In the name of God
Probability and Statistics

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1 Introduction

Genghis' third semester is over and he wants to go on a trip to relieve the fatigue of exams. He doesn't know much about tourist places and decides to leave his destiny in the hands of destiny and start his journey. Due to his great interest in the sweet lesson of statistics and engineering probability, Genghis acts randomly in all the steps of this journey, and we want to get information about the characteristics of his movements and the events that may happen to him. In the world of statistics and probability, a model called random walk is used to describe phenomena such as Genghis' movement. In this project, we want to get to know the beautiful world of random walkers. But before that, we need to increase our information a little.

Figure 1: Changiz

Question 1:

- prove markove's inequality.

As stated in theorem 1-2 x is positive continuous and $a > 0$,

As stated in theorem

$$P[x \geqslant a] \leqslant \frac{E[x]}{a}.$$

$$E[x] = \int_{-\infty}^{+\infty} f_x(x)dx = \int_0^{\infty} xf_x(x)dx = \int_0^a xf_x(x)dx + \int_a^{\infty} xf_x(x)dx$$

We know: $\int_0^a xf_x(x)dx + \int_a^{+\infty} xf_x(x)dx \geqslant \int_a^{+\infty} xf_x(x)dx \geqslant \int_a^{\infty} af_x(x)dx.$

$$= ap(x \geqslant a)$$

So $p[x \geqslant a] \leqslant \frac{E[x]}{a}$ Question 2: Part 1, Using Markove's theorem:

$$P[x \geqslant 85] \leqslant \frac{E[x]}{85}, E[x] = 75$$
$$\to P[x \geqslant 85] \leqslant \frac{75}{85} = \frac{15}{17}$$

part 2)

$$65 \leqslant x \leqslant 85 \to -10 \leqslant x - E[x] \leqslant 10 \to |x - 75| \leqslant 10$$

first we have to prove chebyshevés inequality using Markove's inequality:

Proof of the Chebyshev inequality (continuous case): Given: X a real continuous random variables with $E(X) = \mu, V(X) = \sigma^2$, real number $\epsilon > 0$.

To show: $P(|X - \mu| \geq \epsilon) \leq \frac{\sigma^2}{\epsilon^2}$. Then

$$\begin{aligned}
\sigma^2 &= V(X) \\
&= \int_{-\infty}^{\infty} (t-\mu)^2 f_X(t)dt \\
&\geq \int_{-\infty}^{\mu-\epsilon} (t-\mu)^2 f_X(t)dt + \int_{\mu+\epsilon}^{\infty} (t-\mu)^2 f_X(t)dt,
\end{aligned}$$

where the last line is by restricting the region over which we integrate a positive function. Then this is

$$\geq \int_{-\infty}^{\mu-\epsilon} \epsilon^2 f_X(t)dt + \int_{\mu+\epsilon}^{\infty} \epsilon^2 f_X(t)dt,$$

since $t \leq \mu - \epsilon \Longrightarrow \epsilon \leq |t - \mu| \Longrightarrow \epsilon^2 \leq (t-\mu)^2$. But we rearrange and use the definition of the density function to get

$$\begin{aligned}
&= \epsilon^2 \left(\int_{-\infty}^{\mu-\epsilon} f_X(t)dt + \int_{\mu+\epsilon}^{\infty} f_X(t)dt \right) \\
&= \epsilon^2 P(X \leq \mu - \epsilon \text{ or } X \geq \mu + \epsilon) \\
&= \epsilon^2 P(|X - \mu| \geq \epsilon).
\end{aligned}$$

Thus,

$$\sigma^2 \geq \epsilon^2 P(|X - \mu| \geq \epsilon),$$

and dividing through by ϵ^2 gives the desired.

So: $p(|x-\mu| \geq k) \leqslant \frac{\sigma^2}{k^2} \quad \operatorname{Var}(x)=\sigma^2=25$

$$\rightarrow p(|x-\mu| \geqslant 10) \leqslant \frac{25}{100}=\frac{1}{4}=0.25$$

$$\rightarrow p(|x-\mu|<10)=1-p(|x-\mu| \geqslant 10)=1-\frac{1}{4}=\frac{3}{4}=0.75$$

part 3) $70 \leqslant \bar{x} \leqslant 80 \longrightarrow -5 \leqslant \bar{x}-75 \leqslant 5$

$$\rightarrow P\left[\left|\sum_{i=1}^n x_i-n\mu\right| \geqslant 5n\right]$$

so: $p[70 \leqslant \bar{x} \leqslant 80]=1-P\left[\sum_{i=1}^n x_i-n\mu>5n\right]=1-\frac{25n}{25n^2} \geqslant 0,9$

$$\rightarrow 0/1 \geqslant \frac{1}{n} \rightarrow n \geqslant 10$$

So $\min\{n\}$ is 10 . Question 3: $x \sim \operatorname{poisson}(\lambda)$ part 1)

$$T=n\tau \quad, K>1$$

$P\left[z_n \geqslant kn\lambda\right]$ is desired.

$$z_n=\sum_{i=1}^n x_i \rightarrow E\left[z_n\right]=E\left[\sum_{i=1}^n x_i\right]=\left[E\left[x_i\right]=\lambda n\right.$$

$$P\left[z_n \geqslant kn\lambda\right] \leqslant \frac{E\left[z_n\right]}{kn\lambda}=\frac{\lambda n}{kn\lambda}=\frac{1}{k}$$

