Title: An Overview of DGH's Proof of Uniform Mordell.
Plan?

- Statement of theorem (specificying simpliciation due to Kühne)
- Uniformity in families by working over $S = \mathbb{M}_g$ (w/ symplectic level structure)
- Height Machine and Vojta's approach/relative statement (Rémond?)
- New gap principle (bounded number of balls because $\max\{1, h(s)\}$)
- non-degenerate subvarities and height bound?

  NON-DEGENERACY $\rightsquigarrow$ line bundle big $\rightsquigarrow$ height lower bound (away from base locus?)
- amplification/fibered powers
- Something about equidistribution?

♠♠♠ Niven: [Go fast, have people interrupt with questions.]

1 Uniformity Intro

Theorem 1 (Dimitrov-Gao-Habegger + Kühne)**.** *Let K be a number field, and let C/K be a smooth curve of genus $g \geq 2$. Then,*

$$\#C(K) \leq c(g, \cancel{[K:\mathbb{Q}]})^{1+\operatorname{rank}\operatorname{Jac}(C)(K)}.$$

Remark 2. DGH really prove that, for $P_0 \in C(\overline{\mathbb{Q}})$ and $\Gamma \leq \operatorname{Jac}(C)(\overline{\mathbb{Q}})$ (e.g.[1] $\Gamma = \operatorname{Jac}(C)(K)$),

$$h(\operatorname{Jac}(C)) \geq c_1(g) \implies \#(C(\overline{\mathbb{Q}}) - P_0) \cap \Gamma \leq c_2(g)^{1+\operatorname{rank}\Gamma}.$$

Kühne handles curves of small modular height. {□ *Where the (modular) height here is the height of the corresponding point on the moduli space of abelian varieties (w.r.t to some choice of line bundle on a compactification of the coarse space)* □} ○

Remark 3. There is a different proof by Yuan-Zhang[2] using "adelic line bundles" of which I know nothing about. ○

Conjecture 4 (Unclear how many people believe this)**.** $\#C(K) \leq c(g, K)$

Remark 5. This strong uniformity conjecture would be implied by the weak Lang conjecture[3] or by boundedness of ranks of Jacobians. ○

Remark 6. There are also some other uniformity results based on Chabauty–Coleman(–Kim). Let $\rho = \operatorname{rank}\operatorname{Jac}(C)(\mathbb{Q})$.

- Katz–Rabinoff–Zureick-Brown: $\#C(\mathbb{Q}) \leq 84g^2 - 98g + 28$ if $\rho \leq g - 3$.
- Betts–Corwin–Leonhardt[4]: $\#C(\mathbb{Q}) \leq$ (huge, but explicit, bound depending on g, ρ, the reduction type of C (e.g. number of componets in each special fiber), and choice of prime p of good reduction) if one assumes the Tate-Shafarevich and Bloch-Kato conjectures. ○

[1] finite rank $\not\Rightarrow$ finitely generated, so also applies to $\Gamma = \operatorname{Jac}(C)(\overline{\mathbb{Q}})_{\mathrm{tors}}$.
[2] Xinyi Yuan, Shou-Wu Zhang
[3] Varieties of general type have non-Zariski dense sets of rational points. The implication weak Lang $\implies$ strong uniform is due to Caparaso–Harris–Mazur.
[4] Alex Betts, David Corwin, Marius Leonhardt

♠♠♠ Niven: [Mention Katz-Rabinoff-Zureick-Brown and/or Betts?]

{☐ *Today, I want to focus on DGH's contribution (e.g. getting a uniform bound for curves of large modular height).* ☐}

{☐ *Their approach is based on Vojta's approach to the Mordell conjecture, so the basic idea is to prove enough height inequalities to deduce that points on C "repel each other" in $\mathrm{Jac}(C)$ in some controlled way. This will make more sense when I state the main ingredients of Vojta's proof of Mordell.* ☐}

2 The Big Picture

♠♠♠ Niven: [Ask if you should recall the height machinery]

Recall 7 (Height Machinery)**.** Let $V/\overline{\mathbb{Q}}$ be a projective variety. There is a unique homomorphism

$$h\colon \mathrm{Pic}(V) \longrightarrow \mathrm{Func}(V(\overline{\mathbb{Q}}), \mathbb{R})/O(1)$$

such that

- If $V = \mathbb{P}^N$, $\mathscr{L} = \mathscr{O}(1)$, and $P \in \mathbb{P}^N(K)$ (for some number field K), then
$$h_{\mathbb{P}^N,\mathscr{L}}(P) = \frac{1}{[K:\mathbb{Q}]}\sum_v \log\max\{\|P_0\|_v, \dots, \|P_N\|_v\} + O(1).$$
- If $\varphi : V \to W$ is a morphism, then
$$h_{V,\varphi^*\mathscr{L}} = h_{W,\mathscr{L}} \circ \varphi + O(1)$$
- $h_{V,\mathscr{L}} \ge O(1)$ away from the base locus[5] of $\mathscr{L}$.
- Assume $\mathscr{L}$ is ample, and let K_0 be a number field over which X is defined. Then, for any $d, B \ge 1$,
$$\#\left\{x \in V(K)\colon [K:K_0] \le d, h_{V,\mathscr{L}}(x) \le B\right\} < \infty.$$

