My research interests lie in **arithmetic geometry** and **arithmetic statistics**, especially as they relate to the **arithmetic of curves** (both scheme-y and stacky curves). My research has typically leaned towards finiteness results, obtained by applying a mix of techniques from **number theory**, **algebraic geometry**, and **topology**. My current and future work relates to the following topics:

(**Arithmetic statistics of Selmer groups**, Section 1) In [Ach23], I complete the proof that, over *any* global field, the average rank of elliptic curves is finite. I propose to improve known upper bounds in the function field case (see Problems 1 and 2) as well as to study the theoretical limits of the most commonly used strategy for obtaining such bounds (see Problem 4).

(**Brauer groups of stacky curves**, Section 2) I have developed techniques for computing Brauer groups of stacky curves [ABJ+24, Ach24]. I propose to use these to compute the integral étale-Brauer obstruction set for the moduli stack $\mathcal{Y}(1)$ of elliptic curves (see Problem 5) with the hopes of giving a new proof that there is no elliptic scheme over $\mathbb{Z}$.

(**Arithmetic of high-dimensional varieties**, Section 3) In [AM23], we verified a consequence of Vojta's conjecture on integral points by proving that a nice high-dimensional variety X – if it has infinite étale fundamental group – supports infinitely many *irreducible* divisors D such that $X - D$ has only finitely many integral points. The novelty here is that most such results require D to have many irreducible components.

1. Arithmetic Statistics of Selmer Groups

Arithmetic statistics is a field concerned with understanding the distributions, suitably defined, of various objects of number theoretic interest (finite extensions of $\mathbb{Q}$, class groups of number fields, Selmer groups of elliptic curves, etc.). After ordering the objects under consideration by a suitable defined height function, one is often interested in getting an asymptotic count for them as their heights tend towards infinity and/or in computing average sizes or moments of invariants attached to them.

Of particular relevance to my own work is the study of statistics of ranks of elliptic curves. Recall that an elliptic curve E over a field F is a smooth, projective genus 1 curve equipped with chosen point $0 \in E(F)$ defined over F. This choice of point defines a group law on $E(F)$ with 0 serving as the identity. When F is a global field – i.e. a number field ($\mathbb{Q}$ or a finite extension thereof) or a global function field ($\mathbb{F}_q(t)$, for some prime power q, or a finite extension thereof) – the abelian group $E(F)$ is finitely generated and so one can speak of the *rank* of E.

In this context, the main conjectures are as follows.

Conjecture 1.1 (Minimalist Conjecture, folklore)**.** *Fix any global field K. Half of elliptic curves E/K have rank* 0 *and half have rank* 1*.*

Most recent progress on this conjecture comes, not from directly studying ranks of elliptic curves, but from studying sizes of their Selmer groups. Here, for every $n \geq 1$ and any elliptic curve E/K, the *n-Selmer group* $\mathrm{Sel}_n(E/K)$ is a specific finite $\mathbb{Z}/n\mathbb{Z}$-module which admits an injection $E(K)/nE(K) \hookrightarrow \mathrm{Sel}_n(E/K)$. One consequence of this injection is that $n^{\operatorname{rank} E(K)} \leq \# \mathrm{Sel}_n(E/K)$, so Selmer groups allow one to bound ranks. Correspondingly, the distribution of sizes of Selmer groups has received much attention; in particular, a conjectural formula for this distribution has been proposed in [BKL+15] (building off of previous heuristics of Poonen-Rains [PR12]). One major prediction of this conjectural distribution is the following.

Conjecture 1.2 ([BKL+15])**.** *Fix any global field K and any $n \geq 1$. The average size $\mathbb{E}[\# \mathrm{Sel}_n(E/K)]$ of n-Selmer groups of elliptic curves E/K is the sum of the divisors of n.*

Proving Conjecture 1.2, in addition to understanding the distribution of parities of ranks of elliptic curves (which, under BSD, is related to the root number of the curve's L-function), would prove Conjecture 1.1. Evidence for Conjecture 1.2 goes back to work of de Jong [dJ02, Theorem 1.2] who computed that $\mathbb{E}[\# \mathrm{Sel}_3(E/\mathbb{F}_q(t))] \leq 4 + o(1)$ as $q \to \infty$. Studying Conjecture 1.2 gained even more popularity after work of Bhargava and Shankar [BS15a, BS15b, BS13a, BS13b] verified that $\mathbb{E}[\#\operatorname{Sel}_n(E/\mathbb{Q})] = \sum_{d|n} d$ for $n = 1, 2, 3, 4, 5$. Since then, many other authors have verified Conjecture 1.2, or variations of it, in a variety of settings, both over number fields and function fields; see e.g. [Sha13, HLHN14, Tho19, Lan21a, FLR23, PW23, EL24]. One common feature of work on this conjecture in the function field case is that authors either usually restrict the characteristics of the function fields over which they work or only work over genus 0 function fields, but neither of these restrictions appear in Conjecture 1.2. In contrast, in my preprint [Ach23], I was able to prove the following theorem for any global function field.

