Ordinary Differential Equations

Practice Problems 07

Solutions.

1. $y'' - xy' + 2y = 0 \quad \Rightarrow \quad P(x) = -x$ and $Q(x) = 2$.

This DE does not have a singular point between $-\infty$ and $+\infty$. Thus, $x=0$ is an ordinary point. Assuming that the solution is of the form

$$y = \sum_{n=0}^{\infty} a_n x^n$$

$$\Rightarrow \quad y' = \sum_{n=0}^{\infty} n a_n x^{n-1}$$

$$y'' = \sum_{n=0}^{\infty} n(n-1) a_n x^{n-2}$$

$$\Rightarrow \sum_{n=0}^{\infty} n(n-1) a_n x^{n-2} - x\left(\sum_{n=0}^{\infty} n a_n x^{n-1}\right) + 2\left(\sum_{n=0}^{\infty} a_n x^n\right) = 0$$

$$\underbrace{\sum_{n=2}^{\infty} n(n-1) a_n x^{n-2}}_{k=n-2} - \sum_{n=1}^{\infty} n a_n x^n + 2\underbrace{\sum_{n=0}^{\infty} a_n x^n}_{k=n} = 0$$

$$\sum_{k=0}^{\infty} (k+2)(k+1) a_{k+2} x^{k+2-2} - \sum_{n=1}^{\infty} n a_n x^n + 2\sum_{k=0}^{\infty} a_k x^k = 0$$

$$\sum_{k=0}^{\infty} \left[(k+2)(k+1) a_{k+2} + 2a_k\right] x^k - \underbrace{\sum_{n=1}^{\infty} n a_n x^n}_{k=n-1} = 0$$

$$\sum_{k=0}^{\infty} \left[(k+2)(k+1) a_{k+2} + 2a_k\right] x^k - \sum_{k=0} (k+1) a_{k+1} x^{k+1} = 0$$ (Note that the 2nd term is ahead of the first term by $k=1$.)

$\Rightarrow k=0 \quad 2a_2 + 2a_0 = 0$ coefficients of x^0

$k=1 \quad (3)(2)a_3 + 2a_1 - a_1 = 0$ coefficients of x

$k=2 \quad (4)(3)a_4 + 2a_2 - 2a_2 = 0$ coefficients of x^2

$k=3 \quad (5)(4)a_5 + 2a_3 - 3a_3 = 0$ coefficients of x^3

$\vdots \qquad \vdots$

$$(k+2)(k+1)a_{k+2} - (k-2)a_k = 0$$

$$a_{k+2} = \frac{k-2}{(k+2)(k+1)} a_k$$

2. $2x^2y'' + 7x(x+1)y' - 3y = 0$

Dividing both sides by $2x^2$:

$$y'' + \frac{7x}{2x^2}(x+1)y' - \frac{3}{2x^2}y = 0$$

$$\Rightarrow P(x) = \frac{7(x+1)}{2x} \quad \& \quad Q(x) = \frac{-3}{2x^2}$$

Here $P(x)$ and $Q(x)$ are undefined when $x=0$. Thus, $x=0$ is not an ordinary point. It's a singular point.

3. $y'' + (t-1)y' + (2t-3)y = 0 \quad \Rightarrow \quad P(t) = t-1 \;\; \& \;\; Q(t) = 2t-3$ have values at $t=0$.

Assume that the solution is of the form

$$y = \sum_{n=0}^{\infty} a_n t^n$$

$$\Rightarrow y' = \sum_{n=0}^{\infty} n a_n t^{n-1} = \sum_{n=1}^{\infty} n a_n t^{n-1}$$

$$y'' = \sum_{n=0}^{\infty} (n)(n-1) a_n t^{n-2} = \sum_{n=2}^{\infty} (n)(n-1) a_n t^{n-2}$$

$$\Rightarrow \underbrace{\sum_{n=2}^{\infty} (n)(n-1) a_n t^{n-2}}_{K=n-2} + \underbrace{\sum_{n=1}^{\infty} n a_n t^{n}}_{K=n} - \underbrace{\sum_{n=1}^{\infty} n a_n t^{n-1}}_{K=n-1} + 2\underbrace{\sum_{n=0}^{\infty} a_n t^{n+1}}_{K=n+1} - 3\underbrace{\sum_{n=0}^{\infty} a_n t^{n}}_{K=n} = 0$$

