Étale Cohomology and the Weil Conjectures Notes

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These are my course notes for "Étale Cohomology and the Weil Conjectures" at the University of Georgia. Each lecture will get its own 'chapter'. These notes are live-texed and so likely contain many mistakes. Furthermore, they reflect my understanding (or lack thereof) of the material as the lecture was happening, so they are far from mathematically perfect.[1] Despite this, I hope they are not flawed enough to distract from the underlying mathematics. With all that taken care of, enjoy and happy mathing.

The instructor for this class is Daniel Litt, and the course website can be found by clicking this link. These notes are based off of the lecture recordings uploaded to Youtube.

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[1] In particular, if things seem confused/false at any point, this is me being confused, not the speaker

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1 Lecture 1

1.1 Course info

What are the prerequisites?

- Homological algebra (abelian categories, derived functors, spectral sequences)
- sheaf theory (sheaf cohomology)
- schemes (Hartshorne II + III)

What are the goals of this course?

- Basics of étale cohomology
  - étale morphisms
  - Grothendieck topologies, étale topology
  - étale cohomology
  - some computations: étale cohomology of curves

  (Following Milne's "Lectures on Étale Cohomology" supplemented with Frietals-Kiehl).[2]
- Prove the Weil conjectures, hopefully more than one way
- If time, more advanced topics. Potentially...
  - Weil II; or
  - formality; or
  - other topics (e.g. monodromy)

  (Katz's AWS notes, among other references)

This will be fairly fast-paced since there is a lot to cover. We still won't have time to prove every single theorem we will need though.

1.2 What is étale cohomology?

Let X be a variety over $\mathbb{C}$.

Definition 1.1. For us, a **variety** is a geometrically integral, finite type, separated scheme. ◇

Can associate to X, the singular cohomology of its analytic space

$$X(\mathbb{C})^{an} \rightsquigarrow \mathrm{H}^i(X(\mathbb{C})^{an}; \mathbb{Z}).$$

What's special about these groups?

- There are f.g. $\mathbb{Z}$-modules.

[2] Milne has another book called "Étale Cohomology" which is also good and covers more

- $\mathrm{H}^i(X(\mathbb{C})^{an};\mathbb{C})$ has extra structure (mixed Hodge structure)
- cycle classes
- ...

The goal of étale cohomology is to do a similar thing for more general nice schemes. To X a "nice scheme" we can associated $\mathrm{H}^i(X_{\text{ét}}, \mathbb{Z}/\ell^n\mathbb{Z})$. Taking an inverse limit over all n, you get ℓ-adic cohomology $\mathrm{H}^i(X_{\text{ét}}, \mathbb{Z}_\ell)$. Tensoring with $\mathbb{Q}_\ell$ gives $\mathrm{H}^i(X_{\text{ét}}, \mathbb{Q}_\ell)$. Can also take cohomology with "twisted coefficients."

What kind of schemes are nice?

- $X = \operatorname{Spec} \mathscr{O}_K$ with K a number field
- X a variety over an algebraically closed field
- X a variety over a non-algebraically closed field.

Remark 1.2. The last two cases behave differently, e.g. cohomology of $X/\overline{k}$ vanishes in degree above $2 \dim X$, but this is not true for varieties over general fields. ○

We will also see related invariants. Given $(X, \overline{x})$ (here, $\overline{x}$ a geometric point), we'll associated $\pi_1^{\text{ét}}(X, \overline{x})$, a certain profinite group. There are even more invariants (beyond the scope of this course)

- higher homotopy groups (in some cases)
- an entire homotopy type.

Note that the cohomology theory we construct will have to be weird.

Theorem 1.3 (Serre)**.** *There does not exist a cohomology theory for schemes over $\overline{\mathbb{F}}_q$ with the following properties:*

(1) *functorial*

(2) *Kunneth*

(3) $\mathrm{H}^1(E) = \mathbb{Q}^2$

Slogan. There's no cohomology theory with $\mathbb{Q}$-coefficients.

Proof. Let E be a super singular elliptic curve. Then, $\operatorname{End}(E) \otimes \mathbb{Q}$ is a non-split quaternion algebra R.

Fact. There are no algebra homomorphisms $R \to M_{2\times 2}(\mathbb{Q})$.[3]

Exercise. functoriality + Kunneth imply that the action $\operatorname{End}(E) \curvearrowright E$ gives an action $\operatorname{End}(E) \curvearrowright \mathrm{H}^1(E)$, i.e. a morphism $\operatorname{End}(E) \to M_{2\times 2}(\mathbb{Q})$, a contradiction.

■

Exercise. Prove the same thing for $\mathbb{Q}_p$ coefficients where $p \mid \operatorname{char} k$.

[3] Like, literally none at all

1.3 Weil Conjectures

Let X be a variety over $\mathbb{F}_q$ (geom integral, f.t., separated). We associate to X its **zeta function**

$$\zeta_X(t) = \exp\left(\sum_{n>0} \frac{\#X(\mathbb{F}_{q^n})}{n} t^n\right).$$

Note that $\partial_t \log \zeta_X(t)$ is a (usual) generated function for $\#X(\mathbb{F}_{q^n})$.

Slogan. The locations of zeros/poles of a meromorphic function control the growth rate of the coefficients of the Taylor series of its log-derivative.

This is saying ζ_X is a function whose zeros/poles control the growth rate of the number of $\mathbb{F}_{q^n}$ points of X.

Exercise. Make this precise for rational functions (i.e. ratios of polynomials).

Conjecture 1.4 (**Weil Conjectures**).

(1) *$\zeta_X(t)$ is a rational function.*

(2) *(**functional equation**) Suppose X is smooth, proper of dimension n. Then,*

$$\zeta_X(q^{-n}t^{-1}) = \pm q^{\frac{nE}{2}} t^E \zeta_X(t),$$

where E is the "Euler characteristic" (to be defined).

(3) *(**Riemann hypothesis**) All roots/poles of $\zeta_X(t)$ have absolute value $q^{i/2}$ for $i \in \mathbb{Z}$.*

(4) *Suppose again that X is smooth and proper. The number of roots/poles with absolute value $q^{-i/2}$ is equal to the ith Betti number of $X_{\overline{\mathbb{F}}_q}$ (to be defined[4]).*

Proof. **(1)** Dwork proved this using p-adic methods. It also follows from finite dimensionality of étale cohomology groups.

I think his proof is given in the last chapter of Koblitz

(2) Grothendieck (will follow from Poincaré duality).

See (sub)section 24.3 for **(1)** and a bit on **(2)**

(3–4) Deligne. ∎

The first proof we give of the Weil conjectures will be close to Deligne's original proof.

1.3.1 Euler product

Notation 1.5. We use $|X|$ to denote the set of closed points of X.

Note that

$$\zeta_X(t) = \exp\left(\sum_{n>0} \frac{\#X(\mathbb{F}_{q^n})}{n} t^n\right) = \prod_{x \in |X|} \exp\left(t^{\deg x} + \frac{t^{2\deg x}}{2} + \dots\right).$$

If $x \in |X|$ is a point of degree n (i.e. $\kappa(x) = \mathbb{F}_{q^n}$), then how many $\mathbb{F}_{q^k}$ points does it contribute? Equivalently, how many morphisms are there $\operatorname{Spec} \mathbb{F}_{q^k} \to \operatorname{Spec} \kappa(x) = \operatorname{Spec} \mathbb{F}_{q^n}$ (over $\operatorname{Spec} \mathbb{F}_q$), i.e.

[4] Already defined if X lifts to characteristic 0, since we know what the Betti numbers of a complex manifold are

how many morphisms are there $\mathbb{F}_{q^n} \to \mathbb{F}_{q^k}$ (over $\mathbb{F}_q$). By Galois theory, the answer is 0 if $k \nmid n$ and $n = \#\operatorname{Gal}(\mathbb{F}_{q^n}/\mathbb{F}_q)$ if $k \mid n$. This reasoning gives the last equality above. However, why stop there? Recognizing the Taylor series of $\log(1-x)$, we continue

$$\begin{aligned}
\zeta_X(t) &= \prod_{x\in|X|} \exp\left(t^{\deg x} + \frac{t^{2\deg x}}{2} + \dots\right) \\
&= \prod_{x\in|X|} \exp\left(-\log(1-t^{\deg x})\right) \\
&= \prod_{x\in|X|} \frac{1}{1-t^{\deg x}} \\
&= \prod_{x\in|X|} \left(1 + t^{\deg x} + t^{2\deg x} + \dots\right) \\
&= \sum_{n\geq 0} \left(\#\text{Galois stable subsets of } X(\overline{\mathbb{F}}_q) \text{ of size } n\right) t^n \\
&= \sum_{n\geq 0} \#\operatorname{Sym}^n(X)(\mathbb{F}_q)\cdot t^n
\end{aligned}$$

Above, $\operatorname{Sym}^n X = X^n/\Sigma_n$. To get the second to last equality, partition a Galois-stable subset of $X(\overline{\mathbb{F}}_q)$ into its Galois orbits (each will correspond to some closed point[5], if I understood correctly). For the last equality, an $\overline{\mathbb{F}}_q$-point of $\operatorname{Sym}^n(X)$ is just n choices of $\overline{\mathbb{F}}_q$-points of X, without an ordering, and it comes from an $\mathbb{F}_q$ point exactly when these choices of Galois-stable.

1.3.2 The case of curves

We can now prove the first Weil conjecture for curves.

Theorem 1.6. *Suppose X is a smooth, proper curve over $\mathbb{F}_q$. Then, $\zeta_X(t)$ is rational.*

Proof. Note that there is a map $\operatorname{Sym}^n X \to \operatorname{Pic}^n X$ sending $D \mapsto \mathscr{O}(D)$. What are the fibers of this map? The fiber above $\mathscr{O}(D)$ is the (complete) linear system of divisors linearly equivalent to D, i.e. it is $\mathbb{P}\Gamma(X, \mathscr{O}(D))$. Note that

$$\dim \mathbb{P}\Gamma(X,\mathscr{O}(D)) = \deg(D) + 1 - g + \dim \mathrm{H}^1(X,\mathscr{O}(D)) - 1$$

by Riemann-Roch. If $\deg D \gg 0$ (in fact, $\deg D > 2g-2$), then $\mathrm{H}^1(X,\mathscr{O}(D)) = 0$ by Serre duality (+ the fact that line bundles of negative degree have no nonzero global sections). This, if $n > 2g-2$, the fibers of $\operatorname{Sym}^n X \to \operatorname{Pic}^n X$ are isomorphic to $\mathbb{P}^{n-g}$.

Exercise. We may assume, WLOG, that $X(\mathbb{F}_q) \neq \emptyset$.

With this reduction made, $\operatorname{Pic}^n(X) \cong \operatorname{Pic}^{n+1}(X)$ for all n (via tensoring with $\mathscr{O}(p)$ for some $\mathbb{F}_q$-point p). This tells us that

$$\#\operatorname{Sym}^n(X)(\mathbb{F}_q) = \#\mathbb{P}^{n-g}(\mathbb{F}_q)\cdot\#\operatorname{Pic}^0(X)(\mathbb{F}_q)$$

[5] of degree equal to the size of the orbit

for all $n > 2g - 2$. Thus,

$$\zeta_X(t) = \mathrm{poly}(t) + \sum_{n > 2g-2} \#\operatorname{Pic}^0(X)(\mathbb{F}_q) \cdot \left(1 + q + q^2 + \cdots + q^{n-g}\right) t^n.$$

Exercise. Show that this is a rational function (hint: geometric series).

■

What does the functional equation say in the place of curves?

Theorem 1.7.

$$\zeta_X(q^{-1}t^{-1}) = \pm q^{\frac{2-2g}{2}} t^{2-2g} \zeta_x(t).$$

Proof. Exercise. Use Serre duality. ■

We'll do RH for curves later in the course (and then RH for all varieties).

1.3.3 Some Proof Sketches + Serre's Analogue

Let's sketch the proof of rationality in general.

Theorem 1.8 (Dwork)**.** *Suppose $X/\mathbb{F}_q$ is a variety. Then, $\zeta_X(t)$ is a rational function.*

Proof. (Following Grothendieck) The idea is to take a Frobenius map $\mathrm{Frob} : X \to X$, and realize $X(\mathbb{F}_q) =$ fixed points of Frob on $X_{\overline{\mathbb{F}}_q}$. Then use the "Lefschetz fixed point formula"

$$\#X(\mathbb{F}_{q^n}) = \sum_{i=0}^{2\dim X} (-1)^i \operatorname{Tr}\left(\mathrm{Frob}^n \curvearrowright \mathrm{H}^i_c(X_{\overline{\mathbb{F}}_q}, \mathbb{Q}_\ell)\right).$$

Lemma 1.9. *If V is a f.d. vector space and $F : V \to V$ is a linear map, then*

$$\exp\left(\sum_{n>0} \frac{\operatorname{Tr}(F^n)}{n} t^n\right)$$

is rational.

Proof. Appealing to eigenvalues, suffices to treat the case where $\dim V = 1$. Then,

$$\exp\left(\sum \frac{\alpha^n}{n} t^n\right) = \exp(-\log(1 - at)) = \frac{1}{1 - \alpha t}$$

which is rational. ■

Plugging Lefschetz into the definition of the zeta function gives an alternating product of things of the form given in the lemma, so shows the zeta function is rational. ■

Key inputs above include

- Lefschetz fixed point formula
- finite dimensionality of étale cohomology (with compact support)

Exercise. Try to figure out how Poincaré duality should give functional equation. Try lemma on vector space where V has a bilinear form and F preserves it up to scaling.

Seems we won't have time to do the Kähler analogue of RH, but maybe we'll state it at least.

Theorem 1.10 (Serre)**.** *Let X be a smooth, projective variety over $\mathbb{C}$ and $[H] \in \mathrm{H}^2(X(\mathbb{C})^{an}; \mathbb{Z})$ is a hyperplane class. Now, suppose $F : X \to X$ is an endomorphism such that $F^*[H] = q[H]$ for some $q \in \mathbb{Z}_{>0}$. We define the **Lefschetz number***

$$L(F^n) := \sum_{i=0}^{2\dim X} (-1)^i \operatorname{Tr}((F^*)^n \curvearrowright \mathrm{H}^i(X, \mathbb{Q})).$$

We also define

$$\zeta_{X,F}(t) = \exp\left(\sum_{n\geq 1} \frac{L(F^n)}{n} t^n\right).$$

Then, $\zeta_{X,F}(t)$ satisfies the Riemann hypothesis, i.e. zeros/poles are half-integer powers of q (equivalently, eigenvalues of F^* acting on $\mathrm{H}^i(X, \mathbb{C})$ all have absolute value $q^{i/2}$).

The proof of this will look different from our eventual proof(s) of RH for varieties, but still good to know it.

Next time: prove this + étale morphisms.

2 Lecture 2

2.1 Finishing Serre's Analogue

We left off last time with a statement of a theorem of Serre. We start off this time by proving it.

Theorem 2.1 (Serre)**.** *Let X be a smooth, projective variety over $\mathbb{C}$ and $[H] \in \mathrm{H}^2(X(\mathbb{C})^{an}; \mathbb{Z})$ is a hyperplane class. Now, suppose $F : X \to X$ is an endomorphism such that $F^*[H] = q[H]$ for some $q \in \mathbb{Z}_{>0}$. Then the eigenvalues of F^* on $\mathrm{H}^i(X, \mathbb{C})$ all have absolute value $q^{i/2}$.*

Same i as the cohomology degree

Let's recall some properties of singular cohomology which will be useful for the proof.

Fact.

- We have cup products. In particular, we name

$$\begin{array}{rccc} L: & \mathrm{H}^i(X, \mathbb{C}) & \longrightarrow & \mathrm{H}^{i+2}(X, \mathbb{C}) \\ & \alpha & \longmapsto & \alpha \smile [H] \end{array}$$

- the **Hard Lefschetz theorem** says, among other things, that

$$\mathrm{H}^j(X, \mathbb{C}) \simeq \operatorname{im} L \oplus \mathrm{H}^j_{prim}$$

with the decomposition canonical gives a choice of hyperplane class. This H^j_{prim} is "primitive cohomology" and decomposes canonically as

$$\mathrm{H}^j_{prim} = \bigoplus_{p+q=j} \mathrm{H}^{p,q}_{prim}$$

(i.e. it interacts well with Hodge decomposition).

- There's also the **Hodge index theorem**: given $\alpha, \beta \in \mathrm{H}^k(X)_{prim}$, there is a natural pairing

$$\langle \alpha, \beta \rangle = \left(\sqrt{-1}\right)^k \int_X \alpha \wedge \overline{\beta} \wedge [H]^{n-k}.$$

  This bilinear form is definite[6] on each $\mathrm{H}^{p,q}_{prim}$.

The upshot is that cohomology comes from two pieces, one piece ($\operatorname{im} L$) coming from lower degree terms (so maybe can control inductively), and one piece (H^j_{prim}) which decomposes into further pieces which carry canonical, definite bilinear forms.

Proof Sketch of Theorem 2.1. We want to show that the eigenvalues of $F^* \curvearrowright \mathrm{H}^k(X, \mathbb{C})$ have absolute value $q^{k/2}$. Note that it suffices to do this for H^k_{prim}. Indeed, if $\alpha \in \mathrm{H}^{k-2}(X, \mathbb{C})$, we can inductively assume its eigenvalue has absolute value $q^{(k-2)/2}$, and then

$$F^*(\alpha \smile [H]) = F^*\alpha \smile F^*[H] = \lambda\alpha \smile q[H] = q\lambda(\alpha \smile [H])$$

so $\alpha \smile [H]$ has eigenvalue with absolute value $q^{k/2}$.

This reduces us to the primitive case. Let $\alpha \in \mathrm{H}^k_{prim}$ be an F^*-eigenvector. Since F^* preserves the Hodge decomposition, we may assume that $\alpha \in \mathrm{H}^{p,q}_{prim}$ s.t. $p + q = k$. This pairing is sesquilinear, so

$$\begin{aligned}
|\lambda|^2 \langle \alpha, \alpha \rangle &= \langle F^*\alpha, F^*\alpha \rangle \\
&= i^k \int F^*\alpha \wedge F^*\overline{\alpha} \wedge [H]^{n-k} \\
&= \frac{i^k}{q^{n-k}} \int F^*(\alpha \wedge \overline{\alpha} \wedge [H]^{n-k}) \\
&= \frac{q^n i^k}{q^{n-k}} \int \alpha \wedge \overline{\alpha} \wedge [H]^{n-k} \\
&= q^k \langle \alpha, \alpha \rangle
\end{aligned}$$

Above, we have used the fact that $F^* \curvearrowright \mathrm{H}^{2n}(X, \mathbb{C})$ (top degree cohomology) via multiplication by q^n. You can show this using Poincaré duality (top cohomology is generated by $[H]^n$) or Lefschetz fixed point. Since $\langle \alpha, \alpha \rangle \neq 0$ (the form is definite on $H^{p,q}_{prim}$), we conclude that $q^k = |\lambda|^2$ as desired. ∎

Slogan. Structures on cohomology $\implies$ RH (or, this analogue of it).

We'll want to do something similar algebraically. We will not succeed entirely. There will be no analogue of the Hodge decomposition, and we don't know the analogue of the Hodge index theorem, but we will still see "shadows" of these structures.

[6] i.e. $\langle \alpha, \alpha \rangle = 0 \implies \alpha = 0$

2.2 Étale morphisms

Definition 2.2. Let $f : X \to Y$ be a morphism of schemes. We say that f is **étale** if it is locally of finite presentation, flat, and unramified. $\diamond$

Recall 2.3. f above is **unramified** if it satisfies any (hence all) of the equivalent conditions

- $\Omega^1_{X/Y} = 0$
- all residue field extensions are separable.[7]
- it is smooth of relative dimension 0
- it is **formally étale**. For any nilpotent ideal $I \subset A$ in a ring A (i.e. $I^n = 0$ for some n), the lifting problem

$$\begin{tikzcd} \operatorname{Spec} A/I \arrow[r] \arrow[d] & X \arrow[d, "f"] \\ \operatorname{Spec} A \arrow[r] \arrow[ur, dashed, "\exists!"] & Y \end{tikzcd}$$

  always has a unique solution.

  Can think about this as saying you can lift tangent vectors/infitesimal thickenings.

- locally **standard étale**. For each $x \in X$ (set $y = f(x)$), there exists a $U \ni x$ and $V \ni y$ such that $f(U) \subset V$, and we have

$$V = \operatorname{Spec} R \text{ and } U = \operatorname{Spec}(R[x]_h/g)$$

  with g' (the derivative) a unit in $R[x]_h$ and g monic.

  To interpret this, note that $\operatorname{Spec} R[x] = \mathbb{A}^1_R$ is the affine line over R. Quotienting out by g just gives the vanishing set $V(g) = \operatorname{Spec} R[x]/(g)$, and inverting h corresponds to removing the points at which h vanishes.[8] The fact that g' is a unit tells us that g has no double roots in the fibers (above $\operatorname{Spec} R$). All in all, we've taken some hypersurface in $\mathbb{A}^1_R$ and then removed all double roots of this hypersurface (maybe plus some other points).

  Exercise. Check that standard étale morphisms are étale.

$\odot$

Example. multiplication by $[n] : E \to E$ on an elliptic curve E if n is invertible in the base (e.g. E/k and $\operatorname{char} k \nmid n$). $\triangle$

Example. $\mathbb{G}_m \to \mathbb{G}_m$ given by $t^n \mapsfrom t$, where $\mathbb{G}_m = \operatorname{Spec} k[t, t^{-1}]$, if n prime to $\operatorname{char} k$.

Exercise. Show this is étale (hint: $\frac{\partial t^n}{\partial t} = nt^{n-1}$ is a unit)

$\triangle$

Example. $\mathbb{G}_m \hookrightarrow \mathbb{A}^1$ via $k[t] \to k[t, t^{-1}]$ is étale. It is visible locally of finite presentation, flat, and $\Omega^1_{\mathbb{G}_m/\mathbb{A}^1} = 0$. $\triangle$

[7] I think this should really say if $x \mapsto y$, then the local rings satisfy $f^*(\mathfrak{m}_y)\mathscr{O}_{X,x} = \mathfrak{m}_x$ where $f^* : \mathscr{O}_{Y,y} \to \mathscr{O}_{X,x}$ is the local ring map

[8] Maybe I should have said these last two in the other order

In fact,

Proposition 2.4. *Any open immersion is étale.*

Example (An étale morphism which is not finite onto its image)**.** $\mathbb{G}_m \setminus \{1\} \to \mathbb{G}_m$ via the squaring map $t^2 \leftarrow\!\shortmid t$ (working in characteristic $\neq 2$). This is an étale surjection, but not proper (so not finite). We've deleted one point in the fiber above 1. △

Example. Any finite separable field extension is étale. △

Non-example. Set $X = \operatorname{Spec} k[x,y]/(xy)$. Then its normalization $\widetilde{X} \to X$ is not étale since it is not flat. ▽

Non-example. $\mathbb{A}^1 \xrightarrow{f} \mathbb{A}^1$ via $t^2 \leftarrow\!\shortmid t$ is not étale (ramified at 0). The relative Kähler differentials here are $\Omega^1_f = k[t]\mathrm{d}t/\mathrm{d}(t^2) = k[t]\mathrm{d}t/2t dt$ which is supported at 0 (if char $\neq 2$).

This is finite flat but not étale. ▽

Non-example. Consider $\mathbb{A}^1 \xrightarrow{F} \mathbb{A}^1$ via $t \mapsto t^p$ in characteristic p. Then,

$$\Omega^1_F = k[t]\mathrm{d}t/\mathrm{d}(t^p) = k[t]\mathrm{d}t,$$

so this is finite, flat with non-torsion Ω^1, but still not étale. This map is called **relative Frobenius**. ▽

Example. Say we have $f = (f_1, \ldots, f_m) : \mathbb{A}^m \to \mathbb{A}^m$. Then f is étale in a neighborhood of $(a_1, \ldots, a_m)$ if

$$\det \left(\left. \frac{\partial f_i}{\partial x_j} \right|_{(a_1,\ldots,a_m)} \right) \text{ is a unit.}$$

△

Proposition 2.5. *Properties of étale morphisms*

(1) Open immersions are étale

(2) compositions of étale morphisms are étale[9]

(3) base change of étale is étale

(4) (2 out of 3) If $\varphi \circ \psi$ and φ are étale, then so is ψ (exercise).

Proposition 2.6. *Étale morphisms on varieties over $k = \overline{k}$ induce isomorphisms on complete local rings at closed points.*

Proof. Exercise (hint: use criterion for formall étaleness). ■

Corollary 2.7 (informal)**.** *Any property that can be checked at the level of complete local rings is true for the source of an étale morphism if it is true for the target.*

Exercise. Find φ, ψ such that $\varphi \circ \psi$ and ψ are étale, but φ is not.

[9]Hint: use cotangent exact sequence for $\Omega^1_{X/Y}$

2.3 Sites

Sites will be a generalization of topological spaces. In particular, they will allow us to generalize the notion of a sheaf.

Question 2.8. *What parts of the definition of a topological space do you need to define the notion of a sheaf?*

- Need open sets and inclusions (just to define a presheaf), i.e. you need a "category of open sets." This is actually all you need to define a *presheaf.*
- Need to make sense of the sheaf condition: this says that a section to a sheaf is determined by its values on a *cover*, and furthermore, you can glue sections which agree on *intersections.*
  - Need a collection a morphisms which form "covers".
  - Need existence of certain fiber products (i.e. "intersections").[10]

Let's end with a "pre-definition" of a Grothendieck topology.

Definition 2.9. A category C with a collection of "covering families" $\{X_\alpha \xrightarrow{f_\alpha} X\}_{\alpha\in A}$ satisfying some axioms which we will give next time is called a site. The collection of families is the topology. $\diamond$

Warning 2.10. There are multiple (> 2) definitions/conventions of a 'site' which are different from each other. We use the least general but easiest to work with. $\bullet$

Example. If X is a topological space, and C is its category of open sets (whose morphisms are inclusions), then $\{U_\alpha \to U\}$ is a covering family if U_α covers U in the usual sense. This defines a site. $\triangle$

Example. Say M is a manifold. Let C be the category of manifolds $M' \xrightarrow{f} M$ over M s.t. f is locally on M' an isomorphism. Say $\{M_\alpha \to M'\}$ is a covering family their images cover M'. $\triangle$

Example. Let X be a scheme. Let $X_{\text{ét}}$ be the category of étale morphisms $Y \to X$. We call $\{X_\alpha \xrightarrow{f_\alpha} X'\}$ a covering family if $\bigcup \operatorname{im}(f_\alpha) = X$. $\triangle$

Remark 2.11. Étale morphisms are always open maps. $\circ$

We'll say more next time.

3 Lecture 3

3.1 Last time

Last time we proved Serre's Kähler analogue of RH, introduced éale morphisms, and gave motivation for sites. Our goal today is to introduce sites and so generalize the notions of topological spaces and of sheaves of spaces.

[10] Note that if $U, V \subset X$ are open subsets of X, then $U \times_X V = U \cap V$

3.2 Sites + (pre)sheaves

Definition 3.1 (**Grothendieck topology on a category $\mathcal{C}$/site**)**.** The data of, for each $X \in \mathrm{ob}(\mathcal{C})$, a collection of sets of morphisms $\{X_\alpha \to X\}_\alpha$, called **covering families**, such that

(1) (intersections exist) If $X_\alpha \to X$ appears in a covering family, and $Y \to X$ is arbitrary, then the fiber product $X_\alpha \times_X Y$ exists.

(2) (intersecting w/ a cover gives a cover) If $\{X_\alpha \to X\}$ is a covering family, and $Y \to X$ is arbitrary, then $\{Y \times_X X_\alpha \to Y\}$ is a covering family.

(3) (composition of covers are covers) If $\{X_\alpha \to X\}_\beta$ is a covering family, and for each α, $\{X_{\alpha\beta} \to X_\alpha\}_\beta$ is a covering family, then

$$\{X_{\alpha\beta} \to X_\alpha \to X\}_{\alpha\beta}$$

is a covering family.

(4) (iso are covers) If $f : X \xrightarrow{\sim} Y$ is an isomorphism, then $\{X \xrightarrow{f} Y\}$ is a covering family.

$\diamond$

Example. Let X be a topological space, and let $\mathcal{C} = \mathrm{Open}(X)$, so the objects are open subsets of X, and the morphisms are inclusions (in particular, there is at most one morphism between any pair of objects). We say $\{U_\alpha \to U\}$ is a covering family if $\bigcup_\alpha U_\alpha = U$. $\triangle$

Example. Let X be a scheme.

- The **small étale site** $X_{\text{ét}}$ is the category whose objects are étale morphisms $Y \to X$, and whose morphisms are X-morphisms, i.e. diagrams

  g below is étale by 2 out of 3

$$\begin{tikzcd} Y_1 \arrow[rr, "g"] \arrow[dr] & & Y_2 \arrow[dl] \\ & X & \end{tikzcd}$$

  A family $\{Y_\alpha \xrightarrow{f_\alpha} Y\}$ is a covering family if $\bigcup \mathrm{im}(f_\alpha) = Y$.

- The **big étale site** $X_{\text{Ét}}$ is the category whose objects are all X-schemes with morphisms maps over X. We say $\{U_\alpha \xrightarrow{f_\alpha} U\}$ is a covering family if all f_α are étale and $\bigcup \mathrm{im}(f_\alpha) = U$.

$\triangle$

Example. Let X be a complex analytic space. The **analytic étale site** $X_{\text{an-ét}}$ has objects complex analytic spaces $Y \xrightarrow{f} X$ s.t. locally on Y, f is an analytic isomorphism, and whose morphisms are morphisms over X. Here, covers are what you expect. $\triangle$

Remark 3.2. $\mathrm{Sh}(X_{\text{an-ét}}) \xrightarrow{\sim} \mathrm{Sh}(X^{top})$, the category of sheaves on the analytic étale site of some complex analytic space is canonically equivalent to the category of sheaves on the underlying topological space. Proving this will be an exercise (once we have the definitions).[11] $\circ$

[11] The point is that a cover in the analytic étale site can always be refined to an honest cover in the usual topology

Example. The **fppf topology** (fidelment plat de presentation finie[12]) is the site X_{fppf} whose objects are fppf morphisms $Y \to X$, and whose morphisms are morphisms over X. The covers are what you expect. △

Example. One can also define/study the Nisnevich, Crystalline, infinitesimal, cdh, arc, ... sites. △

Onto sheaves...

Definition 3.3. Let $\mathcal{C}, \mathcal{D}$ be categories. A $\mathcal{D}$**-valued presheaf** on $\mathcal{C}$ is a contravariant functor $F : \mathcal{C} \to \mathcal{D}$. ◇

Remark 3.4. If X is a topological space, then a $\mathcal{D}$-valued presheaf on X is the same as a presheaf on $\mathrm{Open}(X)$. ○

Definition 3.5. Let $\mathcal{C}$ be a site (i.e. a category with a Grothendieck topology). A **sheaf** $\mathscr{F}$ is a presheaf such that

$$\mathscr{F}(U) \to \prod_{\alpha} F(U_\alpha) \rightrightarrows \prod_{(\alpha,\alpha')} \mathscr{F}(U_\alpha \times_U U_{\alpha'})$$

is an equalizer diagram for all covering families $\{U_\alpha \to U\}$. ◇

Remark 3.6. The two arrows $\prod_\alpha F(U_\alpha) \rightrightarrows \prod_{(\alpha,\alpha')} \mathscr{F}(U_\alpha \times_U U_{\alpha'})$ are induced by the projections $U_\alpha \times_U U_{\alpha'} \rightrightarrows U_\alpha, U_{\alpha'}$. This being an equalizer says two things (suppose $\mathscr{F}$ is valued in Set for concreteness)

- ("exactness on the left") $\mathscr{F}(U) \to \prod_\alpha \mathscr{F}(U_\alpha)$ is injective.
- ("exactness on the right") sections $s_\alpha \in \mathscr{F}(U_\alpha)$ agreeing on overlaps (i.e. $s_\alpha|_{U_\alpha \times_U U_\beta} = s_\beta|_{U\alpha \times_U U_\beta}$) glue to some global section $s \in \mathscr{F}(U)$ (with $s|_{U_\alpha} = s_\alpha$).

○

Definition 3.7. A morphism $\mathscr{F}_1 \to \mathscr{F}_2$ of (pre)sheaves is simply a natural transformation. ◇

Let's give some examples os sheaves on $X_{\text{Ét}}$.

Theorem 3.8. *Any representable functor is a sheaf on $X_{\acute{E}t}$ (in fact, any representable functor is a sheaf on the big fppf site X_{Fppf}).*

Example. μ_n, represented by $\operatorname{Spec} k[t]/(t^n - 1)$, is the sheaf

$$\mu_n(U) = \{f \in \mathscr{O}_U(U) : f^n = 1\}.$$

△

Example. $\mathscr{O}_X^{\text{ét}}(U) = \mathscr{O}_U(U)$ is a sheaf represented by $\mathbb{A}^1_X$. △

Example. The constant sheaf $\underline{\mathbb{Z}/\ell^n\mathbb{Z}}$ is represented by the constant group scheme $(\mathbb{Z}/\ell\mathbb{Z}^n) \times X$. Here,

$$\underline{\mathbb{Z}/\ell^n\mathbb{Z}}(U) = \mathrm{Hom}_{cts}(U^{top}, \mathbb{Z}/\ell^n\mathbb{Z}).$$

△

[12] faithfully flat of finite presentation

Example. $\mathbb{G}_m(U) = \mathscr{O}_U(U)^\times$ is represented by the group scheme $\mathbb{G}_{m,X} = \operatorname{Spec}\mathbb{Z}[t,t^{-1}] \times_{\mathbb{Z}} X$. △

Example. The functor $\mathbb{P}^n : U \mapsto \operatorname{Hom}_X(U, \mathbb{P}^n_X)$ is a (set-valued) sheaf. △

These are sheaves, but we have not proved that yet. We will later. In the meantime, let's see a new definition(ish).

Definition 3.9 (sorta-kinda)**.** Let's take for granted for the moment that $\underline{\mathbb{Z}/\ell^n\mathbb{Z}}$ is a sheaf on $X_{\text{Ét}}$, and that the category of abelian sheaves on $X_{\text{ét}}$ is abelian with enough injectives.[13] Let $\Gamma_X : \operatorname{Ab}(X_{\text{ét}}) \to \operatorname{Ab}$ be the global sections functor, $\Gamma_X(\mathscr{F}) = \mathscr{F}(X)$. We define **étale cohomology** with coefficients in $\mathbb{Z}/\ell^n\mathbb{Z}$ to be the derived functors of global sections

$$\mathrm{H}^i(X_{\text{ét}}, \underline{\mathbb{Z}/\ell^n\mathbb{Z}}) := R^i\Gamma_X(\underline{\mathbb{Z}/\ell^n\mathbb{Z}}).$$

◇

Warning 3.10. Even just showing cokernels exist in the category of abelian sheaves on a site is non-trivial (try as exercise). •

Example. Consider the map $\mathbb{G}_m \xrightarrow{t^n \leftarrow t} \mathbb{G}_m$ and suppose n is invertible on the base. This gives a map of sheaves. On the Zariski topology, this is

$$X_{\text{Zar}} : \mathscr{O}^\times \xrightarrow{f \mapsto f^n} \mathscr{O}^\times.$$

Similarly, on the étale topology, it is

$$X_{\text{ét}} : \mathscr{O}^\times_{\text{ét}} \xrightarrow{f \mapsto f^n} \mathscr{O}^\times_{\text{ét}}.$$

Claim 3.11. *This map is in general not an epimorphism on the X_{Zar}, but it is an epimorphism on $X_{ét}$.*

Proof. First we show it is not an epimorphism on the Zariski site. Take $X = \operatorname{Spec}\mathbb{R}$ and $n = 2$. In this case, we're just asking if $\mathbb{R}^\times \xrightarrow{t \mapsto t^2} \mathbb{R}^\times$ is surjective? The answer is no.

Now onto the étale site. Given $f \in \mathbb{G}_m(U)$, we want an étale cover of U such that f obtains an nth root on that cover. Form the fiber product

$$\begin{tikzcd} U \times_{\mathbb{G}_m} \mathbb{G}_m \arrow[r] \arrow[d, "\text{ét}"'] & \mathbb{G}_m \arrow[d, "z \leftarrow z^n"] \\ U \arrow[r, "f"] & \mathbb{G}_m \end{tikzcd}$$

Now, f has an nth root upstairs. This construction is equivalent to considering $V(z^n - f) \subset \mathbb{A}^1_U$, where z is the coordinate on $\mathbb{A}^1$; here, z is the nth root.

Exercise. Check the details.

■

△

Remark 3.12. $\mathbb{G}_m \xrightarrow{t \mapsto t^n} \mathbb{G}_m$ will always be an epimorphism in $\operatorname{Sh}(X_{\text{fppf}})$. The point is that this map is always flat (only étale when n invertible in the base). ○

[13] If you are a sheaf on $X_{\text{Ét}}$, then you restrict to a sheaf on $X_{\text{ét}}$.

Definition 3.13. Let $\mathcal{T}_1, \mathcal{T}_2$ be sites. A **continuous map** $f : \mathcal{T}_1 \to \mathcal{T}_2$ is a functor $F : \mathcal{T}_2 \to \mathcal{T}_1$ which preserves fiber products, and sends covering families to covering families. ⋄

Example. Given $f : X \to Y$ a continuous map of spaces, we can define a functor

$$\begin{array}{ccc} \mathrm{Open}(Y) & \longrightarrow & \mathrm{Open}(X) \\ U & \longmapsto & f^{-1}(U) \end{array}$$

which is a continuous map of sites. △

4 Lecture 4

Last time we defined sites, sheaves, and morphisms on sites. Today we talk about descent!

Recall 4.1 (morphism of sites)**.** If $\mathcal{T}_1, \mathcal{T}_2$ are sites, then a morphism (continuous map) $\mathcal{T}_1 \to \mathcal{T}_2$ of sites is a functor $f^{-1} : \mathcal{T}_2 \to \mathcal{T}_1$ such that

(1) f^{-1} preserves fiber products; and

(2) f^{-1} sends covering families to covering families.

⊙

Example. Let X be a scheme. Then there are natural continuous maps

$$X_{\mathrm{Fppf}} \to X_{\acute{\mathrm{E}}\mathrm{t}} \to X_{\acute{\mathrm{e}}\mathrm{t}} \to X_{\mathrm{zar}}.$$

In every case, the morphism is given by the natural inclusion going the other way. △

Warning 4.2. There are many conventions on what gets called a "Grothendieck topology" or "site". What we defined is sometimes called a "Grothendieck pre-topology" (can make sense of a topology without requiring fiber products using what are called 'sieves') People also sometimes talk about topos (topoi?) which are categories equivalent to the category of sheaves on some site. Lots of what we do can be done using the language of topoi instead, but we won't use it. •

"I don't know if you saw this over here... I put three exclamation points by descent because it's maybe one of my favorite topics in all mathematics."

4.1 Descent!!!

We have defined sheaves on sites and written down a bunch of functors, but we haven't actually proven anything is a sheaf yet.

Question 4.3. *How do you check if some functor is a sheaf on $X_{\acute{e}t}/X_{fppf}$? How do you construct sheaves?*

Theorem 4.4.

(1) *If Y is an X-scheme, then the functor*

$$Z \mapsto \mathrm{Hom}_X(Z, Y)$$

is a sheaf on X_{Fppf} (hence on $X_{\acute{E}t}, X_{\acute{e}t}$, ...).

(2) *Given $\mathscr{F} \in \mathrm{QCoh}(X)$, the functor*

$$\left(Z \xrightarrow{f} X\right) \mapsto \Gamma(X, f^*\mathscr{F})$$

is a sheaf on X_{Fppf} (hence on $X_{\acute{E}t}, X_{\acute{e}t}, \dots$).

Notation 4.5. In this case, we write $\mathscr{F}^{\text{ét}}$ for the associated sheaf on $X_{\text{ét}}$.

Question 4.6 (Audience)**.** *Is the (big) fppf site the finest topology for which this holds?*

Answer. No. There's a notion of a morphism of effective descent, and you can use those to define a topology. There's also the fpqc topology which is arguably finer, but then runs into set theoretic issues. So it's not clear there is a finest topology since you may eventually run into set-theoretic difficulties (which can possibly be avoided if you make different choices in how you set things up). ⋆

Let's start with **(2)**. We'll actually prove something a little more general. Say $U = \bigsqcup U_i \to X$ is an fppf cover of X.

Question 4.7. *Suppose $\mathscr{F} \in \mathrm{QCoh}(U)$. When does it come from a quasi-coherent sheaf on X? More precisely, what extra structure do you need to "descend it" to $\mathrm{QCoh}(X)$?*

Question 4.8. *Given $\mathscr{F}_1, \mathscr{F}_2 \in \mathrm{QCoh}(X)$ and a morphism $f : \mathscr{F}_1|_U \to \mathscr{F}_2|_U$, when does f come from X?*

Example. Say $U = \bigsqcup U_i \to X$ is a Zariski cover. In this case, the data we need to get a sheaf on X is a collection of isomorphisms ("gluing data")

$$\mathscr{F}_i|_{U_i \cap U_j} \xrightarrow{\sim} \mathscr{F}_j|_{U_i \cap U_j}$$

satisfying a cocycle condition on triple intersections. What about morphisms? A morphism $\mathscr{F} \to \mathscr{G}$ is the same as morphisms $\mathscr{F}|_{U_i} \to \mathscr{G}|_{U_i}$ which commute with the gluing data. △

Definition 4.9. Say $U \xrightarrow{f} X$ is some morphism. **Descent data for a qcoh sheaf** on U/X is

(1) A qcoh sheaf $\mathscr{F} \in \mathrm{QCoh}(U)$ on U

(2) gluing data $\varphi : \pi_1^*\mathscr{F} \xrightarrow{\sim} \pi_2^*\mathscr{F}$ where $\pi_1, \pi_2 : U \times_X U \to U$ the two projections.

(3) a cocycle condition

$$\pi_{23}^*\varphi \circ \pi_{12}^*\varphi = \pi_{13}^*\varphi$$

where $\pi_{ij} : U \times_X U \times_X U \to U \times_X U$ is projective from the ith factor onto the first factor and from the jth factor onto the second factor.

◇

Exercise. Unpack this definition in the case of a Zariski cover $U = \bigsqcup U_i \to X$.

Definition 4.10. Given Descent data $(\mathscr{F}, \varphi)$ and $(\mathscr{G}, \psi)$, a **morphism** $(\mathscr{F}, \varphi) \to (\mathscr{G}, \psi)$ is a map $h : \mathscr{F} \to \mathscr{G}$ such that

$$\begin{tikzcd} \pi_1^*\mathscr{F} \arrow[r, "\pi_1^* h"] \arrow[d, "\varphi"'] & \pi_1^*\mathscr{G} \arrow[d, "\psi"] \\ \pi_2^*\mathscr{F} \arrow[r, "\pi_2^* h"] & \pi_2^*\mathscr{G} \end{tikzcd}$$

commutes. ⋄

Let's state the main theorem now.

Theorem 4.11 (Descent for quasi-coherent sheaves). *Suppose $U \xrightarrow{f} X$ is fppf. Then, f^* induces an equivalence of categories between* $\mathrm{QCoh}(X)$ *and descent data on U/X.*

Remark 4.12. Given a $\mathscr{F} \in \mathrm{QCoh}(X)$, how do we get descent data on U? Consider $f^*\mathscr{F} \in \mathrm{QCoh}(U)$ along with gluing data

$$(f \circ \pi_1)^*\mathscr{F} \xrightarrow{\sim} (f \circ \pi_2)^*\mathscr{F}$$

on $U \times_X U$ coming from the fact that $f \circ \pi_1 = f \circ \pi_2 : U \times_U \to U \to X$, so we get an iso from pulling back the identity $\mathrm{id} : \mathscr{F} = \mathscr{F}$. ○

Example. Say $U = \bigsqcup U_i$ is a Zariski cover of X. What's a vector bundle? Take $\mathscr{O}_{U_i}^{\oplus n} \in \mathrm{QCoh}(U_i)$. To glue to a vector bundle on X, we need isos

$$\varphi_{ij} : \mathscr{O}_{U_i \cap U_j}^{\oplus n} \xrightarrow{\sim} \mathscr{O}_{U_i \cap U_j}^{\oplus n}$$

such that

$$\varphi_{jk}|_{U_i \cap U_j \cap U_k} \circ \varphi_{ij}|_{U_i \cap U_j \cap U_k} = \varphi_{ik}|_{U_i \cap U_j \cap U_k}.$$

△

Example. Say L/K is a Galois extension with Galois group G. Then, $\operatorname{Spec} L \to \operatorname{Spec} K$ is an étale cover. Descent data on $\operatorname{Spec} L / \operatorname{Spec} K$ is a qcoh sheaf on $\operatorname{Spec} L$ (i.e. an L-vector space V) with an isomorphism

$$\varphi : \pi_1^* V \xrightarrow{\sim} \pi_2^* V$$

satisfying the cocycle condition. Let's unpack this a little...

$$\operatorname{Spec} L \times_{\operatorname{Spec} k} \operatorname{Spec} L = \operatorname{Spec} L \otimes_k L = \bigsqcup_{\sigma : L \xrightarrow{\sim} L} \operatorname{Spec} L = \bigsqcup_{\mathrm{Gal}(L/k)} \operatorname{Spec} L$$

(a trivial[14] torsor for the Galois group).

Exercise. Convince yourself that descent data in this setting is the same as Galois descent data.

△

Let's prove the theorem now.

Proof of Theorem 4.11. We need to show that f^* is fully faithful and essentially surjective.

good reference: ch. 6 of Néron models by BLR

[14] The identity map gives a basepoint

(full faithfulness) Given $\mathscr{F}_1, \mathscr{F}_2 \in \mathrm{QCoh}(X)$, we have

$$\mathrm{Hom}_X(\mathscr{F}_1, \mathscr{F}_2) \xrightarrow{f^*} \mathrm{Hom}_U(f^*\mathscr{F}_1, f^*\mathscr{F}_2) \underset{\pi_2^*}{\overset{\pi_1^*}{\rightrightarrows}} \mathrm{Hom}_{U\times_X U}(q^*\mathscr{F}_1, q^*\mathscr{F}_2)$$

where $q = f \circ \pi_1 = f \circ \pi_2 : U \times_X U \to X$. Full faithfulness amounts to the claim that this is an equalizer diagram. This follows from

Claim 4.13. *$g \in \mathrm{Hom}_U(f^*\mathscr{F}_1, f^*\mathscr{F}_2)$ is a morphism of descent data if it maps to the same thing under π_1^*, π_2^*.*

Verifying this claim is left as an exercise. With this claim, we're reduced to showing this is an equalizer diagram. Note that if $\mathscr{F}_1 = \mathscr{O}$, then this exactly shows that $\mathscr{F}^{\text{ét}}$ (or even $\mathscr{F}^{fppf}$) is a sheaf. In order to prove this is an equalizer, we'll make use of the following lemma.

Lemma 4.14. *Suppose $R \to S$ is a faithfully flat ring morphism, and let N be an R-module. Then,*

$$N \xrightarrow{n \mapsto n\otimes 1} N \otimes_R S \underset{\mathrm{id}\otimes 1 \otimes \mathrm{id}}{\overset{\mathrm{id}\otimes\mathrm{id}\otimes 1}{\rightrightarrows}} N \times_R S \times_R S$$

is an equalizer diagram.[15]

Proof. Here's the big trick: WLOG the map $R \to S$ splits, i.e. there's a back map $\sigma : S \to R$ so $S \to R \to S$ is the identity. The point is that this is an equalizer diagram iff it is after faithful flat base change,[16] so we can replace $R \to S$ with $S \to S \otimes_R S$ which now has a section given by multiplication.

Now suppose $R \xrightarrow{f} S$ splits via $r : S \to R$. First, $N \to N \otimes_R S$ is injective since this map now splits via $\mathrm{id} \otimes r : N \otimes_R S \to N$. This just leaves exactness in the middle. We define $\widetilde{r} : S \otimes_R S \to S$ via $s_1 \otimes s_2 \mapsto s_1 \cdot f(r(s_2))$. Now note that

$$\mathrm{id}_N \otimes \widetilde{r}(n \otimes s \otimes 1 - n \otimes 1 \otimes s) = n \otimes s - n \otimes f(r(s)) = n \otimes s - n \cdot r(s) \otimes 1,$$

so if $n \otimes s \otimes 1 - n \otimes 1 \otimes s = 0$ (in kernel of differential), then $n \otimes s = n \otimes r(s) \otimes 1$ (in image of differential). This is the proof of pure tensors (we've shown if in kernel, then in image). Doing it for general tensors is left as an exercise. ■

We're out of time, so we will finish the proof next lecture...

■

One last remark

Remark 4.15. Say $R \to S$ is faithfully flat. Then,

$$N \to N \otimes S \to N \otimes S \otimes S \to \cdots \to N \otimes S^{\otimes r} \to \cdots$$

is always exact. Here, the maps are the usual alternating sum thing. ○

[15] This is the case $\mathscr{F}_1 = \mathscr{O}, \mathscr{F}_2 = \widetilde{N}$, $U = \mathrm{Spec}\, S$, and $X = \mathrm{Spec}\, R$

[16] Being an equalizer is equivalent to $0 \to N \to N \otimes_R S \to N \otimes_R S \otimes_R S$ being exact, where the last map is the difference of the two maps appearing in the (claimed) equalizer diagram

5 Lecture 5

Last time we started fppf descent, but did not finish. Recall that the ultimate goal was to show that qcoh sheaves on X and representable functors both give sheaves on $X_{\text{ét}}, X_{\text{fppf}}$. We were in the midst of proving a descent theorem for qcoh sheaves last time when class ended.

5.1 Jumping back in where we left off...

Recall that we were proving

Theorem 5.1 (**Descent for quasi-coherent sheaves**). *Suppose $U \xrightarrow{f} X$ is an fppf covver. Then, f^* induces an equivalence of categories*

$$f^* : \mathrm{QCoh}(X) \xrightarrow{\sim} \left\{ \begin{matrix} \textit{descent data for} \\ \textit{qcoh sheaves on } U/X \end{matrix} \right\}.$$

Proof.