(Using Markove's theorem) part 2, $\lambda=1, \quad k=1,25, \quad n=20$ Using central limit theorem:

$$p\left[\frac{z_n-n\lambda}{\sqrt{n\lambda}} \geqslant \frac{k\lambda n-\lambda n}{\sqrt{n\lambda}}\right]=\int_{k\sqrt{n\lambda}-\sqrt{\lambda n}}^{+\infty} \frac{e^{\frac{-x^2}{2}}}{\sqrt{2\pi}} dx=0.104107$$

Question 4:

$$z_n = \sum_{i=1}^{n} x_i, \psi_{z_n}(s) = n\psi_x(s)$$

X_i are same distributed and independent. (I)

$$\Psi_x(s) = \ln E\left[e^{sx}\right], \Phi_x(s) = E\left[e^{sx}\right]$$
$$\Psi_{Z_n}(s) = E\left[e^{sZ_n}\right] = E\left[e^{s\sum_{i=1}^{n} x_i}\right] = E\left[e^{sx_1+sx_2+\cdots+sx_n}\right]$$
$$= E\left[e^{sx_1}e^{sx_2}\dots e^{sx_n}\right] \overset{(I)}{=} \left(E\left[e^{sx}\right]\right)^n$$
$$\to \psi_{Z_n}(s) = \ln E\left[e^{sx}\right] \times n = n\psi_x^{(s)}$$

Question 5: Suppose that $\beta > E[x], s \geqslant 0$. We have to prove that:

$$P\left[z_n \geqslant \beta n\right] = P\left[e^{sz_n} \geqslant e^{s\beta_n}\right] \leqslant e^{-rn}$$

where $r = \sup\left\{s\beta - \psi_x(s)\right\}$

$$z_n \geqslant n\beta \to sz_n \geqslant n\beta s \to e^{sz_n} \geqslant e^{n\beta s}$$
$$p\left[z_n \geqslant n\beta\right] = p\left[e^{sz_n} \geqslant e^{s\beta}\right]$$
$$\text{(using Markove's theorem.)}$$

(Using Markove's theorem.)

$$\to P\left[z_n \geqslant n\beta\right] \leqslant e^{-n(s\beta - \psi_x(s))}$$

where $r = \sup_{s\geqslant 0}\left\{s\beta - \psi_x(x)\right\}$

Question 6: part 1) $\psi_x(s) = \ln E\left[e^{sx}\right] \Phi_x(s) = E\left[e^{sx}\right] = \int e^{sx} \times \frac{1}{\sigma\sqrt{2\pi}} e^{\frac{-(x-\mu)^2}{2\sigma^2}} dx = e^{\mu s + \frac{s^2\sigma^2}{2}}$

$$\rightarrow \psi_x(s) = \ln \phi_x(s) = \mu_s + \frac{s^2\sigma^2}{2}$$

part 2) $E[x] = 0, |x| < M$ prove that: $\psi_X(s) \leqslant \frac{1}{2}M^2s^2$

$$|x| < M \rightarrow x^2 < \mu^2 \rightarrow -s > 0$$
$$\rightarrow sx^2 \geqslant -sM^2 \rightarrow e^{-sx_2} \geqslant e$$
$$P\left[e^{-sx^2} \geqslant e^{-sm^2}\right], \quad \leqslant \frac{E\left[e^{-sx^2}\right]}{e^{-s\mu^2}} = \frac{\mu^2 s^2}{2}$$

(I) Using markove's inequality. part 3) $Z_n = \sum_{i=1}^{n} x_i$ Where x_i are independent and same distributed.

$$|x_i| \leqslant M \quad \beta > E[X]$$
$$p\left[Z_n \geqslant \beta n\right] \leqslant c^{-\frac{\beta^2 n}{2M^2}}$$

According to part (2) we have: $\psi_x(s) \leqslant \frac{1}{2}M^2s^2$

$$\text{(I) } s\beta - \psi_x(s) \geqslant s\beta - \frac{1}{2}M^2s^2$$
$$\rightarrow \frac{d}{ds}\left(s\beta - \frac{1}{2}M^2s^2\right) = 0 \rightarrow M^2 s = \beta \quad \text{so} \quad s = \frac{\beta}{M^2}$$
$$P\left[z_n \geqslant n\beta\right] \leqslant \frac{E\left[z_n\right]}{n\beta} \longrightarrow P\left[z_n \geqslant n\beta\right] \leqslant e^{\frac{-n\beta^2}{2M^2}}$$

Question 7:

$$s_1, s_2, s_3, \dots.$$

P ; H so rides the subway. 1-P ; T so doesn't ride the subway. First we have to remember that the number of stations has started from 1. So:

$$Z_n \sim \text{Binomial}(n, p)$$
$$p\left[z_n = s\right] = \left(\begin{array}{c} n \\ s-1 \end{array} \right) p^{s-1}(1-p)$$

part 1) Finding PMF of z_n. As calculated: $\left(\begin{array}{c} n \\ s-1 \end{array} \right) p^{s-1}(1-p)^{n-s+1}$

part 2) $\begin{cases} p & 1 \\ 1-p & 0 \end{cases}$

$$E\left[x_i\right] = P + (1-p)0 = p$$
$$E\left[x_i^2\right] = 0(1-p) + 1 \times p = p$$
$$\text{Var}\left(x_i\right) = E\left[x_i^2\right] - \left(E\left[x_i\right]\right)^2 = p - p^2$$

part 3) $Z_n = \sum_{i=1}^{n} x_i \to E\left[Z_n\right] = E\left[\sum_{i=1}^{n} x_i\right] = \sum_{i=1}^{n} E\left[x_i\right]$

$$= nE\left[x_i\right] = np$$

$= nE\left[X_i\right] = np$ (because X_i are independent and same distributed)

$$\to E\left[z_n\right] = 1 + nE\left[x_i\right] = 1 + np$$
$$E\left[Z_n^2\right] = n^2p^2 - np^2 + 3np + 1$$

So:

$$\operatorname{Var}(z_n) = E\left[Z_n^2\right] - (E[Z_n])^2 = n^2p^2 - np^2 + 3np + 1 - 1 - n^2p^2 - 2np$$
$$= np - np^2 = n\left(p - p^2\right)$$

Part 4) In fact, the variance of the final station is equal to the sum of the individual variances of the same n previous stations.

