Let $U_1, U_2, \ldots U_n$ be n iid uniformly distributed random variables on the interval $(0, t)$

Let the set $S = \{U_{(1)}, U_{(2)}, U_{(3)} \ldots U_{(n)}\}$ be the sorted order of random variables. $U_{(i)}$ will be the ith order statistic of $U_1, U_2 \ldots U_n$.

let $g(s_1, s_2, s_3 \ldots s_n)$ be the joint density function of $\{U_1, U_2 \ldots U_n\}$

$$\therefore g(s_1, s_2, s_3 \ldots s_n) = \frac{1}{t^n}, \quad s_i \in (0, t)$$

[each U_i is uniformly distributed & are indp.]

Now given any increasing sequence $0 < s_1 < s_2 \ldots < s_n < t$
~~so~~ So joint distribution of order statistics will be given by. $(U_{(1)}, U_{(2)}, U_{(3)}, \ldots U_{(n)})$

$$f(s_1, s_2, s_3 \ldots s_n) = \frac{n!}{t^n} \quad \text{—— (1)}$$

[its because there are $n!$ permutations of the sequence $(s_1, s_2 \ldots s_n)$ & All of them have same order statistic]

Now we need to show that the joint distribution of the n arrival times is same as the joint distribution of $U_{(1)}, \ldots U_{(n)}$ i.e., the order statistic of n Uniformly distributed r.u.s ; ~~So~~

So ~~wee~~ we need to find;

$$\mathbb{P} = P(t_1 = s_1, t_2 = s_2, \ldots t_n = s_n \mid N(t) = n) = \frac{P(t_1 = s_1, \ldots t_n = s_n, N(t) = n)}{P(N(t) = n)}$$

In terms of inter arrival times.

$$\cancel{\mathbb{P} = P(X_1 = s_1, X_2 = s_2 - s_1, X_3 = s_3 - s_2 \ldots X_n = s_n - s_{n-1},}$$

$$\mathbb{P} = \frac{P(X_1 = s_1, X_2 = s_2 - s_1, X_3 = s_3 - s_2 \ldots, X_n = s_n - s_{n-1}, X_{n+1} > t - s_n)}{P(N(t) = n)}$$

$$\mathbb{P} = \frac{\lambda e^{-\lambda s_1} \cdot \lambda e^{-\lambda (s_2 - s_1)} \cdots \lambda e^{-\lambda(s_n - s_{n-1})} \cdot \left(1 - (1 - e^{-\lambda(t - s_n)})\right)}{P(N(t) = n)}$$

$$\mathbb{P}_2 = \frac{\lambda^n e^{s_n} \cdot e^{-\lambda(t - s_n)}}{P(N(t)=n)} = \frac{\lambda^n e^{-\lambda t}}{P(N(t)=n)}$$

$$\mathbb{P} = \frac{\lambda^n e^{-\lambda t}}{\frac{e^{-\lambda t} (\lambda t)^n}{n!}} = \frac{n!}{t^n} \quad \text{— } (2)$$

Since (1) & (2) are equal

Hence it proves the uniformity in previous event times.

Sunday 02