(fully faithful) Recall that we had reduced full faithfulness to the claim that, given $\mathscr{F}_1, \mathscr{F}_2 \in \mathrm{QCoh}(X)$, the diagram

$$\mathrm{Hom}_X(\mathscr{F}_1, \mathscr{F}_2) \xrightarrow{f^*} \mathrm{Hom}_U(f^*\mathscr{F}_1, f^*\mathscr{F}_2) \underset{\pi_2^*}{\overset{\pi_1^*}{\rightrightarrows}} \mathrm{Hom}_{U\times_X U}\left(q^*\mathscr{F}_1, q^*\mathscr{F}_2\right),$$

where $\pi_1, \pi_2 : U \times_X U \to U$ are the projections and $q = f \circ \pi_1 = f \circ \pi_2 : U \times_X U \to X$, is an equalizer. Towards this goal, we had proven the following lemma.

Lemma 5.2. *Suppose $R \to S$ is a faithfully flat ring morphism, and let N be an R-module. Then,*

$$N \xrightarrow{n \mapsto n \otimes 1} N \otimes_R S \underset{\mathrm{id}\otimes 1 \otimes \mathrm{id}}{\overset{\mathrm{id}\otimes\mathrm{id}\otimes 1}{\rightrightarrows}} N \times_R S \times_R S$$

is an equalizer diagram.

Let's use this to prove full faithfulness. We first reduce to the case where $U \to X$ is affine. We leave this as an exercise.[17] Now we're in the affine case $R \to S$ faithfully flat ($U = \operatorname{Spec} S, X = \operatorname{Spec} R$) and N, M are R-modules. We want

$$\mathrm{Hom}_R(M, N) \to \mathrm{Hom}_S(M \otimes_R S, N \otimes_R S) \rightrightarrows \mathrm{Hom}_{S\otimes_R S}(M \otimes_R S \otimes_R S, N \otimes_R S \otimes_R S)$$

to be an equalizer diagram. Injectivity of the first map (exactness on the left) follows from injectivity of $N \to N \otimes_R S$. Similarly, exactness in the middle follows from exactness in the middle of the diagram in the lemma. Secretly, this is all just left exactness of $\mathrm{Hom}_R(M, -)$.

(essentially surjective) Say we have an fppf cover $U \xrightarrow{f} X$ and we're given descent data $(\mathscr{F}, \varphi)$ on U/X. We want some $\mathscr{G} \in \mathrm{QCoh}(X)$ such that $\mathscr{G}^* \xrightarrow{\sim} \mathscr{F}$ (and this iso respects gluing data). Again,

[17] Need to use that the map is of finite presentation (even just that it's quasicompact)

we reduce to the affine case, and by "we" I mean "you, the reader" since this is left as an exercise. Now we have $R \xrightarrow{f} S$ and an S-module M with descent data

$$\varphi : M \otimes_R S \xrightarrow{\sim} S \otimes_R M,$$

an iso of $S \otimes_R S$ modules (satisfying a cocycle condition). We have two maps

$$M \rightrightarrows S \otimes M$$

given by $m \mapsto 1 \otimes m$ and $m \mapsto \varphi(m \otimes 1)$. Now, set

$$K = \mathrm{eq}(M \rightrightarrows S \otimes M),$$

the equalizer of this diagram. We claim that the natural map $K \otimes_R S \to M$ is an isomorphism (compatible with descent data). As before, we reduce to the case that $R \to S$ has a section, and then this case is easy (pull back along section).

■

Corollary 5.3. *If $\mathscr{F} \in \mathrm{QCoh}(X)$, then the presheaf $\mathscr{F}^{\text{ét}}$ on $X_{\text{ét}}$,*

$$\mathscr{F}^{\text{ét}}(U \xrightarrow{\pi} X) = (\pi^* \mathscr{F})(U),$$

is a sheaf on $X_{\text{ét}}$.

Proof. We actually only need full faithfullness above. Say $U \xrightarrow{f} V$ is an étale cover. We want

$$\mathscr{F}(V) \to \mathscr{F}(U) \rightrightarrows \mathscr{F}(U \times_V U)$$

to be an equalizer diagram. This is precisely the diagram

$$\mathrm{Hom}_V(\mathscr{O}_V, \mathscr{F}) \to \mathrm{Hom}_U(\mathscr{O}_U, f^*\mathscr{F}) \rightrightarrows \mathrm{Hom}_{U \times_V U}(\mathscr{O}_{U \times_V U}, q^* \mathscr{F})$$

from before. ■

Example. $\mathscr{O}_X^{\text{ét}} : (U \to X) \mapsto \Gamma(U, \mathscr{O}_U)$ is a sheaf. △

If you think back to the beginning of last class, we still need to show that representable functors are sheaves. This is acheived in the following theorem.

Theorem 5.4. *Say $p : U \to X$ is an fppf cover. Then the functor*

$$p^* : \mathrm{Sch}/X \to \left\{ \begin{array}{c} \textit{Descent data for} \\ \textit{schemes on } U/X \end{array} \right\}$$

is fully faithful.

Proof. As an exercise, reduce to the case where *everything* is affine (in fact, enough to simply reduce to the case of AffSch/X).

Let Y, Z be X-schemes. We need to show that

$$\mathrm{Hom}_X(Y,Z) \to \mathrm{Hom}_U(p^*Y, p^*Z) \rightrightarrows \mathrm{Hom}_{U\times_X U}(q^*Y, q^*Z)$$

is an equalizer (here $q : U \times_X U \to U \to X$). By the exercise, we may assume $Y = \mathbf{Spec}\,\mathscr{O}_Y$ and $Z = \mathbf{Spec}\,\mathscr{O}_Z$ where $\mathscr{O}_Y, \mathscr{O}_Z$ are qcoh sheaves of $\mathscr{O}_X$-algebras. Now, this diagram is

$$\mathrm{Hom}_{\mathscr{O}_X\text{-alg}}(\mathscr{O}_Z, \mathscr{O}_Y) \to \mathrm{Hom}_{\mathscr{O}_U\text{-alg}}(p^*\mathscr{O}_Z, p^*\mathscr{O}_Y) \rightrightarrows \mathrm{Hom}_{\mathscr{O}_{U\times_X U}\text{-alg}}(q^*\mathscr{O}_Z, q^*\mathscr{O}_Y)$$

which is indeed an equalizer by descent for qcoh sheaves.[18] ■

Corollary 5.5. *If $Z \in \mathrm{Sch}/X$, then* $\mathrm{Hom}(-,Z)$ *is a sheaf on $X_{fppf}, X_{\acute{E}t}, X_{\acute{e}t}$, etc.*

Remark 5.6. p^* is *not* essentially surjective in general for schemes. Descent data for scheme relative to an étale cover U/X is called an **algebraic space**. When this pullback functor is an equivalence of categories, one calls it **effective descent**. Descent is effective for affine schemes as well as for **polarized schemes** (i.e. schemes with a choice of (relatively) ample line bundle). ○

We now have a bunch of examples we *know* are sheaves.

Example.

- $\mathbb{G}_m : U \mapsto \mathscr{O}_U(U)^\times$
- $\mu_\ell : U \mapsto \left\{f \in \mathscr{O}_U(U) : f^\ell = 1\right\}$
- $\underline{\mathbb{Z}/\ell\mathbb{Z}} : U \to \mathrm{Hom}_{cts}(U, \mathbb{Z}/\ell\mathbb{Z})$
- $\mathrm{Hilb}^{p(t)}(\mathbb{P}^n)$
- $\mathbb{P}^n$

are all sheaves. △

Exercise. Work out Galois descent from this point of view.

5.2 Étale Cohomology

We "defined" étale cohomology earlier, but there were a few missing details we needed to fill in. These were

- The category of abelian sheaves in $X_{\text{ét}}$ is abelian.
- This category has enough injectives

We won't have time to do this today (only like 10 minutes left), but we can mention the crucial ingredients.

Remark 5.7. Both of these facts are true for the category of abelian sheaves on *any* site. We won't prove them in this generality, but this is still good to know. ○

The crucial ingredient will be the following theorem.

[18] Need to check descent works with maps of algebras, not just modules, but this is easy

Theorem 5.8. *Say* τ *is a site. Then the forgetful functor*

$$\mathrm{Sh}(\tau) \to \mathrm{Psh}(\tau)$$

has a left adjoint, which we call ***sheafification****.*

We'll prove this for $\tau = X_{\text{ét}}$. We need some preliminaries. Say $f : \tau_1 \to \tau_2$ is a continuous morphism of sites (so really $f^{-1} : \tau_2 \to \tau_1$).

- (pushforward) Given $\mathscr{G} \in \mathrm{Sh}(\tau_1)$, we define $f_*\mathscr{G} \in \mathrm{Sh}(\tau_2)$ via

$$(f_*\mathscr{G})(U) := \mathscr{G}(f^{-1}(U)).$$

  Exercise. $f_*\mathscr{G}$ is a sheaf, not just a presheaf.

We're out of time, so next time it's stalks, sheafification, $\mathrm{Sh}(X_{\text{ét}})$ being abelian, etc.

6 Lecture 6

Last time we talked about fppf descent. Today, it's cohomology.

We begin with some remarks that came up last time/in the discord.

- Let X be a scheme. We showed last time that there is an equivalence of categories

$$\mathrm{QCoh}(X_{\mathrm{zar}}) \xrightarrow{\sim} \mathrm{QCoh}(X_{\text{ét}}) \xrightarrow{\sim} \mathrm{QCoh}(X_{\mathrm{fppf}})$$

  between the categories of quasi-coherent sheaves on "small" sites above X. What about the big sites? Well,

$$\mathrm{QCoh}(X_{\mathrm{Zar}}) \xrightarrow{\sim} \mathrm{QCoh}(X_{\text{Ét}}) \xrightarrow{\sim} \mathrm{QCoh}(X_{\mathrm{Fppf}}).$$

- We need a small correction from last time. We claimed that étale descent data for schemes was the same thing as an algebraic space, but this is not quite true. It is true that étale descent data for schemes gives an algebraic space, but not all arise in thi s fashion.

What's the goal for today? Fill in some remaining gaps, so we can actually define étale cohomology.

Goal. The category of abelian sheaves on $X_{\text{ét}}$ is abelian with enough injectives.

Once we have this, we can define étale cohomology as the right derived functor(s) of global sections. Recall that the crucial ingredient in proving this category is abelian is the following.

Theorem 6.1. *Let* τ *be any site. The forgetful functor* $\mathrm{Sh}(\tau) \to \mathrm{Psh}(\tau)$ *has a left adjoint, called* ***sheafification*** *(we'll prove only for* $X_{\acute{e}t}$*).*

6.1 Sheaf Operations

It'll be useful to define some sheaf operations.

Recall 6.2. Let $f : \tau_1 \to \tau_2$ be a continuous map of sites. For $\mathscr{G} \in \mathrm{Sh}(\tau_1)$, we define the *pushforward sheaf*

$$f_*\mathscr{G} : U \mapsto \mathscr{G}(f^{-1}(U)).$$

Showing this is a sheaf (and not just a presheaf) is left as an exercise. $\odot$

Example. Let $f : X \to Y$ be a map of schemes. Then we get a (continuous) map $f : X_{\text{ét}} \to Y_{\text{ét}}$ between small étale sites given by $f^{-1}(U/Y) := U \times_Y X/X$. Hence, we get a functor $f_* : \mathrm{Sh}(X_{\text{ét}}) \to \mathrm{Sh}(Y_{\text{ét}})$. $\triangle$

Example. Let k be an algebraically closed field, and let $\iota_{\overline{x}} : \operatorname{Spec} k \to X$ be a geometric point of X. Note that $\mathrm{Sh}((\operatorname{Spec} k)_{\text{ét}}) = \mathrm{Set}$ (exercise[19]). Given $\mathscr{F} \in \mathrm{Sh}(\operatorname{Spec} k_{\text{ét}}) = \mathrm{Set}$, we have

$$(\iota_{\overline{x}})_*\mathscr{F}(U \to X) = \mathscr{F}(U \times_X \overline{x}) = \mathscr{F}\left(\bigsqcup \operatorname{Spec} k\right) = \prod \mathscr{F}(\operatorname{Spec} k)$$

where the (co)products above are over preimages of $\operatorname{Spec} k$ in U. $\triangle$

Warning 6.3. Sheaves on $\operatorname{Spec} k_{\text{ét}}$ when k is *not* algebraically closed are not just sets. We'll see later that they're discrete G-modules, where $G = \mathrm{Gal}(k^s/k)$. $\bullet$

Definition 6.4. A sheaf of the form $(\iota_{\overline{x}})_*\mathscr{F}$ as in the previous example is called a **skyscraper sheaf**. $\diamond$

What about pullbacks? We won't be able to define them yet (since we don't know how to sheafify), but we can define pullbacks to geometric points (a sheaf on $\operatorname{Spec} \overline{k}$ is a set, so easy to sheafify).

Definition 6.5. Let $\iota_{\overline{x}} : \operatorname{Spec} k \to X$ be a geometric point (so $k = \overline{k}$), and let $\mathscr{F} \in \mathrm{Sh}(X)$ be a sheaf on X. The **pullback of $\mathscr{F}$ to $\overline{x}$**, or **stalk of $\mathscr{F}$ at $\overline{x}$**, is the set

$$\mathscr{F}_{\overline{x}} := \varinjlim_{(U,\overline{u})} \mathscr{F}(U)$$

where the direct limit is taken over diagrams

$$\begin{tikzcd} \overline{u} \arrow[r, "\text{geom pt}"] \arrow[d] & U \arrow[d, "\text{ét}"] \\ \overline{x} \arrow[r, "\iota_{\overline{x}}"] & X \end{tikzcd}$$

$\diamond$

Remark 6.6. Note we don't have to be at a closed point for the above definition to work. We could take e.g. a geometric generic point. $\circ$

Example. Take $\mathscr{F} = \underline{\mathbb{Z}/\ell\mathbb{Z}}$ and $\overline{x} \hookrightarrow X$ any geometric point. Then,

$$\iota_{\overline{x}}^*\underline{\mathbb{Z}/\ell\mathbb{Z}} = \mathbb{Z}/\ell\mathbb{Z}.$$

In short, the $(U, \overline{u})$ with U connected are cofinal and on each of them $\mathscr{F}(U) = \mathbb{Z}/\ell\mathbb{Z}$. $\triangle$

[19]Hint: the point is that an étale cover of $\operatorname{Spec} k$ is a bunch of disjoint copies of $\operatorname{Spec} k$ (note $k = \overline{k}$), so any sheaf is determined by its value on $\operatorname{Spec} k$

Example. Take $\mathscr{F} = \mathscr{O}_X^{\text{ét}}$. Then,

$$\iota_{\overline{x}}^* \mathscr{O}_X^{\text{ét}} = \mathscr{O}_{X,\overline{x}}^{sh},$$

the strict Hensalization of the usual Zariski stalk. △

Lemma 6.7. *Suppose $\mathscr{F}, \mathscr{G}$ are sheaves of abelian groups on $X_{ét}$. Then, TFAE*

(1) *$\mathscr{F} \to \mathscr{G}$ is an epimorphism*

(2) *$\mathscr{F} \to \mathscr{G}$ is* ***locally surjective****, i.e. given $s \in \mathscr{G}(U)$, there exists an étale cover $U' \to U$ such that $s|_{U'}$ is in the image of $\mathscr{F}(U')$.*

(3) *$\mathscr{F}_{\overline{x}} \to \mathscr{G}_{\overline{x}}$ is surjective for all geometric points $\overline{x} \to X$*

Proof. (**(2)** $\implies$ **(1)**) Say we have

$$\mathscr{F} \xrightarrow{f} \mathscr{G} \underset{b}{\overset{a}{\rightrightarrows}} \mathscr{H}$$

such that the two compositions agree. We want to show $a = b$. Say we have some $s \in \mathscr{G}(U)$. Then there is a cover $U' \to U$ and a $t \in \mathscr{F}(U')$ such that $f(t) = s|_{U'}$, so $a(s|_{U'}) = a(f(t)) = b(f(t)) = b(s|_{U'}) \in \mathscr{H}$, and so $a(s) = b(s)$ since U' was a cover of U (sheaf condition). Hence, $a = b$, so f is epic.

(**(1)** $\implies$ **(3)**) (Contrapositive) Suppose $\mathscr{F}_{\overline{x}} \to \mathscr{G}_{\overline{x}}$ is *not* surjective for some geom point $\overline{x}$, and let $\Lambda = \operatorname{coker}(\mathscr{F}_{\overline{x}} \to \mathscr{G}_{\overline{x}})$. Consider the diagram

$$\mathscr{F} \to \mathscr{G} \overset{0}{\rightrightarrows} (\iota_{\overline{x}})_* \Lambda,$$

where the bottom map from $\mathscr{G}$ is the natural map into that skyscraper sheaf. By definition, both compositions are 0, but the two arrows on the right don't agree, so $\mathscr{F} \to \mathscr{G}$ is not epic.

(**(3)** $\implies$ **(2)**) Fix some $s \in \mathscr{G}(U)$. We want to find $U' \to U$ so that $s|_{U'}$ comes from $\mathscr{F}(U')$. Choose $\overline{x} \in U$. We know $\mathscr{F}_{\overline{x}} \to \mathscr{G}_{\overline{x}}$ is surjective. By definition, this means there exists some étale neighborhood $(V, \overline{v})$ of $\overline{x}$ so that $s|_V$ is in the image of $\mathscr{F}$. Now choose $\overline{x}'$ not in the image of V, and keep going... ■

Lemma 6.8. *Suppose $0 \to \mathscr{F} \to \mathscr{G} \to \mathscr{H}$ is a sequence of abelian sheaves on $X_{ét}$. Then, TFAE*

(1) *The sequence is exact*[20]

(2) *$0 \to \mathscr{F}(U) \to \mathscr{G}(U) \to \mathscr{H}(U)$ is exact for all U.*

(3) *$0 \to \mathscr{F}_{\overline{x}} \to \mathscr{G}_{\overline{x}} \to \mathscr{H}_{\overline{x}}$ is exact for all geometric points $\overline{x}$.*

Proof. Exercise. ■

Corollary 6.9. *Sheafification exists for* $\operatorname{Psh}(X_{ét})$

Proof. We essentially reproduce the proof for topological given e.g. in Hartshorne. The first step is to construct an "espace étalé"

For each $x \in X$, choose a geometric point $\overline{x}$ lying over x. Given a presheaf $\mathscr{F} \in \operatorname{Psh}(X_{\text{ét}})$, we define

$$\operatorname{Esp}(\mathscr{F}) = \prod_x (\iota_{\overline{x}})_* \mathscr{F}_{\overline{x}}.$$

[20] i.e. $\mathscr{F} \to \mathscr{G}$ is a monomorphism, and is the kernel of $\mathscr{G} \to \mathscr{H}$, where 'kernel' means equalizer of $\mathscr{G} \underset{0}{\overset{\mathscr{H}}{\rightrightarrows}}$

Note that this is a sheaf. There is a natural map of presheaves $\mathscr{F} \to \mathrm{Esp}(\mathscr{F})$. We let $\mathscr{F}^a$ be the sub*sheaf* of $\mathrm{Esp}(\mathscr{F})$ generated by $\mathscr{F}$, i.e.

$$\mathscr{F}^a(U) = \{s \in \mathrm{Esp}(\mathscr{F})(U) : s \text{ locally in image of } \mathscr{F}\}.$$

Note that $\mathscr{F}^a$ is indeed a sheaf. Checking that $\mathscr{F} \mapsto \mathscr{F}^a$ is left adjoint to the forgetful functor is left as an exercise. ■

Corollary 6.10. *Colimits exist in* $\mathrm{Sh}(X_{\acute{e}t})$.

Proof. Colimits exist for presheaves (compute pointwise) and left adjoints send colimits to colimits. In particular,

$$\mathrm{colim}_{i\in I} \mathscr{F}_i = (\mathrm{colim}_{\mathrm{Psh},i} \mathscr{F}_i)^a.$$

■

Corollary 6.11. $\mathrm{Sh}^{ab}(X_{\acute{e}t})$ *is an abelian category.*

Proof.

- limits exist (defined pointwise)
- cokernels exist (cokernels are colimits)
- images are coimages by checking on stalks ($\mathrm{coker}\ker = \ker\mathrm{coker}$)

■

Out of time. Pick up next time.

7 Lecture 7

Last time: stalks, sheafification, $\mathrm{Sh}(X_{\text{ét}})$ is abelian.

Remark 7.1. We clear up an issue from before. We earlier claimed that there is a morphism $X_{\text{fppf}} \to X_{\text{ét}}$, but this is actually no such thing. The point is that X_{fppf} consists of schemes which (among other things) are of finite presentation above X, while schemes in $X_{\text{ét}}$ are only required to be *locally* of finite presentation above X. What is true is that we have a zig-zag

$$X_{\text{fppf}} \to X_{\text{ét, fin pres}} \leftarrow X_{\text{ét}}$$

and the right arrow induces an equivalence on $\mathrm{Sh}(-)$ under pushforward. So there's still an adjoint pair of functors between sheaves on $X_{\text{ét}}$ and sheaves on X_{fppf}. ○

7.1 Enough Injectives

Recall we were working towards the definition of sheaf cohomology. We still need to have enough injectives.

Theorem 7.2. $\mathrm{Sh}^{ab}(X_{\acute{e}t})$ *has enough injectives.*

Proof. Let $\mathscr{F} \in \mathrm{Sh}^{ab}(X_{\text{ét}})$ be an abelian sheaf. We want an injective sheaf $\mathscr{I}$ w/ $\mathscr{F} \hookrightarrow \mathscr{I}$. For each $x \in X$, choose a geometric point $\overline{x} \to x \to X$, and let $I(\overline{x})$ be an injective abelian group with a map $\mathscr{F}_{\overline{x}} \hookrightarrow I(\overline{x})$. Now, we claim that $\mathscr{I} := \prod_x (\iota_{\overline{x}})_* I(\overline{x})$ works.

The map $\mathscr{F} \hookrightarrow \mathscr{I}$ is given by sending sections to their germs. To check this is a monomorphism, one just checks on stalks. Checking that $\mathscr{I}$ is injective is left as an exercise. ∎

We now know that the category of abelian sheaves on $X_{\text{ét}}$ has enough injective.

Remark 7.3. Thi sis true for abelian sheaves on any site, but the proof in general is substantially harder. ○

7.2 Inverse images

Let $f : X \to Y$ be a map of schemes.

Definition 7.4. The **presheaf inverse image** is the functor $f^{-1} : \mathrm{Psh}(Y_{\text{ét}}) \to \mathrm{Psh}(X_{\text{ét}})$ given by

$$(f^{-1}\mathscr{F})(V \xrightarrow{\text{ét}} X) = \varinjlim \mathscr{F}(U \to X)$$

with limit taken over diagrams

$$\begin{array}{ccc} V & \longrightarrow & U \\ {\scriptstyle \text{ét}}\downarrow & & \downarrow{\scriptstyle \text{ét}} \\ X & \longrightarrow & Y \end{array}$$

This same definition actually works for any map of sites. ◇

Fact (Exercise)**.** f^{-1} is left adjoint to the pushforward

$$f^{-1} : \mathrm{Psh}(T_{\text{ét}}) \rightleftarrows \mathrm{Psh}(X_{\text{ét}}) : f_*.$$

Definition 7.5. The **sheaf inverse image** is the functor $f^*\mathscr{F} := (f^{-1}\mathscr{F})^a$, the sheafification of the presheaf inverse image. ◇

Remark 7.6. sheafification is a left adjoint, and left adjoints preserve left adjoints, so f^* is left adjoint to f_*. ○

Example. If $\iota : \overline{x} \hookrightarrow X$ is a geometric point, then $\iota^*\mathscr{F} = \mathscr{F}_{\overline{x}}$. △

Example. If $f : X \to Y$ is any morphism, then $f^*\underline{\mathbb{Z}/\ell\mathbb{Z}} = \underline{\mathbb{Z}/\ell\mathbb{Z}}$. △

Example. In general, if you have $Y \xrightarrow{f} X$ and $\mathscr{F} = \underline{\mathrm{Hom}}_X(-, Z)$, then

$$f^*\mathscr{F} = \underline{\mathrm{Hom}}_Y(-, Y \times_X Z).$$

△

7.3 Étale Cohomology

Recall 7.7. Given an abelian sheaf $\mathscr{F} \in \mathrm{Sh}(X_{\text{ét}})$, its **étale cohomology** is

$$\mathrm{H}^i(X_{\text{ét}}, \mathscr{F}) = R^i\Gamma(X, \mathscr{F}).$$

$\odot$

How do we compute this? Choose an injective resolution

$$\mathscr{F} \to \mathscr{I}^0 \to \mathscr{I}^1 \to \cdots$$

and then

$$\mathrm{H}^i(X_{\text{ét}}, \mathscr{F}) = \mathrm{H}^i(\Gamma(X, \mathscr{I}^\bullet)).$$

Remark 7.8. This is how you compute (right) derived functors in general. For example,

$$R^i\pi_*\mathscr{F} = \mathrm{H}^i(\pi_*\mathscr{I}^\bullet)$$

which are sheaves on $Y_{\text{ét}}$. Here, $\pi : X_{\text{ét}} \to Y_{\text{ét}}$. $\circ$

Exercise. $L^i\pi^*\mathscr{G} = 0$ if $i > 0$. Pullback is exact.

Basic properties of étale cohomology:

(1) $\mathrm{H}^0(X_{\text{ét}}, \mathscr{F}) = \mathscr{F}(X) = \Gamma(X, \mathscr{F})$

(2) $\mathrm{H}^i(\mathscr{I}) = 0$ for $i > 0$ when $\mathscr{I}$ injective.

(3) Given a short exact sequence $0 \to \mathscr{F}_1 \to \mathscr{F}_2 \to \mathscr{F}_3 \to 0$ of sheaves on $X_{\text{ét}}$, we get a (natural) long exact sequence

$$\cdots \to \mathrm{H}^{i-1}(X_{\text{ét}}, \mathscr{F}_1) \to \mathrm{H}^{i-1}(X_{\text{ét}}, \mathscr{F}_2) \to \mathrm{H}^{i-1}(X_{\text{ét}}, \mathscr{F}_3) \to \mathrm{H}^i(X_{\text{ét}}, \mathscr{F}_1) \to \cdots$$

Example. Let k be a field. Then, $\mathrm{Sh}((\operatorname{Spec} k)_{\text{ét}})$ is equivalent to the category of discrete G-modules (choose sep closure k^s and then $G = \mathrm{Gal}(k^s/k)$). The natural functor is

$$\begin{array}{rccc} \iota : & \mathrm{Sh}((\operatorname{Spec} k)_{\text{ét}}) & \longrightarrow & \text{Discrete } G\text{-modules} \\ & \mathscr{F} & \longmapsto & \varinjlim_{k^s/L/k} \mathscr{F}(\operatorname{Spec} L) \end{array}$$

Think of as "evaluation on separable closure." The claim is that this is well-defined (i.e. really lands in discrete G-modules), and is an equivalence of categories.

Let's sketch a proof of this. We can describe the inverse functor. Given étale $V \to \operatorname{Spec} k$, we can write $V = \bigsqcup_i \operatorname{Spec} k_i$ with k_i/k separable, so we send a discrete G-module M to the sheaf

$$V \mapsto \prod_i M^{\mathrm{Gal}(k^s/k_i)}.$$

Check that this works. $\triangle$

Corollary 7.9.

$$\mathrm{H}^i((\operatorname{Spec} k)_{ét}, \mathscr{F}) = \mathrm{H}^i(G, \iota\mathscr{F}).$$

Proof. We know that

$$\Gamma(\operatorname{Spec} k, \mathscr{F}) = (\iota\mathscr{F})^G,$$

so $\mathrm{H}^0 \leftrightarrow \iota$ invariants, so étale cohomology is the derived functor of invariants (i.e. group cohomology). ■

Note important above that we were dealing with discrete G-modules. Continuous group cohomology is not actually a derived functor. This is related to why people don't take étale cohomology directly with sheaves like $\underline{\mathbb{Z}_\ell}$.

7.4 Čech Cohomology

Recall there's a more computable version of cohomology which is often useful, so let's introduce it in present contexts.

Warning 7.10. Čech cohomology does not always compute étale cohomology (this is already true for sheaf cohomology on bad spaces). •

Warning 7.11. Čech cohomology is not actually computable, but it will still be useful. This is because, in general, acyclic covers do not exist.[21] •

Say we have an étale cover $U = \bigsqcup_i U_i \to X$, and let $\mathscr{F} \in \operatorname{Sh}(X_{\text{ét}})$ be an abelian sheaf. From this, we get the diagram (a simplicial X-sheaf?)

$$\begin{tikzcd} X & U \arrow[l] & U \times_X U \arrow[l, shift left] \arrow[l, shift right] & U \times_X U \times_X U \arrow[l, shift left=2] \arrow[l] \arrow[l, shift right=2] & \cdots \arrow[l, shift left=3] \arrow[l, shift left=1] \arrow[l, shift right=1] \arrow[l, shift right=3] \end{tikzcd}$$

(think of as double, triple, etc. intersections). Applying our sheaf, we get a diagram (a cosimplicial sheaf of abelian groups)

$$\begin{tikzcd} \mathscr{F}(U) \arrow[r, shift left, "\mathrm{d}^0"] \arrow[r, shift right, "\mathrm{d}^1"'] & \mathscr{F}(U \times_X U) \arrow[r, shift left=2] \arrow[r] \arrow[r, shift right=2] & \mathscr{F}(U \times_X U \times_X U) \arrow[r, shift left=3] \arrow[r, shift left=1] \arrow[r, shift right=1] \arrow[r, shift right=3] & \cdots \end{tikzcd}$$

The usual alternating sum construction flattens this into a chain complex, called the **Čech complex**,

$$\check{C}^\bullet(U/X, \mathscr{F}) : 0 \to \mathscr{F}(U) \to \mathscr{F}(U \times_X U) \to \cdots$$

whose nth differential

$$\mathrm{d}_n = \sum_{i=0}^n (-1)^i d^i$$

is given by the alternating sum of the differentials in the diagram from before. The **total Čech complex** is

$$\check{C}^\bullet(X_{\text{ét}}, \mathscr{F}) := \varinjlim_{\{U_i \to X\}_i} \check{C}^\bullet(U/X, \mathscr{F}),$$

[21] This is a main difference between étale cohomology and say singular cohomology of manifolds (which are locally contractible). We may see later that schemes are locally $K(\pi, 1)$'s, so étale cohomology can be computed in terms of group cohomology. This is due to Artin (if I heard correctly)

with direct limit taken over all covering families.

Definition 7.12. We define **Čech cohomology** as

$$\check{\mathrm{H}}^i(U/X, \mathscr{F}) = \mathrm{H}^i(\check{C}^\bullet(U/X, \mathscr{F}))$$

and

$$\check{\mathrm{H}}^i(X_{\text{ét}}, \mathscr{F}) = \mathrm{H}^i(\check{C}^\bullet(X_{\text{ét}}, \mathscr{F})).$$

$\diamond$

Warning 7.13. There are set-theoretic issues with taking a direct limit over all covers. There are (at least) two ways to resolve this. We resolve by (implicitly) working with $X_{\text{ét, fin. pres}}$ $\bullet$

Proposition 7.14. $\check{\mathrm{H}}^0(U/X, \mathscr{F}) = \check{\mathrm{H}}^0(X_{ét}, \mathscr{F}) = \mathrm{H}^0(X_{ét}, \mathscr{F})$

Proof. This follows directly from the sheaf condition.

$$\mathscr{F}(X) \to \mathscr{F}(U) \to \mathscr{F}(U \times_X U)$$

is exact. This equates first and last terms. For middle, use that directed limits are exact. ■

Proposition 7.15. $\check{\mathrm{H}}^i(U/X, \mathscr{I}) = \check{\mathrm{H}}^i(X_{ét}, \mathscr{I}) = 0$ *if* $i > 0$ *and* $\mathscr{I}$ *is injective.*

Proof. We'll give a different description of Čech cohomology. It is enough to show that $\check{C}^\bullet(U/X, \mathrm{d})$ is exact away from 0. Let $\mathbb{Z}_U = \mathbb{Z}[\mathrm{Hom}_X(-, U)]$, i.e. $\mathbb{Z}_U(V)$ is the free abelian group on $\mathrm{Hom}_X(V, U)$.

Claim 7.16. *We can rewrite the Čech complex as*

$$\check{C}^\bullet(U/X, \mathscr{I}) : \mathrm{Hom}(\mathbb{Z}_U, \mathscr{I}) \to \mathrm{Hom}(\mathbb{Z}_{U \times_X U}, \mathscr{I}) \to \mathrm{Hom}(\mathbb{Z}_{U \times_X U \times_X U}, \mathscr{I}) \to \cdots$$

This is basically just Yoneda's lemma. Note the above comes from a cosimplicial diagram

$$\begin{tikzcd} \mathbb{Z}_U \arrow[r, shift left, "\mathrm{d}^0"] \arrow[r, shift right, "\mathrm{d}^1"'] & \mathbb{Z}_{U \times_X U} \arrow[r, shift left=2] \arrow[r] \arrow[r, shift right=2] & \mathbb{Z}_{U \times_X U \times_X U} \arrow[r, shift left=3] \arrow[r, shift left=1] \arrow[r, shift right=1] \arrow[r, shift right=3] & \cdots \end{tikzcd}$$

which is independent of the choice of sheaf. Now it is enough to show that

$$\mathbb{Z} \to \mathbb{Z}_U \to \mathbb{Z}_{U \times_X U} \to \cdots$$

is exact. This is because $\mathrm{Hom}(-, \mathscr{I})$ is an exact functor. This is actually a special case of the following fact: given a set S,

$$\mathbb{Z} \to \mathbb{Z}^S \to \mathbb{Z}^{S \times S} \to \mathbb{Z}^{S \times S \times S} \to \cdots$$

is always exact for any set S.

Proof. Base change to $\mathbb{Z}^S$ (flat $\mathbb{Z}$-module supported everywhere), and then there's a natural homotopy. ■

This finishes the proof. ■

This shows that Čech cohomology agrees with étale cohomology in degree 0, and they both vanish on injectives. For them to agree, Čech cohomology would need to be a δ-functor.

Theorem 7.17. *If for all short exact sequences* $0 \to \mathscr{F}_1 \to \mathscr{F}_2 \to \mathscr{F}_3 \to 0$ *in* $\mathrm{Sh}(X_{\acute{e}t})$*, the induced sequence*

$$0 \to \check{C}(X_{\acute{e}t}, \mathscr{F}_1) \to \check{C}(X_{\acute{e}t}, \mathscr{F}_2) \to \check{C}(X_{\acute{e}t}, \mathscr{F}_3) \to 0$$

is also exact, then $\check{\mathrm{H}}^i(X_{\acute{e}t}, \mathscr{F}) \xrightarrow{\sim} \mathrm{H}^i(X_{\acute{e}t}, \mathscr{F})$ *for all* $i, \mathscr{F}$.

Proof. We'll see this next time using the Čech-to-derived spectral sequence. ■

Remark 7.18. The above sequence of Čech complexes is always left exact, but not always right exact. ○

Theorem 7.19 (Milne, III)**.** *The above condition holds if X is quasi-compact, and if any finite subset of X is contained in an affine (e.g. X quasi-projective).*

Remember: Čech cohomology computes étale cohomology when X quasi-compact and any finite subset is contained in an affine (e.g. X quasi-projective)

8 Lecture 8

Recall that our current goal is to understand how to compute Étale cohomology. Last time we introduced Čech cohomology. Given an étale cover $U \overset{\text{ét}}{\twoheadrightarrow} X$, this is the cohomology of the complex

$$\check{C}(U/X, \mathscr{F}) : \mathscr{F}(U) \to \mathscr{F}(U \times_X U) \to \mathscr{F}(U \times_X U \times_X U) \to \cdots,$$

or, for something more intrinsic to $X_{\text{ét}}$, cohomology of the complex

$$\check{C}(X_{\text{ét}}, \mathscr{F}) = \varinjlim_{U/X} \check{C}(U/X, \mathscr{F}),$$

with colimit taken over covers of X.

Warning 8.1. $\check{\mathrm{H}}^i(X_{\text{ét}}, \mathscr{F})$ is not in general isomorphic to derived functor cohomology. •

Theorem 8.2 (Milne, Étale Cohomology Sect. III)**.** *Čech cohomology is canonically isomorphic to derived functor cohomology if X is quasi-compact and satisfies: any finite subset of X ic contained in an affine (true e.g. if X is quasi-projective).*

Remark 8.3. There is a version of Čech cohomology with covers replaced by "hypercovers" and this does always compute derived functor cohomology. ○

8.1 Čech-to-derived spectral sequence

Start with an injective resolution

$$\mathscr{F} \to \mathscr{I}^0 \to \mathscr{I}^1 \to \mathscr{I}^2 \to \cdots$$

Given a cover[22] $U \to X$, we get a sequence of complexes (a double complex)

$$\check{C}(U/X, \mathscr{I}^0) \to \check{C}(U/X, \mathscr{I}^1) \to \cdots.$$

[22] Can also make this work for covers consisting of mulitple objects over X

The horizontal differential is coming from our injective resolution, and the vertical differentials are coming from the Čech complex. Let's expand this out a bit

$$\begin{array}{ccccc} \vdots & & \vdots & & \\ \uparrow & & \uparrow & & \\ \mathscr{I}^0(U \times_X U) & \longrightarrow & \mathscr{I}^1(U \times_X U) & \longrightarrow & \cdots \\ \uparrow & & \uparrow & & \\ \mathscr{I}^0(U) & \longrightarrow & \mathscr{I}^1(U) & \longrightarrow & \cdots \\ \hline \check{C}^\bullet(U/X, \mathscr{I}^0) & & \check{C}^\bullet(U/X, \mathscr{I}^1) & & \cdots \end{array}$$

To a double complex, one can associate two spectral sequences.

- In one case, you start by computing horizontal cohomology and then vertical cohomology. Doing this gives a spectral sequence with E_2-page

$$E_2^{i,j} = \check{\mathrm{H}}^i(U, \mathscr{H}^j(\mathscr{F}))$$

  where $\mathscr{H}^j(\mathscr{F})$ is the presheaf $V \mapsto \mathrm{H}^j_{\text{ét}}(V, \mathscr{F})$.

- In the other case, you start by computing cohomology in the vertical direction, and then in the horizontal direction. Doing this gives a spectral sequence with E_2-page

$$E_2^{i,j} = \begin{cases} \mathrm{H}^i(X, \mathscr{F}) & \text{if } j = 0 \\ 0 & \text{otherwise.} \end{cases}$$

  (recall Čech cohomology of an injective sheaf vanishes in positive degree). Note that this sequence visibly degenerates with $E_2 = E_\infty$.

Both of these spectral sequences have the same E_∞-page, so we see that the first case is really a spectral sequence

$$E_2^{r,s} = \check{\mathrm{H}}^r(U, \mathscr{H}^s(\mathscr{F})) \implies \mathrm{H}^{r+s}(X_{\text{ét}}, \mathscr{F}).$$

Exercise. Last time, we claimed that if $\check{C}^\bullet(X_{\text{ét}}, -)$ is exact on $\mathrm{Sh}^{ab}(X_{\text{ét}})$, then $\check{\mathrm{H}}^* = \mathrm{H}^*$. Prove this using the Čech-to-derived spectral sequence.

8.1.1 Mayer-Vietoris

Let $U = U_0 \cup U_1$ be a Zariski-open cover of U.

Proposition 8.4 (Mayer-Vietoris sequence). *There exists a functorial long exact sequence*

$$\cdots \longrightarrow \mathrm{H}^s(U, \mathscr{F}) \to \mathrm{H}^s(U_0, \mathscr{F}) \oplus \mathrm{H}^s(U_1, \mathscr{F}) \to \mathrm{H}^s(U_0 \cap U_1, \mathscr{F}) \to \mathrm{H}^{s+1}(U, \mathscr{F}) \to \cdots$$

Proof. Apply Čech-to-derived s.s. to the cover $U_0 \sqcup U_1 \to U$.

Exercise. Show that the Čech complex

$$\mathscr{F}(U_0 \sqcup U_1) \to \mathscr{F}((U_0 \sqcup U_1)^{\times_U 2}) \to \mathscr{F}((U_0 \sqcup U_1)^{\times_X 3}) \to \cdots$$

is quasi-isomorphic to

$$\mathscr{F}(U_0 \sqcup U_1) \to \mathscr{F}(U_0 \cap U_1).$$

Given this, the Čech to derived E_2-page vanishes except for the first 2 columns. The differentials in this s.s. gives the Mayer-Vietoris sequence. In general, if you have an E_2-spectral sequence with only two non-vanishing columns, then that data is the same as that of some long exact sequence. ■

Remark 8.5. The exercise in the previous proof uses that we've taken a Zariski cover. ○

8.2 Computing Étale Cohomology

Theorem 8.6. *Say X is a scheme, and $\mathscr{F} \in \mathrm{QCoh}(X)$. Then,*

$$\mathrm{H}^i(X, \mathscr{F}) = \mathrm{H}^i(X_{\acute{e}t}, \mathscr{F}^{\acute{e}t}) = \mathrm{H}^i(X_{fppf}, \mathscr{F}^{fppf}).$$

Remark 8.7. Already a non-trivial version of something like this appeared in Hartshorne:

$$\mathrm{Ext}^i_{\mathrm{QCoh}(X)}(\mathscr{O}_X, \mathscr{F}) = \mathrm{H}^i(\mathrm{QCoh}(X), \mathscr{F}) = \mathrm{H}^i(X_{\mathrm{Zar}}, \mathscr{F}) = \mathrm{Ext}^i_{\mathrm{Sh}(X_{\mathrm{Zar}})}(\underline{\mathbb{Z}}, \mathscr{F}).$$

This (the middle equality) was because injective qcoh sheaves are flasque. ○

Proof of Theorem 8.6. We'll prove the theorem only in the special case that X is quasi-compact, separated, and Čech cohomology computes derived functor cohomology.

First, we claim every cover can be refined to a finite cover by affines. This follows from quasi-compactness. Now, suppose X is affine, and $U \to X$ is an fppf affine cover (so U affine). Then, $\check{C}(U/X, \mathscr{F})$ is exact if $\mathscr{F} = \widetilde{M}$ is quasi-coherent. Indeed, if $U = \operatorname{Spec} B$ and $X = \operatorname{Spec} A$, then this complex is just our old friend

$$M \otimes B \to M \otimes B \otimes B \to M \otimes B \otimes B \otimes B \to \cdots,$$

the **Amitsar complex**. This is exact by the usual faithful flat base change to get a section argument. This tells us that

$$\check{\mathrm{H}}^i(U/X, \mathscr{F}) = \begin{cases} \mathscr{F}(X) & \text{if } i = 0 \\ 0 & \text{otherwise.} \end{cases}$$

for $\mathscr{F}$ qcoh, and U, X both affine. From this, we see that indeed

$$\check{\mathrm{H}}^i(X_{\mathrm{\acute{e}t}}, \mathscr{F}^{\mathrm{\acute{e}t}}) = \begin{cases} \mathscr{F}(X) & \text{if } i = 0 \\ 0 & \text{otherwise.} \end{cases}$$

when X affine, since affine covers are cofinal.

Now say X is quasi-compact, separated, and Čech cohomology computes derived functor cohomology. Take an affine cover $\mathcal{U} \to X$, and use the Čech-to-derived spectral sequence to finish (note that intersections of affines are affine). ■

Example. Say $X = \mathbb{P}^n$ and $\mathscr{F} = \mathscr{O}_X$. Then,

$$\mathrm{H}^i(\mathbb{P}^n_{\text{ét}}, \mathscr{O}^{\text{ét}}_{\mathbb{P}^n}) = \begin{cases} k & \text{if } 0 \\ 0 & \text{otherwise.} \end{cases}$$

△

Example. Say $X/\mathbb{F}_p$ is a quasi-projective variety. We can compute $\mathrm{H}^i(X_{\text{ét}}, \underline{\mathbb{F}_p})$. We claim the sequence

$$0 \to \underline{\mathbb{F}_p} \to \mathbb{G}_a \xrightarrow{x^p - x} \mathbb{G}_a$$

of sheaves is exact. This is true at the level of representing objects (or just check by hand). In fact, we claim that

$$0 \to \underline{\mathbb{F}_p} \to \mathbb{G}_a \xrightarrow{x^p - x} \mathbb{G}_a \to 0$$

is exact, i.e. that $\mathbb{G}_a \to \mathbb{G}_a$ above is an epimorphism, e.g. given $f \in \mathscr{O}_U(U) = \mathbb{G}_a(U)$, we need to solve $x^p - x = f$ étale-locally on U. Consider the base change

$$\begin{tikzcd} \mathbb{G}_a \times_{\mathbb{G}_a} U \arrow[r] \arrow[d] & \mathbb{G}_a \arrow[d, "x^p - x"] \\ U \arrow[r, "f"] & \mathbb{G}_a. \end{tikzcd}$$

Since $x^p - x$ is étale (it's derivative -1 is invertible), the left vertical map is étale too, so win.

The upshot is we have a long exact sequence

$$0 \to \mathrm{H}^0(X_{\text{ét}}, \mathbb{F}_p) \to \mathrm{H}^0(X_{\text{ét}}, \mathbb{G}_a) \to \mathrm{H}^0(X_{\text{ét}}, \mathbb{G}_a) \to \mathrm{H}^1(X_{\text{ét}}, \mathbb{F}_p) \to \mathrm{H}^1(X_{\text{ét}}, \mathbb{G}_a) \to \mathrm{H}^1(X_{\text{ét}}, \mathbb{G}_a) \to \mathrm{H}^2(X, \mathbb{F}_p) \to \cdots$$

where $\mathrm{H}^i(X_{\text{ét}}, \mathbb{G}_a) = \mathrm{H}^i(X, \mathscr{O}_X)$, and the maps $\mathrm{H}^i(X, \mathscr{O}_X) \to \mathrm{H}^i(X, \mathscr{O}_X)$ above are all $x \mapsto x^p - x$. △

Example. If $X = \mathbb{A}^1 = \operatorname{Spec} \mathbb{F}_p[t]$ in the previous example, we have

$$0 \to \mathrm{H}^0(\mathbb{A}^1_{\text{ét}}, \mathbb{F}_p) \to \mathbb{F}_p[t] \xrightarrow{t \mapsto t^p - t} \mathbb{F}_p[t] \to \mathrm{H}^1(\mathbb{A}^1_{\text{ét}}, \mathbb{F}_p) \to 0,$$

so $\mathrm{H}^0(\mathbb{A}^1_{\text{ét}}, \mathbb{F}_p) = \mathbb{F}_p$, but $\mathrm{H}^1(\mathbb{A}^1_{\text{ét}}, \mathbb{F}_p)$ is the cokernel which is huge. △

9 Lecture 9

Last time:

- Čech-to-derived s.s.
- Mayer-Vietoris

- étale cohomology of qcoh sheaves
- étale cohomology of $\underline{\mathbb{F}}_p$ in char. p

Recall 9.1. Say $X/\mathbb{F}_p$. Get an **Artin-Schreier exact sequence** of sheaves on $X_{\text{ét}}$

$$0 \longrightarrow \underline{\mathbb{F}}_p \longrightarrow \mathscr{O}_X^{\text{ét}} \xrightarrow{t \mapsto t^p - t} \mathscr{O}_X^{\text{ét}} \longrightarrow 0.$$

Can get something similar over any base of characteristic p. This gives a long exact sequence

$$\cdots \to \mathrm{H}^{i-1}(X, \mathscr{O}_X) \to \mathrm{H}^i(X_{\text{ét}}, \underline{\mathbb{F}}_p) \to \mathrm{H}^i(X, \mathscr{O}_X) \to \mathrm{H}^i(X, \mathscr{O}_X) \to \cdots .$$

It can still be tricky to compute cohomology of $\underline{\mathbb{F}_p}$, but we can at least work things out in special cases. $\odot$

Example. Say $X = \operatorname{Spec} A$ is affine, so $\mathrm{H}^i(X, \mathscr{O}_X) = 0$ for $i > 0$ which means $\mathrm{H}^i(X_{\text{ét}}, \mathbb{F}_p) = 0$ for $i > 1$. We're left with

$$0 \to \mathrm{H}^0(X, \mathbb{F}_p) \to A \xrightarrow{t \mapsto t^p - t} A \to \mathrm{H}^1(X_{\text{ét}}, \mathbb{F}_p) \to 0,$$

so $\mathrm{H}^0(X, \mathbb{F}_p) = \mathbb{F}_p^{\pi_0(X)}$, as always, and $\mathrm{H}^1(X_{\text{ét}}, \mathbb{F}_p)$ is the cokernel of the Artin-Schrier map which is not finitely generated in general. $\triangle$

Remark 9.2. If $X/\mathbb{F}_p$ is proper, then $\mathrm{H}^i(X_{\text{ét}}, \mathbb{F}_p)$ is finite dimensional. This is by proper pushforward for coherent cohomology (i.e. $\mathrm{H}^i(X, \mathscr{O}_X)$ is finite dimensional). $\circ$

Example. Say E an elliptic curve over $k = \overline{k}$ of characteristic p. Then it's not too hard to show that

$$\mathrm{H}^1(E, \mathbb{F}_p) = \begin{cases} \mathbb{F}_p & \text{if } E \text{ ordinary} \\ 0 & \text{if } E \text{ supersingular} \end{cases}$$

One may expect instead for this to have been 2-dimensional. $\triangle$

Étale cohomology with $\mathbb{F}_p$-coefficients in characteristic p may not work to prove the Weil conjectures, but it still gives something interesting/useful.

Example. Recall that $\mathrm{H}^i((\operatorname{Spec} k)_{\text{ét}}, \mathscr{F})$ is Galois cohomology of $\mathscr{F}(k^s) := \varinjlim_{k^s/L/k} \mathscr{F}(L)$. Underlying this is the equivalence of categories between $\mathrm{Sh}^{\mathrm{ab}}(\operatorname{Spec} k_{\text{ét}})$ and discrete G-modules. We can make this explicit using Čech cohomology.