Question 8) We notate stations by $S.50$:

$$s_1, s_2, s_3, s_4 \ldots$$
$$E[x_i] = \frac{\sum_{i=1}^{10} x_i}{10} = \frac{1+2+\cdots+10}{10} = \frac{55}{10} = 5.5$$
$$E\left[x_i^2\right] = \frac{1+4+\cdots+100}{10} = 38.5$$
$$\operatorname{Var}(x_i) = E\left[x_i^2\right] - (E[x_i])^2 = 38.5 - (5.5)^2 = 8.25$$
$$\sum_{i=1}^{79} x_i \leqslant 300,5 \quad (\text{arrecting } 301 \text{ not } be \in -\mathbb{Z})$$
$$\rightarrow P\left[\sum_{i=1}^{79} x_i \leqslant 30015\right] = p\left[\frac{\sum x_i - 434.5}{\sqrt{79 \operatorname{var}(x_i)}} \leqslant \frac{300.5 - 434.5}{\sqrt{79 \operatorname{Var}(x_i)}}\right)$$
$$K = \frac{300.5 - 434.5}{\sqrt{79 \times 38.5}} = \frac{-134}{55.149} = -2.429$$
$$\rightarrow P\left[\sum_{i=1}^{79} x_i \leqslant 300.5\right] = \phi(-2.429) = 1 - \phi(2.429)$$

Question 9: part 1)

$$P_{z_n}(z) = p\left[z_n = z\right]$$
$$P_{z_{n-1}}\left(z_{+1}\right) = p\left[z_{n-1} = z+1\right]$$
$$P_{z_{n-1}}(z-1) = p\left[z_{n-1} = z-1\right]$$

suppose that we are at $Z-1$ in step z_{n-1}. So the possibility of returning to Z_n is $\frac{1}{2}$. Now suppose that we are at place $z+1$ in step z_{n-1}^2. So the probability of returning to Z_n is equal to $\frac{1}{2}$. Now we can conclude that:

$$P_{z_n}(z) = p\left[z_n = z\right] = \begin{cases} z_{n-1} = z_{+1} \rightarrow -1; p = \frac{1}{2} \\ z_{n-1} = z_{-1} \longrightarrow +1; p = \frac{1}{2} \end{cases}$$

so $P_{z_n}(z) = p\left[z_n = z\right] = \frac{P_{z_{n-1}}(z-1)+P_{z_{n-1}}(z+1)}{2}$ part 2) prove that : $p(z) = p\left[z_n = z\right] = \frac{n!}{\left(\frac{n-z}{2}\right)!\left(\frac{n+z}{2}\right)!}\left(\frac{1}{2}\right)^n$ we notate forward movement by x and backward movement by $y\cdot$ so:

$$\left.\begin{array}{l} z = x-y \\ n = x+y \end{array}\right\} \longrightarrow x = \frac{z+n}{2}, y = \frac{n-z}{2}$$

$$\rightarrow P(z) = P\left[z_n = z\right] = \left(\begin{array}{c} n \\ \frac{z-n}{2} \end{array}\right)\left(\frac{1}{2}\right)^{\frac{n+z}{2}} \times \left(\frac{1}{2}\right)^{\frac{n-z}{2}} =$$

$$\frac{n!}{\left(\frac{n-z}{2}\right)!\left(\frac{n+z}{2}\right)!}\left(\frac{1}{2}\right)^n$$

part 3) The only difference of this part and last part is that we should replace " p " and " $1-p$ " by $\frac{1}{2}$. so: $p_{z_n}(z) = p\left[z_n = z\right] = \left(\begin{array}{c} n \\ \frac{n+z}{2} \end{array}\right) p^{\frac{n+z}{2}} \cdot (1-p)^{\frac{n-z}{2}}$

Question 10, Zoo is placed $at + 2$ and park at -1 .

$$\begin{cases} +1; p = \frac{1}{2} \\ -1; p = \frac{1}{2} \end{cases}$$

$P_{z_n} = \dfrac{P_{z_{n+1}} + P_{z_{n-1}}}{2}$ As calculated in Question 9 part 1:

$z^n = \dfrac{z^{n+1} + z^{n-1}}{2} \rightarrow z^2 - 2z + 1 = 0$

$Z = 1$

so:

$P_{z_n} = x_0 + x_n n$ if $z = 2$ impossibility $P_2 = 0$

$x_0 - x_n = 1,$ if $z = -1$ $p_{-1} = 0$

$$\left.\begin{array}{l} x_0 + x_n = 1 \\ x_0 + 2x_n = 0 \end{array}\right\} \longrightarrow x_0 = \frac{2}{3}, x_n = \frac{-1}{3}$$

