}, \mathbb{G}_m)$ of $\mathcal{X}$ can be reduced to understanding the Brauer group of X along with understanding the Picard groups of the fibers of c (this is what $\mathrm{R}^1 c_* \mathbb{G}_m$ keeps track of); as shown in the various examples appearing in this paper, these can, in practice, be tractable objects to study. ○

The strategy alluded to in Remark 1.1 is used in Section 9 to carry out the following example computations.

Theorem B. *Let $\mathcal{Y}(1)$ (resp. $\mathcal{X}(1)$, resp. $\mathcal{Y}_0(2)$) denote the moduli stack of elliptic curves (resp. generalized elliptic curves, resp. elliptic curves equipped with a subgroup of order 2), and let S be a noetherian $\mathbb{Z}[1/2]$-scheme. Then,*

(1) $\mathrm{Br}' \mathcal{X}(1)_S \simeq \mathrm{Br}' \mathbb{P}^1_S \simeq \mathrm{Br}' S$.

(2) If S is regular, there is an explicit isomorphism

$$\mathrm{Br}' \mathbb{A}^1_S \oplus \mathrm{H}^1(S, \mathbb{Z}/12\mathbb{Z}) \xrightarrow{\sim} \mathrm{Br}' \mathcal{Y}(1)_S.$$

(3) If S is regular, there is an explicit isomorphism

$$\mathrm{Br}'(\mathbb{A}^1_S \setminus \{0\}) \oplus \mathrm{H}^1(S, \mathbb{Z}/4\mathbb{Z}) \oplus \mathrm{H}^0(S, \mathbb{Z}/2\mathbb{Z}) \xrightarrow{\sim} \mathrm{Br}' \mathcal{Y}_0(2)_S.$$

Proof. Part **(1)** is proved in Corollary 9.1.9 (note that $\mathrm{Br}' \mathbb{P}^1_S \simeq \mathrm{Br}' S$ by [Gab81, Chapter II, Theorem 2]), part **(2)** is proved in Theorem 9.2.5 and Proposition 9.2.7, and part **(3)** is proved in Theorem 9.3.8 and Proposition 9.3.9. ■

Remark 1.2. Part **(2)** of Theorem B extends results of [AM20, Mei18, LP22], at least away from characteristic 2, and part **(3)** extends [ABJ+24, Theorem 1.1]. ◦

Remark 1.3. This paper focuses mainly on stacks which are tame, or, at least, which are generically tame, as is the case for $\mathcal{Y}(1), \mathcal{X}(1)$ in characteristic 3. However, it would be interesting to also understand how one can effectively compute certain classes of wild stacks. For example, Shin [Shi19] and Di Lorenzo–Pirisi [LP22] were able to compute $\mathrm{Br}'\,\mathcal{Y}(1)_k$ for characteristic two fields k; they show it is always an extension $0 \to \mathrm{Br}'\,\mathbb{A}^1_k \oplus \mathrm{H}^1(k, \mathbb{Z}/12\mathbb{Z}) \to \mathrm{Br}'\,\mathcal{Y}_0(2)_k \to \mathbb{Z}/2\mathbb{Z} \to 0$. In light of Theorem B**(2)**, we view this $\mathbb{Z}/2\mathbb{Z}$ as a 'wild piece of the Brauer group' and think it would be interesting to know how to predict/compute such wild pieces more general. ◦

The generality of Theorem A allows it to readily apply not just to specific examples like those considered above, but also to more general classes of stacks.

Theorem C. *Let k be an algebraically closed field, let $\mathcal{Y}$ be a multiply rooted stack (i.e. of the form (7.1)) over a smooth k-curve X, let G/k be a finite commutative linearly reductive group, and let $\mathcal{X} \to \mathcal{Y}$ be a G-gerbe. If $\mathcal{X}$ is locally Brauerless (in the sense of Definition 4.13), then* $\mathrm{H}^2(\mathcal{X}, \mathbb{G}_m) \simeq \mathrm{H}^1(X, G^\vee)$.

Proof. This is a special case of Proposition 7.10. ■

Remark 1.4. In [ABJ+24, Remark 1.3], my co-authors and I remarked that Tsen's theorem (that Brauer groups of curves over algebraically closed fields vanish) continues to hold for some tame stacky curves, but fails for others (e.g. $\mathrm{Br}\,\mathcal{Y}(1)_{\overline{k}} = 0$ but $\mathrm{Br}\,\mathcal{Y}_0(2)_{\overline{k}} = \mathbb{Z}/2\mathbb{Z}$ by Theorem B**(2,3)**). However, at the time, we were unable to determine what governed this. Theorem C helps clarify the situation. ◦

The results quoted thus far ultimately come about from computing various spectral sequences. However, when computing Brauer groups of schemes, one often times makes use of residue maps in addition to (or in lieu of) spectral sequences. Along these lines, separate from Theorem A, we study Brauer residue maps on stacks in Section 8. Of note, this study allows us to compute *Picard groups* of stacky curves using their coarse space and generic residual gerbe, and it allows us to obtain a stacky version of the usual Faddeev exact sequence used for computing Brauer groups of (not necessarily proper) rational curves.

Theorem D (= Corollary 8.12)**.** *Let $\mathcal{X}$ be a regular, integral, noetherian stacky curve with coarse moduli space $c\colon \mathcal{X} \to X$. Let $j\colon \mathcal{G} \hookrightarrow \mathcal{X}$ denote the residual gerbe of the generic point of $\mathcal{X}$. Then, there is a short exact sequence*

$$0 \longrightarrow \mathrm{Cl}\,\mathcal{X} \longrightarrow \mathrm{Pic}\,\mathcal{X} \xrightarrow{j^*} \mathrm{Pic}\,\mathcal{G} \longrightarrow 0,$$

where $\mathrm{Cl}\,\mathcal{X}$ is the (Weil) divisor class group of $\mathcal{X}$ (see Corollary 8.12 for a precise definition).

Theorem E (Stacky Faddeev, Proposition 8.33)**.** *Let k be a perfect field, let $\mathbb{P}^1 = \mathbb{P}^1_k$, and let $x_1, \ldots, x_r \in \mathbb{P}^1(k)$ be distinct points. Choose pairwise coprime integers $e_1, \ldots, e_r > 1$, and let*

$$\mathcal{X} \coloneqq \sqrt[e_1]{x_1/\mathbb{P}^1} \times_{\mathbb{P}^1} \ldots \times_{\mathbb{P}^1} \sqrt[e_r]{x_r/\mathbb{P}^1},$$

be $\mathbb{P}^1$ rooted at the $x_1, \ldots, x_r$ by degrees $e_1, \ldots, e_r$. Then,

- $\mathrm{Br}\,\mathcal{X} \simeq \mathrm{Br}\,\mathbb{P}^1 \simeq \mathrm{Br}\,k$.
- *For any closed point $x \in \mathbb{P}^1$, set $e_x = 1$ if $x \notin \{x_1, \ldots, x_r\}$ and set $e_x = e_i$ if $x = x_i$. Let $N \coloneqq \prod_{i=1}^r e_i = \prod_x e_x$. There is an exact sequence*

$$0 \to \mathrm{Br}\,k \longrightarrow \mathrm{Br}\,k(\mathbb{P}^1) \xrightarrow{\bigoplus_x e_x \cdot \mathrm{res}_x} \bigoplus_x \mathrm{H}^1(\kappa(x), \mathbb{Q}/\mathbb{Z}) \xrightarrow{\mathrm{Cor}_{\mathcal{X}}} \mathrm{H}^1(k, \mathbb{Q}/\mathbb{Z}) \longrightarrow 0, \tag{1.2}$$

where $\mathrm{res}_x\colon \mathrm{Br}\,k(\mathbb{P}^1) \to \mathrm{H}^1(\kappa(x), \mathbb{Q}/\mathbb{Z})$ is the usual Brauer residue map and $\mathrm{Cor}_{\mathcal{X}} = \sum_x \frac{N}{e_x} \mathrm{Cor}_{\kappa(x)/k}$.

There are more general results on residues maps of stacky curves (and their relation to residue maps on the coarse space) in Section 8.

Paper Organization. We begin in Section 2 by briefly introducing the background material used throughout the paper. In particular, it is here that we include the definitions of tameness and of Brauer groups. Following this, we begin our march towards Theorem A in earnest in Section 3. As we will later mention, the automorphism groups of tame stacks are (locally) extensions of tame étale group schemes by diagonalizable group schemes. Related to this, before tackling Theorem A directly, we find it helpful to first compute the Brauer groups of classifying stacks of certain locally diagonalizable groups schemes in Section 3. In Section 4, we prove Theorem A. Recall that the goal of this theorem is to show vanishing of some $\mathrm{R}^2c_*\mathbb{G}_m$. Because such vanishing can be checked on stalks, after appealing to the local structure of tame stacks (Corollary 2.14), this amounts to computing cohomology groups of the form $\mathrm{H}^2([X/G], \mathbb{G}_m)$ with G an extension of a tame constant group by a diagonalizable group. We ultimately compute such groups by combining earlier work of Meier on quotients by tame[1] constant groups [Mei18] with (a slight generalization of) the work of Section 3 on quotients by diagonalizable groups. Once Theorem A is proven, we work out its consequences for gerbes and root stacks in Sections 5 and 6. Recalling that a stacky curve can often be factored as a gerbe over a root a stack, in Section 7, we use the results of the previous two sections in order to study Brauer groups of stacky curves. In particular, we prove Theorem C and state some consequences for Brauer groups of stacky curves over non-perfect fields (see e.g. Corollary 7.12). With this completed, we then turn to the study of residue maps in Section 8; it is here that we prove Theorems D and E. Finally, in Section 9, we aim to show the utility of our work by carrying out a few example computations; namely, we prove Theorem B.

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1.1. **Conventions.** We end the introduction by very quickly establishing some of the conventions, many of which are standard, that we use throughout this paper.

- By default, given a scheme X (resp. algebraic stack $\mathcal{X}$), we work in the small étale site over X (resp. lisse-étale site over $\mathcal{X}$). As such, all unadorned cohomology groups should be interpreted as étale cohomology.
  - On occasion, we will make use of the big fppf site on X (resp. flat-fppf site on $\mathcal{X}$). In such circumstances, we will make use of the subscript $_{\mathrm{fppf}}$, writing e.g. $\mathrm{H}_{\mathrm{fppf}}$ for fppf cohomology.
  - We will often use, without reference, the fact [Gro68, Theorem 11.7] that étale cohomology and fppf cohomology agree with coefficients in a smooth group scheme.
- Given an algebraic stack $\mathcal{X}$, we write $\mathrm{Sch}_{\mathcal{X}}$ to denote the 1-category of schemes over $\mathcal{X}$ (this is equivalent to the underlying category of $\mathcal{X}$).
- Given an algebraic stack $\mathcal{X}$ and an object $x \in \mathcal{X}(S)$ over some scheme S, we write $\underline{\mathrm{Aut}}(\mathcal{X}, x)$ (or $\underline{\mathrm{Aut}}_{\mathcal{X}}(x)$) for the automorphism group functor of x. When $\mathcal{X}$ is clear from context, we will sometimes denote this simply as $\underline{\mathrm{Aut}}(x)$.
- A '`geometric point`' of a scheme S is a map $\overline{s}\colon \operatorname{Spec}\Omega \to S$ from the spectrum of a *separably* closed field Ω. We will often simply write $\overline{s} \to S$ and use $\kappa(\overline{s})$ to denote the implicitly chosen field Ω.

[1]The word 'tame' does not appear in [Mei18], but realizing that this notion was lurking in the background of his work was one piece of motivation for starting this project.

- By a '**gerbe**' over an algebraic stack $\mathcal{X}$, we mean an 'fppf gerbe', i.e. a morphism $\mathcal{Y} \to \mathcal{X}$ of algebraic stacks realizing $\mathcal{Y}$ as a gerbe over $\mathrm{Sch}_{\mathcal{X}}$ equipped with the flat-fppf topology.
- By a '**group**' we generally mean a 'group scheme'. When we want to emphasize that we mean a 'group in the classical sense of group theory', we will call such a thing an '**abstract group**'.
  - By a '**finite group**' we generally mean a 'finite, flat, finitely presented group scheme'.
- Given a group G and an integer N, we write $G[N]$ for its N-torsion subgroup.
- Given an abstract group G and a prime p, we write $G\{p\}$ for its p-primary torsion subgroup.
- Given a complex $\mathscr{C}^\bullet$ with differentials $\mathrm{d}^n\colon \mathscr{C}^n \to \mathscr{C}^{n+1}$, we set
$$\pi_n(\mathscr{C}^\bullet) \coloneqq \frac{\ker \mathrm{d}^n}{\operatorname{im} \mathrm{d}^{n-1}}$$
and call these the '**homotopy groups**' of the complex. This notation/terminology is chosen to distinguish them from the (hyper)cohomology groups of $\mathscr{C}^\bullet$ which are computed as the homotopy groups of the relevant derived functor applied to $\mathscr{C}^\bullet$.
- We write $|\mathcal{X}|$ for the underlying topological space of an algebraic stack, and we write $|\mathcal{X}|_1$ for its set of codimension 1 points.
- We call a scheme S '**local**' if $S \cong \operatorname{Spec} R$ for a local ring R, and we call it (and also R) '**strictly local**' if furthermore R is strictly henselian.
- For us, a '**stacky curve**' is a separated, noetherian algebraic stack which is of pure dimension 1 and has finite inertia.
- Let k be a field. We say a k-scheme X is **nice** if it is smooth, projective, and geometrically connected.

2. Background & Preliminaries

2.1. Hochschild Cohomology. We will make use of Hochschild cohomology throughout this work, so here we collect some of its basic definitions and properties. See [Mil17, Chapter 15] for more information.

Definition 2.1. Let S be a scheme (or algebraic stack), let G be a group-valued functor on $\mathrm{Sch}_S^{\mathrm{op}}$, and let M be a G-module, i.e. a commutative group functor on which G acts by group homomorphisms. We define a complex $C^\bullet(G,M)$ with $C^n(G,M) \coloneqq \mathrm{Nat}(G^n, M)$, the set of natural transformations (of *set-valued* functors) from G^n to M and whose differential $\mathrm{d}^n\colon C^n(G,M) \to C^{n+1}(G,M)$ is defined as usual (see [Mil17, Section 15.b]). The **Hochschild cohomology** of $G \curvearrowright M$ is defined to be the homotopy of this complex:
$$\mathrm{H}_0^n(G,M) \coloneqq \pi_n(C^\bullet(G,M)) \coloneqq \frac{\ker \mathrm{d}^n}{\operatorname{im} \mathrm{d}^{n-1}}. \qquad \diamond$$

Remark 2.2. In low degrees, Hochschild cohomology has interpretations analogous to those used for classical group cohomology. Let G, M, S be as in Definition 2.1.

- $\mathrm{H}_0^0(G,M) = M^G(S) = M(S)^G$ computes G-invariants.
- $\mathrm{H}_0^1(G,M) = \{\text{crossed homomorphisms } G \to M\}/\{\text{principal crossed homomorphisms}\}$, with (principal) crossed homomorphisms defined completely analogously as they are in the classical group cohomology setting.

  Example 2.3. If $G \curvearrowright M$ trivially, then $\mathrm{H}_0^1(G,M) = \mathrm{Hom}(G,M)$ is the group of homomorphisms from G to M. △

- $\mathrm{H}_0^2(G,M)$ classifies isomorphisms classes of **Hochschild extensions**, i.e. exact sequences
$$0 \longrightarrow M \longrightarrow E \longrightarrow G \longrightarrow 0$$
of *group functors/presheaves* such that there exists a map $s\colon G \to E$ of *set-valued* functors splitting the sequence on the right and such that the conjugation action of $G \curvearrowright M$ is the given action of $G \curvearrowright M$. ○

Remark 2.4. If Q is an abstract group with associated constant sheaf $\underline{Q}$ over a scheme (or algebraic stack) S, then, for any $\underline{Q}$-module M, $\mathrm{H}^n_0(\underline{Q}, M)$ is the classical group cohomology of the abstract group Q acting on $M(S)$. Indeed, the complex $C^\bullet(\underline{Q}, M)$ of Definition 2.1 is simply the bar complex of $Q \curvearrowright M(S)$. For this reason, we will continue to use H_0 to denote classical group cohomology, writing e.g. $\mathrm{H}^n_0(\underline{Q}, M) \simeq \mathrm{H}^n_0(Q, M(S))$. ○

For us, Hochschild cohomology will most commonly come up in the context of computing the (étale) cohomology of quotient stacks, so the modules M we encounter will usually be of the following form.

Notation 2.5. Let S be a scheme, let X/S be an S-scheme equipped with a sheaf $\mathscr{F}$. For any $j \geq 0$, we define the functor

$$\begin{array}{rccc} \underline{\mathrm{H}}^j(X/S, \mathscr{F}) : & \mathrm{Sch}_S^{\mathrm{op}} & \longrightarrow & \mathrm{Set} \\ & T/S & \longmapsto & \mathrm{H}^j(X_T, \mathscr{F}_T), \end{array}$$

where $X_T \coloneqq X \times_S T$ and $\mathscr{F}_T \coloneqq \mathscr{F}|_{X_T}$.

Lemma 2.6 (Descent spectral sequence)**.** *Let S be a scheme, and let G/S be an S-group scheme. Suppose that G acts on some S-scheme X, and set $\mathfrak{X} \coloneqq [X/G]$. Then, for any étale sheaf $\mathscr{F}$ on $\mathfrak{X}$, there is a spectral sequence*

$$E_2^{ij} = \mathrm{H}^i_0(G, \underline{\mathrm{H}}^j(X/S, \mathscr{F}_X)) \implies \mathrm{H}^{i+j}(\mathfrak{X}, \mathscr{F}).$$

Proof. This is a special case of the 'spectral sequence of the covering' $X \to \mathfrak{X}$; see [Ols16, 2.4.26, 9.2.4, and/or 11.6.3]. Writing $X_n \coloneqq X \times_{\mathfrak{X}} \ldots \times_{\mathfrak{X}} X$ ($n+1$ factors), this is the spectral sequence

$$E_1^{ij} = \mathrm{H}^j(X_i, \mathscr{F}_i) \implies \mathrm{H}^{i+j}(\mathfrak{X}, \mathscr{F}),$$

where $\mathscr{F}_j$ is simply the pullback of $\mathscr{F}$ to X_j. Note that, for any $n \geq 0$, $X_n \simeq G \times_S \times \ldots \times_S G \times_S X$ (with n copies of G), so the E_1-page of our spectral sequence is

$$E_1^{ij} = \mathrm{H}^j(X_i, \mathscr{F}_i) = \mathrm{H}^j(G_S^{\times i} \times_S X, \mathscr{F}_i) = \mathrm{Nat}\big(G_S^{\times i}, \underline{\mathrm{H}}^j(X/S, \mathscr{F}_X)\big) =: C^i(G, \underline{\mathrm{H}}^j(X/S, \mathscr{F}_X)).$$

The differentials on this page match those used to define Hochschild cohomology, so $E_2^{ij} = \pi_i\left(E_1^{\bullet j}\right) = \mathrm{H}^i_0(G, \underline{\mathrm{H}}^j(X/S, \mathscr{F}_X))$. ■

Lemma 2.7. *Let R be a strictly Henselian local ring, let X be a finite R-scheme, and let $\mathscr{F}$ be an étale sheaf on X. Then, for any finite R-scheme Y,*

$$\mathrm{Nat}(Y_R^{\times i}, \underline{\mathrm{H}}^j(X/R, \mathscr{F})) = 0$$

for any $i \geq 0$ and $j \geq 1$.

Proof. By Yoneda, $\mathrm{Nat}(Y_R^{\times i}, \underline{\mathrm{H}}^j(X/R, \mathscr{F})) = \underline{\mathrm{H}}^j(X/R, \mathscr{F})(Y_R^{\times i}) = \mathrm{H}^j(X \times_R Y_R^{\times i}, \mathscr{F})$. Note that $X \times_R Y_R^{\times i}$ is still finite over R (by definition, $Y_R^{\times 0} \coloneqq \operatorname{Spec} R$) and so [Sta21, Tag 03QJ] tells us that $X \times_R Y_R^{\times i}$ is a finite disjoint union of Spec's of strictly henselian local rings, and so $\mathrm{H}^j(X \times_R Y_R^{\times i}, \mathscr{F}) = 0$. ■

Corollary 2.8. *Let R be a strictly henselian local ring, let G/R be a finite group scheme, and suppose G acts on a finite R-scheme X. Let $\mathfrak{X} \coloneqq [X/G]$, and let $\mathscr{F}$ be any étale sheaf on $\mathfrak{X}$. Then, $\mathrm{H}^n(\mathfrak{X}, \mathscr{F}) = \mathrm{H}^n_0(G, \mathscr{F}_{X/R})$ for any $n \geq 0$.*

Proof. Consider the cohomological descent spectral sequence of Lemma 2.6:

$$E_2^{ij} = \mathrm{H}^i_0(G, \underline{\mathrm{H}}^j(X/R, \mathscr{F})) \implies \mathrm{H}^{i+j}(\mathfrak{X}, \mathscr{F}).$$

It follows from Lemma 2.7 that $E_2^{ij} = 0$ whenever $j > 0$. Thus, the spectral sequence is concentrated in the $j = 0$ row, from which the corollary follows. ■

2.2. **Tame Stacks.** We review here the basics of tame stacks in the sense of [AOV08].

Definition 2.9. Let S be a scheme. A group scheme G/S is `linear reductive` if it is flat and finitely presented over S, and the functor $\mathrm{QCoh}^G(S) \to \mathrm{QCoh}(S)$, $F \mapsto F^G$, from G-equivariant quasi-coherent sheaves to all quasi-coherent sheaves, is exact. ◇

Definition/Proposition 2.10 ([AOV08, Definition 3.1 and Theorem 3.2])**.** Let S be a scheme. An algebraic stack $\mathcal{X}/S$ is `tame` if it is locally of finite presentation and has finite inertia over S – so it admits a coarse moduli space $c\colon \mathcal{X} \to X$ [KM97, Con05] – and either of the following equivalent conditions hold:

- The functor $c_*\colon \mathrm{QCoh}(\mathcal{X}) \to \mathrm{QCoh}(X)$ is exact.
- For any object $\xi \in \mathcal{X}(k)$ over an algebraically closed field k, the k-group scheme $\underline{\mathrm{Aut}}(\mathcal{X}, \xi)$ is linearly reductive. ◇

Recall that a finite locally free commutative group scheme G/S is `diagonalizable` if its Cartier dual $G^\vee := \underline{\mathrm{Hom}}(G, \mathbb{G}_m)$ is constant. In [AOV08, Section 2], the authors classify finite linearly reductive group schemes (i.e. the automorphism/stabilizer groups of tame stacks). We will implicitly make use of their classification throughout this paper:

Proposition 2.11 ([AOV08, Theorem 2.19])**.** *Let S be a scheme, and let G/S be a finite flat group scheme of finite presentation. Then, the following are equivalent*

(a) *G is linearly reductive.*
(b) *There exists an an fpqc cover $\{S_i \to S\}_{i\in I}$ such that, for all i, the group scheme $G\times_S S_i$ is a semidirect product $\Delta \rtimes \underline{Q}$, where Δ is diagonalizable and $\underline{Q}$ is a tame constant group (i.e. $\#Q \in \Gamma(S_i, \mathscr{O}_{S_i})^\times$).*
(c) *The fibers of $G \to S$ are linearly reductive.*

Example 2.12. Let R be a strictly local ring. It follows from Proposition 2.11 that, for any finite linearly reductive group G/R, its connected–étale sequence [Tat97, Section (3.7)] exhibits G as an extension of a tame constant group scheme $\underline{Q}$ by a diagonalizable group scheme. △

Lemma 2.13. *Let $\mathcal{X}$ be a tame algebraic stack over a scheme S, with coarse moduli space $c\colon \mathcal{X} \to X$. Fix a point $x \in X$. Then, there exists all of the following data.*

- *A finite separable field extension $k/\kappa(x)$ and a point $x' \in \mathcal{X}(k)$ lifting $x \in X$.*
- *An étale neighborhood $(U, u) \to (X, x)$ of x, with $u \in U(k)$.*
- *A finite linearly reductive group G/U such that $G_u \cong \underline{\mathrm{Aut}}(\mathcal{X}, x')$ as k-group schemes.*
- *A finite and finitely presented U-scheme V/U equipped with a G-action and an isomorphism*
$$\mathcal{X} \times_X U \cong [V/G]$$

Proof. This is a consequence of the proof, though not the statement, of [AOV08, Theorem 3.2(d)]. First, [AOV08, Proposition 3.7] guarantees the existence of k and $x' \in \mathcal{X}(k)$ as in the first bullet point. Now, [Sta21, Tag 02LF] guarantees the existence of an étale neighborhood $(U, u) \to (X, x)$ as in the second bullet point. By replacing X with U (and $\mathcal{X}$ with $\mathcal{X} \times_X U$), we may and do assume that x lifts to some $x' \in \mathcal{X}(\kappa(x))$. At this point, the rest of the lemma follows from [AOV08, Step 1 of the proof of Proposition 3.6]; in particular, the third bullet point is guaranteed by their use of [AOV08, Proposition 2.18] to obtain the linearly reductive group G appearing in their argument. ■

Corollary 2.14. *Let $\mathcal{X}$ be a tame algebraic stack over a scheme S, with coarse moduli space $c\colon \mathcal{X} \to X$. Let $\operatorname{Spec} \mathscr{O}_{X,\overline{x}} \to X$ be the (strictly henselian) local ring at some geometric point $\overline{x} \to X$. Then,*
$$\mathcal{X} \times_X \mathscr{O}_{X,\overline{x}} \cong [\operatorname{Spec} R/G]$$
for some strictly henselian local ring R which is finite and finitely presented over $\mathscr{O}_{X,\overline{x}}$ and is acted upon by some finite linearly reductive group $G/\mathscr{O}_{X,\overline{x}}$ such that $G_{\overline{x}} \cong \underline{\mathrm{Aut}}(\mathcal{X}, \overline{x})$. Furthermore, $G_{\overline{x}}$ acts trivially on the residue field of R.

Proof. By Lemma 2.13, by passing to an étale neighborhood of $\overline{x}$, we may and do assume that $\mathcal{X} = [V/G]$ for some finite and finitely presented V/X and some finite linearly reductive group G/X such that $G_{\overline{x}} \cong \underline{\mathrm{Aut}}(\mathcal{X}, \overline{x})$. Write $\operatorname{Spec} R := V \times_X \mathscr{O}_{X,\overline{x}}$. The claim will follow as soon as we show that R is local (equivalently, $\operatorname{Spec} R$ is connected).

Note that R is a finite product of local rings, so $\operatorname{Spec} R$ is connected if its special fiber $Y := (\operatorname{Spec} R)_{\overline{x}}$ is connected. Furthermore, $\mathcal{X} \times_X \overline{x} \simeq [Y/G_{\overline{x}}]$ is connected because [AOV08, Corollary 3.3(a)] shows its coarse space is $\operatorname{Spec} \kappa(\overline{x})$, so $G_{\overline{x}}$ must act transitively on the connected components of Y. After possibly extending the field of definition $\kappa(\overline{x})$ of $\overline{x}$, we may lift it to a geometric point $\overline{y} \to Y$, and then observe that

$$G_{\overline{x}} = \underline{\mathrm{Aut}}(\mathcal{X}, \overline{x}) \simeq \mathrm{Stab}_{G_{\overline{x}}}(\overline{y}) \leq G_{\overline{x}}$$

from which we conclude that $G_{\overline{x}}$ acts trivially on $\overline{y}$. Since we saw earlier that $G_{\overline{x}}$ acts transitively on Y's connected components, we deduce that Y (and so also $\operatorname{Spec} R$) must be connected. ■

2.3. Brauer Groups.

We now set our conventions concerning Brauer groups. In particular, we differentiate between the (Azumaya) Brauer group Br, the cohomological Brauer group Br′, and $\mathrm{H}^2(-, \mathbb{G}_m)$.

Definition 2.15. Fix an algebraic stack $\mathcal{X}$. An `Azumaya algebra` over $\mathcal{X}$ is a quasi-coherent $\mathscr{O}_{\mathcal{X}}$-algebra $\mathscr{A}$ which is étale-locally isomorphism to $M_n(\mathscr{O}_{\mathcal{X}})$, the sheaf of $n \times n$-matrices over $\mathscr{O}_{\mathcal{X}}$, for some $n \geq 1$. Two Azumaya algebras $\mathscr{A}, \mathscr{B}$ are `Brauer equivalent` if there exists vector bundles $\mathscr{E}, \mathscr{E}'$ on $\mathcal{X}$ such that

$$\mathscr{A} \otimes \mathscr{E}nd(\mathscr{E}) \simeq \mathscr{B} \otimes \mathscr{E}nd(\mathscr{E}').$$

The `(Azumaya) Brauer group` $\mathrm{Br}(\mathcal{X})$ is the group of Brauer equivalence classes of Azumaya algebras, under tensor products. ◇

Definition 2.16. The `cohomological Brauer group` of an algebraic stack $\mathcal{X}$ is $\mathrm{Br}'(\mathcal{X}) := \mathrm{H}^2(\mathcal{X}, \mathbb{G}_m)_{\mathrm{tors}}$. ◇

Remark 2.17. For any algebraic stack $\mathcal{X}$, there always exists a natural injection

$$\alpha_{\mathcal{X}} \colon \operatorname{Br} \mathcal{X} \hookrightarrow \operatorname{Br}' \mathcal{X};$$

see [Gro66a, (2.1)] and/or [Shi19, (2.2.1)]. ○

For comparing the various groups $\operatorname{Br} \mathcal{X}, \operatorname{Br}' \mathcal{X}, \mathrm{H}^2(\mathcal{X}, \mathbb{G}_m)$, one has the following results.

Lemma 2.18. *Let $f \colon \mathcal{X} \to \mathcal{Y}$ be a finite, flat, finitely presented surjective morphism of algebraic stacks. Then,*

- ***(1)*** *a class $\beta \in \mathrm{H}^2(\mathcal{Y}, \mathbb{G}_m)$ is in the image of $\alpha_{\mathcal{Y}}$ if and only if $f^*\beta \in \mathrm{H}^2(\mathcal{X}, \mathbb{G}_m)$ is in the image of $\alpha_{\mathcal{X}}$.*
- ***(2)*** *if $\alpha_{\mathcal{X}}$ is an isomorphism, so is $\alpha_{\mathcal{Y}}$.*
- ***(3)*** *if $\mathrm{Br}'(\mathcal{X}) = \mathrm{H}^2(\mathcal{X}, \mathbb{G}_m)$, so too does $\mathrm{Br}'(\mathcal{Y}) = \mathrm{H}^2(\mathcal{Y}, \mathbb{G}_m)$.*

Proof. Parts **(1,2)** are [Shi19, Proposition 2.5 + Corollary 2.6]. For part **(3)**, the norm map $f_*\mathbb{G}_{m,X} \to \mathbb{G}_{m,Y}$ gives a morphism $\mathrm{H}^2(X, \mathbb{G}_m) = \mathrm{H}^2(Y, f_*\mathbb{G}_{m,X}) \to \mathrm{H}^2(Y, \mathbb{G}_m)$ such that the composition

$$\mathrm{H}^2(Y, \mathbb{G}_m) \xrightarrow{f^*} \mathrm{H}^2(X, \mathbb{G}_m) \longrightarrow \mathrm{H}^2(Y, \mathbb{G}_m)$$

is multiplication by $\deg f$. Hence, if the middle group is torsion, the outer group must be as well. ■

Theorem 2.19 (Gabber, [CTS21, Theorem 4.2.1])**.** *Let X be a (quasi-compact, separated) scheme which admits an ample line bundle. Then, α_X is an isomorphism.*

At times, we will implicitly use the following facts about Brauer groups.

Lemma 2.20. *Let $\mathcal{X}$ be a regular stacky curve over a field k. Then, $\operatorname{Br} \mathcal{X} \xrightarrow{\sim} \mathrm{H}^2(\mathcal{X}, \mathbb{G}_m)$.*

Proof. Let $c \colon \mathcal{X} \to X$ be its coarse space map; note that X is normal (and so k-smooth) because $\mathcal{X}$ is. By [EHKV01, Theorem 2.7], there exists a finite surjection $Z \to \mathcal{X}$ from a (not necessarily separated) scheme

Z. We may replace Z with its normalization and then by any one of its (connected) components which dominates X in order to assume that it is normal and irreducible. The composition $Z \to \mathcal{X} \xrightarrow{c} X$, being both proper and quasi-finite, is therefore finite, so Z is a k-curve. Since Z was assumed normal, it is in fact regular. Now, $Z \to \mathcal{X}$ is a finite morphism from a regular scheme to a regular algebraic stack of the same dimension. It follows from [Mat89, Theorem 23.1] (applied to $Z_U \to U$ for $U \to \mathcal{X}$ a smooth cover by a scheme) that $Z \to \mathcal{X}$ must be flat as well. Finally, Z admits an ample line bundle [Sta21, Tag 09NZ] and $\mathrm{Br}'(Z) = \mathrm{H}^2(Z, \mathbb{G}_m)$ [Poo17, Proposition 6.6.7], so we conclude by Theorem 2.19 and Lemma 2.18. ∎

Proposition 2.21. *Let $\mathcal{X}$ be a regular, noetherian algebraic stack, and let $\mathcal{U} \subset \mathcal{X}$ be a dense open substack. Then, the restriction map $\mathrm{H}^2(\mathcal{X}, \mathbb{G}_m) \to \mathrm{H}^2(\mathcal{U}, \mathbb{G}_m)$ is injective.*

Proof. One can argue complete analogously to [Lie08, Lemma 3.1.3.3]. For the sake of completeness, we include details below.

Let $\mathcal{G} \to \mathcal{X}$ be a $\mathbb{G}_m$-gerbe, and suppose that $\mathcal{G}_\mathcal{U}$ is trivial. Then, there exists a 1-twisted line bundle $\mathscr{L}_\mathcal{U}$ on $\mathcal{G}_\mathcal{U}$ [Ols16, 12.3.10]. Let $i\colon \mathcal{G}_\mathcal{U} \hookrightarrow \mathcal{G}$ be the natural inclusion. Then, $i_*\mathscr{L}_\mathcal{U}$ is a quasi-coherent 1-twisted sheaf on $\mathcal{G}$. Since any subsheaf of a 1-twisted sheaf is 1-twisted, [LMB00, Proposition 15.4] shows that $i_*\mathscr{L}_\mathcal{U}$ is a colimit of its coherent 1-twisted subsheaves. Consequently, there exists some coherent 1-twisted sheaf $\mathscr{L}$ on $\mathcal{G}$ extending $\mathscr{L}_\mathcal{U}$; replacing $\mathscr{L}$ by $(\mathscr{L}^\vee)^\vee$, if necessary, we may and do suppose that $\mathscr{L}$ is reflexive. Now, any reflexive sheaf of rank 1 on a regular algebraic stack is invertible (follows from [Har80, Proposition 1.9]), so $\mathscr{L}$ is a 1-twisted line bundle. Since $\mathcal{G}$ supports a 1-twisted line bundle, it follows from [Ols16, Proposition 12.3.11] that $[\mathcal{G}] = 0 \in \mathrm{H}^2(\mathcal{X}, \mathbb{G}_m)$. ∎

Corollary 2.22. *Let $\mathcal{X}$ be a regular, noetherian algebraic stack with coarse moduli space $c\colon \mathcal{X} \to X$. If c is an isomorphism over an open $U \subset X$ such that $Z \coloneqq X \setminus U$ has codimension ≥ 2 in X, then $c^*\colon \mathrm{Br}\, X \to \mathrm{Br}\, \mathcal{X}$ is an isomorphism.*

Proof. Let U, Z be as in the proposition statement, and consider the commutative diagram

$$\begin{array}{ccc} \mathrm{Br}\, X & \xrightarrow{c^*} & \mathrm{Br}\, \mathcal{X} \\ \downarrow\wr & & \downarrow \\ \mathrm{Br}\, U & \xrightarrow{\sim} & \mathrm{Br}\, \mathcal{X}_U, \end{array}$$

whose left arrow is an isomorphism because $\operatorname{codim}_X Z \geq 2$ (see [Čes19, Theorem 1.1] and/or [CTS21, Theorem 3.7.6]) and whose bottom arrow is an isomorphism because $\mathcal{X}_U \xrightarrow{\sim} U$. The claim follows from commutativity of this diagram. ∎

Proposition 2.23. *Let $\mathcal{X}$ be either*

- *a tame algebraic stack; or*
- *a separated DM stack.*

In either case, let $c\colon \mathcal{X} \to X$ denote its coarse space. Let ℓ be a prime such that $\ell \nmid \#\mathrm{Aut}(\mathcal{X}, x)$ for every $x \in |\mathcal{X}|$. Then, for any $i \geq 1$, the pullback map

$$c^*\colon \mathrm{H}^i(X, \mathbb{G}_m) \otimes \mathbb{Z}_{(\ell)} \longrightarrow \mathrm{H}^i(\mathcal{X}, \mathbb{G}_m) \otimes \mathbb{Z}_{(\ell)}$$

is an isomorphism. In particular, $\mathrm{Br}'(X)\{\ell\} \xrightarrow{\sim} \mathrm{Br}'(\mathcal{X})\{\ell\}$.