Theorem 1.3 ([Ach23, Theorem B])**.** *Let $K = \mathbb{F}_q(B)$ be the function field of a nice*[1] *curve $B/\mathbb{F}_q$, and let ζ_B be its zeta function. Then,*

$$\mathbb{E}[\#\operatorname{Sel}_2(E/K)] \le 1 + 2\zeta_B(2)\zeta_B(10),$$

This implies $\lim_{n\to\infty} \mathbb{E}[\#\operatorname{Sel}_2(E/\mathbb{F}_{q^n}K)] \le 3$.

This is the first paper proving such a result for a truly arbitrary global function field. Combined with work of Arul Shankar, this has the following aesthetically pleasing consequence.

Theorem 1.4 ([Ach23, Sha13])**.** *Let K be any global field. Then, $\mathbb{E}[\operatorname{rank} E(K)]$ is finite.*

The above Theorem 1.4 is not a priori obvious, and in fact was not known for certain (high genus, low characteristic) function fields before [Ach23]. It is a priori conceivable that there exists enough elliptic curves E/K of unbounded ranks to force the average to be infinite. In fact, it is known [Ulm02] that ranks of elliptic curves over a global function field are unbounded. However, Theorem 1.3 tells us that curves with truly large ranks are rare enough for the average to remain bounded.

1.1. **Proposed Project(s).** The first project I propose is to improve the elliptic curve average rank bounds I obtained in [Ach23], especially over function fields of characteristic 2. A first very natural continuation of [Ach23] would be to strengthen Theorem 1.3 to state that $\mathbb{E}[\#\operatorname{Sel}_2(E/K)] \le 3$ over any fixed K. This would mainly involve computing local densities for 'hyper-Weierstrass curves' [Ach23, Definition 4.1.2] which are minimal (have at worst rational singularities) and locally solvable.

Problem 1. *Prove that $\mathbb{E}[\#\operatorname{Sel}_2(E/K)] \le 3$ for K as in Theorem 1.3.*

A further extension of this would would be to prove analogous bounds for the average size of n-Selmer, at least for $n \le 5$ (as in the work of Bhargava and Shankar). This should be doable by combining ideas introduced in [Ach23] with those present in [dJ02, BS15b, BS13a, BS13b]. In the case $n = 3$, [dJ02, Theorem 1.2] proves an analogue of Theorem 1.3 for $\mathbb{F}_q(t)$, but the $n \ge 3$ case is still open for higher genus function fields. The cases for $n \ge 4$, as far as I am aware, remain completely open in the function field setting.

Problem 2. *Prove an analogue of Theorem 1.3 for $n = 3, 4, 5$.*

Remark 1.5. As explained in [Cow21, Section 4], Problem 2 (at least for $K = \mathbb{F}_q(t)$) is also related to Alex Cowan's conjecture that 100% of elliptic *surfaces* over $\mathbb{Q}$ have rank 0 (when viewed as elliptic curves over $\mathbb{Q}(t)$). Spelled out slightly, solving Problem 2 (combined with the relatively easier task of producing a large family of elliptic curves in which the parity of the rank is provably equidistributed) would allow one to conclude that $\mathbb{E}[\operatorname{rank} E(K)] \le 1 - \delta$ for some $\delta > 0$. Once one has a statement like this, [Cow21, Section 4] shows how a Chinese remainder theorem argument almost let's one deduce that 100% of elliptic surfaces over $\mathbb{Q}$ have rank 0, his [Cow21, Conjecture 1.1]. In any case, proving that $\mathbb{E}[\operatorname{rank} E(K)] < 1$ (as would be feasible if one solves Problem 2) would give strong evidence towards [Cow21, Conjecture 1.1]. ◦

The previously suggested problems could be tackled using the parameterize-and-count meta-strategy often employed in arithmetic statistics. However, for handling cases of Conjecture 1.2 in the function field