$P_{zoo} = \dfrac{1}{3}$, $P_{\text{park}} = \dfrac{2}{3}$

Question 11)

$$\begin{cases} +1; & P \\ -1; & 1-P \end{cases}$$

part 1)

$$\begin{aligned} &p_{z_n} = p \cdot p_{z_{n-1}} + (1-p) {p_{z_{n+1}}}^{n+1} \\ &p_{z_n} = z^n \to z^n = pz^{n-1} + (1-p)z^n \\ &\to z = p + (1-p)z^2 \end{aligned}$$

So:

$$\begin{aligned} &(1-p)z^2 - z + p = 0 \\ &Z = \frac{1 \pm \sqrt{1-4p+4p^2}}{2(1-p)} = \begin{cases} 1 \\ \frac{p}{1-p} \end{cases} \\ &z_n = x_0 + {x_n}^n \\ &\to p_n = \frac{\left(\frac{p}{1-p}\right)^n - 1}{\left(\frac{p}{1-p}\right)^L - 1} \end{aligned}$$

part 2)

$$\begin{aligned} &t_n = nz_n = p\left(1 + t_{n-1}\right) + (1-p)\left(1 + t_{n+1}\right) = 1 + t_{n+1} - pt_{n+1} + pt_{n-1} \\ &= 1 + (1-p)t_{n+1} + pt_{n-1} = 1 + pz_n(n-1) + (1-p)z_n(n+1) \\ &nz_n = 1 + z_n(n + 2p - 1) \to 1 = z_n(1-2p) \to z_n = \frac{1}{1-2p} \\ &t_0 = 0 \\ &t_L = 0 \end{aligned}$$

Question 12)
part 1) $p_n = p \times p_{n+1} + q \times p_{n-1}$

$$p\alpha^2 - \alpha + q = 0$$
$$\alpha = \frac{1 \pm \sqrt{1-4pq}}{2p} = \frac{1 \pm |1-2p|}{2p}$$
$$\rightarrow \alpha_1 = \frac{1+|1-2p|}{2p} / \alpha_2 = \frac{1-|1-2p|}{2p}$$
$$\Rightarrow p_n = a\alpha_1^n + b\alpha_2^n$$

$p \leq \frac{1}{2} \rightarrow |1-2p| = 1-2p$:

$$p_n = a\left(\frac{1-p}{p}\right)^n + b^n = a\left(\frac{1-p}{p}\right)^n + b$$

$p_0 = 1$ so $a+b=1$. $q > \frac{1}{2}$ $p_\infty = 1$ so $\Rightarrow b = 1$ and $a = 0$.

$$p_n = 1$$

$p > \frac{1}{2} \rightarrow |1-2p| = 2p-1$:

$$p_n = a1^n + b\left(\frac{1-p}{p}\right)^n = a + b\left(\frac{1-p}{p}\right)^n$$

$a+b=0$ and $p_\infty = 0$, so $\rightarrow a = 0$ and $b = 1$

$$p_n = \left(\frac{q}{p}\right)^n$$

if $p > \frac{1}{2}$ $n > N$ part 2) t_n, get t_n. $q = p$ or $q > p$. $\quad T_n = 1 + pT_{n+1} + qT_{n-1}$

$$T_n - T_{n-1} = 1 + p\left(T_{n+1} - T_n + T_n - T_{n-1}\right) \text{ define } : S_n = T_n - T_{n-1}$$
$$S_n = 1 + p\left(S_{n+1} + S_n\right) \rightarrow (1-p)S_n = 1 + pS_{n+1} \text{ define: } K_n = S_n + \frac{1}{1-2p}$$
$$qK_n = pK_{n+1} \rightarrow K_{n+1} = \frac{q}{p}K_n \Rightarrow K_n = \left(\frac{q}{p}\right)^{n-1} K_1$$
$$T_n = \left(\frac{\left(\frac{q}{p}\right)^n - 1}{\frac{q}{p} - 1}\right)\left(T_1 + \frac{1}{1-2p}\right) - \frac{n}{1-2p}$$

$p = q = \frac{1}{2}$ there is no T_n.

$$t_n = \frac{1}{1-2p}\left[n + L\left(\frac{1-\left(\frac{p}{1-p}\right)^n}{1-\left(\frac{p}{1-p}\right)^2}\right)\right]$$

$$\mathrm{maxt}_n : \frac{dt_n}{dn} = 0 \to n = \frac{\left(\frac{p}{1-p}\right) - 1}{L\left(\frac{p}{1-p}\right)^n \cdot L \cdot n}$$

$\to \begin{cases} \frac{L}{2} & \in \text{ Even} \\ \frac{L+1}{2} & \in \text{ Odd} \end{cases}$ Question 13: part 1) In better words we can say:

$$F_n = \sum_{i=1}^{n} x_i$$

part 2) $p_k = P\left[\exists n \in \mathbb{N} : Z_n = k\right]$

$$P_1 = p + qP_2$$

- if he is at station 0 : $P_0 \times P = 1 \times P = P \quad (P_0 = 1)$ $P_1 =$ if he is at station 1: $P_2(1-p) = p_2 q$

$$\begin{cases} p; +1 \\ 1 - p = q; -1 \end{cases}$$

$$\to p_1 = p + p_2 q$$

part 3)

$$p_1 = p + p_2 q$$
$$p_2 = p_1^2$$

$$P_1 = p + P_1^2(1-P)$$
$$\to (1-p)P_1^2 - P_1 + P = 0$$
$$P_1 = \frac{1 \pm \sqrt{1-4p+4P^2}}{2(1-p)} = \frac{1 \pm |1-2p|}{2(1-p)}$$
$$\to P_1 = \begin{cases} 1 & ;\ \text{if } p \leqslant \frac{1}{2} \\ \frac{p}{1-p}; & \text{if } p > \frac{1}{2} \end{cases}$$