Proof. By considering the Leray spectral sequence, we see it suffices to show that $\mathrm{R}^i c_* \mathbb{G}_m \otimes \mathbb{Z}_{(\ell)} = 0$ for all $i \geq 1$. Since this can be verified on the level of stalks, appealing to the local structure theorem for tame stacks (see Corollary 2.14) or for separated DM stacks (see [LMB00, Théorème 6.2] and/or [Ols06, Theorem 2.12]), we may assume that $X = \operatorname{Spec} R$ is strictly local ant that $\mathcal{X} = [\operatorname{Spec} A/G]$ for some finite R-algebra A and some finite R-group G for which $\ell \nmid \#G$. In this case, it suffices to compute that $\mathrm{H}^i(\mathcal{X}, \mathbb{G}_m) \otimes \mathbb{Z}_{(\ell)} = 0$ for all $i \geq 1$. As in the beginning of the proof of Proposition 4.3, the existence of the G-torsor $\operatorname{Spec} A \to \mathcal{X}$

with $\mathrm{H}^i(A, \mathbb{G}_m) = 0$ for all $i \geq 1$ (because A is a product of strictly local rings) suffices to deduce that $\mathrm{H}^i(\mathcal{X}, \mathbb{G}_m)$ is $\#G$-torsion for all $i \geq 1$. Since $\ell \nmid \#G$, we conclude that $\mathrm{H}^i(\mathcal{X}, \mathbb{G}_m) \otimes \mathbb{Z}_{(\ell)} = 0$ for all $i \geq 1$. ■

Remark 2.24. If $\mathcal{X}$ in Proposition 2.23 is DM, then the conclusion

$$c^* \colon \mathrm{H}^i(X, c_*\mathscr{F}) \otimes \mathbb{Z}_{(\ell)} \xrightarrow{\sim} \mathrm{H}^i(\mathcal{X}, \mathscr{F}) \otimes \mathbb{Z}_{(\ell)} \text{ for all } i \geq 1$$

holds for any étale sheaf $\mathscr{F}$. Indeed, once one reduces to the case of $\mathcal{X} = [\operatorname{Spec} A/G]$ with G not an abstract group (because $\mathcal{X}$ is DM) of cardinality not divisible by ℓ, the Hochschild–Serre/descent spectral sequence

$$E_2^{ij} = \mathrm{H}_0^i(G, \mathrm{H}^j(A, \mathscr{F})) \implies \mathrm{H}^{i+j}(\mathcal{X}, \mathscr{F})$$

shows that $\mathrm{H}^i(\mathcal{X}, \mathscr{F})$ is $\#G$-torsion for all $i \geq 1$ (so $\mathrm{H}^i(\mathcal{X}, \mathscr{F}) \otimes \mathbb{Z}_{(\ell)} = 0$).

To get this more general conclusion in the case of algebraic $\mathcal{X}$ in Proposition 2.23, it would suffice to know that $\mathrm{H}_0^i(G, M)$ is $\#G$-torsion for any finite (linearly reductive) group scheme G acting on any G-module M. The author knows neither a counterexample to nor a proof of this statement. ∘

3. Brauer Groups of Classifying Stacks for locally diagonalizable groups

In this section, we compute the Brauer (and Picard) groups of classifying stacks for certain finite group schemes.

Recall 3.1. A finite, flat, finitely presented commutative group scheme G/S is `locally diagonalizable` if its Cartier dual $G^\vee := \underline{\mathrm{Hom}}(G, \mathbb{G}_m)$ is S-étale. ⊙

Definition 3.2. Let G/S be an étale group scheme. Call G `cyclic` if, étale-locally on S, it is isomorphic to $\underline{\mathbb{Z}/n\mathbb{Z}}$ for some $n \in \mathbb{Z}$. Call a locally diagonalizable group G/S `cyclic` if $G^\vee$ is cyclic. ⋄

Definition 3.3. We say a finite group scheme G/S is `connected` if, for every geometric point $\bar{s} \to S$, the scheme $G_{\mathscr{O}_{S,\bar{s}}}$ is connected. ⋄

Example 3.4. μ_n (for any n) is a cyclic locally diagonalizable group. Any commutative group G/S of order invertible on S is locally diagonalizable. △

Lemma 3.5. *Let R be a strictly henselian local ring, fix some $n \geq 1$, and let*

$$1 \longrightarrow \mathbb{G}_m \longrightarrow E \longrightarrow \mu_n \longrightarrow 1$$

be an extension of abelian fppf sheaves on R. Then, E is commutative and represented by a scheme.

Proof. It is clear that E is a $\mathbb{G}_m$-torsor over μ_n, so effective descent for affine schemes shows that E is represented by a scheme. Hence, we only need to show that E is commutative, which one can do by arguing as in [he]. The commutator pairing $\mu_n \times \mu_n \to \mathbb{G}_m$ gives rise to a map $\varphi \colon \mu_n \to \underline{\mathrm{Hom}}(\mu_n, \mathbb{G}_m)$. Since the target is étale, φ uniquely factors through the maximal étale quotient $\mu_n^{\text{ét}}$ of μ_n. Note that $\#\mu_m$ is coprime to $\#\mu_n^\circ$, the order of the identity component of μ_n. For any $\zeta \in \mu_n$, $\varphi(\zeta) \in \underline{\mathrm{Hom}}(\mu_n, \mathbb{G}_m)$ is killed by $\#\mu_n^{\text{ét}}$ (since φ factors through $\mu_n^{\text{ét}}$), so $\varphi(\zeta)$ must kill μ_n°; that is, φ factors as

$$\varphi \colon \mu_n \twoheadrightarrow \mu_n^{\text{ét}} \to \underline{\mathrm{Hom}}(\mu_n^{\text{ét}}, \mathbb{G}_m) \hookrightarrow \underline{\mathrm{Hom}}(\mu_n, \mathbb{G}_m).$$

In other words, we may assume that $\mu_n = \mu_n^{\text{ét}}$ is étale. Now, $\varphi \colon \mu_n \to \underline{\mathrm{Hom}}(\mu_n, \mathbb{G}_m)$ comes from the commutator pairing $\mu_n \times \mu_n \to \mathbb{G}_m$, so for any R-scheme S and any $\zeta \in \mu_n(S)$, we have $\varphi(\zeta)(\zeta) = 1$. Since μ_n is étale, $\mu_n(S)$ is cyclic for any S/R, so by choosing $\zeta \in \mu_n(S)$ to be a generator, we conclude that $\varphi = 0$. ■

Lemma 3.6. *Let G/S be a finite locally free commutative group over a scheme S, and let $\mathcal{X}/S$ be an algebraic stack. Then, there are canonical isomorphisms*

$$\mathrm{H}^n_{\mathrm{fppf}}(\mathcal{X}, G^\vee) \xrightarrow{\sim} \mathrm{Ext}^n_{\mathcal{X}}(G, \mathbb{G}_m),$$

where the latter group is Ext *in the category of abelian sheaves on the big fppf site $\mathcal{X}_{\mathrm{fppf}}$.*

Proof. This was proven by Shatz [Sha69] (and independently by Waterhouse [Wat71] in the case $n = 1$) when $\mathcal{X}$ is a scheme, and the general case follows from his argument. In brief, one has the local global spectral sequence $\mathrm{H}^i_{\mathrm{fppf}}(\mathcal{X}, \mathscr{E}xt^j_{\mathcal{X}}(G, \mathbb{G}_m)) \implies \mathrm{Ext}^{i+j}_{\mathcal{X}}(G, \mathbb{G}_m)$. If $X \to \mathcal{X}$ is a smooth cover by a scheme, then [Sha69] shows that $\mathscr{E}xt^j_X(G, \mathbb{G}_m) = \mathscr{E}xt^j_{\mathcal{X}}(G, \mathbb{G}_m)|_X$ vanishes for all $j > 0$, so also $\mathscr{E}xt^j_{\mathcal{X}}(G, \mathbb{G}_m) = 0$. Thus, the sequence immediately collapses and gives isomorphisms

$$\mathrm{H}^n_{\mathrm{fppf}}(\mathcal{X}, G^\vee) = \mathrm{H}^n_{\mathrm{fppf}}(\mathcal{X}, \mathscr{H}om_{\mathcal{X}}(G, \mathbb{G}_m)) \xrightarrow{\sim} \mathrm{Ext}^n_{\mathcal{X}}(G, \mathbb{G}_m),$$

as claimed. ∎

Lemma 3.7. *Let R be a strictly henselian local ring, and let G/R be a finite group. Then,*

$$\mathrm{H}^0(BG_R, \mathbb{G}_m) = \mathbb{G}_m(R)$$
$$\mathrm{H}^1(BG_R, \mathbb{G}_m) = G^\vee(R).$$

If, furthermore, G is locally diagonalizable and either cyclic or connected, then also

$$\mathrm{H}^2(BG_R, \mathbb{G}_m) = 0.$$

Proof. The first equality holds because $\operatorname{Spec} R$ is BG_R's coarse space. For the second, Corollary 2.8 shows that $\mathrm{H}^1(BG_R, \mathbb{G}_m) \simeq \mathrm{H}^1_0(G, \mathbb{G}_m) = \mathrm{Hom}_{\mathrm{GrpSch}_R}(G, \mathbb{G}_m)$. Now assume that G is locally diagonalizable and either cyclic or connected. Corollary 2.8 shows that $\mathrm{H}^2(BG_R, \mathbb{G}_m) \simeq \mathrm{H}^2_0(G, \mathbb{G}_m)$ which classifies central extensions

$$1 \longrightarrow \mathbb{G}_m \longrightarrow E \longrightarrow G \longrightarrow 1.$$

For any such extension, E is commutative either by Lemma 3.5 if G is cyclic or by [Mil17, Theorem 15.39] (see also [AOV08, Lemma 2.14]) if G is connected. Thus, $\mathrm{H}^2_0(G, \mathbb{G}_m) \simeq \mathrm{Ext}^1(G, \mathbb{G}_m)$ (Ext here is computed in the category of abelian fppf sheaves on $\operatorname{Spec} R$). Finally, Lemma 3.6 shows that $\mathrm{Ext}^1(G, \mathbb{G}_m) \simeq \mathrm{H}^1_{\mathrm{fppf}}(R, G^\vee)$, but because $G^\vee$ is étale, $\mathrm{H}^1_{\mathrm{fppf}}(R, G^\vee) \simeq \mathrm{H}^1(R, G^\vee) = 0$. ∎

Proposition 3.8. *Let G/S be a finite group scheme, and let $c\colon BG_S \to S$ be the structure map. Then,*

$$c_*\mathbb{G}_m \simeq \mathbb{G}_m \text{ and } \mathrm{R}^1 c_*\mathbb{G}_m \simeq G^\vee.$$

If, furthermore, G is locally diagonalizable and either cyclic or connected, then also $\mathrm{R}^2 c_*\mathbb{G}_m = 0$.

Proof. The first of these holds because c is a coarse space map, the second holds because Lemma 3.7 shows that the natural map $\mathrm{R}^1 c_*\mathbb{G}_m \to G^\vee$ (induced from the maps $\mathrm{H}^1(BG_T, \mathbb{G}_m) \to G^\vee(T)$, for any scheme T/S, sending a G-equivariant line bundle on T to the character corresponding to its G-action) is an isomorphism on stalks, and the last holds because Lemma 3.7 shows that the stalks of $\mathrm{R}^2 c_*\mathbb{G}_m$ all vanish in this case. ∎

Proposition 3.9. *Let G/S be a finite locally diagonalizable group over some base scheme S, and consider the classifying stack $c\colon BG_S \to S$. Assume that G is either cyclic or connected. Then, there are split exact sequences*

$$\begin{array}{ccccccccc} 0 \longrightarrow & \mathrm{Pic}(S) & \xrightarrow{c^*} & \mathrm{Pic}(BG_S) & \longrightarrow & G^\vee(S) & \longrightarrow 0 \\ 0 \longrightarrow & \mathrm{H}^2(S, \mathbb{G}_m) & \xrightarrow{c^*} & \mathrm{H}^2(BG_S, \mathbb{G}_m) & \longrightarrow & \mathrm{H}^1(S, G^\vee) & \longrightarrow 0. \end{array} \tag{3.2}$$

The latter sequence restricts to analogous split exact sequences when $\mathrm{H}^2(-, \mathbb{G}_m)$ is replaced by Br' or Br.

Proof. Consider the Leray spectral sequence $E_2^{ij} = \mathrm{H}^i(S, \mathrm{R}^j\pi_*\mathbb{G}_m) \implies \mathrm{H}^{i+j}(BG_S, \mathbb{G}_m)$, pictured in Fig. 1 with terms computed using Proposition 3.8. Let $\pi\colon S \to BG_S$ be the section of c given by the universal torsor. Then, π^* splits all the edge maps $\mathrm{H}^i(S, \mathbb{G}_m) \to \mathrm{H}^i(BG_S, \mathbb{G}_m)$, so the displayed differentials in Fig. 1 must both vanish. Thus, the spectral sequence gives rise to the claimed (split) exact sequences. Because (3.2) is split, it remains exact when passing to torsion subgroups (so remains split exact with $\mathrm{H}^2(-, \mathbb{G}_m)$ replaced by Br'), and by Lemma 2.18, for any $\alpha \in \mathrm{Br}'(BG_S)$, we have $\pi^*\alpha \in \mathrm{Br}(S) \iff \alpha \in \mathrm{Br}(BG_S)$, so (3.2) remains split exact with $\mathrm{H}^2(-, \mathbb{G}_m)$ replaced by Br. ∎

$$\begin{tikzcd}
0 & & & \\
G^\vee(S) \arrow[drr] & \mathrm{H}^1(S, G^\vee) \arrow[drr] & & \\
\mathbb{G}_m(S) & \mathrm{Pic}(S) & \mathrm{H}^2(S, \mathbb{G}_m) & \mathrm{H}^3(S, \mathbb{G}_m)
\end{tikzcd}$$

FIGURE 1. The E_2-page of the Leray spectral sequence for $BG_S \to S$.

Remark 3.10. In the case that $G = \mu_n$ and $S = \operatorname{Spec} L$ for a field L of characteristic not dividing n, $\mathrm{H}^2(B\mu_{n,L}, \mathbb{G}_m)$ was computed earlier by Lieblich [Lie11, Proposition 4.1.4]. ◦

Remark 3.11. The conclusion of Proposition 3.9 – i.e. that $\mathrm{H}^2(BG_S, \mathbb{G}_m) \cong \mathrm{H}^2(S, \mathbb{G}_m) \oplus \mathrm{H}^1_{\mathrm{fppf}}(S, G^\vee)$ – holds for more finite group schemes than its statement accounts for. For example, [AM20, Proposition 3.2] proves the same statement when $G = \underline{\mathbb{Z}/n\mathbb{Z}}$ is a constant, cyclic group scheme. Note that, in this case, G is *not* locally diagonalizable if n is not invertible on the base. As another example, if k is a separably closed field of characteristic p, then

$$\mathrm{H}^2(k, \mathbb{G}_m) \oplus \mathrm{H}^1_{\mathrm{fppf}}(k, \alpha_p^\vee) \cong \mathrm{H}^1_{\mathrm{fppf}}(k, \alpha_p) \cong k/k^p \overset{*}{\cong} \mathrm{H}^2_0(\alpha_p, \mathbb{G}_m) \overset{\text{Corollary 2.8}}{\cong} \mathrm{H}^2(B\alpha_{p,k}, \mathbb{G}_m),$$

where the isomorphism labelled with $*$ holds by [Mil17, Propositions 15.35 and 15.36]. Finally, Remark 5.13 generalizes Proposition 3.9 to certain additional finite linearly reductive groups G.

However, if $G = \mu_\ell \times \mu_\ell$, for some prime ℓ, over a field k with $\operatorname{char} k \neq \ell$, then this statement fails for G [Lie11, Proposition 4.3.2]. Additionally, if G is an abelian variety over a field, then the statement generally fails [Shi24, Lemma 4.2]. ◦

As in [AM20, Lemma 3.3], we can compute an explicit right-splitting of (3.2), though our proof was moreso inspired by [HS13, Proof of Theorem 1.4].

3.1. Constructing a right-splitting of (3.2).

Setup 3.12. Let G/S be a finite locally diagonalizable group over some base scheme S, and assume that G is cyclic or connected. Let $f\colon BG_S \to S$ denote its classifying stack, and let $\sigma \in \mathrm{H}^1_{\mathrm{fppf}}(BG_S, G)$ denote the class of a universal G-torsor $\pi\colon S \to BG_S$.

Remark 3.13. Above, we refer to σ as the class of "a" universal G-torsor instead of "the" universal G-torsor because, in general, BG_S can support multiple non-isomorphic universal G-torsors. For example, if σ is a universal G-torsor, then so is $\sigma \cdot f^*(\tau)$ for any $\tau \in \mathrm{H}^1(S, G)$ as is $\lambda_*(\sigma)$ for any $\lambda \in \mathrm{Aut}_S(G)$. ◦

Our goal in this section is to show that the map

$$\begin{array}{ccc} \mathrm{H}^1(S, G^\vee) & \longrightarrow & \mathrm{H}^2(BG_S, \mathbb{G}_m) \\ \alpha & \longmapsto & \sigma \smile f^*\alpha \end{array}$$

is a right-splitting to (3.2).

In the remainder of this section, we work in fppf topology instead of the étale topology since we will need to cohomology with coefficients in G. Let $\mathscr{C} := (\tau_{\leq 2}\mathrm{R}_{\mathrm{fppf}} f_*\mathbb{G}_m)[1]$ be the indicated shift of the indicated truncation of the derived pushforward of $\mathbb{G}_m$. Note that, by Proposition 3.8, $\mathscr{C} \simeq (\tau_{\leq 1}\mathrm{R}_{\mathrm{fppf}} f_*\mathbb{G}_m)[1]$ and sits in a distinguished triangle

$$\mathbb{G}_m[1] \longrightarrow \mathscr{C} \longrightarrow G^\vee \xrightarrow{+1}. \tag{3.3}$$

Furthermore, the above distinguished triangle (3.3) is split; indeed, the map $\mathbb{G}_m[1] \to (\tau_{\leq 2}\mathrm{R}_{\mathrm{fppf}} f_*\mathbb{G}_m)[1] = \mathscr{C}$ is split by (the shift of the derived pushforward) of the map $\mathbb{G}_{m,BG_S} \to \mathrm{R}_{\mathrm{fppf}}\pi_*\mathbb{G}_{m,S}$ coming from the section

π of f. Now, applying $\mathrm{Hom}_S(G^\vee, -)$ to (3.3), we obtain a short exact sequence

$$0 \longrightarrow \mathrm{Hom}_S(G^\vee, \mathbb{G}_m[1]) \longrightarrow \mathrm{Hom}_S(G^\vee, \mathscr{C}) \xrightarrow{\chi} \mathrm{Hom}_S(G^\vee, G^\vee) \longrightarrow 0.$$

Using that, in general, $\mathrm{Hom}(\mathscr{A}, \mathscr{B}[n]) = \mathrm{Ext}^n(\mathscr{A}, \mathscr{B})$ along with Lemma 3.6 allows us to see that $\mathrm{Hom}(G^\vee, \mathbb{G}_m[1]) \simeq \mathrm{Ext}^1(G^\vee, \mathbb{G}_m) \simeq \mathrm{H}^1_{\mathrm{fppf}}(S, G)$. Similarly,

$$\begin{aligned}
\mathrm{Hom}_S(G^\vee, \mathscr{C}) &= \mathrm{Hom}_S(G^\vee, (\tau_{\leq 2}\mathrm{R}_{\mathrm{fppf}} f_* \mathbb{G}_m)[1]) \\
&\simeq \mathrm{Ext}^1_S(G^\vee, \tau_{\leq 2}\mathrm{R}_{\mathrm{fppf}} f_* \mathbb{G}_m) \\
&\simeq \mathrm{Ext}^1_S(G^\vee, \mathrm{R}_{\mathrm{fppf}} f_* \mathbb{G}_m) \\
&\simeq \mathrm{Ext}^1_{BG_S}(G^\vee, \mathbb{G}_m) \\
&\simeq \mathrm{H}^1_{\mathrm{fppf}}(BG_S, G). \qquad (3.4)
\end{aligned}$$

Thus, our earlier short exact sequence can be rewritten as

$$0 \longrightarrow \mathrm{H}^1_{\mathrm{fppf}}(S, G) \xrightarrow{f^*} \mathrm{H}^1_{\mathrm{fppf}}(BG_S, G) \xrightarrow{\chi'} \mathrm{Hom}_S(G^\vee, G^\vee) \longrightarrow 0. \qquad (3.5)$$

Lemma 3.14. *Let $\tau \in \mathrm{H}^1_{\mathrm{fppf}}(BG_S, G)$ be a class mapping to the identity $\mathrm{id}_{G^\vee} \in \mathrm{Hom}_S(G^\vee, G^\vee)$ under the right map in (3.5). Then, τ is (the class of) a universal G-torsor.*

Proof. Consider the commutative diagram

$$\begin{tikzcd}
\mathrm{Hom}_S(G,G) \times \mathrm{H}^1_{\mathrm{fppf}}(BG_S, G) \arrow[r, "\mathrm{id} \times \chi'"] \arrow[d, "\wr"] \arrow[ddd, bend right=60, "(\lambda,\alpha)\mapsto\lambda_*(\alpha)"'] & \mathrm{Hom}_S(G,G) \times \mathrm{Hom}_S(G^\vee, G^\vee) \arrow[d, equal] \\
\mathrm{Hom}_S(G,G) \times \mathrm{Hom}_S(G^\vee, \mathscr{C}) \arrow[r, "\mathrm{id} \times \chi"] \arrow[d, "(\lambda,\varphi)\mapsto\varphi\circ\lambda^\vee"'] & \mathrm{Hom}(G,G) \times \mathrm{Hom}(G^\vee, G^\vee) \arrow[d, "(\lambda,\psi)\mapsto\psi\circ\lambda^\vee"] \\
\mathrm{Hom}_S(G^\vee, \mathscr{C}) \arrow[r, "\chi"] \arrow[d, "\wr"] & \mathrm{Hom}_S(G^\vee, G^\vee) \arrow[d, equal] \\
\mathrm{H}^1_{\mathrm{fppf}}(BG_S, G) \arrow[r, "\chi'"] & \mathrm{Hom}_S(G^\vee, G^\vee).
\end{tikzcd} \qquad (3.6)$$

Let $\lambda := \chi'(\sigma)^\vee \in \mathrm{Hom}_S(G, G)$. Then commutativity of (3.6) implies that $\chi'(\lambda_*(\tau)) = \mathrm{id}_G \circ \lambda^\vee = \chi'(\sigma)$. By (3.5), possibly after twisting τ by an element of $f^* \mathrm{H}^1_{\mathrm{fppf}}(S, G)$, we may assume wlog that $\lambda_*(\tau) = \sigma$. Note that λ may also be regarded as an S-morphism $\lambda\colon BG_S \to BG_S$; indeed this is the morphism sending a G-torsor T over an S-scheme X to the G-torsor $\lambda_*(T)$ over X. Let $g_\tau\colon BG_S \to BG_S$ denote the unique (up to isomorphism) S-morphism such that $g_\tau^*(\sigma) = \tau$, and consider the commutative diagram

$$\begin{tikzcd}
BG_S \arrow[r, "g_\tau"'] \arrow[rr, bend left, "h"] & BG_S \arrow[r, "\lambda"'] & BG_S.
\end{tikzcd}$$

Note that, by definition of λ, we have

$$h^*(\sigma) = \lambda_*(g_\tau^*(\sigma)) = \lambda_*(\tau) = \sigma.$$

It follows from this that $h \simeq \mathrm{id}_{BG_S}$, so g_τ is an isomorphism. Thus, by transfer of structure, $\tau = g_\tau^*(\sigma)$ must be universal as well. ■

Proposition 3.15. *There exists a choice $\sigma \in \mathrm{H}^1_{\mathrm{fppf}}(BG_S, G)$ of class of universal G-torsor such that the map*

$$\begin{array}{rccc}
s: & \mathrm{H}^1(S, G^\vee) & \longrightarrow & \mathrm{H}^2(BG_S, \mathbb{G}_m) \\
 & \alpha & \longmapsto & f^*\alpha \smile \sigma
\end{array}$$

is a right-splitting to (3.2).

Proof. By Lemma 3.14, we may and do choose σ so that $\chi'(\sigma) = \mathrm{id}_{G^\vee}$. Let $\Sigma \in \mathrm{Hom}_S(G^\vee, \mathscr{C})$ be the morphism corresponding to σ under the identification (3.4). Note that the short exact sequence

$$0 \longrightarrow \mathrm{H}^2(S, \mathbb{G}_m) \longrightarrow \mathrm{H}^2(BG_S, \mathbb{G}_m) \xrightarrow{r} \mathrm{H}^1(S, G^\vee) \longrightarrow 0$$

obtained from applying $\mathrm{H}^1(S, -)$ to the split distinguished triangle (3.3) (note $\mathrm{H}^1(S, \mathscr{C}) \simeq \mathrm{H}^2(BG_S, \mathbb{G}_m)$ by definition of $\mathscr{C}$) is the sequence (3.2) obtained from the Leray spectral sequence. Write $\rho\colon \mathscr{C} \to G^\vee$ for the map in (3.3), and consider the commutative diagram (commutativity follows from [Mil80, Proposition V.1.20])

$$\begin{tikzcd}
\mathrm{H}^1(S, G^\vee) \times \mathrm{Hom}_S(G^\vee, \mathscr{C}) \arrow[r] \arrow[d, "\wr"] & \mathrm{H}^1_{\mathrm{fppf}}(S, \mathscr{C}) \arrow[r, "\rho_*"] \arrow[d, "\wr"] & \mathrm{H}^1(S, G^\vee) \arrow[d, equal] \\
\mathrm{H}^1(S, G^\vee) \times \mathrm{H}^1_{\mathrm{fppf}}(BG_S, G) \arrow[r, "f^*(-)\smile(-)"] & \mathrm{H}^2(BG_S, \mathbb{G}_m) \arrow[r, "r"] & \mathrm{H}^1(S, G^\vee).
\end{tikzcd}$$

Commutativity of this diagram shows that

$$r(s(\alpha)) = r(f^*(\alpha) \smile \sigma) = \rho_*(\Sigma_*(\alpha)) = (\rho \circ \Sigma)_*(\alpha) = \chi(\Sigma)_*(\alpha) = (\mathrm{id}_{G^\vee})_*(\alpha) = \alpha,$$

where the third- and second-to-last equalities hold by definition of χ and our assumption on σ. ∎

4. A Cohomological Vanishing Result

In this section, we aim to prove a version of [Mei18, Theorem 6] which holds for tame algebraic stacks (see Theorem 4.14). As an immediate application, we then use it to compute the Brauer group of the moduli stack $\mathcal{X}(1)$ of generalized elliptic curves over any noetherian strictly Henselian local ring (see Example 4.17).

To prove Theorem 4.14, one needs to guarantee the vanishing of groups of the form $\mathrm{H}^2([\operatorname{Spec} R/G], \mathbb{G}_m)$, where R is a strictly henselian local ring and G is a finite linearly reductive group. Since such G are built from tame étale groups and connected locally diagonalizable groups, the first steps are to understand such cohomology groups when G falls into one of these categories.

Lemma 4.1. *Let R be a strictly henselian local ring with residue field k, and let G be a finite abstract group acting on R. Assume that $p := \operatorname{char} k \nmid \#G$ and set $\mathcal{X} := [\operatorname{Spec} R/G]$. Then, $\mathrm{H}^n(\mathcal{X}, \mathbb{G}_m) \simeq \mathrm{H}^n_0(G, \mathbb{G}_m(R))$ for all $n \geq 1$ and $\mathrm{H}^n_0(G, \mathbb{G}_m(R)) \simeq \mathrm{H}^n_0(G, \mathbb{G}_m(k))$ for all $n \geq 2$. This latter isomorphism holds for $n = 1$ as well if R has no nontrivial p-power roots of unity.*

Proof. This follows from the proof of [Mei18, Lemma 9] (which assumed that R was a domain, but did not use this assumption in its proof) coupled with the observation that $\mathrm{H}^n(\mathcal{X}, \mathbb{G}_m) = \mathrm{H}^n_0(G, \mathbb{G}_m(R))$ is $\#G$-torsion. To assure the reader that R need not be a domain, we include a full proof below.

First note that Corollary 2.8 shows that $\mathrm{H}^n(\mathcal{X}, \mathbb{G}_m) \simeq \mathrm{H}^n_0(G, \mathbb{G}_m(R))$ for all $n \geq 0$. Let $M := \ker(\mathbb{G}_m(R) \to \mathbb{G}_m(k))$. Because R is strictly Henselian, Hensel's lemma shows that M is a $\mathbb{Z}_{(p)}$-module. Thus, $M[1/p]$ (if $p = 0$, by this, we simply mean M) is a $\mathbb{Q}$-vector space, so taking cohomology of the exact sequence $0 \to M[1/p] \to \mathbb{G}_m(R)[1/p] \to \mathbb{G}_m(k)[1/p] \to 0$ shows that

$$\mathrm{H}^n_0(G, \mathbb{G}_m(R)[1/p]) \xrightarrow{\sim} \mathrm{H}^n_0(G, \mathbb{G}_m(k)[1/p]) \text{ for all } n \geq 1.$$

Finally, because $p \nmid \#G$, for any G-module N, we have $\mathrm{H}^n_0(G, N) \xrightarrow{\sim} \mathrm{H}^n_0(G, N[1/p])$ for all $n \geq 2$ (because the kernel and cokernel of $N \to N[1/p]$ are both p-power torsion) and furthermore $\mathrm{H}^1_0(G, N) \xrightarrow{\sim} \mathrm{H}^1_0(G, N[1/p])$ if $N\{p\} = 0$ (because then $N \to N[1/p]$ is injective). ∎

4.1. Cohomology of quotients by locally diagonalizable groups.

Lemma 4.2. *Let $\mathcal{X}$ be a noetherian tame algebraic stack with coarse space morphism $c\colon \mathcal{X} \to X$. Assume that X is an affine scheme. Then, for every $i \geq 1$, the natural map*

$$\mathrm{H}^i(\mathcal{X}, \mathbb{G}_m) \longrightarrow \mathrm{H}^i(\mathcal{X}_{\mathrm{red}}, \mathbb{G}_m)$$

is an isomorphism.

Proof. We can factor $\mathcal{X}_{\mathrm{red}} \hookrightarrow \mathcal{X}$ into a sequence

$$\mathcal{X}_{\mathrm{red}} = \mathcal{X}_0 \hookrightarrow \mathcal{X}_1 \hookrightarrow \dots \hookrightarrow \mathcal{X}_n = \mathcal{X}$$

of square-zero thickenings, so there are quasi-coherent ideal sheaves $\mathscr{I}_i \subset \mathscr{O}_{\mathcal{X}_i}$, for $i = 1, \dots, n$, such that $\mathscr{I}_i^2 = 0$ and $\mathscr{O}_{\mathcal{X}_i}/\mathscr{I}_i \simeq \mathscr{O}_{\mathcal{X}_{i-1}}$. For each i, the map $\mathscr{I}_i \to \mathbb{G}_{m,\mathcal{X}_i}$, $x \mapsto 1 + x$, sits in the following exact sequence

$$0 \longrightarrow \mathscr{I}_i \longrightarrow \mathbb{G}_{m,\mathcal{X}_i} \longrightarrow \iota_*\mathbb{G}_{m,\mathcal{X}_{i-1}} \longrightarrow 1 \tag{4.1}$$

of sheaves on $(\mathcal{X}_i)_{\text{lis-ét}}$, where $\iota\colon \mathcal{X}_{i-1} \hookrightarrow \mathcal{X}_i$ is the natural immersion. Now, by considering the long exact sequence in cohomology associated to (4.1) and inducting on i, one sees that, to prove the lemma, it suffices to show that $\mathrm{H}^j(\mathcal{X}_i, \mathscr{I}_i) = 0$ for all $j \geq 1$ and all $i = 1, \dots, n$. For fixed i, j, let $r\colon \mathcal{X}_i \hookrightarrow \mathcal{X}$ be the natural closed immersion. Because r is a closed immersion r_* is exact. At the same time, because $\mathcal{X}$ is tame, c_* is exact on qcoh sheaves (recall Definition/Proposition 2.10), so

$$\mathrm{H}^j(\mathcal{X}_i, \mathscr{I}_i) \simeq \mathrm{H}^j(\mathcal{X}, r_*\mathscr{I}_i) \simeq \mathrm{H}^j(X, c_*r_*\mathscr{I}_i),$$

but this latter group vanishes because X is affine (and $j \geq 1$). ∎

Proposition 4.3. *Let R be a strictly henselian noetherian local ring with residue field k, let Δ/R be a connected diagonalizable group, suppose Δ acts on some finite, connected R-scheme X, and set $\mathcal{Y} := [X/\Delta]$. Write $X = \operatorname{Spec} A$ for a strictly henselian local ring A, say with residue field k'. Assume that Δ_k acts trivially on k'. Then,*

- $\mathrm{H}^i(\mathcal{Y}, \mathbb{G}_m)$ *is $\#\Delta$-torsion for all $i \geq 1$;*
- $\mathrm{H}^i(\mathcal{Y}, \mathbb{G}_m) \xrightarrow{\sim} \mathrm{H}^i(B\Delta_{k'}, \mathbb{G}_m)$ *for all $i \geq 2$; and*
- $\operatorname{Pic} \mathcal{Y} \twoheadrightarrow \operatorname{Pic} B\Delta_{k'} = \Delta^\vee(k')$ *is surjective.*

It follows from the second bullet point and Lemma 3.7 that $\mathrm{H}^2(\mathcal{Y}, \mathbb{G}_m) = 0$.