Let $U = \operatorname{Spec} L$ with L/k a separable field extension (so U an étale cover of $\operatorname{Spec} k$). Then, we get a Čech complex

$$\check{C}(U/\operatorname{Spec} k, \mathscr{F}) = \mathscr{F}(U) \to \mathscr{F}(U \times_k U) \to \cdots .$$

Now assume L/K Galois with $G = \operatorname{Gal}(L/K)$. Then we can rewrite this complex as

$$\check{C}(U/\operatorname{Spec} k, \mathscr{F}) = \mathscr{F}(U) \to \mathscr{F}(G \times U) \to \mathscr{F}(G \times G \times U) \to \cdots .$$

Exercise. This complex is the same as the standard complex computing $\mathrm{H}^i(G(L/k), \mathscr{F}(U))$.

Taking direct limits, one then sees that $\check{C}((\operatorname{Spec} k)_{\text{ét}}, \mathscr{F})$ is quasi-isomorphic to the usual complex computing Galois cohomology. $\triangle$

Question 9.3 (Audience)**.** *When can you compute étale cohomology as some kind of group cohomology?*

Answer. You can do this if your space is a $K(\pi, 1)$. This is kind of a non-answer since in algebraic geometry, the definition of a $K(\pi, 1)$ is essentially that it is a space whose étale cohomology is group cohomology of its fundamental group. ⋆

The goal of the next few classes is to actually compute étale cohomology of some spaces. Specifically...

Goal. We cant to compute the étale cohomology of curves over $k = \overline{k}$, i.e.

$$\mathrm{H}^i(C_{\text{ét}}, \mathbb{Z}/\ell^n\mathbb{Z})$$

where $\ell \neq \operatorname{char} k$.

Today we'll try to do this for $i = 0, 1$.

9.1 Cohomology of Curves (Really, G-torsors and interpretation of H^1)

Remark 9.4.

$$\mathrm{H}^0(C_{\text{ét}}, \underline{\mathbb{Z}/\ell^n\mathbb{Z}}) = \mathbb{Z}/\ell^n\mathbb{Z}.$$

○

Computing H^1 will be a little more involved. In order to do this, we will give an interpretation of $\mathrm{H}^1(X_{\text{ét}}, \mathscr{F})$ in terms of torsors. These will formalize the notion of a principal, homogeneous space (think principal G-bundles in topology).

Definition 9.5. Let G be a sheaf of (not necessarily abelian) groups on $X_{\text{ét}}$. The idea to defining a G-torsor will be to let it be a sheaf $\mathscr{F} \in \mathrm{Sh}(X_{\text{ét}})$ (of sets) with a G-action s.t. G acts simply transitively on every fiber.

Here's the actual definition. A ***G*-torsor** is a sheaf $T \in \mathrm{Sh}(X_{\text{ét}})$ with an action $G \times T \xrightarrow{a} T$ such that

$$\begin{array}{cccc} (a, \pi_2): & G \times_X T & \longrightarrow & T \times_X T \\ & (g, t) & \longmapsto & (gt, t) \end{array}$$

is an isomorphism.[23] All fiber products in this definition are over X. ◇

Remark 9.6. $T \times T \xrightarrow{\sim} G \times T$, so if you pull back to T, you get a trivial torsor. ○

Example. G is a G-torsor (trivial torsor). △

Example. Suppose G is a finite group, and $\underline{G} \in \mathrm{Sh}(X_{\text{ét}})$ is the corresponding constant sheaf. Then a $\underline{G}$-torsor is a finite étale cover with Galois group G (there's an equivalence of categories here). △

Example. $\mathbb{G}_m : \mathcal{U} \mapsto \mathscr{O}_{\mathcal{U}}(\mathcal{U})^\times$ is the sheaf represented by $\operatorname{Spec} k[t, t^{-1}]$. A line bundle minus its 0 section will give an example of a $\mathbb{G}_m$-torsor. Say $\mathscr{L}$ is a line bundle. Then you get a $\mathbb{G}_m$-torsor

$$\mathbf{Spec}_X \left(\bigoplus_{n \in \mathbb{Z}} \mathscr{L}^{\otimes n} \right).$$

We'll see in a bit that there's a natural bijection between $\mathbb{G}_m$-torsors and line bundles. △

[23] monic = free and pointwise surjective = transitive

Example. $G = \underline{\mathrm{GL}_n}$. The claim is that GL_n-torsors are in natural bijection with rank n vector bundles. To a rank n vector bundle $\mathcal{E}$, associate the bundle of frames $\mathrm{Isom}_{X_{\text{ét}}}(\mathscr{O}^{\oplus n}, \mathcal{E})$, a sheaf whose value on some cover U are isomorphisms between $\mathscr{O}_U^{\oplus n}$ and $\mathcal{E}|_U$.

Given a GL_n-torsor T, one way of recovering a vector bundle is taking $(T \times \mathscr{O}_X^{\oplus n})/\underline{\mathrm{GL}_n}$ (quotient taken in the category of étale sheaves). Not obvious this is a vector bundle, so something to check here. Also, the GL_n-action is the diagonal one. △

Definition 9.7. A G-torsor T is **split by a cover** $U \to X$ if $T|_U$ is isomorphic to $G|_U$ (as a torsor). ⋄

This is being "locally trivial."

Remark 9.8. Suppose T is representable, and $T \to X$ is a cover. Then, T is split by T. ○

Example. Suppose G is a finite étale group scheme over X, and T is a G-torsor split by some $\mathcal{U} \to X$. Then,

(1) T is representable.

(2) T is split by T.

Proof. (**(1)** $\implies$ **(2)**) $T \times_X \mathcal{U} \to \mathcal{U}$ is a cover because it is finite, étale (here, assuming T representable), so $T \to X$ is itself a cover. Previous remark then shows that T splits itself.

(1) Observation: $T|_{\mathcal{U}_{\text{ét}}}$ is representable since it is isomorphic to $G|_{\mathcal{U}_{\text{ét}}}$ (definition of being split by $\mathcal{U} \to X$). We now appeal to effectivity of descent for affine schemes (finite maps are affine) to conclude that $T|_{\mathcal{U}_{\text{ét}}}$ descends to a (unique up to iso) representable sheaf T over $X_{\text{ét}}$. ■

△

Remark 9.9. $\check{\mathrm{H}}^1(\mathcal{U}/X, G)$ makes sense for G an sheaf of groups, even it they're non-abelian, with the same definition. ○

Proposition 9.10. *There is a bijection*

$$\left\{ \begin{matrix} G\text{-}torsors\ T \\ split\ by\ \mathcal{U} \to X \end{matrix} \right\} \longleftrightarrow \check{\mathrm{H}}^1(\mathcal{U}/X, G).$$

Proof. Say $\varphi : T|_{\mathcal{U}_{\text{ét}}} \xrightarrow{\sim} G|_{\mathcal{U}_{\text{ét}}}$ as torsors. Let $\pi_1, \pi_2 : \mathcal{U} \times_X \mathcal{U} \rightrightarrows \mathcal{U}$ be the two projections. Since T is a sheaf on $X_{\text{ét}}$, we get a commutative diagram

$$\begin{tikzcd} \pi_1^* T|_{\mathcal{U}_{\text{ét}}} \arrow[r, "\sim"] \arrow[d, "\pi_1^*\varphi"'] & \pi_2^* T|_{\mathcal{U}_{\text{ét}}} \arrow[d, "\pi_2^*\varphi"] \\ \pi_1^* G \arrow[r, "\sim"] & \pi_2^* G. \end{tikzcd}$$

The bottom arrow in the above diagram is an element of $\Gamma(\mathcal{U} \times_X \mathcal{U}, G)$.[24] The claim is that the cocycle condition implies that this is an element of $\ker \mathrm{d}$. The other claim is that if $T_1 \simeq T_2$, then the cocycles you get differ by coboundary. These two claims are left as exercises. ■

Corollary 9.11. $\{$*locally trivial G-torsors*$\}/\sim \xrightarrow{\sim} \check{\mathrm{H}}^1(X_{\acute{e}t}, G)$.

[24]Automorphism of a trivial G-torsor is an element of G

Proposition 9.12. *This gives*[25]

$$\check{\mathrm{H}}^1(\tau, \mathscr{F}) \xrightarrow{\sim} \left\{ \begin{matrix} \textit{locally trivial} \\ \mathscr{F}\textit{-torsors} \end{matrix} \right\} \xrightarrow{\sim} \mathrm{H}^1(\tau, \mathscr{F})$$

where τ *is any site, and* $\mathscr{F}$ *is a sheaf of groups (abelian groups to get the isomorphism to derived functor cohomology).*

Remark 9.13. It is always that case that Čech $\check{\mathrm{H}}^1$ computes derived functor H^1, and this is via the same proof as usual with sheaf cohomology. ○

Theorem 9.14 (**Hilbert 90**)**.**

$$\check{\mathrm{H}}^1(X_{zar}, \underline{\mathrm{GL}}_n) \xrightarrow{\sim} \check{\mathrm{H}}^1(X_{\acute{e}t}, \underline{\mathrm{GL}}_n) \to \check{\mathrm{H}}^1(X_{fppf}, \underline{\mathrm{GL}}_n)$$

and these are all bijections.

Proof. Need to show that (locally split) torsors are the same. A locally split $\underline{\mathrm{GL}}_n$-torsor is fppf descent data for a vector bundle (element of Čech $\check{\mathrm{H}}^1$), so we win by fppf descent for vector bundles. ■

10 Lecture 10

Question 10.1 (Audience)**.** *If you have a sheaf which is representable after an étale base change, was it representable to begin with?*

Answer. In general, no, since descent for sheaves is not always effective. It will be representable by an algebraic space though. ⋆

Remark 10.2. What we've been calling a torsor, some sources (e.g. Stacks project) call a pseudo-torsor. What we've been calling a locally trivial torsor, some sources instead call a torsor. ○

Recall we ended last time with a proof of "Hilbert 90"

Theorem 10.3. *The maps*

$$\mathrm{H}^1(X_{zar}, \mathrm{GL}_n) \leftarrow \mathrm{H}^1(X_{\acute{e}t}, \mathrm{GL}_n) \leftarrow \mathrm{H}^1(X_{fppf}, \mathrm{GL}_n)$$

are all bijections.

Theorem 10.4. *Let* $\tau \in \{X_{zar}, X_{\acute{e}t}, X_{fppf}\}$*. The data of a* GL_n*-torsor split by some* τ*-cover* $U \to X$ *is the same as descent data for a vector bundle relative to* U/X*. We have*

$$U \times_X U \overset{\pi_1}{\underset{\pi_2}{\rightrightarrows}} U \xrightarrow{\pi} X,$$

and $\pi^*T \simeq \pi^*G$*. Hence the natural iso* $\pi_1^*\pi^*T \xrightarrow{\sim} \pi_2^*\pi^*T$ *corresponds to an iso*

$$G|_{U\times_X U} = \pi_1^*\pi^*G \xrightarrow{\sim} \pi_2^*\pi^*G = G|_{U\times_X U}.$$

[25]Locally trivial torsors are those split by some cover

This is exactly given by a section of G. Now, a section of GL_n (an invertible matrix) on double intersections of a cover (satisfying a cocycle condition) is precisely descent data for a vector bundle. Finally, fppf descent tells us that descent data for vector bundles is always effective. Hence,

$$\mathrm{H}^1(\tau, \mathrm{GL}_n) \simeq \textit{iso classes of rank } n \textit{ vector bundles.}$$

Exercise. Find other groups for which Hilbert 90 is (or is not) true.

Remark 10.5. Suppose G is an affine, flat X-group scheme. Are all G-torsors representable by a X-scheme? Yes, using affineness (by same proof as last time). ○

Question 10.6. *Given a G-torsor T, which is fppf-locally trivial, is it étale-locally trivial.*

Answer. In general no, but yes if G is smooth. Here's a proof sketch: start with the pullback

$$\begin{tikzcd} T \times_X T \arrow[r] \arrow[d] & T \arrow[d] \\ T \arrow[r] & X \end{tikzcd}$$

Then, $T \times_X T \to T$ is a trivial G-torsor, and $T \to X$ is smooth (since $G \to X$ is and $T \to X$ has the same fibers). The hard part is to show one can find some closed $U \hookrightarrow T$ such that $U \hookrightarrow T \to X$ is étale. ⋆

Let's go back to thinking about Hilbert 90.

Example. $X = \operatorname{Spec} k, n = 1$. Note that we have

$$\mathrm{H}^1(\mathrm{Gal}(k^s/k), (k^s)^\times) = \mathrm{H}^1((\operatorname{Spec} k)_{\text{ét}}, \mathbb{G}_m) = \mathrm{H}^1(\operatorname{Spec} k_{\text{zar}}, \mathbb{G}_m) = \operatorname{Pic} k = 0.$$

So this Hilbert 90 does give what's more often called Hilbert 90. △

Example. Let X be any scheme, and set $n = 1$. Then,

$$\mathrm{H}^1(X_{\text{ét}}, \mathbb{G}_m) = \operatorname{Pic} X.$$

△

Example. Let ℓ be a number invertible on X (i.e. $\ell \in \mathscr{O}_X(X)^\times$). Let's compute $\mathrm{H}^1(X_{\text{ét}}, \mu_\ell)$. We use the **Kummer sequence** $1 \to \mu_\ell \to \mathbb{G}_m \xrightarrow{z \mapsto z^\ell} \mathbb{G}_m \to 1$. This gives us

$$0 \to \mathrm{H}^0(X_{\text{ét}}, \mu_\ell) \to \mathrm{H}^0(X_{\text{ét}}, \mathbb{G}_m) \xrightarrow{(-)^\ell} \mathrm{H}^0(X_{\text{ét}}, \mathbb{G}_m) \to \mathrm{H}^1(X_{\text{ét}}, \mu_\ell) \to \operatorname{Pic}(X) \xrightarrow{[\ell]} \operatorname{Pic}(X) \to \mathrm{H}^2(X_{\text{ét}}, \mu_\ell) \to \cdots$$

Now suppose $\mathrm{H}^0(X, \mathscr{O}_X) = k = \overline{k}$. Then,

$$\mathrm{H}^0(X_{\text{ét}}, \mu_\ell) = \mu_\ell(k) \text{ and } \mathrm{H}^1(X_{\text{ét}}, \mu_\ell) = \operatorname{Pic}(X)[\ell].$$

△

Example. Can we compute $\mathrm{H}^1(X_{\text{ét}}, \underline{\mathbb{Z}/\ell\mathbb{Z}})$ where $X/k = \overline{k}$ and $\operatorname{char} k \nmid \ell$. In this case, we have $\underline{\mathbb{Z}/\ell\mathbb{Z}} \simeq \mu_\ell$ via picking a primitive ℓth root of unity. Hence, $\mathrm{H}^i(X_{\text{ét}}, \underline{\mathbb{Z}/\ell\mathbb{Z}}) \simeq \mathrm{H}^i(X_{\text{ét}}, \mu_\ell)$ (not Galois equivariant since iso depends on choice) whenever k contains a primitive ℓth root of unity. △

$\underline{\mathbb{Z}/\ell\mathbb{Z}}$ can be represented by $\operatorname{Spec} k[t]/(t(t-1)\ldots(t-(\ell-1)))$

10.1 Geometric Interpretation

Suppose X is an affine scheme over a field $k = \overline{k}$. We have computed $\mathrm{H}^1(X_{\text{ét}}, \mathbb{F}_p) = \operatorname{coker}(\mathscr{O}_X(X) \xrightarrow{t \mapsto t^p - t} \mathscr{O}_X(X))$ where $p = \operatorname{char} k$. We have also computed $\mathrm{H}^1(X_{\text{ét}}, \mathbb{Z}/\ell\mathbb{Z})$ in terms of a long exact sequence. This groups means something: one gives $\mathbb{F}_p$-torsors and the other $\mathbb{Z}/\ell\mathbb{Z}$-torsors.

Question 10.7. *How would one explicitly write down the associated torsors?*

Say we have $[Y] \in \mathrm{H}^1(X_{\text{ét}}, \mathbb{F}_p) = \operatorname{coker}(\mathscr{O}_X(X) \xrightarrow{t \mapsto t^p - t} \mathscr{O}_X(X))$. Here, $Y = \{y^p - y = a\}$ where $a \in \mathscr{O}_X(X)$, called an **Artin-Schrierer covering**.

Now say $\ell \neq \operatorname{char} k$ and $\operatorname{Pic}(X) = 0$, and we have $[Z] \in \mathrm{H}^1(X_{\text{ét}}, \mu_\ell) = \operatorname{coker}(\mathscr{O}_X^\times \xrightarrow{t \mapsto t^\ell} \mathscr{O}_X^\times)$. Then, $Z = \left\{z^\ell = f\right\}$ with $f \in \mathscr{O}_X^\times$.

Remark 10.8. Explicitly writing down covers belongs to the theory of explicit geometric class field theory, which gives recipes for writing down abelian covers of curves. ∘

10.2 Cohomology of Curves (mostly for real, this time)

Goal. Let X be a smooth curve over $k = \overline{k}$. Then,

'curve' means geometrically integral, separated, finite type

$$\mathrm{H}^i(X_{\text{ét}}, \mathbb{G}_m) = \begin{cases} \mathscr{O}_X(X)^\times & \text{if } i = 0 \\ \operatorname{Pic}(X) & \text{if } i = 1 \\ 0 & \text{if } i > 1 \end{cases}$$

(note we've done $i = 0, 1$ already)

Corollary 10.9. *Say X is a smooth, proper, connected curve over $k = \overline{k}$, and $\ell \neq \operatorname{char} k$, then*

$$\mathrm{H}^i(X_{\acute{e}t}, \underline{\mathbb{Z}/\ell^n\mathbb{Z}}) = \begin{cases} \mathbb{Z}/\ell^n\mathbb{Z} & \text{if } i = 0 \\ \operatorname{Pic}(X)[\ell^n] = (\mathbb{Z}/\ell^n\mathbb{Z})^{2g} & \text{if } i = 1 \\ \mathbb{Z}/\ell^n\mathbb{Z} & \text{if } i = 2 \end{cases}$$

and it vanishes if $i > 2$.

Proof. We need a black box from the theory of abelian varieties. The Jacobian $\operatorname{Jac}(X) = \operatorname{Pic}^0(X)$ of a curve is a g-dimensional abelian variety, so $\operatorname{Jac}(X)[\ell^n] = (\mathbb{Z}/\ell^n\mathbb{Z})^{2g}$. Also, this Jacobian sits in a short exact sequence $0 \to \operatorname{Jac}(X) \to \operatorname{Pic}(X) \xrightarrow{\deg} \mathbb{Z} \to 0$, so $\operatorname{Pic}(X)[\ell^n] = \operatorname{Jac}(X)[\ell^n]$. Lastly, we need to know that $\operatorname{Jac}(X)(k)$ is a divisible group, so $\operatorname{coker}(\operatorname{Pic}(X) \xrightarrow{[\ell^n]} \operatorname{Pic}(X)) = \operatorname{coker}(\mathbb{Z} \xrightarrow{[\ell^n]} \mathbb{Z}) = \mathbb{Z}/\ell^n\mathbb{Z}$.[26]

Now, the Kummer sequence $1 \to \mu_{\ell^n} \to \mathbb{G}_m \to \mathbb{G}_m \to 1$ (note $\mu_{\ell^n} = \underline{\mathbb{Z}/\ell^n\mathbb{Z}}$ since we're over $k = \overline{k}$ with $\ell \neq \operatorname{char} k$) gives

$$0 \to \mathrm{H}^1(X_{\text{ét}}, \underline{\mathbb{Z}/\ell^n\mathbb{Z}}) \to \operatorname{Pic}(X) \xrightarrow{[\ell^n]} \operatorname{Pic}(X) \to \mathrm{H}^2(X_{\text{ét}}, \underline{\mathbb{Z}/\ell^n\mathbb{Z}}) \to 0.$$

Thus, $\mathrm{H}^1(X_{\text{ét}}, \underline{\mathbb{Z}/\ell^n\mathbb{Z}}) = \operatorname{Pic}(X)[\ell^n] = (\mathbb{Z}/\ell^n\mathbb{Z})^{2g}$, and $\mathrm{H}^2(X_{\text{ét}}, \underline{\mathbb{Z}/\ell^n\mathbb{Z}}) = \operatorname{coker}(\operatorname{Pic}(X) \xrightarrow{[\ell^n]} \operatorname{Pic}(X)) = \mathbb{Z}/\ell^n\mathbb{Z}$. ∎

[26] snake lemma

Remark 10.10. These isomorphisms will not be Galois equivariant (the Galois action on $\underline{\mathbb{Z}/\ell^n\mathbb{Z}}$ is trivial, but the actions on the RHS here will be via the cyclotomic character). ○

Ok, so we need to show that $\mathrm{H}^i(X_{\text{ét}}, \mathbb{G}_m) = 0$ for $i > 1$. The proof will have three ingredients

(1) Leray spectral sequence

(2) Divisor exact sequence

(3) Brauer groups

10.3 Pushforwards and Leray s.s.

Recall 10.11. If $f : X \to Y$ is a morphism of schemes, we get a pushforward functor $f_* : \mathrm{Sh}(X_{\text{ét}}) \to \mathrm{Sh}(Y_{\text{ét}})$. Given $\mathscr{F} \in \mathrm{Sh}(X_{\text{ét}})$, we have

$$f_*\mathscr{F}(U \to Y) = \mathscr{F}(U \times_Y X).$$

Recall that this functor is left exact, so we get right derived functors

$$R^i f_* : \mathrm{Sh}^{ab}(X_{\text{ét}}) \to \mathrm{Sh}^{\mathrm{ab}}(Y_{\text{ét}}).$$

⊙

Intuition. Think of these derived functors as taking cohomology of the fibers. This is not literally true in general.

Remark 10.12. In general, $R^i f_*\mathscr{F}$ is the sheafification of the presheaf $V \mapsto \mathrm{H}^i(f^{-1}(V), \mathscr{F})$. ○

Proposition 10.13. *If f is a finite morphism (e.g. a closed immersion), then $R^i f_* = 0$ for $i > 0$.*

Proof. We show that f_* is right exact in this case. We leave this as an exercise (show stalk of $f_*\mathscr{F}$ at $\overline{y}$ in Y is just $\bigoplus_{\overline{x} \in f^{-1}(\overline{y})} \mathscr{F}_{\overline{x}}$). ■

Proposition 10.14. *For any morphism $f : X \to Y$, f_* preserves injectives.*

Proof. This is true for any functor with an exact left adjoint (exercise). ■

Corollary 10.15 (Leray spectral sequence). *Say $f : X \to Y$ and $g : Y \to Z$ be morphisms of schemes. Then there is an E_2-spectral sequence*

$$R^i g_* \circ R^j f_* \mathscr{F} \implies R^{i+j}(g \circ f)_* \mathscr{F}.$$

As a special case, if $Z = \operatorname{Spec} k$ and $k = \overline{k}$, then we get

$$\mathrm{H}^i(Y, R^j f_* \mathscr{F}) \implies \mathrm{H}^{i+j}(X, \mathscr{F}).$$

Proof. Spectral sequence of a composition of functors (Tohoku). In general, given two functors between abelian categories with the first one sending injectives to injectives, you get a similar sequence.

Explicitly, say $\mathscr{F} \to \mathscr{I}^\bullet$ is an injective resolution. Then, $R^i f_*\mathscr{F} = \mathscr{H}^i(f_*\mathscr{I}^\bullet)$ and $f_*\mathscr{I}^\bullet$ is a complex of injectives. We want

$$\mathcal{H}^{i+j}(g_*f_*\mathscr{I}^\bullet) = R^{i+j}(g \circ f)_*\mathscr{F}.$$

Take spectral sequence of filtered complex $f_*\mathscr{I}^\bullet$ with filtration given by the truncations $\tau_{\le p} f_*\mathscr{I}^\bullet$. ■

11 Lecture 11

Last time we introduced the Leray spectral sequence. Given a composition $X \xrightarrow{f} Y \xrightarrow{y} Z$, we obtained a spectral sequence

$$R^i g_* R^j f_*(-) \implies R^{i+j}(g \circ f)_*(-).$$

11.1 Continuing with pushforwards

Example. Say we have $\pi : X \to \operatorname{Spec} k$ with k some field. The composition π_* followed by global sections gives a spectral sequence

$$\mathrm{H}^i(k, R^j\pi_*\mathscr{F}) \implies \mathrm{H}^{i+j}(X_{\text{ét}}, \mathscr{F}).$$

On the left, we have Galois cohomology, and the Galois module corresponding to $R^j\pi_*\mathscr{F}$ is $R^j\pi_*\mathscr{F}(k^s) = \mathrm{H}^j(X_{k^s}, \mathscr{F})$. Hence, the spectral sequence is really

$$\mathrm{H}^i(k, \mathrm{H}^j(X_{\text{ét},k^s}, \mathscr{F})) \implies \mathrm{H}^{i+j}(X_{\text{ét}}, \mathscr{F}).$$

△

Remark 11.1. Say k is a finite field, and X/k is a smooth projective variety. Galois cohomology of a finite field is simple

$$\mathrm{H}^i(k, V) = \begin{cases} V^G & \text{if } i = 0 \\ V_G & \text{if } i = 1 \\ 0 & \text{if } i > 1 \end{cases}$$

∘

Remark 11.2. Say $X \xrightarrow{\pi} Y$ is a smooth, proper morphism (say, for now, of varieties over $\mathbb{C}$). There's a leray spectral sequence over the analytic site giving

$$\mathrm{H}^i(Y, R^j\pi_*\underline{\mathbb{Q}}) \implies \mathrm{H}^{i+j}(X, \underline{\mathbb{Q}}).$$

Fact (Deligne). This spectral sequence degenerates at E_2.

First proof of this uses Weil II. ∘

Proposition 11.3. *Say we have $\pi : X \to Y$. Then, $R^i\pi_*\mathscr{F}$ is the sheaf associated to the presheaf*

$$U \mapsto \mathrm{H}^i(\pi^{-1}(U)_{\acute{e}t}, \mathscr{F}).$$

Proof. Let $\mathscr{F} \to \mathscr{I}^\bullet$ be an injective resolution. Then, $\mathscr{H}^i(\pi_*\mathscr{I}) =: R^i\pi_*\mathscr{F}$. Now, note that we have a commutative diagram

$$\begin{tikzcd} \mathrm{Sh}(X_{\text{ét}}) \arrow[r, "\pi_*"] \arrow[d, "\text{forget}"'] & \mathrm{Sh}(Y_{\text{ét}}) \\ \mathrm{Psh}(X_{\text{ét}}) \arrow[r, "\pi_*"] & \mathrm{Psh}(Y_{\text{ét}}), \arrow[u, "a"'] \end{tikzcd}$$

where a denotes sheafification. Furthermore, note that π_* is exact as a functor $\mathrm{Psh}(X_{\text{ét}}) \to \mathrm{Psh}(Y_{\text{ét}})$, and sheafification is also exact. Hence,

$$R^i\pi_*\mathscr{F} = \mathscr{H}^i(\pi_*\mathscr{I}^\bullet) = \mathscr{H}^i(a \circ \pi_* \circ \mathrm{forget}(\mathscr{I}^\bullet)) = a \circ \pi_* \left(\mathscr{H}^i(\mathrm{forget}(\mathscr{I}^\bullet)\right).$$

By definition, $\mathscr{H}^i(\mathrm{forget}(\mathscr{I}^\bullet))$ is the functor sending $U \mapsto \mathrm{H}^i(U, \mathscr{F})$. Thus, the above equality is exactly what we wanted to prove. ■

Remark 11.4. This is true more generally for arbitrary morphisms of sites. ○

Example. Say X is an integral scheme, with generic point $\iota : \eta \hookrightarrow X$. Let $\mathscr{F} \in \mathrm{Sh}(\eta_{\text{ét}})$; we want to understand $R^i\iota_*\mathscr{F}$. At least, let's try to compute its stalks. Say $\overline{x} \to X$ is a geometric point. Then,

$$\left(R^i\iota_*\mathscr{F}\right)_{\overline{x}} = \varinjlim_{(U,\overline{u})} (R^i\iota_*\mathscr{F})(U) = \varinjlim_{(U,\overline{u})} \mathrm{H}^i(U_\eta, \mathscr{F}|_{U_\eta}).$$

Exercise. Define $\mathscr{O}_{X,\overline{x}}$ to be the stalk of $\mathscr{O}_X$ at $\overline{x}$, and define $K_{\overline{x}} = \mathrm{Frac}(\mathscr{O}_{X,\overline{x}})$. Then,

$$(R^i\iota_*\mathscr{F})_{\overline{x}} = \mathrm{H}^i(K_{\overline{X}}, \mathscr{F}|_{K_{\overline{x}}}).$$

△

Recall that we have the following goal.

Goal. Understand $\mathrm{H}^i(X, \mathbb{G}_m)$ where X is a curve over $k = \overline{k}$ (and $i > 1$).

We want to relate this question to questions in Galois cohomology.

11.2 Cohomology of curves (we meet again)

Proposition 11.5. *Let X be a regular variety over k (maybe to sep closed), and suppose $\eta \hookrightarrow X$ is the generic point. Then there is a short exact sequence of sheaves on $X_{\text{ét}}$:*

$$0 \to \mathbb{G}_m \xrightarrow{res} \eta_*\mathbb{G}_m \xrightarrow{\mathrm{div}} \bigoplus_{\substack{Z \subset X \\ codim\ 1}} \iota_{Z*}\underline{\mathbb{Z}} \to 0.$$

Proof. We first show $\mathbb{G}_m \to \eta_*\mathbb{G}_m$ is injective. We claim $\mathbb{G}_m(U) \to \mathbb{G}_m(U_\eta)$ is injective. This is precisely $\mathscr{O}_U^\times \to \bigoplus_i \mathscr{O}_{\eta_i}$ where η_i runs over the generic points of U. This map is indeed injective.

Now exactness in the middle. Say we have $f \in \eta_*\mathbb{G}_m(U)$ s.t. $\mathrm{div}(f) = 0$. We want f to come from $\mathbb{G}_m(U)$. It is enough to show that f is regular (by the same argument, f^{-1} will also be regular). This boils down to

$$A = \bigcap_{\mathfrak{p}\ \mathrm{ht}\ 1} A_{\mathfrak{p}}$$

(by normality, which is implied by regularity).

Now we show $\eta_*\mathbb{G}_m \to \bigoplus_{\substack{Z\subset X\\ \text{codim } 1}} \iota_{Z*}\underline{\mathbb{Z}}$ is surjective. We claim that every Weil divisor is locally principal, i.e. is Cartier. This is true by regularity. ■

Corollary 11.6. *There is a long exact sequence*

$$\cdots \to \mathrm{H}^{i-1}(X_{\acute{e}t}, \bigoplus_{\substack{Z\subset X\\ codim\ 1}} \iota_{Z*}\underline{\mathbb{Z}}) \to \mathrm{H}^i(X_{\acute{e}t}, \mathbb{G}_m) \to \mathrm{H}^i(X_{\acute{e}t}, \eta_*\mathbb{G}_m) \to \mathrm{H}^i(X_{\acute{e}t}, \bigoplus_{\substack{Z\subset X\\ codim\ 1}} \iota_{Z*}\underline{\mathbb{Z}}) \to \cdots$$

The terms involving pushforwards can be computed using the Leray spectral sequence.

Proposition 11.7. *Say X is a curve over $k = k_s$. Then,*

$$\mathrm{H}^i(X_{\acute{e}t}, \bigoplus_{\substack{Z\subset X\\ codim\ 1}} \iota_{Z*}\underline{\mathbb{Z}}) = 0 \ \textit{for}\ i > 0.$$

Proof. It is enough to show that for $z \in X$ of codim 1 (a closed point), $\mathrm{H}^i(X_{\text{ét}}, \iota_{z*}\underline{\mathbb{Z}}) = 0$ for $i > 0$. There is a Leray spectral sequence

$$\mathrm{H}^i(X_{\text{ét}}, R^i\iota_{z*}\underline{\mathbb{Z}}) \implies \mathrm{H}^{i+j}(z_{\text{ét}}, \underline{\mathbb{Z}}).$$

Note that

$$R^j\iota_{z*}\underline{\mathbb{Z}} = \begin{cases} \iota_{z*}\underline{\mathbb{Z}} & \text{if } j = 0 \\ 0 & \text{otherwise.} \end{cases}$$

since pushforward along closed immersions are exact. Also,

$$\mathrm{H}^s(z_{\text{ét}}, \underline{\mathbb{Z}}) = \begin{cases} \mathbb{Z} & \text{if } i = 0 \\ 0 & \text{otherwise.} \end{cases}$$

since z is spec of a separable closed field. Hence, we have a spectral sequence with just one column, so it's degenerate, and we see that

$$\mathrm{H}^i(X_{\text{ét}}, \iota_{z*}\underline{\mathbb{Z}}) = \mathrm{H}^i(z_{\text{ét}}, \underline{\mathbb{Z}}) = \begin{cases} \mathbb{Z} & \text{if } i = 0 \\ 0 & \text{if } i > 0 \end{cases}.$$

■

Corollary 11.8. *Say X is a smooth curve over $k = k_s$. Then,*

$$\mathrm{H}^i(X_{\acute{e}t}, \mathbb{G}_m) \xrightarrow{\sim} \mathrm{H}^i(X_{\acute{e}t}, \eta_*\mathbb{G}_m) \ \textit{for}\ i > 1.$$

Let's compute this now. We use the Leray spectral sequence once more:

$$\mathrm{H}^i(X_{\text{ét}}, R^j\eta_*\mathbb{G}_m) \implies \mathrm{H}^{i+j}(\eta, \mathbb{G}_m).$$

Recall from a previous example/exercise that

$$(R^j\eta_*\mathbb{G}_m)_{\overline{x}} = \mathrm{H}^j(K_{\overline{x}}, \mathbb{G}_m).$$

We will win if we can prove...

Theorem 11.9. *Suppose*

- *K is the function field of a curve over a separably closed field; or*
- *$K = K_{\overline{x}} = \mathrm{Frac}(\mathscr{O}_{X,\overline{x}})$ is the strictly Henselian field associated to a geometric point of a curve over a separably closed field.*

Then, $\mathrm{H}^i(K, \mathbb{G}_m) = 0$ *for* $i > 0$.

Remark 11.10. This shows that $R^j\eta_*\mathbb{G}_m = 0$ for $j > 0$ since all its stalks vanish. This then gives $\mathrm{H}^i(X_{\text{ét}}, \eta_*\mathbb{G}_m) = \mathrm{H}^i(\eta, \mathbb{G}_m)$. Applying the theorem once more tells us that this vanishes if $i > 0$. ◦

Thus, we've reduced computation of étale cohomology of curves to Galois cohomology.

Lots of what we've done has been fairly formal, but this new goal theorem is not formal. It'll take a while to prove, and will involve introducing new notions.

11.3 Brauer groups

Let X be a scheme.

Definition 11.11 (Cohomological Brauer group).

$$\mathrm{Br}^{coh}(X) = \mathrm{H}^2(X_{\text{ét}}, \mathbb{G}_m)_{tors}.$$

⋄

We will understand this geometrically in terms of PGL_n-torsors.

Claim 11.12. *There is a natural map*

$$\bigcup_n \left\{ \begin{matrix} \textit{étale-locally split} \\ \mathrm{PGL}_n\textit{-torsors} \end{matrix} \right\} \longrightarrow \mathrm{H}^2(X_{\acute{e}t}, \mathbb{G}_m).$$

The main point is that there is a short exact sequence $1 \to \mathbb{G}_m \to \mathrm{GL}_n \to \mathrm{PGL}_n \to 1$ of sheaves of groups on $X_{\text{ét}}$.[27] The idea then is the use the associated long exact sequence which includes $\mathrm{H}^1(X_{\text{ét}}, \mathrm{PGL}_n) \to \mathrm{H}^2(X_{\text{ét}}, \mathbb{G}_m)$, but this does not quite make sense since PGL_n is not abelian. However, we can be hands on.

Say we have T which is a locally trivial PGL_n-torsor. We want to produce $[T] \in \mathrm{H}^2(X_{\text{ét}}, \mathbb{G}_m)$. We start with $[T] \in \check{\mathrm{H}}^1(X_{\text{ét}}, \mathrm{PGL}_n)$. Choose a trivializng $\mathcal{U} \to X$ s.t. $T|_{\mathcal{U}} = \mathrm{PGL}_n|_{\mathcal{U}}$. From this, we get a cocycle in $\mathrm{PGL}_n(\mathcal{U} \times_X \mathcal{U})$. What next? Find out next time...

[27]It if right exact since $\mathrm{GL}_n \to \mathrm{PGL}_n$ is smooth, and so has sections étale locally. Alternatively, $\mathrm{GL}_n \to \mathrm{PGL}_n$ is a $\mathbb{G}_m$-torsor and so (Zariski)-locally trivial, so it's even an epimorphism in the Zariski topology

12 Lecture 12

Today, we talk about Brauer groups.

Recall that our current goal is to prove the following.

Goal. Let C be a smooth curve over $k = \overline{k}$. We have computed $\mathrm{H}^0(C, \mathbb{G}_m) = \mathscr{O}_C^\times(C)$ and $\mathrm{H}^1(C, \mathbb{G}_m) = \mathrm{Pic}(C)$. We still want to show that $\mathrm{H}^i(C, \mathbb{G}_m) = 0$ if $i > 1$.

Recall 12.1. We have reduced the $i = 2$ case to questions about Galois cohomology; in particular, to understanding $\mathrm{H}^i(k(C), \mathbb{G}_m)$ and $\mathrm{H}^i(k_{\overline{x}}, \mathbb{G}_m)$, Galois cohomology of the function field and of the strictly henselian local rings. $\odot$

Recall 12.2. For a scheme X, we defined the **cohomological Brauer group**

$$\mathrm{Br}^{coh}(X) = \mathrm{Br}'(X) = \mathrm{H}^2(X_{\text{ét}}, \mathbb{G}_m)_{tors}.$$

We started defining the Brauer group. We claimed there was a map

$$\bigcup_n \left\{ \begin{array}{c} \text{étale-locally split} \\ \mathrm{PGL}_n\text{-torsors} \end{array} \right\} \xrightarrow{\delta} \mathrm{H}^2(X_{\text{ét}}, \mathbb{G}_m),$$

and we defined the **Brauer group** $\mathrm{Br}(X)$ to be the image of this map. $\odot$

How do we define δ above? It is the boundary map

$$\bigcup_n \mathrm{H}^1(X_{\text{ét}}, \mathrm{PGL}_n) \xrightarrow{\delta} \mathrm{H}^2(X_{\text{ét}}, \mathbb{G}_m)$$

arising from the short exact sequences

$$1 \longrightarrow \mathbb{G}_m \longrightarrow \mathrm{GL}_n \longrightarrow \mathrm{PGL}_n \longrightarrow 1$$

of sheaves of groups on $X_{\text{ét}}$.

Note 1. A good reference for (partial) LES of cohomology of sheaves of *non-abelian* groups is Giraud's book 'Cohomologie Non-Abelienne'

See e.g. section III.3 and/or IV.4

What one gets is a "long exact sequence of pointed sets" terminating at $\mathrm{H}^2(X_{\text{ét}}, \mathbb{G}_m)$ (uses that $\mathbb{G}_m$ lands in the center of GL_n).

We can make δ explicit in terms of Čech cohomology. Start with $[T] \in \mathrm{H}^1(X_{\text{ét}}, \mathrm{PGL}_n)$, so T is a PGL_n-torsor split by some cover $\mathcal{U} \to X$. On $\mathcal{U} \times_X \mathcal{U}$ the descent data (so satisfies cocycle) is given by some section in $\Gamma(\mathcal{U} \times_X \mathcal{U}, \mathrm{PGL}_n)$. After refining $\mathcal{U}$, we can lift this descent data to a section $\Gamma(\mathcal{U} \times_X \mathcal{U}, \mathrm{GL}_n)$ (no longer necessarily satisfies cocycle). Consider

$$\pi_{23}^* s \cdot \pi_{12}^* s \cdot (\pi_{13}^* s)^{-1} \in \Gamma(\mathcal{U} \times_X \mathcal{U} \times_X \mathcal{U}, \mathrm{GL}_n).$$

By construction, this becomes one 1 when pushed to PGL_n, so really it lives in the kernel $\Gamma(\mathcal{U} \times_X \mathcal{U} \times_X \mathcal{U}, \mathbb{G}_m)$. This is a 2-cocycle representing an element of $\mathrm{H}^2(X_{\text{ét}}, \mathbb{G}_m)$ (exercise).

Slogan. $\delta([T])$, the **Brauer class of** T, is the obstruction to lifting T to a GL_n-torsor.

We've defined the map δ. We don't yet know that its image is a group, or that its image lies in the cohomological Brauer group.

12.1 Geometric interpretations of PGL_n-torsors + Brauer classes

General principle: Suppose $T \in \mathrm{Sh}(X_{\text{ét}})$ (e.g. T a scheme) and $G = \underline{\mathrm{Aut}}(T)$. Then, there is a natural bijection between

$$\left\{\begin{matrix}\text{locally split}\\ G\text{-torsors}\end{matrix}\right\} \longleftrightarrow \left\{\begin{matrix}\text{forms of}\\ T\end{matrix}\right\}.$$

By a form[28] of T, we mean a sheaf on $X_{\text{ét}}$ which is locally isomorphic to T.

Starting with a form F or T, the Isom sheaf $\underline{\mathrm{Isom}}(T, F)$ gives a locally split (right) $G = \underline{\mathrm{Aut}}(T)$-torsor. Conversely, given a locally split G-torsor τ, the sheaf quotient $(\tau \times T)/G$ will be a form of T.

Think principal G-bundles and associated bundle construction

Warning 12.3. This is a bijection of sets. It is not, in general, an equivalence of categories. •

Question: Is it always an equivalence of groupoids, though?

Example. (locally split) GL_n-torsors are vector bundles. △

Say $G = \mathrm{PGL}_n$. What are some objects with Aut $= \mathrm{PGL}_n$? The principal example is $\mathrm{Aut}_X(\mathbb{P}^{n-1}_X) = \mathrm{PGL}_n$ (exercise[29]).

Corollary 12.4. *There's a natural bijection between*

$$\{\mathrm{PGL}_n\text{ -torsors}\} \longleftrightarrow \{\text{forms of } \mathbb{P}^{n-1}\}.$$

Elements of the RHS are called ***Severi-Brauer Schemes****.*

Theorem 12.5 (Noether-Skolem). *Let $R = \mathrm{Mat}_{n\times n}(A)$. Then, $\mathrm{Aut}(A) = \mathrm{PGL}_n$.*

Corollary 12.6. *Bijection*

$$\{\mathrm{PGL}_n\text{ -torsors}\} \longleftrightarrow \{\text{forms of } \mathrm{Mat}_{n\times n}\}.$$

Elements of the RHS are called ***Azumaya algebras****, up to spelling.*

12.2 Twisted Sheaves

Let $\mathcal{U} \to X$ be an étale cover, and choose $\alpha \in \Gamma(\mathcal{U} \times_X \mathcal{U} \times_X \mathcal{U}, \mathbb{G}_m)$ representing some $[\alpha] \in \mathrm{H}^2(X_{\text{ét}}, \mathbb{G}_m)$.

Definition 12.7. An **α-twisted sheaf** is a qcoh sheaf $\mathscr{F}$ on $\mathcal{U}$ and an isomorphism $\varphi : \pi_1^*\mathscr{F} \xrightarrow{\sim} \pi_2^*\mathscr{F}$ which satisfies the cocycle condition up to α, i.e.

$$\pi_{23}^*\varphi \circ \pi_{12}^*\varphi = \alpha \cdot \pi_{13}^*\varphi.$$

◇

Example. Go back and look at our construction of the map δ. △

[28]Sometimes also called a twist
[29]non-trivial. Need something like theorem on formal functions at some point

Notation 12.8. We let $\mathrm{QCoh}(X, \alpha)$ denote the category with

- objects: α-twisted sheaves
- morphisms: morphisms of sheaves on U which commute w/ φ

Example. $\mathrm{QCoh}(X, 1) \simeq \mathrm{QCoh}(X)$ via étale descent for quasi-coherent sheaves. △

Proposition 12.9. *Suppose α, α' are 2-cocycles for $\mathbb{G}_m$.*

(1) $[\alpha] \in \mathrm{Br}(X) \iff \exists$ *an α-twisted vector bundle.*

Basically showed this when defining δ

(2) $\mathrm{QCoh}(X, \alpha)$ *is an Abelian category with enough injectives (if X is 'nice').*

(3) *There's a tensor functor* $\otimes : \mathrm{QCoh}(X, \alpha) \times \mathrm{QCoh}(X, \alpha') \to \mathrm{QCoh}(X, \alpha\alpha')$. *There's also a Hom functor* $\mathrm{Hom} : \mathrm{QCoh}(X, \alpha) \times \mathrm{QCoh}(X, \alpha') \to \mathrm{QCoh}(X, \alpha'\alpha^{-1})$

(4) *There are functors*

$$\mathrm{Sym}^n, \bigwedge^n : \mathrm{QCoh}(X, \alpha) \to \mathrm{QCoh}(X, \alpha^n).$$

(5) $\mathrm{QCoh}(X, 1) \xrightarrow{\sim} \mathrm{QCoh}(X)$

Proof. For **(1)**, if you have an α-twisted vector bundle, projectivizing it will give an honest-to-God PGL_n-torsor, so $[\alpha] \in \mathrm{Br}(X)$. Other direction immediate from construction of δ.

Exercise: Try to prove **(3,4)**. ■

Corollary 12.10. $\mathrm{Br}(X)$ *is a group.*

Proof. Say we have $\alpha, \alpha' \in \mathrm{Br}(X)$. Let $\mathcal{E}$ be an α-twisted vector bundle, and similarly for $\mathcal{E}'$. Then, $\mathcal{E} \otimes \mathcal{E}'$ is an $\alpha\alpha'$-twisted vector bundle, so $\alpha\alpha' \in \mathrm{Br}(X)$. For inversion, $\mathcal{E}^\vee$ is an α^{-1}-twisted vector bundle, so $\alpha^{-1} \in \mathrm{Br}(X)$. ■

Proposition 12.11. *Suppose α is a 2-cocycle for $\mathbb{G}_m$. Then,*

$$[\alpha] \text{ is trivial} \iff \exists \alpha\text{-twisted line bundle.}$$

Proof. ($\rightarrow$) If $[\alpha]$ is trivial, then $\mathscr{O}_X$ is an α-twisted line bundle. There's an implicit lemma here: if $[\alpha] = [\alpha']$, then $\mathrm{QCoh}(X, \alpha) \cong \mathrm{QCoh}(X, \alpha')$ (easy exercise).

($\leftarrow$) Say there's an α-twisted line bundle $\mathscr{L}$. Descent data for $\mathscr{L}$ is the same as a global section $\beta \in \Gamma(\mathcal{U} \times_X \mathcal{U}, \mathbb{G}_m)$, and $\delta(\beta) = \alpha$ which exactly says it's trivial in the Brauer group.

Here, α coming from short exact sequence $1 \to \mathbb{G}_m \to \mathbb{G}_m \to \mathrm{PGL}_1 \to 1$ and $\mathrm{PGL}_1 = 1$.

■

Corollary 12.12. *Suppose $\mathcal{E}$ is an α-twisted vector bundle of rank n. Then, $[\alpha] \in \mathrm{H}^2(X_{\acute{e}t}, \mathbb{G}_m)$ is n-torsion.*

Proof. $\bigwedge^n \mathcal{E}$ is an α^n-twisted line bundle, so $[\alpha^n] = [\alpha]^n$ is twisted. ■

Corollary 12.13. $\mathrm{Br}(X) \leq \mathrm{Br}'(X) = \mathrm{H}^2(X_{\acute{e}t}, \mathbb{G}_m)_{tors}$.

Let's give some examples of Brauer classes.

Example. $X = \{x^2 + y^2 + z^2 = 0\}$ over $\mathbb{R}$ is a smooth conic with no rational points. So $X \not\simeq \mathbb{P}^1_{\mathbb{R}}$, but $X_{\mathbb{C}} \simeq \mathbb{P}^1_{\mathbb{C}}$. Hence, X is a non-trivial twisted form of $\mathbb{P}^1_{\mathbb{R}}$, and in fact, $\delta([X])$ generates $\mathrm{Br}(\mathbb{R}) = \mathbb{Z}/2\mathbb{Z}$. △

Example. The Hamilton Quaternions are a central division algebra over $\mathbb{R}$ (hence a Azumaya algebra) which also represents the generator of $\mathrm{Br}(\mathbb{R}) = \mathbb{Z}/2\mathbb{Z}$. △

For going between severi-brauer varieties and azumaya algebras, consider the diagram

$$\begin{tikzcd} & \text{Severi-Brauer} \\ \alpha\text{-Twisted sheaf } \mathcal{E} \arrow[ur, "\mathbb{P}(\cdot)"] \arrow[dr, "\mathrm{End}(\cdot)"'] & \\ & \text{Azumaya algebras} \arrow[uu, dashed, "\text{moduli of certain ideals}"'] \end{tikzcd}$$

Remark 12.14. Descent data for schemes is not always effective, so there's extra argumentation needed to know that $\mathbb{P}(\mathcal{E})$ is a scheme. The main point is that the anticanonical bundle of projective space is (very) ample, and polarized descent data for schemes is effective. ○

Example. $\mathrm{Br}(\mathbb{Q}_p) = \mathbb{Q}/\mathbb{Z}$ △

Remark 12.15. Over a field k, any 2-torsion Brauer class is represented by a 'quaternion algebra'. ○

Example. There is a map

$$0 \longrightarrow \mathrm{Br}(\mathbb{Q}) \longrightarrow \bigoplus_v \mathrm{Br}(\mathbb{Q}_v) \xrightarrow{\Sigma} \mathbb{Q}/\mathbb{Z} \longrightarrow 0$$

In particular, if $\alpha \in \mathrm{Br}(\mathbb{Q})$, then $\alpha|_{\mathbb{Q}_v} = 0$ for almost all v (exercise[30]). △

Let's interpret multiplication.