part 4)

$$\text{if } k > 0 \to p_k = p_1^k$$
$$\text{if } k = 0 \to P = 1$$
$$\text{if } k < 0 \to p_k = p_{-1}^k$$

part 5)

$$\begin{array}{cc} \text{if } p \geqslant \frac{1}{2} & p_k = \left(\frac{p}{1-p}\right)^k \\ 1 & k \leqslant 0 \\ \text{if } p > 0 & \end{array}$$

if $p < \frac{1}{2} \to p_k = \left(\frac{p}{1-p}\right)^k \quad k \geqslant 0$ part 6)

$$1 \quad k < 0$$

if $p = 013 \to p_5 = \frac{(0/3)^5}{(017)^5} = \left(\frac{3}{7}\right)^5 = \frac{243}{16807} = 0.0144582.$

Question 14:

$$\begin{cases} +1; & p \\ -1; & 1-p \end{cases}$$

part 1, prove that $E\left[T_K\right] = KE\left[T_1\right]$

$$\begin{aligned}
&E\left[T_2\right] = E\left[T_1\right] + E\left[T_1\right] = 2E\left[T_1\right] \\
&E\left[T_3\right] = E\left[T_2\right] + E\left[T_1\right] = 3E\left[T_1\right] \\
&\cdots \\
&E\left[T_K\right] = E\left[T_{K-1}\right] + E\left[T_1\right] = (K-1)E\left[T_1\right] + E\left[T_1\right] = KE\left[T_1\right]
\end{aligned}$$

part 21

$$\begin{aligned}
&E\left[T_1\right] = (1-P)\left(E\left[T_2\right] + 1\right) + P \\
&E\left[T_2\right] = 2E\left[T_1\right] \to E\left[T_1\right] = (1-P)\left(2E\left[T_1\right] + 1\right) + P \\
&\to E\left[T_1\right] = \frac{1}{2P-1}
\end{aligned}$$

part 3)

$$E\left[T_k\right] = kE\left[T_1\right] = \frac{k}{2p-1} \qquad \text{(Using part 1) } E\left[T_k\right] = kE\left[T_1\right]\ )$$

par 4)

$$\begin{aligned}
&E\left[T_{50}\right] = 50E\left[T_1\right] \\
&P = 0155 \quad E\left[T_1\right] = \frac{1}{2P-1} = \frac{1}{0/1} = 10 \\
&E\left[T_{50}\right] = 50E\left[T_1\right] = 50 \times 10 = 500
\end{aligned}$$

Question 15)

$$Z_0 = 100$$
$$E[X_i] = 0$$
$$\operatorname{Var}[X_i] = 1$$

X_i are independent and same distributed.

$$z_n = \sum_{i=1}^{n} x_i + z_0$$
$$z_n > 105 \to z_n - 100 > 5$$

$\operatorname{Var}(x) = 10 \operatorname{Var}[x_i] = 10$

$$P\left[\frac{\sum_{i=1}^{10} x_i}{\sqrt{}} > \frac{5}{\sqrt{\operatorname{Var}(x)}}\right] = P\left[\frac{\sum_{i=1}^{10} x_i}{\sqrt{10}} > \frac{5}{\sqrt{10}}\right] 1 - \phi(7.5811)$$

Question 16) part 1) prove that $f(x_1, x_2, x_3, \ldots, x_n) = f(x_n, x_{n-1}, \ldots, x_1)$

$$f(x_n, x_{n-1}, \ldots, x_1) = \frac{f(x_1, x_2, x_3, \ldots, x_n)}{\|J\|}$$

||J||=1

so $f(x_n, x_{n-1}, \ldots, x_1) = f(x_1, x_2, \ldots, x_n)$

part 2)

$$
\begin{aligned}
&E\left[X_i\right]>0\\
&N=\min\left\{n: x_1+x_2+x_3+\cdots+x_n>0\right\}
\end{aligned}
$$

prove that : $E\left[x_i\right]<\infty$ n is minimum number in which the sequence gets positive. So:

$$
\begin{aligned}
&x_1+x_2+\cdots+x_{n-1}<0\\
&x_1+x_2+\cdots+x_{n-1}+x_n>0\\
&\text{ so }: x_n>0, x_n>\left|x_1+x_2+\cdots+x_n\right|\\
&E[N]=E[n]<\infty
\end{aligned}
$$

part 3)

$$
z_n=z_0+\sum_{i=1}^n x_i
$$

Z_n is the location of car in step n.

$$
\lim_{n\to\infty}\frac{E\left[R_n\right]}{n}=\frac{E\left[z_0+z_1+\cdots+Z_n\right]}{n}=1-P[\text{ return to } 0\text{ }]
$$

$=P\left[\text{ never return to } 0'\right]$ Question 17,

$$
\begin{gathered}
\begin{cases}+1; p=\frac{1}{2}\\ -1; p=\frac{1}{2}\end{cases}\\
z_0=0
\end{gathered}
$$