1st term, K starts from 0

2nd ", K starts from 1

3rd ", K starts from 0

4th ", K starts from 1

5th ", K starts from 0

$$\Rightarrow \sum_{K=0}^{\infty} \left[(K+2)(K+1)a_{K+2} - (K+1)a_{K+1} - 3a_K\right] t^K + \sum_{K=1}^{\infty} K a_K t^K + 2\sum_{K=1}^{\infty} a_{K-1} t^K = 0$$

$$(K+2)(K+1)a_{K+2} - (K+1)a_{K+1} - 3a_K + Ka_K + 2a_{K-1} = 0$$

$$a_{K+2} = \frac{K+1}{(K+2)(K+1)} a_{K+1} \frac{-(K-3)a_K}{(K+2)(K+1)} - \frac{2}{(K+2)(K+1)} a_{K-1}$$

4. From the recurrence relation from #3

$a_2 = \frac{1}{2}a_1 + \frac{3}{2}a_0$ (Note that a_{-1} does not exist since K starts from 0 or 1.)

$$a_3 = \frac{1}{3}a_2 + \frac{1}{3}a_1 - \frac{1}{3}a_0 = \frac{1}{3}\left(\frac{1}{2}a_1 + \frac{3}{2}a_0\right) + \frac{1}{3}a_1 - \frac{1}{3}a_0 = \frac{1}{2}a_1 + \frac{1}{6}a_0$$

$$a_4 = \frac{1}{4}a_3 + \frac{1}{12}a_2 - \frac{1}{6}a_1 = \frac{1}{4}\left(\frac{1}{2}a_1 + \frac{1}{6}a_0\right) + \frac{1}{12}\left(\frac{1}{2}a_1 + \frac{3}{2}a_0\right) - \frac{1}{6}a_1 = \frac{1}{6}a_0$$

Thus,

$$y(t) = a_0\left(1+\frac{3}{2}t^2+\frac{1}{6}t^3+\frac{1}{6}t^4+\cdots\right) + a_1\left(t+\frac{1}{2}t^2+\frac{1}{2}t^3+0t^4+\cdots\right)$$

5. $(x^4+4)y'' + xy = x+2$

$$y'' + \frac{x}{x^4+4}y = \frac{x+2}{(x^4+4)}$$

at $x=0$, no singularity exists. Thus, we can use the power series solution.

Assume

$$y = \sum_{n=0}^{\infty} a_n x^n \Rightarrow y' = \sum_{n=0}^{\infty} n a_n x^{n-1} = \sum_{n=1}^{\infty} n a_n x^{n-1}$$

$$y'' = \sum_{n=1}^{\infty} n(n-1)a_n x^{n-2} = \sum_{n=2}^{\infty} n(n-1)a_n x^{n-2}$$

$$(x^4+4)\sum_{n=2}^{\infty} n(n-1)a_n x^{n-2} + x\sum_{n=0}^{\infty} a_n x^n = x+2$$

$$\sum_{n=2}^{\infty} n(n-1)a_n x^{n+2} + 4\sum_{n=2}^{\infty} n(n-1)a_n x^{n-2} + \sum_{n=0}^{\infty} a_n x^{n+1} = x+2 \qquad (*)$$

$$\left[2a_2x^4 + 6a_3x^5 + 12a_4x^6 + \cdots + n(n-1)a_nx^{n-2} + \cdots\right] + \left[8a_2 + 24a_3x + 48a_4x^2 + \cdots + 4n(n-1)a_nx^{n-2} + \cdots\right] +$$
$$a_0x + a_1x^2 + a_2x^3 + \cdots + a_nx^{n+1} + \cdots] = x+2$$

