AWS 2022 Notes

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April 24, 2022

These are my course notes for "Class name" at School name. Each lecture will get its own "chapter." These notes are live-texed or whatever, so there will likely to be some (but hopefully not too much) content missing from me typing more slowly than one lectures. They also, of course, reflect my understanding (or lack thereof) of the material, so they are far from perfect.[1] Finally, they contain many typos, but ideally not enough to distract from the mathematics. With all that taken care of, enjoy and happy mathing.

The instructor for this class is Prof name, and the course website can be found by clicking this link. Extra extra read all about it

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[1] In particular, if things seem confused/false at any point, this is me being confused, not the speaker

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1 Ellen Eischen: Algebraicity and automorphic forms on unitary groups

1.1 Lecture 1

Note 1. A few minutes late

Plan for series: introduce automorphic forms on unitary groups and some strategies for studying algebraic aspects of L-functions, especially in the setting of unitary groups.

Venkatesh talked some about a 'fairy tale,' but sometimes you like to think of fairy tales as being based on true stories. We'd like to get a hint of the underlying true story, in the context of unitary groups.

Goal (Today). Move from modular forms to automorphic forms. Motivations + fundamental definitions.

1.1.1 Motivation from Modular Forms

Example. Euler showed long ago that

$$\zeta(2k) = (-1)^k \pi^{2k} \frac{2^{2k-1}}{(2k-1)!} \underbrace{\left(\frac{-B_{2k}}{2k}\right)}_{\zeta(1-2k)} \text{ where } \frac{te^t}{e^t - 1} = \sum_{n \geq 0} B_n \frac{t^n}{n!}$$

for $k \in \mathbb{Z}_{>0}$. Thus, $\zeta(2k)$ is a rational ($B_n \in \mathbb{Q}$) up to some well-understood transcendental factor.

Example. Eisenstein series

$$G_{2k}(q) = \zeta(1-2k) + 2\sum_{n \geq 1} \sigma_{2k-1}(n) q^n \text{ where } q = e^{2\pi i z} \text{ and } \sigma_{2k-1}(n) = \sum_{d|n} d^{2k-1}.$$

More generally, can prove rationality (up to power of π) of

- Dedekind ζ-functions
$$\zeta_K(s) = \sum_{\mathfrak{a} \subset \mathscr{O}_K} \frac{1}{N(\mathfrak{a})^s}$$
For K totally real, Kleinan (spelling?) and Siegal (based on work of Hecke) proved rationality of these values (for appropriate s) by realizing them as constant terms in the Fourier expansion of an Eisenstein series and exploiting properties of the space of modular forms.
- This approach extends to $L(s, \chi)$ with χ a Hecke character of a totally real field.
- More generally, could ask about algebraicity of certain values of L-functions attached to modular forms.

Remark 1.1.1. The algebraic number showing up in special values of L-functions (conjecturally) have arithmetic meaning. For example, in Kummer's work on Fermat, he showed that p dividing the order of the class group of $\mathbb{Q}(\zeta_p)$ can be determined by checking whether or not p divides the numerators of the Bernoulli numbers B_n for $n \leq p-3$ (assuming I heard correctly).

Remark 1.1.2. All the L-functions considered so far agree with corresponding Artin L-functions $L(s,\rho)$ attached to Galois representations ρ. In the one-dimensional case, this comes from class field theory. In the two dimensional case, you can e.g. associate a Galois representation to a nice cusp form.

There are conjectures on

- Meaning of these L-functions.

  Related to Deligne's conjectures about motives. Suggests these values should be related to certain periods of some motives, whatever that means.

- Connection b/w certain Galois representations ρ and certain "automorphic representations" (Langlands)

Above discussion mainly focused on GL_1 with some of GL_2. One might be tempted to next look at $\mathrm{GL}_3, \mathrm{GL}_4$, etc., but this turns out to not be the next nicest case. Instead, we'll look at unitary groups.

1.1.2 Automorphic forms on Unitary Groups

Fix a CM field K/K^+ (K^+ totally real, K/K^+ quadratic imaginary) as well as a vector space V/K along with a nondegenerate Hermitian pairing $\langle -,- \rangle$.

Remark 1.1.3. Can extend $\langle -,- \rangle$ linearly to $V \otimes_{K^+} R$ for any K^+-algebra R.

Definition 1.1.4. The **general unitary group** $\mathrm{GU}(V, \langle -,- \rangle)$ is the algebraic group G/K^+ whose R-points, for each K^+-algebra R, are given by

$$G(R) = \left\{ (g,\nu) \in \mathrm{GL}_{K\otimes_{K^+}R}(V \otimes_{K^+} R) \times \mathbb{G}_m(R) : \langle gv, gw \rangle = \nu \langle v, w \rangle \text{ for all } v, w \in V \right\}.$$

The subgroup for which $\nu = 1$ is the **unitary group** $U(V, \langle\, ,\, \rangle)$.

Example. If $R = \mathbb{R}$ and we choose an ordered basis for V, we can write

$$\langle v, w \rangle = vA^t\overline{w}$$

for some matrix A. In fact, we can choose the basis so that

$$A = \begin{pmatrix} 1_a & 0 \\ 0 & -1_b \end{pmatrix}.$$

The pair (a,b) is called the **signature** of the form. We denote the corresponding unitary groups by $\mathrm{GU}(a,b)$ and $U(a,b)$.

Assumption. Assume $K^+ = \mathbb{Q}$. This let's us avoid keeping track of the embeddings $K^+ \hookrightarrow \mathbb{R}$.

We now want to see several definitions of automorphic forms, specialized to the case of unitary groups. The various definitions lend themselves more to different perspectives, but all our equally valued. We'll also try to relate each to modular forms.

(1) Interpretation as holomorphic functions

- The first perspective on modular forms (of weight k and level Γ) that one usually sees is that they're holomorphic functions $f : \mathfrak{H} \to \mathbb{C}$ on the upper half-plane satisfying the transformation law
$$f(z) = (cz+d)^{-k} f(\gamma z) \text{ for all } \gamma = \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in \Gamma \subset \mathrm{SL}_2(\mathbb{Z})$$
and that f is "holomorphic at the cusps."

- Have analogue of upper half plane, identified with G/K_∞ (e.g. $U(n,m)(\mathbb{R})/U(n) \times U(m)$).

  Example ($U(n,n)$ case)**.** An automorphic form on $U(n,n)$ is a holomorphic function
$$f : \mathfrak{H}_n \longrightarrow V$$
  (Above, V is a representation $\rho : \mathrm{GL}_n(\mathbb{C}) \times \mathrm{GL}_n(\mathbb{C}) \longrightarrow \mathrm{GL}(V)$) such that
$$f(z) = \rho(Cz+D, \overline{c}^t z + \overline{D})^{-1} f(\gamma z) \text{ for all } \gamma = \begin{pmatrix} A & B \\ C & D \end{pmatrix} \in \Gamma \subset U(n,n)(\mathscr{O}_K).$$
  Above, $\gamma z = (Az+B)\,(Cz+D)^{-1}$ and
$$\mathfrak{H}_n := \left\{ z \in M_{n\times n}(\mathbb{C}) : i({}^t\overline{z} - z) > 0 \right\}.$$
  (Hermitian upper half space)

(2)

- A second perspective on modular forms is to consider them as certain functions $\varphi_f : \mathrm{SL}_2(\mathbb{R}) \to \mathbb{C}$. This is coming from $\mathrm{SL}_2(\mathbb{R})$ acting transitively on $\mathfrak{H}$ with stabilizer $\mathrm{SO}_2(\mathbb{R})$ at i. Think: $\varphi_f(g) = j(g,i)^{-k} f(g,i)$. This gives a function
$$\varphi_f : \Gamma\backslash G(\mathbb{R}) \longrightarrow \mathbb{C}.$$
  Note that
$$\varphi_f\left(g \begin{pmatrix} \cos\theta & \sin\theta \\ -\sin\theta & \cos\theta \end{pmatrix}\right) = e^{ki\theta}\varphi_f(g).$$
  Can extend to $\varphi : \Gamma Z(G)\backslash G(\mathbb{R}) \to \mathbb{C}$ for $G = \mathrm{GL}_2, \mathrm{SL}_2, \mathrm{GL}_2^+$.

- This looks like something that can be generalized to other groups. Consider $\Gamma Z(G)\backslash \mathrm{GU}(n,m) \supset K_\infty = U(n) \times U(m) = U(n,0) \times U(0,m)$ (K_∞ analogue of $\mathrm{SL}_2(\mathbb{Z})$, I think). Can analogously define functions
$$\Gamma Z(G)\backslash G(\mathbb{R}) \longrightarrow \mathbb{C} \text{ where } G = \mathrm{GU}(n,m).$$

(3) Adelic interpretation

- Start with $\mathrm{GL}_2(\mathbb{A}) = \mathrm{GL}_2(\mathbb{Q}) \cdot \mathrm{GL}_2^+(\mathbb{R}) \times \prod_p K_p$ where $K_p \subset \mathrm{GL}_2(\mathbb{Q}_p)$ a certain compact open subgroup with $\det = \mathbb{Z}_p^\times$ and $K_p = \mathrm{GL}_2(\mathbb{Z}_p)$ for almost all p. Let $\Gamma = \mathrm{GL}_2(\mathbb{Q}) \cap (\mathrm{GL}_2^+(\mathbb{R}) \times K)$ (where $K = \prod K_p$). Then, we get an identification
$$\Gamma\backslash \mathrm{GL}_2(\mathbb{R}) \longleftrightarrow \mathrm{GL}_2(\mathbb{Q})\backslash \mathrm{GL}_2(\mathbb{A})/K.$$

Thus, we can get a function

$$\psi_f : \mathrm{GL}_2(\mathbb{Q})\backslash \mathrm{GL}_2(\mathbb{A}) \longrightarrow \mathbb{C} \text{ via } \psi_f(\gamma g_\infty (g_p)_p) = \varphi_f(g_\infty).$$

- On the unitary side, we have $G(\mathbb{A}_f) = \bigsqcup_i G(\mathbb{R})^+ g_i K$ which suggests we can do an analogous reformulation in this setting.

Next time we'll give the geometric interpretation of these objects.

1.2 Lecture 2

Goal (by end of lecture 3). More on automorphic forms + an approach to studying $L(s,\pi)$ with π a cuspidal automorphic representation of a unitary group

Recall 1.2.1. Saw several perspectives on automorphic forms on unitary groups yesterday. In particular, they can be realized as

- functions on some generalization of the upper half plane
- functions on $G(\mathbb{R})$ or $G(\mathbb{A})$, where G denotes a unitary group

We now want to look at automorphic forms on unitary groups as sections of a line (or vector) bundle over certain moduli spaces.

Example (Modular forms)**.** Let M be a modular curve (parameterizing elliptic curves w/ some level structure). Over M, there is a universal elliptic curve $\mathcal{E} \xrightarrow{\pi} M$. Let $\Omega^1_{\mathcal{E}/M}$ be the sheaf of relative 1-forms, and let $\omega := \pi_*\Omega^1_{\mathcal{E}/M}$. Then, a modular form of weight k is an element of $\mathrm{H}^0(M, \omega^{\otimes k})$.

Equivalently, this is a rule that sends a pair (E,ω) of an elliptic curve E with an (invariant) differential $\omega \in \mathrm{H}^0(E,\Omega_E)$ to values $F(E,\omega)\in\mathbb{C}$ so that

$$F(E,\lambda\omega) = \lambda^{-k}F(E,\omega).$$

Equivalently, could consider this as a rule $\widetilde{F}$ that maps an e.c. E to an element $\omega \in (\Omega^1_E)^{\otimes k}$ (connection is $\widetilde{F}(E) = F(E,\omega)\cdot\omega^{\otimes k}$).

The connection with last time is that given (E,ω) integrating along ω gives a lattice $L_{(E,\omega)}$ which gives rise to some $\mathbb{Z}\oplus\mathbb{Z}\tau$ and this let's us identify $F(E,\omega)$ with some $f_F(\tau)$.

We now want to explain that automorphic forms on unitary groups similarly arise as global sections of a vector bundle over some unitary Shimura variety $\mathcal{M}$. These will parameterize AVs w/ a polarization, endomorphism, and level structure.

Remark 1.2.2. The $\mathbb{C}$-points of $\mathcal{M}$ can be identified with $G(\mathbb{Q})\backslash G(\mathbb{A})/KK_\infty$ for appropriate K, K_∞. This double coset turns out to be a finite disjoint union of copies of a symmetric space (e.g. $\mathfrak{H}_n$) for our unitary group.

Can view an automorphic form as a function F s.t. $F(\underline{A}, g\ell) = \rho(tg)^{-1}F(\underline{A},\ell)$ ($\underline{A}$ an AV with level structure, and ℓ an ordered basis for $\Omega_{A/\mathbb{C}}$), where $g \in \mathrm{GL}_a\times\mathrm{GL}_b$ (signature (a,b)). Can also reformulate in terms of lattices and identify with an automorphic form on a symmetric space (e.g. $\mathfrak{H}_n$). As before, we can also consider universal abelian variety $\pi : \mathcal{A}\to\mathcal{M}$, consider $\underline{\omega} := \pi_*\Omega_{\mathcal{A}/\mathcal{M}}$ and then use this to build a sheaf $\underline{\omega}^\rho$ of automorphic forms.

Goal. Introduce approach to studying certain L-functions, with an emphasis on the "doubling method"

Example. Say $f(q)=\sum_{n\geq 1} a_n q^n$ is a weight k cusp form, and say $g(q)=\sum_{n\geq 0} b_n q^n$ is any weight ℓ modular form. Suppose $a_n, b_n \in \overline{\mathbb{Q}}$. Then, their **Rankin-Selberg product** is

$$D(s,f,g)=\sum_{n\geq 1}\frac{a_n b_n}{n^s}.$$

Shimura proved that (can replace $\overline{\mathbb{Q}}$ with number field where coefficients lie)

$$\frac{D(m,f,g)}{\langle f,f\rangle_{\mathrm{Pet}}}\in \pi^k\overline{\mathbb{Q}}$$

when $\ell<k$ and $k+\ell-2<2m<2k$ ($m\in\mathbb{Z}$). The proof of this relies on realization that

$$D(k-1-r,f,g)=c\pi^k\left\langle \widetilde{f}, g\delta_\lambda^{(r)}E\right\rangle_{\mathrm{Pet}},$$

where $\delta_\lambda^{(r)}$ is some Mass-Shimura differential operator, E is an Eisenstein series (of weight $\lambda:=k-\ell-2r$), $\widetilde{f}(q)=\sum_{n\geq 1}\overline{a_n}q^n$, and c is some explicit constant. Finally, the **petersson pairing** is

$$\langle f,g\rangle_{\mathrm{Pet}}:=(*)\int \overline{f}(z)g(z)y^{k-2}\mathrm{d}x\mathrm{d}y$$

with integral over a fundamental domain for the level (and some normalizing factor out front).

We want some kind of recipe for proving algebraicity results of certain values of L-functions.

(1) Find a Petersson-style pairing of automorphic forms (integrated against some Eisenstein series E) that

- factors into an Euler product
- has a functional equation
- can be meromorphically continued to $\mathbb{C}$

(2) Prove "appropriate" rationality or algebraicity results for E

(3) Express a familiar automorphic L-function in terms of this pairing from **(1)**.

Remark 1.2.3. These are all hard steps. Depending on your setting, these might not have been done yet.

Setup. Let $K/\mathbb{Q}$ be an imaginary quadratic (more generally, K some CM field). Let V/K be an n-dimensional vector space with a nondegenerate hermitian pairing $\langle\ ,\ \rangle_V$. Let $G=U(V,\langle -,-\rangle_V)$ be the associated unitary group. Let $W=V\oplus V$ with pairing

$$\langle (u,v),(u',v')\rangle_W:=\langle u,u'\rangle_V-\langle v,v'\rangle_V$$

(this is the 'double' in 'doubling method') and set $H:=U(W,\langle\rangle_W)$. We get an embedding

$$G\times G=U(V,\langle\ ,\ \rangle)\times U(V,-\langle\ ,\rangle_V)\longrightarrow H.$$

If V has signature (a,b), then the second G above has signature (b,a) and H has signature $(a+b, a+b) = (n,n)$ where $n := a+b$.

Next time we'll introduce the doubling integral (after producing some Eisenstein series).

1.3 Lecture 3

Some references in the bibliography of the manuscript.

Let's recall where we left off

Setup. Have $K/\mathbb{Q}$ quad imag and V/K an n-dimensional nondegenerate Hermitian form. We set $G = U(V, \langle -,- \rangle_V)$. We doubled using $W = V \oplus V$ with Hermitian form

$$\langle (u,v),(u',v') \rangle := \langle u,u' \rangle_V - \langle v,v' \rangle_V .$$

Set $H = U(W, \langle -,- \rangle_W)$ and note we have $G \times G \hookrightarrow H$ where first G has signature (a,b), second has signature (b,a) (use $-\langle -,- \rangle_V$), and H has signature $(a+b, a+b)$.