Try to find a sequence

$$
\begin{aligned}
&Z_n = z_0 + \sum_{i=1}^{n} x_i \\
&E[x] = \frac{1}{2} \times 1 + E[x] \times \frac{1}{2} \rightarrow E[x] = 1 \\
&E[x] = \frac{1}{4} + \frac{E[x]}{2} + \frac{E[x]}{4} \rightarrow E[x] = 1 \\
&E[x] = \frac{1}{8} + \frac{E[x]}{2} + \frac{E[x]}{4} + \frac{E[x]}{8} \rightarrow E[x] = 1 \\
&\vdots \\
&E[x] = \frac{1}{2^k} + \sum_{n=1}^{k} \frac{E[x]}{2^n}
\end{aligned}
$$

we have to calculate sum of a geometric progression:

$$a_0 = \frac{1}{2}, q = \frac{1}{2}$$

So we have $\sum_{n=1}^{k} \frac{E[x]}{2^n} = 1 - \left(\frac{1}{2}\right)^k$

$$\rightarrow E[x] = \frac{1}{2^k} + \sum_{n=1}^{k} \frac{E[x]}{2^n} = 1 - \left(\frac{1}{2}\right)^k + \left(\frac{1}{2}\right)^k = 1$$

Question 19: $\theta \sim \text{uniform}(0, 2\pi) \to f_\theta(\theta) = \frac{1}{2\pi}$

$$x_i = \begin{bmatrix} r\cos\theta \\ r\sin\theta \end{bmatrix}$$

$$z_n = \sum_{i=1}^{n} x_i$$

part 1)

$$E[z_n] = E\left[\sum_{i=1}^{n} x_i\right] = \sum_{i=1}^{n} E[x_i]$$

$$E[\cos\theta] = \int_0^{2\pi} \frac{1}{2\pi}\cos\theta d\theta = 0$$

$$E[\sin\theta] = \int_0^{2\pi} \frac{1}{2\pi}\sin\theta d\theta = 0$$

$$\to E[z_n] = 0$$

part 2)

$$E\left[\|z_n\|^2\right] = E\left[\sum_{i=1}^{n} \|x_i\|^2\right] = \sum_{i=1}^{n} E\left[r^2\right] = \sum_{i=1}^{n} r^2 = nr^2$$

part 3)

part 3)

$$\|z_n\|^4 = r^4\left(\left(\sum_{i=1}^{n}\cos\theta_i\right)^4 + \left(\sum_{i=1}^{4}\sin\theta_i\right)^4 + 2\left(\sum_{i=1}^{n}\cos\theta_i\right)^2\left(\sum_{i=1}^{n}\sin\theta_i\right)\right.$$

where:

where:

$$E[\cos\theta_i]\,E\left[\cos^2\theta_j\right] = \frac{1}{4} \text{ (cause } E\left[\cos^2\theta_i\right] = \frac{1}{2\pi}\int_0^{2\pi}\cos^2\theta d\theta = \frac{1}{2}.$$

$$E\left[\cos^4\theta_i\right] = \frac{3}{8}$$

$$\to E\left[\|z_n\|^4\right] = nr^4 \times \frac{7n-3}{4} = \frac{7n^2r^4 - 3nr^4}{4}$$

Question 20:

$$R = \{\exists n \in \mathbb{N} : Z_n = 0\}$$
$$\begin{cases} +1; p = \frac{1}{2} \\ -1; p = \frac{1}{2} \end{cases}$$

If Changiz is on station -1 he returns to 0 $\quad p = \frac{1}{2}$ and if he is on station $+1$, he returns to 0 $\quad p = \frac{1}{2}$. So:

$$P[R] = 1$$

According to question 11. $E[N] = \sum_{n=1}^{\infty}(P[R])^n n = \sum_{n=1}^{\infty} n$ doesn't converge Question 21:

$$P\left[X_i = x\right] = \begin{cases} \frac{2}{3} & x = 1 \\ \frac{1}{3} & x = -1 \\ 0 & 0.\omega \end{cases}$$
$$P[R] = 2\left(1 - \frac{2}{3}\right) = \frac{2}{3} \qquad \text{where:}$$
$$P[x = 1] = \frac{1 - P}{P}$$
$$P[x = -1] = 1$$
$$E[N] = \sum_{i=1}^{\infty} iP[R] = \frac{2}{3} + \frac{8}{9} + \frac{24}{27} + \cdots = \frac{2}{3}\left(1 + \frac{4}{3} + \cdots\right) = 6$$

Question 22: Using weak law of large numbers

$$p\left[|\bar{x}_n - \mu| < \varepsilon\right] \geqslant 1 - p\left[|\bar{x}_n - \mu| \geqslant \varepsilon\right] \geqslant 1 - \frac{\sigma^2}{n\varepsilon^2}$$

so if n converges to ∞ the probability converges to 1 .

Question 23:

$$f(x) = \begin{cases} \frac{1}{2}f(x+1) \\ \frac{1}{2}f(x-1) \end{cases}$$
$$\to f(x) = \frac{1}{2}[f(x+1) + f(x-1)]$$

Because if he starts from $x+1$ he should move one step back with $p = \frac{1}{2}$ and if he start from $x-1$ he should move one step for w ard with possibility $\frac{1}{2}$.

$$f(-1) = 1$$
$$f(M) = 0$$

We have a linear movement and we try to find m and c :

$$f(x) = mx + C$$
$$m = \frac{-1}{M+1} \to C = \frac{M}{M+1}$$
$$f(x) = \frac{-x}{M+1} + \frac{M}{M+1}$$

Question 24) part 1,

$$f(x) = \frac{-x}{M+1} + \frac{M}{M+1}$$

So:

$$f(0) = \frac{M}{M+1}$$

part 2) Using maximum power theory because n converges to ∞ :

$$\lim_{m\to\infty} f(0) = \frac{m}{M} = 1$$

part 3) If changiz is on x the possibility of forward and backward movement is equal to $\frac{1}{2}$.