Combining coefficients of the same x^n:

$$8a_2 + (24a_3+a_0)x + (2a_2+48a_4+a_1)x^2 + (6a_3+80a_5+a_2)x^3 + \cdots = 2+x$$

$$\Rightarrow \quad 8a_2 = 2 \qquad 24a_3 + a_0 = 1 \qquad 2a_2 + 48a_4 + a_1 = 0 \qquad 6a_3 + 80a_5 + a_2 = 0 \ldots$$

$$a_2 = \frac{1}{4} \qquad a_3 = \frac{1}{24} - \frac{1}{24}a_0$$

In general, you do shifting on $(*)$ as long as $n\neq 0$ and $n\neq 1$:

$$k(k-1)a_k + 4(k+2)(k+1)a_{k+2} + a_{n-1} = 0 \qquad (n=2,3,4,\cdots)$$

$$a_{k+2} = -\frac{k(k-1)}{4(k+2)(k+1)}a_k - \frac{1}{4(k+2)(k+1)}$$

Evaluating the coefficients:

$$a_4 = -\frac{1}{24}a_2 - \frac{1}{48}a_1 = -\frac{1}{24}\left(\frac{1}{4}\right) - \frac{1}{48}a_1 = \frac{-1}{96} - \frac{1}{48}a_1$$

$$a_5 = -\frac{3}{40}a_3 - \frac{1}{80}a_2 = -\frac{1}{40}\left(\frac{1}{24} - \frac{1}{24}a_0\right) - \frac{1}{80}\left(\frac{1}{4}\right) = \frac{-1}{160} + \frac{1}{320}a_0$$

$$\vdots \qquad\qquad \vdots$$

$$\Rightarrow y = a_0 + a_1x + \frac{1}{4}x^2 + \left(\frac{1}{24} - \frac{1}{24}a_0\right)x^3 + \left(-\frac{1}{96} - \frac{1}{48}a_1\right)x^4 + \left(\frac{-1}{160} + \frac{1}{320}a_0\right)x^5 + \cdots$$

Combing "coeffients" of a_0 and a_1 :

$$y = \left(1 - \frac{1}{24}x^3 + \frac{1}{320}x^5 + \cdots\right)a_0 + \left(x - \frac{1}{48}x^4 + \cdots\right)a_1 + \left(\frac{1}{4}x^2 + \frac{1}{24}x^3 - \frac{1}{96}x^4 - \frac{1}{160}x^5 + \cdots\right)$$

Note that the 3rd term does not contain a constant. Thus, this must be the particular solution to the original DE.

6. $\frac{d^2y}{dt^2} + ty = e^{t+1}$ $\Rightarrow P(t) = 0$ and $Q(t) = t$ } analytic at $t=0$. Thus, we can use the power series method.

Assume that $y = \sum_{n=0}^{\infty} a_n t^n$ $\Rightarrow$ $y' = \sum_{n=0}^{\infty} n a_n t^{n-1} = \sum_{n=1}^{\infty} n a_n t^{n-1}$

$$y'' = \sum_{n=1}^{\infty} n(n-1) a_n t^{n-2} = \sum_{n=2}^{\infty} n(n-1) a_n t^{n-2}$$

$$\sum_{n=2}^{\infty} n(n-1) a_n t^{n-2} + t \sum_{n=0}^{\infty} a_n t^n = e^{t+1} = e\left(e^t\right)$$

$$\sum_{n=2}^{\infty} n(n-1) a_n t^{n-2} + \sum_{n=0}^{\infty} a_n t^{n+1} = e\left(\sum_{n=0}^{\infty} \frac{t^n}{n!}\right)$$

Note that $e^t = \sum_{n=0}^{\infty} \frac{t^n}{n!}$

Shifting coefficents and equating coefficients of the same t^n :

$$(k+2)(k+1)a_{k+2} + a_{k-1} = \frac{e}{k!}$$

$$a_{k+2} = \frac{-1}{(k+2)(k+1)} a_{k-1} + \frac{e}{(k+2)(k+1)k!} \quad \text{for } k = 1, 2, 3, \ldots$$

When $k=0$:

$$(0+2)(0+1)a_2 = \frac{e}{0!} \Rightarrow 2a_2 = e$$

When $k=1$:

$$6a_3 + a_0 = \frac{e}{1!}$$

When $k=2$:

$$12a_4 + a_1 = \frac{e}{2!}$$

When $k=3$

$$(5)(4)a_5 + a_2 = \frac{e}{3!}$$

and so on ...