- Suppose $\mathcal{A}_1, \mathcal{A}_2$ are Azumaya algebras representing α_1, α_2. Then, $\mathcal{A}_1 \otimes \mathcal{A}_2$ is an Azumaya algebra representing $\alpha_1\alpha_2$.
- Suppose $P_1 = \mathbb{P}(\mathcal{E})$ and $P_2 = \mathbb{P}(\mathcal{E}')$ are SB's representing α_1, α_2. Then, $\mathbb{P}(\mathcal{E} \otimes \mathcal{E}')$ represents $\alpha_1\alpha_2$.

Remark 12.16. Severi-Brauer varieties (I think) come with a map to projective space via the anticanonical bundle. ○

I'm actually not so sure this is true

Open Question 12.17 (Period-index question). *Given $\alpha \in \mathrm{Br}(X)$, what is the minimum rank (or* gcd *of the ranks) of an α-twisted vector bundle?*

Next time: we'll use stuff from today to understand $\mathrm{H}^i(k, \mathbb{G}_m)$.

Theorem 12.18.

(1) *Say $k(C)$ is function field of a curve over $k = \overline{k}$. Then,*

$$\mathrm{H}^2(k(C), \mathbb{G}_m) = 0.$$

[30] Use SB interpretation

(2) Say $K_{\overline{x}}$ is strictly Henselian dvr. Then,

$$\mathrm{H}^2(K_{\overline{x}}, \mathbb{G}_m) = 0.$$

13 Lecture 13

Goal. Finish talking about Brauer groups and finally compute the cohomology of curves.

13.1 Last time

We discussed definition of Brauer group, and showed that it is a group.

Given a 2-cocycle α representing a class $[\alpha] \in \mathrm{Br}(X)$, we discussed the category of α-twisted sheaves which led us to two other ways of interpreting the Brauer group: Severi-Brauer schemes, and Azumaya algebras. Recall these were connected via

$$\begin{tikzcd}
 & \text{Severi-Brauer} \\
\alpha\text{-Twisted sheaf } \mathcal{E} \arrow[ur, "\mathbb{P}(\cdot)"] \arrow[dr, "\mathrm{End}(\cdot)"] & \\
 & \text{Azumaya algebras.} \arrow[uu, dashed, "\text{moduli of certain ideals}"']
\end{tikzcd}$$

We want to use this geometric interpretation to prove the following

Theorem 13.1 (Tsen's theorem). *Suppose k is a C_1 **field** (AKA a **quasi-algebraically closed field**), i.e. for any homogeneous polynomial $f \in k[x_1, \ldots, n]$ with $\deg < n$, f has a non-trivial zero. Then, $\mathrm{Br}(k) = 0$.*

We will also prove

Theorem 13.2. *Suppose K is either*

(1) the function field of a curve over an algebraically closed field; or

(2) the fraction field of a strictly Henselian dvr

Then K is quasi-algebraically closed.

Hence, K as above has trivial Brauer group.

13.2 This time (Finishing computation of cohomology of curves)

Definition 13.3. Let $\mathcal{E}$ be an α-twisted sheaf, so $\mathrm{End}(\mathcal{E})$ is an Azumaya algebra. The **reduced norm** $\mathrm{Nm} : \mathscr{E}nd(\mathcal{E}) \to \mathscr{E}nd(\bigwedge^{\mathrm{top}} \mathcal{E}) = \mathscr{O}_X$ is given by functoriality of $\bigwedge$, e.g. take top exterior power of an endomorphism. $\diamond$

Proposition 13.4. *Given $f \in \mathscr{E}nd(\mathcal{E})$, f is invertible if and only if $\mathrm{Nm}(f)$ is a unit.*

"A matrix is invertible iff its determinant is a unit."

Proof. Invertibility can be checked locally, and locally $\mathscr{E}nd(\mathcal{E})$ is a matrix algebra. This fact holds for matrix algebras. ■

Now, we can proof Tsen's Theorem.

Proof of Theorem 13.1. Let k be quasi-algebraically closed. Given $[\alpha] \in Br(k)$, we want an α-twisted line bundle. We have some α-twisted vector bundle $\mathcal{E}$. We want some non-trivial subbundle of $\mathcal{E}$ if $\operatorname{rank}\mathcal{E} > 1$ (and then induct). How do we do this? Could have an endomorphism of $\mathcal{E}$ with nontrivial (co)kernel, i.e. find $f \in \mathscr{E}nd(\mathcal{E})$ such that $\mathrm{Nm}(f) = 0$.

Note that reduced norm is a function $\mathrm{Nm} : \mathscr{E}nd(\mathcal{E}) \to k$ whose source is a $\operatorname{rank}(\mathcal{E})^2$ dimensional affine space, so Nm is a (homogeneous) polynomial in $\operatorname{rank}(\mathcal{E})^2$ variables.[31] At the same time, $\deg(\mathrm{Nm}) = \operatorname{rank}\mathcal{E}$ (since $\mathrm{Nm} = \det$ after suitable extension). Thus, we get our desired f (when $\operatorname{rank}\mathcal{E} > 1$) since k is C_1. Set $\mathcal{E}' = \ker(f)$, and α-twisted vector bundle of strictly lower rank. Rince, wash and repeat to get an α-twisted line bundle, but the existence of such a thing implies $[\alpha] = 0 \in \mathrm{Br}$. ■

Corollary 13.5. *If k is quasi-algebraically closed, then* $\mathrm{H}^2(k, \mathbb{G}_m) = 0$.

Proof idea. For a field, $\mathrm{H}^2(k, \mathbb{G}_m) = \mathrm{Br}(k)$. This equality is actually true in a lot of situations (though not for general schemes). To show this for fields, you can write down an explicit central simple algebra using a 2-cocycle (done e.g. in Serre's 'Local Fields'). ■

We next want to show that certain fields are quasi-algebraically closed, i.e prove Theorem 13.2. We will actually only prove half of it here, and give a reference for the other case.

Proof of first case of Theorem 13.2. Let $k = k(C)$ be the function field of some curve C over an algebraically closed field L. Given $f \in k(C)[x_1, \ldots, x_n]$ homogeneous with $\deg f < n$, we want a non-trivial zero of f in $k(C)^n$.

Idea: choose an ample divisor D on C, and consider f as a function

$$f : \Gamma(C, \mathscr{O}(mD))^n \longrightarrow \Gamma(C, \mathscr{O}((\deg f)mD + D'))$$

($m \in \mathbb{Z}$ some integer) with D' coming from the poles of the coefficients of f. Hence, f gives a map of irreducible affine spaces over $L = \overline{L}$.

What are the dimensions of these spaces? Using Riemann-Roch, we know

$$\dim \Gamma(C, \mathscr{O}(mD))^n \sim nm \;\text{ and }\; \dim \Gamma(C, \mathscr{O}((\deg f)mD + D')) \sim (\deg f)m.$$

Recall $\deg f < n$ by assumption, so for $m \gg 0$, we get a map $X \to Y$ w/ $\dim X > \dim Y$. Thus, the dimension of any non-empty fiber is positive. Note that $f^{-1}(0)$ is non-empty since it is homogeneous so $0 \in f^{-1}(0)$. Thus, $\dim f^{-1}(0) > 0$ so $f^{-1}(0)$ contains some non-trivial L point and we win. ■

Corollary 13.6. $\mathrm{Br}(k(C)) = \mathrm{H}^2(k(C), \mathbb{G}_m) = 0$.

Remark 13.7. Fraction fields K of strictly Henselian dvrs are also C_1. This is proved e.g. in Lang's thesis.

Exercise. Prove this in the case that K is equicharacteristic 0.

[31] Take some field extension under which $\mathcal{E}$ splits, and then Nm is just the determinant

Hence, $\mathrm{Br}(K) = \mathrm{H}^2(K, \mathbb{G}_m) = 0$ for $K = \mathrm{Frac}(\text{strictly Henselian dvr})$. ○

Theorem 13.8 ((Less general version of) **Tate's theorem).** *Let K be as above (e.g. $k(C)$ or $K_{\overline{x}}$, function field of curve over $L = \overline{L}$ or frac field of strictly Henselian dvr). Then, $\mathrm{H}^i(k, \mathbb{G}_m) = 0$ for all $i > 0$.*

Proof. The goal is $\mathrm{H}^i(\mathrm{Gal}(L/k), L^\times) = \mathrm{H}^i(L/K, \mathbb{G}_m) = 0$ for all finite, separable Galois extensions L/K (recall Čech cohomology is exactly computing Galois cohomology in this case).

(1) We know this vanishes for $i = 1, 2$. $i = 1$ is Hilbert 90 and $i = 2$ follows from Tsen's theorem (for $L = K$) + the inflation-restriction exact sequence.

(2) Now suppose L/K is cyclic. Cohomology of cyclic groups is 2-periodic, so we win by the $i = 1, 2$ cases.

(3) Now suppose L/K is nilpotent. Let $C \leq \mathrm{Gal}(L/K)$ be normal + cyclic (exists since L/K nilpotent[32]) so we get a short exact sequence

$$1 \longrightarrow C \longrightarrow \mathrm{Gal}(L/K) \longrightarrow G' \longrightarrow 1$$

with G' nilpotent as well. We know vanishing for C and for G' (by induction), so we get the result for $\mathrm{Gal}(L/K)$ by inflation-restriction.

(4) Now say L/K general.

Recall 13.9. p-groups are nilpotent. The class equation tells you that p-groups always have nontrivial center, so keep quotient by centers. ⊙

For $G_p \leq \mathrm{Gal}(L/K)$ a p-Sylow, $\mathrm{H}^i(G_p, L^\times) = 0$ by part **(3)**. Now consider the map

$$\mathrm{H}^i(\mathrm{Gal}(L/K), L^\times) \hookrightarrow \bigoplus_p \mathrm{H}^i(G_p, L^\times) = 0.$$

This map is injective, so we win. Why is it injective. Recall the (co)restriction maps

$$\mathrm{res} : \mathrm{H}^i(\mathrm{Gal}(L/K), L^\times) \rightleftarrows \mathrm{H}^i(G_p, L^\times) : \mathrm{cor}$$

satisfy $\mathrm{res} \circ \mathrm{cor} = [G : G_p]$ which is prime to p, so res is injective away from prime-to-p torsion.

■

Corollary 13.10. *Let C be a smooth curve over an algebraically closed field. Then,*

$$\mathrm{H}^i(C_{\acute{e}t}, \mathbb{G}_m) = \mathrm{H}^i(k(C), \mathbb{G}_m) \text{ for } i > 1.$$

Proof. Before, we showed $\mathrm{H}^i(C_{\text{ét}}, \mathbb{G}_m) \simeq \mathrm{H}^i(C_{\text{ét}}, \eta_* \mathbb{G}_m)$ for $i > 1$ (by divisor exact sequence). We now claim $\mathrm{H}^i(C_{\text{ét}}, \eta_* \mathbb{G}_m) = \mathrm{H}^i(k(C), \mathbb{G}_m)$. The Leray spectral sequence tells us that to show this claim it suffices to know that $R^i \eta_* \mathbb{G}_m = 0$ for $i > 0$, but the stalks of this sheaf are $\mathrm{H}^i(K_{\overline{x}}, \mathbb{G}_m) = 0$. ■

[32] being solvable is enough

Corollary 13.11.

$$\mathrm{H}^i(C_{ét}, \mathbb{G}_m) = \mathrm{H}^i(k(C), \mathbb{G}_m) = 0 \text{ for } i > 1$$

Proof. Combine previous corollary with Tate's theorem. ∎

Corollary 13.12.

$$\mathrm{H}^i(C_{ét}, \mathbb{G}_m) = \begin{cases} \mathbb{G}_m(C) & \text{if } i = 0 \\ \mathrm{Pic}(C) & \text{if } i = 1 \\ 0 & \text{if } i > 1 \end{cases}$$

Corollary 13.13.

$$\mathrm{H}^i(C_{ét}, \mu_n) = \begin{cases} \mu_n & \text{if } i = 0 \\ \mathrm{Pic}(C)[n] & \text{if } i = 1 \\ \mathbb{Z}/n\mathbb{Z} & \text{if } i = 2 \text{ and } C \text{ proper} \\ 0 & \text{otherwise} \end{cases}$$

where n is prime to $\operatorname{char} k$.

Proof. Apply the Kummer sequence $1 \to \mu_n \to \mathbb{G}_m \to \mathbb{G}_m \to 1$. For the $i = 2$ case, need to know that $\mathrm{Pic}^0(C)$ is divisible when C proper and $\mathrm{Pic}(C)$ is itself divisible when C is not proper (exercise). ∎

Remark 13.14. The description in the last corollary is Galois equivariant. For a non-Galois equivariant description, $\operatorname{Pic} C[n] = (\mathbb{Z}/n\mathbb{Z})^{2g}$ when C proper. ○

Remark 13.15. Above theorem uses more than Brauer group vanishes. It uses that Brauer group vanishes for all finite extensions. This is in fact true more generally for C_1 fields (all finite extensions are C_1). ○

Question 13.16 (Audience)**.** *For torsors, if you have a section, then you're trivial? Is something similar true more general for twists? Like, a Brauer-Severi is often trivial if it has a rational point, and a rank n vector bundle with n linearly independent sections is trivial, so maybe there is something you can say in general involving sections? (paraphrase)*

Answer. The case of Brauer-Severis is a little special. If one has a section, then its Brauer class vanishes, but this does not mean that is is a trivial form of $\mathbb{P}^n$; it only means it is a *Zariski-locally*[33] trivial form of $\mathbb{P}^n$. Over something like $\operatorname{Spec} k$, Zariski-locally trivial = trivial.k

Let's give a proof of 'Severi-Brauer with section has trivial Brauer class.' A Severi-Brauer is a projectivization $\mathbb{P}(\mathcal{E})$ of a twisted vector bundle $\mathcal{E}$. What is a section of $\mathbb{P}(\mathcal{E})$? Well, it is a twisted subline bundle (or twisted quotient line bundle, depending on conventions), and we've said that having a twisted line bundle means the corresponding Brauer class vanishes.

In general, not sure what a general criterion for a form to be trivial should be. ⋆

Remark 13.17. This computation was the first sign that étale cohomology is a good cohomology theory (looks like singular cohomology). The second big piece of evidence was Artin computing the étale cohomology of $\mathbb{A}^2 \setminus 0$ which looks like the 3-sphere (punctured $\mathbb{C}^2$). ○

We've achieved our first big goal: cohomology of curves. Next time, we start building towards Poincaré duality, cohomology with compact support, ...

[33] Trivial Brauer class means it comes from an element of $\mathrm{H}^1(X_{\text{ét}}, \mathrm{GL}_n) = \mathrm{H}^1(X_{\text{zar}}, \mathrm{GL}_n)$, i.e. that your Severi-Brauer is $\mathbb{P}(\mathcal{E})$ for some *not twisted* vector bundle $\mathcal{E}$ on X. This vector bundle is (Zariski) locally trivial, so $\mathbb{P}(\mathcal{E})$ is (Zariski) locally $\mathbb{P}(\mathscr{O}_X^n) = \mathbb{P}_X^n$

14 Lecture 14

Last time we computed the étale cohomology of curves. Today we will talk about compactly supported cohomology and Gysin sequences. We want to focus on computational aspects of étale cohomology until we get to Weil conjecture stuff. There will be some big theorems we will need along the way, but we will not be proving them.

14.1 Extension by zero

Let $j : U \hookrightarrow X$ be an open embedding.

Definition 14.1. **Extension by zero** is the functor $j_! : \mathrm{Sh}^{ab}(U) \to \mathrm{Sh}^{\mathrm{ab}}(X)$ ("j **lower shriek**") defined as the sheafification of

$$(V \to X) \mapsto \begin{cases} \mathscr{F}(V \times_X U) & \text{if } \mathrm{im}(V) \subset U \\ 0 & \text{otherwise.} \end{cases}$$

◇

Remark 14.2. We will later see 'lower shriek' for more general morphisms, but it requires input from Nagata compactification. ○

Proposition 14.3 (Exercise)**.** *$j_!$ is left adjoint to j^*.*

Proof idea. Check the adjoint property on presheaves, and then use adjoint property of sheafification. ■

Proposition 14.4.

$$(j_!\mathscr{F})_{\overline{x}} = \begin{cases} \mathscr{F}_{\overline{x}} & \text{if } \overline{x} \in \mathrm{im}\, j \\ 0 & \text{if } \overline{x} \notin \mathrm{im}\, j \end{cases}$$

This is immediate from the definition.

Corollary 14.5. *$j_!$ is exact.*

(check on stalks).

This is useful for "excision"

Proposition 14.6. *Suppose $\mathscr{F} \in \mathrm{Sh}^{\mathrm{ab}}(X_{\acute{e}t})$ and inclusions $U \overset{j}{\hookrightarrow} X \overset{\iota}{\hookleftarrow} Z = X \setminus U$ (with U Zariski open and Z Zariski closed in X). Then there is an exact sequence*

$$0 \longrightarrow j_!j^*\mathscr{F} \longrightarrow \mathscr{F} \longrightarrow \iota_*\iota^*\mathscr{F} \longrightarrow 0$$

(the above maps are the (co)units from the relevant adjoint pairs).

Proof. Check on stalks. Choose a geometric point $\overline{x} \in X$.

- First suppose $\overline{x} \in U$. This sequence then becomes

$$0 \longrightarrow \mathscr{F}_{\overline{x}} \xrightarrow{\mathrm{id}} \mathscr{F}_{\overline{x}} \longrightarrow 0 \longrightarrow 0$$

  which is exact.

- Now suppose $\overline{x} \in Z$. This sequence then becomes

$$0 \longrightarrow 0 \longrightarrow \mathscr{F}_{\overline{x}} \xrightarrow{\text{id}} \mathscr{F}_{\overline{x}} \longrightarrow 0$$

which is exact. ■

Definition 14.7. Suppose $\mathscr{F} \in \mathrm{Sh}^{\mathrm{ab}}(U_{\text{ét}})$ and $j : U \hookrightarrow X$ is an open embedding with X *proper*. Then, **Cohomology with compact support** is defined as

$$\mathrm{H}_c^i(U_{\text{ét}}, \mathscr{F}) := \mathrm{H}^i(X_{\text{ét}}, j_!\mathscr{F}).$$

◇

Remark 14.8. To define compactly support cohomology for a (smooth) manifolds, one usually defines it as the cohomology of the de Rham complex of compactly-support differential forms, i.e. forms which vanish outside a compact set. if X is a smooth manifold and $X \hookrightarrow M$ with M a compact smooth manifold, then any compactly support form on X give rise to a compactly support form on M via extension by 0. Since M is compact, all forms on it are compact, so compactly supported (de Rham) cohomology of M is just usual cohomology of M. The upshot is that compactly supported cohomology on X can be studied in terms of usual cohomology of M, and this is the idea we're trying to capture in the above algebraic definition. ○

Question 14.9. ***(1)*** *Why does such an X exist, and* ***(2)*** *why does this definition not depend on j, X?*

There's an obvious construction to the existence of such an X. Proper schemes are always separated, so if U is not separated, then it cannot be open inside a proper scheme. This is all that can go wrong.

Theorem 14.10 (Nagata)**.** *If $U \to S$ is a separated S-scheme, then there exists a universally closed S-scheme $X \to S$, and an open embedding $U \hookrightarrow X$ over S. When $U \to S$ is finite type, then we can take X to be proper over S.*

Applying this when S is a point resolves question **(1)**. Resolving question **(2)** requires proper base change which will give independence for torsion sheaves.

Proposition 14.11. *Let U be a connected regular curve over $k = \overline{k}$, and assume* $\operatorname{char} k \nmid n$. *Then, there is a canonical iso* $\mathrm{H}_c^2(U_{\acute{e}t}, \mu_n) \simeq \mathbb{Z}/n\mathbb{Z}$.

Proof. Let $j : U \hookrightarrow X$ be the open embedding into the canonical regular compactification, and let $Z = X \setminus U \overset{\iota}{\hookrightarrow} X$ be inclusion of the complement. To understand $\mathrm{H}^i(X, j_!\mu_n)$, we use the short exact sequence

$$\begin{tikzcd} 0 \arrow[r] & j_!j^*\mu_n \arrow[r] \arrow[d, equal] & \mu_n \arrow[r] & i_*i^*\mu_n \arrow[r] \arrow[d, equal] & 0 \\ & j_!\mu_n & & \bigoplus \text{skyscraper sheaves} & \end{tikzcd}$$

This gives a long exact sequence

$$\cdots \longrightarrow \mathrm{H}_c^i(U, \mu_n) \longrightarrow \mathrm{H}^i(X, \mu_n) \longrightarrow \mathrm{H}^i(X, \iota_*\iota^*\mu_n) \longrightarrow \mathrm{H}_c^{i+1}(U, \mu_n) \longrightarrow \cdots$$

First note that

$$\mathrm{H}^i(X, \iota_*\iota^*\mu_n) = \begin{cases} \bigoplus_{\text{pts}} \mu_n(k) & \text{if } i = 0 \\ 0 & \text{if } i > 0 \end{cases}$$

since $\iota_*\iota^*\mu_n$ is a sum of skyscraper sheaves[34] support of the finite set of points Z. Hence, we have the sequences (below, we're assuming $U \neq X$)

$$\begin{tikzcd} 0 \arrow[r] & \mathrm{H}^0_C(U,\mu_n) \arrow[r] \arrow[d, equal] & \mathrm{H}^0(X,\mu_n) \arrow[r, "\Delta"] \arrow[d, equal] & \bigoplus_{z \in Z} \mu_n(k) \arrow[r] & \mathrm{H}^1_c(U,\mu_n) \arrow[r] & \mathrm{H}^1(X,\mu_n) \arrow[r] \arrow[d, equal] & 0 \\ & \mathrm{H}^0(X, j_!\mu_n) \arrow[d, equal] & \mu_n(k) & & & \mathrm{Pic}(X)[n] & \\ & 0 & & & & & \end{tikzcd}$$

and

$$\begin{tikzcd} 0 \arrow[r] & \mathrm{H}^2_c(U,\mu_n) \arrow[r] & \mathrm{H}^2(X,\mu_n) \arrow[r] \arrow[d, equal] & 0 \\ & & \mathbb{Z}/n\mathbb{Z} & \end{tikzcd}$$

This finishes the computation of $\mathrm{H}^2_c(U, \mu_n)$. What is H^1? It's some extension

$$0 \longrightarrow \frac{\bigoplus \mu_n(k)}{\mu_n(k)} \longrightarrow \mathrm{H}^1_c(U,\mu_n) \longrightarrow \mathrm{Pic}(X)[n] \longrightarrow 0$$

with $\mu_n(k)$ embedded diagonally. ■

14.2 Proper Base Change + Finiteness results

Definition 14.12. Suppose $\mathscr{F} \in \mathrm{Sh}(X_{\text{ét}})$ is a sheaf (of sets) on $X_{\text{ét}}$. We say it is **constructible** if both

(1) For every $i : Z \hookrightarrow X$, a closed embedding, there exists a non-empty open $U \subset Z$ such that $(i^*\mathscr{F})|_U$ is **locally constant**, i.e. there exists a cover $V \to U$ such that $i^*\mathscr{F}|_V$ is constant.

(2) The stalks of $\mathscr{F}$ are finite.

⋄

Example. If $U \hookrightarrow X$ is open, then $j_!\mu_n$ is constructible. The stalks are either 0 or μ_n so finite. Given $i : Z \hookrightarrow X$, if $Z \cap U = \emptyset$, then $i^*j_!\mu_n = 0$ which is constant. If $Z \cap U \neq \emptyset$, then $(i^*j_!\mu_n)|_{Z\cap U}$ is locally constant; it is isomorphic to μ_n on $U_{\text{ét}}$ which is not constant because of the Galois action, but which becomes constant after taking the cover where you adjoin all nth roots of unity. △

Example. Suppose $\mathscr{F}$ is represented by a quasi-finite X-scheme. △

Theorem 14.13 (Hard)**.** *Let $f : X_{\acute{E}t} \to X_{\acute{e}t}$ be the obvious morphism, and let $\mathscr{F} \in \mathrm{Sh}(X_{\acute{E}t})$ be a sheaf on the big étale site such that*

[34] Use Leray spectral sequence $\mathrm{H}^i(X, R^j\iota_*\iota^*\mu_n) \implies \mathrm{H}^{i+j}(Z, \mu_n)$ to get vanishing higher cohomology. The point is $R^j\iota_*\iota^*\mu_n = 0$ for $j > 0$ since ι_* is exact as it's the pushforward along a finite map

(1) The natural map $\mathscr{F} \leftarrow f^* f_* \mathscr{F}$ is an isomorphism; and

(2) $f_*\mathscr{F}$ is constructible.

Then $\mathscr{F}$ is represented by a quasi-finite X-scheme.

So any constructible sheaf on the small étale site is representable (by a scheme in $X_{\text{Ét}}$. A quasi-finite X-scheme may not live in small étale site).

Theorem 14.14. *Suppose $\pi : X \to S$ is a proper morphism and $\mathscr{F} \in \mathrm{Sh}^{\mathrm{ab}}(X_{ét})$ is a constructible abelian sheaf on the small étale site of X. Then, $R^i\pi_*\mathscr{F}$ is constructible for $i \geq 0$, with stalks*

$$\left(R^i\pi_*\mathscr{F}\right)_{\overline{s}} \xrightarrow{\sim} \mathrm{H}^i(X_{\overline{s}}, \mathscr{F}|_{X_{\overline{s}}})$$

for all geometric points $\overline{s} \in S$.

Corollary 14.15. *Suppose X is proper over $k = k^s$, and suppose $\mathscr{F} \in \mathrm{Sh}^{\mathrm{ab}}(X_{ét})$ is constructible. Then,*

(1) $\mathrm{H}^i(X_{ét}, \mathscr{F})$ is finite.

(2) If $k \subset L$ each a separably closed field, then the natural map

$$\mathrm{H}^i(X_{ét}, \mathscr{F}) \xrightarrow{\sim} \mathrm{H}^i(X_{L,ét}, \mathscr{F}|_{X_L,ét})$$

is an iso.

Proof. Constructible sheaves on $\operatorname{Spec} k$ are finite groups (recall $k = k^s$), so we get **(1)**. For **(2)**, both groups are stalks of the constant sheaf $(R^i\pi_*\mathscr{F})$ (at the geom points $\operatorname{Spec} k = \operatorname{Spec} k$ and $\operatorname{Spec} L \to \operatorname{Spec} k$). ■

Non-example. Take $X = \mathbb{A}^1_{\mathbb{F}_p}$ and let $\mathscr{F} = \mathbb{Z}/p\mathbb{Z}$. We showed before (End of section 8.2) that $\mathrm{H}^1(X_{\text{ét}}, \mathbb{Z}/p\mathbb{Z})$ is infinite. ▽

Corollary 14.16 (Proper base change theorem). *Say we have a Cartesian diagram*

$$\begin{tikzcd} X' \arrow[r, "f'"] \arrow[d, "\pi'"'] & X \arrow[d, "\pi"] \\ T \arrow[r, "f"] & S. \end{tikzcd}$$

For any $\mathscr{F} \in \mathrm{Sh}^{\mathrm{ab}}(X_{ét})$, there is a natural map

$$f^*\left(R^i\pi_*\mathscr{F}\right) \longrightarrow R^i\pi_* f^*\mathscr{F}.$$

If π is proper and $\mathscr{F}$ is torsion, then this map is an isomorphism.

Proof idea. Construct this map using adjointness. Then check it's an isomorphism on stalks where it reduces to the previous theorem. ■

Warning 14.17. Above idea works for *constructible* sheaves, but theorem stated for *torsion* sheaves, so one needs more input. •

15 Lecture 15: Proper base change (continued)

Last time we started discussing proper base change. Let's continue down this path (though we won't prove it). We recall the statement

Theorem 15.1. *Say $\pi : X \to S$ is a proper morphism, and $\mathscr{F} \in \mathrm{Sh}^{\mathrm{ab}}(X_{\acute{e}t})$ is a constructible sheaf of abelian groups on the small étale site of X. Then, $R^i\pi_*\mathscr{F}$ is constructible for $i \geq 0$ with stalks*

$$(R_i\pi_*\mathscr{F})_{\overline{s}} \simeq \mathrm{H}^i(X_{\overline{s},\acute{e}t}, \mathscr{F}|_{X_{\overline{s},\acute{e}t}})$$

for all geom pts $\overline{s} \hookrightarrow S$.

Corollary 15.2. *For $\pi : X \to S$ proper, the formation of $R^i\pi_*\mathscr{F}$ ($\mathscr{F}$ torsion) commutes with base change.*

Remark 15.3. To go from constructible to torsion, use that any torsion sheaf is a filtered colimt of constructible sheaves. ∘

How does one prove this stuff. Here are some of the key ideas

- Reduce to the case where π is a relative curve.

  Image X is quasi-projective. After blowing up, you can factor π as a sequence of maps, each of which is a relative curve. Then use Leray spectral sequence.

- Use Devissage to reduce to the case where $\mathscr{F} = \mu_n$.

  On a big open, $\mathscr{F}$ is locally constant, so honestly constant after taking some cover. Then some push-pull argument gets you to the μ_n case.

- Now $\pi : X \to S$ is a relative curve, and $\mathscr{F} = \mu_n$. Get long exact sequence

  $$0 \to \pi_*\mu_n \to \pi_*\mathbb{G}_m \to \pi_*\mathbb{G}_m \to R^1\pi_*\mu_n \to R^1\pi_*\mathbb{G}_m \to R^1\pi_*\mathbb{G}_m \to R^2\pi_*\mu_n \to 0$$

  (secretly, $R^i\pi_*\mathbb{G}_m = 0$ for $i \geq 2$) coming from Kummer sequence. Goal is to show that $\pi_*\mu_n, R^1\pi_*\mu_n$, and $R^2\pi_*\mu_n$ are represented by quasi-finite S-schemes.

  Key geometric input (Grothendieck): in this situation (i.e. proper relative curve), $R^1\pi_*\mathbb{G}_m = \operatorname{Pic} X/S$ is representable by an S-scheme which is locally of finite type. Note that $\operatorname{Pic} X/S$ is the sheafification of the functor

  $$T \mapsto \operatorname{Pic}(X_T)/p^* \operatorname{Pic} T \text{ where } p : X_T \to T.$$

  Then get

  $$R^1\pi_*\mu_n = \ker\left(\operatorname{Pic} X/X \xrightarrow{[n]} \operatorname{Pic} X/S\right)$$

  and understanding properties of the Picard functor then shows that $R^1\pi_*\mu_n$ is a quasi-finite S-scheme. Also that $R^2\pi_*\mu_n = \operatorname{coker}(\operatorname{Pic} X/S \xrightarrow{[n]} \operatorname{Pic} X/S)$ is quasi-finite.

How does one actually use proper base change?

Example. Say we have $\pi : X \to S$ a smooth, proper curve, and we want to compute $\mathrm{H}^i(X_{\text{ét}}, \mathbb{Z}/n\mathbb{Z})$ (assume n invertible on S). We have a Leray spectral sequence

$$\mathrm{H}^r(S_{\text{ét}}, R^s\pi_*\mathbb{Z}/n\mathbb{Z}) \implies \mathrm{H}^{r+s}(X_{\text{ét}}, \mathbb{Z}/n\mathbb{Z}).$$

We know $R^s\pi_*\mathbb{Z}/n\mathbb{Z}$ is constructible by proper base change, and we know the stalks

$$(R^s\pi_*\mathbb{Z}/n\mathbb{Z})_{\overline{s}} = \mathrm{H}^s_{\text{ét}}(X_{\overline{s}}, \mathbb{Z}/n\mathbb{Z}).$$

The ranks of these talks does not depend on $\overline{s}$. One we know more, we'll be able to say that this is a locally constant sheaf, and that there are techniques for computing ranks of cohomology for locally constant sheaves on curves. $\triangle$

Example. Say $\pi : X \to S$ is a proper curve. We know over the open locus in S where π is smooth, $R^s\pi_*\mathbb{Z}/n\mathbb{Z}$ is locally constant. $\triangle$

Warning 15.4. If you have a locally constant sheaf on S, the Galois action on the fibers may "vary." For example, if you have an elliptic curve over a base, then the n-torsion of the curve is a locally constant sheaf over the base, but the Galois action really depends on the specific elliptic curves showing up. $\bullet$

Proposition 15.5. *Suppose U is a separated scheme, and $\mathscr{F}$ is a constructible sheaf on U. Then,*

$$\mathrm{H}^i_c(U, \mathscr{F}) := \mathrm{H}^i(X_{\acute{e}t}, j_!\mathscr{F}) \text{ where } j : U \overset{open}{\hookrightarrow} X = proper$$

does not depend on X.

Proof. Choose two compactifications $j_1 : U \hookrightarrow X_1$ and $j_2 : U \hookrightarrow X_2$. We want $\mathrm{H}^j(X_i, j_{i!}\mathscr{F})$ to be independent of i. Consider $(j_1, j_2) : U \to X_1 \times X_2$. This may not be open, so set $X = \overline{\mathrm{im}(j_1, j_2)} \subset X_1 \times X_2$. This reduces the case of

$$\begin{tikzcd} U \arrow[r, hook, "j"] \arrow[dr, hook, "j_1"'] & X \arrow[d, "\pi"] \arrow[r, equal] & \text{proper} \\ & X_1 \arrow[r, equal] & \text{proper} \end{tikzcd}$$

We want to compare the cohomology of $j_!\mathscr{F}$ on X with the cohomology of $j_{1!}\mathscr{F}$ on X_1. We use the Leray spectral sequence

$$\mathrm{H}^r\left(X_1, R^s\pi_*j_!\mathscr{F}\right) \implies \mathrm{H}^{r+s}(X, j_!\mathscr{F}).$$

Let's compute the derived pushforwards above. Since π is proper (map between proper schemes), we have

$$(R^s\pi_*j_!\mathscr{F})_{\overline{x}} = \mathrm{H}^s(\pi^{-1}(\overline{x})_{\text{ét}}, j_!\mathscr{F}) = 0 \text{ if } s > 0$$

π is an isomorphism over the image of U

(if $\overline{x} \in \mathrm{im}\, j$, above is cohomology of a (geometric) point. If not, $\mathscr{F}$ is 0). Hence the spectral sequence degenerates immediately, and we get an equality

$$\mathrm{H}^r(X_1, \pi_*j_!\mathscr{F}) = \mathrm{H}^r(X, j_!\mathscr{F}).$$

Exercise. $\pi_*j_!\mathscr{F} \simeq j_{1!}\mathscr{F}$. Construct map and check on stalks.

This exercise finishes the proof. ■

Remark 15.6. Usually one uses the proper base change theorem to compute stalks. ○

Proposition 15.7.

(1) *Given short exact sequence $0 \to \mathscr{F} \to \mathscr{G} \to \mathscr{H} \to 0$ of constructible abelian sheaves on $U_{\text{ét}}$, we get a LES in $\mathrm{H}^i_c(U, -)$.*

(2) *If $\mathscr{F}$ is constructible, $\mathrm{H}^i_c(U_{\text{ét}}, \mathscr{F})$ is finite.*

Proof. **(1)** Want LES $\mathrm{H}^i(X, j_!\mathscr{F}) \to \mathrm{H}^i(X, j_!\mathscr{F}) \to \mathrm{H}^i(X, j_!\mathscr{G}) \to \mathrm{H}^{i+1}(X, j_!\mathscr{H})$ and so on, where $j : U \hookrightarrow X$ with X proper. Recall that $j_!$ is exact, so this arise from short exact sequence $0 \to j_!\mathscr{F} \to j_!\mathscr{G} \to j_!\mathscr{H} \to 0$.

Remember: $j_!$ exact because it's stalks are simple

(2) Enough to show that $j_!\mathscr{F}$ is constructible on X because then $\mathrm{H}^i(X_{\text{ét}}, j_!\mathscr{F})$ is finite by proper base change theorem. To show it is constructible, there are two things to check.

- The stalks are finite.

  This is because the stalks of $\mathscr{F}$ were already finite.

- Given $T \subset X$ closed, $j_!\mathscr{F}|_T$ is locally constant on an open of T.

  Consider $T \cap U$. If nonempty, it's an open subset of T which is closed in U, so it has an open subset on which $\mathscr{F}$ is locally constant. If $T \cap U = \emptyset$, then $j_!\mathscr{F}|_T = 0$ which is already constant.

■

15.1 Purity, Gysin sequence, cohomology w/ supports

Let $\Lambda = \underline{\mathbb{Z}/n\mathbb{Z}}$ (always assume n invertible on the base). We let Sh^Λ denote (the category of) sheaves of Λ-modules.

Example. μ_n is a sheaf of Λ-modules. △

Notation 15.8. Given $\mathscr{F} \in \mathrm{Sh}^\Lambda$, we set

$$\mathscr{F}(r) := \mathscr{F} \otimes_\Lambda \mu_n^{\otimes r}.$$

When we study Gysin sequences, we'll see some twists appearing.

Here's a general way of relating cohomology on an open to cohomology on the complement: cohomology with supports. Let $Z \overset{\text{closed}}{\subset} X$ with complement $U \overset{\text{open}}{\subset} X$. Consider the functor

$$\begin{array}{rccc} \Gamma_Z : & \mathrm{Sh}^{\mathrm{ab}}(X_{\text{ét}}) & \longrightarrow & \mathrm{Ab} \\ & \mathscr{F} & \longmapsto & \ker\left(\Gamma(X, \mathscr{F}) \to \Gamma(U, \mathscr{F})\right) \end{array}$$

i.e.

$$\Gamma_Z(X, -) = \ker\left(\Gamma(X, -) \to \Gamma(U, -)\right).$$

We call this **global sections support on** Z.

Exercise. This functor is left exact.

Definition 15.9. We define **cohomology with support** to be the right derived functors $\mathrm{H}^*_Z(X,-)$ of Γ_Z. ◇

Theorem 15.10. *There is a functorial long exact sequence (here, $- \in \mathrm{Sh}^{\mathrm{ab}}(X_{ét})$)*

This is a Hartshorne exercise

$$\cdots \to \mathrm{H}^i_Z(X_{ét},-) \to \mathrm{H}^i(X_{ét},-) \to \mathrm{H}^i(U_{ét},-) \to \mathrm{H}^{i+1}_Z(X_{ét},-) \to \cdots$$

Proof. Set the notation $U \xrightarrow[\text{open}]{j} X \xhookleftarrow{\iota} Z$ with $X \setminus U = Z$. Recall the short exact sequence

$$0 \to j_! j^* \underline{\mathbb{Z}} \to \underline{\mathbb{Z}} \to \iota_* \iota^* \underline{\mathbb{Z}} \to 0.$$

We want to reinterpret the LES coming from this sequence.

Claim 15.11.

$$\mathrm{Hom}(\iota_*\iota^*\underline{\mathbb{Z}}, \mathscr{F}) \simeq \Gamma_Z(X_{ét}, \mathscr{F}).$$

This is because we have an exact sequence

$$\begin{tikzcd}
0 \arrow[r] & \mathrm{Hom}(\iota_*\iota^*\underline{\mathbb{Z}}, \mathscr{F}) \arrow[r] & \mathrm{Hom}(\underline{\mathbb{Z}}, \mathscr{F}) \arrow[r] \arrow[d, equal] & \mathrm{Hom}(j_!j^*\underline{\mathbb{Z}}, \mathscr{F}) \arrow[d, equal] \\
 & & \Gamma(X, \mathscr{F}) & \mathrm{Hom}(j^*\underline{\mathbb{Z}}, j^*\mathscr{F}) \arrow[d, equal] \\
 & & & \Gamma(U, \mathscr{F}|_U)
\end{tikzcd}$$

so $\mathrm{Hom}(\iota_*\iota^*\underline{\mathbb{Z}}, \mathscr{F})$ is precisely the kernel of this restriction map, as claimed.

As a consequence of the claim, we see that

$$\mathrm{H}^i_Z(X_{\text{ét}}, \mathscr{F}) \simeq \mathrm{Ext}^i_{\mathrm{Sh}^{\mathrm{ab}}(X_{\text{ét}})}(\iota_*\iota^*\underline{\mathbb{Z}}, \mathscr{F}),$$

so our desired long exact sequence is simply the LES of Ext applied to our initial short exact sequence. ■

Theorem 15.12. *Suppose $Z \overset{\text{closed}}{\subset} X$ are k-schemes for some field k, and suppose Z, X are smooth. Also assume Z is of pure codimension c in X. Then for $\mathscr{F} \in \mathrm{Sh}^{\mathrm{ab}}(X_{ét})$* ***locally constant constructible*** *(i.e. locally constant with finite stalks), there is a canonical isomorphism*

Question: Is this like a Thom isomorphism?

Answer: Yes. A bit more on this next lecture.

$$\mathrm{H}^{r-2c}(Z, \mathscr{F}(-c)) \xrightarrow{\sim} \mathrm{H}^r_Z(X, \mathscr{F})$$

for all $r \geq 0$.

Example. Say $Z = pt \subset \mathbb{A}^1$ over $k = \overline{k}$ with $\operatorname{char} k \nmid n$ so $c = 1$. Then,

$$\mathrm{H}^{r-2}(pt, \mathbb{Z}/n\mathbb{Z}(-1)) \simeq \mathrm{H}^r_{pt}(\mathbb{A}^1_{\text{ét}}, \mathbb{Z}/n\mathbb{Z}) = \begin{cases} \mathbb{Z}/n\mathbb{Z}(-1) & \text{if } r = 2 \\ 0 & \text{otherwise.} \end{cases}$$

We can compute this even without the theorem.

We have a long exact sequence

$$\mathrm{H}^i_{pt}(\mathbb{A}^1, \mathbb{Z}/n\mathbb{Z}) \to \mathrm{H}^i(\mathbb{A}^1, \mathbb{Z}/n\mathbb{Z}) \to \mathrm{H}^i(\mathbb{G}_m, \mathbb{Z}/n\mathbb{Z}) \to \mathrm{H}^{i+1}_{pt}(\mathbb{A}^1, \mathbb{Z}/n\mathbb{Z})$$

We know[35]

$$\mathrm{H}^i(\mathbb{A}^1, \mathbb{Z}/n\mathbb{Z}) = \begin{cases} \mathbb{Z}/n\mathbb{Z} & \text{if } i = 0 \\ 0 & \text{if } i > 0 \end{cases}$$

We have a LES for computing cohomology of $\mathbb{G}_m$

$$\mathrm{H}^i(\mathbb{G}_m, \mu_n) \to \mathrm{H}^i(\mathbb{G}_m, \mathbb{G}_m) \to \mathrm{H}^i(\mathbb{G}_m, \mathbb{G}_m) \to \mathrm{H}^{i+1}(\mathbb{G}_m, \mu_n)$$

The beginning part looks like

$$0 \to \mu_n(k) \to k[t, t^{-1}]^\times \xrightarrow{[n]} k[t, t^{-1}]^\times \to \mathrm{H}^1(\mathbb{G}_m, \mu_n) \to \mathrm{Pic}(\mathbb{G}_m) \xrightarrow{[n]} \mathrm{Pic}(\mathbb{G}_m).$$

Note that $\mathrm{Pic}(\mathbb{G}_m) = 0$ and that $k[t, t^{-1}]^\times = k^\times \times t^{\mathbb{Z}}$. Hence get short exact sequence

Remember: If $Z \subset X$ is an irreducible subset in codimension 1, you get an exact sequence $\mathbb{Z} \to \mathrm{Pic}(X) \to \mathrm{Pic}(U) \to 0$ (assuming X 'nice'. See Hartshorne section 2.6)

$$0 \to \mathbb{Z}/n\mathbb{Z} \to \mathrm{H}^1(\mathbb{G}_m, \mu_n) \to 0.$$

Thus, $\mathrm{H}^1(\mathbb{G}_m, \mu_n) = \mathbb{Z}/n\mathbb{Z}$ so $\mathrm{H}^1(\mathbb{G}_m, \mathbb{Z}/n\mathbb{Z}) = \mathbb{Z}/n\mathbb{Z}(-1)$. Combining this with previous calculation (for $\mathbb{A}^1$) reproves the result. $\triangle$

We'll sketch the proof of the theorem next time, and use it to compute the cohomology of projective space.

16 Lecture 16

Last time we talked about purity, the Gysin sequence, and cohomology with supports.

16.1 Picking up where we left off

Assumption. Always assume $\mathscr{F}$ **order prime to** $\operatorname{char} k$, meaning its stalks have orders prime to $\operatorname{char} k$.

We were trying to understand the following theorem.

Theorem 16.1. *Let $Z \hookrightarrow X$ be a closed immersion of smooth k-schemes, and let $c = \operatorname{codim}_X Z$. Then for $\mathscr{F}$ lcc (i.e. locally constant constructible, i.e. locally constant w/ finite stalks),*

$$\mathrm{H}^{r-2c}(Z, \mathscr{F}(-c)) \xrightarrow{\sim} \mathrm{H}^r_Z(X, \mathscr{F})$$

for all $r \geq 0$.

Think of this as a computation of cohomology with supports. Recall we worked out an example of this last time.

[35] Have Kummer sequence $\mathrm{H}^i(\mathbb{A}^1, \mu_n) \to \mathrm{H}^i(\mathbb{A}^1, \mathbb{G}_m) \to \mathrm{H}^i(\mathbb{A}^1, \mathbb{G}_m) \to \mathrm{H}^{i+1}(\mathbb{A}^1, \mathbb{G}_m)$. Use $\mathrm{Pic}(\mathbb{A}^1) = 0$

Corollary 16.2 (**Gysin sequence**)**.** *Suppose we have X, Z as above, and let $U = X \setminus Z$ be the (open) complement of Z. Then, for $0 \le r < 2c-1$, the restriction map*

$$\mathrm{H}^r(X_{\acute{e}t}, \mathscr{F}) \to \mathrm{H}^r(U_{\acute{e}t}, \mathscr{F})$$

is an isomorphism, where $\mathscr{F}$ is any lcc sheaf. For big r, there is a long exact sequence

$$0 \longrightarrow \mathrm{H}^{2c-1}(X, \mathscr{F}) \longrightarrow \mathrm{H}^{2c-1}(U, \mathscr{F}|_U) \longrightarrow \mathrm{H}^0(Z, \mathscr{F}(-c)) \longrightarrow \mathrm{H}^{2c}(X, \mathscr{F}) \longrightarrow \mathrm{H}^{2c}(U, \mathscr{F}) \longrightarrow \mathrm{H}^1(Z, \mathscr{F}(-c)) \longrightarrow \cdots$$

Remark 16.3. Theorem 16.1 is referred to as "**purity**" since it's saying[36] that cohomology doesn't change if you remove something in high codimension; "it's supported *purely* in low dimension" ◦

Remark 16.4 (Topological situation)*.* What are these maps $\mathrm{H}^{2c-1+i}(U, \mathscr{F}|_U) \to \mathrm{H}^i(Z, \mathscr{F}(-c))$? Let $\widetilde{Z}$ be a deleted neighborhood of Z (e.g. take a small ε-ball around it and then delete Z), so $\pi : \widetilde{Z} \to Z$ is homotopic to a sphere bundle over Z. The Leray spectral sequence[37] only has two nonzero rows, so gives way to a long exact sequence

$$\cdots \to \mathrm{H}^{2c-1+i}(Z, \mathscr{F}) \to \mathrm{H}^{2c-1+i}(\widetilde{Z}, \mathscr{F}) \to \mathrm{H}^i(Z, \mathscr{F}) \to \mathrm{H}^{2c+i}(Z, \mathscr{F}) \to \cdots$$

(Thom-Gysin exact sequence). ◦

Let's prove the corollary now.

Proof of Corollary 16.2, assuming Theorem 16.1. In the LES of cohomology with supports, replace $\mathrm{H}^r_Z(X, \mathscr{F})$ with $\mathrm{H}^{r-2c}(Z, \mathscr{F}(-C))$ (note this vanishes if $r < 2c$). ■

Example (Cohomology of projective space)**.** Fix a field $k = \overline{k}$. Assume $\operatorname{char} k \nmid n$ (assume this throughout the lecture). Recall

$$\mathrm{H}^i(\mathbb{A}^1, \mu_n) = \begin{cases} \mu_n & \text{if } i = 0 \\ 0 & \text{if } i > 0 \end{cases}$$

Fact (By Kunneth)**.** We'll see the Kunneth theorem later. It implies that

$$\mathrm{H}^i(\mathbb{A}^n, \mu_n) = \begin{cases} \mu_n & \text{if } i = 0 \\ 0 & \text{if } i > 0 \end{cases}$$

To compute the cohomology of projective space, we use the Gysin sequence for $\mathbb{P}^{n-1} \hookrightarrow \mathbb{P}^n$ (codim $c = 1$ here). This gives $\mathrm{H}^r(\mathbb{P}^n, \mathbb{Z}/n\mathbb{Z}) = \mathrm{H}^r(\mathbb{A}^n, \mathbb{Z}/n\mathbb{Z})$ for $0 \le r < 1$ (which is super useful), and also a long exact sequence (note $\mu_n(-1) = \underline{\mathbb{Z}/n\mathbb{Z}}$)

$$0 \to \mathrm{H}^1(\mathbb{P}^n, \mu_n) \to \mathrm{H}^1(\mathbb{A}^n, \mu_n) \to \mathrm{H}^0(\mathbb{P}^{n-1}, \mathbb{Z}/n\mathbb{Z}) \to \mathrm{H}^2(\mathbb{P}^n, \mu_n) \to \mathrm{H}^2(\mathbb{A}^n, \mu_n) \to \mathrm{H}^1(\mathbb{P}^{n-1}, \mathbb{Z}/n\mathbb{Z}) \to \cdots$$

We see immediately that $\mathrm{H}^1(\mathbb{P}^n, \mu_n) = 0$ and

$$\mathrm{H}^i(\mathbb{P}^n, \mu_n) \simeq \mathrm{H}^{i-2}(\mathbb{P}^{n-1}, \mathbb{Z}/n\mathbb{Z}) \text{ for } i \ge 2.$$

[36] First part of the Gysin seq corollary

[37] maybe call it the Serre spectral sequence in the topological setting

By induction, we see that

$$\mathrm{H}^r(\mathbb{P}^n, \mathbb{Z}/n\mathbb{Z}) = \begin{cases} (\mathbb{Z}/n\mathbb{Z})\left(-\frac{r}{2}\right) & \text{if } r \text{ even and } r \leq 2n \\ 0 & \text{otherwise.} \end{cases}$$

△

Now let's sketch the proof of purity.