1.3.1 Eisenstein Series

Let P be the parabolic subgroup of H preserving $V^\Delta := \{(v,v) : v \in V\}$. With respect to the decomposition $W = V^\Delta \oplus V_\Delta$ (where $V_\Delta = \{(v,-v) : v \in V\}$), we have

$$P = \left\{ \begin{pmatrix} A & * \\ 0 & {}^t\overline{A}^{-1} \end{pmatrix} : A \in \mathrm{GL}_n \right\}.$$

Given $\chi : K^\times \backslash \mathbb{A}_K^\times \to \mathbb{C}$ a Hecke character, we can view this as acharacter of P via

$$P \twoheadrightarrow \mathrm{GL}_n \xrightarrow{\det} \mathbb{A}_K^\times \xrightarrow{\chi} \mathbb{C}$$

Fix $s \in \mathbb{C}$, and let

$$f_{s,\chi} \in \mathrm{Ind}_{\mathbb{P}(\mathbb{A})}^{H(\mathbb{A})} \left(\chi| \cdot |^{-s} \right) = \left\{ f : H(\mathbb{A}) \to \mathbb{C} \mid f(ph) = \chi(p)\,|p|^{-s}\, f(h) \right\}.$$

Definition 1.3.1. The **Siegal Eisenstein series** is

$$E_{f_{s,\chi}}(g) := \sum_{\gamma \in P(\mathbb{Q}) \backslash H(\mathbb{Q})} f_{s,\chi}(\gamma g).$$

This converges for $\mathrm{Re}\, s \gg 0$. This is a meromorphic function of s that's automorphic in g. It also satisfies a functional equation relating $s \leftrightarrow 1-s$.

Doubling Integral Let π be a cuspidal automorphic representation of G. Let $\widetilde{\pi}$ be the contragradient (dual) representation of π. Take some $\varphi \in \pi$ and $\widetilde{\varphi} \in \widetilde{\pi}$. Set

Think: Tate's thesis

$$Z(\varphi, \widetilde{\varphi}, f_{s,\chi}) := \int_{[G\times G]} E_{f_{s,\chi}}(g_1,g_2)\varphi(g_1)\widetilde{\varphi}(g_2)\chi^{-1}(\det g_2) \mathrm{d}g_1 \mathrm{d}g_2$$

(using Haar with appropriate normalization, but we're not gonna stress about the details on that here). These will inherit analytic properties of our Eisenstein series $E_{f_{s,\chi}}$. In particular, these Z's should have functional equations and meromorphic continuations to all of $\mathbb{C}$.

Remark 1.3.2. In the case of $G = \mathrm{GU}(1)$ (or more generally, $\mathrm{GU}(n)$ definite unitary group), if we choose levels appropriately, we can express this integral as a finite sum.

Theorem 1.3.3.

$$Z(\varphi, \widetilde{\varphi}, f_{s,\chi}) = \int_{G(\mathbb{A})} f_{s,\chi}((g,1)) \langle \pi(g)\varphi, \widetilde{\varphi} \rangle \, \mathrm{d}g,$$

where

$$\langle \varphi, \widetilde{\varphi} \rangle = \int_{[G]} \varphi(g)\widetilde{\varphi}(g)\mathrm{d}g$$

is a G-invariant pairing, and so unique up to constant multiple.

Corollary 1.3.4. *If* $\pi = \bigotimes_v' \pi_v$ *and* $\widetilde{\pi} = \bigotimes_v' \widetilde{\pi}_v$ *and* $\mathrm{Ind}(\chi|\cdot|^{-s}) = \bigotimes_v' I_v$ *with* $\varphi = \otimes_v' \varphi_v$, $\widetilde{\varphi} = \otimes_v' \widetilde{\varphi}_v$, *and* $f_{s,\chi} = \otimes_v' f_v$, *then*

$$Z(\varphi, \widetilde{\varphi}, f_{s,\chi}) = \prod_v Z_v(\varphi_v, \widetilde{\varphi}_v, f_v),$$

where

$$Z_v(\varphi_v, \widetilde{\varphi}_v, f_v) = \int_{G(\mathbb{Q}_v)} f_v((g,1)) \langle \pi_v(g)\varphi_v, \widetilde{\varphi}_v \rangle \, \mathrm{d}g.$$

Proof Sketch. Uniqueness of the invariant pairing will us that

$$\langle \varphi, \widetilde{\varphi} \rangle = \prod_v \langle \varphi_v, \widetilde{\varphi}_v \rangle .$$

Plug this into the integral and expand. ∎

Remark 1.3.5. Shimura noted that if f factors, then the Fourier coefficients of $E_{f_{s,\chi}}$ also factor.

Proof Outline of Theorem 1.3.3. The theorem will follow from an analysis of the orbits of $G \times G$ acting on $X := P\backslash H$. For each $\gamma \in X$, we write $[G \times G]^\gamma = \{(g,h) \in G \times G : P\gamma(g,h) = P\gamma\}$ for the stabilizer of γ in $G \times G$ (acting on the right). For each $\gamma \in X$, write $[\gamma]$ for the orbit of $P\gamma$ under the action of $G \times G$. Write

$$E_{f_{s,\chi}}(h) = \sum_{[\gamma] \in P(\mathbb{Q})\backslash H(\mathbb{Q})/(G\times G)(\mathbb{Q})} \sum_{[G\times G]^\gamma(\mathbb{Q})\backslash (G\times G)(\mathbb{Q})} f_{s,\chi}(\gamma h),$$

and insert into doubling integral to get

$$\sum\sum \int_{(G\times G)(\mathbb{Q})\backslash (G\times G)(\mathbb{A})} f_{s,\chi}(\gamma g h)\varphi(g)\widetilde{\varphi}(h)\chi^{-1}(\det h)\mathrm{d}g\mathrm{d}h = \sum_{[\gamma] \in P(\mathbb{Q})\backslash H(\mathbb{Q})/(G\times G)(\mathbb{Q})} I(\gamma)$$

with

$$I(\gamma) := \int_{[G\times G]^\gamma(\mathbb{Q})\backslash (G\times G)(\mathbb{A})} f_{s,\chi}(\gamma(g,h))\varphi(g)\widetilde{\varphi}(h)\chi^{-1}(\det h)\mathrm{d}g\mathrm{d}h.$$

Have two cases ($\gamma = 1$ or $\gamma \neq 1$). The upshot is that, for $\gamma = 1$, $I(\gamma)$ gives RHS of theorem. For $\gamma \neq 1$, $I(\gamma) = 0$.

Note $[G\times G]^1 = P\cap (G\times G) = \{(g,g) : g\in G\} =: G^\Delta$. Furthermore,

$$f_{s,\chi}(1\cdot (g,h)) = f_{s,\chi}(g,h) = f_{s,\chi}((h,h)\cdot (h^{-1}g,1)) = \chi(\det h) f_{s,\chi}(h^{-1}g,1),$$

so

$$I(1) = \int_{G^\Delta(\mathbb{Q})\backslash (G\times G)(\mathbb{A})} f_{s,\chi}(h^{-1}g,1)\varphi(g)\widetilde{\varphi}(g)\mathrm{d}g\mathrm{d}h.$$

Observe $G\times G\cong G^\Delta \times (G\times 1)\cong G\times G$ via

$$(g,h)\leftrightarrow (h,h)\cdot (h^{-1}g,1)\leftrightarrow (g,h^{-1}g).$$

Let $g_1 := h^{-1}g$. So have

$$I(1) = \int_{G(\mathbb{A})}\int_{G(\mathbb{Q})\backslash G(\mathbb{A})} f_{s,\chi}(g_1,1)\pi(g_1)\varphi(h)\widetilde{\varphi}(h)\mathrm{d}h\mathrm{d}g_1 = \int_{G(\mathbb{A})} f_{s,\chi}(g_1,1)\left\langle \pi(g_1)\varphi,\widetilde{\varphi}\right\rangle \mathrm{d}g_1.$$

For all the other orbits $[\gamma]\neq [1]$, $I(\gamma)$ decomposes to get product including terms of the form

$$\int_{N_i(\mathbb{Q})\backslash N_i(\mathbb{A})} \varphi_i(n\cdot g)\mathrm{d}n \ \text{ with } \ i=1,2,\ \ \varphi_1=\varphi,\ \ \varphi_2=\widetilde{\varphi},$$

and N_i the unipotent radical of a parabolic subgroup of G that is nontrivial for at least one i. Since $\varphi,\widetilde{\varphi}$ are cuspidal, this guarantees that at least one factor in the product vanishes. ■

Can choose $f_{s,\chi},\varphi,\widetilde{\varphi}$ so that you get (nice multiplies) of Langlands L-functions $L(s,\pi,\chi)$ (Relies on reducing computations of local integral to ones computed by Godement and Jacquet for GL_n)

2 Wee Teck Gan: Automorphic forms and the theta correspondence

2.1 Lecture 1

Conjecture 2.1.1 (**Ramanujan-Petersson Conjecture**). *Let $f : \mathfrak{H} \to \mathbb{C}$ be a holomorphic cusp form of weight k, level 1. Suppose further that f is an eigenvector or each Hecke operator T_p. Write*

$$f = \sum_{n>0} a_n(f)q^n, \quad q = e^{2\pi i z}.$$

We normalize $a_1(f) = 1$, so $T_p(f) = a_p(f)f$. Given this, one conjectures

$$|a_p(f)| \le 2p^{\frac{k-1}{2}}.$$

This is now a theorem of Deligne, coming from his work on the Weil conjectures.

There's an analogue of this for Maass forms (certain eigenvectors for the hyperbolic Laplacian). Both holomorphic modular forms and Maass forms give examples of automorphic forms (for GL_2). Let's take this transition for granted, and move to the perspective of automorphic forms.

Setup. k will be a number field. We'll let v denote a place/prime of k with completion k_v. We let $\mathbb{A} := \prod'_v k_v$ be the adeles, and we embed $k \hookrightarrow \mathbb{A}$ diagonally (note $k\backslash\mathbb{A}$ is compact). Let G/k be a reductive group (e.g. SL_N, U_N). Then, one gets an analogous embedding

$$G(k) \hookrightarrow G(\mathbb{A}) = \prod_v{}' G(k_v)$$

(restricted direct product taken w.r.t. family $\{K_v\}$ of open compact subgroups). We set

$$[G] := G(k)\backslash G(\mathbb{A}).$$

If G is semisimple (or if its center has no split tori), then $[G]$ has finite volume (assuming I heard correctly). Note that $[G] \curvearrowleft G(\mathbb{A})$ by right translations.

This is the setting for automorphic forms.

Definition 2.1.2. An **automorphic form** on G is a function $f : [G] \to \mathbb{C}$ satisfying some (mild) regularity and finiteness conditions. Regularity conditions include being smooth and of uniform moderate growth (polynomial growth for all derivatives w/ the same exponent near infinite, in an appropriate sense). For finiteness conditions, one often requires K-finiteness ($K = \prod_v K_v$), but we don't want to suppose this. Instead, we'll ask for $Z(\mathfrak{g})$-finiteness.[2]

Notation 2.1.3. We let $A(G)$ denote the space of automorphic forms. Note $G(\mathbb{A}) \curvearrowright A(G)$ by right translation: $(g_0 \cdot f)(g) = f(gg_0)$.

We don't impose K-finiteness because if you impose it, you don't get an action of the full adelic

[2] Missed the explanation for this but think e.g. of modular forms being holomorphic (satisfying Cauchy-Riemann differential equations)

group. Instead, you get an action of the finite adeles, and at the archimedean places, you instead get a $(\mathfrak{g}, K)$-module. We don't want to have to deal with this.

Definition 2.1.4. An **irreducible subquotient** of $A(G)$ is called an **automorphic representation**.

Remark 2.1.5. $A(G)$ is not semisimple

Let's consider two subrepresentations of $A(G)$ which are a little less wild.

- Cusp forms

  Definition 2.1.6. $f \in A(G)$ is **cuspidal** if for all parabolic subgroups $P = M \cdot N$ (MN is the Levi decomposition), **the constant term of** f **along** N is 0, i.e.

  $$f_N(g) = \int_{[N]} f(ng)\mathrm{d}n.$$

  Note that the subspace $A_0(G) \subset A(G)$ of cusp forms is $G(\mathbb{A})$-stable since $G(\mathbb{A})$ acts by right translation.

  Remark 2.1.7. Could take a non-trivial character $\psi : [N] \to \mathbb{C}^\times$ and look at the (N, ψ)**-Fourier coefficient** of f:

  $$f_{N,\psi}(g) = \int_{[N]} \overline{\psi(n)} f(ng)\mathrm{d}n$$

  (c.f. Pollack's lectures)

  Remark 2.1.8. Uniform moderate growth + cuspidality implies that cusp forms $f \in A_0(G)$ are rapidly decreasing at ∞. This then implies that they are L^2, i.e.

  $$\int_{[G]} |f|^2 < \infty.$$

- Square integrable automorphic forms $A_2(G)$. Note we have a chain

  $$A_0(G) \subset A_2(G) \subset A(G)$$

  of $G(\mathbb{A})$-modules. Note that $A_2(G)$ completes to a Hilbert space. Given this, it is perhaps unsurprising that you get a semi-simple decomposition

  $$A_2(G) = \bigoplus_{\pi \in \operatorname{Irr} G(\mathbb{A})} m_2(\pi)\pi.$$

  Consequently, we also have

  $$A_0(G) = \bigoplus_{\pi} m_0(\pi)\pi.$$

  Remark 2.1.9. These $m(\pi)$'s are finite. This is nontrivial, but apparently follows from Harish-Chandra.

Question 2.1.10. *For which π is $m_\bullet(\pi) > 0$ ($\bullet \in \{0, 2\}$)?*

Question 2.1.11. *What/How does $\pi \in \operatorname{Irr} G(\mathbb{A})$ look like?*

Recall 2.1.12. $G(\mathbb{A}) = \prod'_v G(k_v)$

Given this, it is perhaps not too surprising that every irrep π will be of the form

$$\pi = \bigotimes_v{}' \pi_v \text{ with } \pi_v \in \operatorname{Irr} G(k_v).$$

The prime $'$ above means that $\pi_v^{K_v} \neq 0$ (i.e. π_v is K_v**-unramified** or **spherical**) for all but finitely many v.

Question 2.1.13. *What do we know about unramified representations?*

2.1.1 Unramified representations

We'll work locally.

Setup. Assume that G_v is **unramified**, i.e. quasi-split over k_v and split by an unramified extension of k_v. Inside of here, there's a hyper-special maximal compact $K_v \subset G_v$.

Remark 2.1.14. Above, **quasi-split** means there's a Borel $B_v \subset G_v$ defined over k_v. This splits as $B_v = T_v N_v$ with T_v a maximal torus and N_v unipotent.

Theorem 2.1.15. *K_v-unramified irreps of G_v are in bijection with $\{$unramified characters of $T_v\}/$ Weyl group.*

Given some unramified character $\chi : T_v \to \mathbb{G}_m$, how do you get the K_v-unramified irrep? Start with the induced representation $I(\chi) = \operatorname{Ind}_{B_v}^{G_v} \chi$ (via parabolic induction). This will always contain a unique unramified subquotient.

By Langlands, the unramified characters of T_v (up to action by the Weyl group W) are in bijection with semisimple conjugacy classes in $G^\vee(\mathbb{C})$, the $\mathbb{C}$-points of the Langlands dual.

Say $\bigotimes'_v \pi_v = \pi \subset A_0(G)$ is a cuspidal representation. From this, we get a collection $\{s_{\pi_v}\}_{v \in S} \subset G^\vee$ of semisimple conjugacy classes in $G^\vee$ for all v outside a finite set S. This can be used to define automorphic L-functions.

Definition 2.1.16. Say you have cuspidal $\pi \subset A_0(G)$ along with $R : G^\vee \to \mathrm{GL}_N(\mathbb{C})$. This allows us to define a (partial) **automorphic L-function**

$$L^S(s, \pi, R) = \prod_{v \notin S} L(s, \pi_v, R) \text{ where } L(s, \pi_v, R) = \frac{1}{\det\left(1 - q_v^{-s} R(s_{\pi_v})\right)}$$

where q_v is the order of the residue field at v.

This will converge for $\operatorname{Re} s \gg 0$ (maybe even $\operatorname{Re} s > 1$ suffices)

Remark 2.1.17. This automorphic L-functions generalize Hecke L-functions as well as L-functions of modular forms.

TODO: work out example of L-function of a modular form

2.1.2 Tempered Representations

Definition 2.1.18 (Ad-hoc)**.** Let $\chi : T_v \to \mathbb{C}^\times$ be a character. This gives rise to a K_v-unramified irrep π_χ. We say that π_χ is **tempered** if χ is unitary, i.e. $|\chi| = 1$ (i.e. $\chi : T_v \to S^1 \subset \mathbb{C}^\times$).

Tempered representations are those weakly contained in $L^2(G_v)$ (think of $L^2(G_v)$ as the 'regular representation' of G_v. If G is a compact Lie group (e.g. a finite group), then $L^2(G)$ will contain very irrep).

Non-example. Note that the trivial representation is not contained in $L^2(\mathrm{SL}_2(\mathbb{R}))$ since the constant function 1 is not L^2 (since $\mathrm{SL}_2(\mathbb{R})$ doesn't have finite volume).

We can now describe a reformulation of the RP conjecture we started with.

Conjecture 2.1.19. *Assume that G is quasi-split. Say $\pi = \bigotimes_v' \pi_v$ is cuspidal. Then, π_v is tempered for almost all v.*

(It's non-obvious that this is a reformulation)

In Corvallis, Howe-PS constructed counter-example for $G = \mathrm{Sp}_4$. In this course, we will construct a counterexample on $G = U_3$.

What's the correct conjecture then?

Conjecture 2.1.20. *If $\pi \subset A_0(G)$ is 'globally generic', then π_v is tempered for almost all v.*

For GL_n, all cusp forms are globally generic. This is why this condition was missed in the beginning.