Proof. Let $p = \operatorname{char} k$ and write $\#\Delta = p^n$. Fix some $i \geq 1$. The existence of the Δ-torsor $f\colon X \to \mathcal{Y}$ with $\mathrm{H}^i(X, \mathbb{G}_m) = 0$ is enough to deduce that $\mathrm{H}^i(\mathcal{Y}, \mathbb{G}_m) = \mathrm{H}^i(\mathcal{Y}, \mathbb{G}_m)[p^n]$; indeed, the norm map $f_*\mathbb{G}_{m,X} \to \mathbb{G}_{m,\mathcal{Y}}$ gives rise to a composition

$$\mathrm{H}^i(\mathcal{Y}, \mathbb{G}_m) \longrightarrow \underbrace{\mathrm{H}^i(\mathcal{Y}, f_*\mathbb{G}_{m,X})}_{\simeq \mathrm{H}^i(X,\mathbb{G}_m) = 0} \longrightarrow \mathrm{H}^i(\mathcal{Y}, \mathbb{G}_m)$$

which equals multiplication by p^n (see [CTS21, Section 3.8] for more details on this norm map). With this in mind, define Γ via the follow exact sequence of sheaves on $\mathcal{Y}_{\text{lis-ét}}$:

$$1 \longrightarrow \mu_{p^n} \longrightarrow \mathbb{G}_m \xrightarrow{p^n} \mathbb{G}_m \longrightarrow \Gamma \longrightarrow 1.$$

Let $\mathcal{Y}_0$ denote the special fiber of $\mathcal{Y} \to \operatorname{Spec} R$, and consider the two term complexes $\mathscr{C} := [\mathbb{G}_{m,\mathcal{Y}} \xrightarrow{p^n} \mathbb{G}_{m,\mathcal{Y}}]$ and $\mathscr{C}_0 := [\mathbb{G}_{m,\mathcal{Y}_0} \xrightarrow{p^n} \mathbb{G}_{m,\mathcal{Y}_0}]$. The distinguished triangle $\mu_{p^n} \to \mathscr{C} \to \Gamma[-1] \xrightarrow{+1}$ (along with its analogue over $\mathcal{Y}_0$) gives rise to the following homomorphism of exact sequences:

$$\begin{array}{ccccccccc}
\mathrm{H}^{i-2}(\mathcal{Y}, \Gamma) & \longrightarrow & \mathrm{H}^i(\mathcal{Y}, \mu_{p^n}) & \longrightarrow & \mathrm{H}^i(\mathcal{Y}, \mathbb{G}_m \xrightarrow{p^n} \mathbb{G}_m) & \longrightarrow & \mathrm{H}^{i-1}(\mathcal{Y}, \Gamma) & \longrightarrow & \mathrm{H}^{i+1}(\mathcal{Y}, \mu_{p^n}) \\
\downarrow & & \downarrow & & \downarrow & & \downarrow & & \downarrow \\
\mathrm{H}^{i-2}(\mathcal{Y}_0, \Gamma) & \longrightarrow & \mathrm{H}^i(\mathcal{Y}_0, \mu_{p^n}) & \longrightarrow & \mathrm{H}^i(\mathcal{Y}_0, \mathbb{G}_m \xrightarrow{p^n} \mathbb{G}_m) & \longrightarrow & \mathrm{H}^{i-1}(\mathcal{Y}_0, \Gamma) & \longrightarrow & \mathrm{H}^{i+1}(\mathcal{Y}_0, \mu_{p^n}).
\end{array}$$

Since μ_{p^n} and Γ are both torsion sheaves and $\mathcal{Y} \to \operatorname{Spec} R$ is proper, proper base change [Ols05, Theorem 1.3] (see also [Mil80, Corollary VI.2.7]) followed by an application of the five lemma implies that all vertical maps above are isomorphisms. Now, the distinguished triangle $\mathbb{G}_m[-1] \to \mathscr{C} \to \mathbb{G}_m \xrightarrow{+1}$ (along with its analogue over $\mathcal{Y}_0$) gives rise to the following homomorphism of short exact sequences:

$$\begin{tikzcd} 0 \arrow[r] & \mathrm{H}^{i-1}(\mathcal{Y},\mathbb{G}_m)/p^n \arrow[r] \arrow[d] & \mathrm{H}^i(\mathcal{Y},\mathbb{G}_m \xrightarrow{p^n} \mathbb{G}_m) \arrow[r] \arrow[d, "\wr"] & \mathrm{H}^i(\mathcal{Y},\mathbb{G}_m)[p^n] \arrow[r] \arrow[d] & 0 \\ 0 \arrow[r] & \mathrm{H}^{i-1}(\mathcal{Y}_0,\mathbb{G}_m)/p^n \arrow[r] & \mathrm{H}^i(\mathcal{Y}_0,\mathbb{G}_m \xrightarrow{p^n} \mathbb{G}_m) \arrow[r] & \mathrm{H}^i(\mathcal{Y}_0,\mathbb{G}_m)[p^n] \arrow[r] & 0. \end{tikzcd} \tag{4.2}$$

Noting that $(\mathcal{Y}_0)_{\mathrm{red}} \simeq B\Delta_{k'}$, we have a surjection

$$\mathrm{H}^i(\mathcal{Y},\mathbb{G}_m) = \mathrm{H}^i(\mathcal{Y},\mathbb{G}_m)[p^n] \twoheadrightarrow \mathrm{H}^i(\mathcal{Y}_0,\mathbb{G}_m)[p^n] = \mathrm{H}^i(\mathcal{Y}_0,\mathbb{G}_m) \underbrace{\simeq}_{\text{Lemma 4.2}} \mathrm{H}^i(B\Delta_{k'},\mathbb{G}_m).$$

When $i=1$, this (combined with Lemma 3.7) proves the third bullet point of the claim. Assume now that $i \geq 2$. Lemma 4.2 allows us to rewrite (4.2) as

$$\begin{tikzcd} 0 \arrow[r] & \mathrm{H}^{i-1}(\mathcal{Y},\mathbb{G}_m)/p^n \arrow[r] \arrow[d] & \mathrm{H}^2(\mathcal{Y},\mathbb{G}_m \xrightarrow{p^n} \mathbb{G}_m) \arrow[r] \arrow[d, "\wr"] & \mathrm{H}^2(\mathcal{Y},\mathbb{G}_m)[p^n] \arrow[r] \arrow[d] & 0 \\ 0 \arrow[r] & \mathrm{H}^{i-1}(B\Delta_{k'},\mathbb{G}_m)/p^n \arrow[r] & \mathrm{H}^2(\mathcal{Y}_0,\mathbb{G}_m \xrightarrow{p^n} \mathbb{G}_m) \arrow[r] & \mathrm{H}^2(B\Delta_{k'},\mathbb{G}_m)[p^n] \arrow[r] & 0. \end{tikzcd} \tag{4.3}$$

We claim the left vertical arrow in (4.3) is surjective. Indeed, applying the argument so far to $i-1$ in place of i shows that $\mathrm{H}^{i-1}(\mathcal{Y},\mathbb{G}_m)$ is p^n-torsion (as is $\mathrm{H}^{i-1}(B\Delta_{k'},\mathbb{G}_m)$) and that

$$\mathrm{H}^{i-1}(\mathcal{Y},\mathbb{G}_m)/p^n = \mathrm{H}^{i-1}(\mathcal{Y},\mathbb{G}_m) = \mathrm{H}^{i-1}(\mathcal{Y},\mathbb{G}_m)[p^n] \twoheadrightarrow \mathrm{H}^{i-1}(B\Delta_{k'},\mathbb{G}_m)[p^n] = \mathrm{H}^{i-1}(B\Delta_{k'},\mathbb{G}_m) = \mathrm{H}^{i-1}(B\Delta_{k'},\mathbb{G}_m)/p^n$$

is surjective. Now, (4.3) shows that $\mathrm{H}^i(\mathcal{Y},\mathbb{G}_m)[p^n] \to \mathrm{H}^i(B\Delta_{k'},\mathbb{G}_m)[p^n]$ is injective and so an isomorphism. Both $\mathrm{H}^i(\mathcal{Y},\mathbb{G}_m)$ and $\mathrm{H}^i(B\Delta_{k'},\mathbb{G}_m)$ are p^n-torsion, so we conclude the second bullet of the claim. ■

4.2. Vanishing of $\mathrm{R}^2c_*\mathbb{G}_m$.

Proposition 4.4. *Let R be a strictly henselian noetherian local ring with residue field k. Let G/R be a finite linearly reductive group, suppose G acts on some finite R-scheme X, and set $\mathcal{X} \coloneqq [X/G]$. Suppose further that*

- *X is the spectrum of a strictly henselian local ring A with residue field k'; and*
- *G_k acts trivially on $\operatorname{Spec} k'$.*

Write

$$0 \longrightarrow \Delta \longrightarrow G \longrightarrow \underline{Q} \longrightarrow 0$$

for G's connected-étale sequence, with Q an abstract group. Then, there is an exact sequence

$$0 \to \mathrm{H}^1_0(Q,\mathbb{G}_m(k')) \to \operatorname{Pic}\mathcal{X} \to \mathrm{H}^1_0(\Delta,\mathbb{G}_{m,X/R})^Q \to \mathrm{H}^2_0(Q,\mathbb{G}_m(k')) \to \mathrm{H}^2(\mathcal{X},\mathbb{G}_m) \to 0.$$

In particular, if $\mathrm{H}^2_0(Q,\mathbb{G}_m(k'))=0$, then $\mathrm{H}^2(\mathcal{X},\mathbb{G}_m)=0$.

Proof. Set $\mathcal{Y} \coloneqq [X/\Delta]$, so $\mathcal{Y} \to \mathcal{X}$ is a Q-torsor, and consider the Hochschild–Serre spectral sequence

$$E_2^{ij} = \mathrm{H}^i_0(Q,\mathrm{H}^j(\mathcal{Y},\mathbb{G}_m)) \implies \mathrm{H}^{i+j}(\mathcal{X},\mathbb{G}_m).$$

Below, we calculate many of the terms of this spectral sequence. In every argument below, keep in mind that Q is a tame abstract group – so $p \coloneqq \operatorname{char} k \geq 0$ does *not* divide $\#Q$ – and Δ is a finite connected diagonalizable R-group.

(1) Claim: The ring A^Δ of Δ-invariants is strictly henselian and has k' as its residue field. Furthermore,

$$E_2^{n0} = \mathrm{H}^n_0(Q,\mathbb{G}_m(\mathcal{Y})) \simeq \mathrm{H}^n_0(Q,\mathbb{G}_m(k')) \text{ for all } n \geq 1.$$

Note that, since R is noetherian, $A^\Delta \subset A$ is a finite R-algebra as well. It is therefore a product of strictly henselian local rings [Sta21, Tag 03QJ]. At the same time, being a subset of A, it contains no nontrivial idempotents, so A^Δ must itself be strictly henselian local. Furthermore, because Δ

is linear reductive, taking Δ-invariants is an exact functor (on the category of R-modules with Δ-action) and so commutes with quotients. It follows that A^{Δ} has $(k')^{\Delta} = k'$ as a quotient, so k' must be its residue field. Now, $\mathbb{G}_m(\mathcal{Y}) = \mathbb{G}_m(A^{\Delta})$ by construction, and Lemma 4.1 tells us that $\mathrm{H}_0^n(Q, \mathbb{G}_m(A^{\Delta})) \simeq \mathrm{H}_0^n(G, \mathbb{G}_m(k'))$ for all $n \geq 1$, as claimed.

(2) Claim: $E_2^{01} = \mathrm{H}_0^0(Q, \operatorname{Pic} \mathcal{Y}) \simeq \mathrm{H}_0^1(\Delta, \mathbb{G}_{m,X/R})^Q$ and $E_2^{ij} = 0$ for all $i, j \geq 1$.

By Corollary 2.8, $\operatorname{Pic} \mathcal{Y} \simeq \mathrm{H}_0^1(\Delta, \mathbb{G}_{m,X/R})$ so the claimed computation of E_2^{01} holds. Fix now some $i, j \geq 1$. Then, Proposition 4.3 shows that $\mathrm{H}^j(\mathcal{Y}, \mathbb{G}_m)$ is $\#\Delta$-torsion. Thus, $E_2^{ij} = \mathrm{H}_0^i(Q, \mathrm{H}^j(\mathcal{Y}, \mathbb{G}_m))$ is both $\#\Delta$-torsion and $\#Q$-torsion. Since $\gcd(\#\Delta, \#Q) = 1$, we conclude that $E_2^{ij} = 0$.

(3) Claim: $\mathrm{H}^2(\mathcal{Y}, \mathbb{G}_m) \simeq 0$, so $E_2^{02} = \mathrm{H}_0^0(Q, \mathrm{H}^2(\mathcal{Y}, \mathbb{G}_m)) \simeq 0$.

This follows from Proposition 4.3.

The claimed exact sequence now follows from considering the low degree exact sequence associated to our spectral sequence E_2^{ij}. ■

Definition 4.5. Let R be a strictly local ring, and let G/R be a finite group scheme with connected-étale sequence

$$0 \longrightarrow \Delta \longrightarrow G \longrightarrow \underline{Q} \longrightarrow 0. \tag{4.4}$$

We say that G/R is Brauerless if Δ is diagonalizable and $\mathrm{H}_0^3(Q, \mathbb{Z}) = 0$, when Q acts trivially on $\mathbb{Z}$. More generally, if S is a scheme, we will say a finite group scheme G/S is Brauerless if $G_{\mathscr{O}_{S,\overline{s}}}$ is Brauerless for every geometric point $\overline{s} \to S$. ◇

Proposition 4.6. *Let G/S be a finite linearly reductive S-group scheme. Then, the following are equivalent*

(a) G is Brauerless.

(b) For every map $\operatorname{Spec} R \to S$ from a strictly local scheme, G_R is Brauerless.

(c) For every map $\operatorname{Spec} R \to S$ from a strictly local scheme, $\mathrm{H}_0^3(G_R, \underline{\mathbb{Z}}_R) = 0$.

(d) For every geometric point $\overline{s} \to S$, $G_{\overline{s}}$ is Brauerless.

Proof.

(**(a)** $\iff$ **(b)**) Let $\operatorname{Spec} R \to S$ be a map from a strictly local scheme. Let $\overline{s} \in \operatorname{Spec} R$ denote the closed point, so $\overline{s} \hookrightarrow \operatorname{Spec} R \to S$ is a geometric point of S. Then, $\operatorname{Spec} R \to S$ factors as $\operatorname{Spec} R \to \operatorname{Spec} \mathscr{O}_{S,\overline{s}} \to S$, so $G_R \simeq G_{\mathscr{O}_{S,\overline{s}}} \times_{\mathscr{O}_{S,\overline{s}}} R$. It is now clear from definitions that $G_{\mathscr{O}_{S,\overline{s}}}$ is Brauerless if and only if G_R is as well.

(**(b)** $\iff$ **(c)**) Assume that $S = \operatorname{Spec} R$ is strictly local. It suffices to show that $\mathrm{H}_0^3(G, \underline{\mathbb{Z}}) = 0$. Use notation as in (4.4) for G's connected-étale sequence. Since the group scheme $\underline{\mathbb{Z}}$ is étale, every map $G \to \underline{\mathbb{Z}}$ (of schemes) is constant on connected components of G and so factors through $\underline{Q}$. A completely analogous statement holds with G replaced by G^n; hence,

$$C^n(\underline{Q}, \underline{\mathbb{Z}}) = \operatorname{Nat}(\underline{Q}^n, \underline{\mathbb{Z}}) \xrightarrow{\sim} \operatorname{Nat}(G^n, \underline{\mathbb{Z}}) = C^n(G, \underline{\mathbb{Z}}) \text{ for all } n,$$

and so $\mathrm{H}_0^n(\underline{Q}, \underline{\mathbb{Z}}) \xrightarrow{\sim} \mathrm{H}_0^n(G, \underline{\mathbb{Z}})$ for all n. Thus, G is Brauerless if and only if $\mathrm{H}_0^3(G, \underline{\mathbb{Z}}) = 0$.

(**(a)** $\iff$ **(d)**) Assume **(b)**. Let $\overline{s} \to S$ be a geometric point, so $\overline{s}$ factors as $\overline{s} \to G_{\mathscr{O}_{S,\overline{s}}} \to S$. The connected-étale sequence (4.4) of $G_{\mathscr{O}_{S,\overline{s}}}$ restricts to the connected-étale sequence of $G_{\overline{s}}$ from which one sees that $G_{\overline{s}}$ is Brauerless if and only if $G_{\mathscr{O}_{S,\overline{s}}}$ is. ■

Corollary 4.7. *Let G, S be as in Proposition 4.6. Then, G/S is Brauerless if and only if for every geometric point $\overline{s} \to S$, $\mathrm{H}_0^3(G_{\overline{s}}, \underline{\mathbb{Z}}) = 0$.*

Remark 4.8. In the case that G is an abstract group, Meier [Mei18, Definition 3] used the term 'poor' for essentially the same notion, but I felt that 'Brauerless' was more evocative of the utility of this definition in this context. ○

Example 4.9. For every n, both $\mathbb{Z}/n\mathbb{Z}$ and μ_n are Brauerless. △

Remark 4.10. An abstract finite abelian group is Brauerless if and only if it is cyclic; this is a consequence of the Künneth formula [Wei94, Exercise 6.1.8]. Consequently, a finite commutative group scheme G over a strictly local ring R is Brauerless if and only if its maximal étale quotient is cyclic. ◦

Example 4.11. By [Mei18, Example 4], any geometric automorphism group of the moduli space $\mathcal{Y}(1)$ of elliptic curves is Brauerless. △

Lemma 4.12. *If a finite abstract group Q is Brauerless, then for any separably closed field k of characteristic $p \nmid \#Q$, $\mathrm{H}^2_0(Q, \mathbb{G}_m(k)) = 0$ when Q acts trivially on $\mathbb{G}_m(k)$.*

Proof. This was essentially proven in [Mei18, Lemma 5], except k there was assumed algebraically closed. Below, we show how to modify Meier's argument.

Let $M = \mathbb{G}_m(k)[1/p]$ (by which, we simply mean $\mathbb{G}_m(k)$ if $p = 0$), and note that $\mathrm{H}^2_0(Q, M) \simeq \mathrm{H}^2_0(Q, \mathbb{G}_m(k))$ since $p \nmid \#Q$. Because k is separably closed of characteristic p, M is divisible. It follows from the structure theorem for divisible groups [Fuc70, Theorem 23.1] that M is a subgroup (and so direct summand) of some group of the form $(\mathbb{Q}/\mathbb{Z})^{\oplus I} \oplus \mathbb{Q}^{\oplus J}$ for some sets I, J. Thus, it suffices to show $\mathrm{H}^2_0(Q, \mathbb{Q}/\mathbb{Z}) = 0$ and $\mathrm{H}^2_0(Q, \mathbb{Q}) = 0$. The latter of these holds simply because Q is finite. For the former, the exact sequence $0 \to \mathbb{Z} \to \mathbb{Q} \to \mathbb{Q}/\mathbb{Z} \to 0$ identifies $\mathrm{H}^2_0(Q, \mathbb{Q}/\mathbb{Z}) \simeq \mathrm{H}^3_0(Q, \mathbb{Z})$ which vanishes by assumption. ■

Definition 4.13. We say a tame algebraic stack $\mathcal{X}$ is `locally Brauerless` if all of its geometric automorphism groups are Brauerless (see Definition 4.5). ⋄

Theorem 4.14. *Let $\mathcal{X}$ be a locally Brauerless tame algebraic stack, with coarse moduli map $\pi\colon \mathcal{X} \to X$. Then, $\mathrm{R}^2\pi_*\mathbb{G}_m = 0$.*

Proof. Let $\overline{x} \to X$ be a geometric point. By Corollary 2.14, $\mathcal{X} \times_X \mathscr{O}_{X,\overline{x}} \simeq [\operatorname{Spec} R/G]$ for some strictly henselian local ring R acted upon by some finite linearly reductive group $G/\mathscr{O}_{X,\overline{x}}$ satisfying $G_{\overline{x}} \cong \underline{\mathrm{Aut}}(\mathcal{X}, \overline{x})$. Thus, $(\mathrm{R}^2\pi_*\mathbb{G}_m)_{\overline{x}} \simeq \mathrm{H}^2([\operatorname{Spec} R/G], \mathbb{G}_m)$. Write $\underline{Q}_{\mathscr{O}_{X,\overline{x}}}$ for the maximal étale quotient of G, and write k for R's residue field. By Proposition 4.4, $\mathrm{H}^2([\operatorname{Spec} R/G], \mathbb{G}_m) = 0$ if $\mathrm{H}^2_0(Q, \mathbb{G}_m(k))$ vanishes. To see this, first note that $G_{\kappa(\overline{x})}$ is Brauerless by assumption, so G is as well by Corollary 4.7. Thus, Lemma 4.12 shows that $\mathrm{H}^2_0(Q, \mathbb{G}_m(k)) = 0$ and so we conclude that every stalk of $\mathrm{R}^2\pi_*\mathbb{G}_m$ vanishes. ■

Remark 4.15. The DM case of Theorem 4.14, which was essentially proven in [Mei18, Theorem 6], only requires Lemma 4.1 instead of the more difficult Proposition 4.4. ◦

Corollary 4.16. *Let $\mathcal{X}$ be as in Theorem 4.14. Then, there is an exact sequence*

$$\begin{tikzcd}
0 \arrow[r] & \operatorname{Pic} X \arrow[r, "\pi^*"] & \operatorname{Pic} \mathcal{X} \arrow[r] & \mathrm{H}^0(X, \mathrm{R}^1\pi_*\mathbb{G}_m) \arrow[dlll, "\mathrm{d}_2^{0,1}"] & \\
\mathrm{H}^2(X, \mathbb{G}_m) \arrow[r, "\pi^*"] & \mathrm{H}^2(\mathcal{X}, \mathbb{G}_m) \arrow[r] & \mathrm{H}^1(X, \mathrm{R}^1\pi_*\mathbb{G}_m) \arrow[r, "\mathrm{d}_2^{1,1}"] & \ker\left(\mathrm{H}^3(X, \mathbb{G}_m) \xrightarrow{\pi^*} \mathrm{H}^3(\mathcal{X}, \mathbb{G}_m)\right).
\end{tikzcd}$$

Proof. This is the exact sequence of low degree terms in the Leray spectral sequence $E_2^{ij} = \mathrm{H}^i(X, \mathrm{R}^j\pi_*\mathbb{G}_m) \implies \mathrm{H}^{i+j}(\mathcal{X}, \mathbb{G}_m)$. ■

In order to ease difficulties which may arise in attempting to compute the cohomology groups $\mathrm{H}^i(X, \mathrm{R}^1\pi_*\mathbb{G}_m)$ appearing above, it may be most useful to apply Corollary 4.16 only to stacks over separably closed fields (or, more generally, strictly Henselian local rings). Despite this, it can still serve as one ingredient in a larger computation of Brauer groups over more general bases; for example, below we will compute $\operatorname{Br} \mathcal{X}(1)_R$ when R is a strictly Henselian local ring, and then later extend this computation to other bases in Section 9.1.

Example 4.17 ($\operatorname{Br} \mathcal{X}(1)_R$)**.** Let R be a noetherian strictly Henselian local ring, and let $\mathcal{X} = \mathcal{X}(1)/R$ be the moduli space of generalized elliptic curves, sometimes also denoted $\overline{\mathcal{M}}_{1,1}$, and let $\pi\colon \mathcal{X} \to \mathbb{P}^1$ be its coarse moduli space. Assume that $6 \in R^\times$, so $\mathcal{X}/R$ is tame. We will apply Corollary 4.16 in order to compute that $\operatorname{Br}\mathcal{X} = \operatorname{Br}\mathbb{P}^1_R = 0$. First, one produces the exact sequence

$$0 \longrightarrow \underline{\mathbb{Z}/3\mathbb{Z}}_0 \oplus \underline{\mathbb{Z}/2\mathbb{Z}}_{1728} \longrightarrow \mathrm{R}^1\pi_*\mathbb{G}_m \longrightarrow \underline{\mathbb{Z}/2\mathbb{Z}} \longrightarrow 0 \tag{4.5}$$

of étale sheaves on $\mathbb{P}^1_R$, where $\underline{\mathbb{Z}/n\mathbb{Z}}_x$ denotes the pushforward $i_{x,*}\underline{\mathbb{Z}/n\mathbb{Z}}$ along the inclusion $i_x\colon \operatorname{Spec} R \to \mathbb{P}^1_R$ of $x \in \mathbb{P}^1(R)$. To keep this example relatively tidy, we postpone a derivation of (4.5) until Section 9.1, where we compute $\operatorname{Br}\mathcal{X}(1)_S$ more generally.

The long exact sequence in cohomology associated to (4.5) immediately shows that $\mathrm{H}^1(\mathbb{P}^1_R, \mathrm{R}^1\pi_*\mathbb{G}_m) = 0$ and that $\#\,\mathrm{H}^0(\mathbb{P}^1_R, \mathrm{R}^1\pi_*\mathbb{G}_m) = 12$. It is clear from the construction of (4.5) that the Hodge bundle $\lambda \in \operatorname{Pic}\mathcal{X}$ induces a surjection $\underline{\mathbb{Z}/12\mathbb{Z}} \twoheadrightarrow \mathrm{R}^1\pi_*\mathbb{G}_m$, so $\operatorname{Pic}\mathcal{X} \to \mathrm{H}^0(\mathbb{P}^1_R, \mathrm{R}^1\pi_*\mathbb{G}_m)$ is surjective[2] and $\mathrm{H}^0(\mathbb{P}^1_R, \mathrm{R}^1\pi_*\mathbb{G}_m) \cong \mathbb{Z}/12\mathbb{Z}$. The exact sequence in Corollary 4.16 now yields

$$0 \longrightarrow \mathrm{H}^2(\mathbb{P}^1_R, \mathbb{G}_m) \xrightarrow{\pi^*} \mathrm{H}^2(\mathcal{X}(1), \mathbb{G}_m) \longrightarrow \mathrm{H}^1(\mathbb{P}^1_R, \mathrm{R}^1\pi_*\mathbb{G}_m) = 0,$$

so $\mathrm{H}^2(\mathcal{X}(1), \mathbb{G}_m) \simeq \mathrm{H}^2(\mathbb{P}^1_R, \mathbb{G}_m)$. △

Example 4.18 ($\operatorname{Br}\mathcal{Y}_0(2)_R$)**.** Let R be a *regular* noetherian strictly henselian local $\mathbb{Z}[1/2]$-algebra, and let $\mathcal{X} = \mathcal{Y}_0(2)/R$ be the moduli space of elliptic curves equipped with a cyclic subgroup of order 2. Let $\pi\colon \mathcal{X} \to \mathbb{A}^1_R \setminus \{0\}$ denote its coarse moduli space (see [ABJ+24, Corollary 3.15]) and note that $\mathcal{X}$ is tame (see [ABJ+24, Lemma 3.14]). It was shown in [ABJ+24, Lemma 7.5] that $\mathcal{X}$ is a $\mathbb{Z}/2\mathbb{Z}$-gerbe over $\mathbb{A}^1 \setminus \{0, -1/4\}$ while points above $-1/4$ have automorphism group μ_4. Using this, one can produce an exact sequence

$$0 \longrightarrow \underline{\mathbb{Z}/2\mathbb{Z}}_{-1/4} \longrightarrow \mathrm{R}^1\pi_*\mathbb{G}_m \longrightarrow \underline{\mathbb{Z}/2\mathbb{Z}} \longrightarrow 0 \tag{4.6}$$

of étale sheaves on $\mathbb{A}^1 \setminus \{0\}$, analogous to (4.5). Again, we postpone giving the details of the construction of this sequence until later (see Section 9.3). From this exact sequence, one immediately sees that

$$\mathrm{H}^1(\mathbb{A}^1 \setminus \{0\}, \mathrm{R}^1\pi_*\mathbb{G}_m) \simeq \mathrm{H}^1(\mathbb{A}^1 \setminus \{0\}, \mathbb{Z}/2\mathbb{Z}) = \mathrm{H}^1(\mathbb{A}^1 \setminus \{0\}, \mu_2) \simeq \mathbb{G}_m(\mathbb{A}^1 \setminus \{0\})/2 \simeq \mathbb{Z}/2\mathbb{Z},$$

and that $\#\,\mathrm{H}^0(\mathbb{A}^1 \setminus \{0\}, \mathrm{R}^1\pi_*\mathbb{G}_m) = 4$. As in Example 4.17, one can further deduce that $\operatorname{Pic}\mathcal{X} \to \mathrm{H}^0(\mathbb{A}^1 \setminus \{0\}, \mathrm{R}^1\pi_*\mathbb{G}_m)$ is surjective with image generated by the Hodge bundle, so $\mathrm{H}^0(\mathbb{A}^1 \setminus \{0\}, \mathrm{R}^1\pi_*\mathbb{G}_m) \cong \mathbb{Z}/4\mathbb{Z}$. Hence, the exact sequence in Corollary 4.16 yields

$$0 \longrightarrow \mathrm{H}^2(\mathbb{A}^1_R \setminus \{0\}, \mathbb{G}_m) \xrightarrow{\pi^*} \mathrm{H}^2(\mathcal{Y}_0(2)_R, \mathbb{G}_m) \longrightarrow \mathbb{Z}/2\mathbb{Z} \xrightarrow{\mathrm{d}_2^{1,1}} \mathrm{H}^3(\mathbb{A}^1_R \setminus \{0\}, \mathbb{G}_m).$$

We claim that the differential $\mathrm{d}_2^{1,1}$ above vanishes, so that we have an exact sequence

$$0 \longrightarrow \mathrm{H}^2(\mathbb{A}^1_R \setminus \{0\}, \mathbb{G}_m) \longrightarrow \mathrm{H}^2(\mathcal{Y}_0(2)_R, \mathbb{G}_m) \longrightarrow \mathbb{Z}/2\mathbb{Z} \longrightarrow 0. \tag{4.7}$$

Because $\operatorname{im}\mathrm{d}_2^{1,1} \subset \mathrm{H}^3(\mathbb{A}^1 \setminus \{0\}, \mathbb{G}_m)[2]$, the Kummer sequence $0 \to \mu_2 \to \mathbb{G}_m \to \mathbb{G}_m \to 0$ implies that it suffices to show that $\mathrm{H}^3(\mathbb{A}^1_R \setminus \{0\}, \mu_2) = 0$. Gabber's absolute cohomological purity theorem [Fuj02, Theorem 2.1.1] applied to the closed immersion

$$Y \coloneqq \operatorname{Spec} R \sqcup \operatorname{Spec} R \xrightarrow{0 \sqcup \infty} \mathbb{P}^1_R$$

implies that

$$\mathrm{H}^r(R, \mathbb{Z}/2\mathbb{Z}) \oplus \mathrm{H}^r(R, \mathbb{Z}/2\mathbb{Z}) = \mathrm{H}^r(Y, \mathbb{Z}/2\mathbb{Z}(-1)) \simeq \mathrm{H}^{r+2}_Y(\mathbb{P}^1_R, \mathbb{Z}/2\mathbb{Z}) = \mathrm{H}^{r+2}_Y(\mathbb{P}^1_R, \mu_2)$$

for all r. In particular, since R is strictly henselian, we see that $\mathrm{H}^n_Y(\mathbb{P}^1_R, \mu_2) = 0$ for all $n \geq 3$. It then follows from the usual long exact sequence of cohomology with supports in Y that $\mathrm{H}^n(\mathbb{P}^1_R, \mu_2) \xrightarrow{\sim} \mathrm{H}^n(\mathbb{A}^1_R \setminus \{0\}, \mu_2)$ for all $n \geq 3$. Now, by the proper base change theorem [Mil80, Corollary VI.2.7], $\mathrm{H}^3(\mathbb{P}^1_k, \mu_2) \simeq \mathrm{H}^3(\mathbb{P}^1_R, \mu_2) \simeq$

[2]Alternatively, at least when R is regular, one can instead directly show that $\pi^*\colon \mathrm{H}^2(\mathbb{P}^1, \mathbb{G}_m) \to \mathrm{H}^2(\mathcal{X}(1), \mathbb{G}_m)$ is injective as follows. There exists an open $U \subset \mathbb{P}^1$ which admits a section $s\colon U \to \mathcal{X}(1)_U$ of π (one can take $U = \mathbb{P}^1 \setminus \{0, 1728, \infty\}$; see e.g. [Sil09, The proof of Proposition III.1.4(c)]). Hence, $\mathrm{H}^2(U, \mathbb{G}_m) \xrightarrow{\pi^*} \mathrm{H}^2(\mathcal{X}(1)_U, \mathbb{G}_m)$ is injective. By Proposition 2.21, $\mathrm{H}^2(\mathbb{P}^1, \mathbb{G}_m) \hookrightarrow \mathrm{H}^2(U, \mathbb{G}_m)$ and $\mathrm{H}^2(\mathcal{X}(1), \mathbb{G}_m) \hookrightarrow \mathrm{H}^2(\mathcal{X}(1)_U, \mathbb{G}_m)$ are both injective as well. It follows that $\pi^*\colon \mathrm{H}^2(\mathbb{P}^1, \mathbb{G}_m) \to \mathrm{H}^2(\mathcal{X}(1), \mathbb{G}_m)$ must be injective.

$\mathrm{H}^3(\mathbb{A}^1_R \setminus \{0\}, \mu_2)$, where k is the (separably closed) residue field of R. Finally, $\mathrm{H}^3(\mathbb{P}^1_k, \mu_2) = 0$, e.g. by [Mil80, Theorem VI.1.1]. △

5. Brauer Groups of Gerbes

In this section, we use Theorem 4.14 to study the Brauer groups of (tame, locally Brauerless) gerbes.

Recall 5.1. Recall from Section 1.1 that by 'gerbe' we always mean an 'fppf gerbe'. Consequently, in this section, we use the flat-fppf site over algebraic stacks (resp. big fppf site over schemes) instead of the lisse-étale site (resp. small étale). We signify this by making use of the subscript $_{\mathrm{fppf}}$, where appropriate. ⊙

Lemma 5.2. *Let R be a strictly local ring, let G/R be a finite linearly reductive group, and let $c\colon \mathcal{H} \to \operatorname{Spec} R$ be a G-gerbe over R. Then, $\mathcal{H}(R) \neq \emptyset$ (so $\mathcal{H} \cong BG_R$).*

Proof. Let $0 \to \Delta \to G \to \underline{Q} \to 0$ be G's connected-étale sequence, and let $\mathcal{M} := \mathcal{H} /\!\!/ \Delta$, the rigidification as in [AOV08, Appendix A]. Then, $\mathcal{M}$ is a Q-gerbe over R and $\mathcal{H}$ is a Δ-gerbe over $\mathcal{M}$. Note that $\mathcal{M}(R) \neq \emptyset$ since Q is étale and R is strictly local; fix some section $s\colon \operatorname{Spec} R \to \mathcal{M}$. Set $\mathcal{H}' := \mathcal{H} \times_{\mathcal{M},s} R$, so it suffices to show that $\mathcal{H}'(R) \neq \emptyset$. This $\mathcal{H}'$ is a Δ-gerbe over R; noting that Δ is commutative (even diagonalizable), it suffices to show that $\mathrm{H}^2_{\mathrm{fppf}}(R, \Delta) = 0$. One argues as in [AOV08, Lemma 3.13] (which assume that R is a separably closed field); briefly, Δ is a product of groups of the form μ_n, for various values of n, and the Kummer sequence $1 \to \mu_n \to \mathbb{G}_m \to \mathbb{G}_m \to 1$ shows that $\mathrm{H}^2_{\mathrm{fppf}}(R, \mu_n) = 0$. ■

Lemma 5.3. *Let S be a scheme, and let G/S be a finite linearly reductive group scheme. Let $c\colon \mathcal{H} \to S$ be a G-gerbe over S. Then,*

$$c_*\mathbb{G}_m \simeq \mathbb{G}_m \quad \text{and} \quad \mathrm{R}^1 c_*\mathbb{G}_m \simeq G^\vee.$$

If, furthermore, G is Brauerless, then also $\mathrm{R}^2 c_*\mathbb{G}_m = 0$. Note here that $G^\vee := \underline{\mathrm{Hom}}(G, \mathbb{G}_m)$ is commutative even if G is not.

Proof. $c_*\mathbb{G}_m \simeq \mathbb{G}_m$ simply because c is a coarse space map, and $\mathrm{R}^2 c_*\mathbb{G}_m = 0$ (if G is Brauerless) by Theorem 4.14. To compute $\mathrm{R}^1 c_*\mathbb{G}_m$, we first construct a map $\mathrm{R}^1 c_*\mathbb{G}_m \to G^\vee$. For this, it suffices to construct functorial maps $\mathrm{H}^1(\mathcal{H}_T, \mathbb{G}_m) \to G^\vee(T)$ for any scheme T/S. To ease notation in describing such maps, we may as well assume $T = S$ and so construct a map $\operatorname{Pic} \mathcal{H} = \mathrm{H}^1(\mathcal{H}, \mathbb{G}_m) \to G^\vee(S)$. Such a map is constructed in [Lop23, Beginning of Section 6] (essentially sending a line bundle $\mathscr{L}$ to the induced action of G on fibers of $\mathscr{L}$). Thus, we get a map $\mathrm{R}^1 c_*\mathbb{G}_m \to G^\vee$, which we claim is an isomorphism on stalks. Indeed, if $S = \operatorname{Spec} R$ is the spectrum of a strictly henselian local ring, then $\mathcal{H} \simeq BG_R$ by Lemma 5.2. Thus, $\mathrm{H}^1(BG_R, \mathbb{G}_m) \xrightarrow{\sim} G^\vee(R)$ by Lemma 3.7. ■

Remark 5.4. In the setting of Lemma 5.3, one also has $\mathrm{R}^1_{\mathrm{fppf}} c_*\mathbb{G}_m \simeq G^\vee$ and $\mathrm{R}^2_{\mathrm{fppf}} c_*\mathbb{G}_m = 0$ (if G is Brauerless) since these are the fppf-sheafifications of the corresponding higher étale pushforwards. ◦

Setup 5.5. Fix some choice of S, G/S, and $c\colon \mathcal{H} \to S$ as in Lemma 5.3. Furthermore, let G^{ab}/S denote the abelianization of G.

Notation 5.6. Let $\mathcal{H}^{\mathrm{ab}} \to S$ denote the abelianization of $\mathcal{H}/S$; this is the rigidification $\mathcal{H} /\!\!/ G^{\mathrm{der}}$, where $G^{\mathrm{der}} = [G, G]$ is the derived subgroup of G, and represents the image of $[\mathcal{H}]$ under the natural map $\mathrm{H}^2(S, \mathcal{H}) \to \mathrm{H}^2(S, \mathcal{H}^{\mathrm{ab}})$; see [Lop23, Section 6].