If $V = A$ is an abelian variety, then there is a canonical height machine $\widehat{h}\colon \mathrm{Pic}(A) \to \mathrm{Func}(A(\overline{\mathbb{Q}}), \mathbb{R})$, where you don't need to worry about bounded functions everywhere. ⊙

Setup 8. Fix C/K and a point $P_0 \in C(K)$. Let $C \hookrightarrow J \coloneqq \mathrm{Jac}(C)$ be the associated Abel-Jacobi embedding. Let $\widehat{h}\colon J(\overline{\mathbb{Q}}) \to \mathbb{R}$ be the canonical height associated to $\Theta + [-1]^*\Theta$, where $\Theta = j(C) + \dots + j(C) \in \mathrm{Div}(J)$. Then, $\widehat{h}$ is a quadratic form on $J(\overline{\mathbb{Q}}) \otimes \mathbb{R}$, with associated bilinear form

$$\langle x, y\rangle \coloneqq \frac{1}{2}\left(\widehat{h}(x+y) - \widehat{h}(x) - \widehat{h}(y)\right).$$

Given $x, y \in J(\overline{\mathbb{Q}}) \otimes \mathbb{R}$, write $\cos\theta(x,y) \coloneqq \langle x, y\rangle / |x|\,|y|$.

Theorem 9. *With notation as above, there are constants $B, \kappa \ge 1$ such that for any distinct $P, Q \in C(\overline{K})$ satisfying $|P| \ge |Q| > B$ and*

Big cos means small angle

$$\cos\theta(P,Q) \ge \frac{3}{4},$$

one has

[5] points where every section vanishes

(Mumford's Gap Principle) $2|Q| \leq |P|$. *{☐ Repulsion: points with small angle must have a large difference in magnitude ☐}*

(Vojta's Inequality) $|P| \leq \kappa |Q|$. *{☐ At the same time, large points can't be "too far" away from each other ☐}*

Remark 10. These both ultimately come from some bounds on an appropriate height function on $C \times C$. We'll see something similar at the end of the talk. ∘

♠♠♠ Niven: [Draw picture: (or point to it predrawn on side board?). Explain that this shows that there are at most something like $7^{\operatorname{rank} J(K)}$ large points].

{☐ Earlier work of Rémond allows one to prove a relative version of Vojta's results. Before stating this, let me simplify things by fixing a particular choice of relative curve. ☐}

Notation 11. Fix integers $g \geq 2$ and $\ell \geq 3$.

- Let $S = \mathbb{M}_{g,1}$ be the (fine) moduli space of pointed smooth genus g curves with full (symplectic) level ℓ structure.

  Bit of a lie. Should let $S = \mathbb{M}_g$ instead because of quasi-finiteness worries, but whatever

- Let $\mathcal{C} \to S$ denote the universal curve, with section $P \in \mathcal{C}(S)$ (i.e. $P\colon S \to \mathcal{C}$).
- Let $\mathcal{A} \coloneqq \operatorname{Jac}(\mathcal{C}/S) \xrightarrow{\pi} S$, an abelian scheme of relative dimension g. Embed $\mathcal{C} \hookrightarrow \mathcal{A}$ via the universal section.
- Let $\mathscr{L}$ be a relatively ample, symmetric line bundle on $\mathcal{A}$. This gives rise to a (fiberwise) canonical height $\widehat{h}_{\mathscr{L}} : \mathcal{A}(\overline{\mathbb{Q}}) \to \mathbb{R}$
- Let $\mathscr{M}$ be an ample line bundle on a compactification $\overline{S} \supset S$, with associated height function $h_{\mathscr{M}} : \overline{S}(\overline{\mathbb{Q}}) \to \mathbb{R}$.