2.1.3 Unitary Groups

Let E/F be a quadratic extension with $c \in \mathrm{Gal}(E/F)$ the non-trivial automorphism. Let V be a f.dim E-vector space, and let

$$\langle -, - \rangle : V \times V \longrightarrow E$$

be an ε**-Hermitian** form ($\varepsilon = \pm 1$), i.e.

- $\langle av_1, bv_2 \rangle = a \langle v_1, v_2 \rangle b^c$ for all $a, b \in E$
- $\langle v_2, v_1 \rangle = \varepsilon \langle v_1, v_2 \rangle^c$

$\varepsilon = 1$ are Hermitian forms while $\varepsilon = -1$ are skew-Hermitian.

Remark 2.1.21. Take some $\delta \in E_0^\times = \{x \in E^\times : \mathrm{Tr}(x) = 0\}$. Then, $\delta \langle -, - \rangle$ is $(-\varepsilon)$-Hermitian. Thus, there's not much of a difference between the two.

Our unitary groups will be $U(V) = \mathrm{Aut}(V, \langle -, - \rangle)$, automorphisms of such ε-Hermitian forms. We'd like to say a bit of the classification of such things. Say $\dim V = n$. We get invariants

- **discriminant** $\mathrm{disc}\, V = (-1)^{\binom{n}{2}} \det(V) \in F^\times / \mathrm{Nm}(E^\times)$.

Notation 2.1.22. From now on, V will denote a Hermitian space and W will denote a skew-Hermitian one.

2.2 Lecture 2

Notation 2.2.1.

- V will denote a Hermitian space, W will denote a skew-Hermitian form.

- $\operatorname{disc}(V) = (-1)^{\binom{n}{2}} \det(V) \in F^\times / \operatorname{Nm}(E^\times)$ where $n = \dim V$
- $\operatorname{disc}(W) = \operatorname{disc}(\delta^{-n} V)$ where $\delta \in E_0^\times$.

Fact. Over a p-adic field, $\operatorname{disc}(V)$ and $\dim(V)$ determine V. Over a local field, we have $\omega_{E/F} : F^\times / \operatorname{Nm}(E^\times) \xrightarrow{\sim} \langle \pm 1 \rangle$. We let V^+, V^- denote the two Hermitian spaces of a given dimension.

Fact. Over $\mathbb{R}$, the discriminant does not give a complete classification. Instead, need signature (p, q) $(p + q = n)$. Hermitian matrix can always be conjugated to a diagonal matrix of the form

$$\operatorname{diag}\left(\underbrace{1, \dots, 1}_{p}, \underbrace{-1, \dots, -1}_{q}\right).$$

Note that $\operatorname{disc}(V_{p,q}) = (-1)^{\binom{n}{2}} (-1)^q$.

Fact. Now say we have a number field E/k. The first thing to note is the **local-global principle**

$$\left\{ \begin{matrix} \text{Herm. space} \\ \text{over } k \end{matrix} \right\} \hookrightarrow \prod_v \left\{ \begin{matrix} \text{Hermitian space} \\ \text{over } k_v \end{matrix} \right\}$$

(A Hermitian space is determined by its localizations). Furthermore, given a family $\{V_v\}_v$ of local spaces, it lies in the image of this map (i.e. is **coherent**) iff $\varepsilon(V_v) = +1$ for almost all v and $\prod_v \varepsilon(V_v) = +1$.

Note we can go between Hermitin and and skew-Hermitian by multiplying by a trace 0 element, so they have analogous classifications. Let's look a skew-Hermitian forms over a p-adic field.

Example (Rank 1)**.** Say $W_1^+ = \langle \delta \rangle$ and $W_1^- = \langle \delta \rangle'$ with $E_0^\times / NE^\times = \{\delta, \delta'\}$.

Example (Rank 2)**.** $\mathbb{H} = W_2^+ = Ee_1 + Ee_2$ with $\langle e_i, e_i \rangle = 0$. Say $\langle e_1, e_2 \rangle = 1$. Then, $\langle e_2, e_1 \rangle = -1$. This is the **hyperbolic plane** associated to the matrix

$$\begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}.$$

There's also a W_2^- described by quaternion division algebras.

Example (Rank $2n$)**.** $W_{2n}^+ = \mathbb{H}^{\oplus n}$ and $W_{2n}^- = W_2^- \oplus \mathbb{H}^{\oplus (n-1)}$

Example (Rank $2n+1$)**.** $W_{2n+1}^+ = \langle \delta \rangle \oplus \mathbb{H}^{\oplus n}$ and $W_{2n+1}^- = \langle \delta' \rangle \oplus \mathbb{H}^{\oplus n}$

(all examples above over p-adic field)

2.2.1 Idea of Howe-PS

Say $\dim_E W = 3$ and let $U_3 = U(W)$. We want to product automorphic forms on here. By restriction of scalars, we can treat this as a 6-dimensional k-vector space $\operatorname{Res}_{E/k}(W)$ (k a global field?). This comes equipped with a symplectic form $\operatorname{Tr}_{E/k} \langle -, - \rangle_W$, and so we get an embedding

$$U_3 = U(W) \overset{i}{\hookrightarrow} \operatorname{Sp}\left(\operatorname{Res}_{E/k}(W)\right) = \operatorname{Sp}_6.$$

Start with some $\Omega \subset A_2(\operatorname{Sp}(-))$, and then consider $i^*\Omega$. We'll choose Ω consisting of theta functions.

Remark 2.2.2. The center of $U(W)$ is $E^1 = \{x \in E^\times : \mathrm{Nm}(x) = 1\}$.

We can break up $i^*\Omega$ according to central character:

$$A(U(W)) \supset i^*(\Omega) = \bigoplus_\chi \Omega_\chi$$

(sum over automorphic characters of E^1)

Claim 2.2.3.

- *Each Ω_χ is an irreducible cuspidal representation, with at most one exception*
- *Each Ω_χ violates RP conjecture*

There's a complication that arises: theta functions do not live on Sp_6 but instead on a cover of it. We won't go into detail about how to deal with this.

The upshot is that Howe-PS produces a map

$$\left\{\begin{matrix}\text{auto. chars}\\ \text{on } E^1\end{matrix}\right\} \longrightarrow \left\{\begin{matrix}\text{Aut reps}\\ \text{of } U_3\end{matrix}\right\},$$

taking $\chi \mapsto \Omega_\chi$.

Question 2.2.4. *How do you produce maps* $\mathrm{Irr}\, G \to \mathrm{Irr}\, H$*?*

Simple idea: if you have a $G \times H$-representation Ω, then you get a correspondence between $\mathrm{Irr}\, G$ and $\mathrm{Irr}\, H$, i.e. a subset $\Sigma_\Omega \subset \mathrm{Irr}\, G \times \mathrm{Irr}\, H$ (Recall: $\mathrm{Irr}(G \times H) = \mathrm{Irr}\, G \times \mathrm{Irr}\, H$) defined by

$$\Sigma_\Omega = \{(\pi, \sigma) : \mathrm{Hom}_{G\times H}(\Omega, \pi \otimes \sigma) \neq 0\}.$$

Question 2.2.5. *Is this correspondence a graph? That is, for fixed π, is there a unique corresponding σ?*

If so, you get a function $\mathrm{Irr}\, G \to \mathrm{Irr}\, H$.

Remark 2.2.6. Alternatively, can write

$$\Omega|_{G\times H} = \bigoplus_\pi \bigoplus_\sigma m(\pi,\sigma)\pi \otimes \sigma = \bigoplus_\pi \left(\underbrace{\bigoplus_\sigma m(\pi,\sigma)\sigma}_{\Theta(\pi)} \right) \otimes \pi.$$

Then, the question becomes, is $\Theta(\pi)$ irreducible (or 0)? If so, get $\Theta : \mathrm{Irr}\, G \to \mathrm{Irr}\, H \cup \{0\}$.

How can we find Ω giving an interesting map?

- If $\dim \Omega$ is big, the $\dim \Theta(\pi)$ will need to be big as well, and so will be less likely to be irreducible. So would like to find non-trivial Ω with relatively small dimension.

  Suppose you have $G \times H \to E$. Can try taking smallest non-trivial representation Ω of E.

We would like $\Theta : \mathrm{Irr}\, G \to \mathrm{Irr}\, H$ to be injective (where it's defined) if possible.

Theta correspondence will be an instance of the above idea.

2.2.2 Theta Correspondence

Say F is a p-adic field. Say E/F is a quadratic extension. Let V be a Hermitian space, and let W be skew-Hermitian. Then, $V \otimes_E W$ will be skew-Hermitian (pairing comes from multiplying two forms on V, W). Hence, you get a natural map

$$U(V) \times U(W) \longrightarrow \mathrm{Sp}(V \otimes_E W)$$

(with skew-symmetric form coming from taking trace of the restriction of scalars of $V \otimes_E W$ as before). Now, for Ω, we'd like to take the smallest infinite dimensional representation of Sp. This won't quite work; we'll need something smaller which will come from "taking the square root of the smallest rep." This involves passing to the metaplectic cover.

Metaplectic Group and Weil representations The metaplectic group is an S^1-cover of Sp, i.e. have

$$S^1 \longrightarrow \mathrm{Mp}(V \otimes W) \longrightarrow \mathrm{Sp}(V \otimes W).$$

For each non-trivial character $\psi : F \to \mathbb{C}^\times$, we'll get a representation

$$\Omega = \omega_\psi : \mathrm{Mp}(V \otimes W) \to \mathrm{GL}(S).$$

Each ω_ψ (**Weil representation**) is a "smallest infinite-dimensinoal" representation of Mp. We won't talk about constructing this Mp (could teach a whole course on it), but there are some more details in the lecture notes (sections $2.\{3,4\}$).

Back to theta correspondences We have the picture

$$\begin{tikzcd} & \mathrm{Mp}(V \otimes W) \arrow[d] \\ U(V) \times U(W) \arrow[ur, dashed, "\widetilde{\iota}"] \arrow[r, "\iota"] & \mathrm{Sp}(V \otimes W) \end{tikzcd}$$

In order to pull something back to $U(V) \times U(W)$, we need a lift of this map. This is non-trivial to obtain.

Theorem 2.2.7 (Kudle)**.** *$\widetilde{\iota}$ exists and is determined by a pair (χ_V, χ_W) of characters of $E^\times$, i.e.*

$$\chi_V|_{F^\times} = \omega_{E/F}^{\dim V} \text{ and } \chi_W|_{F^\times} = \omega_{E/F}^{\dim W}.$$

Indeed, χ_V gives $U(W) \to \mathrm{Mp}$ and χ_W gives $U(V) \to \mathrm{Mp}$.

Set

$$\Omega = \Omega_{V,W,\chi_V,\chi_W,\psi} = \widetilde{\iota}^*_{\chi_V,\chi_W}(\omega_\psi).$$

See lecture notes from properties of this (e.g. how it varies with choice of data).

Definition 2.2.8. For $\pi \in \mathrm{Irr}\, U(V)$, we set

$$\Theta(\pi) := (\Omega \otimes \pi^\vee)_{U(V)},$$

$U(V)$-coinvariants of this twisted representation. Note that $U(W) \curvearrowright \Theta(\pi)$. We call $\Theta(\pi)$ the **big θ-lift**.

Remark 2.2.9.
$$\mathrm{Hom}((\Omega \otimes \pi^\vee)_G, \mathbb{C}) = \mathrm{Hom}_G(\Omega \otimes \pi^\vee, \mathbb{C}) = \mathrm{Hom}_G(\Omega, \pi)$$
looks like the multiplicity space of π in Ω.

We would like this $\Theta(\pi)$ to be close to irreducible.

Theorem 2.2.10 (Howe, Kudle)**.**

- $\Theta(\pi)$ *has finite length as a $U(W)$-representation (and so has finitely many irreducible quotients)*

  Warning 2.2.11. There are infinite length representations which have no irreducible quotients. Having finite length here guarantees the existence of some irreducible quotients.

- *For any* (π, σ),
$$\dim \mathrm{Hom}_{U(V)\times U(W)}(\Omega, \pi \otimes \sigma) < \infty.$$

Definition 2.2.12. The **small theta lift** $\theta(\pi)$ is the **cosocle** of $\Theta(\pi)$, i.e. the maximal semisimple quotient.

Theorem 2.2.13 (**Howe Duality**)**.**

- $\theta(\pi)$ *is irreducible if* $\Theta(\pi) \neq 0$.
- *Suppose* $\theta(\pi) = \theta(\pi') \neq 0$. *Then,* $\pi \simeq \pi'$.

Thus, get $\theta : \mathrm{Irr}\, U(V) \to \mathrm{Irr}\, U(W) \cup \{0\}$ injective on $\mathrm{supp}\,\theta$.

Question 2.2.14. *When is $\theta(\pi)$ nonzero?*

2.3 Lecture 3

Note 2. Roughly 7 minutes late.

Recall 2.3.1. Weil rep of $U(V) \times U(W)$: $\Omega = \Omega_{V,W,\chi_V,\chi_W,\psi}$. For $\pi \in \mathrm{Irr}\, U(V)$, defined
$$\Theta(\pi) = (\Omega \otimes \pi^\vee)_{U(V)} \curvearrowleft U(W).$$

Howe duality: if $\Theta(\pi) \neq 0$, then it has a unique irreducible quotient $\theta(\pi)$ (same for $\sigma \in \mathrm{Irr}\, U(W)$). Thus, we get $\theta : \mathrm{Irr}\, U(V) \to \mathrm{Irr}\, U(W) \cup \{0\}$ injective where it's defined (since things are symmetric in V, W).

Question 2.3.2. *When is $\Theta(\pi) \neq 0$?*

Let's assume $\dim W$ is odd. Label them by discriminant: $\dim W_r^\varepsilon = 2r + 1$ with $\varepsilon = \pm 1$. Keep in mind the picture

TODO: Rotate π

$$\begin{tikzcd}
U(W_r^-) & & U(W_r^+) \\
\vdots & U(V) \arrow[ul, "\theta_r^-"'] \arrow[dr, "\theta_1^+"] & \vdots \\
U(W_1^-) & & U(W_1^+) \\
U(W_0^-) & & U(W_0^+)
\end{tikzcd}$$

Remark 2.3.3. $W_{r+1}^+ = W_r^+ \oplus \mathbb{H}$

We get a Θ-lift $\theta_r^\pm$ associated to each $W_r^\pm$ (if I'm following), and we want to know when $\theta_r^\pm(\pi) \neq 0$.

Theorem 2.3.4.

(i) *For $\pi \in \operatorname{Irr} U(V)$ and fixed $\varepsilon = \pm$, there is a smallest $r_0 := r_0^\varepsilon(\pi) \leq \dim V$ s.t.*

$$\theta^\varepsilon_{r_0^\varepsilon(\pi)}(\pi) \neq 0.$$

*This is called the **first occurrence of** π **in the** ε**-tower**.*

(ii) *For all $r > r_0$, $\theta_r^\varepsilon(\pi) \neq 0$*

(iii) *If π is a supercuspidal representation, then $\Theta_r^\varepsilon(\pi)$ is irreducible and supercuspidal at the first occurent (but not after). This is a result of Kudle*

The upshot is that the non-vanishing question is reduced to determining the first occurrences $r_0^\pm(\pi)$.

Remark 2.3.5. If $r \geq \dim V$, then $\Theta_r^\varepsilon(\pi) \neq 0$. This is called the stable range.

Theorem 2.3.6 (**Conservation Relation**, B.Y Sun and C.B. Zhu)**.** $\dim W^+_{r_0^+(\pi)} + \dim W^-_{r_0^-(\pi)} = 2\dim V + 2$ *for all* π.

(building on pervious work of Kudle-Rallis)

Corollary 2.3.7 (**Dichotomy**)**.** *If $\dim W^+ + \dim W^-$ (some member of $+$-tower and some member of $-$-tower), then for any $\pi \in U(V)$, exactly one of $\theta_{W^+}(\pi)$ or $\theta_{W^-}(\pi)$ is nonzero.*

Exercise. Prove corollary.

Example. $U_1 \times U_1 = U(V) \times U(W_0)$. Here, $U(V) = E^1$ is the norm one element; say $\chi \in \operatorname{Irr} E^1$ is an irreducible character. Then,

$$\dim W^+_{r^+(\chi)} + \dim W^-_{r^-(\chi)} = 4,$$

so these dimensions must $\{1, 3\}$ is some order (recall we're assuming $\dim W$ odd). Exactly one of $\theta_0^+(\chi)$ or $\theta_0^-(\chi)$ is nonzero. Furthermore, $\theta_r^\varepsilon(\chi) \neq 0$ for all $r > 0$. Which of $\theta_0^\varepsilon(\chi)$ is nonzer?

Theorem 2.3.8 (Moen, Rogawski, Harris-Kudle-Sweat)**.**

$$\theta_{V, W_{0,\varepsilon}}(\pi) \neq 0 \iff \varepsilon(V) \cdot \varepsilon(W_0) = \varepsilon_F\left(\frac{1}{2}, \chi_W \chi_W^{-1} \cdot \psi(\operatorname{Tr}_{E/F}(\delta -))\right)$$

where we have Tate's local ε-factor on the RHS.

Need to explain above expression. Recall $\chi \in \operatorname{Irr} E^1$ and that $E^\times/F^\times \xrightarrow{\sim} E^1$ via $x \mapsto x/x^c$ (by Hilbert 90). χ_E is the lift of χ along this quotient map. I missed what χ_W is... ψ is an additive character of F. $\delta \in E_0^\times$.

Note 3. Got distracted and missed some discussion.