We study the fppf-Leray spectral sequence

$$E_2^{ij} = \mathrm{H}^i_{\mathrm{fppf}}(S, \mathrm{R}^j_{\mathrm{fppf}} c_*\mathbb{G}_m) \implies \mathrm{H}^{i+j}(\mathcal{H}, \mathbb{G}_m), \tag{5.1}$$

pictured in Fig. 2 (the objects on the E_2-page are identified by using Lemma 5.3). We are in particular interested in computing the differentials $\mathrm{d}_2^{i,1}$ for $i \geq 0$. We claim that these come from cupping with the

$$
\begin{tikzcd}
0 & & & \\
G^{\vee}(S) \arrow[drr, "d_2^{0,1}"] & \mathrm{H}^1_{\mathrm{fppf}}(S, G^{\vee}) \arrow[drr, "d_2^{1,1}"] & & \\
\mathbb{G}_m(S) & \mathrm{Pic}(S) & \mathrm{H}^2(S, \mathbb{G}_m) & \mathrm{H}^3(S, \mathbb{G}_m)
\end{tikzcd}
$$

FIGURE 2. The E_2-page of the Leray spectral sequence for a G-gerbe $\mathcal{H} \to S$.

class $[\mathcal{H}] \in \mathrm{H}^2(S, G)$ of the gerbe $\mathcal{H} \to S$ (even better put, cupping with $[\mathcal{H}^{\mathrm{ab}}] \in \mathrm{H}^2(S, G^{\mathrm{ab}})$). To prove this, we exploit the multiplicative structure relating (5.1) to the Leray spectral sequences

$$F_2^{ij} = \mathrm{H}^i_{\mathrm{fppf}}(S, \mathrm{R}^j_{\mathrm{fppf}} c_* G^{\mathrm{ab}}) \implies \mathrm{H}^{i+j}(\mathcal{H}, G^{\mathrm{ab}}) \text{ and } {}'F_2^{ij} = \mathrm{H}^i_{\mathrm{fppf}}(S, \mathrm{R}^j_{\mathrm{fppf}} c_* G^{\vee}) \implies \mathrm{H}^{i+j}_{\mathrm{fppf}}(\mathcal{H}, G^{\vee})$$

for G^{ab} and $G^{\vee}$.

Notation 5.7. We use $\mathrm{d}_2 = \mathrm{d}_2^{ij}$ to denote the differentials on E_2^{ij}, but use $\delta_2 = \delta_2^{ij}$ and ${}'\delta_2 = {}'\delta_2^{ij}$ to denote the differentials on $F_2^{ij}, {}'F_2^{ij}$, respectively.

The natural evaluation map $G \times G^{\vee} \to \mathbb{G}_m$ (which factors through $G^{\mathrm{ab}} \times G^{\vee} \to \mathbb{G}_m$) induces a product $\smile\colon F_2^{ij} \times {}'F_2^{i'j'} \longrightarrow E_2^{i+i',j+j'}$ compatible with differentials in the sense that they satisfy the Leibniz rule

$$\mathrm{d}_2(\alpha \smile \beta) = \delta_2\alpha \smile \beta + (-1)^{i+j}\alpha \smile {}'\delta_2\beta \tag{5.2}$$

for all $\alpha \in F_2^{ij}$ and $\beta \in {}'F_2^{i'j'}$; see [Bre97, Section IV.6.8] for details on the construction of this product.

Warning 5.8. As explained in [Bre97, Section IV.6.8], the product $\smile\colon F_2^{ij} \times {}'F_2^{i'j'} \longrightarrow E_2^{i+i',j+j'}$ alluded to above is *not* simply the cup product

$$\mathrm{H}^i_{\mathrm{fppf}}(S, \mathrm{R}^j_{\mathrm{fppf}} c_* G^{\mathrm{ab}}) \times \mathrm{H}^{i'}_{\mathrm{fppf}}(S, \mathrm{R}^{j'}_{\mathrm{fppf}} c_* G^{\vee}) \xrightarrow{\cup} \mathrm{H}^{i+i'}_{\mathrm{fppf}}(S, \mathrm{R}^j_{\mathrm{fppf}} c_* G^{\mathrm{ab}} \otimes \mathrm{R}^{j'}_{\mathrm{fppf}} c_* G^{\vee}) \longrightarrow \mathrm{H}^{i+i'}_{\mathrm{fppf}}(S, \mathrm{R}^{j+j'}_{\mathrm{fppf}} c_* \mathbb{G}_m)$$

induced by $G^{\mathrm{ab}} \times G^{\vee} \to \mathbb{G}_m$, but is instead $(-1)^{ji'}$ times the above composition. •

Lemma 5.9. *$c_* G^{\vee} \simeq G^{\vee}$, $c_* G^{\mathrm{ab}} \simeq G^{\mathrm{ab}}$, and $\mathrm{R}^1_{\mathrm{fppf}} c_* G^{\mathrm{ab}} \simeq \underline{\mathrm{Hom}}(G, G^{\mathrm{ab}}) \simeq \underline{\mathrm{End}}(G^{\mathrm{ab}})$.*

Proof. The first two of these holds simply because c is a coarse space map. For the last, we construct a map $\mathrm{R}^1_{\mathrm{fppf}} c_* G^{\mathrm{ab}} \to \underline{\mathrm{Hom}}(G, G^{\mathrm{ab}})$ and verify that it is locally an isomorphism. As in the proof of Lemma 5.3, it suffices to construct functorial maps $\mathrm{H}^1(\mathcal{H}_T, \mathbb{G}_m) \to \underline{\mathrm{Hom}}(G, G^{\mathrm{ab}})(T) = \mathrm{Hom}_T(G_T, G^{\mathrm{ab}}_T)$ for schemes T/S, and even to just construct a suitable map $\mathrm{H}^1(\mathcal{H}, G^{\mathrm{ab}}) \to \mathrm{Hom}_S(G, G^{\mathrm{ab}})$. Fix a G^{ab}-torsor P on $\mathcal{H}$. For any pair $(T/S, t \in \mathcal{H}(T))$, t^*P is a G^{ab}-torsor on T and so one obtains a homomorphism

$$G(T) = \mathrm{Aut}_{\mathcal{H}}(t) \longrightarrow \mathrm{Aut}(t^*P) = G^{\mathrm{ab}}(T).$$

Ranging over such pairs, this defines a homomorphism $G \to G^{\mathrm{ab}}$ (say initially defined over some cover of S, but which then descends to one defined over S). This defines the map $\mathrm{R}^1 c_* G \to \underline{\mathrm{Hom}}(G, G^{\mathrm{ab}})$. To see that this is an isomorphism locally, we may assume that $\mathrm{H}^1_{\mathrm{fppf}}(S, G^{\mathrm{ab}}) = 0$ and that $\mathcal{H} \simeq BG_S$ (since both of these hold fppf-locally on S). Then, a G^{ab}-torsor on $\mathcal{H} = BG_S$ is the data of a (necessarily trivial) G^{ab}-torsor on S equipped with an equivariant G-action, i.e. is the data of a homomorphism $G \to G^{\mathrm{ab}}$. Thus, $\mathrm{H}^1(BG_S, G^{\mathrm{ab}}) \xrightarrow{\sim} \mathrm{Hom}_S(G, G^{\mathrm{ab}})$, concluding the proof. ■

Lemma 5.10. *Viewing* $\mathrm{id} = \mathrm{id}_{G^{\mathrm{ab}}} \in \mathrm{End}_S(G^{\mathrm{ab}}) = F_2^{01}$, $\delta_2(\mathrm{id}) = [\mathcal{H}^{\mathrm{ab}}] \in \mathrm{H}^2(S, G^{\mathrm{ab}}) = F_2^{20}$.

Proof. To simplify notation in this proof, rename $\mathcal{H} := \mathcal{H}^{\mathrm{ab}}$ and $G := G^{\mathrm{ab}}$. By [Gir71, Proposition V.3.2.1], $\delta_2(\mathrm{id})$ is represented by the gerbe D/S whose fiber category $D(T)$ over an S-scheme T/S is the category of G-torsors over $\mathcal{H}_T := \mathcal{H} \times_S T$ whose image under the map

$$\mathrm{H}^1_{\mathrm{fppf}}(\mathcal{H}_T, G) \longrightarrow \mathrm{H}^0(T, \mathrm{R}^1_{\mathrm{fppf}} c_* G) \simeq \mathrm{End}_T(G_T)$$

is the identity endomorphism. To show that $D \simeq \mathcal{H}$, it suffices to construct a morphism $\mathcal{H} \to D$ of G-gerbes over S. Consider an arbitrary T/S and $t \in \mathcal{H}(T)$. Then, we claim that

$$t' : T \xrightarrow{(t,\mathrm{id})} \mathcal{H} \times_S T = \mathcal{H}_T$$

defines an element of $D(T)$. Note that t' trivializes the gerbe $\mathcal{H}_T \simeq BG_T$ and so realizes T as a G-torsor over $\mathcal{H}_T$. Tracing through definitions, one sees that the induced action of $\mathrm{Aut}_{\mathcal{H}_T}(t') = G(T)$ on the (trivial) G-torsor $(t')^* T$ is indeed given by the identity map $G(T) \to G(T)$, so $(t' : T \to \mathcal{H}_T) \in D(T)$ as claimed. This defines our morphism $\mathcal{H} \to D$, which is a morphism of G-gerbes by construction. ■

Proposition 5.11. *Continuing to use notation as in Setup 5.5, there is an exact sequence*

$$0 \to \mathrm{Pic}(S) \xrightarrow{c^*} \mathrm{Pic}(\mathcal{H}) \to \mathrm{H}^0(S, G^\vee) \xrightarrow{[\mathcal{H}^{\mathrm{ab}}]\cup} \mathrm{H}^2(S, \mathbb{G}_m) \xrightarrow{c^*} \mathrm{H}^2(\mathcal{H}, \mathbb{G}_m) \to \mathrm{H}^1_{\mathrm{fppf}}(S, G^\vee) \xrightarrow{-[\mathcal{H}^{\mathrm{ab}}]\cup} \mathrm{H}^3(S, \mathbb{G}_m). \tag{5.3}$$

Above, the last term can be replaced with $\ker\Big(\mathrm{H}^3(S, \mathbb{G}_m) \xrightarrow{c^*} \mathrm{H}^3(\mathcal{H}, \mathbb{G}_m)\Big)$.

Proof. The exact sequence of low degree terms in the Leray spectral sequence (5.1), pictured in Fig. 2, is

$$0 \to \mathrm{Pic}\, S \xrightarrow{c^*} \mathrm{Pic}\,\mathcal{H} \to G^\vee(S) \xrightarrow{\mathrm{d}_2^{0,1}} \mathrm{H}^2(S, \mathbb{G}_m) \xrightarrow{c^*} \mathrm{H}^2(\mathcal{H}, \mathbb{G}_m) \to \mathrm{H}^1_{\mathrm{fppf}}(S, G^\vee) \xrightarrow{\mathrm{d}_2^{1,1}} \ker\Big(\mathrm{H}^3(S, \mathbb{G}_m) \xrightarrow{c^*} \mathrm{H}^3(\mathcal{H}, \mathbb{G}_m)\Big),$$

so it suffices to compute the differentials $\mathrm{d}_2^{0,1}, \mathrm{d}_2^{1,1}$. Fix any $i \geq 0$. The multiplicative structure (5.2) shows that

$$\mathrm{d}_2^{i,1}(\alpha \smile \beta) = \delta_2 \alpha \smile \beta + (-1)^{i+1} \alpha \smile {}'\delta_2 \beta = \delta_2 \alpha \smile \beta$$

for any $\alpha \in F_2^{01} = \mathrm{H}^0(S, \mathrm{R}^1_{\mathrm{fppf}} c_* G^{\mathrm{ab}}) \simeq \mathrm{End}_S(G^{\mathrm{ab}})$ (5.9) and $\beta \in {}'F_2^{i0} = \mathrm{H}^i_{\mathrm{fppf}}(S, G^\vee)$. In particular, taking $\alpha = \mathrm{id}_{G^{\mathrm{ab}}} \in \mathrm{End}_S(G^{\mathrm{ab}})$ (and $\beta \in \mathrm{H}^i_{\mathrm{fppf}}(S, G^\vee) \simeq E_2^{i1}$ arbitrary), and letting $\cup$ denote the usual cup product (see Warning 5.8), we see that

$$\mathrm{d}_2(\beta) = \mathrm{d}_2(\mathrm{id}_G \cup \beta) = (-1)^i \mathrm{d}_2(\mathrm{id}_G \smile \beta) = (-1)^i \delta_2(\mathrm{id}_G) \smile \beta = (-1)^i \delta_2(\mathrm{id}_G) \cup \beta,$$

so it suffices to compute $\delta_2(\mathrm{id}_{G^{\mathrm{ab}}}) \in \mathrm{H}^2(S, G) = F_2^{20}$. We conclude by Lemma 5.10. ■

Remark 5.12. Using notation as in Proposition 3.9, by definition of $G^\vee$ and of cup products, one has a commutative diagram

$$\begin{tikzcd} \mathrm{H}^0(S, G^\vee) \arrow[r, "{[\mathcal{H}^{\mathrm{ab}}]\cup}"] \arrow[d, equal] & \mathrm{H}^2(S, \mathbb{G}_m) \arrow[d, equal] \\ \mathrm{Hom}_S(G, \mathbb{G}_m) \arrow[r, "{(-)_*[\mathcal{H}]}"] & \mathrm{H}^2(S, \mathbb{G}_m). \end{tikzcd}$$

Hence, Proposition 5.11 extends an exact sequence constructed by Lopez in [Lop23, Theorem 1.2]. ○

Remark 5.13. Say G/S is as in Proposition 5.11 and $\mathcal{H} = BG_S$. Then, the section $S \to BG_S$ shows that the differentials in the spectral sequence Fig. 2 vanish and that one has

$$\mathrm{Pic}(BG_S) \cong \mathrm{Pic}\, S \oplus \mathrm{H}^0(S, G^\vee) \quad \text{and} \quad \mathrm{H}^2(BG_S, \mathbb{G}_m) \cong \mathrm{H}^2(S, \mathbb{G}_m) \oplus \mathrm{H}^1(S, G^\vee),$$

generalizing Proposition 3.9. ○

Corollary 5.14 (of Proposition 5.11)**.** *Assume that S is regular and noetherian. If there exists a dense open $U \subset S$ over which $\mathcal{H}$ trivializes (i.e. there exists a section $U \to \mathcal{H}$), then the exact sequence of Proposition 5.11 splits into two exact sequences:*

$$0 \longrightarrow \mathrm{Pic}(S) \xrightarrow{c^*} \mathrm{Pic}(\mathcal{H}) \longrightarrow \mathrm{H}^0(S, G^\vee) \longrightarrow 0$$

$$0 \longrightarrow \mathrm{H}^2(S, \mathbb{G}_m) \xrightarrow{c^*} \mathrm{H}^2(\mathcal{H}, \mathbb{G}_m) \longrightarrow \mathrm{H}^1(S, G^\vee) \xrightarrow{-[\mathcal{H}^{\mathrm{ab}}]\cup} \ker\Big(\mathrm{H}^3(S, \mathbb{G}_m) \xrightarrow{c^*} \mathrm{H}^3(\mathcal{H}, \mathbb{G}_m)\Big)$$

Proof. It suffices to show that $\mathrm{H}^0(S, G^\vee) \xrightarrow{[\mathcal{H}^{\mathrm{ab}}]\cup} \mathrm{H}^2(S, \mathbb{G}_m)$ is the zero map. Note that we have a commutative square

$$\begin{array}{ccc} \mathrm{H}^0(S, G^\vee) & \xrightarrow{[\mathcal{H}^{\mathrm{ab}}]\cup} & \mathrm{H}^2(S, \mathbb{G}_m) \\ \downarrow & & \downarrow \\ \mathrm{H}^0(U, G^\vee) & \xrightarrow{[\mathcal{H}^{\mathrm{ab}}_U]\cup} & \mathrm{H}^2(U, \mathbb{G}_m). \end{array}$$

The claim now follows from the facts that $[\mathcal{H}_U] = 0$ (so $[\mathcal{H}^{\mathrm{ab}}_U] = 0$), by assumption, and that $\mathrm{H}^2(S, \mathbb{G}_m) \to \mathrm{H}^2(U, \mathbb{G}_m)$ is injective, by Proposition 2.21. ■

Lemma 5.15. *Assume that $G = \mu_{n,S}$ for some $n \geq 1$. Then, $c^* \colon \mathrm{H}^2(S, \mathbb{G}_m) \to \mathrm{H}^2(\mathcal{H}, \mathbb{G}_m)$ is injective if and only if $[\mathcal{H}] \in \mathrm{im}\big(\mathrm{H}^1(S, \mathbb{G}_m) \to \mathrm{H}^2_{\mathrm{fppf}}(S, \mu_n)\big)$.*

Proof. By Proposition 5.11 (and Remark 5.12), injectivity of $\mathrm{H}^2(S, \mathbb{G}_m) \to \mathrm{H}^2(\mathcal{H}, \mathbb{G}_m)$ is equivalent to

$$\mathrm{Hom}(\mu_n, \mathbb{G}_m) \xrightarrow{(-)_*[\mathcal{H}]} \mathrm{H}^2(S, \mathbb{G}_m)$$

being the zero map. Note that $\mathrm{Hom}(\mu_n, \mathbb{G}_m) \cong \mathbb{Z}/n\mathbb{Z}$, generated by the natural inclusion $\mu_n \hookrightarrow \mathbb{G}_m$ so the displayed map is the zero map if and only if $[\mathcal{H}] \mapsto 0$ under $\mathrm{H}^2(S, \mu_n) \to \mathrm{H}^2(S, \mathbb{G}_m)$. The Kummer sequence $1 \to \mu_n \to \mathbb{G}_m \to \mathbb{G}_m \to 1$ shows that this is the case if and only if $[\mathcal{H}] \in \mathrm{im}\big(\mathrm{H}^1(S, \mathbb{G}_m) \to \mathrm{H}^2_{\mathrm{fppf}}(S, \mu_n)\big)$. ■

6. Brauer Groups of Root Stacks

In this section, we compute the Brauer groups of root stacks. To begin, we recall their construction, following [Ols16, Section 10.3]. One can also see [Cad07] for more information on root stacks.

Definition 6.1. Let $\mathcal{X}$ be an algebraic stack. A `generalized effective Cartier divisor` on $\mathcal{X}$ is a pair $(\mathscr{L}, \rho)$ where $\mathscr{L}$ is a line bundle on $\mathcal{X}$ and $\rho \colon \mathscr{L} \to \mathscr{O}_{\mathcal{X}}$ is a morphism of $\mathscr{O}_{\mathcal{X}}$-modules. An isomorphism between generalized effective Cartier divisors $(\mathscr{L}', \rho')$ and $(\mathscr{L}, \rho)$ is an isomorphism $\mathscr{L}' \xrightarrow{\sim} \mathscr{L}$ of line bundles which identifies ρ' with ρ. ◇

Example 6.2. Let $\mathcal{D} \hookrightarrow \mathcal{X}$ be an effective Cartier divisor, with ideal sheaf $\mathscr{O}_{\mathcal{X}}(-\mathcal{D})$. Then, the inclusion $\mathscr{O}_{\mathcal{X}}(-\mathcal{D}) \hookrightarrow \mathscr{O}_{\mathcal{X}}$ is a generalized effective Cartier divisor. △

Proposition 6.3. *Let $\mathbb{G}_m \curvearrowright \mathbb{A}^1$ with the usual scaling action $\lambda \cdot t = \lambda t$. Then, the quotient stack $[\mathbb{A}^1/\mathbb{G}_m]$ parameterizes generalized effective Cartier divisors; i.e., given a scheme S, the category $[\mathbb{A}^1/\mathbb{G}_m](S)$ is equivalent to the groupoid of generalized effective Cartier divisors on S.*

Proof. This is [Ols16, Proposition 10.3.7]. ■

Definition 6.4. Let $\mathcal{X}$ be an algebraic stack, and let $(\mathscr{L}, \rho)$ be a generalized effective Cartier divisor on $\mathcal{X}$. For an integer $n \geq 1$, the n`th root stack` $\sqrt[n]{(\mathscr{L}, \rho)/\mathcal{X}}$ is the fibered category over Sch whose objects are tuples $(f \colon S \to \mathcal{X}, (\mathscr{M}, \lambda), \sigma)$ where f is a morphism from a scheme S to $\mathcal{X}$, $(\mathscr{M}, \lambda)$ is a generalized effective Cartier divisor on S, and $\sigma \colon (\mathscr{M}^{\otimes n}, \lambda^n) \xrightarrow{\sim} (f^*\mathscr{L}, f^*\rho)$ is an isomorphism of generalized effective Cartier divisors on $\mathcal{X}$. Morphisms in $\sqrt[n]{(\mathscr{L}, \rho)/\mathcal{X}}$ are as expected; see [Ols16, 10.3.9]. ◇

Notation 6.5. If $\mathcal{D} \hookrightarrow \mathcal{X}$ is an effective Cartier divisor with ideal sheaf $\mathscr{O}_{\mathcal{X}}(-\mathcal{D})$ and if $n \geq 1$ is an integer, we'll write $\sqrt[n]{\mathcal{D}/\mathcal{X}} \coloneqq \sqrt[n]{\left(\mathscr{O}_{\mathcal{X}}(-\mathcal{D}), \mathscr{O}_{\mathcal{X}}(-\mathcal{D}) \hookrightarrow \mathscr{O}_{\mathcal{X}}\right)/\mathcal{X}}$.

Proposition 6.6. *Let $\mathcal{X}$ be an algebraic stack, let $(\mathscr{L}, \rho)$ be a generalized effective Cartier divisor on $\mathcal{X}$, fix some $n \geq 1$, and set $\mathcal{X}_n \coloneqq \sqrt[n]{(\mathscr{L}, \rho)/\mathcal{X}}$. Then,* *(1)* *$\mathcal{X}_n$ is an algebraic stack, and is in fact the fiber product*

$$\begin{tikzcd} \mathcal{X}_n \arrow[r] \arrow[d] & {[\mathbb{A}^1/\mathbb{G}_m]} \arrow[d, "(-)^n"] \\ \mathcal{X} \arrow[r, "(\mathscr{L},\rho)"] & {[\mathbb{A}^1/\mathbb{G}_m]}. \end{tikzcd}$$

(2) *If $\mathscr{L} = \mathscr{O}_{\mathcal{X}}$ with ρ corresponding to $f \in \Gamma(\mathcal{X}, \mathscr{O}_{\mathcal{X}})$, then*

$$\mathcal{X}_n \simeq \left[\frac{\mathbf{Spec}_{\mathcal{X}}(\mathscr{O}_{\mathcal{X}}[T]/(T^n - f))}{\mu_n}\right],$$

where μ_n acts trivially on $\mathscr{O}_{\mathcal{X}}$ and acts on T via $\zeta \cdot T = \zeta T$.

(3) *The natural morphism $\pi \colon \mathcal{X}_n \to \mathcal{X}$ is an isomorphism over the open locus $\mathcal{U} \subset \mathcal{X}$ where ρ is an isomorphism.*

(4) *If n is invertible on $\mathcal{X}$, then $\pi \colon \mathcal{X}_n \to \mathcal{X}$ is a DM morphism in the sense of* [Sta21, Tag 04YW].

Proof. See [Ols16, Theorem 10.3.10] for the case the $\mathcal{X}$ is a scheme, to which the general case reduces. ■

Setup 6.7. For the reminder of the section, we fix the following notation.

(1) $\mathcal{X}$ is a tame algebraic stack over some scheme S. We write $c \colon \mathcal{X} \to X$ for its coarse space map.
We further assume that $\mathcal{X}$ is locally Bruaerless, and that there exists an open locus $U \subset X$, flat over S, above which c is an isomorphism.

(2) Fix an effective Cartier divisor $\iota \colon \mathcal{D} \hookrightarrow \mathcal{X}$ which is flat over S.
We further assume that $\mathcal{D}$ lives in $c^{-1}(U)$, so c maps $\mathcal{D}$ isomorphically onto its image in X.

(3) We fix an integer $n \geq 1$, and write $\mathcal{X}_n \coloneqq \sqrt[n]{\mathcal{D}/\mathcal{X}}$. We let $\pi \colon \mathcal{X}_n \to \mathcal{X}$ denote the natural morphism.

Lemma 6.8. $\mathrm{R}^2\pi_*\mathbb{G}_m = 0$.

Proof. By Proposition 6.6**(3)**, π is an isomorphism away from $\mathcal{D}$, so $\mathrm{R}^2\pi_*\mathbb{G}_m$ is supported along $\mathcal{D}$, i.e. $\mathrm{R}^2\pi_*\mathbb{G}_m = \iota_*\mathrm{R}^2\left(\pi|_{\pi^{-1}(\mathcal{D})}\right)_*\mathbb{G}_m$. At the same time, by Setup 6.7, above $\mathcal{D}$, π is equivalent to the composition $\pi^{-1}(\mathcal{D}) \xrightarrow{\pi} \mathcal{D} \xrightarrow{c} X$. By construction, this composition is the coarse space map of $\pi^{-1}(\mathcal{D})$ and all geometric automorphism groups of $\pi^{-1}(\mathcal{D})$ are of the form μ_n for some n (consequence of Proposition 6.6**(2)**) and so are all Brauerless (Example 4.9). Thus, we conclude by Theorem 4.14. ■

Lemma 6.9. $\mathrm{R}^1\pi_*\mathbb{G}_m \simeq \iota_*\underline{\mathbb{Z}/n\mathbb{Z}}$.

Proof. As in the proof of Lemma 6.8, as a consequence of Proposition 6.6**(3)**, we need only compute the higher pushforward along the composition

$$\pi' \colon \mathcal{D}' \coloneqq \pi^{-1}(\mathcal{D}) \xrightarrow{\pi} \mathcal{D} \xrightarrow[c]{\sim} c(\mathcal{D}) \eqqcolon D.$$

Since π is an nth root stack along $\mathcal{D}$, it follows that $\mathcal{D}'$ is an infinitesimal extension of a stack $\mathcal{H}$ which is a μ_n-gerbe over D, say via $\rho \colon \mathcal{H} \to D$ (see e.g. [Cad07, Below Example 2.4.3]). We claim that the natural map $\varphi \colon \mathrm{R}^1\pi'_*\mathbb{G}_m \to \mathrm{R}^1\rho_*\mathbb{G}_m$ is an isomorphism. This can be checked at the level of stalks, where it follows from Lemma 4.2. Finally, the claim follows from Lemma 5.3 which shows that $\mathrm{R}^1\rho_*\mathbb{G}_m \simeq \underline{\mathbb{Z}/n\mathbb{Z}}$. ■

Remark 6.10. When $\mathcal{X}$ is a curve, an alternative proof of Lemma 6.9 is given by Corollary 8.25. ○

Proposition 6.11. *There are exact sequences*

$$0 \longrightarrow \mathrm{Pic}(\mathcal{X}) \xrightarrow{\pi^*} \mathrm{Pic}(\mathcal{X}_n) \longrightarrow \mathrm{H}^0(\mathcal{D}, \underline{\mathbb{Z}/n\mathbb{Z}}) \longrightarrow 0 \tag{6.1}$$

$$0 \longrightarrow \mathrm{H}^2(\mathcal{X}, \mathbb{G}_m) \xrightarrow{\pi^*} \mathrm{H}^2(\mathcal{X}_n, \mathbb{G}_m) \longrightarrow \mathrm{H}^1(\mathcal{D}, \underline{\mathbb{Z}/n\mathbb{Z}}) \xrightarrow{\delta} \ker\left(\mathrm{H}^3(\mathcal{X}, \mathbb{G}_m) \xrightarrow{\pi^*} \mathrm{H}^3(\mathcal{X}_n, \mathbb{G}_m)\right) \tag{6.2}$$

Proof. Note that $\pi_*\mathbb{G}_m = \mathbb{G}_m$, either by a local computation using Proposition 6.6**(2)** or by realizing that $\mathcal{X}_n$ and $\mathcal{X}$ share X as a coarse space. Consider the Leray spectral sequence $E_2^{ij} = \mathrm{H}^i(\mathcal{X}, \mathrm{R}^j\pi_*\mathbb{G}_m) \implies$

$\mathrm{H}^{i+j}(\mathcal{X}_n, \mathbb{G}_m)$; by making use of Lemmas 6.8 and 6.9, where appropriate, we see that its low degree terms fit in to the exact sequence

$$0 \to \mathrm{Pic}(\mathcal{X}) \xrightarrow{\pi^*} \mathrm{Pic}(\mathcal{X}_n) \to \mathrm{H}^0(\mathcal{D}, \mathbb{Z}/n\mathbb{Z}) \xrightarrow{d_2^{0,1}} \mathrm{H}^2(\mathcal{X}, \mathbb{G}_m) \xrightarrow{\pi^*} \mathrm{H}^2(\mathcal{X}_n, \mathbb{G}_m) \to \mathrm{H}^1(\mathcal{D}, \underline{\mathbb{Z}/n\mathbb{Z}}) \to \mathrm{H}^3(\mathcal{X}, \mathbb{G}_m).$$

Above, the map $\pi^* \colon \mathrm{H}^2(\mathcal{X}, \mathbb{G}_m) \to \mathrm{H}^2(\mathcal{X}_n, \mathbb{G}_m)$ fits into a commutative square

$$\begin{tikzcd} \mathrm{H}^2(\mathcal{X}, \mathbb{G}_m) \arrow[r, "\pi^*"] \arrow[d] & \mathrm{H}^2(\mathcal{X}_n, \mathbb{G}_m) \arrow[d] \\ \mathrm{H}^2(\mathcal{X} \setminus \mathcal{D}, \mathbb{G}_m) \arrow[r, "\sim"] & \mathrm{H}^2(\mathcal{X}_n \setminus \pi^{-1}(\mathcal{D}), \mathbb{G}_m). \end{tikzcd}$$

By Proposition 2.21, this forces π^* above to be injective, from which the claim follows. ∎

Remark 6.12. The Picard groups of root stacks have previously been computed in both [Cad07, Section 3.1] and [Lop23, Section 3]. ◦

Example 6.13 ($\sqrt[n]{\infty/\mathbb{P}^1}$)**.** We construct an example showing that the morphism δ in (6.2) can be nonzero. Fix a field k and an integer $n > 1$. Let $\mathcal{X} := \sqrt[n]{\infty/\mathbb{P}^1_k} \xrightarrow{c} \mathbb{P}^1_k$. Then, Proposition 6.11 yields an exact sequence

$$0 \longrightarrow \mathrm{H}^2(\mathbb{P}^1_k, \mathbb{G}_m) \xrightarrow{c^*} \mathrm{H}^2(\mathcal{X}, \mathbb{G}_m) \longrightarrow \mathrm{H}^1(k, \mathbb{Z}/n\mathbb{Z}) \xrightarrow{\delta} \mathrm{H}^3(\mathbb{P}^1_k, \mathbb{G}_m).$$

We will later show (see Proposition 8.33) that $c^* \colon \mathrm{H}^2(\mathbb{P}^1_k, \mathbb{G}_m) \to \mathrm{H}^2(\mathcal{X}, \mathbb{G}_m)$ is an isomorphism. Thus, in this example, $\delta \colon \mathrm{H}^1(k, \mathbb{Z}/n\mathbb{Z}) \hookrightarrow \mathrm{H}^3(\mathbb{P}^1_k, \mathbb{G}_m)$ must be injective. △

7. Brauer groups of stacky curves

In the present section, we combine the work from the previous two in order to study Brauer groups of stacky curves, especially over algebraically closed fields. Since stacky curves often arise as gerbes over root stacks, we begin with a general result above such spaces.

Lemma 7.1. *Let S be a base scheme. Let $\mathcal{Y}/S$ be an algebraic S-stack with coarse moduli space $\rho \colon \mathcal{Y} \to X$. Let G/S be a finite linearly reductive group, and let $\pi \colon \mathcal{X} \to \mathcal{Y}$ be a G-gerbe, so we have a commutative diagram*

$$\begin{tikzcd} \mathcal{X} \arrow[r, "\pi"'] \arrow[rr, bend left, "c"] & \mathcal{Y} \arrow[r, "\rho"'] & X, \end{tikzcd}$$

where c is $\mathcal{X}$'s coarse space map.

Lemma 7.2. *There is an exact sequence*

$$0 \longrightarrow \mathrm{R}^1\rho_*\mathbb{G}_m \longrightarrow \mathrm{R}^1c_*\mathbb{G}_m \longrightarrow G^\vee \longrightarrow \mathrm{R}^2\rho_*\mathbb{G}_m,$$

of étale sheaves on X.

Proof. We use the Grothendieck spectral sequence $E_2^{ij} = \mathrm{R}^i\rho_*\left(\mathrm{R}^j\pi_*\mathbb{G}_m\right) \implies \mathrm{R}^{i+j}c_*\mathbb{G}_m$. Using Lemma 5.3 to compute $\mathrm{R}^1\pi_*\mathbb{G}_m \simeq G^\vee$, we see that its exact sequence of low degree terms begins with

$$0 \longrightarrow \mathrm{R}^1\rho_*\mathbb{G}_m \longrightarrow \mathrm{R}^1c_*\mathbb{G}_m \longrightarrow \rho_*G^\vee \longrightarrow \mathrm{R}^2\rho_*\mathbb{G}_m.$$

Finally, $\rho_*G^\vee = G^\vee$ (as étale sheaves on X) since $G^\vee$ is a scheme and ρ is a coarse space map. ∎

We now restrict to the curve setting.

Setup 7.3. Let k be a field. Let $\mathcal{Y}/k$ be a regular stacky curve which is generically schemey, and let $\rho \colon \mathcal{Y} \to X$ be its coarse space map. Let G/k be a finite linearly reductive group, and let $\pi \colon \mathcal{X} \to \mathcal{Y}$ be a G-gerbe. Consider the commutative diagram

$$\begin{tikzcd} \mathcal{X} \arrow[r, "\pi"'] \arrow[rr, bend left, "c"] & \mathcal{Y} \arrow[r, "\rho"'] & X, \end{tikzcd}$$

where c is $\mathcal{X}$'s coarse space map.

Example 7.4. One could take $\mathcal{Y}$ above to be a multiply rooted stack over X, i.e. one could fix distinct closed points $x_1, \ldots, x_r \in X$ as well as integers $e_1, \ldots, e_r > 1$ and then set

$$\mathcal{Y} = \sqrt[e_1]{x_1/X} \times_X \ldots \times_X \sqrt[e_r]{x_r/X} \xrightarrow{\rho} X. \tag{7.1}$$

Assuming that X is a regular curve, this is always a tame regular generically schemey stacky curve. △

Remark 7.5. It is well known that every tame DM stacky curve over a field can be written as a gerbe over a multiply rooted stack; see [Lop23, Proposition 1.5] and/or [GS16]. ◦

Example 7.6. Over a field k of characteristic not 2 or 3, one can take $\mathcal{X} = \mathcal{Y}(1)$ with $G = \underline{\mathbb{Z}/2\mathbb{Z}}$ and $\mathcal{Y} \simeq \sqrt[3]{0/\mathbb{A}^1} \times_{\mathbb{A}^1} \sqrt[2]{1728/\mathbb{A}^1}$. △

Example 7.7. Over a field k of characteristic not 2, one can take $\mathcal{X} = \mathcal{Y}_0(2)$ with $G = \underline{\mathbb{Z}/2\mathbb{Z}}$ and $\mathcal{Y} \simeq \sqrt[2]{-\frac{1}{4}/\left(\mathbb{A}^1 \setminus \{0\}\right)}$. This follows from [ABJ+24, Section 7.1]. △

Lemma 7.8. *Assume that k is algebraically closed. Then, there is an exact sequence*

$$0 \to \mathrm{H}^0(X, \mathrm{R}^1\rho_*\mathbb{G}_m) \to \mathrm{H}^0(X, \mathrm{R}^1c_*\mathbb{G}_m) \to \mathrm{H}^0(X, G^\vee) \to \mathrm{H}^0(X, \mathrm{R}^2\rho_*\mathbb{G}_m) \to \mathrm{H}^1(X, \mathrm{R}^1c_*\mathbb{G}_m) \to \mathrm{H}^1(X, G^\vee) \to 0.$$

Furthermore, $\mathrm{H}^i(X, \mathrm{R}^1c_*\mathbb{G}_m) \simeq \mathrm{H}^i(X, G^\vee)$ for all $i \geq 2$.