{☐ This notation is meant to emphasize that the results I quote really apply to more general families of curves embedded in abelian schemes. ☐}

Proposition 12 (Rémond + DGH)**.** *There exists a constant $c = c(\mathscr{L}, \mathscr{M}) \geq 1$ such that for any $s \in S(\overline{\mathbb{Q}})$ and any finite rank $\Gamma \leq \mathcal{A}_s(\overline{\mathbb{Q}})$:*

$$\#\left\{P \in \mathcal{C}_s(\overline{\mathbb{Q}}) \cap \Gamma : \widehat{h}_{\mathscr{L}}(P) > c\max\{1, h_{\overline{S},\mathscr{M}}(s)\}\right\} \leq c^{\operatorname{rank}\Gamma}.$$

{☐ As s varies, $\mathcal{C}_s$ covers every curve over $\overline{\mathbb{Q}}$ ☐}

♠♠♠ Niven: [State relative version. Notion of "large" varies with modular height]

This leaves dealing with "small" points, where the notion of "small" depends on the modular height of your curve.

Theorem 13 (New Gap Principle, DGH + Kühne)**.** *There exist positive constants $c_1, c_2, \cancel{c_3}$ (depending only on $g, \mathscr{L}, \mathscr{M}$) such that for any $s \in S(\overline{\mathbb{Q}})$, we have*

$$\#\left\{Q \in \mathcal{C}_s(\overline{\mathbb{Q}})\colon \widehat{h}_{\mathscr{L}}(Q-P) \leq c_1 \max\{1, h_{\overline{S},\mathscr{M}}(s)\} \cancel{-c_3}\right\} < c_2.$$

{☐ Again, we have points on C repelling each other. Can't have too many in a tight radius. ☐} ♠♠♠ Niven: [Draw more picture: something like $(\log_2\kappa + 1)7^{\operatorname{rank}\Gamma} + c_2\left(\frac{R}{r}\right)^{\operatorname{rank}\Gamma}$ total points]

Remark 14. Because the radius above grows at the same rate our notion of "large" does, we can cover our "small" points with a uniformly bounded number of balls. ∘

{☐ As with Vojta's theorem, this new gap principle will rely on some height inequality for (certain) subvarieties of $\mathcal{A}$. The buzzphrase here is "non-degenerate" subvariety, and it will take a bit to define. ☐}

3 Non-degeneracy and Amplification

Slogan. Non-degeneracy of $X \subset \mathcal{A}$ more-or-less tells you that $\mathscr{L}|_X$ is 'big'. In particular, its powers have lots of sections, so points on X should have heights bounded from below.

{☐ *To define non-degeneracy, we first need to define the Betti map* ☐}

Construction 15. Let[6] ${\mathbb{A}_g}^{\mathrm{an}} = [\mathbb{H}_g / \mathrm{Sp}_{2g}(\mathbb{Z})]$ be the moduli space of g-dimensional PPAVs over $\mathbb{C}$, and let ${U_g}^{\mathrm{an}} \to {\mathbb{A}_g}^{\mathrm{an}}$ denote the universal abelian variety. A point $\tau \in \mathbb{H}_g$ is the same data as a pair $(\mathbb{A}_\tau, \varphi : H_1(A_\tau, \mathbb{Z}) \xrightarrow{\sim} \mathbb{Z}^{2g})$ of a PPAV $A_\tau = U_g(\mathbb{C})_\tau$ and a symplectic isomorphism φ. Note

$$A_\tau \xleftarrow[\exp]{\sim} \mathrm{Lie}(A_\tau)/H_1(A_\tau, \mathbb{Z}) \xleftarrow[\varphi]{\sim} \mathbb{R}^{2g}/\mathbb{Z}^{2g}.$$

Define the `Betti map` to be

$$\begin{array}{rccc} b^{\mathrm{univ}}: & U_g(\mathbb{C}) \times_{\mathbb{A}_g(\mathbb{C})} \mathbb{H}_g & \longrightarrow & \mathbb{R}^{2g}/\mathbb{Z}^{2g} \\ & (x \in A_\tau, \varphi) & \longmapsto & \varphi(x). \end{array}$$

More generally, given a contractible open $\Delta \subset S(\mathbb{C})^{\mathrm{an}}$, the composition $\Delta \to S(\mathbb{C})^{\mathrm{an}} \to \mathbb{A}_g^{\mathrm{an}}$ will factor through $\mathbb{H}_g \to \mathbb{A}_g^{\mathrm{an}}$ (in fancy speak, the local system $R^1\pi_*\mathbb{Z}$ will trivialize), so we can define a `Betti map`

$$b_\Delta : \mathcal{A}(\mathbb{C})_\Delta \to U_g^{\mathrm{an}} \times_{\mathbb{A}_g^{\mathrm{an}}} \mathbb{H}_g \xrightarrow{b^{\mathrm{univ}}} \mathbb{R}^{2g}/\mathbb{Z}^{2g}.$$ ○

