Option pricing via Meixner process

Niyu Jia

1.Introduction

In practice, when we observe the distribution of stock log-returns, we always get a distribution that has a fatter tail than the normal distribution, which means that the Black-Scholes model does not always fit the market data very well. To make an improvement upon the Black-Scholes model, here we want to make the assumptions that the log return of our underlying portfolio follows Meixner Distribution and that our log return process is a Meixner process.

Here comes a question that what Meixner distribution and Meixner process is. The Meixner distribution belongs to the class of the infinitely divisible distributions and as such give rise to a Levy process: The Meixner process. While the Meixner process is a special type of Levy process which originates from the theory of orthogonal polynomials. It is related to the Meixner-Pollaczek polynomials by a martingale relation [1]. The Meixner distribution (Meixner(a, b, d, m)) is given by:

$$f(x;a,b,d,m)=\frac{\left(2\cos\left(\frac{b}{2}\right)\right)^{2d}}{2a\pi\Gamma(2d)}\exp\left(\frac{b(x-m)}{a}\right)\left|\Gamma\left(d+\frac{i(x-m)}{a}\right)\right|^{2}$$

where $a>0,-\pi<b<\pi,d>0,\wedge m\in R$

The characteristic function of Meixner(a, b, d, m) is given by:

$$\phi_1(u)=E\left[\exp(iuM_1)\right]=\left(\frac{\cos\left(\frac{b}{2}\right)}{\cosh\frac{au-ib}{2}}\right)^{2d}\exp(imu)$$

If we arbitrarily choose a positive integer k, and we take the kth power of the characteristic function ϕ_1 .

We can get another characteristic function ϕ_μ given by:

$$\phi_\mu(u)=\phi_1(u)^k=E\left[\exp(iuM_1)\right]^k=\left(\frac{\cos\left(\frac{b}{2}\right)}{\cosh\frac{au-ib}{2}}\right)^{2dk}\exp(imku)$$

We set the distribution law μ has the characteristic function ϕ_{μ}, then we can find that μ is still a Meixner distribution, just of different parameters, a.k.a. Meixner(a, b, dk, mk). Then it begins clear that Meixner distribution is infinitely divisible and thus we can construct a process to describe the log returns of our underlying assets by using Meixner distribution.

In general, Levy processes are made up of three components: a Brownian motion part, a deterministic drift with respect to time and a pure jump process. However, Meixner process does not have a Brownian motion part and the pure jump process is determined by a Levy measure, given by:

$$v(dx) = d\frac{\exp(bx/a)}{x\sinh(\frac{\pi x}{a})}dx$$

$v(dx)$ determines how the jumps of our process occur. If we mark the set that describes the size of the jumps as A, then the jumps' occurrence follows a Poisson process with parameter $\int_A v(dx)$. Since

$\int_{-\infty}^{+\infty} |x| v(dx) = \infty$, by standard Levy process theory[3][4], our process has infinite variation.

2. Parameter Estimation

We have tried two methods while estimating these four parameters in our model.

2.1 Moment Estimation

The moments of all orders of the distribution Meixner(a, b, d, m) exists [2], given by:

Mean	$m + ad\tan\frac{b}{2}$
Variance	$\frac{a^2 d}{2}\cos^{-2}\frac{b}{2}$
Skewness	$\sin b\sqrt{\frac{1}{d(\cos b + 1)}}$
Kurtosis	$3 + \frac{3 - 2\cos^2\frac{b}{2}}{d}$

We can calculate these four moments with our data, and then we encounter a problem with four equations and four unknown parameters. Therefore, we can estimate the parameters by solving the equations.