Proof. This follows from the exact sequence

$$0 \longrightarrow \mathrm{R}^1\rho_*\mathbb{G}_m \longrightarrow \mathrm{R}^1c_*\mathbb{G}_m \longrightarrow G^\vee \longrightarrow \mathrm{R}^2\rho_*\mathbb{G}_m$$

of Lemma 7.2 along with the fact that $\mathrm{R}^i\rho_*\mathbb{G}_m$, for $i = 1, 2$, is supported on a finite subscheme of X and hence they (and all their subsheaves) are acyclic. ■

Remark 7.9. If $\mathcal{Y}$ is as in Example 7.4, then $\mathrm{R}^1\rho_*\mathbb{G}_m$ in Lemma 7.8 is $\mathrm{R}^1\rho_*\mathbb{G}_m \simeq \bigoplus_{i=1}^r \underline{\mathbb{Z}/e_i\mathbb{Z}}_{x_i}$; this follows from repeated use of Lemma 6.9 (or one use of Corollary 8.25 per connected component of X). Furthermore, in this case, $\mathrm{R}^2\rho_*\mathbb{G}_m = 0$ by Theorem 4.14, so one has the exact sequence

$$0 \longrightarrow \bigoplus_{i=1}^r \underline{\frac{\mathbb{Z}}{e_i\mathbb{Z}}}_{x_i} \longrightarrow \mathrm{R}^1c_*\mathbb{G}_m \longrightarrow G^\vee \longrightarrow 0.$$

Compare this with the exact sequence of [AMS24, Proposition 6.9] obtained for $\mathcal{X} = \mathcal{Y}(1)$. ◦

Assumption. Assume from now on that $\mathcal{Y}$ (and so also $\mathcal{X}$) is tame.

Proposition 7.10. *Assume that k is algebraically closed and that $\mathcal{X}$ is locally Brauerless. Then, there is a surjection*

$$\operatorname{Br}\mathcal{X} = \mathrm{H}^2(\mathcal{X}, \mathbb{G}_m) \twoheadrightarrow \mathrm{H}^1(X, G^\vee)$$

which is an isomorphism if $\mathcal{Y}$ is locally Brauerless as well.

Proof. We use the Leray spectral sequence $E_2^{ij} = \mathrm{H}^i(X, \mathrm{R}^jc_*\mathbb{G}_m) \implies \mathrm{H}^{i+j}(\mathcal{X}, \mathbb{G}_m)$. Note that $E_2^{i0} = \mathrm{H}^i(X, \mathbb{G}_m) = 0$ for $i \geq 2$, by [Fu15, Theorem 7.2.7], and $E_2^{02} = 0$ by Theorem 4.14. Thus, one concludes that $\mathrm{H}^2(\mathcal{X}, \mathbb{G}_m) \xrightarrow{\sim} \mathrm{H}^1(X, \mathrm{R}^1c_*\mathbb{G}_m)$. Now, Lemma 7.8 provides the surjection $\mathrm{H}^2(\mathcal{X}, \mathbb{G}_m) \xrightarrow{\sim} \mathrm{H}^1(X, \mathrm{R}^1c_*\mathbb{G}_m) \twoheadrightarrow \mathrm{H}^1(X, G^\vee)$ and shows that it is an isomorphism if $\mathrm{H}^0(X, \mathrm{R}^2\rho_*\mathbb{G}_m) = 0$ (e.g. if $\mathcal{Y}$ is locally Brauerless). Finally, Lemma 2.20 shows that $\operatorname{Br}\mathcal{X} = \mathrm{H}^2(\mathcal{X}, \mathbb{G}_m)$. ■

Corollary 7.11. *Assume that k is a perfect field. Then, there is an exact sequence*

$$\begin{tikzcd} 0 \arrow[r] & \mathrm{H}^1(k, \mathbb{G}_m(\overline{X})) \arrow[r] & \operatorname{Pic}\mathcal{X} \arrow[r] & \left(\operatorname{Pic}\overline{\mathcal{X}}\right)^{G_k} \arrow[dlll, rounded corners] \\ \mathrm{H}^2(k, \mathbb{G}_m(\overline{X})) \arrow[r] & \ker\left(\mathrm{H}^2(\mathcal{X}, \mathbb{G}_m) \to \mathrm{H}^1(\overline{X}, G^\vee)\right) \arrow[r] & \mathrm{H}^1(k, \operatorname{Pic}\overline{\mathcal{X}}) \arrow[r] & \mathrm{H}^3(k, \mathbb{G}_m(\overline{X})). \end{tikzcd}$$

where $\overline{\mathcal{X}} := \mathcal{X}_{\overline{k}}$, $\overline{X} := X_{\overline{k}}$, *and* $G_k = \mathrm{Gal}(\overline{k}/k)$.

Proof. This is the exact sequence of low degree terms in the spectral sequence $E_2^{ij} = \mathrm{H}^i(k, \mathrm{H}^j(\overline{\mathcal{X}}, \mathbb{G}_m)) \implies \mathrm{H}^{i+j}(\mathcal{X}, \mathbb{G}_m)$. ■

Corollary 7.12. *Assume that k is a number field and X is proper, geometrically reduced, and geometrically connected. Use notation as in Corollary 7.11. Then, there is an exact sequence*

$$0 \longrightarrow \mathrm{Pic}\,\mathcal{X} \longrightarrow (\mathrm{Pic}\,\overline{\mathcal{X}})^{G_k} \longrightarrow \mathrm{Br}\,k \longrightarrow \ker\big(\mathrm{H}^2(\mathcal{X}, \mathbb{G}_m) \to \mathrm{H}^1(\overline{X}, G^\vee)\big) \longrightarrow \mathrm{H}^1(k, \mathrm{Pic}\,\overline{\mathcal{X}}) \longrightarrow 0.$$

Proof. For such X, $\mathbb{G}_m(\overline{X}) = \mathbb{G}_m(\overline{k})$. Furthermore, it is known that $\mathrm{H}^3(k, \mathbb{G}_m) = 0$ if k is a number field; see [Poo17, Remark 6.7.10]. Apply Corollary 7.11. ■

Remark 7.13. In [AM20, ABJ+24], it was shown that, for k of characteristic not 2,

$$\mathrm{Br}\,\mathcal{Y}(1)_{\overline{k}} = 0 \ \text{ while } \ \mathrm{Br}\,\mathcal{Y}_0(2)_{\overline{k}} = \mathbb{Z}/2\mathbb{Z}.$$

Proposition 7.10 helps explain why one of these groups vanishes while the other does not; note that[3]

$$\mathrm{H}^1(\mathbb{A}^1_{\overline{k}}, \mu_2) = 0 \ \text{ while } \ \mathrm{H}^1(\mathbb{A}^1_{\overline{k}} \setminus \{0\}, \mu_2) = \mathbb{Z}/2\mathbb{Z}. \qquad \circ$$

Remark 7.14. To justify the tameness assumption in Proposition 7.10, we remark that, in characteristic 2, $\mathrm{Br}\,\mathcal{Y}(1)_{\overline{\mathbb{F}}_2} = \mathbb{Z}/2\mathbb{Z}$, as was shown by Shin [Shi19], but $\mathrm{H}^1_{\mathrm{fppf}}(\mathbb{A}^1_{\overline{\mathbb{F}}_2}, \mu_2) = 0$. Note that $\mathcal{Y}(1)_{\overline{\mathbb{F}}_2}$ is nowhere tame. ∘

Remark 7.15. Furthermore, even for tame $\mathcal{X}$ over $k = \overline{k}$, some additional condition is necessary for one to have $\mathrm{H}^2(\mathcal{X}, \mathbb{G}_m) \simeq \mathrm{H}^1(X, G^\vee)$. For example, if $\mathcal{Y} := \sqrt[2]{0/\mathbb{A}^1_k}$, where k is an algebraically closed field of characteristic not 2, and $\mathcal{X} := B\mu_{2,\mathcal{Y}}$, then

$$\mathrm{H}^2(\mathcal{X}, \mathbb{G}_m) \simeq \mathrm{H}^1(\mathcal{Y}, \mu_2) \simeq \mathbb{Z}/2\mathbb{Z} \not\simeq 0 \simeq \mathrm{H}^1(\mathbb{A}^1_k, \mu_2).$$

Notice that, in this example, $\underline{\mathrm{Aut}}(\mathcal{X}, 0) \simeq \mu_{2,k} \times \mu_{2,k}$ is *not* Brauerless. ∘

8. Residue Exact Sequence

One highly useful tool for understanding Brauer groups of schemes is the residue exact sequence; see, for example, [Poo17, Theorem 6.8.3] or [Gro68, Proposition 2.1]. Our goal in the current section is to explore the extension of this sequence to the stacky setting. One such extension has been obtained previously in [HKT16, Proposition 2], at least for 2-dimensional DM stacks. In this section, we first use a stacky version of Artin's relative cohomological purity theorem to construct residue maps for stacks which are smooth over a scheme. We then explore the case of 1-dimensional stacks, where residue maps can more easily be constructed using the usual divisor exact sequence as in [Mil80, Example III.2.22] or [Gro66b, (2) in Section 1].

Proposition 8.1. *Let S be an affine regular, noetherian scheme of dimension ≤ 1, and fix a prime ℓ which is invertible on S. Let $i\colon \mathcal{D} \hookrightarrow \mathcal{X}$ be a closed immersion of smooth algebraic S-stacks which is pure of codimension 1. Let $j\colon \mathcal{U} \hookrightarrow \mathcal{X}$ be the open complement of $\mathcal{D}$. Then, there is an exact sequence*

$$0 \longrightarrow \mathrm{Br}'(\mathcal{X})\{\ell\} \longrightarrow \mathrm{Br}'(\mathcal{U})\{\ell\} \longrightarrow \mathrm{H}^1(\mathcal{D}, \mathbb{Q}_\ell/\mathbb{Z}_\ell) \longrightarrow \mathrm{H}^3(\mathcal{X}, \mathbb{G}_m)\{\ell\} \longrightarrow \mathrm{H}^3(\mathcal{U}, \mathbb{G}_m)\{\ell\}.$$

Proof. One can argue as in [AM20, Proof of Proposition 2.14], using Laszlo and Olsson's stacky version of Artin's relative cohomological purity theorem [LO08, Proposition 4.9.1] (which was proven only for smooth S-stacks with S as in the proposition statement) in place of [AM20]'s use of Gabber's aboslute purity theorem. ■

When $\mathcal{X}$ is a 'stacky curve', we can and will say more. Residues of Brauer classes on regular curves can be more readily understood using the usual divisor exact sequence (8.1) described below.

[3]Technically, Proposition 7.10 only proves $\mathrm{Br}\,\mathcal{Y}(1)_k \simeq \mathrm{H}^1(\mathbb{A}^1_k, \mu_2)$ when $\mathrm{char}\,k \nmid 6$. However, if you replaces its use of Theorem 4.14 with Corollary 9.1.7, then one can deduce that this equality holds even in characteristic 3.

Setup 8.2. Let $\mathcal{X}$ be a regular, integral noetherian algebraic stack. Note that every point $x \in |\mathcal{X}|$ in $\mathcal{X}$'s underlying topological space has a residual gerbe which is in particular a reduced algebraic stack [Sta21, Tag 0H22].

Notation 8.3. Write $j \colon \mathcal{G} \hookrightarrow \mathcal{X}$ for the residual gerbe of the generic point of $\mathcal{X}$. Write $|\mathcal{X}|_1$ for the set of codimension one points in $\mathcal{X}$; for each $x \in |\mathcal{X}|_1$, write $i_x \colon \mathcal{G}_x \hookrightarrow \mathcal{X}$ for the inclusion of its residual gerbe.

Recall 8.4. Recall that, in this paper, we work in the lisse-étale (resp. small étale) site over a given stack $\mathcal{X}$ (resp. given scheme X). Most of what we do is largely agnostic to the difference between this and the big étale site, but the exact sequence in Proposition 8.5 does not hold on the big sites. Since this section is particularly sensative to our choice of the lisse-étale site, we will often be extra explicit by adding the subscript $_{\text{lis-ét}}$ to some of our notation. ⊙

Proposition 8.5. *There is an exact sequence*

$$0 \longrightarrow \mathbb{G}_{m,\mathcal{X}} \longrightarrow j_*\mathbb{G}_{m,\mathcal{G}} \longrightarrow \bigoplus_{x \in |\mathcal{X}|_1} i_{x,*}\underline{\mathbb{Z}} \longrightarrow 0 \tag{8.1}$$

of sheaves on $\mathcal{X}_{\text{lis-ét}}$.

Proof. The sequence (8.1) is characterized by the property that for any smooth cover $U \to \mathcal{X}$ by a connected scheme U, the pullback of (8.1) along this cover is the usual such exact sequence over a regular locally noetherian scheme, as in [Gro66b, Section 1]. This simultaneously defines all the maps in the sequence and shows that it is exact, since it is known to be exact when restricted to the small étale site $U_{\text{ét}}$ for any smooth cover by a scheme. ■

The exact sequence (8.1) is most useful whenever $\mathcal{X}$ is a curve, so that the inclusions $i_x \colon \mathcal{G}_x \hookrightarrow \mathcal{X}$ are closed immersions, as follows from [Sta21, Tag 0H27].

8.1. **The stacky curve case.**

Assumption. From now on, assume $\mathcal{X}$ is a 'stacky curve' (see Section 1.1) in the sense that it is a one-dimensional algebraic stack with finite inertia, on top of still being regular, integral, and noetherian. It follows from Keel–Mori [KM97, Con05] that $\mathcal{X}$ has a coarse moduli space $c \colon \mathcal{X} \to X$.

Remark 8.6. This is a somewhat more general usage of the phrase 'stacky curve' than is commonly used. For example,

- Lopez [Lop23] requires 'stacky curves' to be DM.
- Ellenberg–Satriano–Zureick-Brown [ESZB23, Appendix B] do not require their 'stacky curves' to be DM, but do require them to have birational coarse space, i.e. to be 'generically scheme-y'. ∘

Our goal in the remainder of this section is to establish some basic facts about the long exact sequence in cohomology associated to (8.1) and to compare this sequence on $\mathcal{X}$ to the analogous sequence on its coarse space X.

Notation 8.7. To ease notation, for $x \in |\mathcal{X}|_1$, set $\underline{\mathbb{Z}}_{\mathcal{G}_x} := i_{x,*}\underline{\mathbb{Z}}$.

Lemma 8.8. *For all* $x \in |\mathcal{X}|_1$,

$$\mathrm{H}^1(\mathcal{X}, \underline{\mathbb{Z}}_{\mathcal{G}_x}) \simeq \mathrm{H}^1(\mathcal{G}_x, \underline{\mathbb{Z}}) = 0.$$

Proof. Because $x \in |\mathcal{X}|_1$ is a closed point, $i_x \colon \mathcal{G}_x \hookrightarrow \mathcal{X}$ is a closed immersion from which we deduce that $\mathrm{H}^1(\mathcal{X}, \underline{\mathbb{Z}}_{\mathcal{G}_x}) \simeq \mathrm{H}^1(\mathcal{G}_x, \underline{\mathbb{Z}})$. Furthermore, because $\underline{\mathbb{Z}}$ is a smooth group scheme, we have $\mathrm{H}^1(\mathcal{G}_x, \underline{\mathbb{Z}}) \simeq \mathrm{H}^1_{\text{fppf}}(\mathcal{G}_x, \underline{\mathbb{Z}})$. To compute this latter group, write $f \colon \mathcal{G}_x \to \operatorname{Spec} \kappa(x)$ for the morphism realizing $\mathcal{G}_x$ as a gerbe over its residue field $\kappa(x)$. Note that, by considering the fppf-Leray spectral sequence associated to f, we have an exact sequence

$$\mathrm{H}^1_{\text{fppf}}(\kappa(x), f_*\underline{\mathbb{Z}}) \longrightarrow \mathrm{H}^1_{\text{fppf}}(\mathcal{G}_x, \underline{\mathbb{Z}}) \longrightarrow \mathrm{H}^0_{\text{fppf}}(\kappa(x), \mathrm{R}^1_{\text{fppf}} f_*\underline{\mathbb{Z}}). \tag{8.2}$$

It suffices to show that both of the outer terms in (8.2) vanish. First, $f_*\underline{\mathbb{Z}} \simeq \underline{\mathbb{Z}}$ since f is a universal homeomorphism [Sta21, Tag 06R9], so the leftmost term in (8.2) is

$$\mathrm{H}^1_{\mathrm{fppf}}(\kappa(x), f_*\underline{\mathbb{Z}}) \simeq \mathrm{H}^1_{\mathrm{fppf}}(\kappa(x), \underline{\mathbb{Z}}) \simeq \mathrm{H}^1(\kappa(x), \underline{\mathbb{Z}}) \simeq \mathrm{Hom}_{\mathrm{cts}}(G_{\kappa(x)}, \mathbb{Z}) = 0.$$

For the rightmost term in (8.2), we claim that $\mathrm{R}^1_{\mathrm{fppf}} f_*\underline{\mathbb{Z}} = 0$, which suffices to finish the proof. It suffices to check this at the level of fppf stalks, so it suffices to prove that $\mathrm{H}^1_{\mathrm{fppf}}(\mathcal{G}, \underline{\mathbb{Z}}) = 0$ when $\mathcal{G} = \mathcal{G}_x \times_{\kappa(x)} A$ for some affine $(\mathrm{Sch}_{\kappa(x)}$,fppf)-local scheme $\operatorname{Spec} A$ [GK15, Remark 1.8 and Theorem 2.3]. Since $\operatorname{Spec} A$ is fppf-local [GK15, Definition 0.1], it follows that $\mathcal{G} \to A$ admits a section, so $\mathcal{G}$ is a neutral gerbe over $\operatorname{Spec} A$. Since $\mathcal{X}$ has finite inertia by assumption, we conclude that $\mathcal{G} \simeq BG_A$ for some finite A-group scheme G. Thus, we are reduced to proving that $\mathrm{H}^1_{\mathrm{fppf}}(BG_A, \underline{\mathbb{Z}}) = 0$. For this, we remark that a $\mathbb{Z}$-torsor on BG is determined by the data of a $\mathbb{Z}$-torsor on $\operatorname{Spec} A$ equipped with an action of G, i.e. with a homomorphism $G \to \underline{\mathbb{Z}}$. Since A is fppf-local, every $\mathbb{Z}$-torsor on it is trivial and since G is finite, every homomorphism $G \to \underline{\mathbb{Z}}$ is trivial. Therefore, $\mathrm{H}^1_{\mathrm{fppf}}(BG_A, \underline{\mathbb{Z}}) = 0$, completing the proof. ∎

Corollary 8.9. $\mathrm{H}^1(\mathcal{X}, j_*\mathbb{G}_{m,\mathcal{G}}) \simeq \operatorname{coker}\left(\bigoplus_{x\in|\mathcal{X}|_1} \mathbb{Z} \longrightarrow \operatorname{Pic}\mathcal{X}\right)$

Proof. This follows from the cohomology exact sequence associated to (8.1). ∎

In words, the above corollary says that every line bundle on $\mathcal{X}$ comes from a divisor if and only if $\mathrm{H}^1(\mathcal{X}, j_*\mathbb{G}_m) = 0$.

Example 8.10. If $\mathcal{X}$ is 'generically scheme-y' in the sense that its generic residual gerbe $\mathcal{G}$ is (the spectrum of) a field K, then $\mathrm{H}^1(\mathcal{X}, j_*\mathbb{G}_m) = 0$; indeed, $\mathrm{H}^1(\mathcal{X}, j_*\mathbb{G}_m) \hookrightarrow \mathrm{H}^1(\mathcal{G}, \mathbb{G}_m) = \operatorname{Pic} K = 0$. This shows that, for such $\mathcal{X}$, $\operatorname{Pic}\mathcal{X}$ is the Weil divisor class group, reproving a result of Voight and Zureick-Brown [VZB22, Lemma 5.4.5]. △

We next show that Example 8.10 is, in some sense, almost the only case in which the Picard group of a stacky curve is equal to its divisor class group. More specifically, we show that this holds if and only if $\operatorname{Pic}\mathcal{G} = 0$.

Lemma 8.11. $\mathrm{R}^1_{\text{lis-ét}} j_*\mathbb{G}_m = 0$. *Consequently,* $\mathrm{H}^1(\mathcal{X}, j_*\mathbb{G}_{m,\mathcal{G}}) = \mathrm{H}^1(\mathcal{G}, \mathbb{G}_m) = \operatorname{Pic}\mathcal{G}$.

Proof. To show that $\mathrm{R}^1_{\text{lis-ét}} j_*\mathbb{G}_m = 0$, it suffices to show that for any smooth cover $S \to \mathcal{X}$ by a (necessarily regular) scheme S and any line bundle $\mathscr{L}$ on the generic fiber $S_{\mathcal{G}}$, there exists an étale cover $S' \twoheadrightarrow S$ such that the pullback $\mathscr{L}|_{S'_{\mathcal{G}}}$ is trivial. With that said, let $S \to \mathcal{X}$ and $\mathscr{L}$ be as just indicated. Note that $\mathcal{G} \hookrightarrow \mathcal{X}$, being a monomorphism into a noetherian stack, is both quasi-compact and quasi-separated (qcqs). Consequently, $\iota\colon S_{\mathcal{G}} \hookrightarrow S$ is also qcqs, so $\iota_*\mathscr{L}$ is a quasi-coherent sheaf on S [Sta21, Tag 03M9]. Since S is regular, it follows that there exists a line bundle $\mathscr{L}' \subset \iota_*\mathscr{L}$ such that $\mathscr{L}'|_{S_{\mathcal{G}}} = \mathscr{L}$. Thus, we may take $S' \twoheadrightarrow S$ to be any étale cover trivializing $\mathscr{L}'$. This proves $\mathrm{R}^1_{\text{lis-ét}} j_*\mathbb{G}_m = 0$.

Now, the exact sequence of low-degree terms for the leray spectral sequence associated to $j\colon \mathcal{G} \hookrightarrow \mathcal{X}$ begins

$$0 \longrightarrow \mathrm{H}^1(\mathcal{X}, j_*\mathbb{G}_{m,\mathcal{G}}) \longrightarrow \mathrm{H}^1(\mathcal{G}, \mathbb{G}_m) \longrightarrow \mathrm{H}^0(\mathcal{X}, \mathrm{R}^1_{\text{lis-ét}} j_*\mathbb{G}_m) = 0,$$

from which the rest of the claim follows. ∎

Corollary 8.12. *Let*

$$\operatorname{Cl}\mathcal{X} \coloneqq \operatorname{coker}\left(\mathbb{G}_m(\mathcal{G}) \longrightarrow \bigoplus_{x\in|\mathcal{X}|_1} \mathbb{Z}\right)$$

be the divisor class group of $\mathcal{X}$. *Then, there is a short exact sequence*

$$0 \longrightarrow \operatorname{Cl}\mathcal{X} \longrightarrow \operatorname{Pic}\mathcal{X} \longrightarrow \operatorname{Pic}\mathcal{G} \longrightarrow 0.$$

Proof. This follows from considering the cohomology exact sequence associated to (8.1) and appealing to Lemmas 8.8 and 8.11. ∎

Remark 8.13. There always exists some generically scheme-y stacky curve $\mathcal{Y}$ under $\mathcal{X}$ such that $\mathcal{X} \to \mathcal{Y}$ is a gerbe; see [Lop23, Proposition 1.5] and [AOV08, Appendix A]. It is possible to identify $\operatorname{Cl}\mathcal{X}$ in Corollary 8.12 with $\operatorname{Pic}\mathcal{Y}$ and so obtain a short exact sequence

$$0 \longrightarrow \operatorname{Pic}\mathcal{Y} \longrightarrow \operatorname{Pic}\mathcal{X} \longrightarrow \operatorname{Pic}\mathcal{G} \longrightarrow 0.$$

This is similar to the exact sequence appearing in [Lop23, Corollary 1.7]. A short exact sequence exactly of the form appearing above was first suggested to me by David Zureick-Brown. ◦

Focusing back in on Brauer groups, taking the long exact sequence in cohomology associated to (8.1) and appealing to Lemma 8.8 produces the exact sequence

$$0 \longrightarrow \mathrm{H}^2(\mathcal{X}, \mathbb{G}_m) \longrightarrow \mathrm{H}^2(\mathcal{X}, j_*\mathbb{G}_{m,\mathcal{G}}) \longrightarrow \bigoplus_{x \in |\mathcal{X}|_1} \mathrm{H}^2(\mathcal{G}_x, \mathbb{Z}) \longrightarrow \mathrm{H}^3(\mathcal{X}, \mathbb{G}_m) \longrightarrow \dots.$$

Above, one can further compute $\mathrm{H}^2(\mathcal{G}_x, \mathbb{Z})$.

Lemma 8.14. $\mathrm{H}^i(\mathcal{G}_x, \mathbb{Q}) = 0$ *for* $i = 1, 2$ *and* $x \in |\mathcal{X}|_1$. *Consequently,* $\mathrm{H}^2(\mathcal{G}_x, \mathbb{Z}) \simeq \mathrm{H}^1(\mathcal{G}_x, \mathbb{Q}/\mathbb{Z})$.

Proof. Once the first part of the lemma is established, the second follows from the exact sequence $0 \to \mathbb{Z} \to \mathbb{Q} \to \mathbb{Q}/\mathbb{Z} \to 0$. For the first part, write $f\colon \mathcal{G}_x \to \operatorname{Spec} k$ for the morphism realizing f as a gerbe over (the spectrum of) a field k. One can argue as in Lemma 8.8 (i.e. use the fppf-Leray spectral sequence for f) to show that $\mathrm{H}^1(\mathcal{G}_x, \mathbb{Q}) = 0$ and that $\mathrm{R}^1_{\mathrm{fppf}} f_*\underline{\mathbb{Q}} = 0$. Since $\mathrm{H}^2_{\mathrm{fppf}}(k, f_*\underline{\mathbb{Q}}) = \mathrm{H}^2(k, \mathbb{Q}) = 0$, to prove that $\mathrm{H}^2(\mathcal{G}_x, \mathbb{Q}) = 0$, it now suffices to show that $\mathrm{H}^0(k, \mathrm{R}^2_{\mathrm{fppf}} f_*\underline{\mathbb{Q}}) = 0$. Let k'/k be a field extension over which $\mathcal{G}_x$ admits a point, so $\mathcal{G}_{x,k'} \simeq BG$ for some finite k'-group scheme G. By replacing k' with a further finite extension if necessary, we may and do assume that the maximal étale quotient $G^{\text{ét}}$ of G is a constant group scheme. To show $\mathrm{H}^0(k, \mathrm{R}^2_{\mathrm{fppf}} f_*\underline{\mathbb{Q}}) = 0$, it suffices to show that $\mathrm{H}^2_{\mathrm{fppf}}(\mathcal{G}_{x,k'}, \mathbb{Q}) = \mathrm{H}^2(BG, \mathbb{Q})$ vanishes. For this, one uses the descent spectral sequence

$$E_2^{ij} = \mathrm{H}^i_0(G, \underline{\mathrm{H}}^j(k'/k', \mathbb{Q})) \implies \mathrm{H}^{i+j}(BG, \mathbb{Q}).$$

Note that $E_2^{02} = 0$ since $\mathrm{H}^2(k', \mathbb{Q}) = 0$, and that $E_2^{11} = \operatorname{Hom}(G, \underline{\mathrm{H}}^1(k'/k', \mathbb{Q})) = 0$ since there are no non-trivial homomorphisms from a finite group to a $\mathbb{Q}$-vector space. Finally, because $\underline{\mathbb{Q}}$ is étale, every morphism $G^n \to \underline{\mathbb{Q}}$ factors through $(G^{\text{ét}})^n$, so $E_2^{20} = \mathrm{H}^2_0(G, \mathbb{Q}) = \mathrm{H}^2_0(G^{\text{ét}}, \mathbb{Q})$ which vanishes since $G^{\text{ét}}$ is a finite (abstract) group. ■

Proposition 8.15. *There is a long exact sequence*

$$0 \longrightarrow \mathrm{H}^2(\mathcal{X}, \mathbb{G}_m) \longrightarrow \mathrm{H}^2(\mathcal{X}, j_*\mathbb{G}_{m,\mathcal{G}}) \longrightarrow \bigoplus_{x \in |\mathcal{X}|_1} \mathrm{H}^1(\mathcal{G}_x, \mathbb{Q}/\mathbb{Z}) \longrightarrow \mathrm{H}^3(\mathcal{X}, \mathbb{G}_m) \longrightarrow \dots. \tag{8.3}$$

Furthermore, $\mathrm{H}^2(\mathcal{X}, j_*\mathbb{G}_{m,\mathcal{G}}) \simeq \ker\big(\mathrm{H}^2(\mathcal{G}, \mathbb{G}_m) \longrightarrow \mathrm{H}^0(\mathcal{X}, \mathrm{R}^2_{\text{lis-ét}} j_*\mathbb{G}_m)\big)$.

Proof. Taking the long exact sequence associated to (8.1), and appealing to Lemmas 8.8 and 8.14, we obtain the exact sequence (8.3). The last part of the claim follows from the exact sequence

$$\mathrm{H}^0(\mathcal{X}, \mathrm{R}^1_{\text{lis-ét}} j_*\mathbb{G}_m) \to \mathrm{H}^2(\mathcal{X}, j_*\mathbb{G}_m) \to \ker\big(\mathrm{H}^2(\mathcal{G}, \mathbb{G}_m) \to \mathrm{H}^0(\mathcal{X}, \mathrm{R}^2_{\text{lis-ét}} j_*\mathbb{G}_m)\big) \to \mathrm{H}^1(\mathcal{X}, \mathrm{R}^1_{\text{lis-ét}} j_*\mathbb{G}_m)$$

of low degree terms in the lisse-étale Leray spectral sequence. Above, $\mathrm{R}^1_{\text{lis-ét}} j_*\mathbb{G}_m = 0$ by Lemma 8.11. ■

This shows that stacky curves enjoy a residue exact sequence (8.3) reminiscent of the one attached to schemey curves; see [CTS21, Theorem 3.6.1]. In the following section, after adding the additional assumption that $\mathcal{X}$ is generically schemey, we compare this residue sequence to the analogous one on its coarse space.

8.2. The tame generically schemey case.

Assumption. Assume now that there exists distinct closed points $x_1, \dots, x_r \in X$ and positive integers $e_1, \dots, e_r > 1$ such that

$$\mathcal{X} \simeq \sqrt[e_1]{x_1/X} \times_X \dots \times_X \sqrt[e_r]{x_r/X}. \tag{8.4}$$

In particular, $\mathcal{X}$ is tame and 'generically schemey' in the sense that there exists a dense open $U \subset X$ (e.g. $U = X \setminus \{x_1, \ldots, x_r\}$) over which the coarse space map $c\colon \mathcal{X} \to X$ is an isomorphism, but is not necessarily DM.

It is known that $\mathcal{X}$ will be of the form (8.4) if it is a tame, DM, and generically schemey; see [GS16, Theorem 1] and/or [Lop23, 2.24].

Definition 8.16. We call an $\mathcal{X}$ as in (8.4) a `multiply rooted stack`. ◇

Remark 8.17. Recall that Lemma 2.20 shows that $\operatorname{Br}\mathcal{X} \simeq \mathrm{H}^2(\mathcal{X}, \mathbb{G}_m)$. ○

Notation 8.18. The generic residual gerbe $j\colon \mathcal{G} \hookrightarrow \mathcal{X}$ is now the spectrum of a field, so we write $\mathcal{G} = \operatorname{Spec} K$. We write $\eta\colon \operatorname{Spec} K \to X$ for the corresponding map to X, so we have a commutative diagram

$$\begin{tikzcd} \operatorname{Spec} K \arrow[r, "j"'] \arrow[rr, bend left, "\eta"] & \mathcal{X} \arrow[r, "c"'] & X. \end{tikzcd}$$

For a closed point $x \in |X|_1 = |\mathcal{X}|_1$, we write $\iota_x\colon \operatorname{Spec}\kappa(x) \hookrightarrow X$ for the corresponding closed immersion, we have a commutative (but not necessarily Cartesian) diagram

$$\begin{tikzcd} \mathcal{G}_x \arrow[r, "i_x"] \arrow[d] & \mathcal{X} \arrow[d, "c"] \\ \operatorname{Spec}\kappa(x) \arrow[r, "\iota_x"] & X. \end{tikzcd} \tag{8.5}$$

Furthermore, write $\underline{\mathbb{Z}}_x \coloneqq \iota_{x,*}\underline{\mathbb{Z}}$.

We wish to compare residue maps on $\mathcal{X}$ to those on X. A priori the exact sequences (8.1) for $\mathcal{X}$ and X live in different categories. In order to remedy this, we first apply c_* to the exact sequence (8.1) over $\mathcal{X}$.

Recall 8.19. While we use the lisse-étale site over the stack $\mathcal{X}$, we use the small étale site over the scheme X. ⊙

Lemma 8.20. $\mathrm{R}^1_{\text{lis-ét}} c_*(j_*\mathbb{G}_{m,K}) = 0$

Proof. Let $U \to X$ be an étale X-scheme. Consider the following commutative diagram, all of whose squares are Cartesian:

$$\begin{tikzcd} U_K \arrow[r, "j'"] \arrow[d, "\text{ét}"] & \mathcal{U} \arrow[r] \arrow[d] & U \arrow[d, "\text{ét}"] \\ \operatorname{Spec} K \arrow[r, "j"] & \mathcal{X} \arrow[r, "c"] & X. \end{tikzcd}$$

In order to prove the theorem, it suffices to show the second equality in

$$\mathrm{H}^1(U \times_X \mathcal{X}, (j_*\mathbb{G}_{m,K})|_U) = \mathrm{H}^1(\mathcal{U}, j'_*\mathbb{G}_{m,U_K}) = 0.$$

For this, we remark that $\mathrm{H}^1(\mathcal{U}, j'_*\mathbb{G}_{m,U_K}) \hookrightarrow \mathrm{H}^1(U_K, \mathbb{G}_m) = \operatorname{Pic} U_K$ and that $\operatorname{Pic} U_K = 0$ since U_K is étale over a field. ■

Corollary 8.21. *The sequence (8.1) pushes forward to the exact sequence*

$$0 \longrightarrow \mathbb{G}_{m,X} \longrightarrow \eta_*\mathbb{G}_{m,K} \longrightarrow \bigoplus_{x \in |X|_1} \underline{\mathbb{Z}}_x \longrightarrow \mathrm{R}^1_{\text{lis-ét}} c_*\mathbb{G}_{m,\mathcal{X}} \longrightarrow 0 \tag{8.6}$$

of sheaves on X.

Proof. Given Lemma 8.20, the only part of this claim which is not immediate is the computation $(c \circ i_x)_*\underline{\mathbb{Z}} = \iota_{x,*}\underline{\mathbb{Z}} =: \underline{\mathbb{Z}}_x$ for $x \in |X|_1 = |\mathcal{X}|_1$. Write $f\colon \mathcal{G}_x \to \operatorname{Spec}\kappa(x)$ for the natural morphism. By (8.5), to prove $(c \circ i_x)_*\underline{\mathbb{Z}} = \underline{\mathbb{Z}}_x$, it suffices to show that $f_*\underline{\mathbb{Z}} = \underline{\mathbb{Z}}$.

For this, set $\mathcal{Z} := \kappa(x) \times_X \mathcal{X}$ and note that $\mathcal{Z} \to \operatorname{Spec}\kappa(x)$ is a universal homeomorphism which follows, e.g., from [Alp14, Proposition 5.2.9(3)]. In particular, its underlying topological space $|\mathcal{Z}|$ is a singleton, and so [Sta21, Tag 06MT] implies that $\mathcal{Z}_{\mathrm{red}}$ is the residual gerbe $\mathcal{G}_x$ of $\mathcal{X}$ at x. In particular, f factors as $\mathcal{G}_x \simeq \mathcal{Z}_{\mathrm{red}} \hookrightarrow \mathcal{Z} \to \operatorname{Spec}\kappa(x)$, making it a universal homeomorphism. This implies that $f_*\underline{\mathbb{Z}} = \underline{\mathbb{Z}}$, as desired. ∎

Note that (8.6) sits in the following homomorphism of exact sequences:

$$\begin{tikzcd} 0 \arrow[r] & \mathbb{G}_{m,X} \arrow[r] \arrow[d, equal] & \eta_*\mathbb{G}_{m,K} \arrow[r, "\Delta_X"] \arrow[d, equal] & \bigoplus_{x\in|X|_1} \underline{\mathbb{Z}}_x \arrow[r] \arrow[d, "\varphi"] & 0 \arrow[r] \arrow[d] & 0 \\ 0 \arrow[r] & \mathbb{G}_{m,X} \arrow[r] & \eta_*\mathbb{G}_{m,K} \arrow[r, "\Delta_{\mathcal{X}}"] & \bigoplus_{x\in|\mathcal{X}|_1} \underline{\mathbb{Z}}_x \arrow[r] & \mathrm{R}^1_{\text{lis-ét}}c_*\mathbb{G}_{m,\mathcal{X}} \arrow[r] & 0, \end{tikzcd} \tag{8.7}$$

whose top row is the usual divisor exact sequence on the scheme X and whose bottom row is (8.6).