$$P[R] = \frac{1}{2}P[x = 1] + \frac{1}{2}P[x = -1]$$

So the $P[R] = 1$, the movement is loyal. Question 25) Using strong law of large numbers:

$$\mathrm{Lim}_{n\to\infty} \sum_{i=1}^{n} x_i - nE[x] = 0$$

if we start the movement from 0 , so $E[X] = 0$ and we return to our first place.

$$\lim_{N\to\infty} \frac{\sum_{i=1}^{n} x_i}{N} = 0$$

In the question it is said that $E[x_i] \neq 0$. So we reached to a contradiction. So $p < 1$ and it is not loyal. Question 26)

$$m(z) = \sum_{n=0}^{\infty} p[z_n = z]$$

$$\begin{cases} x_i = 1 & \text{if returns to } z \\ x_i = 0 & \text{if doesn't return to 0} \end{cases}$$

$$E[m(z)] = \sum_{i=0}^{\infty} 1 \times p[x_i = 1] + 0 \times p[x_i = 0] = \sum_{i=0}^{\infty} p[z_i = z]$$

Question 27) suppose that $P[$ returnto "O" from an other place $] = P$. so:

$$m(z) = m(0) \times p$$

we know that p is a value less than $1 (p \leqslant 1)$ so it will be maximized at $m(0)$.
Question 28) part 1,

$$\begin{aligned} & P\left[|z_n| \leqslant 2\beta\sqrt{n}\right] \geqslant \frac{1}{2} \\ & E\left[x_i\right] = 0 \\ & |x_i| \leqslant \beta \end{aligned}$$

We Use the generalization of markov theorem to Chebyshev in which was proved at question 2 .

$$\begin{aligned} & \operatorname{Var}(z_n) = E\left[z_n^2\right] - (E\left[z_n\right])^2 = nE\left[x_i^2\right] - 0 = nE\left[x_i^2\right] \\ & P\left[|Z_n| \geqslant 2B\sqrt{n}\right] \leqslant \frac{\operatorname{Var}(z_n)}{4nB^2} \\ & \qquad |x_i| \leqslant B \rightarrow x_i^2 \leqslant B^2 \rightarrow x_i^2 \leqslant 2B^2 \\ & E\left[x_i^2\right] \leqslant 2B^2 \\ & P\left[Z_n \geqslant 2B\sqrt{n}\right] \leqslant \frac{2nB^2}{4nB^2} = \frac{1}{2} \\ & \rightarrow P\left[|Z_n| \leqslant 2B\sqrt{n}\right] \geqslant \frac{1}{2} \end{aligned}$$

part 2)

$$m(z)=\sum_{n=0}^{\infty} P\left[z_n=z\right]$$

from part 1, we have:

$$\begin{aligned}
& P\left[\left|Z_n\right| \leqslant 2 B \sqrt{n}\right] \geqslant \frac{1}{2} \\
& P\left[-2 B \sqrt{n} \leqslant Z_n \leqslant 2 B \sqrt{n}\right] \geqslant \frac{1}{2} \\
& \rightarrow \sum_{Z=-2 B \sqrt{n}}^{+2 B \sqrt{n}} P\left[Z_n=Z\right] \geqslant \frac{1}{2}
\end{aligned}$$

we h ven n repeatation of it. So it will be multiplied ton. So

$$\sum_{Z=-2 B \sqrt{n}}^{z=+2 B \sqrt{n}} m(z) \geqslant n \times \frac{1}{2}=\frac{n}{2}$$

$Z=-2 B \sqrt{n}$ part 3 ,

$$\begin{aligned}
& m(z) \leqslant m(0) \\
& m(z)<\infty
\end{aligned}$$

$$\begin{gathered}
\lim _{n \rightarrow \infty} \frac{\sum_{i=1}^n x_i}{n}=E[x] \\
\text { if } E\left[x_i\right]=0 \rightarrow \sum_{i=1}^n x_i=0
\end{gathered}$$

if $E\left[x_i\right] \neq 0 \rightarrow \sum_{i=1}^{i=1} x_i \neq 0 \quad \dot{X}.$ come to a contradiction.

Question 30)

$$\begin{aligned}
z_n &= x_n + z_{n-1} \\
f_{x_n}(x_n) &= \frac{1}{(2\pi\sigma^2)^{\frac{3}{2}}} \exp\left(\frac{-x_n^\top x_n}{2\sigma^2}\right) \\
z_n &= \sum_{i=1}^{n} x_i + x_0 = \sum_{i=1}^{n} x_i
\end{aligned}$$

Because $X_0 = 0$.

$$\begin{aligned}
&E[z_n] = nE[x_i] = 0 \\
&\mathrm{Var}(z_n) = n\,\mathrm{var}(x_i) = n_\sigma{}^2 \\
&\rightarrow z_n \sim N\left(0, n\sigma^2\right) \\
&f_{z_n}(z_n) = \frac{1}{(2\pi n\sigma^2)^{\frac{3}{2}}} \exp\left(-\frac{z_n z_n}{2n\sigma^2}\right)
\end{aligned}$$

Z_n is also normal distribution. Question 31)

$$\begin{aligned}
&f_{z_{n-1}}(z_{n-1}) = f(z_n - x_n) \\
&f_{z_n}(z_n) = \int f_{x_n}(x_n) f_{z_{n-1}}(z_n - x_n)\, dx_n \\
&f_{z_{n-1}}(z_{n-1}) = f_{z_{n-1}}(z_n) + x_n^\top \vec{\nabla} f_{z_{n-1}}(z_n) + \frac{1}{2} x_n^T H_{z_{n-1}}(z_n)\, x_n \\
&\rightarrow f_{z_n}(z_n) = f_{z_{n-1}}(z_n) + \frac{\sigma^2}{2}\,\mathrm{trace}\left(H\left(z_{n-1}(z_n)\right)\right)
\end{aligned}$$