The parameters we finally get are：

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Moment Estimator parameters(a,b,d,m):
 [ 0.01294108  0.49220911  1.03397087 -0.00280419]
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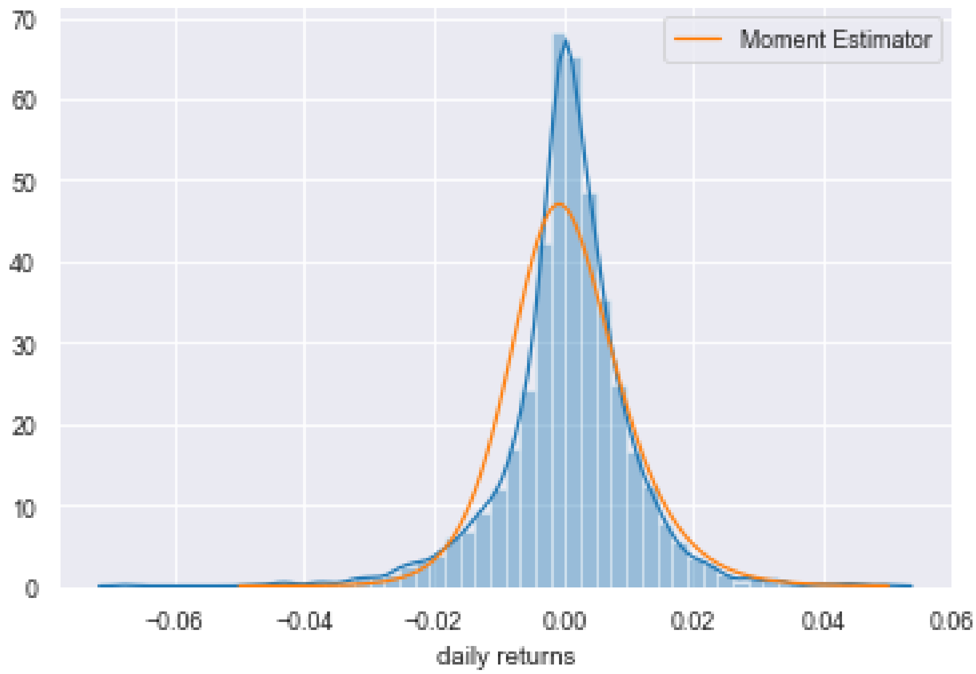
Plot the PDF with respect to the parameters above:



The blue line is the probability density function of the logarithm return curve of the market data and the orange curve is the pdf from moment estimation. From this figure, we can find that the moment estimation fits the real data well in tails area.

Although it performs not good in peak area, we can still choose these parameters as the initial value in the following most likelihood estimation.

2.2 Most Likelihood Estimation

When the type of total distribution is known, the most likelihood estimation is a common estimating method.

Let θ denote the parameter to be estimated in the total distribution. It can take many different values. We need to choose one value from all possibilities to make the probability of the observation become maximum. The value $\hat{\theta}$ that we choose is called the most likelihood estimation of parameter θ .

In the condition that the distribution of X is continuous, let's set the probability density function to be $p(x;\theta)$, where θ is the unknown parameter, $\theta \in \Theta$. Now we get the sample observations $x_1, x_2, \ldots, x_n$ from the population of capacity n, then when $X_1 = x_1, X_2 = x_2, \ldots, X_n = x_{n1}$, the joint density function is $\prod_{i=1}^{n} p(x_i;\theta)$, which is also the function of θ and called likelihood function. We denote it as:

$$L(\theta) = \prod_{i=1}^{n} p(x_i;\theta)$$

For different θ , the value of joint density function of a same sample observation $x_1, x_2, \ldots, x_n$ are still different. So, the most likelihood estimator $\hat{\theta}$ that we choose should satisfy

$$L(\hat{\theta}) = \max_{\theta \in \Theta} L(\theta)$$

Since the analytical form of the PDF of the Meixner distribution is accessible, we can easily get the formula of the joint density function. And we regard it as the objective function in an optimal problem.

The parameters we get from the most likelihood method is:

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Most Likelihood Estimator parameters(a,b,d,m):
 [ 0.03128716 -0.22078567 0.18863356 0.00121117]
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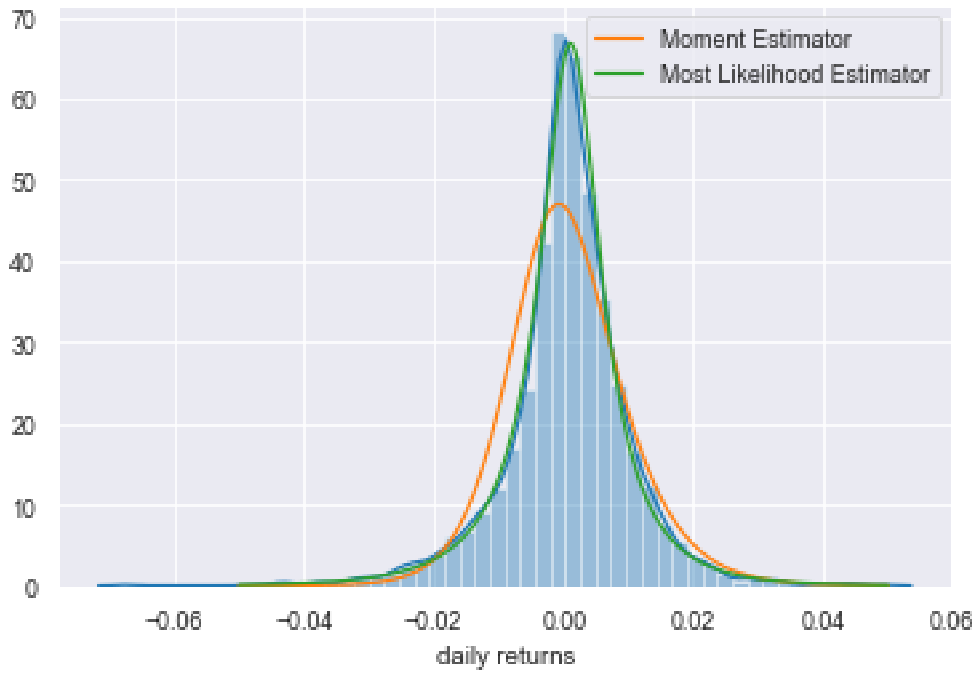
The PDF figure is:



The green line, which denotes the PDF of Meixner with parameters from the MLE method, fits the real market data very well. We will use this model to price the option next.

3.Option Pricing

In our model, we would like to substitute the Brownian motion with the process Meixner(a, b, d, m) as the process that drives the daily log-returns of our underlying asset. Thus we can write the price of the risky asset as below:

$$S_t = S_0 \exp(M_t)$$

We denote that the daily risk-free rate as r and make the assumption that the market consists of one risk-free bond and our underlying risky asset whose price S_t follows a Meixner process. Then we make $G(\{S_u, 0 \le u \le T\})$ denote the payoff function of a derivative underlying our risky assets. Then, by the fundamental theory of asset pricing[10], the arbitrage-free price of this derivative V_t at time $t \in [0, T]$ can be given by:

$$V_t = E^Q\left[e^{-r(T-t)} G(\{S_u, 0 \le u \le T\}) \middle| F_t ¿\right.$$

Where Q is an equivalent local martingale measure, and $F = \{F_t, 0 \le t \le T\}$ is the natural filtration generated by the price process of our underlying asset.

3.1 Equivalent Local Martingale Measure

Since the parameters are calibrated from the market data, which is under the physical probability, we need to change it into the equivalent martingale measure.

An equivalent martingale measure is a probability measure which is equivalent (it has the same null-sets) to the given (historical) probability measure and under which the discounted process $\{e^{-rt} S_t\}$ is a martingale.

One way to obtain an equivalent martingale measure Q is by mean correcting the exponential of the Levy process $S_t = S_0 \exp(M_t)$. According to the mean-correction method, in this case the risk-neutral process is given by [5]

$$S_t^{riskneutral} = S_0 \exp(X_t) \frac{\exp(rt)}{E[\exp(X_t)]}$$

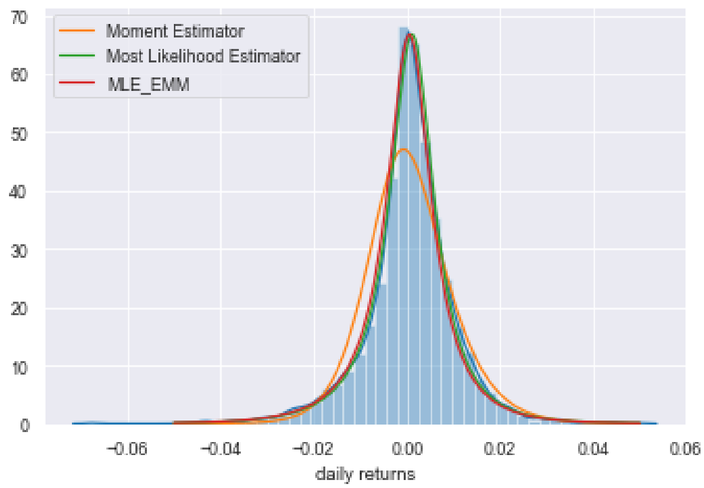
For the Meixner process, the EMM Q now follows a Meixner(a, b, d, $m_{riskneutral}$), with

$$m_{riskneutral} = r - 2d \log\left(\frac{\cos\left(\frac{b}{2}\right)}{\cos\left(\frac{a+b}{2}\right)}\right)$$

According to the reference (*B. Grigelionis. Processes of Meixner type. Lithuanian Mathematical Journal, 39(1):33–41, 1999. MR1711971*), the discounted geometric Meixner process in this Q is indeed a martingale.