[1] smooth, projective, and geometrically connected

setting, there exists a second fruitful approach. It is possible to relate this problem to that of counting $\mathbb{F}_q$-points on certain moduli spaces of n-Selmer elements and then essentially reduce this counting problem to controlling the cohomology groups of the involved moduli spaces. Arguments along these lines have been successfully made by Aaron Landesman and his collaborators [Lan21b, FLR23, EL24], at least in the case of good characteristic function fields. It would be interesting to extend these techniques to bad characteristic function fields, at least starting with the analogue of [Lan21b]. In this case, as mentioned in [Lan21b, Remark 1.10], the main obstacle is to compute the images of certain (wildly ramified) monodromy representations.

Problem 3. *Extend the main result of* [Lan21b] *to characteristic* 2*, say for n-Selmer with n odd.*

As previously alluded to, one generally bounds the average size of n-Selmer via a parameterize-and-count argument where the first step is find a suitable parameterization/description of the n-Selmer elements to count. Most commonly, one (implicitly or explicitly) parameterizes n-Selmer elements – say for $n \geq 3$ – by representing them as (locally solvable) nice degree n, genus 1 curves embedded in $\mathbb{P}^{n-1}$. One expects that these objects *cannot* be parameterized once n is large enough, and it would be interesting to prove this for some explicit value of n. Let $\mathcal{M}_n$ denote the moduli space of "nice degree n, genus 1 curves embedded in $\mathbb{P}^{n-1}$" and let H_n denote the Hilbert scheme of such curves, so $\mathcal{M}_n \simeq [H_n/\operatorname{PGL}_n]$. In practice, a parameterization of such curves (e.g. a description of the equations needed to cut out such a curve in $\mathbb{P}^{n-1}$) amounts to a dominant rational map $f\colon \mathbb{A}^N \dashrightarrow H_n$, for some N.[2] Therefore, if H_n is not unirational, there can be no such parameterization, and so the "parameterize–and–count" strategy may be less feasible for such an n (of course, this would not necessarily rule out the possibility of using some other clever parameterization). It would be interesting, to prove that some H_n is not unirational.[3]

Problem 4. *Find an explicit n such that H_n is not unirational.*

Remark 1.6. Various candidate moduli spaces of Selmer elements have been studied before, e.g. in [dJF08, Section 4] and in [Lan]. Furthermore, rationality problems for moduli spaces of curves have been considered by many authors, including Joe Harris and his collaborators (see e.g. [HM82, Har84, EH84]). The ideas present in these works can serve as a starting point for considering Problem 4. ◦

One can also ask an analogue of Problem 4 in the number field counting setting (e.g. in the setting of questions such as "How many S_n-extensions of $\mathbb{Q}$ are there of discriminant bounded?"). Here, one is interested in parameterizing rank n free $\mathbb{Z}$-algebras, and again, one suspects this is not possible for n sufficiently large, so one can ask the analogous question of proving this for some explicit value of n. The relevant moduli space here has previously been studied in [Poo08].

2. Cohomology of Stacky Curves

An *algebraic stack* is a generalization of a scheme/variety which allows spaces to have stabilizer/automorphism groups attached to each point. They provide a particularly useful context for studying moduli spaces (spaces whose points parameterize other geometric objects of interest) because the objects parameterized often have automorphism. For example, the moduli space $\mathcal{Y}(1)$ of elliptic curves exists as an algebraic stack, but does *not* exist as a variety because an elliptic curve E always has at least one non-trivial automorphism (e.g. negation for its group law).

[2] Note that elements of n-Selmer are represented by *locally solvable* genus 1 curves. For K any global field, any K-point of $\mathcal{M}_n$ representing a Selmer element will lift to a K_v-point of H_n for all places v of K. Because PGL_n-torsors satisfy a local-to-global principal, this means such a K-point lifts to a K-point of H_n, so, for counting Selmer elements, it suffices to find a parameterization of $H_n(K)$ instead of $\mathcal{M}_n(K)$.

[3] In contrast to H_n, one can identify $\mathcal{M}_n \simeq \operatorname{Pic}^n_{\mathcal{C}/\mathcal{M}}$, where $\mathcal{C} \to \mathcal{M}$ is the universal (unpointed) genus 1 curve. This space is unirational because $\mathcal{M}_1$ (the moduli space of elliptic curves) is and the 'multiplication map' $\mathcal{M}_1 = \operatorname{Pic}^1_{\mathcal{C}/\mathcal{M}} \xrightarrow{\times n} \operatorname{Pic}^n_{\mathcal{C}/\mathcal{M}} = \mathcal{M}_n$ is dominant (though far from surjective on $\mathbb{Q}$-points, for example).