$\Rightarrow a_2 = \frac{e}{2}$

$$a_3 = \frac{-1}{6}a_0 + \frac{e}{6}$$

$$a_4 = \frac{-1}{12}a_1 + \frac{e}{24}$$

$$a_5 = \frac{-1}{20}a_2 + \frac{e}{120} = \frac{-1}{20}\left(\frac{e}{2}\right) + \frac{e}{120} = \frac{-e}{120}$$

$\vdots$

$$\Rightarrow y = a_0 + a_1 t + \frac{e}{2}t^2 + \left(-\frac{1}{6}a_0 + \frac{e}{6}\right)t^3 + \left(-\frac{1}{12}a_1 + \frac{e}{24}\right)t^4 + \left(-\frac{e}{60}\right)t^5 + \cdots$$

$$= a_0\left(1 - \frac{1}{6}t^3 + \cdots\right) + a_1\left(t - \frac{1}{12}t^4 + \cdots\right) + e\left(\frac{1}{2}t^2 + \frac{1}{6}t^3 + \frac{1}{24}t^4 - \frac{1}{60}t^5 + \cdots\right)$$

Note that the 3rd term does not contain a constant a_n. Thus, this must be the particular solution to the DE.

7. $y'' - (x-2)y' + 2y = 0$

We can simplify the DE if we let $t = x-2$.

$$\Rightarrow \frac{dy}{dx} = \frac{dy}{dt}\frac{dt}{dx} = \frac{dy}{dt}(1)$$

$$\frac{d^2y}{dx^2} = \frac{d}{dx}\left(\frac{dy}{dx}\right) = \frac{d}{dx}\left(\frac{dy}{dt}\right) = \frac{d}{dt}\left(\frac{dy}{dx}\right) = \frac{d}{dt}\left(\frac{dy}{dt}\frac{dt}{dx}\right) = \frac{d}{dt}\left(\frac{dy}{dt}(1)\right) = \frac{d^2y}{dt^2}$$

Thus, the DE becomes

$$\frac{d^2y}{dt^2} - t\frac{dy}{dt} + 2y = 0$$

When $x=2$, $t=0$. We can solve the DE at $t=0$ since $t=0$ is an ordinary point.

Assume $y = \sum_{n=0}^{\infty} a_n t^n \quad \Rightarrow \quad y' = \sum_{n=0}^{\infty} n a_n t^{n-1} = \sum_{n=1}^{\infty} n a_n t^{n-1}$

$$y'' = \sum_{n=1}^{\infty} n(n-1) a_n t^{n-2} = \sum_{n=2}^{\infty} n(n-1) a_n t^{n-2}$$

$$\Rightarrow \sum_{n=2}^{\infty} n(n-1)a_n t^{n-2} - t\sum_{n=1}^{\infty} n a_n t^{n-1} + 2\sum_{n=0}^{\infty} a_n t^n = 0$$

$$\sum_{n=2}^{\infty} n(n-1)a_n t^{n-2} - \sum_{n=1}^{\infty} n a_n t^n + 2\sum_{n=0}^{\infty} a_n t^n = 0$$

$$\sum_{n=2}^{\infty} n(n-1)a_n t^{n-2} + \sum_{n=0}^{\infty} (2-n)a_n t^n = 0$$

Shifting indices:

$$\sum_{k=0}^{\infty} (k+2)(k+1)a_{k+2}t^k + \sum_{k=0}^{\infty}(2-k)a_k t^k = 0$$

$$\Rightarrow \quad a_{k+2} = \frac{k-2}{(k+2)(k+1)}a_k$$

When $k=0$

$$a_2 = \frac{-2}{2}a_0 = -a_0$$

When $k=1$

$$a_3 = \frac{-1}{3(2)}a_1 = \frac{-1}{6}a_1$$

When $k=2$

$$a_4 = 0$$

When $k=3$

$$a_5 = \frac{1}{20}a_3 = \frac{-1}{120}a_1$$

and so on...