Take the parameters into the new m, we can plot the PDF figure as follows:

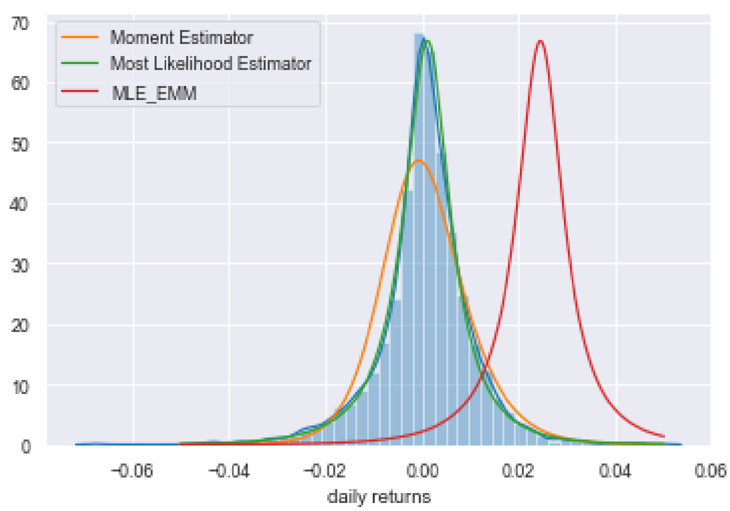
First, we set spot rate is 0. The PDF curve under EMM is:



The EMM curve is very similar to physical PDF curve from MLE.

Next, we choose the three month treasury bill as the spot rate r, which is 2.4%.

By the mean correcting method, the parameters a, b, d don't change. And the new m is: 0.024607499087087092.



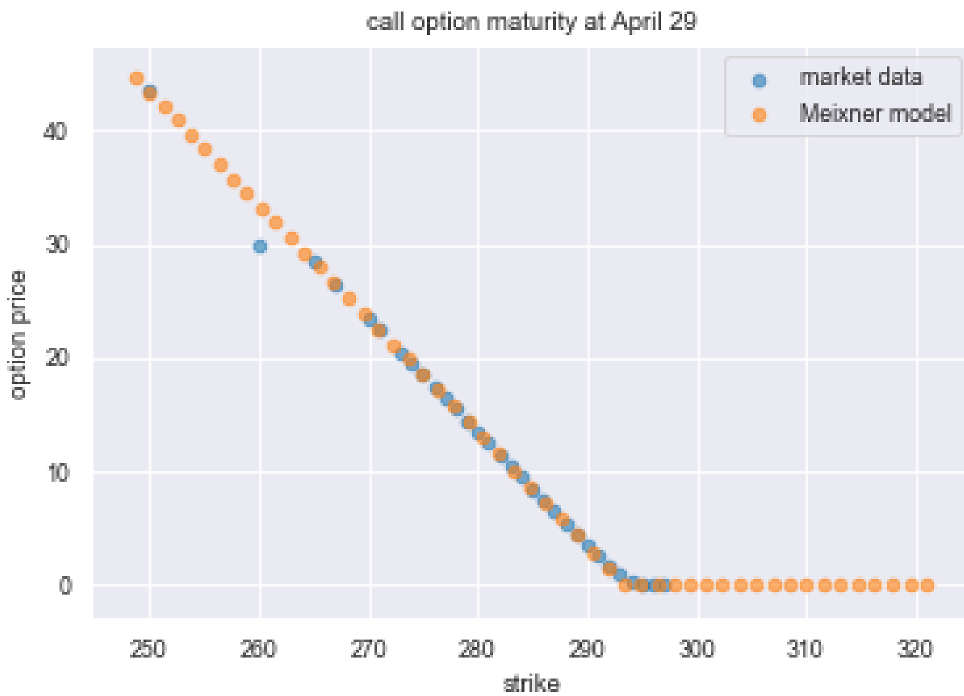
From this figure, we can find the obvious difference between PDFs under physical probability and EMM.