There is much interest in studying moduli spaces, because their geometry has consequences for the objects they parameterize, and in particular, in computing the cohomology of moduli spaces and other algebraic stacks. Of particular relevance to my work, is studying the Brauer groups of these spaces. Brauer groups $\mathrm{Br}(-) \coloneqq \mathrm{H}^2_{\text{ét}}(-, \mathbb{G}_m)_{\text{tors}}$ of fields/schemes/stacks are particular geometric invariants which arise in connection to class field theory, ℓ-adic cohomology, and to obstructions to (rational and integral) points on varieties. In recent years, there has been increasing interest in understanding Brauer groups of algebraic stacks and especially of stacky curves. One can see, for example, the work of Antieau–Meier and others [AM20, Shi19, LP22] computing the Brauer group of $\mathcal{Y}(1)$ and the work of Santens [San23] studying Brauer–Manin obstructions to integral points on stacky curves.

As an example of the results thus far obtained, the stack $\mathcal{Y}(1)$ is arguably the most important stacky curve, and accordingly, its Brauer group has received the most attention. Work of many authors has produced the following result.

Theorem 2.1 ([AM20, Shi19, Mei18, LP22])**.**

(1) Let $R = \mathbb{Z}, \mathbb{Z}[1/2]$, or a field of characteristic not 2. Then,

$$\mathrm{Br}\,\mathcal{Y}(1)_R \simeq \mathrm{Br}\,\mathbb{A}^1_R \oplus \mathrm{H}^1_{\acute{e}t}(R, \mathbb{Z}/12\mathbb{Z}).$$

(2) Let k be a field of characteristic 2. Then, there is a short exact sequence

$$0 \longrightarrow \mathrm{Br}\,\mathbb{A}^1_k \oplus \mathrm{H}^1_{\acute{e}t}(k, \mathbb{Z}/12\mathbb{Z}) \longrightarrow \mathrm{Br}\,\mathcal{Y}(1)_k \longrightarrow \mathbb{Z}/2\mathbb{Z} \longrightarrow 0.$$

(3) Let ℓ be a prime and let $S/\mathbb{Z}[1/\ell]$ be a separated, regular, noetherian scheme. Then,

$$\mathrm{Br}(\mathcal{Y}(1)_S)_{(\ell)} \simeq \mathrm{Br}(S)_{(\ell)} \oplus \mathrm{H}^1_{\acute{e}t}(S, \mathbb{Z}/12\mathbb{Z})_{(\ell)}.$$

In [ABJ+24] – written in collaboration with Deewang Bhamidipati, Aashraya Jha, Caleb Ji, and Rose Lopez – we extended some of the techniques used in Theorem 2.1 to compute the Brauer group of the moduli stack $\mathcal{Y}_0(2)$ of elliptic curves equipped with an étale subgroup of order 2. Namely, we found that

$$\mathrm{Br}\,\mathcal{Y}_0(2)_R \simeq \mathrm{Br}\,Y_0(2)_R \oplus \mathrm{H}^1_{\text{ét}}(R, \mathbb{Z}/4\mathbb{Z}) \oplus \mathbb{Z}/2\mathbb{Z}, \tag{1}$$

where $Y_0(2) = \mathbb{A}^1 \setminus \{0\}$ is the coarse moduli space of $\mathcal{Y}_0(2)$. In later work [Ach24], I studied techniques for computing Brauer groups of (tame) stacky curves more generally. For example, I prove

Theorem 2.2 ([Ach24, Theorem B])**.** *Let S be a noetherian $\mathbb{Z}[1/2]$-scheme.*

(1) $\mathrm{Br}\,\mathcal{X}(1)_S \simeq \mathrm{Br}\,\mathbb{P}^1_S \simeq \mathrm{Br}\,S$, where $\mathcal{X}(1)$ is the moduli stack of generalized elliptic curves.