$$y = a_0 + a_1 t - a_0 t^2 - \frac{1}{6} a_1 t^3 + 0t^4 - \frac{1}{120} a_1 t^5 + \cdots$$

$$= a_0(1-t^2) + a_1\left(t - \frac{1}{6}t^3 - \frac{1}{120}t^5 + \cdots\right)$$

$$= a_0\left[1-(x-2)^2\right] + a_1\left[(x-2) - \frac{1}{6}(x-2)^3 - \frac{1}{120}(x-5)^5 + \cdots\right]$$

8. $y'' + xy' + (2x-1)y = 0$

To simplify the algebra, let $t = x+1$. This implies that

$$\frac{dy}{dx} = \frac{dy}{dt}$$

and $$\frac{d}{dx}\left(\frac{dy}{dx}\right) = \frac{d}{dx}\left(\frac{dy}{dt}\right) = \frac{d}{dt}\left(\frac{dy}{dx}\right) = \frac{d}{dt}\left(\frac{dy}{dt}\right) = \frac{d^2y}{dt^2}$$

$$\Rightarrow \frac{d^2y}{dt^2} + (t-1) + (2t-3)y = 0$$

let $y = \sum_{n=0}^{\infty} a_n t^n \Rightarrow y' = \sum_{n=0}^{\infty} n a_n t^{n-1} = \sum_{n=1}^{\infty} n a_n t^{n-1}$

$$y'' = \sum_{n=1}^{\infty} n(n-1) a_n t^{n-2} = \sum_{n=2}^{\infty} n(n-1) a_n t^{n-2}$$

$$\sum_{n=2}^{\infty} n(n-1) a_n t^{n-2} + (t-1)\sum_{n=1}^{\infty} n a_n t^{n-1} + (2t-3)\sum_{n=0}^{\infty} a_n t^n = 0$$

$$\sum_{n=2}^{\infty} n(n-1) a_n t^{n-2} + \sum_{n=1}^{\infty} n a_n t^n - \sum_{n=1}^{\infty} n a_n t^{n-1} + 2\sum_{n=0}^{\infty} a_n t^{n+1} - 3\sum_{n=0}^{\infty} a_n t^n = 0$$

Shifting:

$$\sum_{k=0}^{\infty} (k+2)(k+1) a_{k+2} t^k + \sum_{k=1}^{\infty} k a_k t^k - \sum_{k=0}^{\infty} (k+1) a_{k+1} t^k + 2\sum_{k=1}^{\infty} a_{k-1} t^k - 3\sum_{k=0}^{\infty} a_k t^k = 0$$

$$\Rightarrow a_{k+2} = \frac{k+1}{(k+2)(k+1)} a_{k+1} - \frac{(k-3)}{(k+2)(k+1)} a_k - \frac{2}{(k+2)(k+1)} a_{k-1} \qquad \text{for } k > 1$$

If $k=0$, we dont consider a_{-1}. Thus

$$a_2 = \frac{1}{2} a_1 - \frac{(-3)}{2} a_0 = \frac{1}{2} a_1 + \frac{3}{2} a_0$$

$$\Rightarrow a_3 = \frac{1}{2} a_1 + \frac{1}{6} a_0$$

$$a_5 = \frac{4}{5(4)} a_4 - \frac{2}{5(4)} a_2 = \frac{1}{30} a_0 - \frac{1}{10}\left(\frac{1}{2}a_1 + \frac{3a_0}{2}\right) = \frac{-7}{60} a_0 - \frac{1}{20} a_1$$

$$a_4 = \frac{1}{6} a_0$$

Thus,

$$y = a_0 + a_1 t + \left(\frac{1}{2}a_1 + \frac{3}{2}a_0\right)t^2 + \left(\frac{1}{2}a_1 + \frac{1}{6}a_0\right)t^3 + \left(\frac{1}{6}a_0\right)t^4 + \cdots$$

$$= a_0\left(1 - \frac{3}{2}t^2 + \frac{1}{6}t^3 + \frac{1}{6}t^4 + \cdots\right) + a_1\left(t + \frac{1}{2}t^2 + \frac{1}{2}t^3 + \cdots\right)$$

$$= a_0\left[1 + \frac{3}{2}(x+1)^2 + \frac{1}{6}(x+1)^3 + \frac{1}{6}(x+1)^4 + \cdots\right] + a_1\left[(x+1) + \frac{1}{2}(x+1)^2 + \frac{1}{2}(x+1)^3 + \cdots\right]$$

9. If $y(-1) = 2$ and $y'(-1) = -2$, then

$$2 = a_0\left[1 + 0 + \cdots\right] + a_1\left[0\right] \qquad \Rightarrow a_0 = 2$$

$$y' = a_0\left[\frac{3}{2}(2)(x+1) + \frac{3}{6}(x+1)^2 + \frac{4}{6}(x+1)^4 + \cdots\right] + a_1\left[1 + \frac{2}{2}(x+1) + \frac{3}{2}(x+1)^2 + \cdots\right]$$

$$-2 = a_0\left(0 + 0 + \cdots\right] + a_1\left[(1) + 0 + 0 \cdots\right]$$

$\Rightarrow a_1 = -2$

Thus

$$y = 2\left[1 + \frac{3}{2}(x+1)^2 + \frac{1}{6}(x+1)^3 + \frac{1}{6}(x+1)^4 + \cdots\right] - 2\left[(x+1) + \frac{1}{2}(x+1)^2 + \frac{1}{2}(x+1)^3 + \cdots\right]$$

10. Taylor series expansion : $y = \sum_{n=0}^{\infty} \frac{y^{(n)}(x_0)}{n!}(x - x_0)^n$

$$\Rightarrow y = \frac{y(x_0)}{0!} + \frac{y'(x_0)}{1!}(x - x_0) + \frac{y''(x_0)}{2!}(x - x_0)^2 + \cdots$$

Here $x_0 = -1$ and $y(x_0) = y(-1) = -2$. Also $y'(x_0) = y'(-1) = -2$.

The y here is the y in problem 9. Differentiating y:

$$y' = 2\left[3(x+1) + \frac{3}{6}(x+1)^2 + \frac{4}{6}(x+1)^3 + \cdots\right] - 2\left[1 + (x+1) + \frac{3}{2}(x+1)^2 + \cdots\right]$$

$$y'' = 2\left[3 + (x+1) + \frac{12}{6}(x+1)^2 + \cdots\right] - 2\left[1 + 3(x+1) + \cdots\right]$$

$$y''(-1) = 2(3) - 2(1) = 4$$

$$y''' = 2\left[1 + \frac{24}{6}(x+1) + \cdots\right] - 2\left[3 + \cdots\right]$$

$$y'''(-1) = 2 - 6 = -4$$

$$y'''' = 2\left[\frac{24}{6} + \cdots\right] - 2\left[\cdots\right] =$$

$$y''''(-1) = 8$$

and so on...

Putting the derivatives at $x=-1$, we get

$$y = 2 + \frac{-2}{1!}(x+1) + \frac{4}{2!}(x+1)^2 + \frac{-4}{3!}(x+1)^3 + \frac{8}{4!}(x+1)^4 + \cdots$$

$$y = 2 - 2(x+1) + 2(x+1)^2 - \frac{2}{3}(x+1)^3 + \frac{1}{3}(x+1)^4 + \cdots$$