Notation 8.22. Recall the inertial degree Ideg of [JL21, Section 2.2]. For a point $x \in |X|_1 = |\mathcal{X}|_1$, set

$$e_x := \operatorname{Ideg}(\mathcal{G}_x \to \operatorname{Spec}\kappa(x)) = \begin{cases} e_i & \text{if } i \in \{1,\dots,r\} \text{ and } x = x_i \\ 1 & \text{otherwise.}\end{cases}$$

Remark 8.23. The results in this section have so far only used that $\mathcal{X}$ is generically schemey. We use that $\mathcal{X}$ is specifically a multiply rooted stack in order to compute φ in the next lemma. ∘

Lemma 8.24. *The morphism φ in (8.7) is $\varphi = \bigoplus \varphi_x$, where $\varphi_x \colon \underline{\mathbb{Z}}_x \to \underline{\mathbb{Z}}_x$ sends $1 \mapsto e_x$.*

Proof. By construction, one can compute φ separately near each point, so it suffices to assume that $X = \operatorname{Spec} R$ is the spectrum of a dvr with closed point x (think: $R = \mathscr{O}_{X,x}$). Since $\mathcal{X}$ is a root stack over X, it follows from Proposition 6.6**(2)** that

$$\mathcal{X} = \sqrt[n]{\pi/R} \simeq \left[\frac{\operatorname{Spec} R[T]/(T^n - \pi)}{\mu_n}\right],$$

where $n := e_x$ and π is a uniformizer for R. Let $K = \operatorname{Frac} R$, so we are attempting to compute the map $\varphi = \varphi_x$ in the commutative diagram

$$\begin{tikzcd} K^\times \arrow[r, "\Delta_X"] \arrow[d, "="'] & \mathbb{Z} \arrow[d, "\varphi_x"] \\ K^\times \arrow[r, "\Delta_{\mathcal{X}}"] & \mathbb{Z} \end{tikzcd}$$

with horizontal maps as in Proposition 8.5. Note that $\Delta_X(\pi) = 1$, so it suffices to show that $\Delta_{\mathcal{X}}(\pi) = n$. From the proof of Proposition 8.5, we see that $\Delta_{\mathcal{X}}$ can be computed on a smooth cover of $\mathcal{X}$, so consider the Cartesian square[4]

$$\begin{tikzcd} \mathbb{G}_{m,R} \arrow[r] \arrow[d, "(-)^n"'] & \mathcal{X} \arrow[d] \\ \mathbb{G}_{m,R} \arrow[r] & \operatorname{Spec} R \end{tikzcd}$$

[4]this is the top face of the Cartesian cube in [Cad07, Proof of Proposition 2.3.5] obtained from pulling back the following Cartesian square (which is the bottom face of the aforementioned cube)

$$\begin{tikzcd} \mathbb{A}^1 \arrow[r, "(-)^n"] \arrow[d] & \mathbb{A}^1 \arrow[d] \\ {[\mathbb{A}^1/\mathbb{G}_m]} \arrow[r, "(-)^n"] & {[\mathbb{A}^1/\mathbb{G}_m]} \end{tikzcd}$$

along $\operatorname{Spec} R \xrightarrow{(\mathscr{O}_R,\pi)} [\mathbb{A}^1/\mathbb{G}_m]$.

whose horizontal arrows are (smooth) $\mathbb{G}_m$-torsors. With this square in mind, it becomes clear that $\Delta_{\mathcal{X}}(\pi) = n\Delta_X(\pi) = n$ since the rational function $\pi \in K = \operatorname{Frac} R$ pulls back to in the top $\mathbb{G}_{m,R}$ is the nth power of the function it pulls back to in the bottom $\mathbb{G}_{m,R}$. ■

Corollary 8.25. $\mathrm{R}^1_{\text{lis-ét}}c_*\mathbb{G}_{m,\mathcal{X}} \simeq \operatorname{coker}\varphi \simeq \bigoplus_{x\in|X|_1} \underline{\mathbb{Z}/e_x\mathbb{Z}}_x$.

In order to better understand (8.6), we introduce the two-term complexes

$$\mathscr{C}_{\mathcal{X}} \coloneqq \left[\eta_*\mathbb{G}_{m,K} \xrightarrow{\Delta_{\mathcal{X}}} \bigoplus_{x\in|\mathcal{X}|_1} \underline{\mathbb{Z}}_x\right] \text{ and } \mathscr{C}_X \coloneqq \left[\eta_*\mathbb{G}_{m,K} \xrightarrow{\Delta_X} \bigoplus_{x\in|X|_1} \underline{\mathbb{Z}}_x\right] \simeq \mathbb{G}_{m,X},$$

both concentrated in degrees 0 and 1. Above, $\mathscr{C}_X$ is quasi-isomorphic to $\mathbb{G}_{m,X}$ because of the top row of (8.7). For $\mathscr{C}_{\mathcal{X}}$, we have the following result.

Proposition 8.26. *$\mathscr{C}_{\mathcal{X}} \simeq \tau_{\le 2}\mathrm{R}_{\text{lis-ét}}c_*\mathbb{G}_m$. In particular,*

$$\mathrm{H}^n(X, \mathscr{C}_{\mathcal{X}}) \simeq \mathrm{H}^n(\mathcal{X}, \mathbb{G}_m) \text{ for } n \le 2.$$

Proof. The exact sequence (8.1) shows that $\mathbb{G}_{m,\mathcal{X}} \simeq [j_*\mathbb{G}_m \to \bigoplus \underline{\mathbb{Z}}_{\mathcal{G}_x}]$ so applying c_* gives a morphism

$$\mathscr{C}_{\mathcal{X}} = \left[c_*j_*\mathbb{G}_m \longrightarrow c_*\left(\bigoplus_{x\in|\mathcal{X}|_1} \underline{\mathbb{Z}}_{\mathcal{G}_x}\right)\right] \to \mathrm{R}_{\text{lis-ét}}c_*\left[j_*\mathbb{G}_m \to \bigoplus_{x\in|\mathcal{X}|_1} \underline{\mathbb{Z}}_{\mathcal{G}_x}\right] \simeq \mathrm{R}_{\text{lis-ét}}c_*\mathbb{G}_m.$$

Furthermore, (8.6) shows that this morphism induces an isomorphism on π_0 and π_1 (while $\pi_i(\mathscr{C}) = 0$ for $i \ge 2$) and so induces a (quasi-)isomorphism $\mathscr{C} \xrightarrow{\sim} \tau_{\le 1}\mathrm{R}_{\text{lis-ét}}c_*\mathbb{G}_m$. Finally,

$$\pi_2(\mathrm{R}_{\text{lis-ét}}c_*\mathbb{G}_m) = \mathrm{R}^2_{\text{lis-ét}}c_*\mathbb{G}_m = 0,$$

with last equality by Theorem 4.14, so $\tau_{\le 1}\mathrm{R}_{\text{lis-ét}}c_*\mathbb{G}_m \simeq \tau_{\le 2}\mathrm{R}_{\text{lis-ét}}c_*\mathbb{G}_m$. ■

Now, we remark that (8.7) induces the following morphism of distinguished triangles:

$$\begin{tikzcd}
 & \mathbb{G}_{m,X} \arrow[d, "\wr"] & & \\
\left(\bigoplus_{x\in|X|_1} \underline{\mathbb{Z}}_x\right)[-1] \arrow[r] \arrow[d, "\varphi"] & \mathscr{C}_X \arrow[r] \arrow[d] & \eta_*\mathbb{G}_{m,K} \arrow[r, "+1"] \arrow[d, equal] & {} \\
\left(\bigoplus_{x\in|\mathcal{X}|_1} \underline{\mathbb{Z}}_x\right)[-1] \arrow[r] & \mathscr{C}_{\mathcal{X}} \arrow[r] \arrow[d, "\wr"] & \eta_*\mathbb{G}_{m,K} \arrow[r, "+1"] & {} \\
 & \tau_{\le 2}\mathrm{R}_{\text{lis-ét}}c_*\mathbb{G}_{m,\mathcal{X}} & &
\end{tikzcd}$$

Therefore, upon taking cohomology, we obtain:

$$\begin{tikzcd}
0 \arrow[r] & \mathrm{H}^2(X,\mathbb{G}_m) \arrow[r] \arrow[d, "c^*"] & \mathrm{H}^2(X,\eta_*\mathbb{G}_m) \arrow[r, "\bigoplus_x \mathrm{res}_{x/X}"] \arrow[d, equal] & \bigoplus_{x\in|X|_1} \mathrm{H}^1(\kappa(x),\mathbb{Q}/\mathbb{Z}) \arrow[d, "\varphi_*"] \\
0 \arrow[r] & \mathrm{H}^2(\mathcal{X},\mathbb{G}_m) \arrow[r] & \mathrm{H}^2(X,\eta_*\mathbb{G}_m) \arrow[r, "\bigoplus_x \mathrm{res}_{x/\mathcal{X}}"] & \bigoplus_{x\in|\mathcal{X}|_1} \mathrm{H}^1(\kappa(x),\mathbb{Q}/\mathbb{Z})
\end{tikzcd} \tag{8.8}$$

whose top row is the usual residue exact sequence associated to X.

Corollary 8.27. *For any* $x \in |X|_1 = |\mathcal{X}|_1$, $\operatorname{res}_{x/\mathcal{X}} = e_x \cdot \operatorname{res}_{x/X}$ *as maps* $\mathrm{H}^2(X, \eta_*\mathbb{G}_m) \to \mathrm{H}^1(\kappa(x), \mathbb{Q}/\mathbb{Z})$.

Proof. This follows from commutativity of (8.8). ■

Finally, we make $\mathrm{H}^2(X, \eta_*\mathbb{G}_m)$ more explicit in certain cases. The below proposition is likely well-known, but I failed to find a statement of it in the literature.

Proposition 8.28. *Assume that X is a nice curve over a field k. Then,* $\mathrm{H}^2(X, \eta_*\mathbb{G}_m) \simeq \ker(\operatorname{Br} k(X) \to \operatorname{Br} k^s(X))$.

Proof. We begin with the leray spectral sequence for the morphism $X \to \operatorname{Spec} k$:

$$E_2^{ij} = \mathrm{H}^i(k, \mathrm{H}^j(\overline{X}, \eta_*\mathbb{G}_m)) \implies \mathrm{H}^{i+j}(X, \eta_*\mathbb{G}_m), \tag{8.9}$$

where $\overline{X} \coloneqq X_{k^s}$. Note that $\mathrm{H}^1(\overline{X}, \eta_*\mathbb{G}_m) = 0$ because it embeds by $\mathrm{H}^1(k^s(X), \mathbb{G}_m) = 0$. Furthermore, on $\overline{X}$, one has the exact sequence

$$0 \longrightarrow \mathrm{H}^2(\overline{X}, \mathbb{G}_m) \longrightarrow \mathrm{H}^2(\overline{X}, \eta_*\mathbb{G}_m) \longrightarrow \bigoplus_{x \in |\overline{X}|_1} \mathrm{H}^1(\kappa(x), \mathbb{Q}/\mathbb{Z}).$$

Above, $\mathrm{H}^2(\overline{X}, \mathbb{G}_m) = 0$ by [Gro68, Corollaire 5.8] and $\mathrm{H}^1(\kappa(x), \mathbb{Q}/\mathbb{Z}) = 0$ for all $x \in |\overline{X}|_1$ since each $\kappa(x)$ is separably closed. Thus, $\mathrm{H}^2(\overline{X}, \eta_*\mathbb{G}_m) = 0$ as well. Thus, (8.9) gives an isomorphism $\mathrm{H}^2(k, k^s(X)^\times) \xrightarrow{\sim} \mathrm{H}^2(X, \eta_*\mathbb{G}_m)$, so we need only compute this former group. For this, we use the spectral sequence

$$F_2^{ij} = \mathrm{H}^i(k, \mathrm{H}^j(k^s(X), \mathbb{G}_m)) \implies \mathrm{H}^{i+j}(k(X), \mathbb{G}_m)$$

whose exact sequence of low degree terms quickly yields $\mathrm{H}^2(k, k^s(X)^\times) \xrightarrow{\sim} \ker(\operatorname{Br} k(X) \to \operatorname{Br} k^s(X))$. ■

8.3. A stacky Faddeev exact sequence.

Remark 8.29. In [San23, Proposition 5.5], Santens gives one description of the Brauer group of a generically scheme-y tame DM stacky $\mathbb{P}^1$. In the present section, we give a different description of such Brauer groups (see Propositions 8.31 and 8.33), one which the author personally finds more amenable to computations, and then show how it can be used to aid in the computations of Brauer groups of (possibly non-proper) stacky rational curves. ○

Setup 8.30. Fix a field k, set $X = \mathbb{P}^1_k$, and use the notation of Section 8.2. In particular, there are distinct closed points $x_1, \ldots, x_r \in \mathbb{P}^1_k$ and integers $e_1, \ldots, e_r > 1$ such that

$$\mathcal{X} = \sqrt[e_1]{x_1/\mathbb{P}^1_k} \times_{\mathbb{P}^1_k} \cdots \times_{\mathbb{P}^1_k} \sqrt[e_r]{x_r/\mathbb{P}^1_k} \xrightarrow{c} \mathbb{P}^1_k.$$

Furthermore, for each $x \in |\mathbb{P}^1_k|_1 = |\mathcal{X}|_1$, we defined its "stacky degree" e_x in Notation 8.22. We also have the morphisms $\varphi_x \colon \underline{\mathbb{Z}}_x \to \underline{\mathbb{Z}}_x$, of sheaves supported on $\operatorname{Spec} \kappa(x) \hookrightarrow X$, given by $1 \mapsto e_x$.

Note that, by Proposition 8.28 and [BPP+23, Theorem 5.1], the top row of (8.8) extends to the Faddeev exact sequence and so (8.8) yields the following commutative diagram with exact rows (note also that $\operatorname{Br} \mathbb{P}^1_k \simeq \operatorname{Br} k$ either as a biproduct of Faddeev or by [Poo17, Propostion 6.9.9]):

$$\begin{tikzcd}
0 \arrow[r] & \operatorname{Br} \mathbb{P}^1_k \arrow[r] \arrow[d, "c^*"] & \ker(\operatorname{Br} k(\mathbb{P}^1) \to \operatorname{Br} k^s(\mathbb{P}^1)) \arrow[r, "\bigoplus_x \operatorname{res}_{x/\mathbb{P}^1_k}"] \arrow[d, equal] & \bigoplus_{x \in |\mathbb{P}^1_k|_1} \mathrm{H}^1(\kappa(x), \mathbb{Q}/\mathbb{Z}) \arrow[r, "\mathrm{Cor}"] \arrow[d, "\varphi_* = \bigoplus_x e_x"] & \mathrm{H}^1(k, \mathbb{Q}/\mathbb{Z}) \arrow[r] & 0 \\
0 \arrow[r] & \operatorname{Br} \mathcal{X} \arrow[r] & \ker\big(\operatorname{Br} k(\mathbb{P}^1) \to \operatorname{Br} k^s(\mathbb{P}^1)\big) \arrow[r, "\bigoplus_x \operatorname{res}_{x/\mathcal{X}}"] & \bigoplus_{x \in |\mathcal{X}|_1} \mathrm{H}^1(\kappa(x), \mathbb{Q}/\mathbb{Z}), & &
\end{tikzcd} \tag{8.10}$$

where $\mathrm{Cor} = \sum_x \mathrm{Cor}_{\kappa(x)/k}$.

Proposition 8.31. *There is an exact sequence*

$$0 \longrightarrow \underbrace{\operatorname{Br}\mathbb{P}^1_k}_{\simeq \operatorname{Br} k} \xrightarrow{c^*} \operatorname{Br}\mathcal{X} \longrightarrow \bigoplus_{i=1}^r \mathrm{H}^1\left(\kappa(x_i), \frac{1}{e_i}\mathbb{Z}\Big/\mathbb{Z}\right) \xrightarrow{\mathrm{Cor}} \mathrm{H}^1(k, \mathbb{Q}/\mathbb{Z})$$

Proof. From a diagram chase or appropriate application of the snake lemma to (8.10), one sees that

$$\operatorname{coker} c^* \simeq \ker\left(\bigoplus_{x\in|\mathbb{P}^1_k|_1} \mathrm{H}^1(\kappa(x), \mathbb{Q}/\mathbb{Z})[e_x] \xrightarrow{\mathrm{Cor}} \mathrm{H}^1(k, \mathbb{Q}/\mathbb{Z})\right).$$

Since $e_x = 1$ unless $x = x_i$ for some i (in which case $e_x = e_i$), it now suffices to show that $\mathrm{H}^1(\kappa(x_i), \mathbb{Q}/\mathbb{Z})[e_i] \simeq \mathrm{H}^1(\kappa(x_i), \frac{1}{e_i}\mathbb{Z}/\mathbb{Z})$. This follows from taking cohomology of the exact sequence $0 \to \frac{1}{e_i}\mathbb{Z}/\mathbb{Z} \longrightarrow \mathbb{Q}/\mathbb{Z} \xrightarrow{e_i} \mathbb{Q}/\mathbb{Z} \longrightarrow 0$. ■

Example 8.32. Fix integers $a, b, c > 0$ and consider the generalized Fermat equation $x^a+y^b = z^c$. To slightly simplify this example, assume that a or b is odd. Primitive integral solutions to this equation correspond to integral points on $S = \operatorname{Spec}\mathbb{Z}[x, y, z]/(x^a+y^b-z^c)\setminus\{(0, 0, 0)\} \subset \mathbb{A}^3_{\mathbb{Z}}$. Note that $\mathbb{G}_m^3 \curvearrowright \mathbb{A}^3$ and let $H \subset \mathbb{G}_m^3$ be the subgroup preserving S; this is the subgroup generated by the image of $\mathbb{G}_m \hookrightarrow \mathbb{G}_m^3, \lambda \mapsto (\lambda^{bc}, \lambda^{ac}, \lambda^{ab})$ and by $\mu_a \times \mu_b \times \mu_c \subset \mathbb{G}_m^3$. Set $\mathcal{X} := [S/H]$. One often studies $S(\mathbb{Z})$ by first studying $\mathcal{X}(\mathbb{Z})$; see [Dar97b, Dar97a] and/or [PSS07]. One can check that $\mathcal{X}$ is isomorphic to $\mathbb{P}^1$ rooted at $0, \infty, -1$ to degrees a, b, c, respectively; the coarse space map $\mathcal{X} \to \mathbb{P}^1$ is descended from the map $S \to \mathbb{P}^1, (x, y, z) \mapsto [x^a : y^b]$. With this description of $\mathcal{X}$, Proposition 8.31 shows that the Brauer group of its generic fiber sits in an exact sequence

$$0 \longrightarrow \operatorname{Br}\mathbb{Q} \longrightarrow \operatorname{Br}\mathcal{X}_{\mathbb{Q}} \longrightarrow \mathrm{H}^1\left(\mathbb{Q}, \frac{1}{a}\mathbb{Z}/\mathbb{Z}\right) \oplus \mathrm{H}^1\left(\mathbb{Q}, \frac{1}{b}\mathbb{Z}/\mathbb{Z}\right) \oplus \mathrm{H}^1\left(\mathbb{Q}, \frac{1}{c}\mathbb{Z}/\mathbb{Z}\right) \xrightarrow{\Sigma} \mathrm{H}^1(\mathbb{Q}, \mathbb{Q}/\mathbb{Z}). \qquad \triangle$$

In the simplest case, where $\mathbb{P}^1$ is rooted at k-points to pairwise coprime degrees, (8.10) can be reduced into a single stacky Faddeev exact sequence.

Proposition 8.33 (Stacky Faddeev)**.** *Assume that each $x_1, \dots, x_r$ is a k-point and that the numbers $e_1, \dots, e_r$ are pairwise coprime. Let $N := \prod_{i=1}^r e_i = \operatorname{lcm}_i(e_i)$. Then, $\operatorname{Br}\mathcal{X} \simeq \operatorname{Br}\mathbb{P}^1_k \simeq \operatorname{Br} k$ and there is an exact sequence*

$$0 \to \operatorname{Br} k \to \ker\big(\operatorname{Br} k(\mathbb{P}^1) \to \operatorname{Br} k^s(\mathbb{P}^1)\big) \xrightarrow{\bigoplus_x e_x\cdot \mathrm{res}_{x/\mathbb{P}^1_k}} \bigoplus_{x\in|\mathcal{X}|_1} \mathrm{H}^1(\kappa(x), \mathbb{Q}/\mathbb{Z}) \xrightarrow{\mathrm{Cor}_{\mathcal{X}}} \mathrm{H}^1(k, \mathbb{Q}/\mathbb{Z}) \to 0,$$

where $\mathrm{Cor}_{\mathcal{X}} := \sum_{x\in|\mathcal{X}|_1} \frac{N}{e_x} \cdot \mathrm{Cor}_{\kappa(x)/k}$.

Proof. Exactness at $\operatorname{Br} k$ is clear. To ease notation below, for any $i \in \{1, \dots, r\}$, we set $N_i := N/e_i \in \mathbb{N}$ and for any $x \in |\mathcal{X}|_1$, we set $N_x := N/e_x \in \mathbb{N}$.

- Exactness at $\ker\big(\operatorname{Br} k(\mathbb{P}^1) \to \operatorname{Br} k^s(\mathbb{P}^1)\big)$.

  By exactness of the top row in (8.10), it suffices to show that, given some $\alpha = (\alpha_x)_{x\in|\mathcal{X}|_1} \in \ker\mathrm{Cor}$ such that $e_x \cdot \alpha_x = 0$ for all x, one must in fact have that $\alpha_x = 0$ for all x. Suppose we have such an α. We first observe that $\alpha_x = 0$ if $x \notin \{x_1, \dots, x_r\}$ (since then $e_x = 1$), so

  $$\sum_{i=1}^r \alpha_{x_i} = -\sum_{x\notin\{x_1,\dots,x_r\}} \mathrm{Cor}_{\kappa(x)/k}\,\alpha_x = 0,$$

  using that each x_i is a k-point. For any fixed $j \in \{1, \dots, r\}$, this implies that

  $$\alpha_{x_j} = -\sum_{\substack{i=1\\ i\neq j}}^r \alpha_{x_i} \in \mathrm{H}^1(k, \mathbb{Q}/\mathbb{Z})[N_j],$$

  using that the e_i's are pairwise coprime. Since α_{x_j} is e_j-torsion by assumption and $\gcd(e_j, N_j) = 1$, we conclude that $\alpha_j = 0$, proving exactness.

We remark that exactness here proves that $\operatorname{Br}\mathcal{X} \simeq \operatorname{Br} k$ since both groups are the kernel of $\bigoplus_x e_x \cdot \operatorname{res}_{x/\mathbb{P}^1_k} = \bigoplus_x \operatorname{res}_{x/\mathcal{X}}$.

- Exactness at $\bigoplus_{x\in|\mathcal{X}|_1} \mathrm{H}^1(\kappa(x), \mathbb{Q}/\mathbb{Z})$.

  Suppose we are given some $\alpha = (\alpha_x)_{x\in|\mathcal{X}|_1} \in \ker \operatorname{Cor}_{\mathcal{X}}$. We need to show there exists some $\beta = (\beta_x)_{x\in|\mathcal{X}|_1} \in \ker \operatorname{Cor}$ such that $\alpha_x = e_x \cdot \beta_x$ for all x. For any $x \notin \{x_1, \dots, x_r\}$ we (necessarily) set $\beta_x = \alpha_x$. For any $j \in \{1, \dots, r\}$, choose integers a_j, b_j such that $a_jN_j + 1 = b_je_j$ and set

$$\beta_{x_j} \coloneqq b_j\alpha_{x_j} + a_j \sum_{x\neq x_j} \frac{N_x}{e_j} \operatorname{Cor}_{\kappa(x)/k} \alpha_x \in \mathrm{H}^1(k, \mathbb{Q}/\mathbb{Z}).$$

  Then,

$$e_j \cdot \beta_{x_j} = (a_jN_j + 1)\alpha_{x_j} + a_j \sum_{x\neq x_j} N_x\alpha_x = \alpha_{x_j} + a_j \operatorname{Cor}_{\mathcal{X}}(\alpha) = \alpha_{x_j}.$$

  Hence, we only need verify that $\beta = (\beta_x)_{x\in|\mathcal{X}|_1}$ satisfies $\sum_x \operatorname{Cor}_{\kappa(x)/k} \beta_x = 0$. At this point we note that, by construction, $\sum_{i=1}^r a_iN_i$ is congruent to $-1 \pmod{e_j}$ for all j and so must be congruent to $-1 \pmod N$ as well. Write $1 + \sum_{i=1}^r a_iN_i = mN$ for some $m \in \mathbb{Z}$. Finally,

$$\begin{aligned}
\sum_x \operatorname{Cor}_{\kappa(x)/k} \beta_x &= \sum_{i=1}^r \left(b_i\alpha_{x_i} + a_i \sum_{x\neq x_i} \frac{N_x}{e_i} \operatorname{Cor}_{\kappa(x)/k} \alpha_x \right) + \sum_{x\notin\{x_1,\dots,x_r\}} \operatorname{Cor}_{\kappa(x)/k} \alpha_x \\
&= \sum_{i=1}^r \left(b_i + \sum_{\substack{j=1\\ j\neq i}}^r \frac{a_jN_i}{e_j} \right) \alpha_{x_i} + \sum_{x\notin\{x_1,\dots,x_r\}} \left(1 + \sum_{i=1}^r a_iN_i\right) \operatorname{Cor}_{\kappa(x)/k} \alpha_x \\
&= \sum_{i=1}^r \left(\frac{a_iN_i + 1}{e_i} + \sum_{\substack{j=1\\ j\neq i}}^r \frac{a_jN_j}{e_i} \right) \alpha_{x_i} + \sum_{x\notin\{x_1,\dots,x_r\}} \left(1 + \sum_{i=1}^r a_iN_i\right) \operatorname{Cor}_{\kappa(x)/k} \alpha_x \\
&= \sum_{i=1}^r \frac{1 + \sum_{j=1}^r a_jN_j}{e_i} \alpha_{x_i} + + \sum_{x\notin\{x_1,\dots,x_r\}} \left(1 + \sum_{i=1}^r a_iN_i\right) \operatorname{Cor}_{\kappa(x)/k} \alpha_x \\
&= m \sum_{i=1}^r N_i \cdot \alpha_{x_i} + m \sum_{x\notin\{x_1,\dots,x_r\}} N \cdot \operatorname{Cor}_{\kappa(x)/k} \alpha_x \\
&= m \operatorname{Cor}_{\mathcal{X}}(\alpha) \\
&= 0.
\end{aligned}$$

- Exactness at $\mathrm{H}^1(k, \mathbb{Q}/\mathbb{Z})$.

  Fix some $\sigma \in \mathrm{H}^1(k, \mathbb{Q}/\mathbb{Z})$. Because Cor is surjective, there exists some $\alpha = (\alpha_x)_x \in \bigoplus_x \mathrm{H}^1(\kappa(x), \mathbb{Q}/\mathbb{Z})$ such that $\sum_x \operatorname{Cor}_{\kappa(x)/k} \alpha_x = \sigma$. If $r = 0$, then $\operatorname{Cor}_{\mathcal{X}}(\alpha) = \operatorname{Cor}(\alpha) = \sigma$ and we win, so assume $r \geq 1$. Then, $(N_1, N_2, \dots, N_r) = (1)$ so there exists integers $a_1, \dots, a_r$ such that $a_1N_1 + \cdots + a_rN_r = 1 - N$. Set $\beta = (\beta_x)_x$ where

$$\beta_x = \begin{cases} \alpha_x & \text{if } x \notin \{x_1, \dots, x_r\} \\ e_i\alpha_{x_i} + a_1\sigma & \text{if } x = x_i \end{cases} \in \mathrm{H}^1(\kappa(x), \mathbb{Q}/\mathbb{Z}).$$

  Then, by construction, $\operatorname{Cor}_{\mathcal{X}}(\beta) = N \operatorname{Cor}(\alpha) + (a_1N_1 + \cdots + a_rN_r)\,\sigma = \sigma$. ■

Example 8.34 ($\operatorname{Br}(\mathcal{Y}(1) /\!\!/ \mu_2)$)**.** Let k be a perfect field with $\operatorname{char} k \nmid 6$, and let $\mathcal{Y}(1)$ denote the moduli stack of elliptic curves. Consider the rigidification $\mathcal{Y} \coloneqq \mathcal{Y}(1)_k /\!\!/ \mu_2$ (defined as in [AOV08, Theorem A.1]), so $\mathcal{Y} \simeq \sqrt[3]{0/\mathbb{A}^1} \times_{\mathbb{A}^1} \sqrt[2]{1728/\mathbb{A}^1}$. Let $\mathcal{X} \coloneqq \sqrt[3]{0/\mathbb{P}^1} \times_{\mathbb{P}^1} \sqrt[2]{1728/\mathbb{P}^1}$, a natural compactification of $\mathcal{Y}$. Then, the bottom row of (8.5) (using that $\mathrm{H}^2(\mathbb{A}^1_k, \eta_*\mathbb{G}_m) \simeq \operatorname{Br} k(\mathbb{A}^1)$ when k is perfect, e.g. by [CTS21, Proof of

Theorem 3.6.1]) shows that $\operatorname{Br}\mathcal{Y} \subset \operatorname{Br} k(\mathbb{A}^1) = \operatorname{Br} k(\mathbb{P}^1)$ consists of the Brauer classes for which the residue maps $\operatorname{res}_{x/\mathcal{X}}$ vanish for all $x \neq \infty \in |\mathcal{X}|_1$. With this in mind, Proposition 8.33 gives rise to the exact sequence

$$0 \longrightarrow \operatorname{Br}\mathcal{Y} \longrightarrow \operatorname{Br} k(\mathbb{P}^1) \xrightarrow{\operatorname{res}_{\infty/\mathbb{P}^1_k}} \mathrm{H}^1(k, \mathbb{Q}/\mathbb{Z}) \xrightarrow{6} \mathrm{H}^1(k, \mathbb{Q}/\mathbb{Z}) \longrightarrow 0$$

from which one deduces that $\operatorname{Br}\mathcal{Y}$ is an extension

$$0 \longrightarrow \operatorname{Br} k \longrightarrow \operatorname{Br}\mathcal{Y} \xrightarrow{\operatorname{res}_{\infty/\mathbb{P}^1_k}} \mathrm{H}^1(k, \mathbb{Z}/6\mathbb{Z}) \longrightarrow 0.$$

This extension is furthermore split since $\mathcal{Y}(k) \neq \emptyset$. △

Question 8.35. *Is there an analogous Faddeev-type sequence when $\mathcal{X}$ is not assumed to be generically scheme-y?*

Remark 8.36. Let k be a field of characteristic not 2 or 3, let $\mathcal{X} = \mathcal{X}(1)_k$ and let $\mathcal{Y} = \mathcal{Y}(1)_k \hookrightarrow \mathcal{X}$. Then, Corollary 9.1.3 and Theorem 9.2.5 show that $\operatorname{Br}\mathcal{X} = \operatorname{Br} k$ and that there is an exact sequence

$$0 \longrightarrow \operatorname{Br}\mathcal{X} \longrightarrow \operatorname{Br}\mathcal{Y} \longrightarrow \mathrm{H}^1(k, \mathbb{Z}/12\mathbb{Z}) \longrightarrow 0.$$

This suggests that, attached to the point $\infty \in \mathcal{X}(k)$, there could be a residue map "$\operatorname{res}_{\infty/\mathcal{X}}$": $\operatorname{Br} B\mu_{2,k(\mathbb{P}^1)} \longrightarrow \mathrm{H}^1(k, \mathbb{Z}/12\mathbb{Z})$. However, I have failed to come up with a satisfactory construction of such a map (let alone to appropriately define such residue maps everywhere in order to attain a Faddeev-type exact sequence). Note that, letting $j\colon B\mu_{2,k(t)} \to \mathcal{X}$ be the inclusion of the generic gerbe, Proposition 8.15 does provide a residue map

$$\mathrm{H}^2(\mathcal{X}, j_*\mathbb{G}_m) \longrightarrow \mathrm{H}^1(B\mu_{2,\kappa(\infty)}, \mathbb{Q}/\mathbb{Z}) \simeq \mathrm{H}^1(k, \mathbb{Q}/\mathbb{Z}) \oplus \operatorname{Hom}(\mu_{2,k}, \mathbb{Q}/\mathbb{Z}) \simeq \mathrm{H}^1(k, \mathbb{Q}/\mathbb{Z}) \oplus \mathbb{Z}/2\mathbb{Z},$$

but it is not a priori obvious to the author that the image of this map should land in $\mathrm{H}^1(k, \mathbb{Z}/12\mathbb{Z})$. ◦

9. Examples

9.1. The Brauer Group of $\mathcal{X}(1)$.

We compute the Brauer group of $\mathcal{X}(1)$, the moduli stack of generalized elliptic curves. We first do this over $\mathbb{Z}[1/6]$-schemes since $\mathcal{X}(1)$ is tame away from characteristics 2 and 3. Afterwards, we show that the techniques of this paper can also be applied to compute $\operatorname{Br}\mathcal{X}(1)_{\mathbb{Z}[1/2]}$ despite the fact that $\mathcal{X}(1)$ is (very mildly) wild in characteristic 3.

9.1.1. *over* $\mathbb{Z}[1/6]$. Fix any noetherian scheme $S/\mathbb{Z}[1/6]$, and let $\mathcal{X} = \mathcal{X}(1)_S$. Let $f\colon \mathcal{X} \to S$ and $g\colon \mathbb{P}^1_S \to S$ denote their structure maps. We will compute $\operatorname{Br}\mathcal{X}$ using the Leray spectral sequence

$$E_2^{ij} = \mathrm{H}^i(S, \mathrm{R}^j f_*\mathbb{G}_m) \implies \mathrm{H}^{i+j}(\mathcal{X}, \mathbb{G}_m). \tag{9.1}$$

Lemma 9.1.1. *$f_*\mathbb{G}_m \simeq \mathbb{G}_m$, $\mathrm{R}^1 f_*\mathbb{G}_m \simeq \underline{\mathbb{Z}}$ and $\mathrm{R}^2 f_*\mathbb{G}_m \simeq \mathrm{R}^2 g_*\mathbb{G}_m$.*

Proof. The fact that $f_*\mathbb{G}_m \simeq \mathbb{G}_m$ follows from the fact that $\mathcal{X}$'s coarse moduli space is $\mathbb{P}^1_S$ (for this, one can either argue as in [FO10, Lemma 4.4] or use the fact that formation of coarse spaces commutes with arbitrary base change in the tame setting [AOV08, Coroarlly 3.3(a)]). The fact that $\mathrm{R}^1 f_*\mathbb{G}_m \simeq \underline{\mathbb{Z}}$ is a consequence of [FO10, Theorem 1.3] which shows that $\operatorname{Pic}\mathcal{X}_T \simeq \mathrm{H}^0(T, \mathbb{Z}) \times \operatorname{Pic}(T)$ for any T/S.

