MACHINE LEARNING - EXERCISE 3 - PART 1

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Question 1

We define the following events:

A: We randomly got a Goldstar from the bar.

B: The box that was moved from the storage to the bar was a Goldstar box.

We need to calculate the probability that a Stella box was moved, given the fact we randomly got a Goldstar at the bar. If a box of Stella was moved to the bar, we got 6 more bottles of Stella at the bar. Thus the probability to randomly get a Goldstar changes:

$$P(A) = \frac{15}{35+6} = \frac{15}{41}$$

To calculate $P(A)$ we will use the law of total probability:

$$P(A) = P(B)P(A|B) + P(\overline{B})P(A|\overline{B}) = \frac{4}{11} \cdot \frac{21}{41} + \frac{7}{11} \cdot \frac{15}{41} = 0.419$$

We need to calculate: $P(\overline{B}|A)$. Thus:

$$P(\overline{B}|A) = \frac{P(\overline{B} \cap A)}{P(A)} = \frac{\frac{7}{11} \cdot \frac{15}{41}}{0.419} = \frac{5}{9}$$

Question 2

(a) Define **A**: the probability that a ship will be detected.

Using the law of total probability we get:

$$P(A) = 0 \cdot 0.8 + 0.2 \cdot 0.7 + 0.3 \cdot 0.6 + 0.5 \cdot 0.5 = 0.57$$

(b) Define **B**: the probability of a ship to be in zone C. Thus: $P(B) = 0.3$.

Using conditional probability we get:

$$P(B|A) = \frac{P(B \cap A)}{P(A)} = \frac{0.3 \cdot 0.6}{0.57} = 0.3157$$

(c) Define **C**: the probability of a ship to be in zone B. Thus: $P(C) = 0.2$.

Using conditional probability we get:

$$P(C|A) = \frac{P(C \cap A)}{P(A)} = \frac{0.2 \cdot 0.7}{0.57} = 0.2456$$

Question 3

We can look at the following situation:

We have 10 balls in a sack: 7 red balls, and 3 yellow balls. We toss a fair coin once:

- If the result is H, we choose a random ball, return it to the sack, and then choose another random ball.
- If the result is T, we choose a random ball, and then choose another random ball (without returning the first one).

Define the following indicator random variables:

$$X = \begin{cases} 1 & \text{red ball on the first pull} \\ 0 & \text{else} \end{cases} \qquad Y = \begin{cases} 1 & \text{red ball on the second pull} \\ 0 & \text{else} \end{cases}$$

$$C = \begin{cases} 1 & \text{the coin showed H} \\ 0 & \text{else} \end{cases}$$

Therefore, the conditions hold:

(a) $X \perp Y | C$:

$$0.49 = \frac{0.7 \cdot 0.7 \cdot 0.5}{0.5} = \frac{P(X = 1, Y = 1, C = 1)}{P(C = 1)} = P(X = 1, Y = 1 | C = 1)$$

$$= P(X = 1 | C = 1) \cdot P(Y = 1 | C = 1)$$

$$= \frac{P(X = 1, C = 1)}{P(C = 1)} \cdot \frac{P(Y = 1, C = 1)}{P(C = 1)} = \frac{0.7 \cdot 0.5}{0.5} \cdot \frac{0.7 \cdot 0.5}{0.5} = 0.49$$

(b) $X \not\perp Y$:

$$\frac{6}{9} = \frac{0.7 \cdot \frac{6}{9}}{(0.7 \cdot 0.7 + 0.3 \cdot 0.7)} = \frac{P(X = 1, Y = 1)}{P(Y = 1)} = P(X = 1 | Y = 1) \neq P(X = 1) = 0.7$$

(c) X, Y, C are binary random variables - since they are indicators

(d) $X \sim B(1, 0.7) \Rightarrow P(X=0) = 1 - 0.7 = 0.3$

$Y \sim B(1, 0.7) \Rightarrow P(Y=0) = 1 - 0.7 = 0.3$

$C \sim B(1, 0.5) \Rightarrow P(X=0) = 0.5$

Calculating the 8 combinations for X, Y, C values we get:

$$P(X=0, Y=0, C=0) = 0.5 \cdot 0.3 \cdot \frac{2}{9} = \frac{1}{30}$$

$$P(X=0, Y=0, C=1) = 0.5 \cdot 0.3 \cdot 0.3 = \frac{9}{200}$$

$$P(X=0, Y=1, C=0) = 0.5 \cdot 0.3 \cdot \frac{7}{9} = \frac{7}{60}$$

$$P(X=0, Y=1, C=1) = 0.5 \cdot 0.3 \cdot 0.7 = \frac{21}{200}$$

$$P(X=1, Y=0, C=0) = 0.5 \cdot 0.7 \cdot \frac{3}{9} = \frac{7}{60}$$

$$P(X=1, Y=0, C=1) = 0.5 \cdot 0.7 \cdot 0.3 = \frac{21}{200}$$

$$P(X=1, Y=1, C=0) = 0.5 \cdot 0.7 \cdot \frac{6}{9} = \frac{7}{30}$$

$$P(X=1, Y=1, C=1) = 0.5 \cdot 0.7 \cdot 0.7 = \frac{49}{200}$$

Question 4

(a) Define a random variable **X**: the number of descent meals at Karnaf during a specific week.

We will treat a week as a series of 5 independent experiences, with $p = 0.7$ to get a descent meal at Karnaf.

Thus:

$$X \sim B(n = 5, p = 0.7)$$

From the binomal distribution we get:

$$P(X = 3) = \binom{5}{3} \cdot 0.7^3 \cdot 0.3^2 = 0.3087$$

(b) Using the binomial distribution again, we get:

$$\begin{aligned} &P(X \geq 2) = \\ &=1 - [P(X = 0) + P(X = 1)] \\ &=1 - [\binom{5}{0} \cdot 0.7^0 \cdot 0.3^5 + \binom{5}{1} \cdot 0.7^1 \cdot 0.3^4] \\ &=1 - [0.00243 + 0.02835] \\ &=0.96922 \end{aligned}$$

(c) Since there are $N \geq 30$ independet samples, and the conditions for the CLE theorem for the binomial distribution holds, we expect the average to hold:

$$\overline{X_n} = \frac{\sum_{i=1}^{100} x_i}{100} \sim B(np, np(1-p))$$

$$\Rightarrow \overline{X_n} \sim B(70, \sqrt{21})$$

Meaning we expect the average (the mean value) to be $\mu = 70$

Question 5

(a) $\forall (x,y) \in D \cap C$, it holds:

1. $x \geq 0$
2. $|x^2 + y^2| \leq 1$

Thus, we can calculate:

$$
\begin{aligned}
&P((x,y) \in D \cap C | (x,y) \in C) \\
&\quad = \frac{P[((x,y) \in D \cap C) \cap ((x,y) \in C)]}{P((x,y) \in C)} \\
&\quad = \frac{P(x \geq 0) \cdot P(|x^2 + y^2 \leq 1|) \cdot P(x \geq 0)}{P(x \geq 0)} \\
&\quad = P(x \geq 0) \cdot P(|x^2 + y^2 \leq 1|) \\
&\quad = \frac{1}{2} \cdot \pi = \frac{\pi}{2}
\end{aligned}
$$

Thus, we can treat the question as a series of independent experiences, with $p = \frac{\pi}{2}$ chance to success.

Meaning:

$$X \sim B(50, \frac{\pi}{2})$$

(b) The CDF of X from 1 to 50: