$$= \exp\left(\mu t + \frac{(\sigma t)^2}{2}\right) \underbrace{\int_{-\infty}^{\infty} \frac{1}{\sqrt{2\pi\sigma^2}} \exp\left(-\frac{(x-(\mu+\sigma^2 t))^2}{2\sigma^2}\right) dx}_{①}$$

① $= f(x; \mu, \sigma^2)$ であるから

$$\int_{-\infty}^{\infty} \frac{1}{\sqrt{2\pi\sigma^2}} \exp\left(-\frac{(x-(\mu+\sigma^2 t))^2}{2\sigma^2}\right) dx = 1$$

よって

$$\underline{M_X(t) = \exp\left(\mu t + \frac{(\sigma t)^2}{2}\right)}_{//}$$

5
$$\begin{cases} E[X] = M_X'(0) \\ M_X'(t) = (\mu + \sigma^2 t)\exp\left(\mu t + \frac{(\sigma t)^2}{2}\right) \end{cases}$$

上記2式より

$$\begin{aligned} E[X] = M_X'(0) &= (\mu + 0)\exp(0+0) \\ &= \mu \end{aligned}$$

$$\begin{cases} E[X^2] = M_X''(0) \\ M_X''(t) = \sigma^2 \exp\left(\mu t + \frac{(\sigma t)^2}{2}\right) + (\mu+\sigma^2 t)^2 \exp\left(\mu t + \frac{(\sigma t)^2}{2}\right) \end{cases}$$

上記2式より

$$\begin{aligned} E[X^2] = M_X''(0) &= \sigma^2 \exp(0+0) + (\mu+0)^2 \exp(0+0) \\ &= \sigma^2 + \mu^2 \end{aligned}$$