Finally, $\mathrm{R}^2 f_*\mathbb{G}_m = \mathrm{R}^2 g_*\mathbb{G}_m$ because Example 4.17 showed that, at the level of stalks, the natural map $\mathrm{R}^2 g_*\mathbb{G}_m \to \mathrm{R}^2 f_*\mathbb{G}_m$, is an isomorphism *except* this argument did not establish the existence of the exact sequence (4.5). To remedy this, set $\mathcal{Y} = \mathcal{X}/\!\!/\mu_2$ and then appeal to Lemma 7.2. Noting that $\mathcal{Y} \simeq \sqrt[3]{0/\mathbb{P}^1} \times_{\mathbb{P}^1} \sqrt[2]{1728/\mathbb{P}^1}$ (because 6 is invertible on the base), one can use Lemma 6.9 twice to compute that $\mathrm{R}^1\rho_*\mathbb{G}_m \simeq \underline{\mathbb{Z}/3\mathbb{Z}}_0 \oplus \underline{\mathbb{Z}/2\mathbb{Z}}_{1728}$ (ρ as in Lemma 7.2) and can use Lemma 6.8 to compute that $\mathrm{R}^2\rho_*\mathbb{G}_m = 0$. Thus, in this case, the exact sequence of Lemma 7.2 exactly produces the exact sequence (4.5). ■

Corollary 9.1.2. *There is an exact sequence*

$$0 \longrightarrow \mathrm{H}^2(S, \mathbb{G}_m) \xrightarrow{f^*} \ker\big(\mathrm{H}^2(\mathcal{X}, \mathbb{G}_m) \to \mathrm{H}^0(S, \mathrm{R}^2 g_*\mathbb{G}_m)\big) \longrightarrow \mathrm{H}^1(S, \mathbb{Z}) \longrightarrow 0$$

Proof. This comes from the exact sequence of low degree terms in the Leray spectral sequence (9.1), using that $f^*\colon \mathrm{H}^i(S,\mathbb{G}_m)\to \mathrm{H}^i(\mathcal{X},\mathbb{G}_m)$ is injective for all i (in particular, for $i=2,3$) since $\mathcal{X}(S)\neq\emptyset$ (the cuspidal point, corresponding to a nodal cubic, is defined over $\mathbb{Z}$ and so over S). ■

Corollary 9.1.3. *For any noetherian scheme $S/\mathbb{Z}[1/6]$,*

$$\mathrm{Br}'\,\mathcal{X}(1)_S\simeq \mathrm{Br}'\,S.$$

Proof. By [Shi21, Lemma A.3], $\mathrm{H}^1(S,\mathbb{Z})$ is torsion-free. As a consequence of [Gab81, Chapter II, Theorem 2], which states that $\mathrm{Br}'\,S\xrightarrow{\sim}\mathrm{Br}'\,\mathbb{P}^1_S$ for any scheme S, one also sees that stalks (and so global sections) of $\mathrm{R}^2g_*\mathbb{G}_m$ are torsion-free. Take torsion in the exact sequence in Corollary 9.1.2. ■

9.1.2. *over* $\mathbb{Z}[1/2]$. Now let S be any noetherian $\mathbb{Z}[1/2]$-scheme. Still let $\mathcal{X}=\mathcal{X}(1)_S\xrightarrow{f}S$. We will compute $\mathrm{Br}\,\mathcal{X}$ by again first computing it in the case that S is strictly local and then leveraging this to compute it for general sense. In the first step (with S strictly local), it will no longer suffice to just apply Corollary 4.16 because $\mathcal{X}$ is no longer tame (if $S_{\mathbb{F}_3}\neq\emptyset$).

Example 9.1.4. Consider the elliptic curve $E_0\colon y^2=x^3-x$ over $S_{\mathbb{F}_3}$. One can check (similarly as in [Sil09, Appendix A]) that $\underline{\mathrm{Aut}}(E_0)\simeq\underline{\mathbb{Z}/3\mathbb{Z}}\rtimes\mu_4$, where $1\in\mathbb{Z}/3\mathbb{Z}$ acts via $(x,y)\mapsto(x+1,y)$ and $\zeta\in\mu_4$ acts via $(x,y)\mapsto(\zeta^2x,\zeta y)$. In particular, $\underline{\mathrm{Aut}}(E_0)$ is a finite étale group of order divisible by 3, so $\mathcal{X}$ is not tame in characteristic 3. △

On the other hand, [Sil09, Proposition A.1.2] shows that the above example captures essentially all of the non-tameness of $\mathcal{X}$.[5] Let $\pi\colon\mathcal{X}\to\mathbb{P}^1_S$ be the coarse space of $\mathcal{X}=\mathcal{X}(1)_S$, given by the j-invariant. Theorem 4.14, applied to the open tame locus of $\mathcal{X}$, shows that $\mathrm{R}^2\pi_*\mathbb{G}_m$ is supported on the closed subscheme $i_0\colon S_{\mathbb{F}_3}\hookrightarrow\mathbb{P}^1_S$ of $\mathbb{P}^1_S$ defined by $j=0=3$. We claim that, in fact, $\mathrm{R}^2\pi_*\mathbb{G}_m=0$; because it is supported along i_0, to prove this it will suffice to show that its stalks over i_0 vanish.

Lemma 9.1.5. *Let M be an abelian group with S_3-action. Then, the natural map*

$$\mathrm{H}^n_0(\mathbb{Z}/2\mathbb{Z},M^{\mathbb{Z}/3\mathbb{Z}})\longrightarrow\mathrm{H}^n_0(S_3,M)$$

is an isomorphism for all $n\equiv1,2\pmod 4$. In particular, $\mathrm{H}^n_0(S_3,M)$ is 2-torsion for such n.

Proof. This will follow from the Hochschild–Serre spectral sequence

$$E_2^{ij}=\mathrm{H}^i_0(\mathbb{Z}/2\mathbb{Z},\mathrm{H}^j_0(\mathbb{Z}/3\mathbb{Z},M))\implies\mathrm{H}^{i+j}_0(S_3,M)$$

once we know that $E_2^{ij}=0$ if $i,j\geq1$ or if $i=0$ and $j\equiv1,2\pmod 4$. When $i,j\geq1$, E_2^{ij} must be both 2-torsion and 3-torsion, so it must vanish. Consider $E_2^{0j}=\mathrm{H}^j_0(\mathbb{Z}/3\mathbb{Z},M)^{\mathbb{Z}/2\mathbb{Z}}$ when $j\equiv1,2\pmod 4$. By [Wei94, Example 6.7.10], $\mathbb{Z}/2\mathbb{Z}$ acts on both $\mathrm{H}^j_0(\mathbb{Z}/3\mathbb{Z},M)$ via multiplication by -1 (if $j\equiv1,2\pmod 4$). Thus, $E_2^{0j}=\mathrm{H}^j_0(\mathbb{Z}/3\mathbb{Z},M)[2]=0$. ■

Proposition 9.1.6. *Let R be a strictly local $\mathbb{F}_3$-algebra, and define $\mathcal{Y}$ via the following pullback square:*

$$\begin{tikzcd}\mathcal{Y}\arrow[r]\arrow[d] & \mathcal{Y}(1)\arrow[d] & \subset & \mathcal{X}(1)\arrow[d,"\pi"]\\ \operatorname{Spec}R\arrow[r,"0"] & \mathbb{A}^1 & \subset & \mathbb{P}^1.\end{tikzcd}$$

Then, $\mathrm{H}^2(\mathcal{Y},\mathbb{G}_m)=0$.

Proof. Note that $\mathcal{Y}$ is 'the moduli stack of elliptic curves with constant j-invariant 0, in characteristic 3.' Following the strategy of [AM20], we compute $\mathrm{Br}\,\mathcal{Y}$ using a presentation/smooth cover for $\mathcal{Y}$ derived from

[5]The cusp of $\mathcal{X}(1)$ represents a Néron 1-gon C (i.e. a nodal cubic viewed as a generalized elliptic curve), and one can check that its automorphism group of $\underline{\mathbb{Z}/2\mathbb{Z}}$, with generator acting by inversion on the smooth locus $\mathbb{G}_m\subset C$.

$$\begin{tikzcd}
0 & & & \\
\mu_2(R) & \mu_2(R) \arrow[rrd, "d_2^{1,1}"] & & \\
\mathbb{G}_m(Y')^{S_3} & \mu_2(R) & 0 & \mathrm{H}^3_0(S_3, \mathbb{G}_m(Y'))
\end{tikzcd}$$

FIGURE 3. The E_2-page of the Hochschild–Serre spectral sequence computing the $\mathbb{G}_m$-cohomology of $\mathcal{Y} = [\mathcal{Y}'/S_3]$.

the Legendre family $\mathbb{A}^1 \setminus \{0, 1\} \to \mathcal{Y}(1)$. That is, we consider the following commutative diagram (all of whose squares are Cartesian):

$$\begin{tikzcd}
 & Y' \arrow[r] \arrow[d] & \mathbb{A}^1_R \setminus \{0,1\} \arrow[d, "y^2=x(x-1)(x-\lambda)"] & =: & Y(2) \arrow[ddd, bend left, "2^8\frac{(\lambda^2-\lambda+1)^3}{\lambda^2(\lambda-1)^2}"] \\
B\underline{\mathbb{Z}/2\mathbb{Z}}_{Y'} \arrow[r, equal] & \mathcal{Y}' \arrow[r] \arrow[d] & \mathcal{Y}(2)_R \arrow[r, equal] \arrow[d, "S_3\text{-torsor}"] & B\underline{\mathbb{Z}/2\mathbb{Z}}_{Y(2)} & \\
 & \mathcal{Y} \arrow[r] \arrow[d] & \mathcal{Y}(1)_R \arrow[d, "\pi"] & & \\
 & \operatorname{Spec} R \arrow[r, "0"] & \mathbb{A}^1_R & =: & Y(1).
\end{tikzcd}$$

Above, one can directly compute that

$$Y' \simeq \operatorname{Spec} R[\lambda]/(\lambda^2 - \lambda + 1)^3 = \operatorname{Spec} R[\varepsilon]/(\varepsilon^6) \text{ where } \varepsilon = \lambda + 1.$$

Furthermore, by [AM20, Below Corollary 4.6], $S_3 \curvearrowright Y(2) = \operatorname{Spec} R[\lambda][1/\lambda(\lambda - 1)]$ via

$$\sigma = (132)\colon \lambda \mapsto \frac{\lambda - 1}{\lambda} \text{ and } \tau = (23)\colon \lambda \mapsto \lambda^{-1}. \tag{9.2}$$

With all of this set up, consider the Hochschild–Serre spectral sequence

$$E_2^{ij} = \mathrm{H}^i_0(S_3, \mathrm{H}^j(\mathcal{Y}', \mathbb{G}_m)) \implies \mathrm{H}^{i+j}(\mathcal{Y}, \mathbb{G}_m). \tag{9.3}$$

Its E_2-page is pictured in Fig. 3, whose contents are justified in claims **(1)**–**(3)** below. To begin computing this spectral sequence, we first remark that $\mathbb{G}_m(\mathcal{Y}') = \mathbb{G}_m(Y')$ sits in a short exact sequence

$$0 \longrightarrow \underbrace{R\varepsilon \oplus R\varepsilon^2 \oplus R\varepsilon^3 \oplus R\varepsilon^4 \oplus R\varepsilon^5}_{R^{\oplus 5}} \longrightarrow \mathbb{G}_m(Y') \xrightarrow{\varepsilon=0} R^\times \longrightarrow 1, \tag{9.4}$$

which becomes S_3-equivariant when $S_3 \curvearrowright R^\times$ trivially.[6]

(1) Claim: $\mathrm{H}^2(\mathcal{Y}', \mathbb{G}_m) = 0$ so $E_2^{i2} = 0$ for all $i \geq 0$.

Applying [AM20, Proposition 3.2] (or Proposition 3.9 since $\mu_2 = \underline{\mathbb{Z}/2\mathbb{Z}}$ over Y') to $\mathcal{Y}' = B\underline{\mathbb{Z}/2\mathbb{Z}}_{Y'}$ shows that

$$\mathrm{H}^2(\mathcal{Y}', \mathbb{G}_m) \simeq \mathrm{H}^2(Y', \mathbb{G}_m) \oplus \mathrm{H}^1(Y', \mu_2). \tag{9.5}$$

[6]The particular S_3-action on $R^{\oplus 5} = R\varepsilon \oplus R\varepsilon^2 \oplus R\varepsilon^3 \oplus R\varepsilon^4 \oplus R\varepsilon^5$ is not directly important for the present argument, but for the sake of completeness, $\sigma = (132)$ and $\tau = (23)$ act on $R^{\oplus 5}$ via the matrices

$$\sigma\colon \begin{pmatrix} 1 & & & & \\ 1 & 1 & & & \\ 1 & -1 & 1 & & \\ 1 & 0 & 0 & 1 & \\ 1 & 1 & 0 & 1 & 1 \end{pmatrix} \text{ and } \tau\colon \begin{pmatrix} -1 & & & & \\ -1 & 1 & & & \\ -1 & -1 & -1 & & \\ -1 & 0 & 0 & 1 & \\ -1 & 1 & 0 & 1 & -1 \end{pmatrix}.$$

Because $Y' \simeq \operatorname{Spec} R[\varepsilon]/(\varepsilon^6)$ is affine, [CTS21, proof of Proposition 3.2.5] (or Lemma 4.2) shows that $\mathrm{H}^2(Y', \mathbb{G}_m) \simeq \mathrm{H}^2(R, \mathbb{G}_m) = 0$ and that $\mathrm{H}^1(Y', \mathbb{G}_m) \simeq \mathrm{H}^1(R, \mathbb{G}_m) = 0$. The Kummer sequence then shows that $\mathrm{H}^1(Y', \mu_2) \simeq \mathbb{G}_m(Y')/2$. Applying the snake lemma to

$$\begin{tikzcd} 0 \arrow[r] & R^{\oplus 5} \arrow[r] \arrow[d, "2"] & \mathbb{G}_m(Y') \arrow[r, "\varepsilon=0"] \arrow[d, "2"] & R^\times \arrow[r] \arrow[d, "2"] & 0 \\ 0 \arrow[r] & R^{\oplus 5} \arrow[r] & \mathbb{G}_m(Y') \arrow[r, "\varepsilon=0"] & R^\times \arrow[r] & 0 \end{tikzcd}$$

shows that $\mathbb{G}_m(Y')/2 = 0$ because $R/2 = 0$ (since $2 \in R^\times$) and $R^\times/2 = 0$ (since R is strictly henselian). Thus, $\mathrm{H}^1(Y', \mu_2) = 0$ as well, so (9.5) shows that $\mathrm{H}^2(\mathcal{Y}', \mathbb{G}_m) = 0$.

(2) Claim: $\mathrm{H}^1(\mathcal{Y}', \mathbb{G}_m) \simeq \mu_2(R)$, so $S_3 \curvearrowright \mathrm{H}^1(\mathcal{Y}', \mathbb{G}_m)$ trivially. Consequently, $E_2^{01} = \mu_2(R) = E_2^{11}$.

As before, [AM20, Proposition 3.2] (or Proposition 3.9) shows that $\mathrm{H}^1(\mathcal{Y}', \mathbb{G}_m) \simeq \mathrm{H}^1(Y', \mathbb{G}_m) \oplus \mathrm{H}^0(Y', \mu_2)$. While proving **(1)**, we showed that $\mathrm{H}^1(Y', \mathbb{G}_m) = 0$, so $\mathrm{H}^1(\mathcal{Y}', \mathbb{G}_m) \simeq \mathrm{H}^0(Y', \mu_2) = \mu_2(R[\varepsilon]/(\varepsilon^6)) = \mu_2(R)$. Finally, the S_3-action is trivial because $\mu_2(R) = \{\pm 1\}$ has no non-trivial automorphisms. Finally, $E_2^{01} = \mu_2(R)^{S_3} = \mu_2(R)$ and $E_2^{11} = \mathrm{H}_0^1(S_3, \mu_2(R)) = \operatorname{Hom}(S_3, \mu_2(R)) = \operatorname{Hom}(\mathbb{Z}/2\mathbb{Z}, \mu_2(R)) \simeq \mu_2(R)$.

(3) Claim: $E_2^{10} = \mu_2(R)$ and $E_2^{20} = 0$.

We first claim that $\mathrm{H}_0^1(S_3, \mathbb{G}_m(Y')) \xrightarrow{\sim} \mathrm{H}_0^1(S_3, R^\times)$ and that $\mathrm{H}_0^2(S_3, \mathbb{G}_m(Y')) \hookrightarrow \mathrm{H}_0^2(S_3, R^\times)$. Taking S_3-cohomology of the exact sequence (9.4) shows that these both follow from knowing that $\mathrm{H}_0^i(S_3, R^{\oplus 5}) = 0$ for $i = 1, 2$. Lemma 9.1.5 implies that $\mathrm{H}_0^i(S_3, R^{\oplus 5})$ is 2-torsion (for $i = 1, 2$), but $R^{\oplus 5}$ itself is 3-torsion, so these groups must be killed by both 2 and 3; thus, they must vanish. Therefore,

$$E_2^{10} = \mathrm{H}_0^1(S_3, \mathbb{G}_m(Y')) \xrightarrow{\sim} \mathrm{H}_0^1(S_3, R^\times) = \operatorname{Hom}(S_3, R^\times) = \operatorname{Hom}(\mathbb{Z}/2\mathbb{Z}, R^\times) = \mu_2(R)$$

and

$$E_2^{20} = \mathrm{H}_0^2(S_2, \mathbb{G}_m(Y')) \hookrightarrow \mathrm{H}_0^2(S_3, R^\times) \overset{\text{Lemma 9.1.5}}{=} \mathrm{H}_0^2(\mathbb{Z}/2\mathbb{Z}, R^\times) = R^\times/2 = 0.$$

By Fig. 3, in order to conclude that $\mathrm{H}^2(\mathcal{Y}, \mathbb{G}_m) = 0$, it suffices to show that the differential

$$d_2^{1,1} = \mathrm{d}_2^{1,1}(R)\colon \mu_2(R) \longrightarrow \mathrm{H}_0^3(S_3, \mathbb{G}_m(Y'))$$

is injective (equivalently, nonzero). Of course, it is enough to know that the composition $\mu_2(R) \xrightarrow{\mathrm{d}_2^{1,1}} \mathrm{H}_0^3(S_3, \mathbb{G}_m(Y')) \to \mathrm{H}_0^3(S_3, R^\times)$ is nonzero. Since S_3 acts trivially on $R^\times$, one knows (see e.g. [AM20, Lemma 5.2]) that $\mathrm{H}_0^3(S_3, R^\times) = \mu_6(R)$. Since $\mu_2(R)$ is 2-torsion, this composition factors through a map

$$\varphi = \varphi_R\colon \mu_2(R) \to \mu_6(R)[2] = \mu_2(R) \subset \mathrm{H}_0^3(S_3, R^\times).$$

Now, the spectral sequence (9.3) (and so also the above map φ_R) is functorial in R by construction. With this in mind, consider the sequence of maps $\mathbb{Q}_3 \leftarrow \mathbb{Z}_3 \to \mathbb{F}_3 \to R$. Functoriality of φ gives rise to a commutative diagram

$$\begin{tikzcd} \mu_2(R) \arrow[r, "\varphi_R"] & \mu_2(R) \\ \mu_2(\mathbb{F}_3) \arrow[r, "\varphi_{\mathbb{F}_3}"] \arrow[u, "\parallel"'] & \mu_2(\mathbb{F}_3) \arrow[u, "\parallel"'] \\ \mu_2(\mathbb{Z}_3) \arrow[r, "\varphi_{\mathbb{Z}_3}"] \arrow[u, "\parallel"'] \arrow[d, "\parallel"'] & \mu_2(\mathbb{Z}_3) \arrow[u, "\parallel"'] \arrow[d, "\parallel"'] \\ \mu_2(\mathbb{Q}_3) \arrow[r, "\varphi_{\mathbb{Q}_3}"] & \mu_2(\mathbb{Q}_3) \end{tikzcd}$$

Thus, it suffices to show that $\varphi_{\mathbb{Q}_3}$ is nonzero. Let $K = \mathbb{Q}_3$. We analyze the analogous spectral sequence $\mathrm{H}^i_0(S_2, \mathrm{H}^j(\mathcal{Y}'_K, \mathbb{G}_m)) \implies \mathrm{H}^{i+j}(\mathcal{Y}_K, \mathbb{G}_m)$ over K. Since $\operatorname{char} K = 0$, we know that $\mathrm{H}^2(\mathcal{Y}_K, \mathbb{G}_m) = 0$ (e.g. because it is a stalk of the sheaf $\mathrm{R}^2\pi_*\mathbb{G}_m$ of Theorem 4.14 applied to the tame stack $\mathcal{X}(1)_K$). This vanishing implies that the differential

$$\mathrm{d}_2^{1,1}(K)\colon\ \mathrm{H}^1_0(S_3, \operatorname{Pic} \mathcal{Y}'_K) \longrightarrow \mathrm{H}^3_0(S_3, \mathbb{G}_m(Y'_K))$$

must be injective. At the same time, because K is a $\mathbb{Q}$-algebra, the exact sequence $0 \to K^{\oplus 5} \to \mathbb{G}_m(Y'_K) \to K^\times \to 1$, analogous to (9.4), shows that $\mathrm{H}^1_0(S_3, \mathbb{G}_m(Y'_K)) \xrightarrow{\sim} \mathrm{H}^1_0(S_3, K^\times) = \mu_2(K)$ and $\mathrm{H}^3_0(S_3, \mathbb{G}_m(Y'_K)) \xrightarrow{\sim} \mathrm{H}^3_0(S_3, K^\times) = \mu_6(K)$. Furthermore, one can argue as in the earlier claim **(2)** to see that $\mathrm{H}^1_0(S_3, \operatorname{Pic} \mathcal{Y}'_K) \simeq \mu_2(K)$. Thus, the composition

$$\mu_2(K) \simeq \mathrm{H}^1_0(S_3, \operatorname{Pic} \mathcal{Y}'_K) \xrightarrow{\mathrm{d}_2^{1,1}(K)} \mathrm{H}^3_0(S_3, \mathbb{G}_m(Y'_K)) \xrightarrow{\sim} \mathrm{H}^3_0(S_3, K^\times)$$

must be injective, and so φ_K is injective as well (recall the above composition factors through φ_K). This completes the proof. ∎

Corollary 9.1.7. $\mathrm{R}^2\pi_*\mathbb{G}_m = 0$.

Proof. As remarked above Lemma 9.1.5, Theorem 4.14 shows that $\mathrm{R}^2\pi_*\mathbb{G}_m$ is supported on the closed subscheme $i_0\colon S_{\mathbb{F}_3} \hookrightarrow \mathbb{P}^1_S$ defined by $j = 0 = 3$. On top of this, Proposition 9.1.6 shows that all stalks of $\mathrm{R}^2\pi_*\mathbb{G}_m$ along this subscheme vanish as well, so $\mathrm{R}^2\pi_*\mathbb{G}_m = 0$. ∎

With this in place, we can now extend Corollary 9.1.3 to $\mathbb{Z}[1/2]$-schemes. Recall that $f\colon \mathcal{X} = \mathcal{X}(1)_S \to S$ and $g\colon \mathbb{P}^1_S \to S$ are their structure morphisms. As before,

$$f_*\mathbb{G}_m \simeq \mathbb{G}_m \ \text{ and } \ \mathrm{R}^1 f_*\mathbb{G}_m \simeq \underline{\mathbb{Z}}$$

because $\mathcal{X}$'s coarse space is $\mathbb{P}^1_S$ and as a consequence of [FO10, Theorem 1.3].

Lemma 9.1.8. $\mathrm{R}^2 f_*\mathbb{G}_m \simeq \mathrm{R}^2 g_*\mathbb{G}_m$.

Proof. It suffices to show that $\mathrm{H}^2(\mathcal{X}(1)_R, \mathbb{G}_m) = \mathrm{H}^2(\mathbb{P}^1_R, \mathbb{G}_m)$ for any noetherian strictly local $\mathbb{Z}[1/2]$-algebra R. Let R be such a ring and let $\pi\colon \mathcal{X}(1)_R \to \mathbb{P}^1_R$ be the coarse space map. Using that $\mathrm{R}^2\pi_*\mathbb{G}_m = 0$ by Corollary 9.1.7, the argument of Example 4.17 goes through for this case as well with the following modification. Let $\mathcal{Y} \coloneqq \mathcal{X}(1)_R /\!\!/ \mu_2 \xrightarrow{\rho} \mathbb{P}^1_R$. The exact sequence (4.5) in Example 4.17 should be replaced with the exact sequence

$$0 \longrightarrow \mathrm{R}^1\rho_*\mathbb{G}_m \longrightarrow \mathrm{R}^1\pi_*\mathbb{G}_m \longrightarrow \mathbb{Z}/2\mathbb{Z} \longrightarrow 0$$

coming from Lemma 7.2. Above, $\mathrm{R}^1\rho_*\mathbb{G}_m$ is supported on the closed subscheme $\operatorname{Spec} R \overset{0}{\hookrightarrow} \mathbb{P}^1_R$ (since ρ is an isomorphism away from this subscheme) and so is acyclic. Hence, one still has $\mathrm{H}^1(\mathbb{P}^1_R, \mathrm{R}^1\pi_*\mathbb{G}_m) \xrightarrow{\sim} \mathrm{H}^1(\mathbb{P}^1_R, \mathbb{Z}/2\mathbb{Z}) = 0$ as is necessary for the rest of the argument of Example 4.17 to apply here. ∎

Corollary 9.1.9. *For any noetherian scheme $S/\mathbb{Z}[1/2]$,*

$$\operatorname{Br}' \mathcal{X}(1)_S \simeq \operatorname{Br}' S.$$

Proof. Using our computations of $\mathrm{R}^i f_*\mathbb{G}_m$ for $i = 0, 1, 2$, we argue exactly as in Corollaries 9.1.2 and 9.1.3. ∎

9.2. **The Brauer Group of $\mathcal{Y}(1)$.** Let $\mathcal{Y}(1)/\mathbb{Z}$ denote the moduli stack of elliptic curves. Let $S/\mathbb{Z}[1/2]$ be a regular noetherian scheme. As our next example, we will compute $\operatorname{Br} \mathcal{Y}(1)_S$, partially generalizing results of [AM20, Mei18, LP22].

Setup 9.2.1. Set $\mathcal{X} \coloneqq \mathcal{Y}(1)_S$ and let $f\colon \mathcal{X} \to S$ be its structure map. Note that, by [FO10, Lemma 4.4], the coarse moduli space of $\mathcal{X}$ is given by the j-invariant $c\colon \mathcal{X} \to \mathbb{A}^1_S$. Let $g\colon \mathbb{A}^1_S \to S$ be its structure map, so we have a commutative triangle

$$\begin{tikzcd} \mathcal{X} \arrow[rr, "c"] \arrow[dr, "f"'] & & \mathbb{A}^1_S \arrow[dl, "g"] \\ & S. & \end{tikzcd}$$

We will compute $\mathrm{H}^2(\mathcal{X}, \mathbb{G}_m)$ by leveraging f's Leray spectral sequence

$$E_2^{ij} = \mathrm{H}^i(S, \mathrm{R}^j f_* \mathbb{G}_m) \implies \mathrm{H}^{i+j}(\mathcal{X}, \mathbb{G}_m). \tag{9.6}$$

Lemma 9.2.2. $f_*\mathbb{G}_m = g_*\mathbb{G}_m = \mathbb{G}_m$.

Proof. The first equality $f_*\mathbb{G}_m = g_*\mathbb{G}_m$ holds simply because $\mathbb{A}^1_S$ is $\mathcal{X}$'s coarse moduli space and this remains true after smooth base change over S. The second equality $g_*\mathbb{G}_m = \mathbb{G}_m$ holds because S is regular and so, at the level of stalks over a geometric point $\overline{s} \to S$, $\mathbb{G}_m(\mathscr{O}_{S,\overline{s}}) \xrightarrow{\sim} \mathbb{G}_m(\mathbb{A}^1_{\mathscr{O}_{S,\overline{s}}})$. ■

Lemma 9.2.3. $\mathrm{R}^1 f_*\mathbb{G}_m \simeq \underline{\mathbb{Z}/12\mathbb{Z}}$ *while* $\mathrm{R}^1 g_*\mathbb{G}_m = 0$. *To fix a particular identification, we insist that the isomorphism* $\mathrm{R}^1 f_*\mathbb{G}_m \xrightarrow{\sim} \underline{\mathbb{Z}/12\mathbb{Z}}$ *maps the class of the Hodge bundle on* $\mathcal{X}$ *to* $1 \in \mathbb{Z}/12\mathbb{Z}$.

Proof. The first claim is a consequence of [FO10, Theorem 1.1] while the second is easily verified on stalks; S is regular so $\operatorname{Pic} \mathbb{A}^1_{\mathscr{O}_{S,\overline{s}}} = 0$ for any geometric point $\overline{s} \to S$. ■

Lemma 9.2.4. $\mathrm{R}^2 g_*\mathbb{G}_m \xrightarrow{\sim} \mathrm{R}^2 f_*\mathbb{G}_m$

Proof. We show that the natural map $\mathrm{R}^2 g_*\mathbb{G}_m \to \mathrm{R}^2 f_*\mathbb{G}_m$ is an isomorphism by checking on stalks, so assume $S = \operatorname{Spec} R$ is a noetherian, regular strictly local $\mathbb{Z}[1/2]$-scheme. In this case, we need to show that

$$\mathrm{H}^2(\mathbb{A}^1_R, \mathbb{G}_m) \xrightarrow{c^*} \mathrm{H}^2(\mathcal{X}, \mathbb{G}_m)$$

is an isomorphism. One can now argue as in Example 4.17, as we briefly explain. Corollary 9.1.7 implies that $\mathrm{R}^2 c_*\mathbb{G}_m = 0$ (because $\mathcal{Y}(1) \overset{\text{open}}{\subset} \mathcal{X}(1)$). Setting $\mathcal{Y} := \mathcal{X} /\!\!/ \mu_2 \xrightarrow{\rho} \mathbb{A}^1_R$, Lemma 7.2 produces an exact sequence

$$0 \longrightarrow \mathrm{R}^1\rho_*\mathbb{G}_m \longrightarrow \mathrm{R}^1 c_*\mathbb{G}_m \longrightarrow \underline{\mathbb{Z}/2\mathbb{Z}} \longrightarrow 0$$

of étale sheaves on $\mathbb{A}^1_R$. Above, $\mathrm{R}^1\rho_*\mathbb{G}_m$ is supported on an R-finite subscheme of $\mathbb{A}^1_R$ and so is acyclic; hence,

$$\mathrm{H}^1(\mathbb{A}^1_R, \mathrm{R}^1 c_*\mathbb{G}_m) \simeq \mathrm{H}^1(\mathbb{A}^1_R, \mathbb{Z}/2\mathbb{Z}) \simeq \mathrm{H}^1(\mathbb{A}^1_R, \mu_2) \simeq \mathbb{G}_m(\mathbb{A}^1_R)/2 \simeq R^\times/2 = 0.$$

Now, the Leray spectral sequence for c yields

$$0 \dashrightarrow \mathrm{H}^2(\mathbb{A}^1_R, \mathbb{G}_m) \xrightarrow{c^*} \mathrm{H}^2(\mathcal{X}, \mathbb{G}_m) \longrightarrow \mathrm{H}^1(\mathbb{A}^1_R, \mathrm{R}^1 c_*\mathbb{G}_m) = 0.$$

Above, c^* is injective e.g. as a consequence of the fact that there exists a dense open subscheme $U \overset{\text{open}}{\subset} \mathbb{A}^1_R$ above which c admits a section. Thus, this exact sequence shows that c^* is an isomorphism, as desired. ■

Theorem 9.2.5. *For any regular noetherian scheme* $S/\mathbb{Z}[1/2]$, *there is a short exact sequence*

$$0 \longrightarrow \mathrm{H}^2(\mathbb{A}^1_S, \mathbb{G}_m) \xrightarrow{c^*} \mathrm{H}^2(\mathcal{Y}(1)_S, \mathbb{G}_m) \xrightarrow{f} \mathrm{H}^1(S, \mathbb{Z}/12\mathbb{Z}) \longrightarrow 0. \tag{9.7}$$

Proof. We compare the Leray spectral sequences for $f\colon \mathcal{X} = \mathcal{Y}(1)_S \to S$ and $g\colon \mathbb{A}^1_S \to S$. Their exact sequences of low degree terms sit in the following commutative diagram

$$\begin{tikzcd}
 & \ker\big(\mathrm{H}^2(\mathbb{A}^1_S,\mathbb{G}_m) \longrightarrow \mathrm{H}^0(S,\mathrm{R}^2g_*\mathbb{G}_m)\big) \arrow[d, phantom, "\shortparallel"] & & \\
0 \arrow[r, dashed] & \mathrm{H}^2(S,\mathbb{G}_m) \arrow[r, "g^*"] \arrow[d, equal] & K_1 \arrow[r] \arrow[d, "c^*"] & 0 & \\
0 \arrow[r, dashed] & \mathrm{H}^2(S,\mathbb{G}_m) \arrow[r, "f^*"] & K_2 \arrow[r] \arrow[d, phantom, "\shortparallel"] & \mathrm{H}^1(S,\mathbb{Z}/12\mathbb{Z}) \arrow[r, "0"] & K \\
 & & \ker\big(\mathrm{H}^2(\mathcal{X},\mathbb{G}_m) \longrightarrow \mathrm{H}^0(S,\mathrm{R}^2f_*\mathbb{G}_m)\big)\,, & &
\end{tikzcd}$$

where $K := \ker\Big(\mathrm{H}^3(S,\mathbb{G}_m) \xrightarrow{f^*} \mathrm{H}^3(\mathcal{X},\mathbb{G}_m)\Big)$. Above, note that $\mathbb{A}^1_S(S) \neq \emptyset$ and also $\mathcal{X}(S) \neq \emptyset$ (e.g. it contains the elliptic curve $y^2 = x^3 - x$), so g^*, f^* are injective and $K = 0$. Hence, the above diagram yields

$$\operatorname{coker}\left(K_1 \overset{c^*}{\hookrightarrow} K_2\right) \simeq \mathrm{H}^1(S,\mathbb{Z}/12\mathbb{Z}). \tag{9.8}$$

We would like to upgrade this to a computation of the cokernel of $\mathrm{H}^2(\mathbb{A}^1_S,\mathbb{G}_m) \to \mathrm{H}^2(\mathcal{X},\mathbb{G}_m)$. Note that, because $\mathcal{X}(S) \neq \emptyset$, there are no nonzero differentials in the Leray spectral sequence (9.6) which map to the $j = 0$ row (and the analogous statement holds for $\mathbb{A}^1_S$). Thus, $E^{0,2}_\infty = \ker\Big(\mathrm{d}_2^{0,2}\colon\ \mathrm{H}^0(S,\mathrm{R}^2f_*\mathbb{G}_m) \to \mathrm{H}^2(S,\mathrm{R}^1f_*\mathbb{G}_m)\Big)$ and the analogous statement holds for $\mathbb{A}^1_S$. Hence, we have the following homomorphism of exact sequences:

$$\begin{tikzcd}
 & K_1 & & & \mathrm{H}^2(S,\mathrm{R}^1g_*\mathbb{G}_m) \arrow[d, equal] \\
0 \arrow[r] & \mathrm{H}^2(S,\mathbb{G}_m) \arrow[r, "g^*"] \arrow[u, "\wr"] \arrow[d] & \mathrm{H}^2(\mathbb{A}^1_S,\mathbb{G}_m) \arrow[r] \arrow[d, "c^*"] & \mathrm{H}^0(S,\mathrm{R}^2g_*\mathbb{G}_m) \arrow[r] \arrow[d] & 0 \\
0 \arrow[r] & K_2 \arrow[r] & \mathrm{H}^2(\mathcal{X},\mathbb{G}_m) \arrow[r] & \mathrm{H}^0(S,\mathrm{R}^2f_*\mathbb{G}_m) \arrow[r, "\mathrm{d}_2^{0,2}"] & \mathrm{H}^2(S,\mathrm{R}^1f_*\mathbb{G}_m).
\end{tikzcd} \tag{9.9}$$

Now, Lemma 9.2.4 shows that $\mathrm{R}^2g_*\mathbb{G}_m \xrightarrow{\sim} \mathrm{R}^2f_*\mathbb{G}_m$, so commutativity shows that $\mathrm{d}_2^{0,2} = 0$ and the above is really a homomorphism of *short* exact sequences. The snake lemma then yields that $c^*\colon\ \mathrm{H}^2(\mathbb{A}^1_S,\mathbb{G}_m) \to \mathrm{H}^2(\mathcal{X},\mathbb{G}_m)$ is injective with cokernel isomorphism to $\operatorname{coker}(K_1 \to K_2)$ which is isomorphic to $\mathrm{H}^1(S,\mathbb{Z}/12\mathbb{Z})$ by (9.8), finishing the proof. ■

To finish this section, we will show that the exact sequence in Theorem 9.2.5 is actually split. Recall that we have fixed a regular, noetherian $\mathbb{Z}[1/2]$-scheme S and that we set $\mathcal{X} = \mathcal{Y}(1)_S \xrightarrow{f} S$. Let $\mathscr{L}$ denote the Hodge bundle on $\mathcal{X}$. Then, the discriminant (of elliptic curves) gives a trivialization $\Delta\colon \mathscr{O}_\mathcal{X} \xrightarrow{\sim} \mathscr{L}^{12}$ of its twelfth power, and we consider the μ_{12}-torsor $\mathcal{T} := \mathcal{X}\Big(\sqrt[12]{\Delta}\Big) \longrightarrow \mathcal{X}$ of 12th roots of this trivialization. We let $\sigma = [\mathcal{T}] \in \mathrm{H}^1_{\mathrm{fppf}}(\mathcal{X},\mu_{12})$ denote its corresponding cohomology class, and we consider the map

$$\begin{array}{cccc} s: & \mathrm{H}^1(S,\mathbb{Z}/12\mathbb{Z}) & \longrightarrow & \mathrm{H}^2(\mathcal{X},\mathbb{G}_m) \\ & \alpha & \longmapsto & f^*\alpha \cup \sigma. \end{array} \tag{9.10}$$