We want to apply these theorems to Howe-PS. Recall we're interested in $U_1 \times U_3$.

Take $V = \langle 1 \rangle = V_0^+$ and some $\chi \in \operatorname{Irr} E^1 = \operatorname{Irr} U(V)$. Say $\dim W = 3$ (so $W = W_1^\varepsilon$). Since $U(V)$ is compact, the associated Weil representation is semisimple:

$$\Omega^\varepsilon = \bigoplus_{\chi \in \operatorname{Irr} E^1} \chi \otimes \Theta^\varepsilon(\chi).$$

- $\Theta^\varepsilon(\chi) \neq 0$ for all χ since $r = 1 \geq 1 = \dim V$.
- $\Theta^\varepsilon(\chi)$ is irreducible by Howe duality + supercuspidality (every rep of a compact group is supercuspidal?)
- If $\varepsilon = \varepsilon_F(1/2, blah)$ as in theorem of [HKS], then $\theta^\varepsilon(\chi)$ is non-supercuspidal and $\theta^{-\varepsilon}(\chi)$ is supercuspidal.

  In fact, $\theta^\varepsilon(\chi) \hookrightarrow \operatorname{Ind}_B^{U(W)} \left(\chi_v|\cdot|^{-1/2} \otimes \chi\right)$ with B the Borel

  $$B = \left\{ \begin{pmatrix} a & & * \\ & b & \\ 0 & & (a^c)^{-1} \end{pmatrix} : a \in E^\times, b \in E^1 \right\}$$

  The exponent of the absolute value in the rep being induced is non-imaginary, so θ^ε is non-tempered.

Let's look at the global setting.

Global setting Say k is a number field. Can imagine getting an abstract lifting

$$\theta = \prod_v \theta_v : \operatorname{Irr} U(V)_{\mathbb{A}} \to \operatorname{Irr} U(W)_{\mathbb{A}}.$$

We would really like this be a lifting of automorphic representations

$$\theta : \left\{ \begin{matrix} \text{auto reps} \\ \text{of } U(V) \end{matrix} \right\} \longrightarrow \left\{ \begin{matrix} \text{auto reps} \\ \text{of } U(W) \end{matrix} \right\}$$

In particular, a lift not of abstract representations but automorphic representations (whose elements are functions).

Question 2.3.9. *How to transfer functions from a space X to a space Y?*

Simple idea: If $K \in C(X \times Y)$, then get

$$\begin{array}{rccc} T_K : & C(X) & \longrightarrow & C(Y) \\ & f & \longmapsto & \left[y \mapsto \int_X K(x,y) f(x) \mathrm{d}x \right]. \end{array}$$

In our case, we had

$$w_\psi$$

$$\begin{tikzcd} & \mathrm{Mp}(V\otimes W)_{\mathbb{A}} \arrow[d] \\ U(V)_{\mathbb{A}}\times U(W)_{\mathbb{A}} \arrow[ur, dashed, "\widetilde{\iota}"] \arrow[r, "\iota"] & \mathrm{Sp}(V\otimes W)_{\mathbb{A}} \end{tikzcd}$$

$$\Omega = \widetilde{\iota}^* w_\psi$$

Have

$$\theta : w_\psi \longrightarrow A_2(\mathrm{Mp}(-)) \xrightarrow{\widetilde{\iota}^*} C([U(V)\times U(W)]).$$

Say $\pi \in A_0(U(V))$. This gives

$$\begin{array}{ccc} w_\psi \otimes \pi & \longrightarrow & A(u(W)) \\ \varphi\otimes f & \longmapsto & \theta(\varphi, f) \end{array}$$

where

$$\theta(\varphi, f)(g) = \int_{[U(V)]} \theta(\varphi)(g,h)\overline{f(h)}\mathrm{d}h.$$

The **global theta lift** of π is

$$\Theta(\pi) = \langle \theta(\varphi, f) : \varphi \in w_\varphi, f\in\pi\rangle \subset A(U(W)).$$

Question 2.3.10. *Is $\Theta(\pi)$ nonzero? Is $\Theta(\pi)\subset A_2(U(W))$ or $A_0(U(W))$ or neither? What is the relation with the local theta correspondence?*

Proposition 2.3.11. *If $\Theta(\pi) \subsetneq A_2(U(W))$, then $\Theta(\pi)$ is either 0 or it is irreducible and*

$$\Theta(\pi) \simeq \bigotimes_v{}' \Theta(\pi_v).$$

Theorem 2.3.12. *Say $\pi\subset A_0(U(V))$ a cuspidal automorphic representation. Then,*

- ***(i)*** *There is a smallest $r_0 = r_0^\varepsilon(\pi) \le \dim V$ s.t. $\Theta^\varepsilon_{r_0}(\pi)\neq 0$, in which case $\Theta^\varepsilon_{r_0}(\pi)\subset A_0(U(W))$.*
- ***(ii)*** *For all $r > r_0$, $\Theta^\varepsilon_r(\pi) \neq 0$ and non-cuspidal*
- ***(iii)*** *For all $r\ge \dim V$, $\Theta^\varepsilon_r(\pi)\subset A_2(U(W))$.*

3 Aaron Pollack: Modular forms on exceptional groups

3.1 Lecture 1

Plan of lectures

(1) (Today) What is G_2, and what are modular forms on G_2?

(2) Fourier expansion of modular forms on G_2

(3) Examples/theorems about modular forms on G_2

(4) Beyond G_2

The first thing we want to do is remember how to go between classical modular forms on the upper half plane and automorphic forms on $\mathrm{SL}_2(\mathbb{R})$.

Recall 3.1.1. Let $f : \mathfrak{H} \to \mathbb{C}$ be a weight $\ell > 0$, level Γ modular form. We define $\varphi_f : \mathrm{SL}_2(\mathbb{R}) \to \mathbb{C}$ via

$$\varphi_f(g) = j(g,i)^{-\ell} f(g \cdot i) \text{ where } j(g,z) = cz + d \text{ if } g = \begin{pmatrix} * & * \\ c & d \end{pmatrix}.$$

Then,

(0) φ_f is of moderate growth

(1) φ_f is left-invariant under Γ, i.e. $\varphi_f(\gamma g) = \varphi_f(g)$ for all $\gamma \in \Gamma \subset \mathrm{SL}_2(\mathbb{Z})$

(2) Let $k_\theta := \begin{pmatrix} \cos\theta & -\sin\theta \\ \sin\theta & \cos\theta \end{pmatrix} \in \mathrm{SO}(2)$. Then,

$$\varphi_f(gk_\theta) = e^{-i\ell\theta}\varphi_f(g)$$

(3) $D_{cR}\varphi_f \equiv 0$ corresponding to the fact that f satisfies the Cauchy-Riemann equations. To define this, write write $\mathfrak{sl}_2(\mathbb{R}) \otimes \mathbb{C} = \mathfrak{k}_0 \otimes \mathbb{C} + \mathfrak{p}_0 \otimes \mathbb{C}$. Here, $\mathfrak{k}_0$ is the antisymettric (traceless) 2×2 matrices and $\mathfrak{p}_0$ is the symmetric ones. Note that $\mathfrak{p}_0 \otimes \mathbb{C} = \mathbb{C}X_+ \oplus \mathbb{C}X_-$, where

$$X_\pm = \begin{pmatrix} 1 & \pm i \\ \pm i & -1 \end{pmatrix}.$$

Now, $D_{cR}\varphi_f = X_-\varphi_f$ (which is 0 identically).

We can go back as well. Suppose $\varphi : \mathrm{SL}_2(\mathbb{R}) \to \mathbb{C}$ satisfies conditions **(0)**-**(3)** above. Then, we can define $f : \mathfrak{H} \to \mathbb{C}$ via

$$f(z) = j(g_z, i)^\ell \varphi(g_z) \text{ where } g_z \in \mathrm{SL}_2(\mathbb{R}) \text{ such that } g_z \cdot i = z.$$

One can check that this is a well-defined, holomorphic, weight ℓ, level Γ modular form.

3.1.1 Modular forms on G_2

Note that G_2 is a simple, non-compact Lie group of dimension 14. Let $K \subset G_2$ be a maximal compact subgroup (so $K \cong (\mathrm{SU}(2) \times \mathrm{SU}(2))/(\pm 1)$). These SU(2)'s correspond to the 'long roots' and the 'short roots'. Note that $K \curvearrowright \mathbb{V}_\ell$ where $V_\ell := \mathrm{Sym}^{2\ell}(\mathbb{C}^2) \boxtimes \mathbf{1}$.

Remark 3.1.2. The diagonal ± 1 acts trivially on $\mathbb{V}_\ell$.

Definition 3.1.3 (Gross-Wallach, Gan-Gross-Savin)**.** Suppose $\Gamma \subset G_2$ is a congruence subgroup (i.e. $\Gamma = G_2(\mathbb{Q}) \cap K_f$ with $K_f \subset G_2(\mathbb{A}_f)$ compact open). Suppose ℓ is a positive integer. A **modular form on G_2 of weight ℓ and level** Γ is a function $\varphi : G_2 \longrightarrow \mathbb{V}_\ell$ satisfying

(0) φ has moderate growth

(1) $\varphi(\gamma g) = \varphi(g)$ for all $\gamma \in \Gamma$

(2) $\varphi(gk) = k^{-1} \cdot \varphi(g)$ for all $k \in K$

(3) $D_\ell \varphi = 0$ for some specific differential operator D_ℓ.

Could also make an adelic definition: $\varphi : G_2(\mathbb{Q})\backslash G_2(\mathbb{A}) \to \mathbb{V}_\ell$ satisfying approparite properties. What do we want to do in this lecture...

(1) What is G_2?

(2) What is D_ℓ?

(3) Examples/theorems about modular forms on G_2 (probably next time)

The upshot will be that modular forms on G_2 have classical Fourier expansions/coefficients, and their Fourier coefficients appear to be very arithmetic.

3.1.2 What is G_2?

We start with its Lie algebra/$\mathbb{Q}$. As a vector space, it is

$$\mathfrak{g}_2 = \mathfrak{sl}_3 \oplus V_3(\mathbb{Q}) \oplus V_3^\vee(\mathbb{Q})$$

where $V_3(\mathbb{Q})$ is the standard representation of $\mathfrak{sl}_3$. We give it a $\mathbb{Z}/3\mathbb{Z}$-grading by putting $\mathfrak{sl}_3$ in degree 0, V_3 in degree 1, and $V_3^\vee$ in degree 2. In particular, if X is in degree i and Y is in degree j, then $[X, Y]$ is in degree $i + j \mod 3$. The bracket is given by the linear extension of the following: suppose $\varphi, \varphi' \in \mathfrak{sl}_3$, $v, v' \in V_3$, and $\delta, \delta' \in V_3^\vee$. Then,

$$[\varphi, \varphi'] = \varphi \circ \varphi' - \varphi' \circ \varphi, \ \ [\varphi, v] = \varphi(v), \ \text{ and } \ [\varphi, \delta] = \varphi(\delta).$$

These are the easy relations. For the others, first observe that $\bigwedge^3 V_3 = \mathbf{1}$ is the trivial representation, so we get identifications $\bigwedge^2 V_3 = V_3^\vee$ and $\bigwedge^2 V_3^\vee = V_3$. Explicitly, fixing a dual bases $v_1, v_2, v_3 \in V_3$ and $\delta_1, \delta_2, \delta_3 \in V_3^\vee$, we can identify

$$v_i \wedge v_{i+1} = \delta_{i-1} \ \text{ and } \ \delta_i \wedge \delta_{i+1} = v_{i-1}.$$

Back to the bracket, we take

$$[v, v'] = 2v \wedge v' \in \bigwedge^2 V_3 = V_3^\vee \text{ and } [\delta, \delta'] = 2\delta \wedge \delta' \in \bigwedge^2 V_3^\vee = V_3.$$

Finally,

$$[\delta, v] = 3v \otimes \delta - \delta(v)\mathbf{1}.$$

Note $v \otimes \delta \in V_3 \otimes V_3^\vee = \mathrm{End}(V_3)$ and subtracting off $\delta(v)\mathbf{1}$ kills its trace.

Proposition 3.1.4. *$\mathfrak{g}_2$ is a simple Lie algebra, i.e. $[-,-]$ satisfies the Jacobi identity, and $\mathfrak{g}_2$ has no nontrivial ideals.*

Define $\mathrm{Aut}(\mathfrak{g}_2) := \{g \in \mathrm{GL}(\mathfrak{g}_2) : [gX, gY] = g[X,Y] \text{ for all } X, Y \in \mathfrak{g}_2\}$ (a Lie group or linear algebraic group depending on tastes). Then, $G_2 = \mathrm{Aut}^\circ(\mathfrak{g}_2)$ is the identity component of the automorphism group.

Remark 3.1.5. In the notes, you can find a completely analogous description of all the exceptional groups.

Root diagram of $\mathfrak{g}_2$ Let $\mathfrak{h} \subset \mathfrak{sl}_3$ be the diagonal elements, so

$$\mathfrak{g} = \{\alpha_1 E_{11} + \alpha_2 E_{22} + \alpha_3 E_{33} : \alpha_1 + \alpha_2 + \alpha_3 = 0\}.$$

Let $r_1, r_2, r_3 : \mathfrak{h} \to \mathbb{Q}$ be given by $r_j \left(\sum_i \alpha_i E_{ii}\right) = \alpha_j$. Note that $r_1 + r_2 + r_3 = 0$. This $\mathfrak{h} \subset \mathfrak{sl}_3 \subset \mathfrak{g}_2$ will be a Cartan subalgebra of $\mathfrak{g}_2$.

What are its weights?

- On V_3, they are r_1, r_2, r_3
- On $V_3^\vee$, they are $-r_1, -r_2, -r_3$
- On $\mathfrak{sl}_3$, they are $\{r_i - r_j\}_{i \neq j}$

Remark 3.1.6. The root diagram looks like the vertices (including some inner vertices) of a star of David. Put the roots r_1, r_2, r_3 at the vertices of an equilateral triangle with center 0.

TODO: Add picture?

3.1.3 The differential operator D_ℓ

Define $\Theta : \mathfrak{g}_2 \otimes \mathbb{R} \to \mathfrak{g}_2 \otimes \mathbb{R}$, a **Cartan involution**, explicitly via

- $\Theta : \mathfrak{sl}_3 \to \mathfrak{sl}_3$ as $X \mapsto -{}^t X$.
- $\Theta : V_3 \longleftrightarrow V_3^\vee$ via $v_j \leftrightarrow \delta_j$

Let $\mathfrak{k}_0 = (\mathfrak{g}_2 \otimes \mathbb{R})^{\Theta = 1}$ be where Θ acts by 1 and let $\mathfrak{p}_0$ be where it acts by -1. Let

$$K = \{g \in G_2 : \mathrm{Ad}(g) \circ \Theta = \Theta \circ \mathrm{Ad}(g)\}.$$

This is the maximal compact. Let $\mathfrak{k} = \mathfrak{k}_0 \otimes \mathbb{C}$ and $\mathfrak{p} = \mathfrak{p}_0 \otimes \mathbb{C}$. Then, $\mathfrak{k} \simeq \mathfrak{sl}_2 \oplus \mathfrak{sl}_2$ and $\mathfrak{p} \simeq V_2 \otimes \mathrm{Sym}^3(V_3)$ as a representation of $\mathfrak{k}$ (via the Lie bracket). Finally, $D_\ell = \mathrm{pr} \circ \widetilde{D}_\ell$ is the composition of two operators. Suppose

$$\varphi : G_2 \longrightarrow \mathbb{V}_\ell = \mathrm{Sym}^{2\ell}(\mathbb{C}^2) \boxtimes \mathbf{1}$$

satisfies $\varphi(gk) = k^{-1}\varphi(g)$ for all $k \in K$. Let $\{X_\alpha\}$ be a basis of φ with dual basis $\{X_\alpha^\vee\}_\alpha$ of $\varphi^\vee$. Then,

$$\widetilde{D}_\ell \varphi = \sum_\alpha X_\alpha \varphi \otimes X_\alpha^\vee \in \mathbb{V}_\ell \otimes \varphi^\vee$$

where $X_\alpha \varphi$ is the differential of the right regular action, i.e. if $X \in \mathfrak{p}_0$ then

$$(X\varphi)(g) = \frac{\partial}{\partial t}\Big|_{t=0} \left(\varphi\left(ge^{tX}\right)\right).$$

As a representation of $\mathfrak{k}$, we have

$$\mathbb{V}_\ell \otimes \varphi^\vee = \left(S^{2\ell} \boxtimes \mathbf{1}\right) \otimes \left(V_2 \boxtimes \operatorname{Sym}^2(V_\ell)\right) = \left(S^{2\ell+1} + S^{2\ell-1}\right) \boxtimes S^3(V_2) \xrightarrow{\text{pr}} S^{2\ell-1}(V_2) \boxtimes S^3(V_2)$$

(pr natural projection).

Remark 3.1.7. You could give this same definition with SL_2 in place of G_2. If you do, then this D_ℓ gets replaced by D_{CR}, so can think of this as a representation theoretic derivation of the Cauchy-Riemann equations.

3.2 Lecture 2

Last time we talked about the group G_2 as well as modular forms on it.

Remark 3.2.1. Let $K \subset G_2$ be its maximal compact subgroup, and set $X = G_2/K$. In the case of holomorphic modular forms, $\mathrm{SL}_2(\mathbb{R})/\mathrm{SO}(2) = \mathfrak{H}$ is the upper half-plane which has an $\mathrm{SL}_2(\mathbb{R})$-invariant complex structure. In contrast, X does *not* have a G_2-invariant complex structure.

Goal. Discuss the Fourier expansion of modular forms on G_2

We'll start by discussing a group-theoretic perspective on the Fourier expansion of classical modular forms...

Say $f(z) = \sum_{n\geq 0} a_f(n) q^n$ is a weight ℓ modular form.

Recall 3.2.2. Defined $\varphi_f : \mathrm{SL}_2(\mathbb{R}) \to \mathbb{C}$ via $\varphi_f(g) = j(g,i)^{-\ell} f(g \cdot i)$.

Define

$$\begin{array}{cccc} W_n : & \mathrm{SL}_2(\mathbb{R}) & \longrightarrow & \mathbb{C} \\ & g & \longmapsto & j(g,i)^{-\ell} e^{2\pi i n (g\cdot i)} \end{array}$$

Note that

- $$W_n\left(\begin{pmatrix} 1 & x \\ 0 & 1 \end{pmatrix} g\right) = e^{2\pi i n x} W_n(g)$$
- $$W_n(gk_\theta) = e^{-i\ell\theta} W_n(g) \text{ where } k_\theta = \begin{pmatrix} \cos\theta & -\sin\theta \\ \sin\theta & \cos\theta \end{pmatrix}.$$
- $X_- W_n \equiv 0$

- $$W_n \begin{pmatrix} y^{1/2} & \\ & y^{-1/2} \end{pmatrix} = y^{\ell/2} e^{-2\pi n y}$$

  is completely explicit.

We write
$$\varphi_f(g) = \sum_{n \geq 0} a_f(n) W_n(g)$$
for the Fourier expansion.

Say we have a modular form $\varphi : \Gamma \backslash G_2 \to \mathbb{V}_\ell$ of weight ℓ. Recall $\mathbb{V}_\ell = \mathrm{Sym}^{2\ell}(\mathbb{C}^2) \boxtimes \mathbf{1}$ is a representation of $K = (\mathrm{SU}(2) \times \mathrm{SU}(2)) / \langle \pm 1 \rangle$. What will happen is that we'll get
$$\varphi(g) \text{``} =\text{''} \sum_f a_\varphi(f) W_f(g)$$
where $a_\varphi(f) \in \mathbb{C}$ the Fourier coefficients, f ranges over binary cubics (?), and $W_f(g)$ satisfies properties reminiscent of those of W_n above.

Recall 3.2.3. $\mathfrak{g}_2 = \mathfrak{sl}_2 \oplus V_3 \oplus V_3^\vee$. Take basis $\{E_{ij}\}$ of $\mathfrak{sl}_3$, v_1, v_2, v_3 of V_3, and $\delta_1, \delta_2, \delta_3$ of $V_3^\vee$ (dual to v_1, v_2, v_3). Imagine the root diagram of G_2 with $E_{12}, E_{13}, E_{23}, v_1, \delta_3$ all in the top half. E_{13} is top vertex, and E_{12}, E_{23} are outer vertices of second row.

G_2 has 2 conjugacy classes of maximal parabolic subgroups. Let P be the parabolic w/ Lie algebra as pictured in the drawing I don't have included in these notes... We can write $P = MN$ with $M \cong \mathrm{GL}_2$ and $N \supset Z = [N, N]$ (Z the center?). Here,
$$Z = \exp(\mathbb{R} E_{13}) \text{ and } N/Z = \exp\left(\underbrace{\mathbb{R} E_{12} + \mathbb{R} v_1 + \mathbb{R} \delta_3 + \mathbb{R} E_{23}}_{W} \right).$$
Note that $M \curvearrowright Z$ as det and $M \curvearrowright N/Z$ as $\mathrm{Sym}^3(V_2) \otimes \det(V_2)^{-1}$. There is a symplectic form $\langle -, - \rangle : W \times W \to \mathbb{R}$ on W defined by
$$[w, w'] = \langle w, w' \rangle E_{13}.$$
Explicitly, if $w = aE_{12} + \frac{b}{2} v_1 + \frac{c}{2} \delta_3 + dE_{23}$ and similarly for w', then
$$\langle w, w' \rangle = ad' - \frac{bc'}{3} + \frac{cb'}{3} - da'.$$
This pairing is M-equivariant: $\langle mw, mw' \rangle = \det(m) \langle w, w' \rangle$.

Characters of N Let's switch to adelic notation. Suppose φ is an automorphic form on $G_2(\mathbb{A})$. Also suppose $\psi : \mathbb{Q} \backslash \mathbb{A} \to \mathbb{C}^\times$ is a nontrivial additive character. Consider some $w \in W(\mathbb{Q})$. Define
$$\varphi_w(g) = \int_{[N]} \psi^{-1}(\langle w, \overline{n} \rangle) \varphi(ng) \mathrm{d}n$$

where $\overline{n} \in N/Z = W$ (identified via exponetial map) is the image of n. These are Fourier coefficients. Also have constant terms

$$\varphi_Z(g) = \int_{[Z]} \varphi(zg)\mathrm{d}z \text{ and } \varphi_N(g) = \int_{[N]} \varphi(ng)\mathrm{d}n$$

along Z and N. Then,

$$\varphi_Z(g) = \varphi_N(g) + \sum_{w \in W(\mathbb{Q})} \varphi_w(g)$$

(expand along abelian N/Z?). For modular forms, we can refine this Fourier expansion.

Proposition 3.2.4. *If $\varphi_Z(g) \equiv 0$, then $\varphi(g) \equiv 0$*

Thus, φ is determined by its Fourier expansion along Z (or something. Not sure if this is the right phrase).

Suppose $\varphi : G_2(\mathbb{Q})\backslash G_2(\mathbb{Q}) \longrightarrow \mathbb{V}_\ell$ is a modular form of weight ℓ. The functions $\varphi_w(g)$ satisfying

(0) they are of moderate growth

(1) $\varphi_w(ng) = \psi(\langle w, \overline{n}\rangle)\varphi_w(g)$

(2) $\varphi_w(gk) = k^{-1}\varphi_w(g)$

(3) $D_\ell \varphi_w \equiv 0$

Definition 3.2.5. A function F satisfying **(0)**–**(3)** above is called a **generalized Whittaker function** of type (w, ℓ).

Such functions will be uniquely determined up to scalar multiple. In fact, we'll be able to say $\varphi_w(g) = \lambda W_w(g)$ for some explicit W_w (capital W for generalized Whittaker function). This will imply that

$$\varphi_Z(g) = \varphi_N(g) + \sum_{w \neq 0} a_\varphi(w) W_w(g),$$

and so give a Fourier expansion for φ a modular form of weight ℓ.

We identify the space W with binary cubics via

$$w := aE_{12} + \frac{b}{3}v_1 + \frac{c}{3}\delta_3 + dE_{23} \longleftrightarrow au^3 + bu^2v + cuv^3 + dv^3 =: f_w.$$

For nonzero $w \in W(\mathbb{R})$, we define $\beta_w(m)$ (for $m \in \mathrm{GL}_2(\mathbb{R})$) as

$$\beta_w(m) := \left\langle w, m \cdot (u - iv)^3 \right\rangle.$$

These appear in the Fourier expansion on G_2.

Proposition 3.2.6. *TFAE*

(1) *$\beta_w(m) \neq 0$ for all $m \in \mathrm{GL}_2(\mathbb{R})$*

(2) *$f_w(z, 1) \neq 0$ for $z \in \mathfrak{H}$*

(3) f_w *splits into linear factors over* $\mathbb{R}$

The positive condition appearing in the Fourier expansion of modular forms on G_2 concerns these properties.

Definition 3.2.7. If w satisfies any of the equivalent properties above, then we say that it is **positive semi-definite** and write $w \geq 0$.

Example. $f_u(u, v) = au^3 \geq 0$
$f_w(u, v) = -u^3 + uv^2 = u(v - u)(v + u) \geq 0$
$f_w(u, v) = u^3 + v^3 \not\geq 0$

Define, for $m \in \mathrm{GL}_2(\mathbb{R}) = M(\mathbb{R})$ (the 'Levi subgroup of the Heisenberg parabolic') and $w \geq 0$,

$$W_w(m) = |\det(m)| \det(m)^\ell \sum_{-\ell < v \leq 0} \frac{|\beta_w(m)|}{\beta_w(m)}^v K_v\left(|\beta_w(m)|\right) \frac{x^{\ell+v} y^{\ell-v}}{(\ell+v)!(\ell-v)!}$$

where K_v is a K-Bessel function and $x^{2\ell}, x^{2\ell-1}y, \ldots, y^{2\ell}$ is a basis of $\mathbb{V}_\ell$. Note

$$K_v(y) = \frac{1}{2} \int_0^\infty e^{-y*t+t^{-1})/2} t^v \frac{\mathrm{d}t}{t}.$$

Remark 3.2.8. K_v diverges at 0, so good thing that $\beta_w(m) \neq 0$.

The functions $W_w : M(\mathbb{R}) \to \mathbb{V}_\ell$ extend uniquely to functions $G_2 \to \mathbb{V}_\ell$ via

$$W_w(ng) = e^{2\pi i \langle w, \overline{n} \rangle} W_w(g) \text{ for all } n \in N(\mathbb{R}) \text{ and } W_w(gk) = k^{-1} W_w(g) \text{ for all } k \in K.$$

Theorem 3.2.9. *Suppose* $w \neq 0$ *and that* F *is a generalized Whittaker function of type* (w, ℓ)*. Then,*

(1) $w \not\geq 0 \implies F \equiv 0$

(2) $w \geq 0 \implies F = \lambda W_w$ *for some* $\lambda \in \mathbb{C}$.

Corollary 3.2.10. *If* φ *is a modular form on* G_2 *of weight* ℓ*, there exists Fourier coefficients* $a_\varphi(w) \in \mathbb{C}$ *s.t.*

$$\varphi_Z(g) = \varphi_N(g) + \sum_{w \geq 0} a_\varphi(w) W_w(g).$$

In fact, the sum only varies over $w \geq 0$ *with* f_w *an integral binary cubic.*

Moreover, φ_N can be described explicitly in terms of holomorphic modular forms of weight 3ℓ on GL_2.

Definition 3.2.11. These $a_\varphi(w)$ are, by definition, the Fourier coefficients of φ.

Goal for next two lectures is to give some examples and to say something about this Fourier coeffs.

Remark 3.2.12. Gan-Gross-Savin previously defined these Fourier coefficients using a multiplicity 1 result of N. Wallach without suing the explicit functions $W_w(g)$ (if $\mathrm{disc}(f_w) \neq 0$ and $\ell \geq 4$).

3.3 Lecture 3: Examples of modular forms on G_2

The simplest example is the degenerate Eisenstein series.

Recall 3.3.1. $G_2 \supset P = MN$ where P the Heisenberg parabolic and $M = \mathrm{GL}_2$ (the Levi subgroup?).

Consider character $\nu : P \to M \xrightarrow{\det} \mathrm{GL}_1$. Suppose $\ell > 0$ is even. Consider $\mathrm{Ind}_P^G(|\nu|^s)$. Let $f_{\ell,\infty}(g,s) \in \mathrm{Ind}_{P(\mathbb{R})}^{G_2(\mathbb{R})}(|\nu|^s) \otimes \mathbb{V}_\ell$ where

$$\mathbb{V}_\ell = \mathrm{Sym}^{2\ell}(V_2) \boxtimes \mathbf{1} \curvearrowleft K \subset G_2.$$

$f_{\ell,\infty}$ is defined by

- $f_{\ell,\infty}(pg,s) = (???)f_{\ell,\infty}(g,s)$ for all $p \in P(\mathbb{R})$ (could not read the ???)
- $f_{\ell,\infty}(gk,s) = k^{-1}f_{\ell,\infty}(g)$ for all $k \in K$
- By Iwasawa decomposition $G_2(\mathbb{R}) = P(\mathbb{R})K$, $f_{\ell,\infty}$ now uniquer determined once we put

$$f_{\ell,\infty}(1) = x^\ell y^\ell \in \mathbb{V}_\ell = \left\langle x^{2\ell}, \ldots, y^{2\ell} \right\rangle.$$

Let $f_{fin}(g,s) \in \mathrm{Ind}_{P(\mathbb{A}_f)}^{G_2(\mathbb{A}_f)}(|\nu|^s)$ be a **flat section**, i.e. $f_{fin}|_{G_2(?)}$ is independent of s. Let $f_\ell(g,s) = f_{fin}(g_f,s)f_{\ell,\infty}(g_\infty,s)$ with $g \in G_2(\mathbb{A})$. Define

$$E_\ell(g,f,s) = \sum_{\gamma \in P(\mathbb{Q})\backslash G_2(\mathbb{Q})} f_\ell(\gamma g, s)$$

which converges if $\mathrm{Re}(s) > 3$. Set $E_\ell(g) := E_\ell(g,f,\ell+1)$.

Theorem 3.3.2. *If $\ell > 0$ is even and $\ell + 1 > 3$, then $E_\ell(g)$ is a quaternionic modular form on G_2 of weight ℓ.*

Proof. Check $f_{\ell,\infty}(g, s = \ell+1)$ is annihilated by the differential operator D_ℓ. Then, $E_\ell(g)$ will also be killed by D_ℓ since the sum converges absolutely. ∎

What's the next simplest example? Suppose π is a cuspidal automorphic representation of $\mathrm{GL}_2 = M$, associated to a holomorphic weight 3ℓ cuspidal modular form. Then, can make inducing sections $f_\pi \in \mathrm{Ind}_{P(\mathbb{A})}^{G_2(\mathbb{A})}(\pi)$ and form an Eisenstein series $E(g,f_\pi) = \sum_{\gamma \in P(\mathbb{Q})\backslash G_2(\mathbb{Q})} f_\pi(\gamma g)$. This will converge absolutely if $\ell \geq 6$ and define a weight ℓ modular form on G_2. If $\ell = 4$, can still make sense $E(g,f_\pi)$, giving a weight 4 modular form on G_2 associated to Δ (weight 12 cusp form).

We'd like to say something about the Fourier coefficients of these things, but to do that, we'll need know a bit about cubic rings.

Fact. If φ is a level 1 quaternionic modular form on G_2, then

- $a_\varphi(w) \neq 0 \implies f_w(u,v)$ is integeral, i.e. is $au^3 + \cdots + dv^3$ with $a,b,c,d \in \mathbb{Z}$.
- $a_\varphi(w \cdot \gamma) = \det(\gamma)^\ell a_\varphi(w)$ for $\gamma \in \mathrm{GL}_2(\mathbb{Z})$

  The Fourier coefficients are (almost) constant on $\mathrm{GL}_2(\mathbb{Z})$-orbits of integral binary cubics.

Fact. There is a canonical bijection

$$\{\text{Integral binary cubic forms}\} / \mathrm{GL}_2(\mathbb{Z}) \longleftrightarrow \{\text{cubic rings}\} / \simeq$$

Corollary 3.3.3. *If ℓ is even and φ is a level 1 weight ℓ modular form on G_2, you get a well-defined Fourier coefficient $a_\varphi(A)$ for any cubic ring A.*

Remark 3.3.4. Last time we introduced a positive semi-definiteness condition on integral binary cubics. If f_w is non-degenerate, the associated cubic ring $A(f_w)$ is totally real $\iff f_w \geq 0$.

Theorem 3.3.5 (Gan-Gross-Savin, Jieng-Rallis)**.** *Suppose A is the maximal order in a totally real cubic étale $\mathbb{Q}$-algebra E. Then, there exists $c_\ell \in \mathbb{C}$ (independent of A) s.t.*

$$a_{E_\ell}(A) = c_\ell \zeta_E(1-\ell)$$

(for ℓ even)

Remark 3.3.6. The constant c_ℓ is not known to be nonzero (This is an exercise on one of the problem sheets)

Open Question 3.3.7. *Can one saying anything about the Fourier coefficients of the Eisenstein series $E(g, f_\pi)$?*

3.3.1 Cusp forms

Theorem 3.3.8. *Suppose $\ell \geq 16$ is even. There exist nonzero cusp forms on G_2 of weight ℓ, all of whose Fourier coefficients are algebraic integers.*

Proof Sketch. Start with a holomorphic Seigal modular form f of weight ℓ on $\mathrm{Sp}(4)$. It's possible to choose $f \neq 0$ with Fourier coefficients in $\overline{\mathbb{Z}}$. Take the θ-lift of f to $\mathrm{SO}(4,4)$; obtain $\theta(f)$. This is

$$\theta(f)(g) = \int_{[\mathrm{Sp}(4)]} \Theta(g,h)\overline{f(h)}\mathrm{d}h$$

for some Θ-function Θ on $\mathrm{SO}(4,4) \times \mathrm{Sp}(4)$ (coming from some Weil representation as in Gan's talks). There exists a notion of quaternionic modular forms on the group $\mathrm{SO}(4,n)$. Can choose $\theta(g,h)$ s.t. $\theta(f)$ is a QMF on $\mathrm{SO}(4,4)$ of weight ℓ. One can express the Fourier coefficients of $\theta(f)$ in terms of the classical Fourier coefficients of f, and in particular see that the F.C.'s of $\theta(f)$ are in $\overline{\mathbb{Z}}$. Finally, there's map $\iota : G_2 \hookrightarrow \mathrm{SO}(4,4)$ and you can consider $\iota^*\theta(f)$ on G_2. This pullback will still be cuspidal (non-obvious) and still has algebraic integer Fourier coefficients. ■

Usually Fourier coefficients don't play this nicely with transferring between groups; that they do here is a testament to the rigidity of quaternionic modular forms.