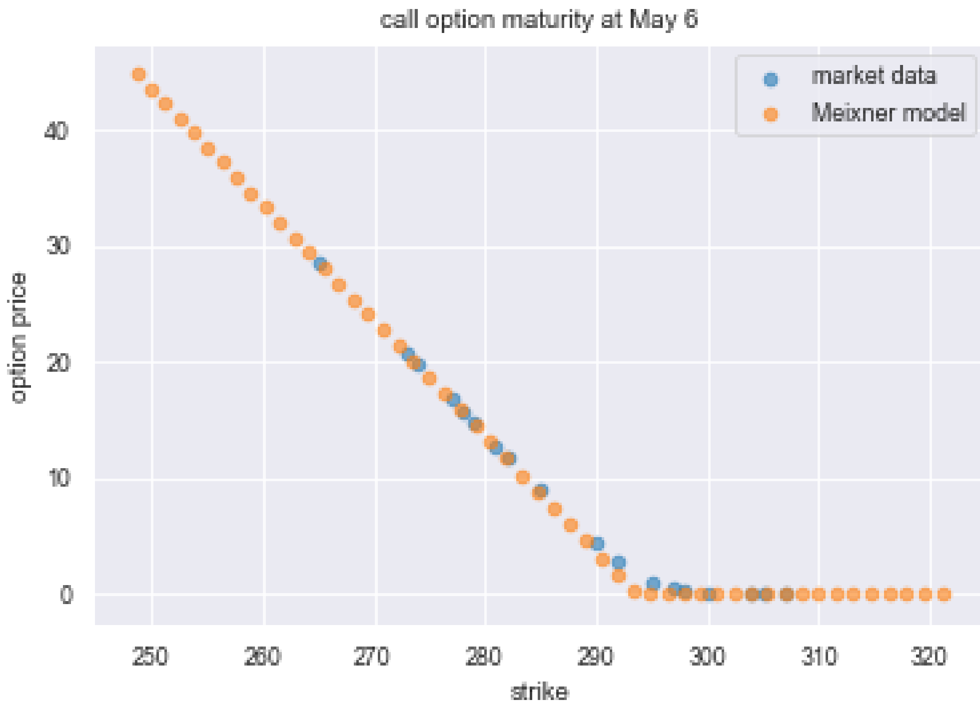
3.2 Pricing the European Call Option via FFT Method

Since we have the beautiful analytical form of the characteristic function, we can use FFT method to price the European call option. Notice the form of the CF for maturity T. It's $\phi_T(u)$ but not $\phi_1(u)$, and the m is under risk neutral measure:

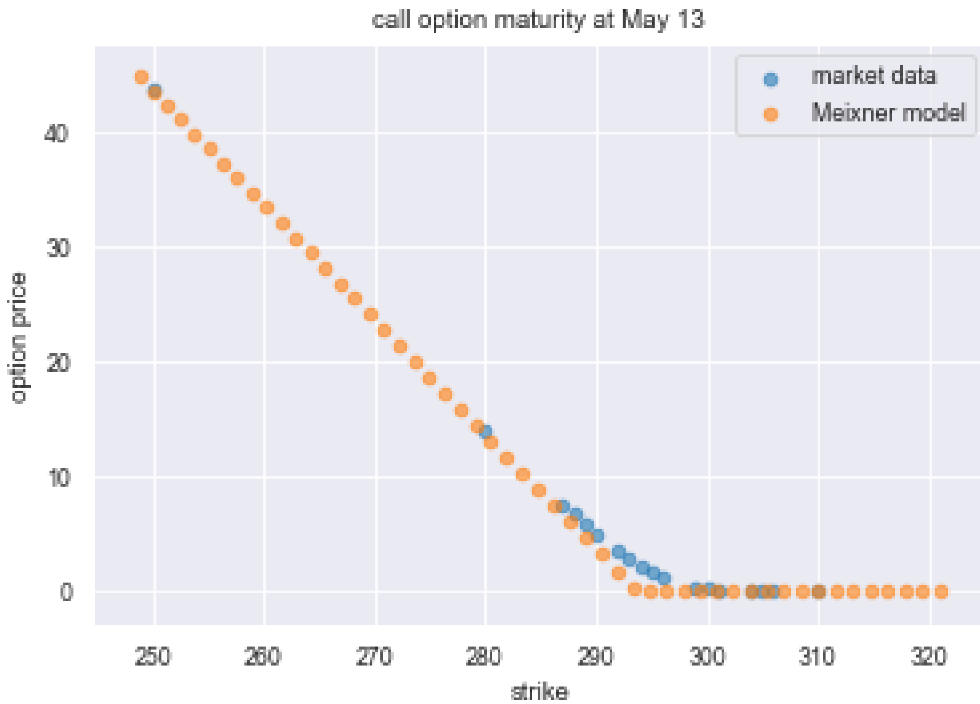
$$\phi_T(u) = E\left[\exp\left(iuM_T\right)\right] = \left(\frac{\cos\left(\frac{b}{2}\right)}{\cosh\frac{au-ib}{2}}\right)^{2dT} \exp\left(i m_{riskneutral} uT\right)$$