We claim that s is a section of (9.7). As in Section 3.1, to prove this, we will need to wander into derived categories. Define the objects

$$\mathscr{C} := (\tau_{\le 2}\mathrm{R}_{\mathrm{fppf}}f_*\mathbb{G}_m)\,[1] \ \text{ and } \ \mathscr{D} := (\tau_{\le 2}\mathrm{R}_{\mathrm{fppf}}g_*\mathbb{G}_m)\,[1]$$

in the (bounded below) derived category of fppf sheaves on S. Note that Lemmas 9.2.2 to 9.2.4 shows that these sit in a distinguished triangle

$$\mathscr{D} \longrightarrow \mathscr{C} \longrightarrow \overbrace{\underline{\mathbb{Z}/12\mathbb{Z}}}^{\mathrm{R}^1_{\mathrm{fppf}} f_* \mathbb{G}_m} \xrightarrow{+1} . \tag{9.11}$$

Applying $\mathrm{Hom}(\underline{\mathbb{Z}/12\mathbb{Z}}, -)$ to this triangle produces the exact sequence

$$0 \longrightarrow \mathrm{Hom}_S(\mathbb{Z}/12\mathbb{Z}, \mathscr{D}) \longrightarrow \mathrm{Hom}_S(\mathbb{Z}/12\mathbb{Z}, \mathscr{C}) \xrightarrow{\chi} \mathrm{Hom}_S(\mathbb{Z}/12\mathbb{Z}, \mathbb{Z}/12\mathbb{Z}).$$

Furthermore, the identifications

$$\begin{aligned}
\mathrm{Hom}_S(\mathbb{Z}/12\mathbb{Z}, \mathscr{C}) &= \mathrm{Hom}_S(\mathbb{Z}/12\mathbb{Z}, (\tau_{\leq 2}\mathrm{R}_{\mathrm{fppf}} f_* \mathbb{G}_m)[1]) \\
&\simeq \mathrm{Ext}^1_S(\mathbb{Z}/12\mathbb{Z}, \tau_{\leq 2}\mathrm{R}_{\mathrm{fppf}} f_* \mathbb{G}_m) \\
&\simeq \mathrm{Ext}^1_S(\mathbb{Z}/12\mathbb{Z}, \mathrm{R}_{\mathrm{fppf}} f_* \mathbb{G}_m) \\
&\simeq \mathrm{Ext}^1_{\mathcal{X}}(\mathbb{Z}/12\mathbb{Z}, \mathbb{G}_m) \\
&\simeq \mathrm{H}^1_{\mathrm{fppf}}(\mathcal{X}, \mu_{12}) \qquad \text{by Lemma 3.6}
\end{aligned} \tag{9.12}$$

and the similarly obtained $\mathrm{Hom}_S(\mathbb{Z}/12\mathbb{Z}, \mathscr{D}) \simeq \mathrm{H}^1_{\mathrm{fppf}}(\mathbb{A}^1_S, \mu_{12})$ allow us to rewrite this exact sequence as

$$0 \longrightarrow \mathrm{H}^1_{\mathrm{fppf}}(\mathbb{A}^1_S, \mu_{12}) \xrightarrow{c^*} \mathrm{H}^1_{\mathrm{fppf}}(\mathcal{X}, \mu_{12}) \xrightarrow{\chi'} \mathrm{Hom}(\underline{\mathbb{Z}/12\mathbb{Z}}, \underline{\mathbb{Z}/12\mathbb{Z}}) \dashrightarrow 0. \tag{9.13}$$

Remark 9.2.6. Recall that the second $\underline{\mathbb{Z}/12\mathbb{Z}}$ in (9.13) is $\mathrm{R}^1 f_* \mathbb{G}_m$, the relative Picard scheme of $\mathcal{X}/S$. Identifying the first $\underline{\mathbb{Z}/12\mathbb{Z}}$ with $\mu_{12}^\vee$ – by sending $1 \in \mathbb{Z}/12\mathbb{Z}$ to the natural inclusion $\mu_{12} \hookrightarrow \mathbb{G}_m$ – one can compute the map

$$\chi' \colon \mathrm{H}^1_{\mathrm{fppf}}(\mathcal{Y}(1), \mu_{12}) \longrightarrow \mathrm{Hom}(\mathbb{Z}/12\mathbb{Z}, \mathbb{Z}/12\mathbb{Z})$$

of (9.13) as follows: for $\beta \in \mathrm{H}^1_{\mathrm{fppf}}(\mathcal{Y}(1), \mu_{12})$, $\chi'(\beta)(1) \in \mathbb{Z}/12\mathbb{Z}$ is the image of β under the composition

$$\mathrm{H}^1_{\mathrm{fppf}}(\mathcal{X}, \mu_{12}) \to \mathrm{H}^1_{\mathrm{fppf}}(\mathcal{X}, \mathbb{G}_m) = \mathrm{Pic}\,\mathcal{X} \twoheadrightarrow \mathrm{Pic}\,\mathcal{X}/\,\mathrm{Pic}\,\mathbb{A}^1_S \xrightarrow{\sim} \mathbb{Z}/12\mathbb{Z},$$

where, as in Lemma 9.2.3, $\mathrm{Pic}\,\mathcal{X}/\,\mathrm{Pic}\,\mathbb{A}^1_S \xrightarrow{\sim} \mathbb{Z}/12\mathbb{Z}$ sends the Hodge bundle $[\mathscr{L}]$ to 1. ○

Recalling the earlier defined $\sigma = [\mathcal{X}(\sqrt[12]{\Delta}) \to \mathcal{X}] \in \mathrm{H}^1_{\mathrm{fppf}}(\mathcal{X}, \mu_{12})$, one consequence of Remark 9.2.6 is that $\chi'(\sigma) = \mathrm{id}_{\mathbb{Z}/12\mathbb{Z}}$ so (9.13) is short exact.

Proposition 9.2.7. *The map s of (9.10) gives a right-splitting of (9.7). In particular, we have an isomorphism*

$$c^* \oplus s \colon \mathrm{H}^2(\mathbb{A}^1_S, \mathbb{G}_m) \oplus \mathrm{H}^1(S, \mathbb{Z}/12\mathbb{Z}) \xrightarrow{\sim} \mathrm{H}^2(\mathcal{Y}(1)_S, \mathbb{G}_m)$$

for any regular noetherian $\mathbb{Z}[1/2]$-scheme S.

Proof. Note that the short exact sequence (9.7) is the exact sequence obtained by applying $\mathrm{H}^1_{\mathrm{fppf}}(S, -)$ to the distinguished triangle (9.11). Write $\rho \colon \mathscr{C} \to \underline{\mathbb{Z}/12\mathbb{Z}}$ for the map appearing in (9.11). Consider the commutative diagram (commutativity follows from [Mil80, Proposition V.1.20])

$$\begin{tikzcd}
\mathrm{H}^1(S, \mathbb{Z}/12\mathbb{Z}) \times \mathrm{Hom}_S(\mathbb{Z}/12\mathbb{Z}, \mathscr{C}) \arrow[r] \arrow[d, "\wr"] & \mathrm{H}^1_{\mathrm{fppf}}(S, \mathscr{C}) \arrow[r, "\rho_*"] \arrow[d, "\wr"] & \mathrm{H}^1(S, \mathbb{Z}/12\mathbb{Z}) \arrow[d, equal] \\
\mathrm{H}^1(S, \mathbb{Z}/12\mathbb{Z}) \times \mathrm{H}^1_{\mathrm{fppf}}(\mathcal{X}, \mu_{12}) \arrow[r, "f^*(-)\cup(-)"] & \mathrm{H}^2(\mathcal{X}, \mathbb{G}_m) \arrow[r, "r"] & \mathrm{H}^1(S, \mathbb{Z}/12\mathbb{Z}).
\end{tikzcd}$$

Let $\Sigma \colon \mathbb{Z}/12\mathbb{Z} \to \mathscr{C}$ be the map corresponding to $\sigma \in \mathrm{H}^1_{\mathrm{fppf}}(\mathcal{X}, \mu_{12}) \simeq \mathrm{Hom}_S(\mathbb{Z}/12\mathbb{Z}, \mathscr{C})$. Recalling that $\chi'(\sigma) = \mathrm{id}_{\mathbb{Z}/12\mathbb{Z}}$, commutativity shows that, for $\alpha \in \mathrm{H}^1(S, \mathbb{Z}/12\mathbb{Z})$

$$r(s(\alpha)) = r(f^*\alpha \cup \sigma) = \rho_*(\Sigma_*(\alpha)) = (\rho \circ \sigma)_*(\alpha) = \chi'(\sigma)_*(\alpha) = \mathrm{id}_{\mathbb{Z}/12\mathbb{Z},*}(\alpha) = \alpha. \qquad \blacksquare$$

9.3. **The Brauer Group of $\mathcal{Y}_0(2)$.** Let $\mathcal{Y}_0(2)/\mathbb{Z}[1/2]$ denote the moduli stack of elliptic curves equipped with an (étale) subgroup of order 2. Let $S/\mathbb{Z}[1/2]$ be any regular noetherian $\mathbb{Z}[1/2]$-scheme. We will compute $\operatorname{Br}\mathcal{Y}_0(2)_S$, generalizing the main theorem of [ABJ+24].

Setup 9.3.1. Set $\mathcal{X} \coloneqq \mathcal{Y}_0(2)_S \xrightarrow{f} S$. As shown in [ABJ+24, Section 3], $\mathcal{X}$ is a tame stack with coarse moduli space $c\colon \mathcal{X} \to \mathbb{A}^1_S \setminus \{0\} = \mathbf{Spec}_S\, \mathscr{O}_S[s, s^{-1}] =: X$ given by the "s-invariant" of [ABJ+24, Section 3]. Let $g\colon X \to S$ be the structure map, so we have a commutative triangle

$$\begin{tikzcd} \mathcal{X} \arrow[rr, "c"] \arrow[dr, "f"'] & & X \arrow[dl, "g"] \\ & S. & \end{tikzcd}$$

Note that we use 's' to denote the coordinate on $X = \mathbb{A}^1_S \setminus \{0\}$.

Remark 9.3.2. It was shown in [ABJ+24, Lemma 7.5] that $\mathcal{X}$ is a μ_2-gerbe over $\mathbb{A}^1 \setminus \{0, -1/4\}$ while points above $-1/4$ has automorphism group μ_4. In particular, $\mathcal{X}$ is locally Brauerless. $\circ$

As is maybe routine by now, we will compute $\mathrm{H}^2(\mathcal{X}, \mathbb{G}_m)$ by leveraging the Leray spectral sequence

$$E_2^{ij} = \mathrm{H}^i(S, \mathrm{R}^j f_* \mathbb{G}_m) \implies \mathrm{H}^{i+j}(\mathcal{X}, \mathbb{G}_m). \tag{9.14}$$

Lemma 9.3.3. $f_*\mathbb{G}_m = g_*\mathbb{G}_m \xleftarrow{\sim} \mathbb{G}_m \oplus \underline{\mathbb{Z}}$.

Proof. The equality $f_*\mathbb{G}_m = g_*\mathbb{G}_m$ holds simply because X is $\mathcal{X}$'s coarse space. The isomorphism $g_*\mathbb{G}_m \simeq \mathbb{G}_m \oplus \underline{\mathbb{Z}}$ holds because S is regular and so, at the level of stalks over a geometric point $\overline{x} \to S$, one has

$$\mathbb{G}_m(\mathscr{O}_{S,\overline{x}}) \oplus s^{\mathbb{Z}} \xrightarrow{\sim} \mathscr{O}_{S,\overline{x}}[s, s^{-1}]^\times = \mathbb{G}_m(\mathbb{A}^1_{\mathscr{O}_{S,\overline{x}}} \setminus \{0\}). \qquad \blacksquare$$

Lemma 9.3.4. $\mathrm{R}^1 f_*\mathbb{G}_m \simeq \underline{\mathbb{Z}/4\mathbb{Z}}$ *(sending the class of the Hodge bundle on $\mathcal{X}$ to $1 \in \mathbb{Z}/4\mathbb{Z}$) while* $\mathrm{R}^1 g_*\mathbb{G}_m = 0$.

Proof. The first is a consequence of [ABJ+24, Theorem A.3] which shows that $\operatorname{Pic}\mathcal{Y}_0(2)_T \simeq \operatorname{Pic} X_T \oplus \mathbb{Z}/4\mathbb{Z}$ (with the second factor generated by the image of the Hodge bundle) for any T/S. The second claim can be verified on stalks; since S is regular, $\operatorname{Pic} X_{\mathscr{O}_{S,\overline{x}}} = 0$ for any geometric point $\overline{x} \to S$. $\blacksquare$

Lemma 9.3.5. *There is an exact sequence*

$$0 \longrightarrow \mathrm{R}^2 g_*\mathbb{G}_m \longrightarrow \mathrm{R}^2 f_*\mathbb{G}_m \longrightarrow \underline{\mathbb{Z}/2\mathbb{Z}} \longrightarrow 0.$$

Proof. As usual, we verify this on the level of stalks, so assume $S = \operatorname{Spec} R$ is a noetherian, regular strictly local $\mathbb{Z}[1/2]$-scheme. Then, Example 4.18 already computed that we have an exact sequence

$$0 \longrightarrow \mathrm{H}^2(\mathbb{A}^1_R \setminus \{0\}, \mathbb{G}_m) \longrightarrow \mathrm{H}^2(\mathcal{Y}_0(2)_R, \mathbb{G}_m) \longrightarrow \mathbb{Z}/2\mathbb{Z} \longrightarrow 0,$$

except it did not establish the existence of the exact sequence (4.6). This exact sequence is obtained by applying Lemma 7.2 to $\mathcal{X}$ with $\mathcal{Y} = \mathcal{X} /\!\!/ \mu_2$. With (4.6) established, we complete the argument of Example 4.18 and so of the present lemma. $\blacksquare$

Proposition 9.3.6. *The exact sequence of Lemma 9.3.5 is split, so* $\mathrm{R}^2 f_*\mathbb{G}_m \simeq \mathrm{R}^2 g_*\mathbb{G}_m \oplus \underline{\mathbb{Z}/2\mathbb{Z}}$.

Proof. Let $\mathscr{L}$ denote the Hodge bundle on $\mathcal{X} = \mathcal{Y}_0(2)_S$. Fix any trivialization $D\colon \mathscr{O}_{\mathcal{X}} \to \mathscr{L}^4$ of the fourth power of $\mathscr{L}$, and consider the μ_4-torsor $\mathcal{X}\left(\sqrt[4]{D}\right) \to \mathcal{X}$ with class $\sigma \in \mathrm{H}^1(\mathcal{X}, \mu_4)$. Let $[s] \in \mathrm{H}^1(\mathcal{X}, \mathbb{Z}/4\mathbb{Z})$ denote the image of $s \in \mathbb{G}_m(\mathcal{X}) = \mathbb{G}_m(X)$ under the composition

$$\mathbb{G}_m(\mathcal{X}) \to \mathbb{G}_m(\mathcal{X})/2 \xrightarrow{\kappa_2} \mathrm{H}^1(\mathcal{X}, \mu_2) = \mathrm{H}^1(\mathcal{X}, \mathbb{Z}/2\mathbb{Z}) \to \mathrm{H}^1(\mathcal{X}, \mathbb{Z}/4\mathbb{Z}), \tag{9.15}$$

where κ_2 is the Kummer map and the last map is induced from the usual inclusion $\mathbb{Z}/2\mathbb{Z} \hookrightarrow \mathbb{Z}/4\mathbb{Z}$. Define

$$\alpha \coloneqq [s] \cup \sigma \in \operatorname{im}\left(\mathrm{H}^2(\mathcal{X}, \mu_4) \to \mathrm{H}^2(\mathcal{X}, \mathbb{G}_m)\right). \tag{9.16}$$

This is the class of a cyclic algebra over $\mathcal{X}$; see [AM20, Section 2] for more details on relating α to cyclic algebras. Note that α is 2-torsion because $[s]$ is. We claim that $\underline{\mathbb{Z}/2\mathbb{Z}} \to \mathrm{R}^2 f_* \mathbb{G}_m$ sending 1 to the class of α (i.e. its image under $\mathrm{H}^2(\mathcal{X}, \mathbb{G}_m) \to \mathrm{H}^0(S, \mathrm{R}^2 f_* \mathbb{G}_m)$) is our desired splitting. To prove this, it suffices to assume that $S = \operatorname{Spec} R$ is a regular strictly local $\mathbb{Z}[1/2]$-scheme and then to show that $\alpha \notin \mathrm{H}^2(X, \mathbb{G}_m) \subset \mathrm{H}^2(\mathcal{X}, \mathbb{G}_m)$, so this is what we do.

Let $\mathcal{Y} \coloneqq \mathcal{Y}(2)_R$ denote the moduli stack of elliptic curves equipped with a basis for their 2-torsion. Then, $\mathcal{Y}$ is a trivial μ_2-gerbe over $Y \coloneqq \mathbb{A}^1_R \setminus \{0,1\} = \operatorname{Spec} R[t, t^{-1}, 1/(t-1)]$, i.e. $\mathcal{Y} \simeq B\mu_{2,Y}$; see [AM20, Proposition 4.5]. Furthermore, the coarse space map $\mathcal{Y} \to Y$ sends the elliptic curve $y^2 = (x-e_1)(x-e_2)(x-e_3)$ – with chosen basis $(e_1, 0), (e_2, 0)$ for its 2-torsion – to the point $t = \frac{e_3 - e_1}{e_2 - e_1} \in Y$ [AM20, Corollary 4.6]. In addition, the natural map $\pi \colon \mathcal{Y} \to \mathcal{X}$, remembering the subgroup generated by the first basis element, induces the map

$$\pi_c = \frac{t}{(t-1)^2} \colon Y \longrightarrow X$$

on coarse spaces [ABJ+24, Lemma 3.10]. As our last piece of setup, the isomorphism $\mathcal{Y} \simeq B\mu_{2,Y}$ allows us to compute, via [AM20, Proposition 3.2] or Proposition 3.9, that

$$\mathrm{H}^2(\mathcal{Y}, \mathbb{G}_m) \simeq \mathrm{H}^2(Y, \mathbb{G}_m) \oplus \mathrm{H}^1(Y, \mu_2) \text{ and } \operatorname{Pic} \mathcal{Y} \simeq \operatorname{Pic} Y \oplus \mu_2(Y) \simeq \operatorname{Pic} Y \oplus \mathbb{Z}/2\mathbb{Z},$$

with the above $\mathbb{Z}/2\mathbb{Z}$ generated by the class of the Hodge bundle $\pi^* \mathscr{L}$. Consider the commutative square

$$\begin{tikzcd} \mathrm{H}^2(X, \mathbb{G}_m) \arrow[r, "c^*"] \arrow[d, "\pi_c^*"'] & \mathrm{H}^2(\mathcal{X}, \mathbb{G}_m) \arrow[d, "\pi^*"] \\ \mathrm{H}^2(Y, \mathbb{G}_m) \arrow[r] & \mathrm{H}^2(\mathcal{Y}, \mathbb{G}_m). \end{tikzcd}$$

To finish the proof, it suffices to show that $\pi^* \alpha \notin \mathrm{H}^2(Y, \mathbb{G}_m) \subset \mathrm{H}^2(\mathcal{Y}, \mathbb{G}_m)$.

To begin, note that $\pi^* \mathscr{L}^2 \simeq \mathscr{O}_{\mathcal{Y}}$ is already trivial and fix some trivialization $D' \colon \mathscr{O}_{\mathcal{Y}} \to \mathscr{L}^2$. Consider the μ_2-torsor $\mathcal{Y}\left(\sqrt[2]{D'}\right) \to \mathcal{Y}$ with class $\sigma' \in \mathrm{H}^1(\mathcal{Y}, \mu_2)$. Write $\varphi \colon \mathrm{H}^1(\mathcal{Y}, \mu_2) \to \mathrm{H}^1(\mathcal{Y}, \mu_4)$ for the map induced by the natural inclusion $\mu_2 \hookrightarrow \mu_4$. Then, $\varphi(\sigma')$ and $\pi^* \sigma$ has the same class when projected onto $\operatorname{Pic}(\mathcal{Y})[4] = \operatorname{Pic} \mathcal{Y}$ (namely, the class of $\pi^* \mathscr{L}$), so, letting κ_4 denote the relevant Kummer map,

$$\varphi(\sigma') = \pi^* \sigma + \kappa_4(\lambda) \text{ for some } \lambda \in \mathbb{G}_m(Y).$$

At the same time, since $\pi^* s = t/(t-1)^2 \in \mathbb{G}_m(\mathcal{Y})$ differs from t by a square, the construction of $[s]$ via (9.15) shows that $\pi^*[s] = [t]$, where $[t]$ is similarly defined as the image of t under the composition $\mathbb{G}_m(\mathcal{Y}) \xrightarrow{\kappa_2} \mathrm{H}^1(\mathcal{Y}, \mu_2) = \mathrm{H}^1(\mathcal{Y}, \mathbb{Z}/2\mathbb{Z}) \to \mathrm{H}^1(\mathcal{Y}, \mathbb{Z}/4\mathbb{Z})$. From this, we conclude that

$$\pi^* \alpha = \pi^*[s] \cup \pi^* \sigma = [t] \cup (\varphi(\sigma') - \kappa_4(\lambda)) = [t] \cup \varphi(\sigma') - [t] \cup \kappa_4(\lambda) = \kappa_2(t) \cup \sigma' - [t] \cup \kappa_4(\lambda) \in \mathrm{H}^2(\mathcal{Y}, \mathbb{G}_m).$$

Since $[t] \cup \kappa_4(\lambda)$ is visibly in the subgroup $\mathrm{H}^2(Y, \mathbb{G}_m) \subset \mathrm{H}^2(\mathcal{Y}, \mathbb{G}_m)$, to show that $\pi^* \alpha$ is *not* in this subgroup (and so to conclude the proof), it suffices to show that $\kappa_2(t) \cup \sigma'$ is *not* in this subgroup. For this, we first observe that σ' is the class of a universal torsor on $\mathcal{Y} = B\mu_{2,Y}$; this follows from [AM20, Proof of Corollary 4.6] where it was shown that one obtains a universal torsor by adjoining the square root of a trivialization of $\pi^* \mathscr{L}^2$. Given this, it now follows from [AM20, Lemma 3.3] (or even Proposition 3.15) that $\kappa_2(t) \cup \sigma' \notin \mathrm{H}^2(Y, \mathbb{G}_m) \subset \mathrm{H}^2(\mathcal{Y}, \mathbb{G}_m)$ since it will have nonzero image under the projection map $\mathrm{H}^2(\mathcal{Y}, \mathbb{G}_m) \twoheadrightarrow \mathrm{H}^1(Y, \mu_2)$. This completes the proof. ∎

Remark 9.3.7. The proof of Proposition 9.3.6 shows that the summand $\underline{\mathbb{Z}/2\mathbb{Z}} \subset \mathrm{R}^2 f_* \mathbb{G}_m$ is generated by the class

$$\alpha = [s] \cup \sigma \in \mathrm{H}^2(\mathcal{X}, \mathbb{G}_m)$$

(see (9.16)) which exists (and is nonzero) over every base S. We abuse notation by writing $\alpha \colon \mathrm{H}^0(S, \mathbb{Z}/2\mathbb{Z}) \to \mathrm{H}^2(\mathcal{X}, \mathbb{G}_m)$ also for the map sending 1 to this class (separately over each connected component of S). ◦

Theorem 9.3.8. *There is an exact sequence*

$$0 \longrightarrow \mathrm{H}^2(X, \mathbb{G}_m) \oplus \mathrm{H}^0(S, \mathbb{Z}/2\mathbb{Z}) \xrightarrow{c^* \oplus \alpha} \mathrm{H}^2(\mathcal{X}, \mathbb{G}_m) \longrightarrow \mathrm{H}^1(S, \mathbb{Z}/4\mathbb{Z}) \longrightarrow 0. \tag{9.17}$$

Proof. We compare the Leray spectral sequences

$$E_2^{ij} = \mathrm{H}^i(S, \mathrm{R}^j f_* \mathbb{G}_m) \implies \mathrm{H}^{i+j}(S, \mathbb{G}_m) \text{ and } F_2^{ij} = \mathrm{H}^i(S, \mathrm{R}^j g_* \mathbb{G}_m) \implies \mathrm{H}^{i+j}(S, \mathbb{G}_m) \tag{9.18}$$

for $f\colon \mathcal{X} = \mathcal{Y}_0(2)_S \to S$ and $g\colon X = \mathbb{A}^1_S \setminus \{0\} \to S$, as in the proof of Theorem 9.2.5. Noting that $X(S) \neq \emptyset \neq \mathcal{X}(S)$ (e.g. because of the elliptic curve $y^2 = x(x-1)(x+1)$ over $\mathbb{Z}[1/2]$), we see that the low degree terms of these spectral sequences sit in the commutative diagram

$$\begin{tikzcd}
 & & \ker\big(\mathrm{H}^2(X, \mathbb{G}_m) \to \mathrm{H}^0(S, \mathrm{R}^2 g_* \mathbb{G}_m)\big) \arrow[d, equal, "!!"] & & \\
0 \arrow[r] & \mathrm{H}^2(S, \mathbb{G}_m) \arrow[r, "g^*"] \arrow[d, equal] & K_1 \arrow[r] \arrow[d, "c^*"] & 0 & \\
0 \arrow[r] & \mathrm{H}^2(S, \mathbb{G}_m) \arrow[r, "f^*"] & K_2 \arrow[r] \arrow[d, equal, "!!"] & \mathrm{H}^1(S, \mathbb{Z}/4\mathbb{Z}) \arrow[r] & 0 \\
 & & \ker\big(\mathrm{H}^2(\mathcal{X}, \mathbb{G}_m) \to \mathrm{H}^0(S, \mathrm{R}^2 f_* \mathbb{G}_m)\big) & &
\end{tikzcd} \tag{9.19}$$

with exact rows. Note that (9.19) shows that $\operatorname{coker}(K_1 \hookrightarrow K_2) \simeq \mathrm{H}^1(S, \mathbb{Z}/4\mathbb{Z})$. Now, because $X(S), \mathcal{X}(S) \neq \emptyset$, there are no nonzero differentials in either of the spectral sequences (9.18) which map to the $j = 0$ row, so

$$E_\infty^{0,2} = \ker\big(\mathrm{H}^0(S, \mathrm{R}^2 f_* \mathbb{G}_m) \to \mathrm{H}^2(S, \mathrm{R}^1 f_* \mathbb{G}_m)\big) \text{ and } F_\infty^{0,2} = \ker\big(\mathrm{H}^0(S, \mathrm{R}^2 g_* \mathbb{G}_m) \to \mathrm{H}^2(S, \mathrm{R}^1 g_* \mathbb{G}_m)\big).$$

Hence, we obtain the following homomorphism of exact sequences:

$$\begin{tikzcd}
 & & & & \mathrm{H}^2(S, \mathrm{R}^1 g_* \mathbb{G}_m) \arrow[d, equal, "\textit{Lemma 9.3.4}"] \\
0 \arrow[r] & K_1 \arrow[r] \arrow[d] & \mathrm{H}^2(X, \mathbb{G}_m) \arrow[r] \arrow[d, "c^*"] & \mathrm{H}^0(S, \mathrm{R}^2 g_* \mathbb{G}_m) \arrow[r] \arrow[d] & 0 \\
0 \arrow[r] & K_2 \arrow[r] & \mathrm{H}^2(\mathcal{X}, \mathbb{G}_m) \arrow[r] & \mathrm{H}^0(S, \mathrm{R}^2 f_* \mathbb{G}_m) \arrow[r, "\mathrm{d}_2^{0,2}"] \arrow[d, equal, "\textit{Proposition 9.3.6}"] & \mathrm{H}^2(S, \mathrm{R}^1 f_* \mathbb{G}_m) \\
 & & & \mathrm{H}^0(S, \mathrm{R}^2 g_* \mathbb{G}_m) \oplus \mathrm{H}^0(S, \mathbb{Z}/2\mathbb{Z}) &
\end{tikzcd} \tag{9.20}$$

Note that commutativity of (9.20) shows that the differential $\mathrm{d}_2^{0,2}$ vanishes when restricted to the summand $\mathrm{H}^0(S, \mathrm{R}^2 g_* \mathbb{G}_m)$ of $\mathrm{H}^0(S, \mathrm{R}^2 f_* \mathbb{G}_m)$. At the same time, Proposition 9.3.6 (see also Remark 9.3.7) shows that the $\mathrm{H}^0(S, \mathbb{Z}/2\mathbb{Z})$ summand must survive to the end of the spectral sequence, so the $\mathrm{d}_2^{0,2}$ differential must vanish when restricted to it as well, so $\mathrm{d}_2^{0,2} = 0$. Hence, (9.20) is really a homomorphism of *short* exact sequences, so the snake lemma produces a short exact sequence

$$0 \longrightarrow \operatorname{coker}(K_1 \to K_2) \longrightarrow \operatorname{coker}\big(c^*\colon \mathrm{H}^2(X, \mathbb{G}_m) \to \mathrm{H}^2(\mathcal{X}, \mathbb{G}_m)\big) \longrightarrow \mathrm{H}^0(S, \mathbb{Z}/2\mathbb{Z}) \longrightarrow 0$$

and shows that c^* is injective. Furthermore, Remark 9.3.7 shows that the composition $\mathrm{H}^2(\mathcal{X}, \mathbb{G}_m) \to \mathrm{H}^2(\mathcal{X}, \mathbb{G}_m)/\mathrm{H}^2(X, \mathbb{G}_m) \to \mathrm{H}^0(S, \mathbb{Z}/2\mathbb{Z})$ is split, so we have an exact sequence

$$0 \longrightarrow \mathrm{H}^2(X, \mathbb{G}_m) \oplus \mathrm{H}^0(S, \mathbb{Z}/2\mathbb{Z}) \xrightarrow{c^* \oplus \alpha} \mathrm{H}^2(\mathcal{X}, \mathbb{G}_m) \longrightarrow \operatorname{coker}(K_1 \to K_2) \longrightarrow 0.$$

Finally, (9.19) shows that $\operatorname{coker}(K_1 \to K_2) \simeq \mathrm{H}^1(S, \mathbb{Z}/4\mathbb{Z})$, completing the proof. ∎

As in the case of $\mathcal{Y}(1)$, the exact sequence of Theorem 9.3.8 is split. Let $\pi\colon \mathcal{X} = \mathcal{Y}_0(2)_S \to \mathcal{Y}(1)_S =: \mathcal{Y}$ denote the natural projection. Let $\mathscr{M}$ denote the Hodge bundle on $\mathcal{Y}$, and let $\mathscr{L} := \pi^*\mathscr{M}$ be the Hodge bundle on $\mathcal{X}$. Fix trivializations $\Delta\colon \mathscr{O}_{\mathcal{Y}} \xrightarrow{\sim} \mathscr{M}^{12}$ and $D\colon \mathscr{O}_{\mathcal{X}} \xrightarrow{\sim} \mathscr{L}^4$. Consider the cohomology classes

$$\sigma := \left[\mathcal{Y}\left(\sqrt[12]{\Delta}\right) \to \mathcal{Y}\right] \in \mathrm{H}^1(\mathcal{Y}, \mu_{12}) \text{ and } \tau := \left[\mathcal{X}\left(\sqrt[4]{D}\right) \to \mathcal{X}\right] \in \mathrm{H}^1(\mathcal{X}, \mu_4).$$

Recall that $f\colon \mathcal{X} = \mathcal{Y}_0(2)_S \to S$ denotes $\mathcal{X}$'s structure map and letting $h\colon \mathcal{Y} = \mathcal{Y}(1)_S \to S$ be $\mathcal{Y}$'s, define the maps

$$\begin{array}{cccc} s: & \mathrm{H}^1(S, \mathbb{Z}/12\mathbb{Z}) & \longrightarrow & \mathrm{H}^2(\mathcal{Y}, \mathbb{G}_m) \\ & \beta & \longmapsto & h^*\beta \cup \sigma \end{array} \text{ and } \begin{array}{cccc} t: & \mathrm{H}^1(S, \mathbb{Z}/4\mathbb{Z}) & \longrightarrow & \mathrm{H}^2(\mathcal{X}, \mathbb{G}_m) \\ & \beta & \longmapsto & f^*\beta \cup \tau. \end{array} \tag{9.21}$$

In Proposition 9.2.7, we showed that s above is a right-splitting of (9.7).

Proposition 9.3.9. *The map t of (9.21) is a right-splitting of (9.17). In particular, we have an isomorphism*

$$c^* \oplus t \oplus \alpha\colon\ \mathrm{H}^2(\mathbb{A}^1_S \setminus \{0\}, \mathbb{G}_m) \oplus \mathrm{H}^1(S, \mathbb{Z}/4\mathbb{Z}) \oplus \mathrm{H}^0(S, \mathbb{Z}/2\mathbb{Z}) \xrightarrow{\sim} \mathrm{H}^2(\mathcal{Y}_0(2)_S, \mathbb{G}_m)$$

for any regular noetherian $\mathbb{Z}[1/2]$-scheme S.

Proof. Write $\pi_c\colon X = \mathbb{A}^1_S \setminus \{0\} \to \mathbb{A}^1_S$ for the map $\pi\colon \mathcal{X} \to \mathcal{Y}$ induces on coarse space. Then, comparing (9.17) to (9.7) shows that π induces the following homomorphism of exact sequences:

$$\begin{tikzcd} 0 \arrow[r] & \mathrm{H}^2(\mathbb{A}^1_S, \mathbb{G}_m) \arrow[r] \arrow[d, "\pi_c^*"] & \mathrm{H}^2(\mathcal{Y}, \mathbb{G}_m) \arrow[r] \arrow[d, "\pi^*"] & \mathrm{H}^1(S, \mathbb{Z}/12\mathbb{Z}) \arrow[r] \arrow[d, "\varphi"] & 0 \\ 0 \arrow[r] & \mathrm{H}^2(X, \mathbb{G}_m) \arrow[r, "c^*"] & \mathrm{H}^2(\mathcal{X}, \mathbb{G}_m) \arrow[r] & \mathrm{H}^1(S, \mathbb{Z}/4\mathbb{Z}) \oplus \mathrm{H}^0(S, \mathbb{Z}/2\mathbb{Z}) \arrow[r] & 0 \end{tikzcd} \tag{9.22}$$

Furthermore, Lemmas 9.2.3 and 9.3.4 show that both the $\mathbb{Z}/12\mathbb{Z}$ and the $\mathbb{Z}/4\mathbb{Z}$ appearing in (9.22) are generated by the class of the Hodge bundle, respectively on $\mathcal{Y}$ and $\mathcal{X}$. Thus, the map

$$\mathrm{H}^1(S, \mathbb{Z}/12\mathbb{Z}) \xrightarrow{\varphi} \mathrm{H}^1(S, \mathbb{Z}/4\mathbb{Z}) \oplus \mathrm{H}^0(S, \mathbb{Z}/2\mathbb{Z}) \twoheadrightarrow \mathrm{H}^1(S, \mathbb{Z}/4\mathbb{Z})$$

is the map induces by the projection $\mathbb{Z}/12\mathbb{Z} \twoheadrightarrow \mathbb{Z}/4\mathbb{Z}$ sending $1 \mapsto 1$. In particular, it is split by the map $\psi\colon\ \mathrm{H}^1(S, \mathbb{Z}/4\mathbb{Z}) \to \mathrm{H}^1(S, \mathbb{Z}/12\mathbb{Z})$ induced by the map $\mathbb{Z}/4\mathbb{Z} \to \mathbb{Z}/12\mathbb{Z}$ sending $1 \mapsto 9$. Since the map s of (9.21) is a right splitting of the top row of (9.22) (by Proposition 9.2.7), commutativity of (9.22) shows that the composition

$$\mathrm{H}^1(S, \mathbb{Z}/4\mathbb{Z}) \xrightarrow{\psi} \mathrm{H}^1(S, \mathbb{Z}/12\mathbb{Z}) \xrightarrow{s} \mathrm{H}^2(\mathcal{Y}, \mathbb{G}_m) \xrightarrow{\pi^*} \mathrm{H}^2(\mathcal{X}, \mathbb{G}_m) \tag{9.23}$$

is a splitting map to the surjection $\mathrm{H}^2(\mathcal{X}, \mathbb{G}_m) \twoheadrightarrow \mathrm{H}^1(S, \mathbb{Z}/4\mathbb{Z})$ in the bottom row of (9.22). Finally, one can check that, by construction, the composition (9.23) is the map t of (9.21). ∎

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