(2) If S is regular, then there is an explicit isomorphism

$$\mathrm{Br}\,\mathbb{A}^1_S \oplus \mathrm{H}^1_{\acute{e}t}(S, \mathbb{Z}/12\mathbb{Z}) \xrightarrow{\sim} \mathrm{Br}\,\mathcal{Y}(1)_S.$$

(3) If S is regular, then there is an explicit isomorphism

$$\mathrm{Br}(\mathbb{A}^1_S \setminus \{0\}) \oplus \mathrm{H}^1_{\acute{e}t}(S, \mathbb{Z}/4\mathbb{Z}) \oplus \mathrm{H}^0_{\acute{e}t}(S, \mathbb{Z}/2\mathbb{Z}) \xrightarrow{\sim} \mathrm{Br}\,\mathcal{Y}_0(2)_S,$$

where $\mathcal{Y}_0(2)$ is the moduli stack of elliptic curves equipped with a subgroup of order 2.

Note that this generalizes both Theorem 2.1 (away from characteristic 2) and (1).

Furthermore, in the paper [ABJ+24], we observed that $\mathrm{Br}\,\mathcal{Y}_0(2)_{\overline{k}} = \mathbb{Z}/2\mathbb{Z}$, when k is a field of characteristic not 2. This gave the first example of a tame stacky curve with nontrivial Brauer group over an algebraically closed fields. For a schemey curve X/k, Tsen's theorem says that $\mathrm{Br}\,X_{\overline{k}} = 0$ and even for the stacky curve $\mathcal{Y}(1)$, [AM20] had shown that $\mathrm{Br}\,\mathcal{Y}(1)_{\overline{k}} = 0$ if $\mathrm{char}\,k \neq 2$ (note that $\mathcal{Y}(1)$ is nowhere tame in characteristic 2). This led us to pose the question of what was governing this behavior; what aspects of the geometry of a stacky curve would let you predict if it has nontrivial Brauer group over an algebraically closed field. This question was addressed in [Ach24].

Theorem 2.3 (special case of [Ach24, Proposition 7.10])**.** *Let k be an algebraically closed field, and let $\mathcal{X}/k$ be a regular cyclotomic*[4] *stacky curve with generic stabilizer group G/k. Then,* $\operatorname{Br}\mathcal{X} \simeq \mathrm{H}^1_{\acute{e}t}(X, G^\vee)$.

Theorem 2.3 applies in particular to all tame modular curves and gives a geometric explanation for the observation in [ABJ+24] that $\mathcal{Y}(1)$ has trivial Brauer group over algebraically closed fields (in tame characteristics) while $\mathcal{Y}_0(2)$ has nontrivial Brauer group for these fields. The relevant geometric difference is that the coarse space $\mathbb{A}^1$ of $\mathcal{Y}(1)$ is simply connected (more specifically, has no non-trivial double covers) while the coarse space of $\mathcal{Y}_0(2)$ is not.

The paper [Ach24] also contains a study of residue maps for Brauer groups of stacks, including some new results on Picard groups of stacky curves [Ach24, Theorem D] and a 'stacky Faddeev sequence' [Ach24, Theorem E], generalizing the usual Faddeev sequence [CTS21, Section 1.5] often used to compute Brauer groups of (affine) rational curves.

2.1. **Proposed Project(s).** A personal motivation for my work [ABJ+24, Ach24] on Brauer groups is the hope of computing (étale) Brauer–Manin obstruction sets on stacky curves. Santens studied this problem when the stacky curve in question contains a dense, open subscheme, but I am not aware of work looking at examples which are everywhere stacky. The main such example I have in mind to study is the moduli space $\mathcal{Y}(1)$ of elliptic curves. In particular, I am interested in understanding Brauer groups of modular curves well enough to compute the integral étale Brauer–Manin set for $\mathcal{Y}(1)$. I anticipate that this set is empty, and if so, showing this would give a new proof of Tate's theorem [Ogg66] that there is no elliptic scheme over $\mathbb{Z}$.

Problem 5. *Compute the integral étale-Brauer obstruction set* $\mathcal{Y}(1)(\widehat{\mathbb{Z}} \times \mathbb{R})^{\acute{e}t\text{-}Br}$.

Carrying out such a computation would also serve as a proof-of-concept showcasing that exploiting Brauer groups of moduli stacks can be effective in determining whether certain classes of geometric objects can have everywhere good reduction. It would be my hope that resolving Problem 5 would motivate more people to study Brauer groups of stacks as well as their associated obstructions.

3. Arithmetic of high-dimensional varieties

The basic problem of arithmetic geometry is computing/understanding integral points on varieties (note that 'integral points' = 'rational points' if the variety is proper). In the case of curves, we have the following amazing result of Faltings and Siegel.