Proof Sketch of Theorem 16.1. We have $j : U \overset{\text{open}}{\hookrightarrow} X \overset{\text{cl}}{\hookleftarrow} Z : \iota$ where $Z = X \setminus U$.

(Step 1: Reduce to a local statement) We first define **upper shriek**

Special case of upper shriek in Verdier duality

$$\iota^! \mathscr{F} = \iota^* \ker(\mathscr{F} \to j_* j^* \mathscr{F}) \in \mathrm{Sh}(Z_{\text{ét}})$$

"sections of $\mathscr{F}$ supported on Z."

Proposition 16.5. *ι_* is left adjoint to $\iota^!$.*

Note ι_* is exact, so believable it has a right adjoint (in addition to its let adjoint ι^*)

Proof of this proposition is left as an (easy?) exercise.

Corollary 16.6. *$\iota^!$ is left exact, and preserves injectives.*

Remember: If you have an exact left adjoint, then you preserve injectives

Claim 16.7 (**Local version of purity**)**.** *Say $Z, X, \mathscr{F}$ as in the theorem. Then,*

$$R^{2c}\iota^! \mathscr{F} = \iota^* \mathscr{F}(-c)$$

and

$$R^r \iota^! \mathscr{F} = 0 \text{ for } r \neq 2c$$

($r = 0$ case clear since $\mathscr{F}$ locally constant constructible).

Let's show that this claim implies the theorem.

- $\Gamma(Z, \iota^! \mathscr{F}) = \Gamma_Z(X, \mathscr{F})$

  This follows from expanding definitions.
- The Grothendieck spectral sequence then gives

$$\mathrm{H}^r(Z, R^s \iota^! \mathscr{F}) \implies \mathrm{H}^{r+s}_Z(X, \mathscr{F}),$$

  i.e. $R\Gamma \circ R\iota^! = R\Gamma_Z$ (uses $\iota^!$ preserves injectives). This spectral sequence has only one column, by the claim, so it degenerates immediately and you get

$$\mathrm{H}^r(Z, \iota^* \mathscr{F}(-c)) = \mathrm{H}^{r+2c}_Z(X, \mathscr{F}).$$

(Step 2: Prove the local statement) It's a local claim. One can reduce to the case $(\mathbb{A}^m, \mathbb{A}^{m-c})$ using the structure theorem for smooth morphisms. Then, do induction on m, c. The base case is $m = 1, c = 1$ which is the example we did at the end of last class. ■

In the end, this pair looks kinda like a sphere bundle

16.2 Comparison Theorems (e.g. Artin comparison) + Elementary fibrations

Definition 16.8. An **elementary fibration** is a diagram

$$\begin{tikzcd} U \arrow[r, hook, "j"] \arrow[dr, "f"'] & Y \arrow[d, "h"] & Z \arrow[l, hook', "i"'] \arrow[dl, "g"] \\ & S & \end{tikzcd}$$

such that

(1) j is a Zariski-open immersion s.t. $j(U)$ is **fiberwise dense** in Y, i.e. for each $s \in S$, $U_s \subset Y_s$ is dense. $Z = Y \setminus U$ is the complement of U with $i : Z \hookrightarrow Y$ a closed embedding

(2) h is smooth and projective with geometrically irreducible fibers, and it has relative dimension 1

(3) g is finite étale

$\diamond$

Usually U is the object of interest. We write it as an open inside a smooth, proper curve bundle over S s.t. the complement of U is finite étale. The key thing is the following.

Slogan. Topology of the fibers of f are constant.

This is meaningless in general, but over $\mathbb{C}$ it is at least saying the fibers are all Riemann surfaces of the same genus with the same number of points removed.

Proposition 16.9 (Artin)**.** *Suppose X is smooth over $k = \overline{k}$. For each scheme-theoretic $x \in X$, there exists a Zariski open $U \ni x$ s.t. U fits into an elementary fibration.*

The proof of geometrically pretty involved. The idea is to puck a U, embed in $\mathbb{P}^n$, and then project down (away from a point or linear subspace?) until the base has dimension $\dim U - 1$. They will be some bad loci which you need to delete.

Theorem 16.10. *Suppose X is a variety over $\mathbb{C}$, and that $\mathscr{F}$ is a constructible, abelian sheaf on $X_{\acute{e}t}$. There are three sites coming into play*

$$\begin{tikzcd} & X^{an\text{-}\acute{e}t} \arrow[dl, "\pi"'] \arrow[dr, "an"] & \\ X^{\mathrm{an}} & & X_{\acute{e}t} \end{tikzcd}$$

(1) π induces an isomorphism on the cohomology of all abelian sheaves (in fact, it induces an equivalence of categories)

$$\pi^* : \mathrm{Sh}(X^{\mathrm{an}}) \xrightarrow{\sim} \mathrm{Sh}(X^{an\text{-}\acute{e}t})$$

(2) $an^* : \mathrm{H}^i(X_{\acute{e}t}, \mathscr{F}) \xrightarrow{\sim} \mathrm{H}^i(X^{an\text{-}\acute{e}t}, an^*\mathscr{F})$.

Recall 16.11 (Maybe this is a defn, not a recall?)**.** X^{an} is the site associated to the Euclidean topology on the X^{an}. i.e. its objects are open sets, the morphisms are inclusions, and covers are open covers.

$X^{\textbf{an-ét}}$ is the category of complex-analytic spaces mapping to X^{an} via local analytic isomorphisms, and covers are covers. $\odot$

Corollary 16.12. *For $\mathscr{F}$ as in the theorem, there is a canonical isomorphism*

$$\mathrm{H}^i(X_{ét}, \mathscr{F}) \xrightarrow{\sim} \mathrm{H}^i(X^{\mathrm{an}}, \mathscr{F}^{\mathrm{an}}).$$

Above, $\mathscr{F}^{\mathrm{an}}$ is $\pi_* an^* \mathscr{F}$.

We will prove this for X smooth and $\mathscr{F}$ lcc via the theory of elementary fibrations.

Exercise. Prove **(1)** of the theorem.

We'll do **(2)** next time.

17 Lecture 17

Today we'll talk about comparison theorems and then hopefully introduce the fundamental group.

17.1 Comparison Theorems, continued

Last time we stated the following theorem.

Theorem 17.1. *Let X be a variety over $\mathbb{C}$. Then, there is a zig-zag of sites*

$$\begin{tikzcd} & X^{an\text{-}ét} \arrow[dl, "\pi"'] \arrow[dr, "an"] & \\ X^{\mathrm{an}} & & X_{ét} \end{tikzcd}$$

such that

(1) *π^* induces an equivalence of categories*

$$\mathrm{Sh}(X^{\mathrm{an}}) \to \mathrm{Sh}(X^{an\text{-}ét})$$

(2) *$an^* : \mathrm{H}^i(X_{ét}, \mathscr{F}) \xrightarrow{\sim} \mathrm{H}^i(X^{an\text{-}ét}, an^*\mathscr{F})$ is an iso when $\mathscr{F}$ constructible.*

Corollary 17.2. *If $\mathscr{F}$ is a constructible abelian sheaf on $X_{ét}$, then there is a canonical iso*

$$\mathrm{H}^i(X_{ét}, \mathscr{F}) \xrightarrow{\sim} \mathrm{H}^i(X^{\mathrm{an}}, \mathscr{F}^{\mathrm{an}}) \text{ where } \mathscr{F}^{\mathrm{an}} = \pi_* an^* \mathscr{F}.$$

We left **(1)** as an exercise last time. Here's how you do it

*Proof of **(1)**.* Given $\mathscr{F} \in \mathrm{Sh}(X^{\text{an-ét}})$, we want $\mathscr{F}^{\mathrm{an}} \in \mathrm{Sh}(X^{\mathrm{an}})$ an isomorphism $\pi^*\mathscr{F}^{\mathrm{an}} \xrightarrow{\sim} \mathscr{F}$. In the other direction, given $\mathscr{F} \in \mathrm{Sh}(X^{\mathrm{an}})$, we want an iso $(\pi^*\mathscr{F})^{\mathrm{an}} \xrightarrow{\sim} \mathscr{G}$.

We can simply take $\mathscr{F}^{\mathrm{an}} := \pi_*\mathscr{F}$. The two desired isos now are the (co)unit of the adjunction between π_*, π^*. These are isos essentially because a local isomorphism can be refined to an honest open cover. ∎

Now let's prove two (when X smooth and $\mathscr{F}$ locally constant).

Claim 17.3. *It is enough to show*

(1) *$\mathscr{F} \to an_* an^* \mathscr{F}$ is an iso*

(2) *$R^i an_* an^* \mathscr{F} = 0$ for $i > 0$*

Proof. You the leray spectral sequence (for the morphism $\text{an} : X^{\text{an-ét}} \to X_{\text{ét}}$)

$$\mathrm{H}^i(X_{\text{ét}}, R^j an_* an^* \mathscr{F}) \implies \mathrm{H}^{i+j}(X^{\text{an-ét}}, an^* \mathscr{F}).$$

The claim implies that the E_2-page only has one column with values $\mathrm{H}^i(X_{\text{ét}}, \mathscr{F})$, so **(2)** of the theorem follows. ■

Let's prove **(1)** from the claim.

*Proof of **(1)**.* We want to show $\mathscr{F} \to \text{an}_* \text{an}^* \mathscr{F}$ is an iso. We will do this in the case that $\mathscr{F}$ is locally constant. This is a local statement, so we may as well assume $\mathscr{F}$ is constant, so $\mathscr{F} = \underline{\Lambda}$ for some Λ. In this case, we need to show that

$$\Gamma(U, \underline{\Lambda}) \to \Gamma(U^{\text{an}}, an^* \underline{\Lambda}) \xrightarrow{\sim} \Gamma(U^{\text{an}}, \underline{\Lambda})$$

is an isomorphism. Thus, it's enough to show that

$$\pi_0(U^{\text{an}}) \to \pi_0(U)$$

is a bijection. Recall we're assuming X (hence U) is smooth.

- We can pass to a connected component to assume that U is connected.
- Further, we may assume U fits into an elementary fibration.

$$\begin{tikzcd} U \arrow[r, hook, "j"] \arrow[dr, "f"'] & Y \arrow[d, "h"] & Z \arrow[l, hook', "i"'] \arrow[dl, "g"] \\ & S & \end{tikzcd}$$

  This is because U has a smaller open fitting into such a thing by Proposition 16.9 and since U is smooth, it is connected $\iff$ a dense open is connected.

- Suppose U^{an} (hence Y^{an}) is not connected, so write $Y = Y_1 \cup Y_2$. Note these Y_i's are unions of fibers of h (to get this, you have to prove the theorem for curves directly[38]). Hence, $h(Y_1), h(Y_2)$ are (unions of) connected components of S.[39] Now we're done by induction on dimension. ■

Now let's prove the second statement: $R^i an_* an^* \mathscr{F} = 0$ for $i > 0$

*Proof of **(2)**.* Again, we only handle to locally constant case. This is local, so we may assume $\mathscr{F}$ is constant. We may also assume that U fits into an elementary fibration. We want $R^i an_* \underline{\Lambda} = 0$ for $i > 0$ where $an : U^{\text{an-ét}} \to U_{\text{ét}}$ for U in an elementary fibration. Recall these derived pushforwards are the sheaficiation of the presheaves of "cohomology of the pullbacks." We claim it suffices to prove the following lemma

[38]The analytification of curves are Riemann surfaces, so this is easy

[39]h is smooth so open. It is proper so closed. Hence is sends connected components to connected components

Lemma 17.4. *Suppose U is connected and smooth over $\mathbb{C}$, and $\mathscr{F}$ is lcc on $U^{an\text{-}ét}$ (i.e. locally constant with finite fibers) and $r > 0$. Then, for any $s \in \mathrm{H}^r(U^{an\text{-}ét}, \mathscr{F})$, there exists an étale cover $\{U_i \to U\}$ s.t. $s|_{U_i^{\mathrm{an}}} = 0$.*

That is, we can kill analytic étale cohomology classes with honest étale covers.

Corollary 17.5. *The stalks $(R^r an_* \mathscr{F})_{\overline{x}} = 0$, so $R^r an_* \mathscr{F} = 0$.*

Thus, we finish by proving this lemma. WLOG U sits in an elementary fibration

$$\begin{tikzcd} U \arrow[r, hook, "j"] \arrow[dr, "f"'] & Y \arrow[d, "h"] & Z \arrow[l, hook', "i"'] \arrow[dl, "g"] \\ & S & \end{tikzcd}$$

We have $s \in \mathrm{H}^r(U^{\text{an-ét}}, \mathscr{F})$ which we compute via leray

$$\mathrm{H}^i(S, R^j f_* \mathscr{F}) \implies \mathrm{H}^r(U^{\text{an-ét}}, \mathscr{F}).$$

Note $R^j \mathscr{F}_*$ is the sheafification of $\left(V \mapsto \mathrm{H}^j(f^{-1}(V), \mathscr{F})\right)$. By induction (we haven't done base case yet), we can kill the contributions coming from $R^j f_* \mathscr{F}$ for $j > 0$. We're left with contributions from (assume $j = 0$ now) $\mathrm{H}^i(S, f_* \mathscr{F})$. Since we're in an elementary fibration, $f_* \mathscr{F}$ is again lcc. Since S has lower dimension than U, we're done by induction hypothesis.

This is a topological computation since things here computed in the analytic site (recall $\mathscr{F}$ lcc on $U^{\text{an-ét}}$ here)

This just leaves the base case, where $\dim U = 1$. We have $\mathscr{F}$ an lcc sheaf on the analytic-étale site on a Riemann surface. We want to kill $s \in \mathrm{H}^i(U^{\text{an-ét}}, \mathscr{F})$ for $i = 1, 2$ (since we're on a Riemann surface) by passing to algebraic étale covers of U.

- Kill $i = 2$. Pass to any affine cover (e.g. delete a finite set of points).
- Kill $i = 1$. We have $s \in \mathrm{H}^1(U^{\text{an-ét}}, \underline{\Lambda})$, and we want $\{U_i \to U\}$ an étale cover s.t. $s|_{U_i^{\text{an-ét}}} = 0$. Note that s corresponds to some Λ-torsor over $U^{\text{an-ét}}$, i.e. a covering space of U^{an} with Galois group Λ.

  Claim 17.6 (Riemann existence theorem). *There is an equivalence of categories*

  $$U_{f\text{-}ét} \xrightarrow{\sim} U^{f\text{-}an\text{-}ét}$$

  between finite, étale covers of U and finite, analytic covers of U^{an} (i.e. finite covering spaces).

  In the present case, $s \in \mathrm{H}^1(U^{\text{an-ét}}, \underline{\Lambda})$ corresponds to some $U_s \to U^{\text{an}}$ which is a finite local analytic isomorphism. Riemann existence then implies that $(U_s \to U) = (V \to U)^{\text{an}}$ for some algebraic $V \to U$. Then, $s|_V = 0$ because torsor kill their corresponding cohomology classes (recall remark 9.6).

■

This concludes the proof of Theorem 17.1, at least when X smooth and $\mathscr{F}$ lcc. This let's now compute many examples of étale cohomology.

Example. Say X is a K3 surface over $\mathbb{C}$. Then,

$$\mathrm{H}^i(X_{\text{ét}}, \mathbb{Z}/n\mathbb{Z}) = \begin{cases} \mathbb{Z}/n\mathbb{Z} & \text{if } i = 0, 4 \\ 0 & \text{if } i \notin \{0, 2, 4\} \\ (\mathbb{Z}/n\mathbb{Z})^{22} & \text{if } i = 2 \end{cases}$$

Why? Because we're over $\mathbb{C}$ so it must be the same as the singular cohomology of an analytic K3. △

Non-example. For $X = \mathbb{G}_{m,\mathbb{C}}$ one has

$$\mathrm{H}^1(X_{\text{ét}}, \underline{\mathbb{Z}}) = 0 \;\text{ but }\; \mathrm{H}^1(X^{\text{an}}, \mathbb{Z}) = \mathbb{Z}.$$

(note that $\underline{\mathbb{Z}}$ is *not* constructible, e.g. since its fibers are not finite). Objects here corresponds to $\mathbb{Z}$-torsors over $\mathbb{G}_m$. There is one analytically, given by the exponential map $\exp : \mathbb{C} \to \mathbb{C}^\times$, but this is not an algebraic morphism (and neither are any of its powers). ▽

17.2 Étale fundamental groups

Definition 17.7. Let X be a locally noetherian scheme. Consider the **finite étale site** $X_{\text{f.ét}}$ whose objects are finite, étale morphisms $Y \to X$; whose morphisms are morphisms over X; and whose covers are topological covers. ⋄

Remark 17.8. If X is connected, all morphisms are covers. ∘

Definition 17.9. Let $\overline{x}$ be a geometric point of X. Then there is a **fiber functor**

$$\begin{array}{rccc} F_{\overline{x}} : & X_{\text{f.ét}} & \longrightarrow & \text{FinSet} \\ & Y/X & \longmapsto & Y_{\overline{x}}. \end{array}$$

The **étale fundamental group** is the group

$$\pi_1^{\text{ét}}(X, \overline{x}) := \mathrm{Aut}(F_{\overline{x}})$$

with topology the coarsest (i.e. fewest open) one such that

$$\pi_1^{\text{ét}}(X, \overline{x}) \to \mathrm{Aut}(F_{\overline{x}}(Y))$$

is continuous for all Y ($\mathrm{Aut}(F_{\overline{x}}(Y))$ is discrete). ⋄

Example. Say $X = \operatorname{Spec} k$. Take $\overline{x} : \operatorname{Spec} \overline{k} \to \operatorname{Spec} k$. Note that

$$X_{\text{f.ét}} = (\text{finite étale } k\text{-algebras})^{\text{op}}.$$

In this case,

$$\pi_1^{\text{ét}}(\operatorname{Spec} k, \overline{x}) = \mathrm{Gal}(k^s/k)$$

(exercise). △

"I was gonna give you some nice examples first, but let me give you some mean examples."

Example. Take $\mathbb{A}^1_k$ where $\operatorname{char} k = p > 0$. Recall (from end of Lecture 8)

$$\mathrm{H}^1(\mathbb{A}^1_{k,\text{ét}}, \mathbb{F}_p) = \operatorname{coker}\left(k[t] \xrightarrow{t \mapsto t^p - t} k[t]\right)$$

which is huge. This is saying that $\mathbb{A}^1_k$ has many $\mathbb{F}_p$-covers. Hence, one should expect $\pi_1^{\text{ét}}(\mathbb{A}^1_k)$ to be really big. In fact,

$$\mathrm{H}^1(\mathbb{A}^1_{k,\text{ét}}, \mathbb{F}_p) = \operatorname{Hom}(\pi_1^{\text{ét}}(\mathbb{A}^1_k, \overline{x})^{\text{ab}}, \mathbb{F}_p)$$

from which we see $\pi_1^{\text{ét}}(\mathbb{A}^1_k)$ is not topologically finitely generated. △

Example. Say E is an elliptic curve over $k = \overline{k}$ with $\operatorname{char} k = p > 0$. Then, we'll see,

$$\pi_1^{\text{ét}}(E) = \varprojlim_n E[n] = \begin{cases} \mathbb{Z}_p \times \prod_{\ell \neq p} \mathbb{Z}_\ell^2 & \text{if } E \text{ ordinary} \\ \prod_{\ell \neq p} \mathbb{Z}_\ell^2 & \text{if } E \text{ supersingular.} \end{cases}$$

Note this is (probably) not the profinite completion of a discrete group. △

Example. Say X is normal $/\mathbb{C}$ and connected. Then,

$$\pi_1^{\text{ét}}(X, \overline{x}) \xrightarrow{\sim} \pi_1(X^{\text{an}}, x)^\wedge$$

where $^\wedge$ is profinite completion. △

18 Lecture 18

Last time we introduced the étale fundamental group.

Let X be a normal variety over a field k.

Recall 18.1. $\mathbf{F\acute{E}t}(X)$ is the category whose objects are finite étale morphisms $Y \to X$ and whose morphisms are X-morphisms. ⊙

Given a geometric point $\overline{x} \to X$, we defined the functor

$$\begin{array}{rccc} F: & \mathrm{F\acute{E}t}(X) & \longrightarrow & \mathrm{Set} \\ & Y/X & \longmapsto & Y_{\overline{x}} \end{array}$$

Recall 18.2. The *étale fundamental group* of X based at $\overline{x}$ is

$$\pi_1^{\text{ét}}(X, \overline{x}) = \operatorname{Aut}(F_{\overline{x}}).$$

⊙

Example. When $X = \operatorname{Spec} k$, this is the absolute Galois group and the choice of basepoint is the same as a choice of algebraic closure. △

18.1 Fundamental Group

Theorem 18.3 (SGA1)**.** *FÉt* $\xrightarrow{F}$ *finite continuous* $\pi^{\acute{e}t}$*-sets is an equivalence of categories.*

Remark 18.4. In present context (normal variety over a field) this is mostly a restatement of Galois theory of the fraction field. ○

Corollary 18.5. *For* $\mathbb{A}^1_k$*, where* $\operatorname{char} k > 0$*,* $\pi_1^{\acute{e}t}(\mathbb{A}^1_k, \overline{x})$ *is not topologically finitely generated.*

Proof. $\mathrm{H}^1(\mathbb{A}^1_k, \mathbb{F}_p)$ is not finitely generated. We claim that

$$\mathrm{Hom}_{cts}(\pi^{\text{ét}}, \mathbb{F}_p) \xrightarrow{\sim} \mathrm{H}^1(\mathbb{A}^1_k, \mathbb{F}_p) = \{\mathbb{F}_p\text{-torsors}\}\,.$$

Note $\mathbb{F}_p \curvearrowright \mathbb{F}_p$ by addition, so we get a map

$$\mathrm{Hom}_{cts}(\pi_1^{\text{ét}}(\mathbb{A}^1_k), \mathbb{F}_p) \to \left\{ \begin{matrix} \text{fin. cts } \pi_1^{\text{ét}}\text{-sets s.t. the action} \\ \text{factors though a map } \pi_1^{\text{ét}}(\mathbb{A}^1_k) \to \mathbb{F}_p \end{matrix} \right\}$$

(an element of the LHS gives an action $\pi_1^{\text{ét}}(\mathbb{A}^1_k) \curvearrowright \mathbb{F}_p$). By theorem from SGA1, these are the same as $\mathbb{F}_p$-torsors. ■

Corollary 18.6. *For any* $\overline{x}_1, \overline{x}_2$ *geometric points of* X*,*

$$\pi_1^{\acute{e}t}(X, \overline{x}_1) \simeq \pi_1^{\acute{e}t}(X, \overline{x}_2)$$

(this uses assumption that X *is a variety, at least that it's connected).*

Proof. The first step is that we have an equiv of categories $\pi_1^{\text{ét}}(X, \overline{x}_1)$-sets $\xleftrightarrow{\sim}$ $\pi_1^{\text{ét}}(X, \overline{x}_2)$ (since both equiv to $\mathrm{FÉt}(X)$). The second step – a nontrivial exercise – is that the category determines the abstract group. ■

In fact, the iso above is well-defined up to inner conjugacy. To choose an iso, choose a sequence of specializations and generalizations

$$\overline{x}_1 \leftsquigarrow \overline{y}_1 \rightsquigarrow \overline{y}_2 \leftsquigarrow \cdots \rightsquigarrow \overline{x}_2.$$

Notation 18.7. $x \rightsquigarrow y$ means that y is a **specialization** of x, i.e. y is in the closure of x.

Claim 18.8. *If* $\overline{x}$ *specializes to* $\overline{y}$*, then there is a natural transformation* $\eta : F_{\overline{x}} \to F_{\overline{y}}$

Proof. We let $\eta(Y/X) : Y_{\overline{x}} \to Y_{\overline{y}}$ be the map $\eta(Y/X)(z) = \overline{z} \cap Y_{\overline{y}}$, where $\overline{z}$ is the closure of z. ■

Theorem 18.9 (**Comparison Theorem for Fundamental Groups**)**.** *Let* X *be a normal variety over* $\mathbb{C}$*, and choose some* $\overline{x} \in X(\mathbb{C})$*. Then, there is a map*

$$\pi_1^{\acute{e}t}(X, \overline{x}) \leftarrow \pi_1(X^{\text{an}}, \overline{x}^{\text{an}})$$

which induces an isomorphism after taking profinite completions.

Proof. Recall $\pi_1^{\text{ét}}(X,\overline{x}) := \mathrm{Aut}(F_{\overline{x}})$. Covering space theory tells us that

$$\pi_1(X^{\mathrm{an}}, \overline{x}^{\mathrm{an}}) = \mathrm{Aut}(F_{\overline{x}}^{\mathrm{an}} : \mathrm{Cov}(X) \to \mathrm{Set}).$$

Hence, it is enough to show that we have a commutative diagram

$$\begin{tikzcd}
\text{FÉt} \arrow[rr, "\mathrm{an}"] \arrow[dr, "F_{\overline{x}}"'] & & \mathrm{Cov}(X) \arrow[dl, "F_{\overline{x}}^{\mathrm{an}}"] \\
& \mathrm{Set} &
\end{tikzcd}$$

where an induces an equivalence $\text{FÉt} \xrightarrow{\sim} \mathrm{FinCov}(X)$. This equivalence is Riemann existence. ■

Corollary 18.10. *If X is a smooth proper curve of genus g over $\mathbb{C}$, then*

$$\pi_1^{\acute{e}t}(X) = \left\langle a_1, b_1, \dots, a_g, b_g \,\middle|\, \prod [a_i, b_i] = 1 \right\rangle^{\wedge}$$

Apparently, this is the only known proof of this (i.e. no known algebraic proof).

What if we're not over $\mathbb{C}$? There's a sequence of maps $X_{\overline{k}} \to X \to \operatorname{Spec} k$. This induces

Theorem 18.11. *The sequence*

$$1 \longrightarrow \pi_1^{\acute{e}t}(X_{\overline{k}}, \overline{x}) \longrightarrow \pi_1^{\acute{e}t}(X, \overline{x}) \longrightarrow \mathrm{Gal}(\overline{k}/k) \longrightarrow 1$$

is short exact.

Remark 18.12. Surjectivity follows from geometric connectedness of X (running assumption that X is a variety). Giving the whole proof of exactness is non-trivial. ○

18.2 Specialization maps

Assumption. Assume X is proper and flat over a complete dvr R w/ geometrically connected fibers.

Notation 18.13. Let $K = \mathrm{Frac}(R)$ and $k = R/\mathfrak{m}$.

Theorem 18.14. *Given $\overline{x} \to X_k$ on the special fiber, the natural map*

$$\pi_1^{\acute{e}t}(X_k, \overline{x}) \to \pi_1^{\acute{e}t}(X, \overline{x})$$

is an isomorphism of topological groups.

"Can anyone tell me why this is true? This is a very non-trivial theorem."

Proof. Need to show that the category of finite étale covers are the same, i.e. that

$$\text{FÉt}(X) \xrightarrow{\mathrm{res}} \text{FÉt}(X_k)$$

is an equivalence of categories.

Let's prove essential surjectivity. Given $Y \to X_k$ finite étale, we want to construct $\mathfrak{Y} \to \widehat{X}$, the formal scheme obtained by completing X at the maximal ideal of the base $\mathfrak{m}$.

We need a digression on deformation theory. We need to lift Y over X mod $\mathfrak{m}^n$ where $\mathfrak{m}$ is our maximal ideal. We'll do this one n at a time. The first step looks like

$$\begin{tikzcd} Y \arrow[r, dashed] \arrow[d, "\text{ét}"] & ? \arrow[d, dashed] \\ X_k \arrow[r, hook] & X \times R/\mathfrak{m}^2 \end{tikzcd}$$

We want to understand existence and uniqueness properties of ?. In general, we want to look at

$$\begin{tikzcd} Y_n \arrow[r, dashed] \arrow[d, "\text{ét}"] & ? \arrow[d, dashed] \\ X \otimes R/\mathfrak{m}^n \arrow[r, hook] & X \times R/\mathfrak{m}^{n+1} \end{tikzcd}$$

Let $\mathscr{I}$ be the ideal defining the embedding $X \otimes R/\mathfrak{m}^n \hookrightarrow X \otimes R/\mathfrak{m}^{n+1}$. The general principal in deformation theory is that the existence of such a ? (an object making the diagram Cartesian and the right arrow flat) is controlled by an obstruction class, usually living in some H^2, and the set of such objects form a torsor for some H^1.

Question: Is this the right embedding?

In the present case, one obtains a construction class in $obs \in \mathrm{Ext}^2_{X_k}(\Omega^1_{Y/X_k}, \mathscr{I})$. If $obs = 0$, then there exists Y_{n+1} flat over $X \otimes R/\mathfrak{m}^{n+1}$ making the diagram Cartesian. The set of such Y_{n+1}, up to iso, is a torsor for $\mathrm{Ext}^1_{X_k}(\Omega^1_{Y/X_k}, \mathscr{I})$. For us, $Y \to X_k$ is étale, so $\Omega^1_{Y/X_k} = 0$ so both of these groups vanish. Hence, there's a unique Y_{n+1}.

Exercise (conormal exact sequence). $Y_{n+1}/X \otimes R/\mathfrak{m}^{n+1}$ is étale.

This gives us a unique $\mathfrak{Y} \to \widehat{X}$ lifting Y. Now we want some $\mathcal{Y} \to X$. Since X is proper, this is immediate from formal GAGA.

The same sort of argument gives full faithfulness (exercise). ■

Where does this deformation theory stuff come from? One uses Čech cohomology. First show that locally on X, you can find the necessary lifts. Smooth maps locally look like affine space and you can lift affine space. Now you need to try to glue these lifts together. Pick some local isomorphisms between them; these typically won't satisfy the cocycle condition, and their failure to do so will be an element of H^2, and really a Čech cocycle representing an element of Ext^2. Then you need to figure out how many ways there are to glue.

Corollary 18.15. *Given X as above (e.g. proper and flat over a dvr) as well as $\overline{\eta} \to X_K$ a geometric over the generic fiber specializing to some $\overline{\xi} \to X_k$ in the special fiber, we get a specialization map*

$$\mathrm{sp} : \pi_1^{\acute{e}t}(X_K, \overline{\eta}) \to \pi_1^{\acute{e}t}(X_k, \overline{\xi})$$

(not an iso in general).

Proof. $(X_K, \overline{\eta}) \to (X, \overline{\eta})$ so we get

$$\pi_1^{\text{ét}}(X_K, \overline{\eta}) \to \pi_1^{\text{ét}}(X, \overline{\eta}) \xrightarrow{\sim} \pi_1^{\text{ét}}(X, \overline{\xi}) \xleftarrow{\sim} \pi_1^{\text{ét}}(X_k, \overline{\xi}),$$

so our map is this composition. ■

Theorem 18.16. *If X is normal, then* sp *is surjective.*

Proof. Given $Y \to X$ finite étale with Y connected, then Y_K is also connected (use Zariski-Zagata purity; Y is the normalization of X in the function field of Y). ■

Corollary 18.17. *X is normal, flat and proper over R. Say we have η, ξ as above. Then,*

$$\pi_1^{ét}(X_{\overline{K}}, \overline{\eta}) \to \pi_1^{ét}(X_{\overline{k}}, \overline{\xi})$$

is also surjective.

Theorem 18.18. *If X is a variety over k alg. closed of char. 0, and L/k is an extension of algebraically closed fields, then*

$$\pi_1^{ét}(X_L) \to \pi_1^{ét}(X)$$

is an iso.

Proof idea. Galois descent (exercise). ■

Remark 18.19. Note above false in characteristic p, e.g. already for the affine line. ○

Example. Suppose X is a smooth, proper curve over $k = \overline{k}$ of characteristic $p > 0$. Then, $\pi_1^{\text{ét}}(X, \overline{x})$ is topologically generated by at most $2g(X)$ elements. △

Proof. The first step is to lift to characteristic 0 (the obstruction to doing so lives in some Ext^2 which vanishes since X is a curve), and then algebrize by formal GAGA. Now, we have a surjective specialization map

$$\pi_1^{\text{ét}}(X_{\overline{K}}) \to \pi_1^{\text{ét}}(X),$$

and the LHS can be computed over $\mathbb{C}$. This finishes the proof. ■

For a singular curve, this is false using geometric genus, but probably true using arithmetic genus (exercise if you want).

Theorem 18.20 (SGA1)**.** *Say X as above. Then,*

$$\pi_1^{ét}(X_{\overline{K}}) \to \pi_1^{ét}(X_{\overline{k}})$$

induces an isomorphism of prime-to-p completions, where $p = \operatorname{char}(k)$.

Remark 18.21. There are analogous theorems for non-proper varieties w/ snc (simple normal crossings) compactification (a good reference is Grothendieck-Murre) for the "tame fundamental group." ○

Proposition 18.22. *There's an equivalence of categories*

$$\{\textit{lcc sheaves on } X_{\acute{e}t}\} \xleftarrow{\sim} \left\{\begin{matrix} \textit{finite cts} \\ \pi_1^{\acute{e}t}\textit{-modules} \end{matrix}\right\}.$$

Proof. lcc sheaves are represented by finite étale covers. ■

Notation 18.23. Given a finite continuous $\pi_1^{\text{ét}}$-module M, we lset $\mathscr{F}_M$ denote the corresponding lcc sheaf on $X_{\text{ét}}$.

Theorem 18.24. *Assume X is connected. There's a canonical map*

$$\mathrm{H}^i_{cts}(\pi_1^{\acute{e}t}(X,\overline{x}), M) \longrightarrow \mathrm{H}^i(X_{\acute{e}t}, \mathscr{F}_M)$$

(LHS is continuous group cohomology) which induces an iso on H^0 *and* H^1*.*

Proof. Morally, if you have a (topological) space X it has a covering space which is a π_1-torsor, so it has a map $X \to B\pi_1$, and this is the pullback on cohomology.

Mathily, we have a functor of sites $f : X_{\text{ét}} \to \text{FÉt}(X)$.

Claim 18.25 (exercise)**.** $\mathrm{Sh}(\textit{FÉt}(X)) = \pi_1^{\acute{e}t}$*-sets and* $\mathscr{F}_M = f^*M$*.*

Furthermore, $R^1 f_* \mathscr{F}_M = 0$. Why? If you have a cohomology class with coefficients in an lcc sheaf, then you can kill it by a finite étale cover (torsor kill themselves). Use leray ss to finish. ■

19 Lecture 19

Last time we talked about the étale fundamental group pretty quickly. We've kinda been speeding up to make sure we have enough time to get to the Weil conjecture by the end.

19.1 Fundamental Group Stuff Reviewed

We arrived last at an equivalence of categories.

Recall 19.1. There's an equivalence of categories

$$\begin{array}{ccc} \{\text{lcc abelian sheaves on } X_{\text{ét}}\} & \longleftrightarrow & \left\{\begin{matrix} \text{finite discrete} \\ \pi_1^{\text{ét}}\text{-modules} \end{matrix}\right\} \\ \mathscr{F}_M & \longleftarrow\!\shortmid & M \end{array}$$

⊙

What is this correspondence again? Say $\mathscr{F}$ is an lcc sheaf of abelian groups on $X_{\text{ét}}$. It is locally constant, so locally representable by constant schemes above X. The sheaf condition gives gluing data, and descent is effective for finite maps, so $\mathscr{F}$ is in fact represented by a finite étale X-scheme. A finite étale X-scheme is a finite $\pi_1^{\text{ét}}$-set, and $\mathscr{F}$ is a module (not just a set) since it's valued in abelian groups.

In the other direction, a finite discrete $\pi_1^{\text{ét}}$-module gives rise to a finite étale X-scheme Y_M, and the sheaf $\mathrm{Hom}(-, Y_M)$ is represents is an lcc sheaf of abelian groups. It is locally constant since it's represented by a finite étale X-scheme (e.g. pullback to Galois closure of covering to split it), and it's abelian since we started with a module.

Recall 19.2. Assume X is connected. There's a canonical map

$$\mathrm{H}^i_{cts}(\pi_1^{\text{ét}}(X,\overline{x}),M) \longrightarrow \mathrm{H}^i(X_{\text{ét}},\mathscr{F}_M)$$

(LHS is continuous group cohomology) which induces an iso on H^0 and H^1. ⊙

Proof. There's a continuous map of sites $X_{\text{ét}} \xrightarrow{\pi} \text{FÉt}(X)$ since finite étale X-schemes are in particular étale X-schemes.

Claim 19.3. *There's an equiv of categories FÉt$(X) \simeq$ Finite cts $\pi_1^{ét}$-sets.*

It's a general fact about group cohomology that

Looking at sheaves on the category of continuous $\pi_1^{\text{ét}}$-sets is apparently closely related to 'anima'

$$\mathrm{Sh}^{\text{fin.ab}}\left(\text{finite cts } \pi_1^{\text{ét}}\text{-sets}\right) \simeq \left\{\text{cts discrete finite } \pi_1^{\text{ét}}\text{-modules}\right\}$$

(think, sheaves on $\operatorname{Spec} k_{\text{ét}}$ are Galois module), sheaves of finite abelian groups on finite continuous $\pi_1^{\text{ét}}$-sets are the same as continuous discrete finite $\pi_1^{\text{ét}}$-modules.

Now, our map of sites in the beginning induces

$$\mathrm{H}^i(\text{FÉt}(X),M) \xrightarrow{\pi^*} \mathrm{H}^i(X_{\text{ét}},\pi^*M).$$

We can show this is an iso in low degrees using the leray spectral sequence. In particular, one shows $R^i\pi_*\pi^*M = 0$ for $i = 1$. ■

Corollary 19.4. $\mathrm{H}^1(\pi_1^{ét}(X,\overline{x});M) = \mathrm{H}^1(X_{ét},\mathscr{F}_M)$.

Remember: $\mathrm{H}^1_{\text{ét}}$ of abelian lcc sheaves is that same as $\mathrm{H}^1(\pi_1^{\text{ét}};-)$

(recall earlier proof that fundamental group of affine line in positive characteristic is not topologically finitely generated).

Remark 19.5. If M is a trivial $\pi_1^{\text{ét}}$-module, then $\mathrm{H}^1(\pi_1^{\text{ét}}(X,\overline{x}),M) = \operatorname{Hom}(\pi_1^{\text{ét}}(X,\overline{x}),M)$, so we can use corollary to compute maps out of $\pi_1^{\text{ét}}$ in some cases. ○

19.2 Finiteness Theorem

This will be one of the big inputs for proving Weil.

Theorem 19.6. *Let X be a k-variety with $k = k^s$ separably closed, and suppose $\mathscr{F}$ is a constructible sheaf on $X_{ét}$. If*

(1) X is proper; or

(2) the stalks of $\mathscr{F}$ all have order prime to $\operatorname{char}(k)$

then $\mathrm{H}^r(X_{ét},\mathscr{F})$ is finite.

Non-example. $\mathrm{H}^1(\mathbb{A}^1_{\text{ét}},\underline{\mathbb{Z}/p\mathbb{Z}})$ is not finite. ▽

Proof Sketch. **(1)** This we already know because of proper base change.

(2) We induct on $\dim X$. It is true in dimensions 0 and 1.[40] We'll sketch the proof when X is smooth (a simplifying assumption to make proof more accessible). We also assume $U = X$ fits into an elementary fibration

$$\begin{tikzcd} U \arrow[r, hook, "i"] \arrow[dr, "f"'] & Y \arrow[d, "h"] & Z \arrow[l, hook', "j"'] \arrow[dl, "g"] \\ & S & \end{tikzcd}$$

(so $Y \to S$ smooth and proper of rel. dim 1, and $Z \to S$ finite étale).

It's not immediately obvious why this reduction is allowed. The idea is that we can cover X by things fitting into elementary fibrations, and then use the Čech-to-derived spectral sequence to get finiteness of cohomology for X. However, the issue is the intersections may not fit into elementary fibrations, but you can still cover the intersections with opens that though and then end up forming a hypercover, and making use of this.

Now one has to do some devissage to reduce to the case that $\mathscr{F}$ is lcc (make some Gysin argument). Now we may assume $\mathscr{F}$ is lcc. Consider leray

One day I'll not be scared by this word, but not today

$$\mathrm{H}^i(S, R^j f_* \mathscr{F}) \implies \mathrm{H}^{i+j}(U, \mathscr{F}).$$

By induction on dimension, it is enough to show that $R^j f_* \mathscr{F}$ is constructible for all $j \geq 0$. Note $f = h \circ i$, so we can use the composition of functors spectral sequence

$$R^s h_* R^t i_* \mathscr{F} \implies R^{s+t} f_* \mathscr{F}.$$

Note that h is a proper morphism, so it's enough to show that $R^j i_* \mathscr{F}$ are constructible (by proper base change).

First consider the case where $\mathscr{F}$ is actually constant, $\mathscr{F} = \underline{\Lambda}$. Then, $i_* \mathscr{F} = \underline{\Lambda}_Y$, which is constructible, since U is a dense open. Further (purity exercise), $R^i i_* \underline{\Lambda} = j_* \underline{\Lambda}(?)$ is a pushforward of some twist of Λ along $j : Z \hookrightarrow Y$. Again by purity, $R^i i_* \underline{\Lambda} = 0$ for $i > 1$. Since one can reduce to the case of $\mathscr{F}$ constant, we're done. ■

Remark 19.7. We use that the order is prime to characteristic in the purity step at the end. ○

Remark 19.8. To reduce to smooth case, can make uses of alterations. ○

19.3 Sheaves of $\mathbb{Z}_\ell$-modules

Recall 19.9. The goal of this course is to count points on varieties over finite fields. We'll do this by taking some Frobenius action on cohomology, and taking some traces. If we want these traces to have a hope of counting something, they better live in characteristic 0. ⊙

Warning 19.10. A 'sheaf of $\mathbb{Z}_\ell$-modules' is not a sheaf. •

Definition 19.11. A **sheaf of $\mathbb{Z}_\ell$-modules** is a sequence of sheaves $(M_n, f_{n+1} : M_{n+1} \to M_n)$ if

[40] Technically, we haven't quite proved it in dimension 1; we've only proved it for smooth curves; and it's a little more work to get it for singular curves (e.g. compare with normalization). However, it secretly suffices to know it in dimension 0 for the sake of this argument.

(1) each M_n is a constructible sheaf of $\mathbb{Z}/\ell^n\mathbb{Z}$-modules.

(2) each f_{n+1} induces an isomorphism

$$M^{n+1}/\ell^n M_{n+1} \xrightarrow{\sim} M_n.$$

$\diamond$

Motivation. Given an ℓ-complete $\mathbb{Z}_\ell$-module N, we have $N = \varprojlim_n N/\ell^n N$, so the data of N is the same as the data of $(N/\ell^n N, \text{transition maps})$. Above, we've made a definition in terms of the 'discrete data' since it's not a priori clear how to incorporate the topology on $\mathbb{Z}_\ell$ into a direct definition.

Example. Taking $M_n = \underline{\mathbb{Z}/\ell^n\mathbb{Z}}$ with $f_{n+1} : \underline{\mathbb{Z}/\ell^{n+1}\mathbb{Z}} \to \underline{\mathbb{Z}/\ell^n\mathbb{Z}}$ given by the natural quotient map gives a canonical example. $\triangle$

Note that in the example above, the stalks are all $\mathbb{Z}/\ell^n\mathbb{Z}$ so the inverse limit of them is a free $\mathbb{Z}_\ell$-module.

Definition 19.12. A **flat sheaf of $\mathbb{Z}_\ell$-modules** $(M_n, f_{n+1} : M_{n+1} \to M_n)$ is a sheaf of $\mathbb{Z}_\ell$-modules such that[41]

$$0 \to M_s \xrightarrow{\ell^n} M_{n+s} \to M_n \to 0$$

is exact (for all s, n?). $\diamond$

Motivation. This exactness characterizes flat ℓ-complete $\mathbb{Z}_\ell$-modules.

Definition 19.13. Suppose (M_n, f_{n+1}) is a sheaf of $\mathbb{Z}_\ell$-modules. We define its **cohomology** to be

$$\mathrm{H}^r(X_{\text{ét}}, M) = \varprojlim_n \mathrm{H}^r(X_{\text{ét}}, M_n) \text{ and } \mathrm{H}^r_c(X_{\text{ét}}, M) = \varprojlim_n \mathrm{H}^r_c(X_{\text{ét}}, M_n).$$

$\diamond$

Example. Suppose X is a smooth proper curve of genus g over a field $k = k^s$ (and $\operatorname{char} k \neq \ell$). Then, (recall Corollary 13.13)[42]

$$\mathrm{H}^i(X_{\text{ét}}, \mathbb{Z}_\ell) = \varprojlim \mathrm{H}^i(X_{\text{ét}}, \mathbb{Z}/\ell^n\mathbb{Z}) = \begin{cases} \mathbb{Z}_\ell & \text{if } i = 0 \\ T_\ell(\operatorname{Jac} X)(-1) & \text{if } i = 1 \\ \mathbb{Z}_\ell(-1) & \text{if } i = 2 \\ 0 & \text{if } i > 2 \end{cases}$$

where $\mathbb{Z}_\ell$ is the example $\mathbb{Z}_\ell$-sheaf we saw before ($(\mathbb{Z}_\ell)_n = \underline{\mathbb{Z}/\ell^n\mathbb{Z}}$). Forgetting the Galois actions, we have

$$\mathrm{H}^0(X_{\text{ét}}, \mathbb{Z}_\ell) = \mathbb{Z}_\ell, \quad \mathrm{H}^1(X_{\text{ét}}, \mathbb{Z}_\ell) = \mathbb{Z}_\ell^{2g}, \text{ and } \mathrm{H}^2(X_{\text{ét}}, \mathbb{Z}_\ell) = \mathbb{Z}_\ell,$$

which is what cohomology of a smooth proper genus g curve *should* look like. $\triangle$

[41] For the first map, picking a lift along $M_{n+s} \to M_s$ and then multiplying by ℓ^n is well-defined.

[42] Twist in $i = 1$ case since we're using $\mathbb{Z}_\ell$ instead of $\mathbb{Z}_\ell(1)$ (i.e. since using $\underline{\mathbb{Z}/\ell\mathbb{Z}}$ instead of μ_ℓ)

Theorem 19.14. *Let M be a flat sheaf of $\mathbb{Z}_\ell$-modules on a variety X/k with k separably closed. If X is proper or $\ell \neq \operatorname{char} k$, then*

(1) $\mathrm{H}^r(X_{\acute{e}t}, M)$ *is a f.g. $\mathbb{Z}_\ell$-module.*

(2) There is a long exact sequence

$$\cdots \to \mathrm{H}^{r-1}(X_{\acute{e}t}, M_n) \to \mathrm{H}^r(X_{\acute{e}t}, M) \xrightarrow{\ell^n} \mathrm{H}^r(X_{\acute{e}t}, M) \to \mathrm{H}^r(X_{\acute{e}t}, M_n) \to \cdots$$

for each n.

Remark 19.15. Should think that the above LES is associated to the "SES" $0 \to M \xrightarrow{\ell^n} M \to M_n \to 0$. In reality, not a sense in which this short exact sequence really exists. One instead builds the LES out of those coming from the exact sequences characterizing flatness. ○

Proof. Exercise. Hint: reduce to the previous finiteness theorem, and then build the LES above out of the LES's in cohomology arising from $0 \to M_s \to M_{n+s} \to M_n \to 0$ by taking limit as $s \to \infty$ (note inverse limits are not exact). ■

Definition 19.16. A $\mathbb{Z}_\ell$-sheaf (M, f_{n+1}) is **locally constant** if each M_n is locally constant (i.e. lcc since $\mathbb{Z}_\ell$-sheaves have finite stalks built in). It is **lisse** if it is flat and locally constant. ◇

Remark 19.17. lisse is a French word meaning smooth, but think of it as saying it is a local system. ○

Warning 19.18. (M, f_{n+1}) being locally constant $\not\Rightarrow$ there exists a cover of X s.t. it is constant. •

19.3.1 $\pi_1^{\text{ét}}$-reps associated to locally constant $\mathbb{Z}_\ell$-sheaves

Suppose $M = (M_n, f_{n+1})_n$ is a locally constant $\mathbb{Z}_\ell$-sheaf. For each n, we get a continuous representation $\rho_n : \pi_1^{\text{ét}}(X, \overline{x}) \to \operatorname{Aut}(M_{n,\overline{x}})$ since M_n is an lcc sheaf which is the same think as a finite, discrete $\pi_1^{\text{ét}}$-module. Moreover, this representations live in a tower

$$\begin{tikzcd} & \operatorname{Aut}(M_{n+1,\overline{x}}) \arrow[d] \\ \pi_1^{\text{ét}}(X, \overline{x}) \arrow[ur, "\rho_{n+1}"] \arrow[r, "\rho_n"] & \operatorname{Aut}(M_{n,\overline{x}}) \end{tikzcd}$$

Hence,

Corollary 19.19. $\{$*locally constant $\mathbb{Z}_\ell$-sheaves*$\} \leftrightarrow$ *continuous representations of $\pi_1^{\acute{e}t}$ on f.g. $\mathbb{Z}_\ell$-modules. Similarly,* $\{$*lisse $\mathbb{Z}_\ell$-sheaves*$\} \longleftrightarrow$ *continuous representations of $\pi_1^{\acute{e}t}$ on f.g. flat (i.e. finite free) $\mathbb{Z}_\ell$-modules.*

Remark 19.20. Usually, M becomes constant after pullback to a cover only when the corresponding representation has finite image (which is a rare occurrence). ○

Exercise. A finitely generated flat (abstract) $\mathbb{Z}_\ell$-module is automatically free (use $\mathbb{Z}_\ell$ noetherian and local). Hence, being flat as a discrete module is the same as being flat as a topological module (at least in the f.g. case).