{☐ *In any case, the Betti map is the (real analytic) map you get by identifying all your fibers with a fixed choice of real torus.* ☐}

Definition 16. Let $X \subset \mathcal{A}$ be an irreducible subvariety, and choose some $x \in X^{\mathrm{sm}}(\mathbb{C})$. The `Betti rank` of X at x is defined to be

$$\mathrm{rank}_{\mathrm{Betti}}(X, x) := \mathrm{rank}_{\mathbb{R}}(db_\Delta|_{X^{\mathrm{sm,an}}})_x\,,$$

for any small enough open neighborhood Δ of $\pi(x) \in S^{\mathrm{an}}$. ◇

{☐ *Think about this as measuring, among other things, how much X varies as you move along the family.* ☐}

Remark 17. Consider the square

$$\begin{tikzcd} \mathcal{A}(\mathbb{C}) \arrow[r, "\iota"] \arrow[d, "\pi"'] & U_g^{\mathrm{an}} \arrow[d] \\ S(\mathbb{C}) \arrow[r] & \mathbb{A}_g^{\mathrm{an}}. \end{tikzcd}$$

The Betti map b_Δ factors through ι, so we easily see that

$$\mathrm{rank}_{\mathrm{Betti}}(X, x) \le \min\{2 \dim \iota(X), 2g\}.$$

{☐ *In particular, X will have small Betti rank if $\dim \iota(X)$ is small, i.e. if it does not vary much.* ☐} ○

{☐ *We call a variety non-degenerate if it achieves this maximum* ☐}

Definition 18. An irreducible subvariety $X \subset \mathcal{A}$ is said to be `non-degenerate` if $\mathrm{rank}_{\mathrm{Betti}}(X, x) = 2\dim X$ for some $x \in X^{\mathrm{sm}}(\mathbb{C})$. By the previous remark, this implies that $\iota|_X$ is generically finite and that $\dim X \le g$. ◇

[6] $\mathbb{H}_g = \{$symmetric complex matrices A with $\mathrm{Im}\, A > 0\}$

Non-example. $\mathcal{C} \subset \mathcal{A}$ has dimension $1 + \dim S = 3g - 1$. We'll fix this later. $\triangledown$

Fact. There is a differential form (the `Betti form`) ω on $\mathbb{A}_g^{\mathrm{an}}$ so that

$$\left(\omega|_X^{\wedge \dim X}\right)_x \neq 0 \iff \mathrm{rank}_{\mathrm{Betti}}(X, x) = 2 \dim X,$$

and so that $[\omega] = c_1(\mathscr{L}^{\mathrm{univ}}) \in H^2(\mathbb{A}_g^{\mathrm{an}}, \mathbb{Z})$.

{☐ *Think: if X is non-degenerate, the top intersection power of a principal polarization is nonzero. Asymptotic Riemann-Roch will tell you this means that powers of the bundle has lots of sections on X, so you should expect a height lower bound for its points.* ☐}

Theorem 19 (DGH, Theorem 1.6)**.** *Let $X \subset \mathcal{A}$ be non-degenerate and assume that it dominates S. There, there are constants $c_1 > 0$ and $c_2 \geq 0$ along with a dense Zariski open $U \subset X$ such that*

$$\widehat{h}_{\mathscr{L}}(x) \geq c_1 h_{\mathscr{M}}(\pi(x)) - c_2 \text{ for all } x \in U(\overline{\mathbb{Q}}).$$

Remark 20. The main point here is to invoke a theorem of Siu in order to show that a certain line bundle (related, but not equal, to $\mathscr{L}|_X$) is big. $\circ$

To deal with the non-example from before, one proves a theorem like the following

Theorem 21 (Poorly stated, see Theorem 6.5 in Gao's survery article)**.** *Consider an irreducible subvariety $X \subset \mathcal{A}$ which dominates S. Then, if X is "not too silly," then $X^{[M]} = X \times_S X \times_S \ldots \times_S X$ will be non-degenerate if $M \gg_{\dim S} 1$.*

{☐ *This will tell you that $\mathcal{C}^{[M]}$ is eventually non-degenerate. Combined with the height inequality, this let's you prove the new gap principle*[7]*, at least over an open on the base. One then uses noetherian induction (restrict to family on complement of open and use same fiber-power trick) to conclude.* ☐}

♠♠♠ Niven: [Draw amplification picture]

[7] The basic idea is that a point of $X^{[M]}$ of large height is essentially M points $Q_1, \ldots, Q_M$ so that $\sum \widehat{h}_{\mathscr{L}}(Q_i - P) \geq c_1 h_{\mathscr{M}}(s)$. If each $\widehat{h}_{\mathscr{L}}(Q_i - P)$ is small, this can't happen