Fact. In weight 20, there's a nonzero cuspidal modular form on G_2 w/ all Fourier coeffs$\in \mathbb{Z}$.

Theorem 3.3.9 (R. Dalal (in the audience))**.** *There exists an explicit dimension formula for the level 1 cuspidal quaternionic modular forms on G_2 of weight ℓ. In particular, the smallest nonzero level 1 cusp form is in weight* 6.

Theorem 3.3.10 ((three names I can't spell)-Hammonds-Pollack-Roy). *Suppose φ is a level 1 cusp QMF on G_2 associated to a cuspidal automorphic representation π on $G_2(\mathbb{A})$. Suppose moreover that $a_\varphi(\mathbb{Z}\times\mathbb{Z}\times\mathbb{Z}) \neq 0$. Then,*

(1) The completed standard L-function of π has functional equation

$$\Lambda(\pi, \mathtt{Std}, s) = \Lambda(\pi, \mathtt{Std}, 1-s).$$

(2) There exists a Dirichlet series for this L-function

$$\sum_{T\subset\mathbb{Z}^3, n\geq 1} \frac{a_\varphi(\mathbb{Z}+nT)}{[\mathbb{Z}^3:T]^{s-\ell+1}n^s} = a_\varphi(\mathbb{Z}^3)\frac{L(\pi, \mathtt{Std}, s-2\ell+1)}{\zeta(s-2\ell+2)^2\zeta(2-4s+2)}$$

(up to typos)

Proof idea. Make a refined analysis of a Rankin-Selberg integral due to Gurnich-Seigal (spelling?) ■

3.3.2 Half-integral weight

Theorem 3.3.11 (Leslie-P.). *There exists a theory of half-integral weight modular forms on G_2. These also have a good notion of F.C.'s, which are elements of $\mathbb{C}/\langle\pm 1\rangle$. Suppose R is a cubic ring contained in a totally real cubic field E. Let Q_R be the square roots of the inverse different of R, in the narrow class group of E.*

Precisely, say a pair (I,μ) is balanced if I is a fractional R-ideal, $\mu \in E^\times_{>0}$ totally positive, $I\mu^2 \subset \partial_R^{-1}$ is contained in the codifferent, and $N(I)^2N(\mu)\operatorname{disc}(R) = 1$.

Remark 3.3.12. If R is the maximal order, then (I,μ) balanced $\iff$ $I^2\mu = \partial_R^{-1}$

We say $(I,\mu)\sim(I',\mu')$ if there's some $\beta\in E^\times$ s.t. $I' = \beta I$, $\mu' = \beta^{-2}\mu$. Q_R is really $\{(I,\mu)$ balanced$\}/\sim$.

Remark 3.3.13. This definition is inspired by work of A. Swaminthan (spelling?)

Q_R can be empty

If $Q_R \neq \emptyset$ and R is the maximal order in E, then $\#Q_R = 4\#\operatorname{Cl}_E^+[2]$.

There exists a weight $\frac{1}{2}$ modular form Θ' on G_2 whose Fourier coefficients include the numbers $\pm\#Q_R$ for R even monogenic (i.e. $R \simeq \mathbb{Z}[y]/(y^3+cy^2+by+a)$ with $a,b,c\in 2\mathbb{Z}$)

4 Zhiwei Yun: Rigidity method for automorphic forms over function fields

4.1 Lecture 1

Today we will talk about automorphic forms over function fields. The next lecture, we'll introduce automorphic data and the notion of rigidity. The remaining will be devoted to constructing examples and trying to realize Langlands correspondence as explicitly as possible.

Setup. Let $k = \mathbb{F}_q$, and let X be a projective smooth geometrically connected curve over k (e.g. $X = \mathbb{P}^1_k$). Let $F = k(X)$ be the function field of X. Let $|X|$ denote the set of closed points of X. For $x \in |X|$, we let

$$\begin{tikzcd} F_x \arrow[d, "\wr"] & \supset & \mathscr{O}_x \arrow[r, two heads] \arrow[d, "\wr"] & k_x \\ k_x(\!(t_x)\!) & & k_x[\![t_x]\!] & \end{tikzcd}$$

denote the completion of F at x, its ring of integers, and the residue field at x.

We will also want to fix a split semisimple group G/k, e.g. $\mathrm{SL}_n, \mathrm{PGL}_n, \mathrm{Sp}_{2n}, G_2, E_8$, dot dot dot.

Let $\mathbb{A} := \prod'_{x\in|X|} F_x$ denote the adele ring. Then, $G(\mathbb{A}) = \prod'_{x\in|X|} G(F_x)$ is a locally compact topological group (almost all components $\in G(\mathscr{O}_x)$). We will also talk about **level groups**

$$K = \prod_{x\in|X|} K_x \text{ with } K_x \subset G(F_x) \text{ compact open}$$

(almost all $K_x = G(\mathscr{O}_x)$). For this lecture, can think about $K = K^\natural := \prod_{x\in|X|} G(\mathscr{O}_x)$.

Definition 4.1.1. The space A_K of **automorphic forms** is the space

$$A_K = c(G(F)\backslash G(\mathbb{A})/K, \mathbb{C})$$

of (continuous?) real-valued functions from this double coset to $\mathbb{C}$.

This is simpler than the number field situation since we don't need to worry about any regularity condition or differential equations. Note that A_K admits a right action of the **Hecke algebra**

$$H_K := \left\{K\backslash G(\mathbb{A})/K \xrightarrow{h} \mathbb{C} \text{ compactly supported}\right\}$$

(where multiplication is convolution and the unit is $\mathbf{1}_K$). The action $H_K \curvearrowright A_K$ is given by

$$(f * h)(x \in G(\mathbb{A})) = \sum_{g\in G(\mathbb{A})/K} f(xg)h(g^{-1})$$

where $f : G(F)\backslash G(\mathbb{A})/K \to \mathbb{C}$ and $h : K\backslash G(\mathbb{A})/K \to \mathbb{C}$. Note that the sum is finite since h is compactly supported.

We want to understand A_K as an H_K-module. We won't actually want to look at all of A_K.

Notation 4.1.2. Let $A_{K,c} \subset A_K$ be the subspace of compactly supported functions.

Let $A_{K,0} = \{f \in A_{K,c} \mid \dim_{\mathbb{C}}(H_K \cdot f) < \infty\}$ be the space of **cusp forms**. In the function field case, this is equivalent to the definition of cusp forms given in Wee Teck's first talk.

Definition 4.1.3. An **eigenform** $f \in A_{K,0}$ is one where for almost all x $(K_x = G(\mathscr{O}_x))$, f is an eigenvector under the **local Hecke algebra** $H_{K_x} = C_c(K_x \backslash G(F_x)/K_x, \mathbb{C})$.

4.1.1 Bun_G

We are interested in the double coset $G(F)\backslash G(\mathbb{A})/K^\natural$. André Weil observed that this double coset space is in natural bijection with G-bundles on X.

Example ($G = \mathrm{GL}_n$)**.** Note that a GL_n-bundle is the same data as a rank n vector bundle. In one direction, associated to a vector bundle $\mathcal{V}$ is its **frame bundle** $\underline{\mathrm{Isom}}(\mathscr{O}^{\oplus n}, \mathcal{V})$ which is now a $(\mathrm{GL}_n = \underline{\mathrm{Aut}}(\mathscr{O}^{\oplus n}))$-torsor over X. Weil gives a natural bijection

$$\mathrm{GL}_n(F)\backslash \mathrm{GL}_n(\mathbb{A})/K^\natural \longleftrightarrow \mathrm{Vect}_n(X)$$

which in fact preserves automorphisms (on the LHS, think of stabilizers of $\mathrm{GL}_n(F) \curvearrowright \mathrm{GL}_n(\mathbb{A})/K^\natural$), i.e. it gives an equivalence of groupoids. In one direction, start with some $(g_x) \in \mathrm{GL}_n(\mathbb{A})$. To keep life simple, assume that $g_x = 1$ for all $x \neq x_0$. To this, first assign the lattice $\Lambda_{x_0} := g_{x_0}\mathscr{O}_{x_0}^{\oplus n} \subset F_{x_0}^{\oplus n}$. Then, glue Λ_{x_0} with $\mathscr{O}_{X\setminus x_0}^{\oplus n}$. How do we do the gluing? Let $j : X \setminus x_0 \hookrightarrow X$ be the inclusion, and consider $j_*\mathscr{O}_{X\setminus x_0}^{\oplus n}$ which is quasi-coherent but not nec. coherent. If $U \subset X$ is affine, the glued bundle should have sections

$$U \mapsto \Gamma(U \setminus x_0, \mathscr{O}^{\oplus n}) \cap \Lambda_{x_0} \subset F_{x_0}^{\oplus n}.$$

Take sections of the trivial bundle, and then control polar behavior at x_0 according to the lattice Λ_{x_0}.

In the other direction, say we have a vector bundle V of rank n. Let $U \subset X$ be a Zariski open s.t. $V|_U \simeq \mathscr{O}_U^{\oplus n}$. For $x \not\in U$, consider the lattice $\Lambda_x = V|_{\mathrm{spec}\,\mathscr{O}_x}$ and write $\Lambda_x = g_x\mathscr{O}_x^{\oplus n}$. Then send $V \mapsto (g_x)_x$ $(g_x = 1$ if $x \in U)$.

Here's a concrete case. Take

$$g_{x_0} = \begin{pmatrix} t_{x_0} & & & \\ & 1 & & \\ & & \ddots & \\ & & & 1 \end{pmatrix}$$

with $t = t_{x_0}$ a uniformizer. This will give the vector bundle $\mathscr{O}(-x_0) \oplus \mathscr{O}^{\oplus(n-1)}$.

Is this fpqc descent? Alternatively, can you spread this lattice out to a small open?

Question: Why does the left factor sit inside $F_{x_0}^{\oplus n}$

Example ($G = \mathrm{Sp}_{2n}$)**.** Here a G-bundle is the same data as a pair (V, ω) with V a rank $2n$ v.b. and $\omega : V \otimes_{\mathscr{O}_X} V \to \mathscr{O}_X$ is a symplectic form (alternating , non-degenerate, billinear).

We have so far described things pointwise. Later we will geometrize the story and talk about Bun_G as a geometric object. It will be an Artin stack with

$$\mathrm{Bun}_G(R) = \{G\text{-bundles on } X_R\}$$

for any k-algebra R.

Example ($X = \mathbb{P}^1$)**.** Let's try and describe the set $\mathrm{Bun}_G(k)/\simeq$ of iso classes of G-bundles.

- Say $G = \mathrm{GL}_n$. Grothendieck tells us that every vector bundle on $\mathbb{P}^1$ splits as a direct sum of line bundles, and it's classical that $\mathrm{Pic}\,\mathbb{P}^1 = \mathbb{Z}$, so get a bijection

$$\mathrm{Vect}_n(k)/\simeq \longleftrightarrow (d_1 \geq d_2 \geq \cdots \geq d_n)\,.$$

  Note that such n-tuples give dominant coweights for GL_n.

- In general, fix a maximal torus $T \subset G$ with Weyl group W. Then, there's a bijection

$$\mathrm{Bun}_{G,\mathbb{P}^1}(k)/ \xrightarrow{\sim}\longleftrightarrow X_*(T)/W$$

  (RHS W-orbits of cocharacter lattice of maximal torus of G)

Note that we can think of $A_{K^\natural}$ as functions on $\mathrm{Bun}_G(k)$ (ignoring groupoid structure). How do we translate the H_K-action to the Bun_G world?

Example. Take $G = \mathrm{GL}_n$ and consider the characteristic function

$$h_x = 1_U \in H_{K_x} \text{ where } U := K_x \begin{pmatrix} t_x & & & \\ & 1 & & \\ & & \ddots & \\ & & & 1 \end{pmatrix}.$$

Say we're given $f : \mathrm{Bun}_G(k) \to \mathbb{C}$. What is $f * h_x : \mathrm{Bun}_G(k) \to \mathbb{C}$? Say $V \in \mathrm{Bun}_G(k)$ is a vector bundle on X. Then,

$$(f * h_x)(V) = \sum_{\substack{V \hookrightarrow V' \\ V'/V \cong k_x}} f(V')$$

(sum over "**elementary upper modifications of** V").

More generally, if we have taken

$$U = K_x \begin{pmatrix} t_x^{\lambda_1} & & \\ & \ddots & \\ & & t_x^{\lambda_n} \end{pmatrix} \text{ where } \lambda = (\lambda_1, \dots, \lambda_n),$$

then

$$(f * h_x)(V) = \sum_{V \underset{\lambda,x}{\dashrightarrow} V'} f(V')$$

where the relative modification of V' relative to V is determined by λ, x.

4.1.2 Level structures

Fix a closed point $x \in |X|$. To get interesting automorphic forms, often need to work with level groups other than $K^\natural$. Common choices come from parahoric subgroups of $G(F_x)$.

Definition 4.1.4. Let $B \leq G$ be a Borel subgroup. An **Iwahori subgroup** is the preimage $I_x \subset G(\mathscr{O}_x)$ of $B(k_x) \subset G(k_x)$ under the reduction map.

(Note all Iwahori subgroups are conjugate)

Example. If $G = \mathrm{GL}_2$, can take B to be the upper triangular matrices. Then get

$$I_x = \left\{ \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in \mathrm{GL}_2(\mathscr{O}_x) : c \in \mathfrak{m}_x \right\}.$$

Analogy:

G/k	G/F_x
Borel subgroups	Iwahori
parabolic subgroups	parahoric
Dynkin diagram	Affine Dynkin diagram

Example ($G = \mathrm{GL}_n$)**.** Fix some lattice $\Lambda_x \subset F_x^{\oplus n}$. Consider the stabilizer

$$\mathrm{Stab}_{G(F_x)}(\Lambda_x) = \{ g \in \mathrm{GL}_n(F_x) : g\Lambda_x = \Lambda_x \}.$$

This is a parahoic subgroup (e.g. if $\Lambda_x = \mathscr{O}_x^{\oplus n}$, then you get $\mathrm{GL}_n(\mathscr{O}_x)$). You can play the same game with multiple lattices, e.g. consider $\Lambda_0 \subset^1 \Lambda_1$ (notation means $\Lambda_1/\Lambda_0 \simeq k_x$). Then, $\mathrm{Stab}(\Lambda_0, \Lambda_1)$ will be parahoric. In general, if you have any chain $\Lambda_0 \subset \Lambda_{a_1} \subset \cdots \subset \Lambda_n \subset \Lambda_{n+a_1}$ where $\Lambda_n = t_x^{-1}\Lambda_0$ and $\Lambda_{n+a_1} = t_x^{-1}\Lambda_{a_1}$, then $\mathrm{Stab}_{\mathrm{GL}(F_x)}(\Lambda_\bullet)$ will be parahoric.[3] This is in fact how all parahoric subgroups of $\mathrm{GL}_n(F_x)$ arise. The group will be Iwahoric precisely when you take a complete chain.

Affine Dynkin diagram Let's see an example.

Example. The Dynkin diagram for G_2 is (point from shorter root to longer root)

$$\bullet \Rrightarrow \bullet$$

The corresponding affine Dynkin diagram is

$$\begin{array}{ccccc} \alpha_0 & & \alpha_1 & & \alpha_2 \\ \bullet & \text{———} & \bullet & \Rrightarrow & \bullet \end{array}$$

The empty set gives an Iwahori subgroup. The subset $\{\alpha_1, \alpha_2\}$ gives $G(\mathscr{O}_x)$. The subset $\{\alpha_0, \alpha_1\}$ gives $P \twoheadrightarrow \mathrm{SL}_3$ (can see the SL_3 in the diagram). The subset $\{\alpha_0, \alpha_2\}$ gives $Q \twoheadrightarrow \mathrm{SO}_4 = (\mathrm{SL}_2 \times \mathrm{SL}_2)/(\pm 1)$. The subdiagram tells you the type of the quotient and the kernel will be a projective limit of unipotent groups (if I heard correctly).

4.2 Lecture 2: Rigidity

Yesterday, we introduced the general setup for automorphic forms over a function field. Today, we want to introduce rigidity. We will begin by describing *Automorphic Data*

[3] subscripts indicate relative size

Fix a finite set $S \subset |X|$ of places of a complete, smooth curve X.

Assumption. Assume all points of S are $k = \mathbb{F}_q$-points

(e.g. $X = \mathbb{P}^1$ and $S = \{0, \infty\}$ or $S = \{0, 1, \infty\}$)

Setup. To each $x \in S$, say we have a compact open $K_x \subset G(F_x)$ (local level structure?) along with a continuous character $\chi_x : K_x \to \mathbb{C}^\times$ (factoring through a finite quotient L_x). This data (K_S, χ_S) constitutes **automorphic data**.

Definition 4.2.1. Given (K_S, χ_S), we can define (K_S, χ_S)**-typical automorphic forms**.

- If $\chi_S = 1$, these are $f \in A_{K,c}$ where $K = K_S \times \prod_{x \notin S} G(\mathscr{O}_x)$.
- In general, these are functions

$$f : G(F)\backslash G(\mathbb{A}) / \prod_{x \notin S} G(\mathscr{O}_x) \to \mathbb{C}$$

  so that

$$f(gk_x) = \chi_x(k_x) f(g) \text{ for all } x \in S, k_x \in K_x, g \in G(\mathbb{A}).$$

We let $A_c(K_s, \chi_S)$ denote the space of compactly supported (K_S, χ_S)-typical automorphic forms.