19.3.2 $\mathbb{Q}_\ell$-sheaves

The category of $\mathbb{Q}_\ell$-sheaves is the localization of the category of $\mathbb{Z}_\ell$-sheaves at ℓ.

Definition 19.21. A $\mathbb{Q}_\ell$**-sheaf** is a $\mathbb{Z}_\ell$-sheaf. A **morphism of** $\mathbb{Q}_\ell$**-sheaves** $M \to N$ is a diagram

Buzzphrase: 'localization of a category at a Serre subcategory' (or something)

$$\begin{tikzcd} & M' \arrow[dl, "f"'] \arrow[dr, "g"] & \\ M & & N \end{tikzcd}$$

such that f, g are morphisms of $\mathbb{Z}_\ell$-sheaves, and $f : M' \to M$ has finite kernel and cokernel. Given a $\mathbb{Q}_\ell$-sheaf M, we define

$$\mathrm{H}^i(X_{\text{ét}}, M) = \left(\varprojlim_n \mathrm{H}^i(X_{\text{ét}}, M_n)\right) \otimes \mathbb{Q}_\ell \quad \text{and} \quad \mathrm{H}^i_c(X_{\text{ét}}, M) = \left(\varprojlim_n \mathrm{H}^i_c(X_{\text{ét}}, M_n)\right) \otimes \mathbb{Q}_\ell.$$

◇

Remark 19.22. A morphism $M' \to M$ with finite kernel and cokernel with induce an isomorphism of $\mathbb{Q}_\ell$-cohomology groups defined above. ○

Given a $\mathbb{Q}_\ell$-sheaf whose underlying $\mathbb{Z}_\ell$-sheaf is locally constant, you get a continuous representation

$$\rho : \pi_1^{\text{ét}}(X, \overline{x}) \to \mathrm{GL}_n(\mathbb{Q}_\ell),$$

and in fact this gives an equivalence of categories. To get a functor in the other direction

Claim 19.23 (Exercise)**.** *ρ as above is conjugate to a representation into* $\mathrm{GL}_n(\mathbb{Z}_\ell)$.

Proof Sketch. Enough to show all maximal compacts in $\mathrm{GL}_n(\mathbb{Q}_\ell)$ are conjugate to $\mathrm{GL}_n(\mathbb{Z}_\ell)$. Consider $\mathbb{Z}_\ell^n \subset \mathbb{Q}_\ell^n$. The stabilizer of $\mathbb{Z}_\ell^n$ in $\pi_1^{\text{ét}}$ is open (so finite index). Now consider $\sum_{g \in \pi_1^{\text{ét}} / \mathrm{Stab}} g \cdot \mathbb{Z}_\ell^n$. This is stable under $\pi_1^{\text{ét}}$ and f.g. so iso to $\mathbb{Z}_\ell^n$ (any f.g. torsion-free $\mathbb{Z}_\ell$-module is free[43]). Picking a basis of $\sum_{g \in \pi_1^{\text{ét}} / \mathrm{Stab}} g \cdot \mathbb{Z}_\ell^n$ gives conjugation into $\mathrm{GL}_n(\mathbb{Z}_\ell)$, so then we win since this is the same as a lisse $\mathbb{Z}_\ell$ sheaf. ■

20 Lecture 20

20.1 Last time

Finished up discussion of étale π_1, and then introduced $\mathbb{Z}_\ell$- and $\mathbb{Q}_\ell$-sheaves. In nice (i.e. 'lisse') cases, these are same as continuous $\pi_1^{\text{ét}}$-representations. This time we introduce a grab-bag of topics needed for Weil stuff.

20.2 Smooth Base Change

Like proper base change, the proof is too complicated to give in a course. We may say a little about it later when we get to vanishing cycles.

[43] modules over a PID

Theorem 20.1 (Smooth Base Change). *Say we have a Cartesian diagram (Below, π f.type separated)*

$$\begin{tikzcd} X' \arrow[r, "f'"] \arrow[d, "\pi'"'] & X \arrow[d, "\pi"] \\ T \arrow[r, "f"] & S \end{tikzcd}$$

If $\mathscr{F}$ is a constructible sheaf on X, $(\mathscr{F})_{\overline{x}}$ always has order invertible on the base, and f is smooth, then

$$f^* \left(R^r \pi_* \mathscr{F}\right) \to R^r \pi'_* f'^* \mathscr{F}$$

is an isomorphism for all $r \geq 0$.

Remark 20.2. Still applies when f is an inverse limit of smooth morphisms.

- f is the inclusion of a generic point
- fraction field of a strict henselization
- ...

$\circ$

Let's say a bit about intuition. In topology, what causes cohomology to fail to compute with base change? This generally happens if the cohomology of the fiber of a point is different from the cohomology above fibers of nearby points. One reason we should expect it to commute e.g. will base change to the generic point is that the generic topology in families is constant.

Theorem 20.3 (smooth + proper base change theorem). *Suppose $\pi : X \to S$ is smooth and proper, $\mathscr{F}$ is an lcc sheaf on X, and $\#\mathscr{F}_{\overline{x}}$ invertible on S. Then, $R^r \pi_* \mathscr{F}$ are lcc.*

It's nonobvious, but this follows from the smooth base change theorem.

Exercise (Important exercise). Find a counterexample if you drop the hypothesis on the orders of the fibers.

Hint: Family of elliptic curves in char. p s.t. generic fiber is ordinary, but at least one fiber is supersingular, and take $\mathscr{F} = \underline{\mathbb{F}}_p$. The claim for this hint is that $(R^1 \pi_* \underline{\mathbb{F}}_p)_{\overline{x}} = \mathbb{F}_p$ if $E_{\overline{x}}$ ordinary, but is 0 if it is supersingular. Use proper base change to conclude that this stalk is

$$\mathrm{H}^1(E_{\overline{x}}, \mathbb{F}_p) = \mathrm{Hom}_{cts}(\pi_1^{\text{ét}}(E_{\overline{x}}), \mathbb{F}_p) = \begin{cases} \mathbb{F}_p & \text{if ordinary} \\ 0 & \text{if semisimple} \end{cases}.$$

Need to show that $\pi_1^{\text{ét}}(E_{\overline{x}}) = \prod_\ell T_\ell(E_{\overline{x}})$ is the **total Tate module**.

Let X be a scheme, with geometric points $\overline{\eta}, \overline{\xi}$ of X. Suppose $\overline{\eta} \rightsquigarrow \overline{\xi}$ ($\overline{\eta}$ specializes to $\overline{\xi}$).

Claim 20.4. *There is a non-canonical specialization map $\mathscr{F}_{\overline{\xi}} \to \mathscr{F}_{\overline{\eta}}$ where $\mathscr{F}$ is any constructible sheaf on $X_{ét}$.*

Construction 20.5. First recall the definition of these stalks: $\mathscr{F}_{\overline{\xi}} = \varinjlim_{(U,\overline{u})} \mathscr{F}(U)$. We want a map

$$\mathscr{F}_{\overline{\xi}} = \varinjlim_{(U,\overline{u})} \mathscr{F}(U) \dashrightarrow \varinjlim_{(V,\overline{v})} \mathscr{F}(V) = \mathscr{F}_{\overline{\eta}}.$$

Note that U as above is an étale X-scheme so $\mathrm{im}(\overline{\xi}) \subset \mathrm{im}(U)$. Since $\overline{\xi}$ generalizes to $\overline{\eta}$ means that $\overline{\eta}$ also factors through $\mathrm{im}(U)$. Thus, U is also an étale neighborhood (in a weak sense) of $\overline{\eta}$. To make U an actual étale neighborhood of $\overline{\eta}$, pick some $\overline{v} \in U$ lifting $\overline{\eta}$ (this is the choice). Once we know we can make these choices compatibly, we get our map $\mathscr{F}_{\overline{\xi}} \to \mathscr{F}_{\overline{\eta}}$.

All the dependence above boils down to a choice of a map $\mathscr{O}_{X,\overline{\xi}} \to \mathscr{O}_{x,\overline{\eta}}$ (above construction applied to sheaf $\mathscr{O}_X^{\text{ét}}$).

In practice, say R is a dvr with reside field k and fraction field K. If $\pi : X \to \operatorname{Spec} R$ is a proper morphism and we take $\mathscr{F} = R^i\pi_*\left(\underline{\mathbb{Z}/\ell\mathbb{Z}}\right)$, then we get a map

$$\mathrm{H}^i(X_k, \mathbb{Z}/\ell\mathbb{Z}) = \mathscr{F}_k \to \mathscr{F}_K = \mathrm{H}^i(X_K, \underline{\mathbb{Z}/\ell\mathbb{Z}}),$$

called a **cospecialization map**.

Remark 20.6. Intuitively, if you have a family $X \to \Delta$ of a disk, then it retracts to its special fiber X_0, so this retraction gives a map from a generic fiber to the special fiber and so a map in cohomology going the other direction. ○

Proposition 20.7 (Exercise)**.** *Let $\mathscr{F}$ be a constructible sheaf. Then, $\mathscr{F}$ is lcc $\iff$ all cospecialization maps are isomorphisms.*

Corollary 20.8 (to Smooth + proper base change theorem)**.** *Let $\pi : X \to S$ be smooth and proper, let $\mathscr{F}$ be an lcc sheaf on $X_{ét}$ s.t. $\#(\mathscr{F}_{\overline{x}})$ is invertible on S. Given $\overline{\eta}, \overline{\xi}$ geometric points of S with specialization $\overline{\eta} \rightsquigarrow \overline{\xi}$, the cospecialization map*

$$\mathrm{H}^i(X_{\overline{\xi}}, \mathscr{F}|_{X_{\overline{\xi}}}) \xrightarrow{\sim} \mathrm{H}^i(X_{\overline{\eta}}, \mathscr{F}|_{X_{\overline{\eta}}})$$

is an isomorphism.

Corollary 20.9. *Let k be a field of characteristic $p > 0$, and X/k be a smooth, proper variety. Then if X lifts to characteristic 0, one can compute is $\mathbb{F}_\ell/\mathbb{Z}_\ell$-cohomology for $\ell \neq k$.*

Let's explain what this is saying.

- **Lift to characteristic** 0

  This means there exists a smooth, proper R-scheme $\mathcal{X}/R$ (R a dvr) s.t. $R/\mathfrak{m}_R = k$ and $\mathcal{X}_k \simeq X$. It's important that $\mathcal{X}$ is a *scheme*.

  Non-example. If $\mathrm{H}^2(X, T_X) = 0$, get a formal list, which is not enough. ▽

  However, in above example, if you can also formally lift an ample line bundle, then formal GAGA implies the existence of a lift.

  Example. If X is projective, $\mathrm{H}^2(X, T_X) = 0$ and $\mathrm{H}^2(X, \mathscr{O}_X) = 0$, then you can lift to char. 0. △

  Example. Curves can be lifted to characteristic 0 since both relevant H^2's vanish.

  Abelian varieties also lift (even though both H^2's don't vanish). This is nonobvious.

  K3s also work (due to Deligne) △

Example. Hypersurfaces and complete intersections in $\mathbb{P}^n$ lift. Just pick lifts of the defining equations. $\triangle$

- "Can compute cohomology." By smooth and proper base change theorem, it is enough to compute cohomology $\mathcal{X}_{\overline{K}}$, where K is the fraction field of a dvr to which X lifts. Consider $L := \mathbb{Q}(\text{coeffs of polys defining } X)$. Then, $\overline{K} \supset \overline{L} \subset \mathbb{C}$. Hence, we have maps

$$\begin{tikzcd} X_{\overline{K}} \arrow[dr] & & X_{\mathbb{C}} \arrow[dl] \\ & X_{\overline{L}} & \end{tikzcd}$$

(1) These maps induce isomorphisms on cohomology

(2) Cohomology of $X_{\mathbb{C}}$ is computable (via Artin comparison)

Why is **(1)** true?

Proposition 20.10. *Let X/k be a variety with $k = \overline{k}$, and let $k \subset L$ be an extension of algebraically closed fields. Then, if $\mathscr{F}$ is a constructible sheaf on $X_{ét}$ s.t. $\#\mathscr{F}_{\overline{x}}$ is invertible in k, then*

$$\mathrm{H}^i(X_{ét}, \mathscr{F}) \to \mathrm{H}^i(X_{L,ét}, \mathscr{F}|_{X_L})$$

is an isomorphism.

Proof. This follows from smooth base change. There's extra to argue in characteristic p, but in characteristic 0, an extension of fields is a direct limit of smooth morphisms (e.g. separable field extensions). In characteristic p, show that étale cohomology is insensitive to inseparable field extensions (e.g. if you have an étale morphism, it descends uniquely through inseparable field extensions). ∎

20.3 Künneth formula + cycle class maps

After this, one more topic (Poincaré duality + Lefschetz trace) before Weil conjectures.

Theorem 20.11 (Künneth formula). *Suppose X, Y are proper k-schemes and $k = \overline{k}$. Suppose $\mathscr{F}$ is a constructible sheaf of $\mathbb{Z}/\ell^n\mathbb{Z}$-modules on $X_{ét}$, and $\mathscr{G}$ is a constructible sheaf on $Y_{ét}$. Then, in the derived category $D^b(\mathrm{Ab})$, the canonical map*

$$R\Gamma(X_{ét}, \mathscr{F}) \overset{L}{\otimes} R\Gamma(Y_{ét}, \mathscr{G}) \to R\Gamma((X \times Y)_{ét}, \mathscr{F} \overset{L}{\otimes} \mathscr{G})$$

is an isomorphism.

Corollary 20.12. *There is a spectral sequence*

$$\bigoplus_{i+j=s} \mathrm{Tor}_{-r}^{\mathbb{Z}/\ell^n\mathbb{Z}}\left(\mathrm{H}^i(X_{ét}, \mathscr{F}), \mathrm{H}^j(Y_{ét}, \mathscr{G})\right) \implies \mathrm{H}^{r+s}((X \times Y)_{ét}, \mathscr{F} \otimes \mathscr{G})$$

if $\mathscr{F}$ or $\mathscr{G}$ is a sheaf of flat $\mathbb{Z}/\ell^n\mathbb{Z}$-modules.

Note this spectral sequence can be infinite (Tor can be nonvanishing in arbitrary negative degree). However, suppose $\mathscr{F} = \mathscr{G} = \underline{\mathbb{Z}/\ell^n\mathbb{Z}}$ and that their cohomologies are free $\mathbb{Z}/\ell^n\mathbb{Z}$-modules; then all Tor's vanish for $r \geq 1$, so you just get that the cohomology of the product is the tensor product of the cohomologies, as a graded ring.

Corollary 20.13. *Over $\mathbb{Z}_\ell$, just get an exact sequence*

$$0 \longrightarrow \bigoplus_{r+s=m} \mathrm{H}^r(X_{\acute{e}t}, \mathbb{Z}_\ell) \otimes \mathrm{H}^s(X_{\acute{e}t}, \mathbb{Z}_\ell) \longrightarrow \mathrm{H}^m((X \times Y)_{\acute{e}t}, \mathbb{Z}_\ell) \longrightarrow \bigoplus_{r+s=m+1} \mathrm{Tor}_1^{\mathbb{Z}_\ell}(\mathrm{H}^r(X_{\acute{e}t}, \mathbb{Z}_\ell), \mathrm{H}^s(X_{\acute{e}t}, \mathbb{Z}_\ell)) \longrightarrow 0.$$

Since tor dimension of $\mathbb{Z}_\ell$ is one (Since $\mathbb{Z}_\ell$ is a PID).

Non-example. Consider $X = Y = \mathbb{A}^1/k$ with $k = \overline{k}$ and $\operatorname{char} k = p > 0$. Then, Künneth is false in this case, even for H^1. To see this, just compare H^1's of $\mathbb{A}^1$ and $\mathbb{A}^2$. $\triangledown$

Where does the map in Kunneth come from? This comes from cup products

$$\mathrm{H}^r(X_{\text{ét}}, \mathscr{F}) \otimes \mathrm{H}^s(Y_{\text{ét}}, \mathscr{G}) \to \mathrm{H}^{r+s}((X \times Y)_{\text{ét}}, \mathscr{F} \otimes \mathscr{G}).$$

These exist always for sheaves of abelian groups on a site. To make things concrete, we'll describe them in Čech cohomology, so say we have $\{\mathcal{U}\} \to X$ and $\{\mathcal{V}\} \to Y$ covers. Then, $\{\mathcal{U} \times \mathcal{V}\} \to X \times Y$ is a cover, and the map of Čech cohomology is simply $(f, g) \mapsto f \otimes g$.

21 Lecture 21

Last time, we introduced cup products and the Kunneth formula.

21.1 Cup products and Künneth

Let's recall cup products in Čech cohomology. Say X, Y are k-schemes with sheaf $\mathscr{F}, \mathscr{G}$ of $\mathbb{Z}/\ell^n\mathbb{Z}$-modules on $X_{\text{ét}}, Y_{\text{ét}}$, respectively. Say $U \to X$ and $V \to Y$ are covers. Then, there's a map

$$\mathrm{H}^i(\check{C}(U/X, \mathscr{F})) \otimes \mathrm{H}^j(\check{C}(V/Y, \mathscr{F})) \longrightarrow \mathrm{H}^{i+j}(\check{C}(U \times V/X \times Y, \mathscr{F} \boxtimes \mathscr{G})),$$

where $\mathscr{F} \boxtimes \mathscr{G}$ is the sheaf $\mathrm{pr}_1^* \mathscr{F} \otimes \mathrm{pr}_2^* \mathscr{G}$ on $X_{\text{ét}} \times Y_{\text{ét}}$. This is induced by a map of complexes

$$\mathrm{Tot}(\check{C}(U/X, \mathscr{F}) \otimes \check{C}(V/Y, \mathscr{F})) \longrightarrow \check{C}(U \times V/X \times Y, \mathscr{F} \boxtimes \mathscr{G}).$$

Given $f \in \check{C}^i(U/X, \mathscr{F})$ and $g \in \check{C}^j(V/Y, \mathscr{G})$, we get

$$f, g \mapsto f \boxtimes g \in \check{C}^{i+j}(U \times V/X \times Y, \mathscr{F} \boxtimes \mathscr{G})$$

with $f \boxtimes g$ defined in the obvious way.

Exercise. Check that this gives a morphism of complexes.

Remark 21.1. Čech cohomology does not always compute derived functor cohomology. Nevertheless, cup products always exist in derived functor sheaf cohomology over a site (see e.g. Iversen's book). $\circ$

This brings us (back) to Künneth.

Proposition 21.2 (**Künneth Formula (simple version)**).

(1) *Let X, Y be proper varieties over k. Let $\Lambda = \underline{\mathbb{Z}/\ell^n\mathbb{Z}}$ or $\Lambda = \underline{\mathbb{Z}_\ell}$ (the latter choice is a $\mathbb{Z}_\ell$-sheaf, not a sheaf).*

If $\mathrm{H}^*(X, \Lambda)$ or $\mathrm{H}^*(Y, \Lambda)$ is a free Λ-module, then the cup product

$$\cup : \mathrm{H}^*(X, \Lambda) \otimes \mathrm{H}^*(Y, \Lambda) \to \mathrm{H}^*(X \times Y, \Lambda)$$

is an isomorphism of graded groups (in fact one of graded rings[44]).

(2) *If X, Y are as above (i.e. proper), then*

$$\cup : \mathrm{H}^*(X, \mathbb{Q}_\ell) \otimes \mathrm{H}^*(Y, \mathbb{Q}_\ell) \to \mathrm{H}^*(X \times Y, \mathbb{Q}_\ell)$$

is an isomorphism of graded rings always.

Recall 21.3 (Fancy Künneth, Theorem 20.11)**.** Let X, Y be proper k-schemes ($k = \overline{k}$)[45], $\mathscr{F} \in \mathrm{Sh}(X_{\text{ét}})$ constructible, and $\mathscr{G} \in \mathrm{Sh}(Y_{\text{ét}})$ constructible. Then, there is a quasi-isomorphism

$$R\Gamma(X_{\text{ét}}, \mathscr{F}) \overset{\mathrm{L}}{\otimes} R\Gamma(Y_{\text{ét}}, \mathscr{G}) \xrightarrow{\sim} R\Gamma((X \times Y)_{\text{ét}}, \mathscr{F} \overset{L}{\boxtimes} \mathscr{G})$$

in $D^b(\mathrm{Ab})$. $\odot$

Remark 21.4. When X, Y are say projective, so Čech cohomology computes derived functor cohomology, can take $R\Gamma$ to be the Čech complex, and then the above map is just the cup product. $\circ$

Example. Let's compute $\mathrm{H}^*(C \times \mathbb{P}^1, \mathbb{Q}_\ell)$ where C a smooth proper curve over $k = \overline{k}$. Recall

$$\mathrm{H}^*(C, \mathbb{Q}_\ell) = \begin{cases} \mathbb{Q}_\ell & \text{if } * = 0 \\ V_\ell(\operatorname{Jac} C)(-1) & \text{if } * = 1 \\ \mathbb{Q}_\ell(-1) & \text{if } * = 2 \end{cases}$$

and vanishes for $* > 2$. Above $V_\ell(\operatorname{Jac} C) = T_\ell(\operatorname{Jac} C) \otimes \mathbb{Q}_\ell$. Hence,

$$\mathrm{H}^*(\mathbb{P}^1, \mathbb{Q}_\ell) = \begin{cases} \mathbb{Q}_\ell & \text{if } * = 0 \\ 0 & \text{if } * = 1 \\ \mathbb{Q}_\ell(-1) & \text{if } * = 2. \end{cases}$$

[44] The ring structure comes from $\mathrm{H}^*(X, \Lambda) \otimes \mathrm{H}^*(X, \Lambda) \xrightarrow{\cup} \mathrm{H}^*(X \times X, \Lambda) \xrightarrow{\Delta^*} \mathrm{H}^*(X, \Lambda)$

[45] $k = k^s$ is enough

Thus, we see

$$\mathrm{H}^*(C \times \mathbb{P}^1, \mathbb{Q}_\ell) = \begin{cases} \mathbb{Q}_\ell & * = 0 \\ V_\ell(\operatorname{Jac} C)(-1) & * = 1 \\ \mathbb{Q}_\ell(-1) \otimes \mathbb{Q}_\ell(-1) & * = 2 \\ V_\ell(\operatorname{Jac} C)(-2) & * = 3 \\ \mathbb{Q}_\ell(-2) & * = 4. \end{cases}$$

$\triangle$

Question 21.5. *What info does* $\mathrm{H}^*(X_{\overline{k},\acute{e}t}, \mathbb{Q}_\ell)$ *carry?*

(1) Over $\mathbb{F}_q$: gives point counts.

(2) Over K a f.g. field: Galois reps on $\mathrm{H}^*(X_{\overline{k},\text{ét}}, \mathbb{Q}_\ell)$ conjecturally carry geometric info (might touch on this if we get to cycle classes today)

(3) In general, they give linear algebraic invariants attached to your space which are in principle computable.

Proof sketch of fancy Künneth for $\mathbb{Z}_\ell$-sheaves. The idea is use leray spectral sequence. To compute something on $X \times Y$, we can project to X and then project to Y. But we'll do this in the derived category instead of directly using spectral sequences, since it makes things a little cleaner.

(Step 1) Projection formula

Claim 21.6 (Projection formula). *Say $f : X \to S$ with $\mathscr{F}$ a flat $\mathbb{Z}_\ell$-sheaf (or flat $\mathbb{Z}/\ell^n\mathbb{Z}$-sheaf) on $X_{\acute{e}t}$, and $\mathscr{G}$ a bounded above complex of abelian sheaves on S. Then there is a natural quasi-isomorphism*

$$(Rf_*\mathscr{F}) \otimes \mathscr{G} \xrightarrow{\sim} Rf_*(\mathscr{F} \otimes f^*\mathscr{G}).$$

Proof idea. If $\mathscr{G} = \mathbb{Z}_\ell$, this is the identity map. In general, reduce to this case. ■

Example. $\mathrm{H}^i(X_{\overline{k}}, \mu_\ell) \simeq \mathrm{H}^i(X_{\overline{k}}, \mathbb{Z}/\ell\mathbb{Z}) \otimes \mu_\ell$. This is the case where $\mathscr{F} = \underline{\mathbb{Z}/\ell\mathbb{Z}}$ and $\mathscr{G} = \mu_\ell$ and $f : X_{\overline{k}} \to \operatorname{Spec} \overline{k}$. $\triangle$

(Step 2) Consider the diagram

$$\begin{tikzcd} & X \times Y \arrow[dl, "p"'] \arrow[dd, "h"] \arrow[dr, "q"] & \\ X \arrow[dr, "f"'] & & Y \arrow[dl, "g"] \\ & \operatorname{Spec} \overline{k} & \end{tikzcd}$$

and now assume $\mathscr{F}, \mathscr{G}$ are constructible sheaves of $\mathbb{Z}/\ell^n\mathbb{Z}$-modules. To keep this simple, let's further assume $\mathscr{F}$ is a sheaf of flat $\mathbb{Z}/\ell^n\mathbb{Z}$-modules. We want to compute $Rf_*\mathscr{F} \otimes Rg_*\mathscr{G}$. By the projection formula, we have an isomorphism

$$Rf_*\mathscr{F} \otimes Rg_*\mathscr{G} \xrightarrow{\sim} Rf_*(\mathscr{F} \otimes f^* Rg_*\mathscr{G})$$

(if $\mathscr{F}$ not flat, use derived tensor product). Proper base change further gives an iso

$$Rf_*(\mathscr{F} \otimes f^* Rg_*\mathscr{G}) \xrightarrow{\sim} Rf_*(\mathscr{F} \otimes Rp_* q^*\mathscr{G}).$$

A second application of the projection formula now says

$$Rf_*(\mathscr{F} \otimes Rp_* q^*\mathscr{G}) \xrightarrow{\sim} Rf_*(Rp_*(p^*\mathscr{F} \otimes q^*\mathscr{G})) \xrightarrow{\sim} Rh_*(\mathscr{F} \boxtimes \mathscr{G}).$$

■

Remark 21.7.

- Above works for an arbitrary base S. It doesn't have to be $\operatorname{Spec} \overline{k}$.
- g being proper would have been enough.
- We also could have assume ℓ invertible on S and f smooth (and then used smooth base change)

○

21.2 Cycle class maps

Assumption. X is a non-singular variety over k, a field a characteristic $p \neq \ell$.

Question: Is 'non-singular' smooth or regular?

Answer: I think smooth since we appeal to purity at some point

Motivation. Over $\mathbb{C}$, if you have some d-dimensional (smooth) subvariety $Z \hookrightarrow X$ and a $2d$-form $\omega \in \mathrm{H}^{2d}_{dR}(X)$, then you can integrate ω over Z. Thus, a choice of Z gives a linear function $\omega \mapsto \int_Z \omega|_Z \in \mathbb{C}$ on H^{2d}_{dR}. By poincaré duality, this is the same thing as a class $[Z] \in \mathrm{H}^{2\dim X - 2d}_{dR}$.

We want something like this in ℓ-adic cohomology.

Notation 21.8. Let $C^r(X)$ be the free abelian group on **prime cycles** (irreducible subvarieties) of codimension r

Goal. Define a map $\mathrm{cl}^r : C^r(X) \to \mathrm{H}^{2r}(X, \Lambda(r))$ where $\Lambda = \mathbb{Z}/\ell^n\mathbb{Z}, \mathbb{Z}_\ell, \mathbb{Q}_\ell$. This map will be functorial, and $\mathrm{cl}^1 : C^1(X) \to \mathrm{H}^2(X, \Lambda(1))$ (when $\Lambda = \mathbb{Z}/\ell^n\mathbb{Z}$) will be the familiar map

$$\mathrm{cl}^1 : C^1(X) \to \operatorname{Pic}(X) = \mathrm{H}^1(X, \mathbb{G}_m) \xrightarrow{\kappa} \mathrm{H}^2(X, \Lambda(1))$$

where κ is the **Kummer map** arising from the short exact sequence

$$1 \longrightarrow \mu_{\ell^n} \longrightarrow \mathbb{G}_m \xrightarrow{\ell^n} \mathbb{G}_m \longrightarrow 1.$$

Definition 21.9. For Z nonsingular of codimension r, we define

$$\mathrm{cl}^r(Z) : \text{image } 1 \in \mathrm{H}^0(Z, \Lambda) \xrightarrow{\sim} \mathrm{H}^{2r}_Z(X, \Lambda(r)) \to \mathrm{H}^{2r}(X, \Lambda(r))$$

(first iso by purity, Theorem 16.1). Then we extend by linearity. $\diamond$

What about for singular Z?

Lemma 21.10. *Let $Z \subset X$ of codimension r. Then,*

$$\mathrm{H}^s_Z(X, \Lambda) = 0 \text{ for } s < 2r.$$

Moreover,

$$\mathrm{H}^{2r}_Z(X, \Lambda) = \mathrm{H}^{2r}_{Z \setminus Z^{sing}}(X \setminus Z^{sing}, \Lambda).$$

Note that $Z \setminus Z^{sing} \subset X \setminus Z^{sing}$ is a smooth pair.

Proof idea. Induction on dimension of Z, filter by singular locus, and appeal to another long exact sequence involving cohomology with supports. ■

Remark 21.11. So we've obtained a map $\mathrm{cl} : C^r(X_{\overline{k}}) \to \mathrm{H}^{2r}(X_{\overline{k}}, \Lambda(r))$. This map is Galois equivariant. Hence it restricts to

$$r\text{-cycles defined}/k \longrightarrow \mathrm{H}^{2r}(X_{\overline{k}}, \Lambda(r))^{G_k}.$$

$\circ$

Conjecture 21.12 (Tate Conjecture). *Suppose X is smooth, projective over k, a f.g. field (e.g. fraction field of a variety). Then, the map*

$$C^r(X) \otimes \mathbb{Q}_\ell \longrightarrow \mathrm{H}^{2r}(X_{\overline{k}}, \mathbb{Q}_\ell(r))^{G_k}$$

is surjective.

Remark 21.13. This is really a rational conjecture. It is known to be false if you replace $\mathbb{Q}_\ell$ with $\mathbb{Z}_\ell$, and is super false if you replace it with $\mathbb{Z}/\ell^n\mathbb{Z}$. $\circ$

Example. Suppose X, Y are smooth, projective varieties of dimension m. Assume $\mathrm{H}^{2m}((X \times Y)_{\overline{k}}, \mathbb{Q}_\ell(m))^{G_k} \neq 0$. This should imply there is a cycle $X \times Y$ w/ non-trivial cycle class α (up to $\mathbb{Q}_\ell$-linear combination). If you think of α as a map on cohomology, then this is saying that the map is induced by a cycle.

If X, Y are abelian varieties, this is saying any map between ℓ-adic Tate modules comes from a map of abelian varieties. Tate proved this over finite fields, and Faltings proved it over arbitrary f.g. fields. $\triangle$

Remark 21.14. cl factors through cycles mod 'rational equivalence' as well as cycles mod 'algebraic equivalence'. If you mod out by the kernel of it, then we call this considering cycles up to 'homological equivalence'. $\circ$

Next time: Chern classes.

22 Lecture 22

Last time: Künneth + cycle class amps.

Today: Chern classes of vector bundles + Poincaré duality.

22.1 Chern Classes

What are Chern classes?

Let X be a smooth projective k-field. These properties are not needed for everything (in particular, not for the definition of Chern classes), but they are needed for some of the nicer properties of Chern classes.

Goal. Given a vector bundle $\mathscr{E}$ on X, we wish to construct cohomology classes $c_i(\mathscr{E}) \in \mathrm{H}^{2i}(X, \mathbb{Z}_\ell(i))$.

Example. When $i = 1$, we want $c_1(\mathscr{E}) \in \mathrm{H}^2(X, \mathbb{Z}_\ell(1))$. We can actually already construct this.

Recall that $\mathbb{Z}_\ell(1)$ is the $\mathbb{Z}_\ell$-sheaf corresponding to the sequence

$$\mathbb{Z}_\ell(1) = (\cdots \longrightarrow \mu_{\ell^n} \longrightarrow \mu_{\ell^{n+1}} \longrightarrow \cdots).$$

The short exact sequence $1 \to \mu_{\ell^n} \to \mathbb{G}_m \xrightarrow{x \mapsto x^{\ell^m}} \mathbb{G}_m \to 1$ give rise to the Kummer maps $\kappa_{\ell^n} : \operatorname{Pic} X \to \mathrm{H}^2(X, \mu_{\ell^n})$. These maps are compatible, so glue to give a map

$$\kappa : \operatorname{Pic} X \to \mathrm{H}^2(X, \mathbb{Z}_\ell(1)) = \varprojlim_n \mathrm{H}^2(X, \mu_{\ell^n}).$$

We define $c_1(\mathscr{E}) := \kappa(\det(\mathscr{E}))$. △

Theorem 22.1. *There exists a unique assignment $\mathscr{E} \rightsquigarrow c_i(\mathscr{E})$ satisfying*

(1) *Functorial under pullbacks, i.e. $f^*(c_i(\mathscr{E})) = c_i(f^*\mathscr{E})$*

(2) $c_1(\mathscr{E}) = \kappa(\det(\mathscr{E}))$

(3) *Multiplicative under exact sequences, i.e. if we define the **total Chern class***

$$c(\mathscr{E}) := 1 + c_1(\mathscr{E}) + c_2(\mathscr{E}) + \cdots \in \bigoplus_i \mathrm{H}^{2i}(X, \mathbb{Z}_\ell(i)),$$

then for any s.e.s. $0 \to \mathscr{E}_1 \to \mathscr{E} \to \mathscr{E}_2 \to 0$, one has $c(\mathscr{E}) = c(\mathscr{E}_1)c(\mathscr{E}_2)$.

Proof Idea. Same as in topology. Analyze the cohomology of $\mathbb{P}(\mathscr{E})$ from which the Chern classes can be extracted. This gives a construction. For uniqueness: use induction on the rank. Consider $\pi : \mathbb{P}(\mathscr{E}) \to X$ and note that

$$0 \to \mathscr{V} \to \pi^*\mathscr{E} \to \mathscr{O}(1) \to 0$$

$\pi^*\mathscr{E}$ has a line bundle quotient, so $c(\mathscr{V})$ and $c(\mathscr{O}(1))$ are determined as is $c(\pi^*(\mathscr{E})) = (1 + c_1(\mathscr{O}(1)))c(\mathscr{V})$. Then use that $\pi^* : \mathrm{H}^*(X) \to \mathrm{H}^*(\mathbb{P}(\mathscr{E}))$ is injective. ■

Remark 22.2. Multiplicativity in short exact sequences means that we actually have a map

$$K^*(X) \xrightarrow{c} \bigoplus_i \mathrm{H}^{2i}(X, \mathbb{Z}_\ell(i)),$$

a (multiplicative) map of sets, where $K^*(X)$ is the free abelian group on iso classes of vector bundles on X modulo $[\mathscr{E}] = [\mathscr{E}_1] + [\mathscr{E}_2]$ if there is a short exact sequence $0 \to \mathscr{E}_1 \to \mathscr{E} \to \mathscr{E}_2 \to 0$.

One can modify these to get an actual ring homomorphism

$$\gamma : K^*(X) \to \bigoplus_i \mathrm{H}^{2i}(X, \mathbb{Q}_\ell(i)).$$

The product on the LHS is tensor product while on the RHS it is cup product. ◦

We can use this to reinterpret the cycle class map. Let $C^*(X)$ denote the **Chow groups** of X, cycles modulo rational equivalence. The compositions

$$C^*(X) \xrightarrow{\mathrm{ch}^{-1}} K^*(X)_{\mathbb{Q}} \xrightarrow{\gamma} \bigoplus \mathrm{H}^{2i}(X, \mathbb{Q}_\ell(i))$$

is the cycle class map from before (in particular, the cycle class map is a ring homomorphism).

Fact. For X smooth projective, ch induces an isomorphism

$$C^*(X)_{\mathbb{Q}} \xrightarrow{\sim} K^*(X)_{\mathbb{Q}}.$$

22.2 Poincaré duality

Fix X a smooth variety over $k = k^s$, and let $\Lambda = \underline{\mathbb{Z}/m\mathbb{Z}}$ with m invertible in k.

Theorem 22.3 (Poincaré duality). *The cup product induces a perfect pairing*

$$\mathrm{H}^i(X_{\acute{e}t}, \Lambda) \times \mathrm{H}_c^{2\dim X - i}(X_{\acute{e}t}, \Lambda(d)) \xrightarrow{\smile} \mathrm{H}_c^{2\dim(X)}(X_{\acute{e}t}, \Lambda(d)) \xrightarrow{\mathrm{Tr}} \mathbb{Z}/m\mathbb{Z}$$

where Tr *is a canonical isomorphism.*

Motivation. Suppose X is an oriented manifold. Integration gives a canonical isomorphism

$$\mathrm{H}_c^{\dim X}(X, \mathbb{R}) \xrightarrow{\int} \mathbb{R},$$

the "trace map." This makes wedge product

$$\begin{array}{ccc} \mathrm{H}^i(X, \mathbb{R}) \times \mathrm{H}_c^{\dim X - i}(X, \mathbb{R}) & \longrightarrow & \mathrm{H}_c^{\dim X}(X, \mathbb{R}) \\ (\omega, \eta) & \longmapsto & \omega \wedge \eta \end{array}$$

a perfect pairing.

In the algebraic setting, our variety X is canonically oriented (analogous to complex manifolds being canonically oriented).

Proof Sketch of Theorem 22.3. Need to construct trace map, and show cup product is a perfect pairing.

Tony Feng's notes are a good reference

(1) We claim something more general. Let $f : X \to S$ be a smooth, **compactifiable**[46] morphism w/ geometrically connected fibers. We claim there exists a canonical map

$$\mathrm{Tr} : R^{2d} f_! f^* \mathscr{F}(d) \to \mathscr{F} \quad \text{where} \quad d = \mathrm{reldim}(f).$$

[46] i.f. f factors as $X \hookrightarrow X' \to S$ with $X \hookrightarrow X'$ an open embedding, and $X' \to S$ proper

Recall 22.4. Say we compactify f as

$$\begin{tikzcd} X \arrow[r, "\iota"'] \arrow[rr, bend left, "f"] & X' \arrow[r, "\widetilde{f}"'] & S \end{tikzcd}$$

Then, we define $R^i f_! \mathscr{G} := R^i \widetilde{f}_* \iota_! \mathscr{G}$. ⊙

This Tr will satisfy

- For curves/separable closed field, agrees with computation we did earlier.[47]
- functorial in $\mathscr{F}$
- compatible with base change
- If f is étale ($d = 0$)

  $$f_! f^* \mathscr{F} \to \mathscr{F}$$

  is the counit of the adjunction between $f_!$ and f_*.
- Compatible w/ compositions.

Tr is the unique natural transformation satisfying all of this.

Proof Sketch. Say we have $X \underset{\text{ét}}{\xrightarrow{g}} \mathbb{A}^n_S \xrightarrow{\pi} S$ (always true locally of any smooth morphism). Compatibility with compositions says it's enough to define Tr for g, π. For g, it is the counit of the adjunction for $g_!, g^*$. For π, things are more tricky. One need to essentially redo the computation of compactly support cohomology of $\mathbb{A}^1$ but now over a base. We can factor π into a sequence $\mathbb{A}^n_S \to \mathbb{A}^{n-1}_S \to \cdots \to \mathbb{A}^1_S \to S$ of relative curves, and then use the curve case.

To do this e.g. use smooth base change (Theorem 20.1) so you can do computation over a point and then pull back to S

This gives a construction in this special ("local") case. It's not obvious it satisfies all the properties we claimed. At least, we can say how to do the construction in general. Mayer-Vietoris let's one reduce to this case.

Why is this well-defined/unique? Uniqueness is not bad; our construction for "standard étale maps" forced by axioms. For well-definedness, we had to choose an open set U and choose an étale map $U \to \mathbb{A}^n_S$. The choice independence of choice of U, take common refinements and argue some diagram(s) commute. For choice of étale map, suppose we have

$$\begin{tikzcd} X \arrow[r, "\text{ét}"] \arrow[d, "\text{ét}"'] & \mathbb{A}^1_S \arrow[d] \\ \mathbb{A}^1_S \arrow[r] & S \end{tikzcd}$$

Can reduce to the case of relative curves, and then we're in luck since the map for curves we computed long ago is very canonical. ■

[47] Apparently we computed compactly support cohomology of μ_n before, and for curves over k^s, basically everything is μ_n

Claim 22.5. *If $X \xrightarrow{f} S$ has geometrically connected fibers, then*

$$R^{2d} f_! f^* \mathscr{F}(d) \longrightarrow \mathscr{F}$$

is an isomorphism.

For this, reduce to the case of relative curves via structure theorem for smooth morphisms. In the case of relative curves, check on stalks.

■

23 Lecture 23

Last time we started talking about Poincaré duality. Let X be a smooth variety over $k = k^s$ of dimension d. We talked about the construction of the trace map

$$\mathrm{H}_c^{2d}(X, \Lambda(d)) \xrightarrow[\sim]{\mathrm{Tr}} \Lambda$$

where $\Lambda = \underline{\mathbb{Z}/\ell^n\mathbb{Z}}$, $\underline{\mathbb{Z}_\ell}$ or $\underline{\mathbb{Q}_\ell}$.

Today we want to sketch the use of this for proving Poincaré duality, and then introduce the Lefschetz fixed point formula. After that, we should have all the preliminaries we need to start proving the Weil conjectures.

23.1 Poincaré Duality, continued

Theorem 23.1. *Let X be a smooth variety over $k = k^s$ of dimension d. The cup product induces a perfect pairing*

$$\mathrm{H}_c^i(X, \Lambda(d)) \times \mathrm{H}^{2d-i}(X, \Lambda) \xrightarrow{\sim} \mathrm{H}_c^{2d}(X, \Lambda(d)) \xrightarrow[\sim]{\mathrm{Tr}} \Lambda$$

when $\Lambda = \underline{\mathbb{Q}_\ell}$.

What's the idea behind the proof of this? Find a relative version (**Verdier duality**). Let $f : X \to Y$ be a morphism of k-varieties. Then there's a map

$$Rf_! : D^c(X) \to D^c(Y)$$

where $D^c(-)$ is the (bounded) **derived category of constructible sheaves** on $-$, i.e. category of bounded complexes of abelian sheaves[48] on the étale site with constructible cohomology sheaves (and with quasi-isos formally inverted).

The idea now is to construct a right adjoint to $Rf_!$. This will be an $f^! : D^c(Y) \to D^c(X)$. One then computes that if f is smooth of pure dimension d, then $f^!(\mathscr{F}) = \mathscr{F}(d)[2d]$. Finally, adjointness will imply that

$$Rf_* R\underline{\mathrm{Hom}}^{D^c(X)}(\mathscr{F}, f^!\mathscr{G}) \xrightarrow{\sim} R\underline{\mathrm{Hom}}^{D^c(Y)}(Rf_!\mathscr{F}, \mathscr{G}),$$

i.e. we have a canonical equivalence of the above bi-functors.

[48] Use abelian sheaves instead of constructible sheaves e.g. so you can take injective resolutions w/o worry

What does the above have to do with Poincaré duality?

Example. Take X a smooth variety, Y a pt, $\mathscr{F}, \mathscr{G} = \Lambda$ (a constant sheaf, think $\underline{\mathbb{Q}_\ell}$). Then we get an isomorphism

$$Rf_*\Lambda(d)[2d] = Rf_*\underline{\mathrm{Hom}}(\Lambda, \Lambda(d)[2d]) = Rf_*\underline{\mathrm{Hom}}(\Lambda, f^!\Lambda) \xrightarrow{\sim} R\,\mathrm{Hom}(Rf_!\Lambda, \Lambda)$$

Looking at the front end, this complex (object of the derived category) computes $\mathrm{H}^{i+2d}(X, \Lambda(d))$. On the other end, we have a spectral sequence

$$\mathrm{Ext}^i(\mathrm{H}^j_c(X, \Lambda), \Lambda) \implies \mathrm{H}^{i+j}(R\,\mathrm{Hom}(Rf_!\Lambda, \Lambda).$$

If $\Lambda = \underline{\mathbb{Q}_\ell}$, the Ext-stuff vanishes, and we're just left with an isomorphism

$$\mathrm{H}^{-j}_c(X, \Lambda)^\vee \simeq \mathrm{H}^{j-2d}(X, \Lambda(d)),$$

which is just *Poincaré duality.* There's still more to do. This gives a paring which one needs to compare with the cup product, for example.

When $\Lambda = \underline{\mathbb{Z}_\ell}$, the spectral sequence is non-trivial (may actually have Ext-terms) since torsion gets involved. △

What's the modern construction of $f^!$ look like? It's due to Neeman, and the idea is to use Brown representability (not classical Brown, but one of it's more categorical analogues). What is the functor $\mathrm{Hom}(-, f^!\mathscr{G})$? Well, it's an adjoint, so this functor must be $\mathrm{Hom}(Rf_*(-), \mathscr{G})$. One checks that this is representable (via Brown), and then now you need to compute it for f smooth. For this,

(i) Reduce to the case $f : X \to S$ is smooth of relative dimension 1, $\mathscr{F} = \Lambda$ (take a resolution + some dévissage[49]), and $\mathscr{G}$ constructible.

Now it's a non-trivial 'direct computation'. One simply trick is that if s is a point, then can reduce to situation over $\mathbb{C}$.

(ii) (locally) write any smooth map as a composition of relative curves.

There are details left out. Things aren't so straightforward.

Warning 23.2. One subtlety we ignored is that we were working with derived categories of sheaves, but also wanting to allow $\Lambda = \mathbb{Z}_\ell$ or $\Lambda = \mathbb{Q}_\ell$, but these aren't literal sheaves. Sounds like one *can* construct e.g. a derived category of $\mathbb{Z}_\ell$-sheaves in some generality, but that this is hard. One thing that helps here is that, for making sense of our statements (e.g. making since of the use of Ext), one mainly only needs the derived category of $\mathbb{Z}_\ell$-sheaves on a point, and this is easier to make sense of. In particular, these are just (continuous) $\mathbb{Z}_\ell$-modules, so you only need to be able to make sense of continuous $\mathbb{Z}_\ell$-modules. Something like this. The upshot is subtly abounds, but things can be done rigorously with work. •

Let's mention some stuff Poincaré duality gets us.

Corollary 23.3. *Let X be a smooth variety of dimension d over $k = k^s$. Then, $\mathrm{H}^i(X, \mathscr{F}) = 0$ for $i > 2d$ (where we assume $\#\mathscr{F}_{\overline{x}}$ prime to $\mathrm{char}\, k$).*

[49] Whatever that means

Proof. dual to something in negative degree. ■

Remark 23.4. True for arbitrary varieties of dimension d. ○

Non-example. Let X be a curve$/\mathbb{F}_q$. Then, $\mathrm{H}^3_c(X, \Lambda(1)) \neq 0$. ▽

23.2 Lefschetz Fixed Point Formula

Let X be a smooth projection over $k = k^s$ with a map $\varphi : X \to X$.

Theorem 23.5 (**Lefschetz Fixed Point Formula**). *Let $\Gamma_\varphi \subset X \times X$ be the graph of φ, and let $\Delta \subset X \times X$ be the diagonal. Then,*

$$\deg(\Gamma_\varphi \cdot \Delta) = \sum_{i=0}^{2d} (-1)^i \operatorname{Tr}\left(\varphi^* \curvearrowright \mathrm{H}^i(X_{\acute{e}t}, \mathbb{Q}_\ell)\right).$$

Remark 23.6. Above, $\Gamma_\varphi \cdot \Delta$ is the intersection of 2 cycles of codim $\dim X$, on $X \times X$. It's degree $\deg \Gamma_\varphi \cdot \Delta$ is then a number. We have not talked much about intersection theory here, so in this class, we'll think of this as

$$\deg \Gamma_\varphi \cdot \Delta = \operatorname{Tr}\left(\operatorname{cl}(\Gamma_\varphi) \smile \operatorname{cl}(\Delta)\right),$$

the trace of the cup products of their cycle classes. ○

Corollary 23.7. $\deg(\Delta \cdot \Delta) = \chi(\mathrm{H}^i(X, \mathbb{Q}_\ell))$*, the Euler characteristic.*

We'll apply to this to $\varphi = \mathrm{Frob}$ in order to count rational point over a finite field.

23.2.1 Gysin maps

Let $\pi : Y \to X$ be a proper map between smooth varieties X, Y. Let's go ahead and assum efor now that X, Y are also proper. Then we get a dual map

$$\pi_* : \mathrm{H}^r(Y, \Lambda) \to \mathrm{H}^{r-2c}(X, \Lambda(-c)),$$

where $c = \operatorname{reldim}(\pi) = \dim(Y) - \dim(X)$. This map is Poincaré dual to π^*. What are some properties of this?

(1) If $y \in \mathrm{H}^r(Y)$ and $x \in \mathrm{H}^{2\dim Y - r}(x)$, then

$$\operatorname{Tr}_X(\pi_*(x) \smile y) = \operatorname{Tr}_Y(X \smile \pi^*(y))$$

(this is what is means to be a dual map)

(2) If π is a closed immersion, then $\pi_*(1) = \operatorname{cl}(Y)$ (unwind definitions)

(3) $(\pi_1 \circ \pi_2)_* = (\pi_1)_* \circ (\pi_2)_*$

(4) (**projection formula**)

$$\pi_*(y \smile \pi^*(x)) = \pi_*(y) \smile x.$$

(5) If π finite of degree d, then

$$\pi_* \circ \pi^* = d \operatorname{Id}.$$

Most of these follow from **(1)**. **(5)** is trickier.

In general (X, Y not proper), get a map

$$\pi_* : \mathrm{H}_c^r(Y, \Lambda) \to \mathrm{H}_c^{r-2c}(X, \Lambda(-c))$$

using cohomology with compact supports. Similarly, if X is smooth but not proper, get Lefschetz formula w/ compactly supported cohomology.