Theorem 3.1 (Siegel [Sie14], Faltings [Fal83])**.** *Let X be a (not necessarily proper) hyperbolic curve. Then, every set of integral points on X is finite.*

When $\dim X > 1$, one has the following conjectural generalization. Informally, one says that a smooth quasi-projective variety U is `arithmetically hyperbolic` if it only has finitely many integral points (even after field extension) and calls it `pseudo-arithmetically hyperbolic` if all but finitely many of its integral points belong to some closed subvariety $Z \subsetneq U$ (see e.g. [Jav20] for precise definitions).

Conjecture 3.2 (Lang–Vojta, [Lan86, Conjecture 5.7] and [Voj87, Proposition 4.1.2])**.** *Let X be a nice projective variety over a number field K, and let $D \subset X$ be a normal crossings divisor. If $\omega_X(D)$ is ample, then $X \setminus D$ is pseudo-arithmetically hyperbolic.*

Faltings [Fal91] proved Conjecture 3.2 when X is an abelian variety (and Vojta [Voj96, Voj99] obtained analogous results for semiabelian varieites). Beyond these cases, most work on Conjecture 3.2 has focused on cases where the divisor D has many irreducible components (see e.g. [CZ04, Lev09, Aut11, RV20, RV21]).

[4]An algebraic stack is cyclotomic if all its stabilizer groups of the form μ_n for some n

Remark 3.3. Conjecture 3.2 suggests that, if $\dim X \geq 2$, then it should support infinitely many irreducible divisors D for which $X \setminus D$ is psuedo-arithmetically hyperbolic. Indeed, given such an X, choose an ample line bundle $\mathscr{L}$ on X such that $\omega_X \otimes \mathscr{L}$ is also ample and $\dim \mathrm{H}^0(X, \mathscr{L}) > 1$. Any divisor $D \in |\mathscr{L}|$ is $\mathscr{L}$'s complete linear system will be connected [Har77, Corollary III.7.9] and a general such D will be smooth (so irreducible) by Bertini. ∘

Perhaps the most famous example of a higher-dimensional X (which is not an abelian variety) with irreducible $D \subset X$ for which $X \setminus D$ is *provably* arithmetically hyperbolic comes from work of Faltings [Fal02] where he constructed such D on $X = \mathbb{P}^2$. In the paper [AM23], Jackson Morrow and I constructed new examples of irreducible divisors D on nice varieties X such that $X \setminus D$ is arithmetically hyperbolic.

Theorem 3.4 ([AM23, Corollary B])**.** *Let X be a smooth projective variety over a number field K. Assume that $\dim X \geq 2$ and that $\pi_1^{\acute{e}t}(X_{\overline{K}})$ is infinite. Then, there exists infinitely many ample irreducible divisors $D \subset X$ such that $X \setminus D$ is arithmetically hyperbolic. If $X(K) \neq \emptyset$, then one can arrange that these D are geometrically irreducible.*

3.1. **Proposed Project(s).** While Theorem 3.4 has fairly permissible hypotheses, it is non-constructive. It gives no easy method of testing whether a given divisor $D \subset X$ has an arithmetically hyperbolic complement. Furthermore, because it requires $\pi_1^{\mathrm{\acute{e}t}}(X_{\overline{K}})$ to be infinite, it does not include Faltings' earlier examples [Fal02] of irreducible divisors on projective space with arithmetically hyperbolic complements.

Building on work of Ru [Ru17] and Ru–Vojta [RV20, RV21], one way of addressing the first of these deficiencies would be to introduce and study an "étale Nevanlinna volume constant $\mathrm{Nev}_{\mathrm{\acute{e}t}}(\mathscr{L}, D)$," an analogue of the Nevanlinna constant $\mathrm{Nev}(D)$ of [Ru17, Definition 1.2] which takes into account information from "étale" covers of X; in order to have a hope of constructing a constant powerful enough to recover Faltings' results [Fal02], one should really allow the covers to be ramified along D. It would be interesting to compare such a constant to the Evertse–Ferretti and birational Nevanlinna constants introduced in [RV20, RV21], which took into account information from proper birational maps to X.

Problem 6. *Define and study an "étale Nevanlinna volume constant $\mathrm{Nev}_{\acute{e}t}(\mathscr{L}, D)$," and compare it to the analogous constants appearing in* [RV20, RV21].

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