We want examples where this space is 1-dimensional, so any nonzero element of it will automatically be a Hecke eigenform. If $\dim A_c(K_S, \chi_S) = 1$, for all $y \notin S$, we have $H_{K_y} \curvearrowright A_c(K_S, \chi_S)$.

Example. Take $G = \mathrm{SL}_2$, $X = \mathbb{P}^1$ and $S = \{0, 1, \infty\}$. Also take $K_x = I_x$ for all $x \in S$, the Iwahori subgroup

$$I_x = \left\{ \begin{pmatrix} a & b \\ c & d \end{pmatrix} : \begin{matrix} a, b, c, d \in \mathscr{O}_x \\ c \in \mathfrak{m}_x \text{ and } ad - bc = 1 \end{matrix} \right\}$$

Consider $I_x \to k^\times$, picking about $\overline{a}$, and compose this with a character $\chi_x : k^\times \to \mathbb{C}^\times$. If we take $\chi_0, \chi_1, \chi_\infty$ in "generic position," then

$$\dim A_c(K_S, \chi_S) = 1.$$

Here, generic means

$$\chi_0^{\pm 1} \chi_1^{\pm 1} \chi_\infty^{\pm 1} \neq 1$$

is never trivial for any choice of signs in the exponents. The 'unique' $f \in A_c(K_S, \chi_S)$ will give a 2-dimensional local system on $\mathbb{P}^1 \setminus \{0, 1, \infty\}$ (this direction of global Langlands for SL_2 is known).

Example. Take $G = \mathrm{PGL}_2$ with $\chi_0 = \chi_1 = 1$ and χ_∞ quadratic. The resulting local system turns out to be the Tate module for the Legendre family $E_t : y^2 = x(x-1)(x-t)$ of elliptic curves over $\mathbb{P}^1 \setminus \{0, 1, \infty\}$. Get rank 2 local system $\{\mathrm{H}^1(E_t)\}$ which is the Langlands parameter for this example. More on this in the last lecture.

Example. Take $G = \mathrm{SL}_2$, $X = \mathbb{P}^1$, $S = \{0, \infty\}$, $K_0 = I_0$, $\chi_0 = 1$,

$$K_\infty = I_\infty^+ := \left\{ \begin{pmatrix} a & b \\ c & d \end{pmatrix} : \begin{matrix} a, d \equiv 1 \mod \mathfrak{m}_x \\ b \in \mathscr{O}_x \text{ and } c \in \mathfrak{m}_x \end{matrix} \right\}$$

(think $\Gamma_1(p)$). For the character at infinity, first note we have

$$\begin{array}{ccc} K_\infty & \longrightarrow & k \\ \begin{pmatrix} a & b \\ c & d \end{pmatrix} & \longmapsto & b + \dfrac{c}{\tau} \mod \tau \end{array}$$

(τ a uniformizer at ∞). χ_∞ is this composed with some $\psi : k \to \mathbb{C}^\times$. In this scenario, $\dim A_s(K_S, \chi_S) = 1$ and the rank 2 local system you get is the famous Kloosterman localy system on $\mathbb{P}^1 \setminus \{0, \infty\}$.

Definition 4.2.2. **Naive rigidity** of automorphic data (K_S, χ_S) is the requirement that $\dim A_c(K_S, \chi_S) = 1$.

When G is not simply connected, this is not a reasonable condition to ask for. Bun_G will have several components and so one should only require the space of automorphic forms to be one-dimensional on each component. Also, sometimes you can have $\dim A_c(K_S, \chi_S) = 1$ over a small field (e.g. $\mathbb{F}_2$), but then this may fail after base change (e.g. over $\mathbb{F}_8$).

Remark 4.2.3 (Base change of automorphic data). Let k'/k be a finite extension. Let $X' = X \otimes k'$. Let S' be the k'-rational points extending the (k-rational) points of S. We take $K_x \otimes_k k'$ for our compact opens. Our characters are

$$\chi'_x : K_x(k') \overset{\mathrm{Nm}}{\dashrightarrow} K_x(k) \xrightarrow{\chi_x} \mathbb{C}^\times$$

(I missed how to make sense of this Nm map in general, In the examples from before, things work as you'd naively expect)

Note 4. Got distracted and missed some remark on character sheaves (rank 1 local systems on K_x)

Now we can base change $(K_S, \chi_S)/k \rightsquigarrow (K'_S, \chi'_S)'_k$ and talk about $A_c(k'; K'_S, \chi'_S)$.

Definition 4.2.4. We say (K_S, χ_S) is **weakly rigid** if $\dim A_c(k'; K'_S, \chi'_S)$ is uniformly bounded for all k'/k (finite extensions).

(The two examples from before are both weakly rigid, with bound 1)

Recall 4.2.5. $G(F)\backslash G(\mathbb{A})/K \longleftrightarrow \mathrm{Bun}_G(K)(k)$ (capital K is level structure)

$f \in A_c(K_S, \chi_S)$ are functions on $\mathrm{Bun}_G(K_S^+)(k)$ where $K_x^+ \triangleleft K_x$ (for $x \in S$) is a finite codim subgroup (i.e. K_x/K_x^+ the k-points of a finite dim gp L_x) s.t. $\chi_x|_{K_x^+} = 1$.

Example. $I_x^+ \triangleleft I_x \to \mathbb{G}_m(k)$

Example. $I_\infty^{++} \triangleleft I_\infty^+ \to k \oplus k$ where $I_\infty^+ = K_\infty \to k \oplus k$ via $\begin{pmatrix} a & b \\ c & d \end{pmatrix} \to (b, c/\tau) \mod \tau$ (so $K_\infty^+ = I_\infty^{++}$)

Note that

$$C(\mathrm{Bun}_G(K_S^+)(k)) \curvearrowleft \prod_{x \in S} L_x(k),$$

and we're interested in the eigenfunctions under this action with eigencharacters $(\chi_x)_{x \in S}$. These are the elements of $A_c(K_S, \chi_S)$.

Remark 4.2.6. If you fix a point $\mathcal{E}$ in $\mathrm{Bun}_G(K_S)$, its fiber in $\mathrm{Bun}_G(K_S^+)$ will be a torsor for $\prod L_x(k)$. At the same time $\mathrm{Aut}(\mathcal{E})$ will act on this fiber (on the left) in a way that commutes with the right actions of L_x. This gives a group homomorphism

$$\mathrm{Aut}(\mathcal{E}) \xrightarrow{\mathrm{ev}_{\mathcal{E}}} \prod_{x \in S} L_x(k),$$

well-defined up to conjugacy.

Definition 4.2.7. A k-point $\mathcal{E} \in \mathrm{Bun}_G(K_S)(k)$ is (K_S, χ_S)**-relevant** if

$$\mathrm{ev}_{\mathcal{E}}^* \left(\prod_{x \in S} \chi_x \right) |_{\mathrm{Aut}(\mathcal{E})^\circ(k)} = 1.$$

Similarly, can define relevant k' points (and $\overline{k}$-relevant points, i.e. those relevant for some finite extension).

Fact. $\dim \mathcal{A}_c(k'; K_S, \chi_S) \le \#(K'_S, \chi'_S)$-relevant k'-points of $\mathrm{Bun}_G(K_S)$

(Would get exact equality if we didn't restrict to identity component earlier. However, in that case, relevance would not be stable under base change)

Corollary 4.2.8. (K_S, χ_S) *is weakly rigid iff there are finitely many* (K_S, χ_S)*-relevant* $\overline{k}$*-points of* $\mathrm{Bun}_G(K_S)$.

Example. $G = \mathrm{SL}_2$, $K_x = I_s$ for $x \in \{0, 1, \infty\} = S$. Here,

$$\mathrm{Bun}_G(K_S)(k) = \left\{ \left(V, \iota : \bigwedge^2 V \xrightarrow{\sim} \mathscr{O}_X, \{\ell_x \subset V_x\}_{x \in S} \right) \right\}.$$

In this example, we have

$$\begin{array}{ccc} I_x & \longrightarrow & \mathbb{G}_m = L_x \\ \begin{pmatrix} a & b \\ c & d \end{pmatrix} & \longmapsto & a \mod \tau \end{array}$$

Given $\mathcal{E} = (V, \iota, \ell_0, \ell_1, \ell_\infty)$, we need to describe $\mathrm{Aut}(\mathcal{E}) \to \prod_{x \in S} \mathbb{G}_m$. An automorphism of $\mathcal{E}$ is some $\gamma : V \xrightarrow{\sim} V$ preserving that data. In particular, γ_0 preserves on ℓ_0, and so acts by a scalar there. This scalar is its image in $\mathbb{G}_m = L_0$. Similarly for the other two factors. The relevancy condition is

$$\prod_{x \in S} \chi_c |_{\mathrm{Aut}(\mathcal{E})^0} \stackrel{?}{=} 1$$

Say $V = \mathscr{O}^2$ and $\ell_0, \ell_1, \ell_\infty \subset k^2$ three lines in generic position (no two coincide). Then, $\mathrm{Aut}(\mathcal{E}) = \{\pm 1\}$ so the identity component is trivial, and there is no condition. This point will be relevant.

All other points, however, are irrelevant. Say $V = L \oplus L'$ s.t. $\ell_x \in L_x$ or $\ell_x \in L'_x$. Then, $\mathbb{G}_m \subset \mathrm{Aut}(V, \dots)$ (scale L by λ and L' by λ^{-1}). This maps non-trivially onto each factor under $\mathbb{G}_m \to \prod_{x \in S} \mathbb{G}_m$ (either by identity or inverse). If the characters are generic (i.e. $\chi_0^{\pm 1} \chi_1^{\pm 1} \chi_\infty^{\pm 1} \neq 1$), then

$$\mathrm{ev}_{\mathcal{E}}^* \left(\prod_{x \in S} \chi_x \right) \bigg|_{\mathrm{Aut}(\mathcal{E})^\circ(k)} \neq 1,$$

so all other points are irrelevant (modulo proving you can always get a nice decomp like this).

4.3 Lecture 3

Keep in mind that Langlands aims to relate automorphic data on G with Galois data on $G^\vee$.

Automorphic G	Galois $G^\vee$
eigenform $f \in A_c(K_S, \chi_S)$	Local system $\pi_1(X \setminus S) \to G^\vee(\overline{\mathbb{Q}}_\ell)$
rigid auto data	rigid local system (Katz)

Want to look at the Galois side some today, and then discuss how to pass from the automorphic side to the Galois side tomorrow.

4.3.1 Designing Rigid Automorphic Data

Numerical Rigidity Essentially want functions on the moduli space $\mathrm{Bun}_G(K_S)(k)$. We want a situation where $A_c(K_S, \chi_S)$ has small dimension independent of the base field. This suggests we want $\dim \mathrm{Bun}_G(K_S) \leq 0$ (continuous moduli would give rise to more points over larger fields).

Fact. $\dim \mathrm{Bun}_G(K_S) = 0 \iff \sum_{x \in S}[G(\mathscr{O}_x) : K_x] = (1-g)\dim G$ ($[- : -]$ relative dimension as algebraic groups, and $g = g(X)$ is genus of base curve).

Example. If $K_x = I_x$ is Iwahori then $[G(\mathscr{O}_x) : I_x] = \dim(G(\mathscr{O}_x)/I_x) = \dim(G/B) = \#\Phi^+$

Remark 4.3.1. If $K_x \not\subset G(\mathscr{O}_x)$, then

$$[G(\mathscr{O}_x) : K_x] := \dim\left(G(\mathscr{O}_x)/G(\mathscr{O}_x) \cap K_x\right) - \dim K_x/G(\mathscr{O}_x) \cap K_x.$$

In the fact above, RHS≥ 0 only if $g \leq 1$. When $g = 1$, we're requiring $K_x \sim G(\mathscr{O}_x)$ for all $x \in S$. In genus > 1, we don't expect any rigid automorphic data. When $g = 1$, we expect few of them. Most examples will live above genus 0.

Example. Say $S = \{0, 1, \infty\} \subset \mathbb{P}^1 = X$. Say K_x is a parahoric subgroup for each x (coming from choices of subsets of affine Dynkin diagrams?). The numerical condition now becomes

$$\sum_{x=0,1,\infty} [G(\mathscr{O}_x) : K_x] = \dim G.$$

Let $K_x \twoheadrightarrow L_x$ be the reductive quotient of K_x (over the residue field k_x). Then,

$$[G(\mathscr{O}_x) : K_x] = \frac{\dim G - \dim L_x}{2}.$$

The numerical condition is now

$$\sum_{x=0,1,\infty} \dim L_x = \dim G.$$

- Say $G = G_2$ with affine Dynkin diagram

$$\alpha_1 \qquad \alpha_2 \qquad \alpha_3 \bullet \!-\!\!-\!\!-\!\! \bullet \Rrightarrow \bullet$$

- L_0. Take $\{\alpha_1, \alpha_3\}$. This has type $\mathrm{SL}_2 \times \mathrm{SL}_2$
- L_1. Take $\emptyset$ (so Iwahori). This has reductive quotient isogenous to $T(2d)$
- L_∞. Take $\{\alpha_1, \alpha_3\}$ again.

Get total dimension $(3+3)+2+(3+3) = 14 = \dim G_2$.

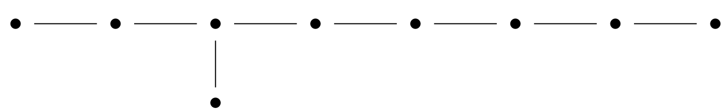
Could also do

- L_0 coming from $\{\alpha_1\}$ so get SL_2
- L_1 associated to $\{\alpha_1, \alpha_2\}$, so get SL_3
- L_∞ associated to $\{\alpha_3\}$ so get SL_2

These dimension again add to 14.

Warning 4.3.2. Above not what Yun did, but if I'm not too confused, I think it works

- Say $G = E_8$ with affine Dynkin diagram



Take

- $L_0 = L_\infty$ corresponding to the D_8 Dynkin diagram obtained by removing the furtherest left vertex. This gives $\mathrm{Spin}(16)/\{\pm 1\}$
- Take L_1 Iwahori $\twoheadrightarrow T$ of dimension 8 ($= \operatorname{rank} E_8$)

Sums to $2(120) + 8 = 248 = \dim E_8$

Yun said something about adding characters in order to get ridigity... Not gonna lie, my attention is running thin... Maybe I should have kept paying attention; he mentioned something about constructing a motive over $\mathbb{Q}$ whose motivic Galois group is E_8...

5 Akshay Venkatesh

5.1 Lecture 1

We only have two lectures, one at 9am on a Saturday and one at the end of four days of lectures, so we won't do anything too heavy. We'll talk about a fairy tale (reference to Langlands article in Corvalis). Akshay also mentioned a specific quote, google "Langland's Elephant".

We want to speak a bit about 'arithmetic topology,' an analogy between number fields and 3-manifolds. In this vein, what (if anything) should be the analogue of automorphic forms on the 3-manifold side? This analogy is a bit of the fairy tale; it won't help in proving things about the Langlands program, but should help to see the field as a whole as well as its interactions with some other parts of mathematics.

Let's start with a bit of a historical introduction to this analogy.

History. Mazur ('63 or '64) wrote a paper on the Alexander polynomial. In it, he writes "$\operatorname{spec}\mathbb{Z}/p\mathbb{Z}$ is like a knot in $\operatorname{spec}\mathbb{Z}$, which is like a simply connected 3-manifold," and attributes this analogy to Mumford. Mazur also mentions Artin and Tate in connection with this.

Where did this idea come from?

- Weil (1949) proposed the Weil conjectures. Reading between the lines, he's suggesting that there should be an "algebraic" cohomology theory for varieties X/k $(k = \overline{k})$ – say $X \rightsquigarrow \mathrm{H}^*(X)$ – such that, for $K = \mathbb{C}$, this recovers singular cohomology of the topological space $X(\mathbb{C})$.

  We want to emphasize that this is a wild idea a priori.

  Example. Say $X = \left\{X^3 + Y^3 + Z^3 = 0\right\}/\mathbb{C}$ projective curve (topologically, a torus). This has various symmetries, e.g. swap x, y, conjugate the coordinates, etc. These will act on $\mathrm{H}^*(X)$. However, if this is an algebraic cohomology theory, it doesn't matter that the automorphisms be continuous. $\mathrm{Aut}(\mathbb{C})$ is really big, but Weil's proposal suggests any (strange, discontinuous) automorphism of it will act on $\mathrm{H}^*(X)$.

  In any case, Weil's proposal was realized by Artin and Grothendieck (at least for finite coefficients) via the theory of étale cohomology.

  Given this, what can ask e.g. "what's the cohomology fo $\operatorname{spec}\mathbb{Z}$? Does it look like that of a 3-manifold?"

- Tate (1962) and Patou (1961), using the theory of Galois cohomology, showed (in modern language) that the étale cohomology of $\operatorname{spec}\mathbb{Z}$ (or of other number rings) has a duality relating $\mathrm{H}^i \leftrightarrow \mathrm{H}^{3-i}$ (Secretly, this was some reformulation of class field theory). This is reminiscent of Poincaré duality for 3-manifolds.

Let's try and get a sense of how different these two things are, e.g. by comparing how they're computed.

- How would you compute H^*(manifold)? Maybe you find some triangulation and then write down a complex using the vertices, edges, faces, etc. of this triangulation, and then compute cohomology of this complex. All in all, the process is some sort of glorified linear algebra.