23.2.2 Example

Example. Say $X = \mathbb{P}^1_{k=k^s}$ (and $\operatorname{char} k = 0$ to keep things simple). Consider $\varphi : x \mapsto x^n$. How many fixed points does this have? There are $n+1$ of them, $\{0, \infty\} \cup \mu_{n-1}$. These all have multiplicity one since $x^n - x = 0$ is separable (similarly check at ∞). Hence, $\#\Gamma_\varphi \cdot \Delta = n+1$. At the same time,

$$\sum_{i \geq 0} (-1)^i \operatorname{Tr}(\varphi | \mathrm{H}^i(\mathbb{P}^1, \mathbb{Q}_\ell)) = \operatorname{Tr}(\varphi^* | \mathrm{H}^0(\mathbb{P}^1, \mathbb{Q}_\ell)) + \operatorname{Tr}(\varphi^* | \mathrm{H}^2(\mathbb{P}^1, \mathbb{Q}_\ell)) = 1 + n.$$

On H^0, $\varphi^* = \operatorname{Id}$ (H^0 is sections to the constant $\mathbb{Q}_\ell$ sheaf, so connected components of $\mathbb{P}^1$). What's going on on H^2? We're on a curve, so recall

$$\mathrm{H}^2(\mathbb{P}^1, \mu_{\ell^n}) = \operatorname{coker}\left(\operatorname{Pic} \mathbb{P}^1 \xrightarrow{\mathscr{L} \mapsto \mathscr{L}^{\ell^n}} \operatorname{Pic} \mathbb{P}^1\right),$$

so (over $k = k^s$ so $\mu_{\ell^n} = \underline{\mathbb{Z}/\ell^n\mathbb{Z}}$)

$$\mathrm{H}^2(\mathbb{P}^1, \mathbb{Q}_\ell) = \left(\varprojlim_n \operatorname{coker}\left(\operatorname{Pic} \mathbb{P}^1 \xrightarrow{[\ell^n]} \operatorname{Pic} \mathbb{P}^1\right)\right) \otimes \mathbb{Q}_\ell.$$

What does φ do to line bundles? It acts by multiplication by $[n]$ on $\operatorname{Pic} \mathbb{P}^1$, and so acts the same way on $\mathrm{H}^2(\mathbb{P}^1, \mathbb{Q}_\ell)$. Hence, it's trace is n (keep in mind $\dim \mathrm{H}^i = 1$ for $i = 0, 2$). $\triangle$

Think about transition functions

23.2.3 Proof

Let's sketch a proof of Lefschetz. We start with a lemma.

Lemma 23.8. *Consider $\varphi : X \to Y$, and suppose we're given $y \in \mathrm{H}^*(Y, \mathbb{Q}_\ell)$. Fix (for all time), an iso $\mathbb{Q}_\ell \xrightarrow{\sim} \mathbb{Q}_\ell(1)$ (so we can ignore twists showing up with cycle classes). Then,*

$$\varphi^*(y) = p_*\left(\operatorname{cl}(\Gamma_\varphi) \smile q^* y\right),$$

where we have projections $X \xleftarrow{p} X \times Y \xrightarrow{q} Y$.

Remark 23.9. To see what happens to y, take the 'preimage', 'intersect' with the graph, and then 'push forward'. $\circ$

Proof. Exercise (and/or look at Milne). The content is the definition of p_* + projection formula. ■

Lemma 23.10. *Suppose e_i^r is a basis of $\mathrm{H}^r(X, \mathbb{Q}_\ell)$, one basis for each r. Let f_i^{2d-r} be the dual basis of $\mathrm{H}^{2d-r}(X, \mathbb{Q}_\ell(d))$. Then,*

$$\mathrm{cl}_{X\times X}(\Gamma_\varphi) = \sum \varphi^*(e_i^r) \otimes f_i^{2d-r},$$

under the Künneth isomorphism

$$\mathrm{H}^*(X \times X, \mathbb{Q}_\ell(d)) \simeq \mathrm{H}^*(X, \mathbb{Q}_\ell) \otimes \mathrm{H}^*(X, \mathbb{Q}_\ell(d)).$$

We'll prove this next time (you can also try it as an exercise if you want).

24 Lecture 24: Last Day of "Foundational" Material

Last time we started talking about the Lefschetz fixed point formula. Let X be a smooth projective variety[50]$/k = k^s$, and let $\varphi : X \to X$ be some map. Lefschetz tells us that

$$\Gamma_\varphi \cdot \Delta = \sum_{i=0}^{2\dim X} (-1)^i \operatorname{Tr}(\varphi | \mathrm{H}^i(X, \mathbb{Q}_\ell)).$$

Example. Say $X = \bigsqcup_{i=0}^n \operatorname{Spec} k$ is a bunch of points, and we have $\varphi : X \to X$. Then,

$$\#\text{fixed points of } \varphi = \operatorname{Tr}(\varphi | \mathbb{Q}_\ell^X).$$

In this case, φ acts by a permutation (matrix) on the finite set X. △

Exercise. Say $f : \mathbb{P}^1 \to \mathbb{P}^1$ is a degree d map (so it'll have $d+1$ fixed points). Check that Lefschetz holds in this case.

24.1 Proof of Lefschetz

We started the preparations for the proof last time. In particular, we stated the following lemmas (X, Y smooth, proper below).

Lemma 24.1. *Consider $\varphi : X \to Y$, and suppose we're given $y \in \mathrm{H}^*(Y, \mathbb{Q}_\ell)$. Fix (for all time), an iso $\mathbb{Q}_\ell \xrightarrow{\sim} \mathbb{Q}_\ell(1)$ (so we can ignore twists showing up with cycle classes). Then,*

$$\varphi^*(y) = p_*\left(\mathrm{cl}(\Gamma_\varphi) \smile q^* y\right),$$

where we have projections $X \xleftarrow{p} X \times Y \xrightarrow{q} Y$.

Exercise. Prove this.

Lemma 24.2. *Suppose e_i^r is a basis of $\mathrm{H}^r(X, \mathbb{Q}_\ell)$, one basis for each r. Let f_i^{2d-r} be the dual basis of $\mathrm{H}^{2d-r}(X, \mathbb{Q}_\ell(d))$. Then,*

$$\mathrm{cl}_{X\times X}(\Gamma_\varphi) = \sum \varphi^*(e_i^r) \otimes f_i^{2d-r},$$

[50] Can drop 'smooth' and 'projective' with extra work

under the Künneth isomorphism

$$\mathrm{H}^*(X \times X, \mathbb{Q}_\ell(d)) \simeq \mathrm{H}^*(X, \mathbb{Q}_\ell) \otimes \mathrm{H}^*(X, \mathbb{Q}_\ell(d)).$$

Example. If $\varphi = \mathrm{id}$, then $\mathrm{cl}(\Gamma_\varphi) = \mathrm{cl}(\Delta) = \sum e_i^r \otimes f_i^{2\dim X - r}$. △

Proof. Write $\mathrm{cl}(\Gamma_\varphi) = \sum a_i \otimes f_i$ for unique $a_i \in \mathrm{H}^*(X, \mathbb{Q}_\ell)$, which is possible since $\mathrm{H}^*(X \times X, \mathbb{Q}_\ell)$ is free as a (right) $\mathrm{H}^*(X, \mathbb{Q}_\ell)$-module.

Goal. Compute a_i's to show $a_i = \varphi^*(e_i)$.

Observe (recall e_i, f_j are dual bases)

$$\begin{aligned}
\varphi^*(e_i) = p_*(\mathrm{cl}(\Gamma_\varphi) \smile q^* e_i) &= p_*\left(\sum_j (a_j \otimes f_j) \smile q^* e_i\right) \\
&= p_*\left(\sum_j ((a_j \otimes f_j) \smile (1 \otimes e_i))\right) \\
&= p_*\left(\sum_j a_j \otimes (f_j \smile e_i)\right) \\
&= p_*\left(\sum_j a_j \otimes \delta_{ij} e_1^{2\dim X}\right) \\
&= p_*(a_i \otimes e_1^{2\dim X}) \\
&= a_i.
\end{aligned}$$

This completes the proof. ■

Proof Sketch of Theorem 23.5. (Assume that the cycle class map sends the intersection product to the cup product.)

We know that (last equality holds up to sign, cup product graded commutative)

$$\mathrm{cl}(\Gamma_\varphi) = \sum \varphi^* e_i \otimes f_i \ \text{ and } \ \mathrm{cl}(\Delta) = \sum e_i \otimes f_i = \sum f_i \otimes e_i.$$

Hence,

$$\mathrm{cl}(\Gamma_\varphi \cdot \Delta) = \mathrm{cl}(\Gamma_\varphi) \smile \mathrm{cl}(\Delta) = \sum (\varphi^* e_i) f_i \otimes e_1^{2\dim X}.$$

We want Tr(the above) $= \mathrm{Tr}(\varphi | \mathrm{H}^*)$, up to sign (we're not keeping track of signs in this argument). This is just linearly algebra

Fact. If e_i is a basis of V, with dual basis $e_i^\vee$, then

$$\mathrm{Tr}(A) = \sum_i e_i^\vee (A e_i).$$

■

Question 24.3. *What is $\Gamma_\varphi \cdot \Delta$?*

Claim 24.4. *$\Gamma_\varphi \cdot \Delta$ is simple the number of fixed points of φ if they each have multiplicity 1.*

If $Y, Z \subset X$ are subvarieties of complimentary dimenion (and X smooth), then $(Y \cdot Z)_p = 1$ if

- *Y, Z smooth at p*
- *$T_pY \cap T_pZ = 0$.*

Lemma 24.5. *This is satisfies for Γ_φ, Δ if X smooth, and 1 is not an eigenvalue of φ-action on T_pX for p any fixed point of φ.*

Proof left as exercise.

Example. Hypotheses of lemma are true for $X/\mathbb{F}_p$ with $\varphi : X \to X$ the absolute Frobenius map. △

24.2 Frobenius maps

There are a few different Frobenii, so let's set straight which is which.

Let X be a variety over $\mathbb{F}_q$.

- (**absolute Frobenius**) This is the $\mathbb{F}_q$-morphism $\mathrm{Frob}^q : X \to X$ sending $f^q \mapsfrom f$ on sheaves. This is a natural endomorphism of the identity functor. To apply Lefschetz, we need to be over an algebraically closed field, so absolute Frobenius won't do.
- (**relative Frobenius**) Consider the Cartesian diagram

$$\begin{tikzcd} X'_{\overline{\mathbb{F}}_q} \arrow[r] \arrow[d] & X_{\overline{\mathbb{F}}_q} \arrow[d] \\ \operatorname{Spec} \overline{\mathbb{F}}_q \arrow[r, "\mathrm{Frob}^q"] & \operatorname{Spec} \overline{\mathbb{F}}_q. \end{tikzcd}$$

  Since X is defined over $\mathbb{F}_q$, there is a canonical isomorphism $X_{\overline{\mathbb{F}}_q} \simeq X'_{\overline{\mathbb{F}}_q}$.

  Example (Affine setting)**.** Say $X = \operatorname{Spec} \mathbb{F}_q[x_1, \ldots, x_N]/I$ so $X_{\overline{\mathbb{F}}_q} = \operatorname{Spec} \overline{\mathbb{F}}_q[x_1, \ldots, x_N]/I$. Then, $X'_{\overline{\mathbb{F}}_q} = \operatorname{Spec} \overline{\mathbb{F}}_q \otimes_{\overline{\mathbb{F}}_q} \overline{\mathbb{F}}_q[x_1, \ldots, x_N]/I$ with the left $\overline{\mathbb{F}}_q$ acted on through Frobeinus. We have an isomorphism

$$\overline{\mathbb{F}}_q \otimes_{\overline{\mathbb{F}}_q} \overline{\mathbb{F}}_q[x_1, \ldots, x_N]/I \xrightarrow{\sim} \overline{\mathbb{F}}_q[x_1, \ldots, x_N]/I$$

  coming from multiplication, $a \otimes f(x_1, \ldots, x_N) \mapsto a^{1/q} f(x_1, \ldots, x_N)$. △

  Back to the case at hand, *relative Frobenius* is the map $F^q_{X_{\overline{\mathbb{F}}_q}/\overline{\mathbb{F}}_q}$ below

$$\begin{tikzcd} X_{\overline{\mathbb{F}}_q} \arrow[dr, "F^q_{X_{\overline{\mathbb{F}}_q}/\overline{\mathbb{F}}_q}"] \arrow[drr, bend left, "\mathrm{Frob}^q"] \arrow[ddr, bend right] & & \\ & X_{\overline{\mathbb{F}}_q} \arrow[r] \arrow[d] \arrow[dr, phantom, "\ulcorner"] & X_{\overline{\mathbb{F}}_q} \arrow[d] \\ & \operatorname{Spec} \overline{\mathbb{F}}_q \arrow[r, "\mathrm{Frob}^q"] & \operatorname{Spec} \overline{\mathbb{F}}_q \end{tikzcd}$$

Claim 24.6. *The fixed points of $F^q := F^q_{X_{\overline{\mathbb{F}}_q}/\overline{\mathbb{F}}_q}$ are precisely the $\mathbb{F}_q$-rational points of X. Furthermore, all fixed points have multiplicity one.*

Proof. First part is not difficult. For the second part, suffices to show F^q induces the zero map on tangent spaces. Consider

$$\operatorname{Spec} \mathbb{F}_q[t]/(t^2) \to X \xrightarrow{F^q} X.$$

We're taking qth power, but $q \geq 2$, so it kills t. ■

24.3 Weil Conjectures

What were these again? Say X a variety over $\mathbb{F}_q$.

Recall 24.7. The Zeta function of X is

$$\zeta_X(t) = \exp\left(\sum_{i\geq 1} \frac{\#X(\mathbb{F}_{q^i})}{i} t^i\right).$$

⊙

We would first like to show that this is a rational function of t. We need Lefschetz for this, so we'll stick to the case of X smooth projective. Later we'll discuss a fancier version of Lefschetz allowing a proof for more general X.

Proof of Rationality of Zeta Function for X Smooth, Projective. We define

$$\operatorname{Tr}(F^q | \operatorname{H}^*(X)) = \sum_{i=0}^{2\dim X} (-1)^i \operatorname{Tr}(F^{q,*} | \operatorname{H}^i(X_{\overline{\mathbb{F}}_q}, \mathbb{Q}_\ell)).$$

We know from Lefschetz that $\#X(\mathbb{F}_{q^i}) = \operatorname{Tr}(F^{q^i} | \operatorname{H}^*(X)) = \operatorname{Tr}((F^q)^i | \operatorname{H}^*(X))$. Hence, the zeta function is

$$\zeta_X(t) = \exp\left(\sum_{n\geq 0} \frac{\operatorname{Tr}((F^q)^n | \operatorname{H}^*(X))}{n} t^n\right).$$

To show this is rational, since Lefschetz involves only finitely many terms, it'll suffice to show that

$$\zeta_{X,r} := \exp\left(\sum_{n\geq 1} \frac{\operatorname{Tr}((F^q)^n | \operatorname{H}^r(X_{\overline{\mathbb{F}}_q}, \mathbb{Q}_\ell) t^n}{n}\right)$$

is a rational function.[51] WLOG, we can write ($\lambda_i \in \overline{\mathbb{Q}}_\ell$)

$$\operatorname{Tr}((F_q)^n | \operatorname{H}^r(X_{\overline{\mathbb{F}}_q}, \mathbb{Q}_\ell)) = \sum \lambda_i^n$$

[51] $\zeta_X(t)$ is an alternating product of these guys

(keep in mind that these cohomology groups are f.dim vector spaces). Thus,

$$\zeta_{X,r} = \exp\left(\sum_{n\geq 1}\sum_{i=1}^{\dim \mathrm{H}^r}\frac{\lambda_i^n t^n}{n}\right) = \exp\left(\sum_{i=1}^{\dim \mathrm{H}^r} -\log(1-\lambda_i t)\right) = \prod \frac{1}{(1-\lambda_i t)} \in \mathbb{Q}_\ell(t)$$

is indeed a rational function. ■

We can strengthen things. We claim that ζ_X is in fact rational with integer coefficients. This is e.g. because

$$\mathrm{d}\log\zeta_X(t) = \sum \#X(\mathbb{F}_{q^n})t^{n-1}$$

has integer coefficients.

We won't go over the functional equation in detail. The point is to plug in Poincaré duality: $\mathrm{H}^i(X,\mathbb{Q}_\ell)$ is dual to $\mathrm{H}^{2\dim X - i}(X,\mathbb{Q}_\ell(d))$, so there is a relationship between $\zeta_{X,r}(t)$ and $\zeta_{X,2\dim X-r}(q^d t)$. Figuring out what this relationship exactly is pops out the functional equation.

24.3.1 Riemann Hypothesis

The remainder of the class will be on this. Let's begin by recalling the statement.

Theorem 24.8. *Say X smooth projective. The Frobenius action $F^{q,*} \curvearrowright \mathrm{H}^*(X_{\overline{\mathbb{F}}_q},\mathbb{Q}_\ell)$ has eigenvalues $\lambda_1,\dots,\lambda_n \in \overline{\mathbb{Q}}$, and for any embedding $\mathbb{Q}(\lambda_1,\dots,\lambda_n)\hookrightarrow\mathbb{C}$, their absolute values are all equal to $q^{r/2}$.*

We already know these eigenvalues are algebraic numbers in many cases, e.g. for (smooth, projective) hypersurfaces.

Example. Say $X\subset\mathbb{P}^n$ is a smooth projective hypersurface. Then,

$$\mathrm{H}^*(X,\mathbb{Q}_\ell) = \mathrm{H}^*(\mathbb{P}^n,\mathbb{Q}_\ell) \text{ for } * < \dim X$$

(because $\mathbb{P}^n\setminus X$ is affine + excision + Poincaré duality). Thus, $\mathrm{H}^*(X,\mathbb{Q}_\ell)$ is a twist of $\mathbb{Q}_\ell$ unless $* = \dim X$. We know (from the proof of rationality) that the zeros/poles of ζ_X are algebraic numbers, and now we know the only thing that can contribute interesting eigenvalues is the middle cohomology, so all those eigenvalues must be algebraic (there can be no "cancellation"). △

Next time we start on Deligne's proof of the Riemann hypothesis.

25 Lecture 25

Last time we discussed the rationality of the zeta function, at least for smooth projective varieties. We left as an exercise showing that PD implies the functional equation. Today, we talk about the Riemann hypothesis.

25.1 More Frobenii

There are like 4 of these in total. We talked about a couple before. Let's finish off the list.

Say X_0 is some variety over $\mathbb{F}_q$, and let $X = (X_0)_{\overline{\mathbb{F}}_q}$ be its basechange to $\overline{\mathbb{F}}_q$.

- (**absolute Frobenius**) $\mathrm{Frob}_{abs} : X \to X$ acting via $f^q \hookleftarrow f$ on functions/sheaves. This is *not* a morphism over $\overline{\mathbb{F}}_q$ (it induces the q-power map on $\mathbb{F}_q$, which is not the identity), but is one over $\mathbb{F}_q$.

  Question 25.1. *This induces a map*

  $$\mathrm{Frob}_{abs}^* : \mathrm{H}^*(X, \mathbb{Q}_\ell) \to \mathrm{H}^*(X, \mathrm{Frob}_{abs}^* \mathbb{Q}_\ell) = \mathrm{H}^*(X, \mathbb{Q}_\ell).$$

  What map is this?

  Answer. This is the identity map. Why? The induced map of sites $\mathrm{Frob}_{abs} : X_{\text{ét}} \to X_{\text{ét}}$ is naturally isomorphic to the identity. As a functor, this sends $U \xrightarrow[\pi]{\text{ét}} X$ to the fiber product $U \times_{X,\mathrm{Frob}} X \to X$. We need to show this fiber product is canonically isomorphic to U over X. Well, take as our isomorphism relative Frobenius

  $$F_{U/X} = (\mathrm{Frob}_{abs,U}, \pi) : U \to U \times_{X,\mathrm{Frob}} X.$$

  Exercise. $F_{U/X}$ above is an isomorphism. Use that $\pi : U \to X$ is étale.

  $\star$

  Hence, absolute Frobenius is probably not what we want to tackle the Weil conjectures.

- (**relative Frobenius**) This is defined via the diagram.

  $$\begin{tikzcd}
X_{\overline{\mathbb{F}}_q} \arrow[dr, "F^q_{X_{\overline{\mathbb{F}}_q}/\overline{\mathbb{F}}_q}"] \arrow[drr, bend left, "\mathrm{Frob}^q_{abs}"] \arrow[ddr, bend right] & & \\
& X_{\overline{\mathbb{F}}_q} \arrow[r] \arrow[d] \arrow[dr, phantom, "\ulcorner", very near start] & X_{\overline{\mathbb{F}}_q} \arrow[d] \\
& \mathrm{Spec}\,\overline{\mathbb{F}}_q \arrow[r, "\mathrm{Frob}^q_{abs}"] & \mathrm{Spec}\,\overline{\mathbb{F}}_q
\end{tikzcd}$$

  Here's a more concrete description: recall $X_0/\mathbb{F}_q$ with it's absolute Frobenius $\mathrm{Frob}^q_{abs,X_0} : X_0 \to X_0$ over $\mathbb{F}_q$. Relative frobenius is just the base change $F_{X/\overline{\mathbb{F}}_q} = \mathrm{Frob}^q_{abs,X_0} \times \overline{\mathbb{F}}_q$.

  Example. Say $X = \mathbb{A}^n_{\overline{\mathbb{F}}_q} = \mathrm{Spec}\,\overline{\mathbb{F}}_q[t_1, \dots, t_n]$. Then

  $$\mathrm{Frob}_{abs} : f \mapsto f^p \text{ and } F_{\mathbb{A}^n/\overline{\mathbb{F}}_q} : t_i \mapsto t_i^p.$$

  $\triangle$

- (**Arithmetic,geometric Frobenii**) Instead of acting on the t_i, we can act only on the coefficients. Consider $F_k := \mathrm{Frob}_{abs} : \overline{\mathbb{F}}_q \to \overline{\mathbb{F}}_q$. We get two more Frobenii (recall $X = \overline{\mathbb{F}}_q \times_{\mathbb{F}_q} X_0$)

  $$F_k \times \mathrm{id}_{X_0} : X \to X \text{ and } F_k^{-1} \times \mathrm{id}_{X_0} : X \to X.$$

  The first (**arithmetic Frobenius**) raises coefficients to pth power, and the second (**geometric Frobenius**) takes pth roots of the coefficients.

Remark 25.2. $\mathrm{Frob}_{abs,X} = (F_k \times \mathrm{id}_X) \circ F_{X/k}$, absolute Frobenius is the composition of arithmetic Frobenius and relative Frobenius. Hence, these two Frobenii act as inverses to each other on ℓ-adic cohomology $\mathrm{H}^*(X, \mathbb{Q}_\ell)$. ○

Assumption. From now on, 'Frobenius' will always mean relative Frobenius $F_{X/k}$.

25.2 Riemann Hypothesis

What's our goal now?

Theorem 25.3 (Riemann Hypothesis). *Let X_0 be a smooth projective variety/$\mathbb{F}_q$, and let $X = X_{0,\overline{\mathbb{F}}_q}$. Then, the eigenvalues of $F_{X/k}^* \curvearrowright \mathrm{H}^i(X, \mathbb{Q}_\ell)$ are algebraic integers, and for any embedding $\mathbb{Q}$(eigenvalues $\alpha_j) \hookrightarrow \mathbb{C}$, one has $|\alpha_j| = q^{i/2}$.*

This statement is false (slash needs to be modified) is you replace relative Frobenius with one of the others.

We know the Riemann Hypothesis already in some cases.

Example (Hartshorne chpt. IV)**.** We know this for curves. △

Example. If $\mathrm{H}^{2i}(X, \mathbb{Q}_\ell(i))$ is spanned by cycle classes, then we know RH for $\mathrm{H}^{2i}(X, \mathbb{Q}_\ell)$.

Proof. Pick a basis of $\mathrm{H}^{2i}(X, \mathbb{Q}_\ell(i))$ consisting of cycle classes. Can extend base field to assume WLOG that all cycles defined over $\mathbb{F}_q$. Then,

$$CH^i(X)_{\mathbb{Q}_\ell} \twoheadrightarrow \mathrm{H}^{2i}(X, \mathbb{Q}_\ell(i))$$

is $F_{X/k}$-equivariant, so it acts on $\mathrm{H}^{2i}(X, \mathbb{Q}_\ell(i))$ trivially. Hence, it acts on $\mathrm{H}^{2i}(X, \mathbb{Q}_\ell) = \mathrm{H}^{2i}(X, \mathbb{Q}_\ell(i))(-i)$ via the $(-i)$th power of the cyclotomic character χ_{cyc}. Now all the eigenvalues are simply $q^i = q^{2i/2}$. ■

△

Example. If S is a cubic surface, then H^* is spanned by cycle classes. △

Remark 25.4. It really is special that the absolute values of the eigenvalues are the same independent of the embedding into $\mathbb{C}$. For example, $\beta = 1 + \sqrt{2}$ does note satisfy this, $\left|1 + \sqrt{2}\right| \neq \left|1 - \sqrt{2}\right|$. ○

Definition 25.5. If $|\alpha|$ is independent of the embedding, and $|\alpha| = q^{i/2}$, then we say α is a **q-Weil number of weight i**. ◇

Let's get to the proof. We will closely follow Deligne's original argument. First some reductions.

(i) We can replace $\mathbb{F}_q$ with $\mathbb{F}_{q^n}$. This has the effect of replacing $F_{X/k}$ with $F_{X/k}^n$ and so raising the eigenvalues α_j to the nth power. Since α_j satisfies the desired properties iff α_j^n does (replacing q w/ q^n), this is A-ok.

(ii) Enough to show a much weaker statement

Theorem 25.6. *Say X smooth and projective of even dimension $2d$, and let α be an eigenvalue of $F_{X/k}$ on $\mathrm{H}^d(X, \mathbb{Q}_\ell)$. Then,*

$$q^{\frac{d}{2}-\frac{1}{2}} < |\alpha| < q^{\frac{d}{2}+\frac{1}{2}}$$

for all embeddings $\mathbb{Q}(\alpha) \hookrightarrow \mathbb{C}$.

Proof that Theorem $\implies$ RH. The idea is to take products ("**Tensor product trick**").

Apparently, Tao has a blog post giving more examples of this sort of trick

Let Y be any smooth projective variety of dimension n (possibly not even). Consider Y^{2M}, a variety of dimension $2nM$. By Künneth, it's middle cohomology is

$$\mathrm{H}^{2nM}(Y, \mathbb{Q}_\ell) = \bigoplus_{\substack{i_1,\dots,i_{2M} \\ \sum i_j = 2nM}} \bigotimes_j \mathrm{H}^{i_j}(Y, \mathbb{Q}_\ell).$$

Taking all $i_j = n$, for example, we see that

$$\mathrm{H}^n(Y, \mathbb{Q}_\ell)^{\otimes 2M} \le \mathrm{H}^{2nM}(Y^{2M}, \mathbb{Q}_\ell).$$

Now the theorem tells us that for any eigenvalue α of $F_{X/k} \curvearrowright \mathrm{H}^n(Y, \mathbb{Q}_\ell)$, we have

$$q^{nM-\frac{1}{2}} < |\alpha|^{2M} < q^{nM+\frac{1}{2}}.$$

Now we take $(2M)$th roots to see that

$$q^{\frac{n}{2}-\frac{1}{4m}} < |\alpha| < q^{\frac{n}{2}+\frac{1}{4M}}.$$

Taking the limit as $M \to \infty$ gives $|\alpha| = q^{n/2}$.

This proves RH for middle cohomology of all varieties. We need it in other degrees. Poincaré duality means it's enough to do it for $\mathrm{H}^r(Y, \mathbb{Q}_\ell)$ for $r > \dim Y$. For this, we note that

$$\mathrm{H}^r(Y, \mathbb{Q}_\ell)^{\otimes A} \otimes \mathrm{H}^0(Y, \mathbb{Q}_\ell)^{\otimes B} \le \mathrm{H}^{Ar+B}(Y^{A+B}, \mathbb{Q}_\ell).$$

Choosing A, B appropriately can arrange that this is in middle cohomology. Since Frobenius acts trivially on H^0, we can repeat the same sort of argument (fill in details as exercise). ■

In order to proceed from here, we'll need a generalization of Lefschetz fixed point formula for

- Non-proper varieties
- Non-constant sheaves
- Sheaves of modules over $\mathbb{Z}/\ell^n\mathbb{Z}$.

Warning 25.7. The cohomology of modules over this ring is not free, so issues can arise in taking a trace. •

25.3 Lefschetz for...

25.3.1 Non-proper varieties

Suppose $U \overset{j}{\hookrightarrow} X \overset{i}{\hookleftarrow} Z = X \setminus U$ with X smooth proper variety and U an open subscheme. Recall the exact sequence

$$0 \longrightarrow j_! \underline{\mathbb{Q}_\ell} \longrightarrow \underline{\mathbb{Q}_\ell} \longrightarrow i_* \underline{\mathbb{Q}_\ell} \longrightarrow 0.$$

This gave rise to the Gysin sequence. Say we have an endomorphism $\varphi : X \to X$ s.t. $\varphi(U) \subset U$ and $\varphi(Z) = Z$. Then (consider LES in cohomology coming from above short exact sequence),

$$\sum (-1)^r \operatorname{Tr}(\varphi^* | \operatorname{H}^r(X, \mathbb{Q}_\ell)) = \sum (-1)^r \operatorname{Tr}(\varphi^* | \operatorname{H}^r_c(U, \mathbb{Q}_\ell)) + \sum (-1)^r \operatorname{Tr}(\varphi^* | \operatorname{H}^*(Z, \mathbb{Q}_\ell)).$$

If Z is smooth proper, can apply Lefschetz to X, Z in order to get that

$$\sum (-1)^r \operatorname{Tr}(\varphi^* | \operatorname{H}^r_c(U, \mathbb{Q}_\ell)) = \# X^\varphi - \# Z^\varphi$$

where these numbers of fixed points are counted with multiplicity.

Question 25.8. $\# X^\varphi - \# Z^\varphi \overset{?}{=} \# U^\varphi$

No, or we wouldn't have asked.

Example. Say $X = \mathbb{P}^1$, $U = \mathbb{A}^1$, and $Z = \infty$. Consider

$$\varphi : [x_0 : x_1] \mapsto [x_0 + x_1 : x_1].$$

Then, $\# U^\varphi = 0$ (this is the map $z \mapsto z + 1$) and $\# Z^\varphi = 1$. However, counted with multiplicity, $\#(\mathbb{P}^1)^\varphi = 2$ (You want $x_0 x_1 = x_1(x_0 + x_1)$, i.e. $x_1^2 = 0$). Note that $2 - 1 \neq 0$.

Also note that φ is homotopic to the identity (consider $[x_0 : x_1] \mapsto [x_0 + t x_1 : x_1]$) so it induces the same action of cohomology, so Lefschetz tells us that $\#(\mathbb{P}^1)^\varphi$ is the Euler characteristic 2. △

The issue is that computing multiplicity does not play nicely with restriction to a (closed) subscheme in general. Hence, we get a Lefschetz fixed point formula $\iff \# U^\varphi + \# Z^\varphi = \# X^\varphi \iff \forall x \in Z^\varphi : \operatorname{mult}_p(x, Z) = \operatorname{mult}_\varphi(x, X)$.

Example. If $\varphi =$Frobenius, this holds since all fixed points have multiplicity 1. △

Corollary 25.9. *Say U is smooth with smooth compactification and smooth complement, then*

$$\# U(\mathbb{F}_{q^n}) = \sum (-1)^r \operatorname{Tr}(F^r_{U/k} | \operatorname{H}^r_c(U, \mathbb{Q}_\ell)).$$

These hypotheses are almost never satisfied, but that's ok, because we can reduce.

Corollary 25.10. *For U_0 a variety over $\mathbb{F}_q$. Then,*

$$\# U_0(\mathbb{F}_{q^m}) = \sum (-1)^r \operatorname{Tr}(F^m_{U/k} | \operatorname{H}^r_c(U, \mathbb{Q}_\ell)).$$

To show this, one does some annoying dévissage induction on dimension. We know it for points. The previous corollary gives it from curves. For surfaces, *if* we had resolution of singularities, we could find a smooth compactification with singular complement built out of curves, and then induct. We don't have resolution of singularities, so this doesn't quite work, but something like it does.

25.3.2 Non-constant sheaves

Let $\mathscr{E}$ be a lisse $\mathbb{Q}_\ell$-sheaf on X. Consider $\varphi : X \to X$. This gives rise to

$$\mathrm{H}^*(X, \mathscr{E}) \to \mathrm{H}^*(X, \varphi^*\mathscr{E}).$$

In order to take traces, we need additionally a map $\varphi_{\mathscr{E}} : \varphi^*\mathscr{E} \to \mathscr{E}$. We then get

$$(\varphi^*, \varphi_{\mathscr{E}})^* : \mathrm{H}^*(X, \mathscr{E}) \xrightarrow{\varphi^*} \mathrm{H}^*(X, \varphi^*\mathscr{E}) \xrightarrow{\varphi_{\mathscr{E}}} \mathrm{H}^*(X, \mathscr{E}).$$

If $x \in X^\varphi$ is a fixed point, can look at stalks:

$$\mathscr{E}_x = \mathscr{E}_{\varphi(x)} = (\varphi^*\mathscr{E})_x \xrightarrow{\varphi_{\mathscr{E}}} \mathscr{E}_x.$$

Question 25.11.

$$\sum_{x \in X^\varphi} \mathrm{Tr}(\varphi_x | \mathscr{E}_x) \stackrel{?}{=} \sum_r (-1)^r \,\mathrm{Tr}((\varphi, \varphi_{\mathscr{E}}) | \,\mathrm{H}^r_c(X, \mathscr{E}))$$

Example. Say X a finite set. We have $\varphi : X \to X$, $\mathbb{Q}_\ell$-vector spaces $\mathscr{E}_x$, and maps $\varphi_{\mathscr{E}} : \mathscr{E}_{\varphi(x)} \xrightarrow{\varphi_x} \mathscr{E}_x$ for all x. Then this says

$$\sum_{x \in X^\varphi} \mathrm{Tr}(\varphi_x) = \mathrm{Tr}\left(\bigoplus \varphi_x : \bigoplus \mathscr{E}_x \to \bigoplus \mathscr{E}_x\right),$$

which is true since this just computing the trace of a block-diagonal matrix. △

26 Lecture 26

Recall 26.1. We're trying to prove the Riemann hypothesis[52]. ⊙

Last time

- Discussed Frobenii
- preliminary reductions

  e.g. to an inequality for the absolute values of eigenvalues of middle degree cohomology on an even dimensional variety
- Lefschetz trace formula. Still a few issues to overcome
  - non-proper varieties
  - non-constant sheaves (e.g. local systems)
  - sheaves over torsion rings (e.g. $\mathbb{Z}/\ell^n\mathbb{Z}$)

[52]for smooth varieties over a finite field

26.1 Non-constant sheaves

For a non-constant sheaf $\mathscr{E}$ on $X_{\text{ét}}$, one gets

$$\varphi^* : \mathrm{H}^i(X_{\text{ét}}, \mathscr{E}) \to \mathrm{H}^i(X_{\text{ét}}, \varphi^*\mathscr{E}).$$

To be able to trace traces, we need a map $\mathrm{H}^i(X_{\text{ét}}, \varphi^*\mathscr{E}) \to \mathrm{H}^i(X_{\text{ét}}, \mathscr{E})$, so say we also are given $\varphi_{\mathscr{E}} : \varphi^*\mathscr{E} \to \mathscr{E}$. Note that this $\varphi_{\mathscr{E}}$ also gives us maps

$$\begin{tikzcd} \mathscr{E}_{\varphi(x)} \arrow[r, "="'] \arrow[rr, bend left, "\varphi_{\mathscr{E},*}"] & \varphi^*(\mathscr{E})_x \arrow[r, "(\varphi_{\mathscr{E}})_*"'] & \mathscr{E} \end{tikzcd} .$$

If $x = \varphi(x)$ is a fixed point, this is $\varphi_{\mathscr{E},x} : \mathscr{E}_x \to \mathscr{E}_x$.

Goal. If X_0 variety over $\mathbb{F}_q$ with $X = (X_0)_{\overline{\mathbb{F}}_q}$. We want a theorem like

$$\sum_{x \in X^\varphi} \mathrm{Tr}(\varphi_* \mid \mathscr{E}_x) = \sum_{i \geq 0} (-1)^i \,\mathrm{Tr}\left((\varphi, \varphi_{\mathscr{E}})^* \mid \mathrm{H}^i_c(X_{\text{ét}}, \mathscr{E})\right).$$

Note the LHS is more 'arithmetic' while the RHS is more 'geometric'.

Exercise. Say X is a finite set. Work out what the goal says in this case.

We'll want to apply this when φ is Frobenius. Where does this $\varphi_{\mathscr{E}}$ come from?

Question 26.2. *When is there a natural map $F_{\mathscr{E}} : F^*\mathscr{E} \to \mathscr{E}$ (F = relative Frobenius).*

Answer. If $\mathscr{E}$ comes from a sheaf on $X_{\text{ét}}$.

Recall 26.3. $F_{abs} : X_{\text{ét}} \to X_{\text{ét}}$ is naturally isomorphic to the identity. $\odot$

Hence, $\mathscr{E}_0$ on $X_0/\mathbb{F}_q$, there is a canonical isomorphism $F^*_{abs}\mathscr{E}_0 \xrightarrow{\sim} \mathscr{E}_0$. Restricting to $X_{\text{ét}}$ gives the desired $F_{\mathscr{E}}$. $\star$

The theorem we actually want is the following

Theorem 26.4. *Let U_0 be a smooth curve over $\mathbb{F}_q$, and let $\mathscr{E}_0$ be a locally constant $\mathbb{Q}_\ell$-sheaf on U_0. Then,*

$$\sum_{x \in U^F} \mathrm{Tr}(F_x \mid \mathscr{E}_x) = \sum_{r \geq 0} (-1)^r \,\mathrm{Tr}\left(F \mid \mathrm{H}^r_c(U, \mathscr{E})\right).$$

Definition 26.5. Let $U_0, \mathscr{E}_0$ as above. The **ζ-function of $\mathscr{E}_0$** is

$$\zeta(U_0, \mathscr{E}_0, t) := \exp\left(\sum_m \sum_{x \in U^{F^m}} \mathrm{Tr}\left(F^m_x \mid \mathscr{E}_x\right) \frac{t^m}{m}\right).$$

Our target theorem (+ finiteness theorems for étale cohomology) implies that $\zeta(U_0, \mathscr{E}_0, t)$ is always rational. $\diamond$

Example. If $\mathscr{E}_x = \mathbb{Q}_\ell$, then $\zeta(U_0, \mathbb{Q}_\ell, t)$ is $\zeta_{U_0}(t)$. $\triangle$

Example. Say $\pi : X_0 \to U_0$ is smooth proper morphism, and set $\mathscr{E}_0^i := R^i\pi_*\mathbb{Q}_\ell$. Then,

$$\prod_{i\geq 0} \zeta(U_0, \mathscr{E}_0^i, t)^{(-1)^i} = \zeta_{X_0}(t)$$

by the Leray spectral sequence for π.[53] $\triangle$

Some remarks

(1) We wrote the target theorem in terms of H_c^i. Using Poincaré duality, we could have written it instead in terms of $\mathrm{H}^{2-i}(X, \mathscr{E}^\vee(1))$.

(2) Formula can be re-interpreted in terms of π_1 (this remark only true for curves).

Recall 26.6.
$$\left\{\begin{matrix}\text{lisse } \mathbb{Q}_\ell\text{-sheaves} \\ \mathscr{E} \text{ on } U_0\end{matrix}\right\} \longleftrightarrow \left\{\begin{matrix}\pi_1^{\text{ét}}(U_0, u)\text{-reps} \\ \text{into } \mathrm{GL}(\mathscr{E}_u)\end{matrix}\right\}.$$
$\odot$

LHS of formula was $\sum_{x\in U^F = U_0(\mathbb{F}_q)} \mathrm{Tr}(F_x \mid \mathscr{E}_x)$. Given $x \in U_0(\mathbb{F}_q)$, this is same as a map $\operatorname{Spec}\mathbb{F}_q \xrightarrow{x} U_0$; hence get a map $\mathrm{Gal}(\overline{\mathbb{F}}_q/\mathbb{F}_q) \to \pi_1^{\text{ét}}(U_0, \overline{x})$ on $\pi_1^{\text{ét}}$'s. Frobenius $F_x \in \mathrm{Gal}(\overline{\mathbb{F}}_q/\mathbb{F}_q)$ gives a well-defined conjugacy class $[F_x] \in \pi_1^{\text{ét}}(U_0, \overline{x}) \simeq \pi_1^{\text{ét}}(U_0, u)$. Thus the LHS has a group-theoretic interpretation:

$$\mathrm{Tr}(F_x \mid \mathscr{E}_x) = \mathrm{Tr}(F_x \in \pi_1^{\text{ét}}(U_0))$$

(Trace in the representation corresponding to the given sheaf).

On the RHS, we had $\mathrm{Tr}(F \mid \mathrm{H}_c^i(U, \mathscr{E})) = \mathrm{Tr}(F^{-1} \mid \mathrm{H}^{2-i}(U, \mathscr{E}^\vee(1)))$. Say U_0 is affine. Then we can give a group-theoretic interpretation of $F^{-1} \curvearrowright \mathrm{H}^{2-i}(U, \mathscr{E}^\vee(1))$ because

$$\mathrm{H}^{2-i}(U, \mathscr{E}^\vee(1)) \simeq \mathrm{H}_{\text{ét}}^{2-i}(\pi_1^{\text{ét}}(U, u), \mathscr{E}^\vee(1)_x)$$

since affine curves are étale π_1's (true for any variety w/ cohomology vanishing above degree 1). The action of F can be described group theoretically as follows: $U_0 \to \operatorname{Spec}\mathbb{F}_q$ induces

$$1 \longrightarrow \pi_1^{\text{ét}}(U, u) \to \pi_1^{\text{ét}}(U_0, u) \to \mathrm{Gal}(\overline{\mathbb{F}}_q/\mathbb{F}_q) \longrightarrow 1.$$

Claim 26.7. *The outer action of* $\mathrm{Gal}(\overline{\mathbb{F}}_q/\mathbb{F}_q) \curvearrowright \pi_1^{\acute{e}t}(U, u)$ *induces action on* $\mathrm{H}^i(\pi_1^{\acute{e}t}(U, u), \mathscr{E}^\vee(1)_x)$ *agreeing w/ geometrically described Frobenius action.*

How will we prove our target theorem?

- Formulation for lcc torsion sheaves
- Pass to cover to reduce to case of constant sheaves
- Pass to inverse limit to get statement for $\mathbb{Q}_\ell$-sheaves

[53] To show this, use that the differentials are Frobenius equivariant

Theorem 26.8. *Let U_0 be a smmoth, geometrically connected curve over $\mathbb{F}_q$. Let $\mathscr{E}_0$ be an lcc sheaf of flat $\mathbb{Z}/\ell^n\mathbb{Z}$-modules ($\ell \neq \operatorname{char}\mathbb{F}_q$). Then,*

$$\sum_{x \in U^F} \operatorname{Tr}(F_x \mid \mathscr{E}_x) = \sum (-1)^r \operatorname{Tr}(F \mid \mathrm{H}^r_c(U, \mathscr{E})).$$

What does the RHS above mean w/o the context of free modules?

Definition 26.9. Let R be a noetherian local ring. A **perfect complex** of R-modules is one that is quasi-isomorphic to a bounded complex of finite free R-modules.[54] $\diamond$

This will allow us to reinterpret the RHS in our theorem to a statement about a bounded complex of finite free R-modules, where (an alternating sum of) traces makes sense.

Proposition 26.10. *Let R be a noetherian local ring, and let $M^\bullet$ be a complex of R-modules. Say $\gamma : P^\bullet \to M^\bullet$ is a quasi-isomorphism w/ $P^\bullet$ bounded complex of f.g. free R-modules (slightly stronger than $M^\bullet$ being perfect). Suppose $\alpha : M^\bullet \to M^\bullet$ is an endomorphism. Then, $\exists \beta : P^\bullet \to P^\bullet$ so that*

$$\begin{tikzcd} P^\bullet \arrow[r, "\beta"] \arrow[d, "\gamma"'] & P^\bullet \arrow[d, "\gamma"] \\ M^\bullet \arrow[r, "\alpha"] & M^\bullet \end{tikzcd}$$

commutes, β is well-defined up to homotopy, and

$$\operatorname{Tr}(\beta \mid P^\bullet) := \sum (-1)^r \operatorname{Tr}(\beta \mid P^r)$$

is independent of all choices.

Proof. Non-trivial homological algebra, but "You'll be able to do it if you sit down and try it." ■

Remark 26.11. Not all complexes are perfect, e.g. note that perfect complexes have finite projective dimension. ○

Non-example. Take $R = \mathbb{Z}/\ell^2\mathbb{Z}$ and $M = \mathbb{Z}/\ell\mathbb{Z}$. We view M as a complex of $(\mathbb{Z}/\ell^2\mathbb{Z})$-modules concentrated in degree zero. It is not q.iso to a bounded complex of f.g. free modules. Why?

Suppose it was, i.e. that we have $P^\bullet : \mathbb{Z}/\ell\mathbb{Z}[0]$. This is a projective resolution, so we would get $\operatorname{Tor}_i^{\mathbb{Z}/\ell^2\mathbb{Z}}(\mathbb{Z}/\ell\mathbb{Z}, -) = 0$ for $i \gg 0$. However, we have an projective resolution

$$\cdots \longrightarrow \mathbb{Z}/\ell^2\mathbb{Z} \xrightarrow{\cdot \ell} \mathbb{Z}/\ell^2\mathbb{Z} \xrightarrow{\cdot \ell} \mathbb{Z}/\ell^2\mathbb{Z} \to \mathbb{Z}/\ell\mathbb{Z} \to 0$$

which implies that

$$\operatorname{Tor}_i^{\mathbb{Z}/\ell^2\mathbb{Z}}(\mathbb{Z}/\ell\mathbb{Z}, \mathbb{Z}/_{\mathbb{Z}}\ell) = \mathbb{Z}/\ell\mathbb{Z} \text{ for all } i.$$

$\triangledown$

Here's a criterion for checking when a complex is perfect

Proposition 26.12 (Mumford)**.** *Say R is a Noetherian local ring, and $M^\bullet$ is a complex of R-modules.*

[54] Without local condition, would want it to be "locally quasi-isomorphic" to such a thing

- *If* $\mathrm{H}^r(M^\bullet)$ *is f.g. for all* r *and* $\mathrm{H}^r(M^\bullet) = 0$ *for* $r > m$*, then there exists quasi-iso* $Q^\bullet \to M^\bullet$ *w/* Q^i *f.g. free s.t.* $Q^r = 0$ *for all* $r > m$*.*
- *If in addition,* $\mathrm{H}^r(Q^\bullet \otimes_R N) = 0$ *for* $r < 0$ *and all f.g.* R*-modules* N*, then there exists a quasi-iso* $Q^\bullet \to P^\bullet$ *w/* $P^\bullet$ *a complex of f.g. free modules supported in degrees* $0, 1, \ldots, m$*.*

Proof Idea for Second Bullet Point. Replace Q_0 w/ $Q_0/\operatorname{im} Q_1$, and then check it's flat. Using Tor criterion for flatness to check. ■

Remember: f.g. module over noetherian local ring is free

Remark 26.13. Projection formula + finiteness theorems $\implies$ for $\mathscr{E}$ flat lcc $\mathbb{Z}/\ell^n\mathbb{Z}$-sheaf, the complex $R\Gamma(U_{\text{ét}}, \mathscr{E})$ satisfies the conditions of the theorem. Thus, $R\Gamma(U_{\text{ét}}, \mathscr{E})$ is a perfect complex, so can make sense of the expression

$$\text{"} \sum_{r \geq 0} (-1)^r \operatorname{Tr}\left(F \mid \mathrm{H}^i(U, \mathscr{E})\right) \text{"}$$

since we can computed it on a quasi-isomorphic bounded complex consisting of f.g. free modules. ○

Won't prove our target theorem since we're low on time, and we want to get to the geometric content of the Weil conjectures.

26.2 Introduction to main geometric ideas of Weil II

Given $X \subset \mathbb{P}^n$ smooth projective variety, we'll roughly want to find a map $\mathrm{Bl}\, X \xrightarrow{\pi} \mathbb{P}^1$ s.t. the fibers have mild singularities and s.t. the "monodromy" on the middle cohomology of the fibers has large image. Say the fibers have dimension d. We'll want to understand $\mathscr{E} := R^d\pi_{sm,*}\mathbb{Q}_\ell \in \mathrm{Sh}^{\mathbb{Q}_\ell}(U)$ where $\pi_{sm} : X^0 \to U$ is the restriction of π to the locus where π is smooth.

We'll axiomatize the properties of sheaves like $\mathscr{E}$ above, and then analyze the consequences for $\mathrm{H}^i(U, \mathscr{E})$. This should allow us to do some sort of induction on dimension.

27 Lecture 27

27.1 Last time: Grothendieck-Lefschetz trace formula

Theorem 27.1. *Let* U_0 *be a curve over* $\mathbb{F}_q$*, and let* $\mathscr{E}_0$ *be a locally constant* $\mathbb{Q}_\ell$*-sheaf on* U_0 *(where* $\ell \nmid q$*). Then,*

$$\sum_{x \in U^F} \operatorname{Tr}(F_x \mid \mathscr{E}_x) = \sum_{r \geq 0} (-1)^r \operatorname{Tr}\left(F \mid \mathrm{H}^r_c(U, \mathscr{E})\right).$$

We did not prove this, but we said what the main strategy was. We then defined

$$\zeta(U_0, \mathscr{E}_0, t) := \exp\left(\sum_m \sum_{x \in U^{F^m}} \frac{\operatorname{Tr}(F_x \mid \mathscr{E}_x) t^m}{m}\right).$$

The theorem implies that this ζ-function is rational, and can be written in terms of the characteristic polynomial of Frobenius F on $\mathrm{H}^i_c(U, \mathscr{E})$.