We show the results of the theoretical option price and market data in form of figures below:





call option maturity at May 6
market data
Meixner model
option price
strike
0
10
20
30
40
250
260
270
280
290
300
310
320


call option maturity at May 13
market data
Meixner model
option price
strike
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30
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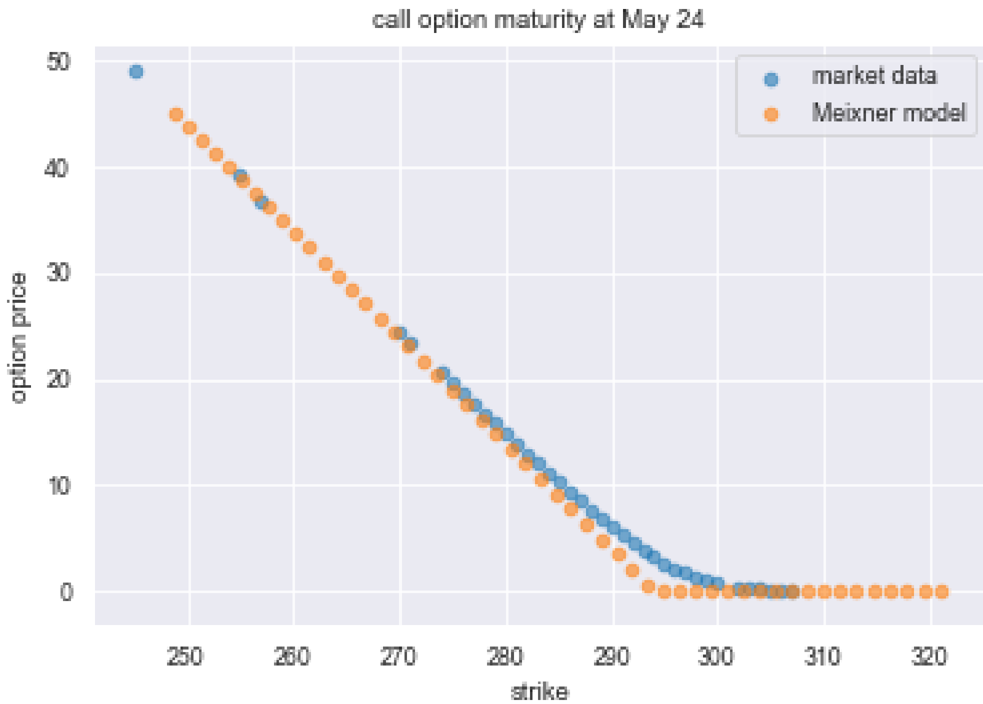
260
270
280
290
300
310
320