- On the other hand, how do you compute $\mathrm{H}^*(\operatorname{spec}\mathbb{Z})$? We'll look at the simplest case $\mathrm{H}^*_{\text{ét}}(\operatorname{spec}\mathbb{Z}[1/2], \mathbb{Z}/2\mathbb{Z})$. One can show that

$$\mathrm{H}^1 \simeq \frac{\text{units in } \mathbb{Z}[1/2]}{\text{squares}} = (\pm 1, \pm 2) \simeq (\mathbb{Z}/2\mathbb{Z})^2 .$$

Furthermore,

$$\mathrm{H}^2 \simeq \left\{ \begin{matrix} \text{quaternion algebras} \\ \text{over } \mathbb{Z}\left[\frac{1}{2}\right] \end{matrix} \right\} \simeq \frac{\mathbb{Z}}{2\mathbb{Z}}$$

(there are two such things, $M_2(\mathbb{Z}[1/2])$ and the Hamilton quaternions $\mathbb{Z}[1/2][i,j]/(i^2 = -1, j^2 = -1, ij = -ji)$). This is all quite algebraic and of a very different flavor from the manifold case. Also, note that $\#\,\mathrm{H}^1 \neq \#\,\mathrm{H}^2$ so the two groups aren't quite dual to each other.

From the above, it should not be clear why/how these two sorts of things behave 'in the same way.'

Ultimately, we'd like an analogy that's more precise than the statement 'a number field is like a 3-manifold'. In this direction, let's compare duality for number rings and 3-manifolds. Here, a 'number ring' is either the S-integers in some number field, or the ring of functions on a smooth (open) curve over $\mathbb{F}_\ell$. In talking about duality for number rings, we'll restrict ourselves to $\mathbb{Z}[1/p]$ for simplicity.

- Consider $\mathbb{Z}[1/p]$. We'll write $\mathrm{H}^i(\mathbb{Z}[1/p], M) = \mathrm{H}^i_{\text{ét}}(\operatorname{spec}\mathbb{Z}[1/p], M)$ for étale cohomology where M is a p-torsion abelian group (e.g. $\mathbb{Z}/p^n\mathbb{Z}$) which may have a Galois action (required to be unramified outside p) if you want. We want to state *Tate duality*. There is a 9-term exact sequence which includes (ignoring Galois actions)

$$\begin{tikzcd} \text{degree } i-1 \text{ stuff} \arrow[d] & & \\ \mathrm{H}^{3-i}(\mathbb{Z}[i/p], M^\vee)^\vee \arrow[r] & \mathrm{H}^i(\mathbb{Z}[1/p], M) \arrow[r] & \mathrm{H}^i(\mathbb{Q}_p, M) \arrow[d] \\ & & \text{degree } i+1 \text{ stuff} \end{tikzcd}$$

Above,

$$M^\vee = \mathrm{Hom}(M, S^1) = \mathrm{Hom}(M, p\text{-power roots of } 1).$$

- Now look a 3-manifold X w/ boundary ∂X. Here, get a sequence (LES of a pair)

$$\begin{tikzcd} \mathrm{H}^i(X, \partial X; M) \arrow[r] \arrow[d, "\wr"] & \mathrm{H}^i(X, M) \arrow[r] & \mathrm{H}^i(\partial X, M) \arrow[r] & \text{degree } i+1 \text{ stuff} \\ \mathrm{H}^{3-i}(X; M^\vee)^\vee & & & \end{tikzcd}$$

Slogan. $\mathbb{Z}[1/p]$ is a like a (nonorientable) 3-manifold with boundary $\mathbb{Q}_p$, which is like a 2-manifold.

(The 'nonorientable' above has to do with needing to twist the Galois action in the first bullet point above in order to things to be technically correct)

Let's try and relate this to Mazur's picture of $\operatorname{spec}\mathbb{Z}/p\mathbb{Z} \hookrightarrow \operatorname{spec}\mathbb{Z}$ being a knot in a 3-manifold. Picture deleting a tube about $\operatorname{spec}\mathbb{Z}/p\mathbb{Z}$ in order to obtain $\operatorname{spec}\mathbb{Z}[1/p]$ as a 3-manifold, now with boundary, and the boundary of that tube being $\operatorname{spec}\mathbb{Q}_p$. In other words

Slogan. $\operatorname{spec}\mathbb{Z}$ is like $\operatorname{spec}\mathbb{Z}[1/p]$ glued along $\operatorname{spec}\mathbb{Q}_p$ to $\operatorname{spec}\mathbb{Z}_p$ (= solid tube)

(Above $\operatorname{spec}\mathbb{Q}_p$ is like the boundary of the tube as well as the boundary of $\operatorname{spec}\mathbb{Z}[1/p]$)
Let's describe a 'Garden of rings'

- "3-dimensional" rings/arithmetic object (those behaving like 3-manifolds)

  $\mathbb{Z}, \mathbb{Z}[1/p], \mathbb{Z}[\sqrt{2}], \mathbb{F}_q(t)$, projective smooth curves over $\mathbb{F}_p$, $\mathbb{Z}_p$, etc.

- "2-dimensional" objects

  $\mathbb{Q}_p, \mathbb{F}_p((t))$, a projective smooth curve over $\overline{\mathbb{F}}_p$

We've made the metaphor between number fields and 3-manifolds a little more precise.

Question 5.1.1. *What do automorphic forms become on the 3-manifold side?*

The topic of automorphic forms is very big, so let's restrict ourselves to one aspect of the theory. Fix some group G (e.g. $G = \mathrm{SL}_n$). Very broadly, studying automorphic forms is studying functions on $G_\mathbb{Z}\backslash G_\mathbb{R}$. Let $A_\mathbb{Z}$ be the vector space of functions on $G_\mathbb{Z}\backslash G_\mathbb{R}$. If we want to work over $\mathbb{Z}[1/p]$ instead, we'd study $A_{\mathbb{Z}[1/p]}$, functions on $G_{\mathbb{Z}[1/p]}\backslash(G_\mathbb{R} \times G_{\mathbb{Q}_p})$.

So at a basic level, to get an analogy, we'd like some associate $M \mapsto A_M$ from 3-manifolds to vector spaces which behaves similarly.

Non-example. The goto way of associating a vector space to a manifold is to use cohomology: $M \mapsto \mathrm{H}^*(M;\mathbb{C})$. This won't do here, it behaves noting like $\mathbb{Z} \rightsquigarrow A_\mathbb{Z}$.

- It has the wrong functoriality

  A double cover of 3-manfolds gives maps on cohomology in both directions. However, $\mathbb{Z} \to \mathbb{Z}[\sqrt{2}]$ doesn't induce natural maps on automorphic forms in either direction.

- $\mathbb{Z} \to \mathbb{Z}[1/p]$

  $A_\mathbb{Z}$ and $A_{\mathbb{Z}[1/p]}$ have a huge difference. However, if you cut a tube out of a manifold, their cohomologies don't have such a drastic difference.

- $\mathrm{H}^*(M \sqcup N) = \mathrm{H}^*(M) \oplus \mathrm{H}^*(N)$. On the other hand, $A_{\mathbb{Z}\oplus\mathbb{Z}} = A_\mathbb{Z} \otimes A_\mathbb{Z}$ (this isn't literally a part of Langlands e.g. since $\mathbb{Z} \oplus \mathbb{Z}$ is not a number ring, but this is the only sensible interpretation of the LHS)

The upshot is that cohomology won't work. However, this is a different sort of object attached to 3-manifolds which e.g. takes disjoint unions to tensor products. These are *topological quantum field theories* (TQFTs). The reasonableness of this proposal comes from a 2006 paper of Kapustin-Witten. Briefly, a TQFT (in dimension 4) is a functor (3-manifolds,bordisms) $\to$ Vect taking disjoint unions to tensor products.

Remark 5.1.2. Can read more about this e.g. is section 2 of Atiyah's paper on TQFTs.

5.2 Lecture 2

5.2.1 $TQFT_4$ (all in Atiyah section 2)

Looks like maybe a 4-dimensional TQFT is a functor from the (symmetric, monoidal?) category of (smooth, oriented) 3-manifolds and bordisms to that of vector spaces (and linear maps). In particular, it sends disjoint unions to tensor products.

This gives invariants of 4-manifolds. If Z is a 4-manifold with boundary M, you get a vector in A_M (linear map $\mathbb{C} \to M$ since Z a bordism from $\emptyset \to M$). A 4-manifold without boundary will give a complex number (i.e. a linear map $\mathbb{C} \to \mathbb{C}$). These invariants will be compatible with cut-and-paste operations.

Example. Imagine a 4-manifold Z with a 3-manifold M inside it. This separates Z into two pieces Z_1, Z_2 (each a 4-manifold w/ boundary M). Then, Z_1 gives rise to a vector in A_M and Z_2 gives rise to one in $A_M^\vee$ (keeping track of orientations). The number Z corresponds to is the pairing of these vectors.

Example (of a TQFT_2)**.** Assign to a circle S^1 the conjugacy-invariant functions in the group algebra $\mathbb{C}G$. A pair of pants then corresponds to multiplication int he group algebra. Finally, a genus g surface will give (#ways to write $e \in G$ as a product of g commutators)$/k!$.

(Look up Dijkgroof-Witten, up to spelling)

Extended TQFT_4, informally These will assign

- 4-manifold $\to \mathbb{C}$
- 3-manifold $\to$ vector spaces
- 2-manifold $\to$ linear categories

For example, if you have a 3-manifold M with boundary S (a surface), then this will give an object in the category A_S associated to S. If you glue two 3-manifolds M_1, M_2 along S (say with $M = M_1 \cup_S M_2$ the result), then A_M should be $\text{Hom}_{A_S}(A_{M_1}, A_{M_2})$

5.2.2 Correction from last time

Last time we said that $\operatorname{spec} \mathbb{Z}[1/p]$ is like a 3-manifold w/ boundary $\operatorname{spec} \mathbb{Q}_p$ (which is itself like a 2-manifold). We motivated this on cohomological grounds. On the other hand, we know that all the places of a global object should be placed on equal footing, and $\mathbb{Z}[1/p]$ is missing two places. Hence, we propose that the right analogy is

Slogan. $\operatorname{spec} \mathbb{Z}[1/p]$ has boundary $\operatorname{spec} \mathbb{R} \cup \operatorname{spec} \mathbb{Q}_p$, and $\operatorname{spec} \mathbb{Z}$ has boundary $\operatorname{spec} \mathbb{R}$

5.2.3 Automorphic forms as extended TQFT_4

Recall 5.2.1. In our analogy

- 3-dimensional objects include

  $\mathbb{Z}, \mathbb{Z}[1/p], X$ smooth projective curve/$\mathbb{F}_p$, $\mathbb{Z}_p$
- 2-dimensional objects include $\mathbb{Q}_p, \mathbb{R}, \overline{X}$ smooth projective curve over $\overline{\mathbb{F}}_p$

(Implicitly fix a group G (e.g. SL_n) so we can talk about automorphic forms)

To rings like $\mathbb{Z}$, we get a space of automorphic forms $A_{\mathbb{Z}}$ given by functions on $G_{\mathbb{Z}}\backslash G_{\mathbb{R}}$. Similarly, $A_{\mathbb{Z}[1/p]}$ is functions on $\mathbb{G}_{\mathbb{Z}[1/p]}\backslash(G_{\mathbb{Q}} \times G_{\mathbb{R}})$.

Warning 5.2.2. 3-manifolds w/ boundaries should be assigned something slightly different than just a vector space. More of that later.

For X smooth projective curve over $\mathbb{F}_p$, A_X gets assigned functions on the set of G-bundles on X. It's less clear what $A_{\mathbb{Z}_p}$ should be.

Let's look at the 2-dimensional objects now; they should be assigned categories. $\mathbb{Q}_p$ will go the to category of $G(\mathbb{Q}_p)$-representations. $\mathbb{R}$ will go to the category of $G(\mathbb{R})$-representations.

Sanity Check 5.2.3. A 3-manifold with boundary should be assigned an object in the category assigned to its boundary. Note that $A_{\mathbb{Z}} = \mathrm{Fun}(G_{\mathbb{Z}}\backslash G_{\mathbb{R}}, \mathbb{C})$ indeed comes with an action of $G(\mathbb{R})$, and so lives the category of $G(\mathbb{R})$-reps. Similarly, $A_{\mathbb{Z}[1/p]}$ comes with an action of $G_{\mathbb{Q}_p} \times G_{\mathbb{R}}$, so things look good there as well.

$\overline{X}$ will be assigned the category of sheaves on G-bundles on $\overline{X}$.

Recall 5.2.4. $\operatorname{spec}\mathbb{Z}$ is obtained by gluing $\operatorname{spec}\mathbb{Z}_p$ to $\operatorname{spec}\mathbb{Z}[1/p]$ along $\operatorname{spec}\mathbb{Q}_p$

Maybe I should actually try learning about the pro-étale topology?

Let's check that the appropriate gluing property holds for this situation (ignore $\operatorname{spec}\mathbb{R}$)

Example. We want that homomorphisms from the left side to the right recover the whole space, i.e.

$$\mathrm{Hom}_{G(\mathbb{Q}_p)}(A_{\mathbb{Z}_p}, A_{\mathbb{Z}[1/p]}) \stackrel{?}{=} A_{\mathbb{Z}}.$$

Note that $A_{\mathbb{Z}}$ = elements of $A_{\mathbb{Z}[1/p]}$ which are unramified at p (invariant under $G(\mathbb{Z}_p)$). By Frobenious reciprocity, this is

$$\mathrm{Hom}_{G(\mathbb{Q}_p)}\left(\begin{matrix}\text{(compactly support)}\\ \text{functions on } G(\mathbb{Q}_p)/G(\mathbb{Z}_p)\end{matrix}, A_{\mathbb{Z}[1/p]}\right).$$

This tells us what $A_{\mathbb{Z}_p}$ should be. Furthermore, this encodes the Hecke operator at p, as endomorphisms on $A_{\mathbb{Z}_p}$ will naturally act on $A_{\mathbb{Z}}$.

5.2.4 What is the Langlands correspondence?

In usual language, this is meant to be a bijection between certain automorphic representations and certain Galois representations. Phrased this way, it's not clear what this is on the TQFT side.

Assumption. Let's pretend we have some gadget $O \to A_O$ (from 'arithmetic rings' to 'vector spaces/-categories') as we imagined above, and call it an "arithmetic field theory."

Let X be a projective smooth curve over $\mathbb{F}_p$ and $G = \mathrm{GL}_n$. Langlands correspondence gives an isomorphism

$$\left\{\begin{matrix}\text{cuspidal functions on } n\text{-dimensional}\\ \text{vector bundles on } X\end{matrix}\right\} \xrightarrow{\sim} \left\{\begin{matrix}\text{functions on } n\text{-dimensional}\\ \text{irreducible Galois representations}\end{matrix}\right\}$$

(up to being careful about GL_n having a non-trivial center).

For each point $x \in X$, get Hecke operator T_x on the LHS and Frobenius F_x on RHS. Langlands correspondence is equivalent to saying the LHS and RHS vector spaces match, identifying T_x and F_x.

This suggests the following viewpoint on the Langlands correspondence: there's a *second* "arithmetic field theory," build out of Galois representations into $G^\vee$ (Langlands dual group) – which we'll call $B^{(G^\vee)}$ – and an "equivalence of arithmetic field theories"

$$A^{(G)} \simeq B^{(G^\vee)}$$

(this should capture local, global, and geometric Langlands all in the same setting, assuming I'm following the spirit correctly)

Much of the power of automorphic forms doesn't come just from having some bijection between automorphic forms and Galois representations, but from being able to match structures/invariants on these sides. There's some whole zoo of matching invariants

Example. Say $E/\mathbb{Q}$ an elliptic curve. Then,

$$\mathbb{Q} \cdot \pi \, \mathrm{area}(E_{\mathbb{C}}) \ni L(\mathrm{Sym}^2 E, 1) := \prod_p \frac{p^2}{\left(1 - \frac{1}{p}\right) \# E(\mathbb{F}_{p^2})}.$$

"Good luck proving this without automorphic forms."

Let's think about numerical invariants of automorphic forms and of Galois representations. Say $\mathscr{O}$ is a 3-dimensional ring of integers over X.

A numerical invariant of Galois representations is an element of $B_{\mathscr{O}}^{(G^\vee)}$ (this is defined to be (something like) functions on Galois representations). Numerical invariants of automorphic forms line in $A_{\mathscr{O}}^{(G)}$. Given some P there, get invariant $\varphi \mapsto \langle P, \varphi \rangle$ (imagining things in L^2).

To find matching invariants, we want to find matching elements of $A_{\mathscr{O}}$ and $B_{\mathscr{O}}$.

Definition 5.2.5. A *boundary condition* in a $TQFT_4$ is (informally) a consistent assignment to every 3-manifold M, some distinguished vector in A_M; and to each 2-manifold S, a distinguished object in the category A_S.

(See e.g. Kapuskin's 2010 ICM address)

We'd like "matching boundary conditions" in $A^{(G)}$ and $B^{(G^\vee)}$.

5.2.5 One word on joint work between Venkatesh and Ben-Zvi, Sakellarids (spelling?)

In this work, they try to develop this point of view. Informally, they say

(1) A G-variety Y gives boundary condition for $A^{(G)}$ and $B^{(G)}$

(2) For suitable choice of Y, this recovers all the familiar invariants of automorphic forms. On the Galois side, it recovers L-functions.

(3) They propose a specific class of dual pairs $(G, Y) \leftrightarrow (G^\vee, Y^\vee)$ which (conjecturally) give matching boundary conditions

6 List of Marginal Comments

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