Question 32) As calculated in previous part:

As calculated in previous part.

$$
\begin{aligned}
&f_{z_n}(z_n) = \frac{\sigma^2}{2}\operatorname{trace}\left(H_{z_{n-1}}(z_n)\right) + f_{z_{n-1}}(z_n) \\
&f_{z_t}(z_t) = \frac{\sigma^2}{2}\operatorname{trace}\left(H_{z_t}(z_t)\right) + f_{z_t}(z_t) - \frac{df_{z_t}(z_t)}{dt}\delta t \\
&\rightarrow \frac{df_{z_t}(z_t)}{dt} = \frac{s}{2}\operatorname{trace}\left(H_{z_t}(z_t)\right)
\end{aligned}
$$

So

$$\alpha = \frac{5}{2}$$

So that:

$$\frac{df_{z_t}(z_t)}{dt} = \alpha\nabla^2 f_{z_t}(z_t) = \frac{s}{2}\nabla^2 f_{z_t}(z_t)$$

Question 33) Fourier Transformation:

$$\frac{d\tilde{F}_{(k)}}{dt} = \frac{-s}{2}k^2\widetilde{F}_k$$

Solving the ODE:

$$
\begin{aligned}
&\tilde{F}_{(k)} = e^{\frac{-sk^2}{2}t} \\
&f_{z_t}(z_t) = \left(\frac{1}{2\pi st}\right)^{\frac{3}{2}} e^{\frac{-\left(x_1^2+x_2^2+x_3^2\right)}{2st}}
\end{aligned}
$$

Question 34)

Question 34)

$$f_{z_t}(r) = \frac{4\pi r^2}{(2\pi st)^{\frac{3}{2}}} \exp\left(\frac{-r^2}{2st}\right)$$

$$E\left[\|z_t(r)\|\right] = \int_0^\infty \frac{4\pi r^3}{(2\pi st)^{\frac{3}{2}}} \exp\left(\frac{-r^2}{2st}\right) dr = \sqrt{8st\pi^{-1}}$$

Using wolfram to solve the integral.

$$E\left[\|z_t(r)\|^2\right] = \int_0^\infty \frac{4\pi r^4}{(2\pi st)^{\frac{3}{2}}} \exp\left(-\frac{r^2}{2st}\right) dr = 3st$$

$\operatorname{Var}(\|z_t\|) = E\left[\|z_t\|^2\right] - \left(E\left[\|z_t\|\right]\right)^2 = 3st - \frac{8st}{\pi}$ So:

$$\operatorname{var}(\|z_t\|) = 3st - \frac{8st}{\pi} = \frac{3st\pi - 8st}{\pi}$$

Question 35) We found the equation in parts above. So:

$$f_{z_n}(z_n) = \left(\frac{1}{2\pi n\sigma^2}\right)^{\frac{3}{2}} \exp\left[\frac{-z_n^2}{2}\right]$$

Question 36,

$$z_n = \sum_{i=1}^{n} x_i + x_0$$
$$x_0 = 0$$
$$E[z_n] = nE[x_i] = 0$$
$$\operatorname{Var}(z_n) = n\sigma^2 + \sigma_0^2$$

So:

$$z_n \sim N\left(0, \sigma_0^2 + n\sigma^2\right)$$

$$f_{z_t}\left(z_t\right) = \frac{1}{\left(2\pi\left(\sigma_0^2 + n\sigma^2\right)\right)^{\frac{3}{2}}} \exp\left(\frac{-z_t^2}{2\left(\sigma_0^2 + n\sigma^2\right)}\right) =$$

$$\frac{1}{\left(2\pi\left(\sigma_0^2 + st\right)\right)^{\frac{3}{2}}} \exp\left(\frac{-z_t^2}{2\left(\sigma_0^2 + st\right)}\right)$$

Question 37,

$$f_{z_n}\left(z_n\right) = \int_{\mathbb{R}^3} f_{x_n}^n\left(x_n\right) f_{z_{n-1}}\left(z_n - x_n\right) dx_n$$

$$f_{z_n}\left(z_n\right) = \int_{-\infty}^{+\infty}\int_{-\infty}^{+\infty}\int_{-\infty}^{+\infty} \pi f_{x_n}^n\left(x_1, x_2, x_3\right) dx_1 dx_2 dx_3$$

Question 38)

$$f_{x_m}^{(m)}\left(x_m\right) = \frac{1}{\left(2\pi m^2\sigma_0^2\right)^{\frac{3}{2}}} \exp\left(\frac{-x_m^\top x_m}{2m^2\sigma_0^2}\right)$$

$f_{z_n}\left(z_n\right)$ is desired.

$$z_n \sim N\left(0, \sigma_\beta^2\right)$$

$$\sigma_\beta^2 = \sigma_0^2 \times \sum_{m=0}^{n} m^2 = \sigma_0^2\left(1 + 4 + 9 + \cdots + n^2\right) = \sigma_0^2 \frac{n(n+1)(2n+1)}{6}$$

$$s_0 :$$

So:

$$f_{z_n}\left(z_n\right) = \frac{1}{\frac{n(n+1)(2n+1)\pi\sigma_0^2}{3}} \exp\left(\frac{-z_n^2}{\frac{\sigma_0^2 n(n+1)(2n+1)}{3}}\right)$$