call option maturity at May 24
market data
Meixner model
option price
strike
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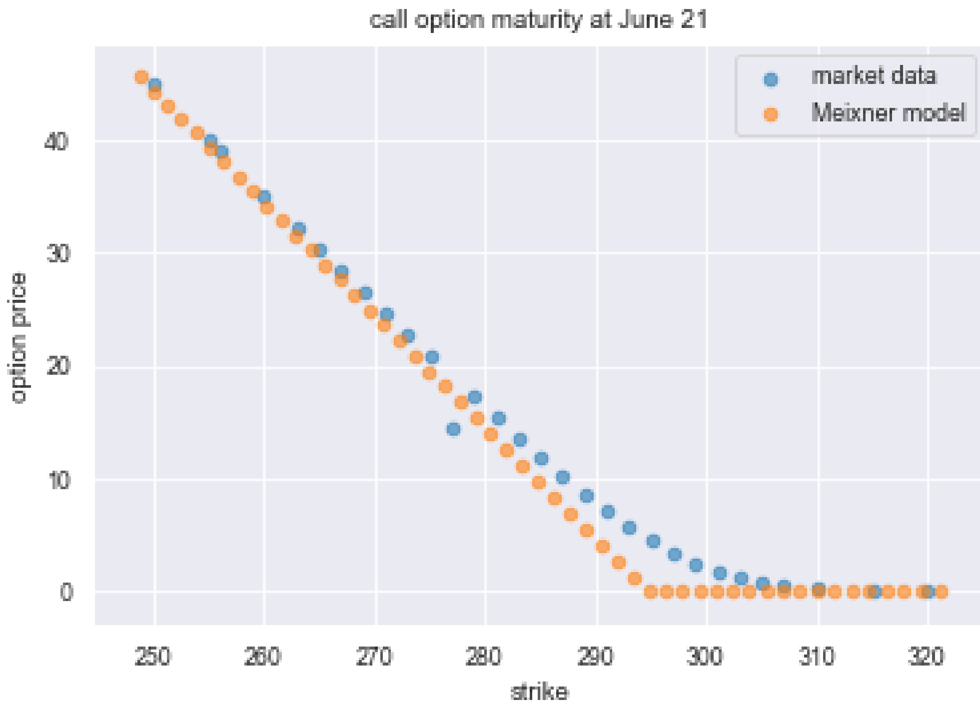
call option maturity at June 21
market data
Meixner model
option price
strike
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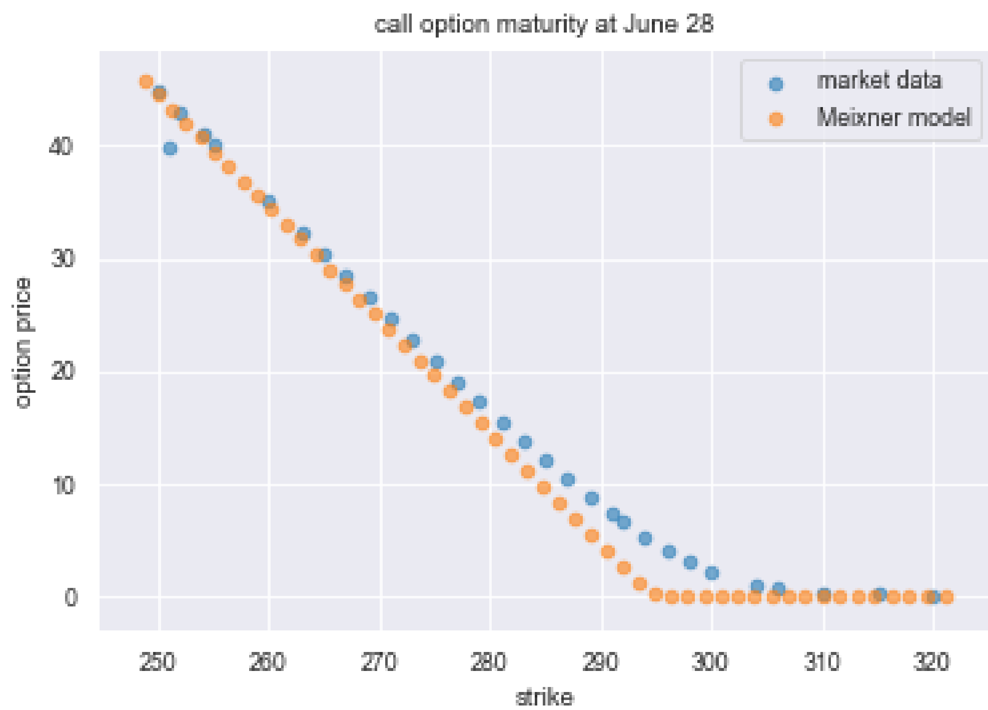

The model performs well in short time period.

3.3 Pricing the Asian Option

For geometric Asian Option, the payoff is $H_t = \left(e^{\frac{1}{T}\int_0^T \ln S_t\, dt} - K\right)^{+¿}$ Let $\acute{S}_t = e^{\frac{1}{T}\int_0^T \ln S_t\, dt}$

,the price can be written as:

$$price = E^{¿}\left[e^{-rT} H_T\right] = E^{¿}\left[e^{-rT}\acute{S}_t 1_{\{\acute{S}_T > K\}}\right] - e^{-rT} K P^{¿}\left(\acute{S}_T > K\right)$$

Where $P^{¿}\left(\acute{S}_T > K\right)$ means the probability that $\acute{S}_T > K$ under risk neutral measure. Therefore the main problem is to find the distribution of $\acute{S}_T$. Moreover, from the mean-correction method we mentioned above, under the risk neutral measure, using mean correction method, the stock price becomes:

$$S_t^{¿} = \frac{S_0 e^{rT} e^{M_t}}{E\left[e^{M_t}\right]}$$

$$\ln S_t^{¿} = \log S_0 e^{rT} e^{M_t} - \log E[e^{M_t}]$$

Let $A = \frac{1}{T}\int_0^T \ln S_t \, dt$, $E[e^{M_t}]$ can be regarded as the characteristic function at point (-i)

$$A = \frac{1}{T}\int_0^T [\ln S_0 e^{rT} + \ln e^{M_t} - \ln E[e^{M_t}]] dt$$

$$¿ \int_0^T [\ln S_0 e^{rT} + M_t - t \ln \phi_{x1}(-i)] dt$$

$$¿ \int_0^T \ln S_0 e^{rT} dt - \frac{1}{2} \ln \phi_{x1}(-i) + \frac{1}{T}\int_0^T M_t \, dt$$