27.2 Today: Study $\zeta(U_0, \mathscr{E}_0, t)$ for very special $\mathscr{E}_0$ arising from Lefschetz fibrations

Note 2. The recording freezes here and then resumes with "... because it's the main lemma that goes into the proof of the Riemann Hypothesis, we'll discuss it in a little bit."

Lemma 27.2 (**MAIN LEMMA**, all caps required)**.** *Let X_0 be a smooth affine genus 0 curve*[55] *over $\mathbb{F}_q$. Let $X = (X_0)_{\overline{\mathbb{F}}_q}$, and let $\mathscr{E}$ be a locally constnat $\mathbb{Q}_\ell$-sheaf on X_0 with corresponding representation E of $\pi_1^{\acute{e}t}(X_0)$. Assume*

(1) *For each $x \in |X|$ (closed points of X), $F_x \curvearrowright \mathscr{E}_x$ the characteristic polynomial has rational coefficients, i.e.* charpoly $\in \mathbb{Q}[t]$.

(2) *There is a non-degenerate skew-symmetric form*[56]

$$\psi : E \times E \to \mathbb{Q}_\ell(-n).$$

(3) *$\rho : \pi_1^{\acute{e}t}(X) \to \mathrm{GL}(E)$ actually lands in $\mathrm{Sp}(E, \psi)$, and $\mathrm{im}(\rho)$ is open in $\mathrm{Sp}(E, \psi)$, i.e. we have "**big monodromy.**"*

Then,

(a) *E has "weight" n, i.e. the eigenvalues α of $F_x \curvearrowright \mathscr{E}_x$ have absolute value $q^{n(\deg x)/2}$ for any $x \in |X|$ for any embedding $\mathbb{Q}(\alpha) \hookrightarrow \mathbb{C}$.*

(b) *$F \curvearrowright \mathrm{H}^1_c(X, \mathscr{E})$ has rational characteristic polynomial, and for all eigenvalues α, $|\alpha| < q^{n/2+1}$ for all embeddings $\mathbb{Q}(\alpha) \hookrightarrow \mathbb{C}$.*

(c) *$F \curvearrowright \mathrm{H}^1(\mathbb{P}^1, j_*\mathscr{E})$ has rational characteristic polynomial w/ all eigenvalues α satisfying*

$$q^{n/2} < |\alpha| < q^{n/2+1}.$$

Imagine $\mathscr{E} = R^i\pi_*\underline{\mathbb{Q}}_\ell$ with $\pi : X \to \mathbb{P}^1$ some family of varieties over $\mathbb{P}^1$, and say $\dim X$ even. You should think that **(c)** above + leray gives contributions to middle dimensional cohomology with a bound of the form we want (recall Theorem 25.6). Let's give some more details.

Question 27.3. *Where does $\mathscr{E}$ come from?*

Definition 27.4. Let X be a smooth projective variety, with embedding $|\mathscr{L}| : X \hookrightarrow \mathbb{P}^n$ via some complete linear system. Let $\ell \subset \check{\mathbb{P}}^n$ be a line in the dual projective space, i.e. ℓ a linear family of hyperplanes H_t. We say this is a **Lefschetz pencil** if

(1) The base locus (or axis) of the pencil $A = \bigcap_t H_t$ intersects X transversely.[57]

(2) $X_t = X \cap H_t$ is smooth for all t in a dense open U of ℓ

[55] in particular, geometrically connected
[56] Note $\pi_1^{\acute{e}t}(X_0)$ surjects on $\mathrm{Gal}(\overline{\mathbb{F}}_q/\mathbb{F}_q)$, so $\mathbb{Q}_\ell(-n)$ makes sense as a $\pi_1^{\acute{e}t}$-rep too
[57] intersection of tangent spaces has dimension $\dim X - 2$?

(3) For $t \notin U$ (where X_t singular), X_t has a unique singular point which is an **ordinary double point**: $\widehat{\mathscr{O}}_{X,p} \simeq k[t_1, \ldots, t_N] / (\text{non-deg quadratic})$ (looks like vertex of a quadric cone).

$\diamond$

Example. $\left\{X^2 + Y^2 + tZ^2 = 0\right\} \subset \mathbb{P}^2$. For $t \neq 0$, get a nice smooth quadric surface, but at $t = 0$ you get the quadric cone which has an ordinary double point at its vertex. $\triangle$

The Lefschetz pencils are nice because it's easy to compute their monodromy.

Theorem 27.5 (Existence of Lefschetz pencils). *Let X be smooth projective over $k = \overline{k}$, and let $\mathscr{L}$ be a very ample line bundle on X. Then, there exists some $\ell \subset \left|\mathscr{L}^{\otimes 2}\right|$ such that ℓ is a Lefschetz pencil.*

(Bertini argument, see SGA7 or Milne's Étale Cohomology book)

Now let's explain the strategy for the proof of RH: Let X_0 be a smooth projective even dimensional variety over $\mathbb{F}_q$, say $\dim X_0 = n + 1$ (so n odd). We want the eigenvalues of F on $\mathrm{H}^{n+1}(X, \mathbb{Q}_\ell)$ (middle cohomology) to satisfy $q^{n/2} \leq |\alpha| \leq q^{n/2+1}$.

(1) WLOG may assume X admits a Lefschetz pencil which descends to X_0 (may need to replace $\mathbb{F}_q$ with a finite extension in order to get the coefficients appearing in the equation for the pencil).

(2) Can replace X_0 with the blowup $\mathrm{Bl}_A X_0 = (X_0 \times \ell) \times_\ell \mathcal{H} \to \ell$ where A is the base locus of the Lefschetz pencil and $\mathcal{H}$ is the family of hyperplanes. This is a family over $\mathbb{P}^1$ w/ fibers $(X_0) \cap H_t$.

Note that the cohomology of the blowup is built out of the cohomology of X, and the cohomology of $X \cap (\text{base locus})$ (which is codim 2 in X). The key claim is $\mathrm{Bl}_A X \to X$ induces injection $\mathrm{H}^*(X) \hookrightarrow \mathrm{H}^*(\mathrm{Bl})$.

So can assume $X \to \mathbb{P}^1$ a "Lefschetz fibration".

(3) Enough to understand eigenvalues of Frobenius on the parts of the leray spectral sequence contributing to the middle cohomology of X, i.e. on

(a) $\mathrm{H}^2\left(\mathbb{P}^1, R^{n-1}\pi_*\mathbb{Q}_\ell\right)$

$R^{n-1}\pi_*\mathbb{Q}_\ell$ will be a constant sheaf (this is not obvious). Hence, this is equal to $\mathrm{H}^{n-1}(X_t, \mathbb{Q}_\ell)(-1)$. Let $Y \subset X_t$ be a smooth hyperplane section (exists by Bertini). The Lefschetz hyperplane theorem will tells us that

$$\mathrm{H}^{n-1}(X_t, \mathbb{Q}_\ell) \hookrightarrow \mathrm{H}^{n-1}(Y, \mathbb{Q}_\ell)$$

and this latter space is the middle cohomology of an even dimensional variety ($\dim Y = n-1$), so can induct.

(b) $\mathrm{H}^1(\mathbb{P}^1, R^n\pi_*\mathbb{Q}_\ell)$

This cases uses the MAIN LEMMA (LEMMA 27.2). The monodromy won't be big in general, but there will be a piece of it that is big. More details next time.

(c) $\mathrm{H}^0(\mathbb{P}^1, R^{n+1}\pi_*\mathbb{Q}_\ell)$

In good situations, this is again a constant sheaf. Win via application of Lefschetz hyperplane + Poincaré duality.

With the strategy written down, let's prove the MAIN LEMMA.

Proof of MAIN LEMMA 27.2. Let $\mathscr{E}_0$ be a locally constant $\mathbb{Q}_\ell$-sheaf on X_0 w/ $F_x \curvearrowright \mathscr{E}_x$ having rational char poly, skew-symmetric non-deg $\psi : E \times E \to \mathbb{Q}_\ell(-n)$ and big monodromy. We want to show that **(a)** E has weight n.

Lemma 27.6 (Lemma 1)**.** $(E^{\otimes(2k)})_{\pi_1^{\text{ét}}(X)} = \mathbb{Q}_\ell(-kn)^{\oplus N}$*, i.e. this tensor-power has simple coinvariants.*

Lemma 27.7 (Lemma 2)**.** *If for all k, $\zeta(X_0, \mathscr{E}_0^{\otimes 2k}, t)$ converges for $t < \frac{1}{q^{kn+1}}$, then E has weight n.*

Let's show these sublemma imply MAIN LEMMA **(a)**. We first remark that lemma 1 gives the hypothesis of lemma 2. To see this, note that

$$\zeta(X_0, \mathscr{E}_0^{\otimes 2k}, t) = \frac{\text{poly coming from } \mathrm{H}^1_c(X, \mathscr{E}^{\otimes 2k})}{\det(1 - F^*t \mid \mathrm{H}^0_c)\det(1 - F^*t \mid \mathrm{H}^2_c)}.$$

Above, $\mathrm{H}^0_c(X, \mathscr{E}^{\otimes 2k}) = 0$ since X affine, so first factor in the denominator is 1. Furthermore,

$$\mathrm{H}^2_c(X, \mathscr{E}^{2k}) \overset{PD}{=} \mathrm{H}^0(X, (\mathscr{E}^\vee)^{\otimes 2k}(1))^\vee = \left(\left((E^\vee)^{\otimes 2k}\right)^{\pi_1^{\text{ét}}}(1)\right)^\vee = E^{\otimes 2k}_{\pi_1^{\text{ét}}}(-1) = \mathbb{Q}_\ell(-kn-1)^{\oplus N}.$$

Hence,

$$\zeta(X_0, \mathscr{E}_0, t) = \frac{\text{poly}}{(1 - q^{kn+1}t)^N}$$

which satisfies the hypotheses of lemma 2. We'll talk about the proofs of these lemmas next time. ■

28 Lecture 28

28.1 Proof of MAIN LEMMA 27.2

Recall we were in the middle of proving the MAIN LEMMA.

Recall 28.1 (MAIN LEMMA 27.2)**.** Let $X_0 \overset{\text{open}}{\subset} \mathbb{P}^1_{\mathbb{F}_q}$ with $X = (X_0)_{\overline{\mathbb{F}}_q}$. Let $\mathcal{E}$ be a locally constant $\mathbb{Q}_\ell$-sheaf on X_0, and let E be the corresponding $\pi_1^{\text{ét}}$-representation. Assume

(1) Frobenius $F_x \curvearrowright \mathcal{E}_x$ acting on the stalks have rational characteristic polynomial for $x \in |X|$

(2) There's a non-deg skew-symmetric form $\psi : E \times E \to \mathbb{Q}_\ell(-n)$

(3) $\pi_1(X) \to \mathrm{Sp}(E, \psi)$ has open image

Then,

(a) E has weight n, i.e. eigenvalues of $F_x \curvearrowright E_x$ have absolute value $|\alpha| = q^{n(\deg x)/2}$ for all embeddings $\mathbb{Q}(\alpha) \hookrightarrow \mathbb{C}$

(b) $F \curvearrowright \mathrm{H}^1_c(X, \mathscr{E})$ has rational characteritic polynomial and eigenvalues β satisfying $|\beta| \leq q^{n/2+2}$

(c) $F \curvearrowright \mathrm{H}^*(\mathbb{P}^1, j_*\mathscr{E})$, $j : X \hookrightarrow \mathbb{P}^1$, has rational char poly w/ eigenvalues γ satisfying

$$q^{n/2} \leq |\gamma| \leq q^{n/2+1}.$$

⊙

Last time we reduced the proof of **(a)** to the following two lemmas...

Lemma 28.2.

$$\left(\bigotimes_{2k} E\right)_{\pi_1(X)} = \bigoplus \mathbb{Q}_\ell(-kn).$$

Proof. First consider the case where $k = 1$. Note that

$$\mathrm{Hom}_{\pi_1(X)}(E \otimes E, \mathbb{Q}_\ell) = \mathrm{Hom}((E \otimes E)_{\pi_1(X)}, \mathbb{Q}_\ell)$$

contains our skew-symmetric form $\psi : E \times E \to \mathbb{Q}_\ell(-n)$. Since π_1 has dense image in $\mathrm{Sp}(E, \psi)$, we see that $(E \otimes E)_{\pi_1(X)} = (E \otimes E)_{\mathrm{Sp}(E,\psi)}$ from which it follows that $\mathrm{Hom}((E \otimes E)_{\pi_1(X)}, \mathbb{Q}_\ell) = \mathrm{span}\,\{\psi\}$.

For general k, we have

$$\mathrm{Hom}((E^{\otimes 2k})_{\mathrm{Sp}(E,\psi)}, \mathbb{Q}_\ell) = \mathrm{span}\left\{\prod_k \psi(v_{i_1}, v_{i_2})\right\}.$$

This is a 'linear algebra statement about coinvariants of the symplectic statement.' Since each copy of ψ contributes a $\mathbb{Q}_\ell(-n)$ and each spanning element above contains k copies of ψ, we see the whole rep is $\mathbb{Q}_\ell(-kn)^{\oplus N}$ for some N. ∎

Lemma 28.3. *If for all k, $\zeta(X, E^{\otimes 2k}, t)$ converges for $t < \frac{1}{q^{kn+1}}$, then E has weight n.*

Proof. We start off with the Euler product of these zeta functions:

$$\zeta(X, \mathscr{E}^{\otimes 2k}, t) = \prod_{x \in |X|} \frac{1}{\det\left(1 - F_x t^{\deg x} \mid \mathscr{E}_x^{\otimes 2k}\right)}.$$

Note that if the product converge, then each factor must converge. Hence, $\det\left(1 - F_x t^{\deg x} \mid \mathscr{E}_x^{\otimes 2k}\right)^{-1}$ converges for $|t| < q^{-(kn+1)}$. Hence each eigenvalue of Frobenius must satisfy $|\alpha| > q^{(\deg x)kn+1}$. An application of the tensor power trick now gives $|\alpha| \geq q^{(\deg x)n/2}$. The pairing then gives the other inequality $|\alpha| \leq q^{(\deg x)n/2}$. ∎

This proves **(a)** of the MAIN LEMMA.

Let's start proving **(b)**, i.e. that $F \curvearrowright \mathrm{H}^1_c(X, \mathscr{E})$ has rational characteristic polynomial and eigenvalues β satisfying $|\beta| \leq q^{n/2+2}$

*Proof of **(b)** of MAIN LEMMA 27.2.* We're trying to understanding the zeta function $\zeta(X_0, \mathscr{E}, t)$; this is controlled by Frob action on $\mathrm{H}^i_c(X, \mathscr{E})$. First recall that $\mathrm{H}^0_c(X, \mathscr{E}) = 0$ since X affine. Furthermore, $\mathrm{H}^2_c(X, \mathscr{E}) = E_{\pi_1(X)}(-1)$ by Poincaré duality. Since π_1 is dense in the symplectic group this is furthermore

$$\mathrm{H}^2_c(X, \mathscr{E}) = E_{\pi_1(X)}(-1) = E_{\mathrm{Sp}(E,\psi)}(-1) = 0$$

(the standard representation of the symplectic group has no coinvariants). Thus,

$$\zeta(X_0, \mathscr{E}, t) = \det\left(1 - F^* t \mid \mathrm{H}^1_c(X, \mathscr{E})\right) = \prod_{x \in |X|} \frac{1}{\det\left(1 - F_x t^{\deg x} \mid \mathscr{E}_x\right)}.$$

These factors have rational coefficients, so their product $\det\left(1 - F^*t \mid \mathrm{H}^1_c(X, \mathscr{E})\right)$ does as well. This gives the first part of **(b)**.

To understand the eigenvalues, it suffices to show this product $\prod \det(1 - F_x t^{\deg x} \mid \mathscr{E}_x)$ converges for $|t| < 1/q^{n/2+1}$.[58] Let $a_{i,x}$ be the eigenvalues of $F_x \curvearrowright \mathscr{E}_x$. It's enough to show that

$$\sum_{i,x} \left|a_{i,x} t^{\deg x}\right| \text{ converges for } |t| < \frac{1}{q^{n/2+1}}.$$

This follows from

(1) $|a_{i,x}| = q^{(\deg x)n/2}$

Follows from MAIN LEMMA **(a)**.

(2) $\#$ closed points of $\deg n$ is $\leq q^n + 1$

X is an (affine open) subset of $\mathbb{P}^1$ and $\mathbb{P}^1(\mathbb{F}_{q^n}) = q^n + 1$. ■

This just leaves part **(c)**: $F \curvearrowright \mathrm{H}^*(\mathbb{P}^1, j_*\mathscr{E})$, $j : X \hookrightarrow \mathbb{P}^1$, has rational char poly w/ eigenvalues γ satisfying

$$q^{n/2} \leq |\gamma| \leq q^{n/2+1}.$$

*Proof of MAIN LEMMA 27.2***(c)***.* Recall the short exact sequence (the third object is a sum of skyscraper sheaves supported on the complement)

$$0 \longrightarrow j_!\mathscr{E} \longrightarrow j_*\mathscr{E} \longrightarrow i_*i^*j_*\mathscr{E} \longrightarrow 0$$

where $X_0 \overset{j}{\hookrightarrow} \mathbb{P}^1 \overset{i}{\hookleftarrow} \mathbb{P}^1 \setminus X_0$. The LES in cohomology looks like

$$\begin{tikzcd}
0 \arrow[r] \arrow[d, equal] & \mathrm{H}^0(\mathbb{P}^1, j_*\mathscr{E}) \arrow[r] & \mathrm{H}^0(\mathbb{P}^1_{\overline{\mathbb{F}}_q}, i_*i^*j_*\mathscr{E}) \arrow[r, "\delta"] & \mathrm{H}^1_c(X, \mathscr{E}) \arrow[r] \arrow[d, equal] & \mathrm{H}^1(\mathbb{P}^1_{\overline{\mathbb{F}}_q}, j_*\mathscr{E}) \arrow[r] & 0 \\
\mathrm{H}^0_c(X, \mathscr{E}) & & & \mathrm{H}^1(\mathbb{P}^1_{\overline{\mathbb{F}}_q}, j_!\mathscr{E}) & &
\end{tikzcd}$$

We first want to show that frobenius action on $\mathrm{H}^1(\mathbb{P}^1, j_*\mathscr{E})$ has rational characteristic polynomial. The main point is that

$$\operatorname{charpoly}\left(F \middle| \mathrm{H}^1(\mathbb{P}^1_{\overline{\mathbb{F}}_q}, j_*\mathscr{E})\right) = \frac{\operatorname{charpoly}\left(F \mid \mathrm{H}^1_c(X, \mathscr{E})\right) \cdot \operatorname{charpoly}(F \mid \mathrm{H}^0(\mathbb{P}^1, j_*\mathscr{E}))}{\operatorname{charpoly}(F | \operatorname{im}\delta)}$$

and the details of checking that the RHS is rational is left as an exercise.

Let α be an eigenvalue of $F \curvearrowright \mathrm{H}^1(\mathbb{P}^1, j_*\mathscr{E})$. Note that MAIN LEMMA **(b)** + the surjection above show that $|\alpha| \leq q^{n/2+1}$. For other pairing, use Poincaré duality, which gives a perfect pairing

$$\mathrm{H}^1(\mathbb{P}^1, j_*\mathscr{E}) \times \mathrm{H}^1(\mathbb{P}^1, j_*\mathscr{E}^\vee(1)) \to \mathrm{H}^2(\mathbb{P}^1, \mathbb{Q}_\ell(1)).$$

This sheaf $\mathscr{E}^\vee(1)$ satisfies hypotheses of lemma, so can apply upper bound to it's eigenvalues, concluding the lower bound we want here. ■

[58] Poles of this are zeros of its reciprocals or something like that

28.2 Understanding $j_*\mathscr{E}$ for $\mathscr{E}$ locally constant on X

Proposition 28.4. *Suppose $U \subset Y$ is an affine open inside a proper geometrically connected curve over $k = \overline{k}$. Let $\Lambda = \mathbb{Z}/\ell^n\mathbb{Z}$ or $\mathbb{Q}_\ell$, $\ell \neq \operatorname{char} k$. Let $\mathscr{F}$ be a sheaf of Λ-modules on Y. Then,*

- ***(a)*** *$\mathscr{F} \xrightarrow{\sim} j_* j^* \mathscr{F}$ is an iso iff*
  - ***(i)*** *For all $s \in Y \setminus U$, the cospecialization map $\mathscr{F}_s \to \mathscr{F}_{\overline{\eta}}$ is injective*
  - ***(ii)*** *image of above cospecialization map is $\mathscr{F}_{\overline{\eta}}^{I_s}$, invariants of inertia group at s.*
- ***(b)*** *For $\mathscr{F}$ a $\mathbb{Q}_\ell$-sheaf satisfying the above and locally constant on U, the cup product pairing*

$$\mathrm{H}^r(Y, j_*\mathscr{F}) \times \mathrm{H}^{2-r}(Y, j_*\mathscr{F}^\vee(1)) \to \mathrm{H}^2(Y, \mathbb{Q}_\ell(1))$$

  is perfect.

(a) is a local computation, and **(b)** comes out of Verdier duality.

28.3 Cohomology of Lefschetz Pencils

"I would love to just teach a whole course on this."

Setup 28.5. Let X be a smooth projective variety with fixed embedding $X \hookrightarrow \mathbb{P}^n$. Let $L \subset \check{\mathbb{P}}^n$ be a Lefschetz pencil. To this data, one associates the family of hyperplane sections

$$\begin{array}{c} \mathcal{X}_t \\ \pi \big\downarrow \\ L = \mathbb{P}^1. \end{array}$$

This satisfies

- generic fiber is smooth
- the singular fibers have unique singularities which are ordinary double points

Recall explicitly that $\mathcal{X}_t = \mathrm{Bl}_{A \cap X} X$ where A is the base locus of L.

Notation 28.6. Let $S \subset \mathbb{P}^1$ be the set of $t \in \mathbb{P}^1$ where $\mathcal{X}_t$ is singular.

Assumption. Assume that the fiber dimension is $n = 2m + 1$ odd.

Notation 28.7. Let $U = \mathbb{P}^1 \setminus S$ be the locus where π is smooth. Let $I - s$ be tame inertia at s, and let $V = (R^n \pi_* \mathbb{Q}_\ell)_{\overline{\eta}}$, the 'monodromy representation of this family.'

Claim 28.8.

- ***(a)*** *For $r \neq n, n+1$, $R^r \pi_* \mathbb{Q}_\ell$ is locally constant (hence constant since $\pi_1^{\acute{e}t}(\mathbb{P}^1) = 0$).* If all fibers were smooth, this would be true for all r by smooth and proper base change (Theorem 20.3
- ***(b)*** *$R^n \pi_* \mathbb{Q}_\ell|_U$ is locally constant and tame (no wild inertia)*
- ***(c)*** *For each $s \in S$, there exists a "vanishing cycle" $\delta_s \in V(m)$, well-defined up to sign, so that span$\{\delta_s\}$ is dual to the cokernel of the cospecialization map $V_s \hookrightarrow V_{\overline{\eta}}$.*

(d) *There's an exact sequence*

$$0 \longrightarrow \mathrm{H}^n(X_s, \mathbb{Q}_\ell) \xrightarrow{\text{cospec.}} \mathrm{H}^n(X_{\overline{\eta}}, \mathbb{Q}_\ell) \xrightarrow{\cup\delta_s} \mathbb{Q}_\ell(m-n) \longrightarrow 0$$

(e) $\sigma_s \in I_s \simeq \prod_{\ell \neq p} \mathbb{Z}_\ell(1) \xrightarrow{t_\ell} \mathbb{Z}_\ell(1)$*. Given* $x \in V$*,* $\sigma_s(x) = x \pm t_\ell(\sigma_s)(s \cup \delta_s)\delta_s$*.*

29 Lecture 29

29.1 Refresher on Lefschetz pencils and inertia

Hopefully we can finish sketching the proof of the Riemann hypothesis today.

Question 29.1 (Audience)**.** *Can you remind us how the inertia subgroup I is constructed?*

Answer. Start by recalling the context. We have a Lefschetz fibration $X \subset \mathbb{P}^n$, a smooth projective variety, along with a line $L \subset \check{\mathbb{P}}^n$ (of hyperplanes) s.t. the base locus (intersection $\bigcap_{H \in L} H$) intersects X transversely; $X_t := X \cap H_t$ is smooth for almost all $t \in L$; and finally, any singular X_t has a unique singular point which is an ordinary double point (analytically-locally looks like cone point of a cone over a non-deg quadratic). Recall that, by a Bertini argument, any smooth projective variety (maybe after changing the projective embedding) admits such a Lefschetz pencil.

Let $\mathcal{X} := \mathrm{Bl}_{\text{base locus}} X$. Get map $\pi : \mathcal{X} \to L$ with $\pi^{-1}(t) = \mathcal{X}_t$. We wanted to understand the monodromy of this family, the higher push forwards of the constant sheaf through π. These will be locally constant $\mathbb{Q}_\ell$-sheaves (on some subset(s) of L), and we wanted to understand how $\pi_1^{\text{ét}}$ acts on them.

The question was about inertia subgroups of $\pi_1^{\text{ét}}$. Let $S \subset L$ be the set of points t so that X_t is singular. Then, $R^i\pi_*\mathbb{Q}_\ell|_{L \setminus S}$ is a locally constant $\mathbb{Q}_\ell$-sheaf for all i, so it corresponds to some representation

$$\pi_1^{\text{ét}}(L \setminus S) \longrightarrow \mathrm{GL}_n(\mathbb{Q}_\ell).$$

For each $s \in S$, can consider the inclusion $\operatorname{Spec}\operatorname{Frac}\widehat{\mathscr{O}}_{L,s} \to L \setminus S$. Base changing to $\overline{k}$, we get a map

If $\operatorname{Spec} A$ is an affine neighborhood of $s \in L$, then $L \setminus S$ will have an affine open of the form $\operatorname{Spec} A_f$, and $A_f \to \operatorname{Frac} A_f = \operatorname{Frac} A = \operatorname{Frac} \mathscr{O}_{L,s}$ (or something like this)

$$\pi_1^{\text{ét}}(\operatorname{Spec}\operatorname{Frac}\widehat{\mathscr{O}}_{L_{\overline{k}},s}) \to \pi_1^{\text{ét}}((L \setminus S)_{\overline{k}}),$$

and the LHS above is called the **inertia group at** $s \in S$, and denoted I_s. Abstractly, this just looks like an absolute Galois group $G_{\overline{k}((t))}$. Note that this is only well-defined up to conjugacy (omitted basepoints in our discussion).

Intuition. $L \setminus S$ is $\mathbb{P}^1$ minus finitely many points. The picture you should have in mind is that I_s is generated by small loops around s. You don't get a normal subgroup, but (if I understood correctly) the normal subgroup it generates is $\ker\left(\pi_1^{\text{ét}}(L \setminus S) \to \pi_1^{\text{ét}}(L \setminus (S \setminus \{s\}))\right)$.

Also, if $\operatorname{char} k = 0$, then this inertia group is just $\widehat{\mathbb{Z}}$ (get extensions by extracting roots of t).

⋆

Example. Consider the family of (affine) elliptic curves

$$E = \left\{y^2 = x(x-1)(x-\lambda)\right\} \longrightarrow \mathbb{A}^1$$

(the coordinate on $\mathbb{A}^1$ is λ). This is smooth over $\mathbb{A}^1 \setminus \{0,1\}$, but the fibers over $0,1$ are nodal cubics.

Let's first consider the picture over $\mathbb{C}$. Note that (topological π_1)

$$\pi_1\left(\mathbb{A}^1_{\mathbb{C}} \setminus \{0,1\}^{\mathrm{an}}\right) = \langle \gamma_0, \gamma_1 \rangle$$

is free on two generators, a loop around 0 and a loop around 1. Note that $R^1\pi_*\mathbb{Q}_\ell$ is a locally constant $\mathbb{Q}_\ell$-sheaf on $\mathbb{A}^1 \setminus \{0,1\}$, so it corresponds to some representation

$$\pi_1(\mathbb{A}^1 \setminus \{0,1\}^{\mathrm{an}}) \to \mathrm{GL}_2(\mathbb{Q}_\ell).$$

One can compute analytically that (after choosing appropriate bases), this representation is

$$\gamma_0 \mapsto \begin{pmatrix} 1 & 2 \\ 0 & 1 \end{pmatrix} \text{ and } \gamma_1 \mapsto \begin{pmatrix} 1 & 0 \\ 2 & 1 \end{pmatrix}.$$

Note that this is trivial mod 2.[59] Also note that these two elements topologically generate an open subgroup of $\mathrm{SL}_2(\mathbb{Q}_\ell)$.

Daniel drew some pictures and explained what was going on, but I didn't take notes on this... $\triangle$

Warning 29.2. Properties of these representations depend on parity of cohomological degree. $\bullet$

Example. Consider quadric surfaces $\{x^2+y^2+z^2+tw^2=0\} \longrightarrow \mathbb{P}^1_t$. Here, inertia will act by reflections on H^2, i.e. $\pi_1(\mathbb{A}^1 \setminus 0) \simeq \mathbb{Z}$ has generator acting on $\mathrm{H}^2(X_t, \mathbb{Z})$ via $\begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}$. $\triangle$

When proving RH, we'll use special properties of the symplectic group, and this is why we wanted to reduce to the case of middle cohomology in even dimensional varieties.

29.2 Claims about Lefschetz fibrations w/ odd-dimensional fibers

Call the fiber dimension $n = 2m-1$.

(1) For $r \neq n, n+1$, the sheaves $R^r\pi_*\mathbb{Q}_\ell$ are locally constant on L (base of Lefschetz fibration).

Example. In elliptic curve example from earlier, we have fibers of dimension $1 = 2(1)-1$. Hence, for $r \neq 1,2$ (i.e. for $r=0$), we expect $R^r\pi_*\mathbb{Q}_\ell$ to be locally constant. Indeed, $R^0\pi_*\mathbb{Q}_\ell = \pi_*\mathbb{Q}_\ell = \mathbb{Q}_\ell$ since all fibers are geometrically connected. $\triangle$

Non-example. To see that $R^2\pi_*\mathbb{Q}_\ell$ need not be locally constant, imagine a family of genus 2 curves degenerating to two elliptic curves meeting at a point. $\triangledown$

(2) $R^n\pi_*\mathbb{Q}_\ell|_{L\setminus S}$ is locally constant (use smooth and proper base change), tame (tame inertia is $\widehat{\mathbb{Z}}$)

(3) For $s \in S$, get co-specialization map

$$0 \longrightarrow \mathrm{H}^n(X_s, \mathbb{Q}_\ell) \longrightarrow \mathrm{H}^n(X_{\overline{\eta}}, \mathbb{Q}_\ell) \xrightarrow{-\cup\delta_s} \mathbb{Q}_\ell(m-n) \longrightarrow 0.$$

[59] Ultimately because this representation has to act trivially on the 2-torsion of your elliptic curve

Above, $\delta_s \in \mathrm{H}^n(X_{\overline{\eta}}, \mathbb{Q}_\ell)^\vee(m-n) = \mathrm{H}^n(X_{\overline{\eta}}, \mathbb{Q}_\ell(n))(m-n) = \mathrm{H}^n(X_{\overline{\eta}}, \mathbb{Q}_\ell(m))$ is the image of 1 under the natural map $\mathbb{Q}_\ell \to \mathrm{H}^n(X_{\overline{\eta}}, \mathbb{Q}_\ell(m))$ dual to $- \cup \delta_s$. Concretely, the span of δ_s is the kernel $\ker\left(\mathrm{H}^n(X_{\overline{\eta}}, \mathbb{Q}_\ell(m))^\vee \to \mathrm{H}^n(X_s, \mathbb{Q}_\ell)^\vee\right)$. This is why these are called **vanishing cycles** (think, 'homology' class vanishing when restricted to special fiber).

(4) $\sigma_s \in I_s$ acts on $x \in \mathrm{H}^n(X_{\overline{\eta}}, \mathbb{Q}_\ell)$ via (**Picard-Lefschetz formula**)

$$\sigma_s(x) = x \pm t(\sigma_s)(x \cup \delta_s)\delta_s \text{ where } t : I_s \to \mathbb{Z}_\ell(1)$$

is the natural map (onto the Galois group of the extension obtained by only adding ℓ-power roots of t).

Example. Returning to the elliptic curve example, we see that the vanishing cycle has cup product 2 with the other loop.[60] Thus, the vanishing cycle isn't quite the loop you might have expected, but is instead twice it. This sort of corresponds to the fact that the lambda line mapping to $\mathcal{M}_{1,1}$ is not an isomorphism, but is instead ramified of degree 2. Don't ask me why (though maybe ask Daniel why). △

Let's quickly recap the road to RH. We want to understand $\mathrm{H}^{n+1}(X, \mathbb{Q}_\ell)$ and will compute using the Leray spectral sequence. The interesting part will be $\mathrm{H}^1(\mathbb{P}^1, R^n\pi_*\mathbb{Q}_\ell)$. Hence, we need to understand this sheaf better.

Goal. Understand π_1 acton on the span of the vanishing cycles in $\mathrm{H}^n(X_{\overline{\eta}}, \mathbb{Q}_\ell(m))$,

Let $E \subset \mathrm{H}^n(X_{\overline{\eta}}, \mathbb{Q}_\ell)$ be the span of the vanishing cycles (get well defined span without twisting since over algebraically closed field).

Proposition 29.3. *E is stable under $\pi_1(L \setminus S)$ with orthocomplement*

$$E^\perp = \mathrm{H}^n(X_{\overline{\eta}}, \mathbb{Q}_\ell)^{\pi_1(L\setminus S)}.$$

Proof. Use Picard-Lefschetz. First observe that $\{I_s\}_{s\in S}$ generate (the tame piece of) $\pi_1^{\text{ét}}(L/S)$ (ultimately b/c I_s generate $\pi_1(\mathbb{P}^1 \setminus S)$ in characteristic 0). Hence, to show E is π_1-stable, it is enough to show that $I_s(\delta_{s'}) \subset E$, but this holds since

$$\sigma_s(\delta_{s'}) = \delta_{s'} \pm t(\sigma_s)(\delta_s \cup \delta_{s'})\delta_s \in E.$$

Now, say $x \in E^\perp$. Then, $\sigma_s(x) = x$ so $E^\perp \subset \mathrm{H}^n(X_{\overline{\eta}}, \mathbb{Q}_\ell)^{\pi_1^{\text{ét}}}$, and the other direction is the same argument. ■

Remark 29.4. We want to study the following filtration of middle cohomology

$$0 \subset E \cap E^\perp \subset E \subset \mathrm{H}^n(X_{\overline{\eta}}, \mathbb{Q}_\ell).$$

Note that $E \cap E^\perp$ is constant (local system) by above. Similarly, $\mathrm{H}^n(X_{\overline{\eta}}, \mathbb{Q}_\ell)/E$ is also constant (by Picard-Lefschetz). The interesting piece is $E/(E \cap E^\perp)$. ○

[60] Picture a torus. Picture a red loop going the long way around, and a blue loop going the short way around. We degenerate by pinching the blue loop (giving a nodal curve, i.e. $\mathbb{P}^1$ with two points glued together). The red loop is the "other loop" in the sentence before this footnote.

We would like to apply MAIN LEMMA 27.2 to $E/(E \cap E^{\perp})$.

Proposition 29.5.

(1) *char poly of Frob on $E/E \cap E^{\perp}$ are rational*

(2) *There's non-deg skew-symmetric pairing*

$$\psi : \frac{E}{E \cap E^{\perp}} \times \frac{E}{E \cap E^{\perp}} \longrightarrow \mathbb{Q}_{\ell}(-n)$$

(3) *The image of*

$$\pi_1(L \setminus S) \to \mathrm{Sp}(E/E \cap E^{\perp}, \psi)$$

is open.

Proof. **(1)** Skip. Non-trivial but "not that bad."

(2) Pairing inherited from cup product. This is obvious skew-symmetric since cup product is. Why non-degenerate? Well, we modded out by the orthocomplement.

(3) First step is to show $E/E \cap E^{\perp}$ is absolutely simple as a π_1-rep. Use Picard-Lefschetz + vanishing cycles all conjugate to each other. Then, use the following result

Theorem 29.6 (Kazhdan-Marguliz)**.** *Let ψ be a non-deg symplectic form on a $\mathbb{Q}_{\ell}$-vector space W, and let $G \subset \mathrm{Sp}(W, \psi)$ be a closed subgroup s.t.*

(a) *W is absolutely simple G-module (i.e. simple over algebraic closure)*

(b) *G is generated by **transvections** $x \mapsto x \pm \psi(x, \delta)\delta$*

Then, G contains an open subgroup of $\mathrm{Sp}(W, \psi)$.

Proof sketch. First show G is an ℓ-adic Lie subgroup. Then it is enough to show that $\mathrm{Lie}\, G = \mathrm{Lie}\, \mathrm{Sp}(W, \psi)$ (and use ℓ-adic exponential). Property **(b)** tells us that $\mathrm{Lie}\, G$ is generated by $x \mapsto \pm\psi(x, \delta)\delta$. At this point, one is reduced to linear algebra. ■

■

Thus, $E/E \cap E^{\perp}$ satisfies the hypotheses of the MAIN LEMMA 27.2.

30 Lecture 30

Plan today

- sketch remainder of proof of RH
- statement of main theorem in Weil II
- + some applications

30.1 Outline of proof of Riemann Hypothesis

We have reduced things to the following

Claim 30.1. *Let X be even dimensional of dimension $n+1$. The eigenvalues α of* Frob $\curvearrowright \mathrm{H}^{n+1}(X_{\overline{k}}, \mathbb{Q}_\ell)$ *satisfy the inequality*

$$q^{n/2} \leq |\alpha| \leq q^{n/2+1}.$$

The proof of this is via induction on n

(1) Find an embedding $X \hookrightarrow \mathbb{P}^N$ s.t. there exists a Lefschetz pencil L.

Over an algebraically closed field, can do this (Theorem 27.5), and the claim is unphased by base extension, so find one over $\overline{k}$ and then extend to a large enough (finite) subfield for this pencil to still be defined.

(2) Enough to show theorem for $\mathrm{Bl}_{\text{base locus}(L)} X$ because $\mathrm{H}^*(X) \hookrightarrow \mathrm{H}^*(\mathrm{Bl}\, X)$, so we will replace X with this blowup.

(3) Have Lefschetz fibration $\pi : \mathrm{Bl}\, X \to \mathbb{P}^1$. We want to understand H^{n+1} of this blowup, so use Leray

$$\mathrm{H}^i(\mathbb{P}^1, R^j\pi_*\mathbb{Q}_\ell) \implies \mathrm{H}^{i+j}(\mathrm{Bl}\, X, \mathbb{Q}_\ell).$$

The interesting groups are (want to show Frobenius eigenvalues satisfy desired inequality in each of these groups)

(i) $\mathrm{H}^2(\mathbb{P}^1, R^{n-1}\pi_*\mathbb{Q}_\ell)$

This sheaf $R^{n-1}\pi_*\mathbb{Q}_\ell$ is actually constant, so this is simply $\mathrm{H}^2(\mathbb{P}^1, \underline{\mathrm{H}^{n-1}(\text{fiber})}) = \mathrm{H}^{n-1}(\text{fiber})(-1)$. The fiber is a variety of odd dimension n. Taking a hyperplane section Z, get a map

$$\mathrm{H}^{n-1}(\text{fiber}) \hookrightarrow \mathrm{H}^{n-1}(Z)$$

(weak Lefschetz for injectivity). The RHS is middle cohomology of an even dimensional variety, so win by induction.

(ii) $\mathrm{H}^1(\mathbb{P}^1, R^n\pi_*\mathbb{Q}_\ell)$

This is the sheaf we studied in terms of vanishing cycles, so recall the subspace $E \subset R^n\pi_*\mathbb{Q}_\ell$ spanned by vanishing cycles. Let $E^\perp$ be its orthocomplement. We break things into the pieces

(a) $R^n\pi_*\mathbb{Q}_\ell/E$

This is a constant sheaf by Picard-Lefschetz. Done by Weak Lefschetz.

(b) $E/E \cap E^\perp$

Saw last time that this satisfies hypotheses of MAIN LEMMA 27.2, so we win in this case.

(c) $E \cap E^\perp$

This is a constant sheaf by Picard-Lefschetz. Done by Weak Lefschetz.

(iii) $\mathrm{H}^0(\mathbb{P}^1, R^{n+1}\pi_*\mathbb{Q}_\ell)$

In good situations, this is again a constant sheaf. Use the same argument as in **(i)** + Poincaré duality.

30.2 Weil II + Applications

(A special case of?) the main theorem of Weil II is the following:

Theorem 30.2 (Deligne)**.** *Suppose $U/\mathbb{F}_q$ is a smooth geometrically connected (possibly non-proper) curve, and let $\mathscr{F}$ be a $\mathbb{Q}_\ell$-sheaf on U which is **pure of weight zero**, i.e. $F_x \curvearrowright \mathscr{F}_x$ has eigenvalues α s.t. $|\alpha| = 1$ (under any embedding). Then, $\mathrm{H}^1_c(U_{\overline{\mathbb{F}}_q}, \mathscr{F})$ is **mixed of weights** ≤ 1, i.e. eigenvalues α of Frobenius satisfy $|\alpha| = q^{i/2}$ where $i \in \mathbb{Z}_{\geq -1}$.*

Can get in this situation e.g. by twisting a sheaf which is pure of some other weight

Corollary 30.3 (by PD)**.** $\mathrm{H}^1(U_{\overline{\mathbb{F}}_q}, \mathscr{F})$ *is mixed of weights* ≥ 1*.*

In particular, the eigenvalues in the corollary can't have absolute value 1 (in more particular, they can't be 1).

30.2.1 Application 1: Semisimplicity of some monodromy

Say $\pi : X \to U$ is a smooth proper morphism of varieties over a field k. For each i, get

$$\rho_i : \pi_1^{\text{ét}}(U_{\overline{k}}, u) \to \mathrm{GL}\left((R^i\pi_*\mathbb{Q}_\ell)_u\right)$$

(monodromy representation on fiber over u, or something like that).

Theorem 30.4 (Deligne)**.** *These representations ρ_i are semisimple, i.e. direct sums of irreducible representations.*

Proof. Step **(i)** reduce to case of finite fields. Note π, X, U are all defined over some f.g. $\mathbb{Z}$-algebra (only finitely many coefficients appear in description of everything). Then specialize to finite field[61]

Step **(ii)**. Let $E \subset R^i\pi_*\mathbb{Q}_\ell$ be a locally constant subsheaf. Want to show it has a complement, i.e. that

$$0 \longrightarrow E \longrightarrow R^i\pi_*\mathbb{Q}_\ell \longrightarrow F \longrightarrow 0$$

splits. This extension gives an element of

$$\mathrm{Ext}^1_{\pi_1^{\text{ét}}(U_{\overline{\mathbb{F}}_q})}(F, E).$$

In fact it lives in a subgroup of this, it lives in the Frobenius invariants

$$\mathrm{Ext}^1_{\pi_1^{\text{ét}}(U_{\overline{\mathbb{F}}_q})}(F, E)^{\mathrm{Frob}}$$

(after replacing k with finite extension so things descend to ground field).

Step **(iii)** show $\mathrm{Ext}^1_{\pi_1^{\text{ét}}(U_{\overline{\mathbb{F}}_q})}(F, E)^{\mathrm{Frob}} = 0$. This group is the same as

$$\mathrm{H}^1(\pi_1^{\text{ét}}(U_{\overline{\mathbb{F}}_q}), \mathrm{Hom}(F, E))^{\mathrm{Frob}} = \mathrm{H}^1_{\text{ét}}(U_{\overline{\mathbb{F}}_q}, \mathrm{Hom}(F, E))^{\mathrm{Frob}}.$$

Note that F, E have the same weight i (both subquotients of $R^i\pi_*\mathbb{Q}_\ell$), so $\mathrm{Hom}(F, E)$ has weight 0. Weil II then tells us that $\mathrm{H}^1(U_{\overline{\mathbb{F}}_q}, \mathrm{Hom}(F, E))$ is mixed of weights ≥ 1 (in fact, of weights $1, 2$), so 1 is not an eigenvalue of Frob and we win. ∎

[61] something something specialization map surjective on prime-to-p fundamental group something something

30.2.2 Application 2: Chebotarev

Theorem 30.5 (Serre)**.** *Suppose X is normal with G-cover $f : Y \to X$, i.e. f finite étale w/ Galois group G. Here, X, Y are geometrically connected $\mathbb{F}_q$-varieties. Let $\rho : \pi_1^{ét}(X) \to G$ be the representation determined by f. Let $C \subset G$ be a conjugacy class. Then,*

Remember: A finite étale X-scheme is a finite set with action of $\pi_1^{ét}(X)$

$$\lim_{n\to\infty} \frac{\#\{x \in X(\mathbb{F}_{q^n}) \mid \mathrm{Frob}_x \in C\}}{\#X(\mathbb{F}_{q^n})} = \frac{\#C}{\#G}$$

(we will even see explicit error terms).

Serre did this using Lang-Weil estimates.

Remember: Image of Frob under $x_* : \pi_1^{ét}(\mathrm{Spec}\,\mathbb{F}_{q^n}) \to \pi_1^{ét}(X)$ is well-defined up to conjugacy, called Frob_x

Proof. Let $\mathbf{1}_C : G \to \mathbb{Q}_\ell$ be the indicator function of C. We want to count

$$\sum_{x \in X(\mathbb{F}_{q^n})} \mathbf{1}_C(\rho(\mathrm{Frob}_x)).$$

This looks somewhat Lefschetz-y. Note that we have a basis of (class) functions on G given by characters, so we can rewrite

$$\sum_{x \in X(\mathbb{F}_{q^n})} \mathbf{1}_C(\rho(\mathrm{Frob}_x)) = \sum_{x \in X(\mathbb{F}_{q^n})} \sum_{\chi \in \mathrm{Ch}(G)} a_i \chi_i(\rho(\mathrm{Frob}_x))$$

where

$$a_i = \frac{1}{\#G} \sum_{\chi_i \in \mathrm{Ch}(G)} \#C \cdot \chi_i([C]).$$

That is,

$$\sum_{x \in X(\mathbb{F}_{q^n})} \mathbf{1}_C(\rho(\mathrm{Frob}_x)) = \frac{\#C}{\#G} \sum_{x \in X(\mathbb{F}_{q^n})} \sum_{\chi_i \in \mathrm{Ch}(G)} \chi_i([C]) \chi_i(\rho(\mathrm{Frob}_x)).$$

We'd like to interpret this sheaf-theoretically. Given χ_i we get a $\mathbb{Q}_\ell$-sheaf $\mathscr{F}_{\chi_i}$ associated to the representation

Remember: A $\mathbb{Q}_\ell$-sheaf is the same thing as a continuous $\mathbb{Q}_\ell$-linear $\pi_1^{ét}$-representation

$$\pi_1^{ét}(X) \xrightarrow{\rho} G \xrightarrow{\rho_{\chi_i}} \mathrm{GL}_{n_i}(\mathbb{Q}_\ell)$$

(χ_i is a character, i.e. trace of a representation ρ_{χ_i}). Unwinding definitions, we have that

$$\chi_i(\rho(\mathrm{Frob}_x)) = \mathrm{Tr}\left(\mathrm{Frob}_x \curvearrowright (\mathscr{F}_{\chi_i})_x\right)$$

(χ_i is the trace of ρ_{χ_i}). Hence, the sum from before becomes

$$\frac{\#C}{\#G} \sum_{\chi \in \mathrm{Ch}(G)} \sum_{x \in X(\mathbb{F}_{q^n})} \chi_i(C)\, \mathrm{Tr}\left(\mathrm{Frob}_x \mid (\mathscr{F}_{\chi_i})_x\right) = \frac{\#C}{\#G} \sum_{\chi_i \in \mathrm{Ch}(G)} \chi_i(C) \sum_{i=0}^{2\dim X} (-1)^i\, \mathrm{Tr}(\mathrm{Frob} \mid \mathrm{H}_c^i(X_{\overline{\mathbb{F}}_q}, \mathscr{F}_{\chi_i})).$$

Note that this representation is weight 0 since it factors through a finite group, so eigenvalues are roots of unity

Two possibilities

(i) χ_i trivial

Get $\sum_{i=0}^{2\dim X} \mathrm{Tr}(\mathrm{Frob} \mid \mathrm{H}_c^i(X_{\overline{\mathbb{F}}_q}, \mathbb{Q}_\ell)) = \#X(\mathbb{F}_{q^n})$.

(ii) χ_i non-trivial. Want

$$T(n,\chi) := \sum_{i=0}^{2\dim X} \mathrm{Tr}\left(\mathrm{Frob} \mid \mathrm{H}^i_c(X_{\overline{\mathbb{F}}_q}, \mathscr{F}_{\chi_i})\right)$$

small, i.e. $T(n,\chi)/\#X(\mathbb{F}_{q^n}) \to 0$ as $n \to \infty$. Now, note that $\mathrm{H}^{2\dim X}_c(X_{\overline{\mathbb{F}}_q}, \mathscr{F}_{\chi_i})$ is dual to $\mathrm{H}^0(X_{\overline{\mathbb{F}}_q}, \mathscr{F}^\vee_{\chi_i}(\dim)) = 0$ because χ_i has no fixed part (non-trivial irreducible). Get contributions to the sum $T(n,\chi)$ from $\mathrm{H}^i_c(X_{\overline{\mathbb{F}}_q}, \mathscr{F}_{\chi_i})$ for $i < 2\dim X$, so the eigenvalues of Frob have absolute value $q^{ni/2}$ for $i < 2\dim X$. Thus,

Question: What?

Note the Betti numbers appearing in the error term here

$$T(n,\chi) \le q^{\frac{2\dim X - 1}{2}\cdot n} \sum \dim \mathrm{H}^i_c(X_{\overline{\mathbb{F}}_q}, \mathscr{F}_{\chi_i}),$$

so $T(n,\chi)/q^{n\dim X} \to 0$ as $n \to \infty$. Since $\#X(\mathbb{F}_{q^n})$ grows like $q^{n\dim X}$, we win.

■

31 List of Marginal Comments

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