$$¿ \ln S_0 + \frac{1}{2} rT - \frac{1}{2} \ln \phi_{x1}(-i) + \frac{1}{T}\int_0^T M_t \, dt$$

$$\acute{S}_T = e^A = e^{\ln S_0 + \frac{1}{2} rT - \frac{1}{2}\ln \phi_{M1}(-i) + \int_0^T M_t dt} = S_0 e^{\frac{1}{2}[rT - \ln \phi_{M1}(-i)]} e^{\frac{1}{T}\int_0^T M_t dt}$$

Now the key point is to find the closed form of $\frac{1}{T}\int_0^T M_t \, dt$ and then we can price the Asian option through the same trick we play in BS model.

$$Y_t = \int_0^t X_s \, ds = \lim_{n \to \infty} \frac{t}{n} \sum_{k=1}^{n} X_{\frac{kt}{n}}$$

Where X_t is a Levy process. let's set $L_k = X_{\frac{kt}{n}}$,

$$\sum_{k=1}^{n} X_{\frac{kt}{n}} = \sum_{k=1}^{n} L_k = n(L_1 - L_0) + (n-1)(L_2 - L_1) + \ldots + 2(L_{n-1} - L_{n-2}) + (L_n - L_{n-1}) = \sum_{k=1}^{n} k(L_k - L_{k-1}) = \sum_{k=1}^{n}$$

Therefore, $Y_t = \lim_{n \to \infty} \sum_{k=1}^{n} k\left(X_{\frac{kt}{n}} - X_{\frac{(k-1)t}{n}}\right)$

$$E\left[e^{iuY_t}\right]=E\left[e^{iu\lim_{n\to\infty}\sum_{k=1}^{n}k\left(X_{\frac{kt}{n}}-X_{\frac{(k-1)t}{n}}\right)}\right]$$

$$¿\exp\left(\log E\left[e^{\frac{iut}{n}\left(X_{\frac{t}{n}}-X_0\right)}\right]+\log E\left[e^{\frac{iu2t}{n}\left(X_{\frac{2t}{n}}-X_{\frac{t}{n}}\right)}\right]+\ldots\right)$$

$$¿\exp\left(\lim_{n\to\infty}\sum_{k=1}^{n}\log E\left[e^{\frac{iukt}{n}X_{\frac{t}{n}}}\right]\right)$$

With the infinitely divisible property of Levy process:

$$\phi_{X_t}(u)=\left[\phi_{X_1}(u)\right]^t$$

We write this Riemann sum as a definite integral:

$$E\left[e^{\frac{iu}{T}\int_0^T X_t\,dt}\right]=\exp\left\{T\int_0^1\log\phi_{X_1}(\mu z)\,dz\right\}$$

This is the characteristic function of $\frac{1}{T}\int_0^T M_t\,dt$, then we can get the distribution of

$\frac{1}{T}\int_0^T M_t\,dt$ and furthermore get the cumulative function of $S_0e^{\frac{1}{2}\left[rT-\ln\phi_{M1}(-i)\right]}e^{\frac{1}{T}\int_0^T M_t\,dt}$ to calculate $P^{¿}(\acute{S}_T>K)$.

4. Conclusion

The Meixner process with MLE parameters can approximate log return distribution with higher accuracy comparing with the BS model. In pricing the European option, Meixner model performs better in the short term than in the long term. One of the reasons may be that Meixner parameters are changing over the time period. To solve this problem, we can make the volatility to be a stochastic process.

Since Meixner is one of the Levy processes and we have derived the pricing form of Geometric Asian option in risk-neutral measure. Then we can price the Asian option with Meixner process.

5. Reference

[1] Schoutens, W., 2002. *The Meixner process: Theory and applications in finance* (pp. 2002-004). Eindhoven: Eurandom.

[2] Grigoletto, M. and Provasi, C., 2008. Simulation and estimation of the Meixner distribution. *Communications in Statistics-Simulation and Computation*, *38*(1), pp.58-77.

[3] Bertoin, J., 1996. *Lévy processes* (Vol. 121). Cambridge: Cambridge university press.

[4] Sato, K.I. and Ken-Iti, S., 1999. *Lévy processes and infinitely divisible distributions*. Cambridge university press.

[5] Grigelionis, B., 1999. Processes of Meixner type. *Lithuanian Mathematical Journal*, *39*(1), pp.33-41.